Analysis of High Dimensional Data

Homework 1 and 2

11 March 2016

```
# First set your working directory (with data file CanadianWeather.rda),
#     and install the required packages
#setwd("~/dropbox/education/AnalysisHighDimensionalData/Homeworks1516/")
#install.packages("ldr")
```

This text contains an introduction to the data and to Functional Data Analysis. The details of the assignment can be found at the end of this text.

1. Introduction

1.1. Data

We have data on average daily rainfall (mm/day) for the 365 days in the year and for 35 Canadian cities.

Note: in this text, we use the word "rainfall" as shorthand for all forms of precipitation (rain, snow, hail, ...).

First we read the data:

```
load("CanadianWeather.rda")

da<-CanadianWeather[[1]]
da<-da[,,"Precipitation.mm"] # precipitation data
head(da)
```

```
##       St. Johns Halifax Sydney Yarmouth Charlottvl Fredericton Scheffervll
## jan01       5.2     6.0    5.3      5.6        4.6         4.0         1.1
## jan02       5.8     5.3    5.2      3.7        4.4         3.2         1.3
## jan03       3.9     2.6    2.1      2.8        2.3         3.3         1.2
## jan04       4.3     5.3    5.0      5.3        4.8         3.3         1.3
## jan05       6.2     6.0    7.3      3.8        5.1         2.7         1.0
## jan06       3.4     2.1    2.2      2.4        1.5         0.8         1.3
##       Arvida Bagottville Quebec Sherbrooke Montreal Ottawa Toronto London
## jan01    2.6         3.0    4.1        2.9      2.9    2.5     1.8    2.4
## jan02    1.2         1.8    2.3        2.9      1.2    1.1     0.9    1.4
## jan03    2.1         1.3    2.6        1.9      1.4    1.3     0.9    1.8
## jan04    2.3         2.5    4.3        2.9      3.6    3.1     1.5    2.9
## jan05    1.7         2.1    2.3        2.1      1.6    1.3     0.8    1.1
## jan06    2.0         1.6    1.5        0.8      1.1    1.3     1.0    1.4
##       Thunder Bay Winnipeg The Pas Churchill Regina Pr. Albert
## jan01         0.7      0.5     0.5       0.5    0.2        0.1
## jan02         1.9      0.6     0.9       0.6    0.3        0.9
## jan03         0.8      0.3     0.7       0.5    0.6        0.6
## jan04         0.3      0.5     0.5       0.4    0.3        0.3
## jan05         0.8      0.4     0.2       0.4    0.8        0.2
## jan06         1.7      0.7     0.9       0.2    0.5        0.3
```

```
##       Uranium City Edmonton Calgary Kamloops Vancouver Victoria Pr. George
## jan01          0.3      0.4     0.3      0.6       5.5      5.3        2.2
## jan02          0.4      0.8     0.1      0.4       6.6      5.2        1.9
## jan03          1.3      1.1     0.3      1.2       6.8      5.4        1.9
## jan04          0.6      1.1     0.6      1.3       5.1      4.5        1.8
## jan05          0.8      1.0     1.0      1.2       3.8      4.6        1.1
## jan06          1.1      0.8     0.2      0.5       2.5      2.6        1.2
##       Pr. Rupert Whitehorse Dawson Yellowknife Iqaluit Inuvik Resolute
## jan01        6.0        0.5    0.9         0.6     1.1    0.8      0.1
## jan02        5.0        0.8    0.6         0.7     0.9    0.9      0.1
## jan03        6.7        1.1    0.8         0.3     0.8    0.8      0.0
## jan04        7.1        0.2    0.8         0.5     0.7    0.4      0.2
## jan05        6.1        0.6    1.0         0.7     0.9    0.8      0.2
## jan06        8.1        0.7    1.0         0.5     0.2    0.4      0.2
```

Here is a list of the 35 cities

```
colnames(da)
```

```
##  [1] "St. Johns"    "Halifax"      "Sydney"       "Yarmouth"
##  [5] "Charlottvl"   "Fredericton"  "Scheffervll"  "Arvida"
##  [9] "Bagottville"  "Quebec"       "Sherbrooke"   "Montreal"
## [13] "Ottawa"       "Toronto"      "London"       "Thunder Bay"
## [17] "Winnipeg"     "The Pas"      "Churchill"    "Regina"
## [21] "Pr. Albert"   "Uranium City" "Edmonton"     "Calgary"
## [25] "Kamloops"     "Vancouver"    "Victoria"     "Pr. George"
## [29] "Pr. Rupert"   "Whitehorse"   "Dawson"       "Yellowknife"
## [33] "Iqaluit"      "Inuvik"       "Resolute"
```

