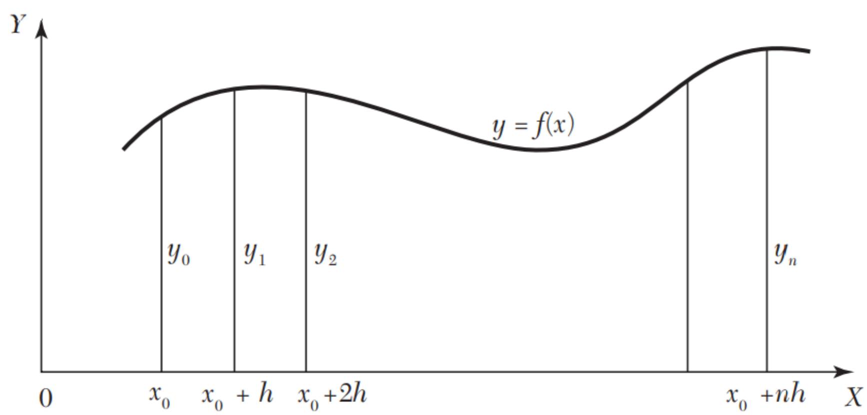
# Numerical Integration

August 23, 2021

## 1 Newton-Cotes Quadrature Formula

Let $I = \int_a^b f(x)dx$
where $f(x)$ takes the values $y_0,\ y_1,\ y_2,\ \ldots, y_n$ for $x_0,\ x_1,\ x_2,\ \ldots,\ x_n$.
Let us divide the interval $(a, b)$ into n sub-intervals of width $h$ so that $x_0 = a,\ x_1 = x_0 + h,\ x_2 = x_0 + 2h,\ \ldots,\ x_1 = x_0 + nh = b$. Then




$$I = \int_{x_0}^{x_0+nh} f(x)dx = h\int_0^n f(x_0 + ph)dp, \quad \text{Putting } x = x_0 + ph,\ dx = hdp$$
$$= h\int_0^n \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots\right]$$

[by Newton's forward interpolation formula]

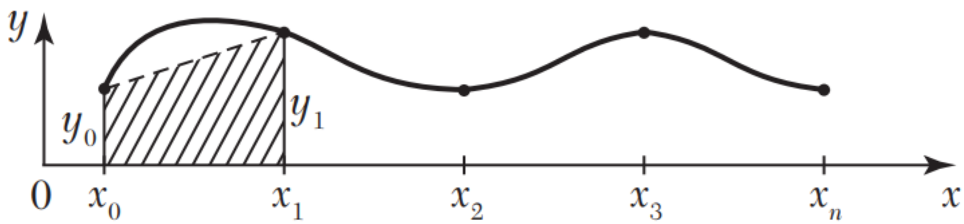
Integrating term by term, we obtain

$$I = \int_{x_0}^{x_0+nh} f(x)dx = nh\Big[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \cdots\Big] \tag{1}$$

This is known as *Newton-Cotes quadrature formula.* We obtained different integration formula for different value of $n$, where $n = 1, 2, 3, \ldots$

## 1.1 Trapezoidal rule

Putting $n = 1$ in (1) and taking the curve through $(x_0,\ y_0)$ and $(x_1,\ y_1)$as a straight line i.e., a polynomial of first order so that differences of order higher than first become zero, we get




$$\int_{x_0}^{x_1} f(x)dx = h\Big[y_0 + \frac{1}{2}\Delta y_0\Big] = \frac{h}{2}[y_0 + y_1]$$

$$\int_{x_1}^{x_2} f(x)dx = h\Big[y_1 + \frac{1}{2}\Delta y_1\Big] = \frac{h}{2}[y_1 + y_2]$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\int_{x_{n-1}}^{x_n} f(x)dx = h\Big[y_{n-1} + \frac{1}{2}\Delta y_{n-1}\Big] = \frac{h}{2}[y_{n-1} + y_n]$$

Combining all these $n$ integrals, we obtain

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}\big[y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n\big] \tag{2}$$