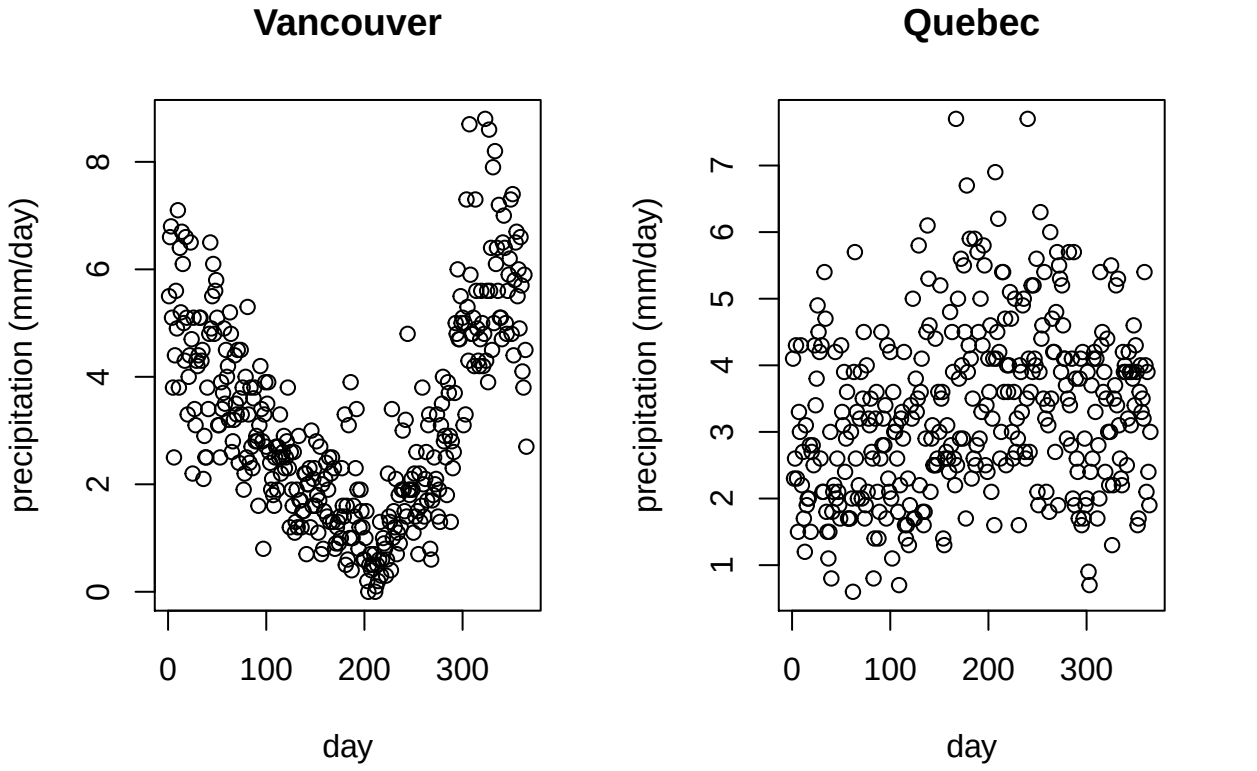
The data set also contains extra information about the cities. For example, the regions, provinces and the coordinates are included.

```
MetaData<-data.frame(city=colnames(da), region=CanadianWeather$region,
                     province=CanadianWeather$province, coord=CanadianWeather$coordinates)
head(MetaData)
```

```
##                    city   region      province coord.N.latitude
## St. Johns     St. Johns Atlantic  Newfoundland           47.34
## Halifax         Halifax Atlantic   Nova Scotia           44.39
## Sydney           Sydney Atlantic   Nova Scotia           46.09
## Yarmouth       Yarmouth Atlantic   Nova Scotia           43.50
## Charlottvl   Charlottvl Atlantic       Ontario           42.48
## Fredericton Fredericton Atlantic New Brunswick           45.58
##             coord.W.longitude
## St. Johns               52.43
## Halifax                 63.36
## Sydney                  60.11
## Yarmouth                66.07
## Charlottvl              80.25
## Fredericton             66.39
```

I show you the data of Vancouver and Quebec.

```
par(mfrow=c(1,2))
plot(1:365,da[,"Vancouver"], main="Vancouver", xlab="day", ylab="precipitation (mm/day)")
plot(1:365,da[,"Quebec"], main="Quebec", xlab="day", ylab="precipitation (mm/day)")
```




```
par(mfrow=c(1,1))
```

1.2. Research Question

The objective is to discover which cities have similar precipitation patterns, and which have dissimilar patterns. We need a 2-dimensional graph that shows each city as a point, such that cities with similar precipitation patters are close to one another. We also want to understand the difference in rainfall patterns: in what sense do they differ (e.g overall more rainfall in Western Canada, or less rain in summer, ...).

1.3. Functional Data Analysis

1.3.1. Introduction

From the research question we conclude that a MDS is an appropriate method.

On way of looking at the data is to consider it as a multivariate data set with $n = 35$ rows (cities) and $p = 365$ columns (days). However, this is not the approach that we take. In this section I will introduce a functional data analysis (FDA) approach.

In FDA we typically consider functions as observations. For example, for each of the 35 cities we have observations on a precipitation function. To take this approach it is necessary for each city to first convert the 365 data entries to a single function. This function will contain parameter estimates (typically less than the

original number of data entries, say $q < p = 365$). Thus each city will have its set of q parameter estimates, and thus an $n \times q$ data matrix can be constructed. These parameter estimates form now the input for the MDS. To give a meaningful interpretation to the results, at the end we need to back-transform our solution from the parameter space to the function space.

1.3.2. Transformation to functions

Let $Y_i(t)$ denote the outcome of observation $i = 1, \ldots, n$ (here: average daily rainfall) at time $t \in [1, 365]$. For observation i we have data on times t_{ij}, $j = 1, \ldots, p_i$. This setting thus allows for irregularly spaces measurements: each observation i may come with its own set of time points t_{ij} with measurements $Y_i(t_{ij})$. Our data set, however, contains regularly spaced data: for each city i precipitations are available for $t = 1, 2, \ldots, 365$.

Consider the non-linear statistical model

$$Y_i(t_{ij}) = f_i(t_{ij}) + \varepsilon_{ij} \quad i = 1, \ldots, n; j = 1, \ldots, p_i$$

for some smooth functions $f_i(\cdot)$ and with ε_{ij} i.i.d. with mean 0 and constant variance σ^2. If $f_i(\cdot)$ is a parametric function, then non-linear least squares can be used to estimate the parameters.

Functional approximation theory (mathematical theory) says that a sufficiently smooth function $f(\cdot)$ can be arbitrary well approximated by the expansion

$$f_i(t) = \sum_{k=0}^{\infty} \theta_{ik} \phi_k(t)$$

where the θ_{ik} are parameters, and the $\phi_k(\cdot)$ form an set of orthonormal basis functions. In practice the expansion (sum) is truncated after a few terms, say m. For this homework it is not important to exactly understand the meaning of an orthonormal basis. Classical examples of orthonormal basis functions: Fourier functions, polynomial basis functions, wavelet functions. For this homework you may choose between a Fourier or a polynomial basis.