This is known as the *trapezoidal rule.*
**Error in Trapezoidal rule:**

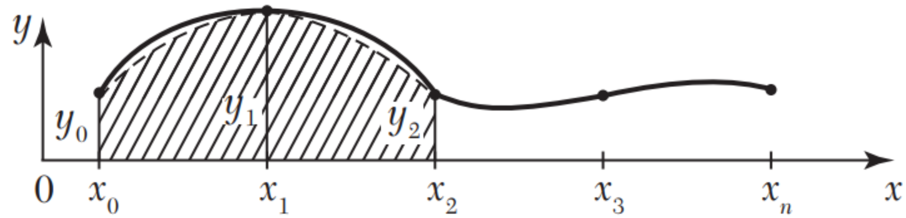
$$E^T_{trap} = -\frac{h^3}{12}f''(x), \quad a < x < b$$

Which is also called truncated error of trapezoidal rule.
**Relative error:**

$$Relative\ error = \left|\frac{I_{exact} - I_{trap}}{I_{exact}}\right| \times 100$$

## 1.2 Simpson's 1/3 rule

Putting $n = 2$ in (1) above and taking the curve through $(x_0,\ y_0),\ (x_1,\ y_1)$, and $(x_2,\ y_2)$ i.e., a polynomial of the second order so that differences of order higher than the second vanish, we get




$$\int_{x_0}^{x_2} f(x)dx = 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right] = \frac{h}{3}[y_0 + 4y_1 + y_2]$$

$$\text{Similarly} \quad \int_{x_2}^{x_4} f(x)dx = \frac{h}{3}[y_2 + 4y_3 + y_4]$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\int_{x_{n-2}}^{x_n} f(x)dx = \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n]$$

Combining all these integrals, we have when $n$ is even

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})] \tag{3}$$

This is known as the *Simpson's* 1/3 *rule* or simply *Simpson's rule* and is most commonly used.

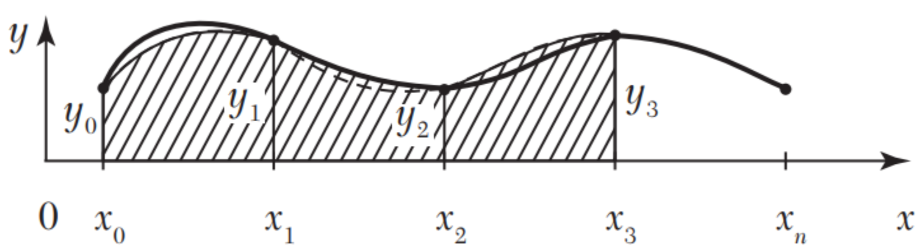
**Note:** *While applying (3), the given interval must be divided into an even number of equal subintervals, since we find the area of two strips at a time.*

**Error in Simpson's 1/3 rule.:**

$$E_{simp}^{T} = -\frac{h^5}{90} f^{iv}(x), \quad a < x < b$$

## 1.3 Simpson's 3/8 rule

Putting $n = 3$ in (1) above and taking the curve through $(x_i,\ y_i)$ : $i = 0,\ 1,\ 2,\ 3$ as a polynomial of the third order so that differences above the third order vanish, we get



$$\int_{x_0}^{x_3} f(x)dx = 3h\big[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\big]$$

$$= \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3]$$

Similarly,

$$\int_{x_3}^{x_6} f(x)dx = \frac{3h}{8}[y_3 + 3y_4 + 3y_5 + y_6]$$

and so on.
Combining all such expressions from $x_0$ to $x_n = x_0 + nh$, where $n$ is a multiple of 3, we obtain

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}\big[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots) + 2(y_3 + y_6 + \cdots)\big] \tag{4}$$

This is known as the *Simpson's* 3/8 *rule.*
**Note:** *While applying (4), the number of sub-intervals should be taken as a multiple of 3.*
**Error in Simpson's 3/8 rule.:**

$$E_{simp}^{T} = -\frac{3h^5}{80} f^{iv}(x), \quad a < x < b$$