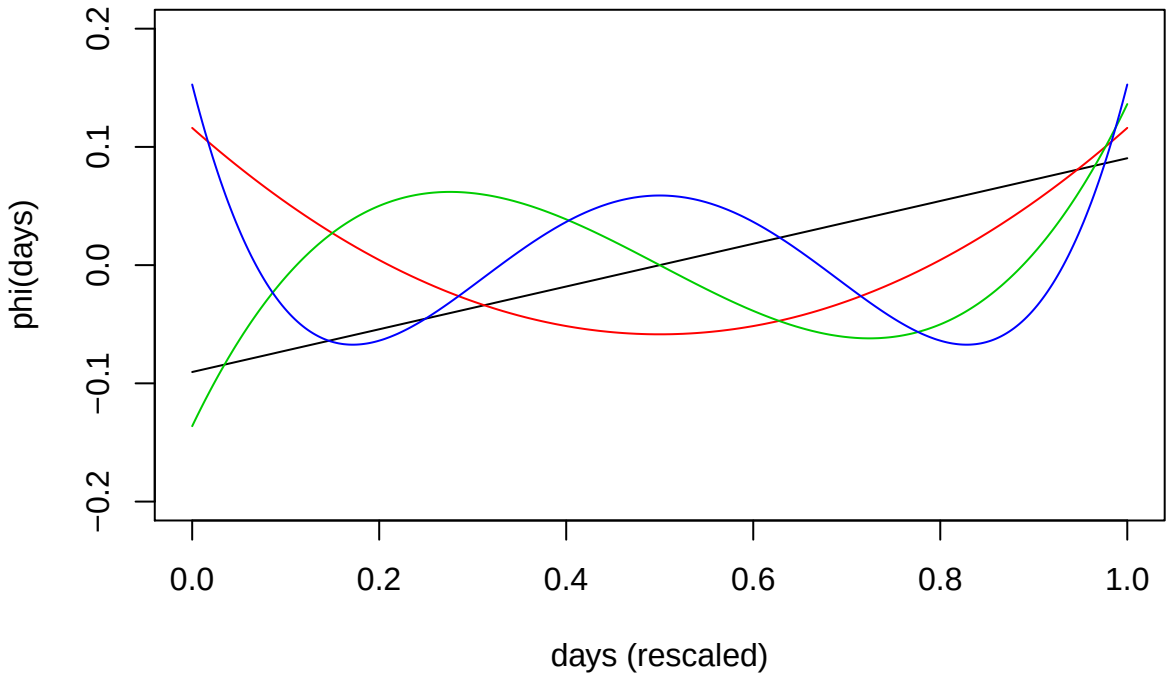
Functions from a polynomial basis are of the form

$$\phi_k(t) = \sum_{j=0}^{k} \alpha_j t^j,$$

i.e. $\phi_k(\cdot)$ is a polynomial function in t of degree k. The α_j's are constants (no need to estimate).

The next chunck of R code illustrates the generation of a polynomial basis. Note that I rescaled the 365 days to the $[0, 1]$ interval. This is only to avoid numerical problems when computing days^j for a large j.

```
days<-1:365
days<-(days-min(days))/(diff(range(days))) # rescaling to [0,1]
phi<-poly(days,degree=4)
# the i-th column of phi contains the i-th order
#    polynomial evaluated in "days"

plot(days,phi[,1],type="l",ylim=c(-0.2,0.2), xlab="days (rescaled)", ylab="phi(days)")
lines(days,phi[,2],type="l",col=2)
lines(days,phi[,3],type="l",col=3)
lines(days,phi[,4],type="l",col=4)
```




```
# the next line demonstrates that the polynomials are normalised
#    (no need to exactly understand this)
colMeans(scale(phi,center=T,scale=F)^2)
```

```
##           1           2           3           4
## 0.002739726 0.002739726 0.002739726 0.002739726
```

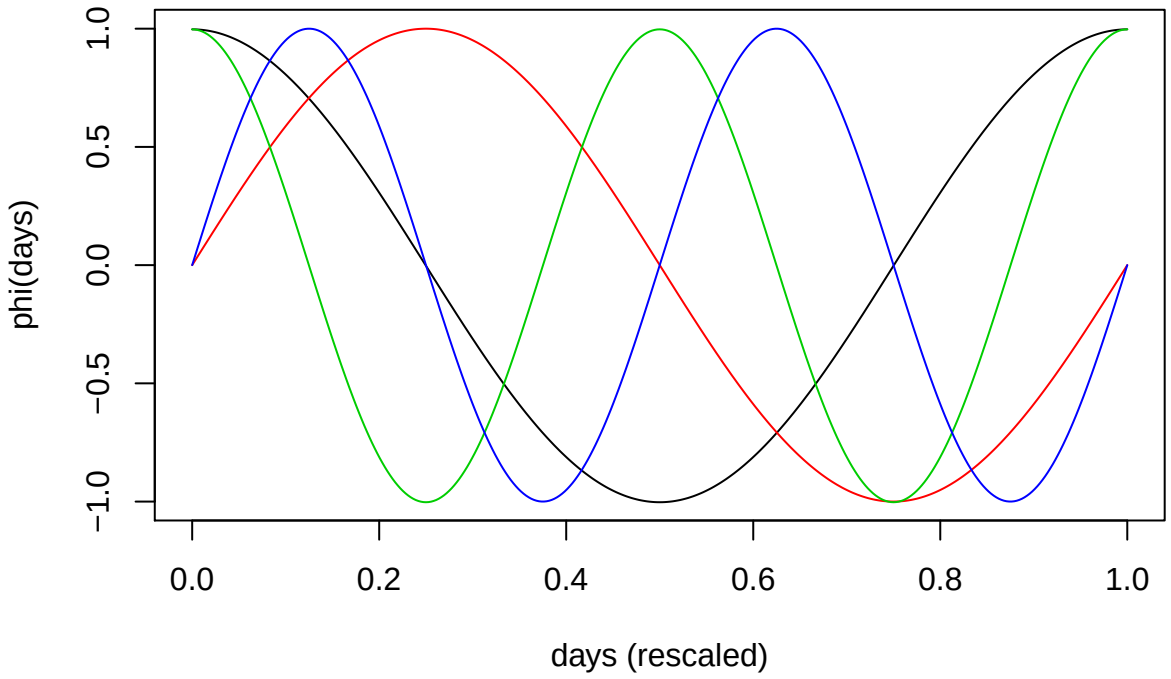
Now an example with the Fourier basis.

```
library(ldr) # needed for the bf function for generation of Fourier basis
```

```
## Loading required package: GrassmannOptim
```

```
## Loading required package: Matrix
```

```
phi<-bf(days,case="fourier",degree=4)
# the i-th column of phi contains the i-th order
#    Fourier basis function evaluated in "days"

plot(days,phi[,1],type="l",ylim=c(-1,1), xlab="days (rescaled)", ylab="phi(days)")
lines(days,phi[,2],type="l",col=2)
lines(days,phi[,3],type="l",col=3)
lines(days,phi[,4],type="l",col=4)
```




```
# the next line demonstrates that the polynomials are normalised
#    (no need to exactly understand this)
colMeans(scale(phi,center=T,scale=F)^2)
```

```
## [1] 0.5013624 0.4986301 0.5013624 0.4986301 0.5013624 0.4986301 0.5013624
## [8] 0.4986301
```

The R code has demonstrated that the poly and bf functions transform a vector with the days at which measurements are available, to a matrix. For a given city (i.e. for a given i), the number of rows of the matrix equals the number of days, and each column corresponds to a basis function. The (j,k)th element of the matrix equals the kth basis function evaluated in the jth day t_{ij}. Let $x_{ijk} = \phi_k(t_{ij})$ denote this element.

We now rewrite the statistical model for $Y_i(t_{ij})$,

$$Y_i(t_{ij}) = \sum_{k=0}^{m} \theta_{ik}\phi_k(t_{ij}) + \varepsilon_{ij}$$

or

$$Y_i(t_{ij}) = \sum_{k=0}^{m} \theta_{ik}x_{ijk} + \varepsilon_{ij}.$$

For a given i, this has the structure of a linear regression model with outcomes $Y_i(t_{ij})$, $j = 1, \ldots, n$, and $q = m + 1$ regressors x_{ijk}.

Finally, we write the statistical model for city i in matrix notation.

$$\boldsymbol{Y}_i = \boldsymbol{X}_i\boldsymbol{\theta}_i + \boldsymbol{\varepsilon}_i$$

with $\boldsymbol{Y}_i$ the vector with the outcomes of observation i (one for each day t_{ij}), $\boldsymbol{\theta}_i$ the vector with the θ_{ik} (one for each basis function k), $\boldsymbol{X}_i$ the matrix with the x_{ijk} (days j in the rows, basis function index k in columns), and $\boldsymbol{\varepsilon}_i$ the vector with the i.i.d. error terms.

The parameters θ_{ik} can be estimated by means of least squares. This is illustrated in the next R code for a single city (i.e. a single i). The process need to be repeated for all cities $i = 1, \ldots, n$. Once the parameters are estimated they can be used to plot the fitted function.

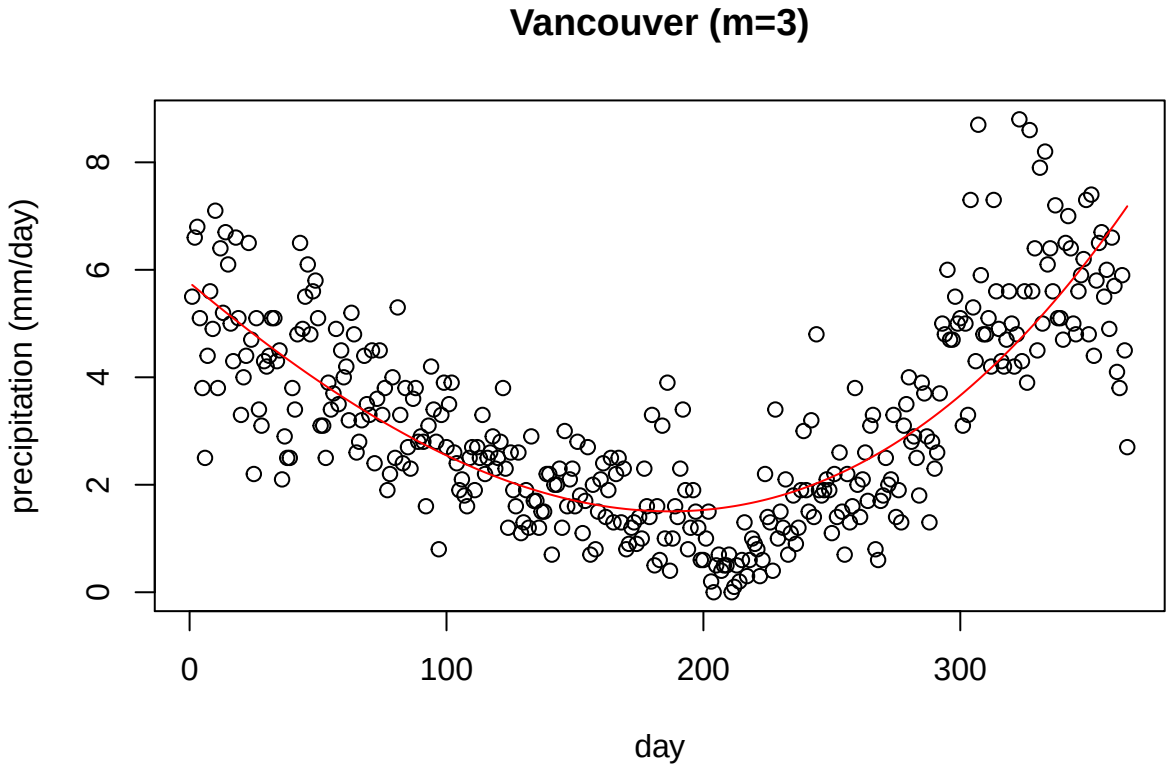
```
# Using the polynomial basis functions up to degree 3

days<-1:365
days<-(days-min(days))/(diff(range(days))) # rescaling to [0,1]
phi<-poly(days,degree=3)
dim(phi)
```

```
## [1] 365   3
```

```
# estimation of the theta parameters for Vancouver
m.Vancouver<-lm(da[,"Vancouver"]~phi)
summary(m.Vancouver)
```

```
##
## Call:
## lm(formula = da[, "Vancouver"] ~ phi)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -4.4831 -0.7312 -0.1068  0.6815  4.7459
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   3.1647     0.0592  53.455  < 2e-16 ***
## phi1          3.9234     1.1311   3.469 0.000586 ***
## phi2         28.3115     1.1311  25.031  < 2e-16 ***
## phi3          2.7691     1.1311   2.448 0.014833 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.131 on 361 degrees of freedom
## Multiple R-squared:  0.641,  Adjusted R-squared:  0.638
## F-statistic: 214.9 on 3 and 361 DF,  p-value: < 2.2e-16
```

```
# plot of fitted function
plot(1:365,da[,"Vancouver"],main="Vancouver (m=3)" ,xlab="day", ylab="precipitation (mm/day)")
lines(1:365,m.Vancouver$fitted.values,type="l", col=2)
```




```
# Using the polynomial basis functions up to degree 10

days<-1:365
days<-(days-min(days))/(diff(range(days))) # rescaling to [0,1]
phi<-poly(days,degree=10)
dim(phi)
```

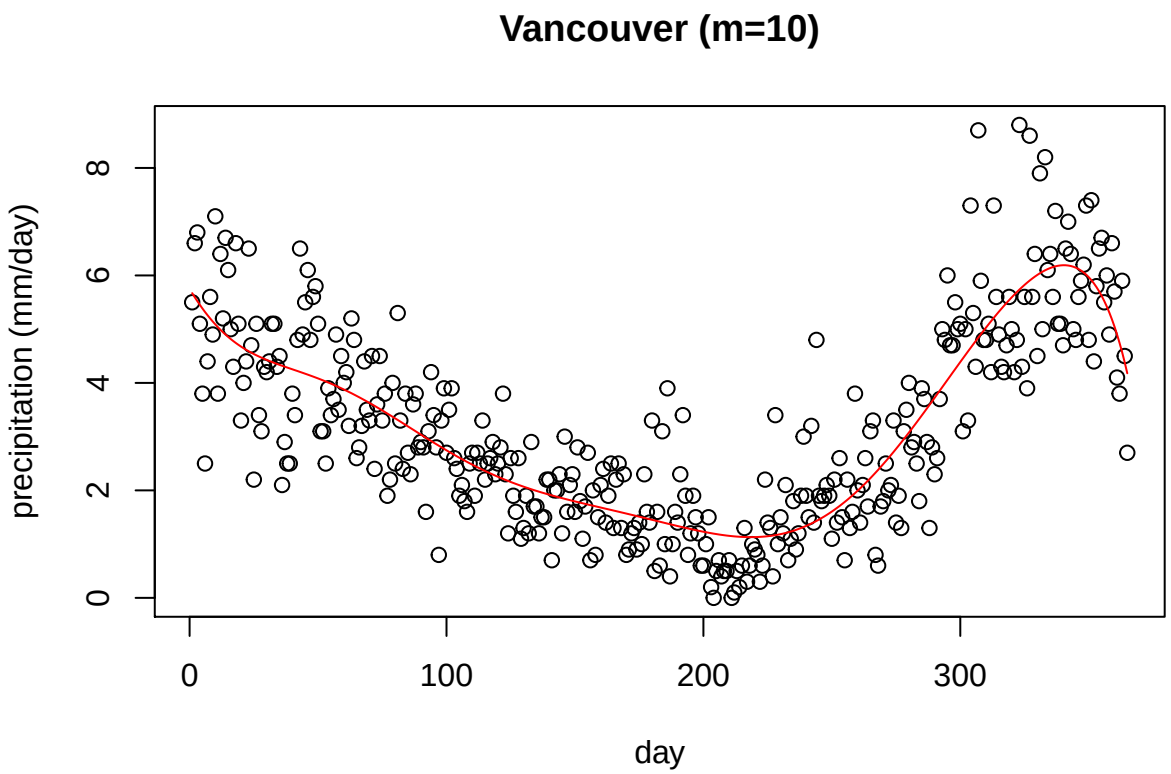
```
## [1] 365  10
```

```
# estimation of the theta parameters for Vancouver
m.Vancouver<-lm(da["Vancouver"]~phi)
summary(m.Vancouver)
```

```
##
## Call:
## lm(formula = da[, "Vancouver"] ~ phi)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -2.8201 -0.6551 -0.0621  0.5794  3.8571
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   3.1647     0.0523  60.511  < 2e-16 ***
## phi1          3.9234     0.9992   3.927 0.000104 ***
## phi2         28.3115     0.9992  28.335  < 2e-16 ***
## phi3          2.7691     0.9992   2.771 0.005878 **
```

```
## phi4         -6.7529     0.9992  -6.758 5.76e-11 ***
## phi5         -6.9005     0.9992  -6.906 2.32e-11 ***
## phi6         -3.4009     0.9992  -3.404 0.000741 ***
## phi7         -1.2774     0.9992  -1.278 0.201913
## phi8          1.2035     0.9992   1.205 0.229201
## phi9         -0.4364     0.9992  -0.437 0.662548
## phi10        -0.6000     0.9992  -0.601 0.548537
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.9992 on 354 degrees of freedom
## Multiple R-squared:  0.7253, Adjusted R-squared:  0.7175
## F-statistic: 93.46 on 10 and 354 DF,  p-value: < 2.2e-16
```

```
# plot of fitted function
plot(1:365,da[,"Vancouver"],main="Vancouver (m=10)", xlab="day", ylab="precipitation (mm/day)")
lines(1:365,m.Vancouver$fitted.values,type="l", col=2)
```



The R code demonstrate that the quality of the fit improves with increasing number of basis functions. Later in the course we will see methods for choosing an appropriate number of basis functions. For this homework I suggest that you choose an $m \leq 20$ that makes sense to you. I'm not claiming that this is a good method, but for now this will do.

1.3.3. Multidimensional Scaling of Functions

Let $\hat{\boldsymbol{\theta}}_i$ denote the vector with the parameter estimates. Given the set of basis functions, $\hat{\boldsymbol{\theta}}_i$ contain the information (shape) of the precipitation functions for observation i. The estimates for all cities can be collected into a single new $n \times (m+1)$ data matrix $\boldsymbol{\Theta}$. The ith row of $\boldsymbol{\Theta}$ is $\hat{\boldsymbol{\theta}}_i^t$. The matrix $\boldsymbol{\Theta}$ contains now all information on the shape of the precipitation functions. Since $\boldsymbol{\Theta}$ has the structure of an ordinal data

matrix, classical multivariate techniques can be applied to it. We will apply a MDS to $\boldsymbol{\Theta}$, so that we can construct a 2-dimensional plot with each point representing a city. The distances between the points in the 2-dimensional MDS space are approximations of the distances between the rows of $\boldsymbol{\Theta}$, and hence can be interpreted as distances between the precipitation functions.

The MDS thus starts from the truncated SVD of $\boldsymbol{\Theta}$,

$$\boldsymbol{\Theta}_k = \boldsymbol{U}_k \boldsymbol{D}_k \boldsymbol{V}_k^t$$

(note that the index k now refers to the number of components in the truncated SVD and not to the index of the basis functions).

2. Assignment of HW1

For this homework you are asked to perform a MDS on the average rainfull functions for the Canadian cities. In particular:

- choose a set of basis functions (polynomial of Fourier basis with some $m \leq 20$)
- transform the data series for each Canadian city to a fitted function (estimated θ parameteres)
- collect all estimated θ parameters into a $\boldsymbol{\Theta}$ matrix
- perform a MDS on the $\boldsymbol{\Theta}$ matrix
- make an informative graph
- interprete the graph (you can e.g. use the metadata)

You are asked to present your results as an R markdown file. Briefly comment your R code, motivate your choices and provide interpretation of your final results.

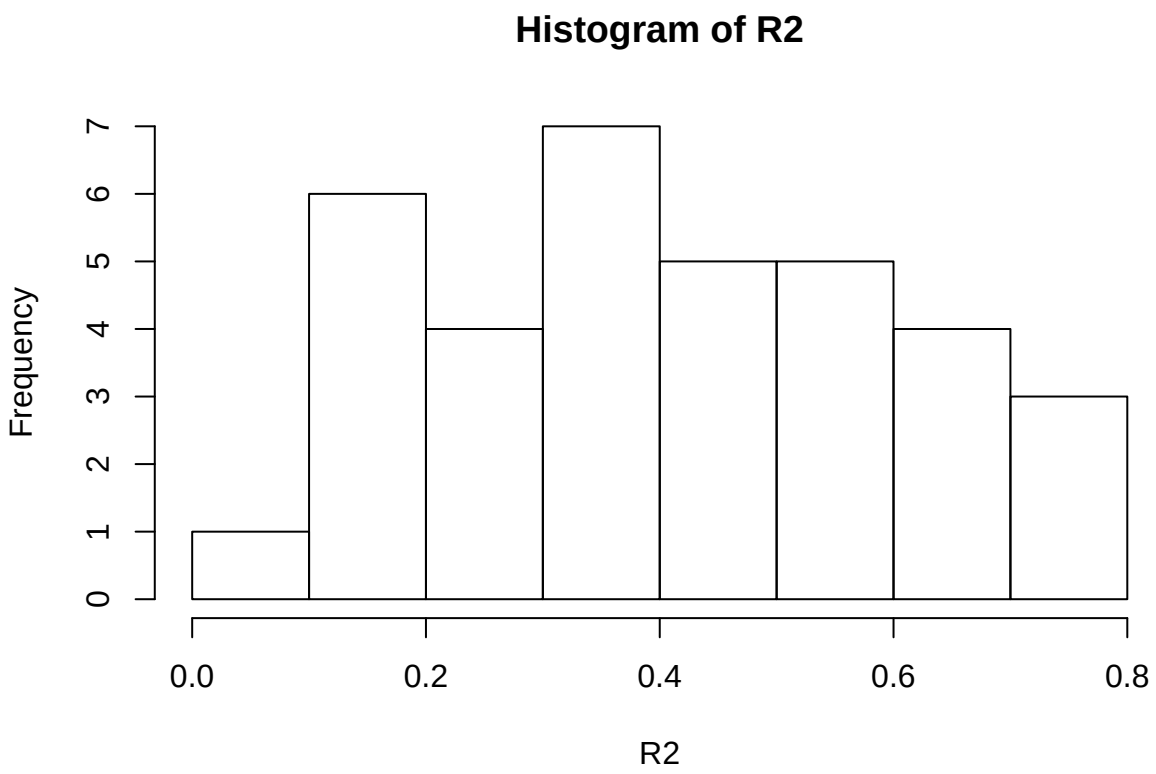
There is an R package that contains functions for an FDA. However, you are explicitly asked NOT to use this package. Using the package will not give you deep insight into how an FDA works.

3. Feedback

We give only one possible solution; other approaches are certainly possible.

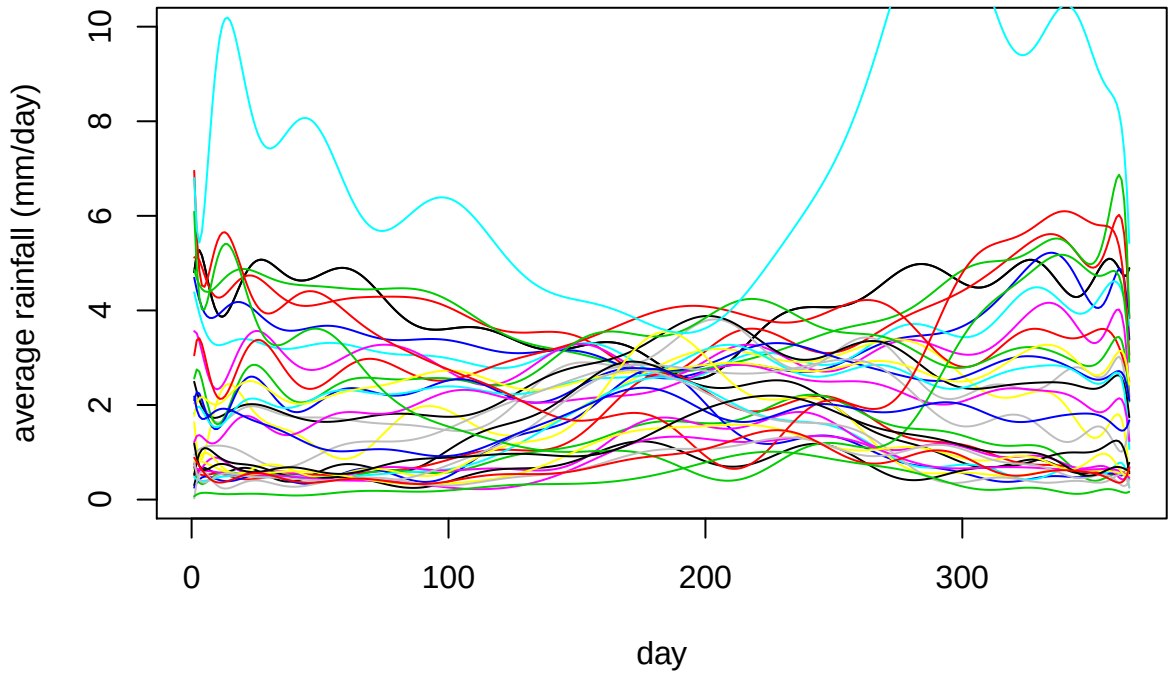
We will use a polynomial basis with m=20:

```
m<-20
days<-1:365
days<-(days-min(days))/(diff(range(days))) # rescaling to [0,1]
#phi<-bf(days,case="poly",degree=m,scale=T)
phi<-poly(days,degree=m)

models<-list()
Theta<-matrix(nrow=dim(da)[2],ncol=m+1)
R2<-c()
for(i in 1:dim(da)[2]) {
  models[[i]]<-lm(da[,i]~phi)
  Theta[i,]<-models[[i]]$coefficients
  R2[i]<-summary(models[[i]])$r.squared
}
hist(R2)
```



From the histogram of the R^2-values, it is clear that there is a large variation between cities: while for some cities the calendar explains up to 80% of the variation in rainfall, for 7 cities this is less than 20%.

```
plot(days*364+1,models[[1]]$fitted.values,type="l", ylim=c(0,10),
     xlab="day", ylab="average rainfall (mm/day)")
for(i in 1:dim(da)[2]) {
  lines(days*364+1,models[[i]]$fitted.values,col=i)
}
```

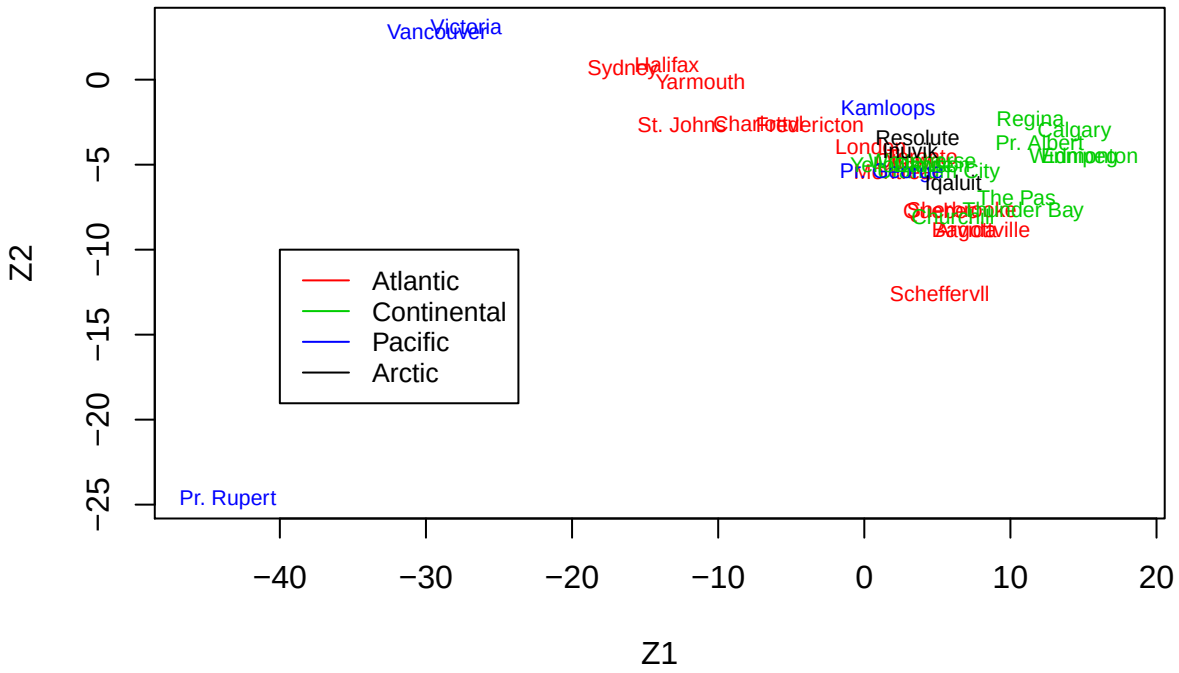


First we look at the results of the SVD using all 20 θ's.

```
Theta.svd<-svd(Theta)
k<-2
Uk<-Theta.svd$u[,1:k]
Dk<-diag(Theta.svd$d[1:k])
Zk<-Uk%*%Dk
rownames(Zk)<-colnames(da)
head(Zk)
```

```
##                   [,1]       [,2]
## St. Johns   -12.482351 -2.6083401
## Halifax     -13.508995  0.8824985
## Sydney      -16.523963  0.5846116
## Yarmouth    -11.217010 -0.1122870
## Charlottvl   -7.249574 -2.5971748
## Fredericton  -3.717962 -2.6255353
```

```
plot(Zk,type="n",xlab="Z1",ylab="Z2",xlim=c(-46,18))
text(Zk,rownames(Zk),cex=0.7, col=as.numeric(MetaData$region))
legend(-40,-10,unique(MetaData$region), col=as.numeric(unique(MetaData$region)),
       cex=0.8, lty=1)
```

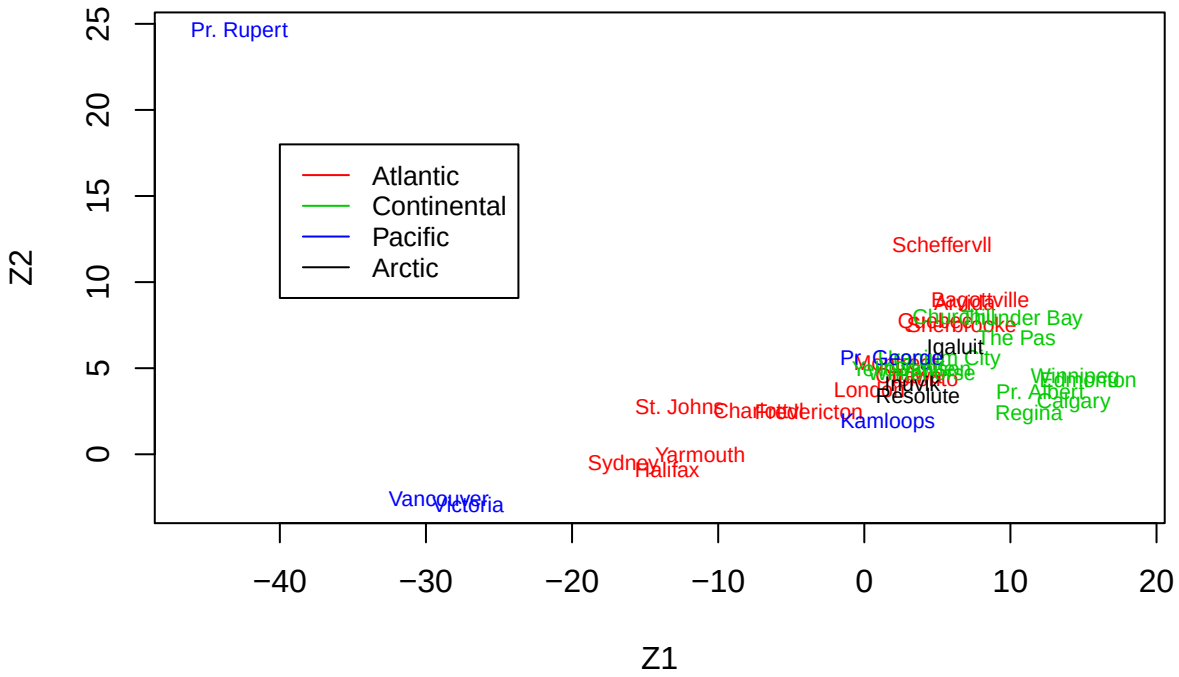


Now based on the first 10 θ's.

```
Theta.svd<-svd(Theta[,1:11])
k<-2
Uk<-Theta.svd$u[,1:k]
Dk<-diag(Theta.svd$d[1:k])
Zk<-Uk%*%Dk
rownames(Zk)<-colnames(da)
head(Zk)
```

```
##                   [,1]        [,2]
## St. Johns   -12.631885  2.76847586
## Halifax     -13.484904 -0.88879732
## Sydney      -16.490555 -0.60276335
## Yarmouth    -11.227157 -0.03065501
## Charlottvl   -7.233493  2.55336381
## Fredericton  -3.782242  2.50799036
```

```
plot(Zk,type="n",xlab="Z1",ylab="Z2",xlim=c(-46,18))
text(Zk,rownames(Zk),cex=0.7, col=as.numeric(MetaData$region))
legend(-40,18,unique(MetaData$region), col=as.numeric(unique(MetaData$region)),
       cex=0.8, lty=1)
```



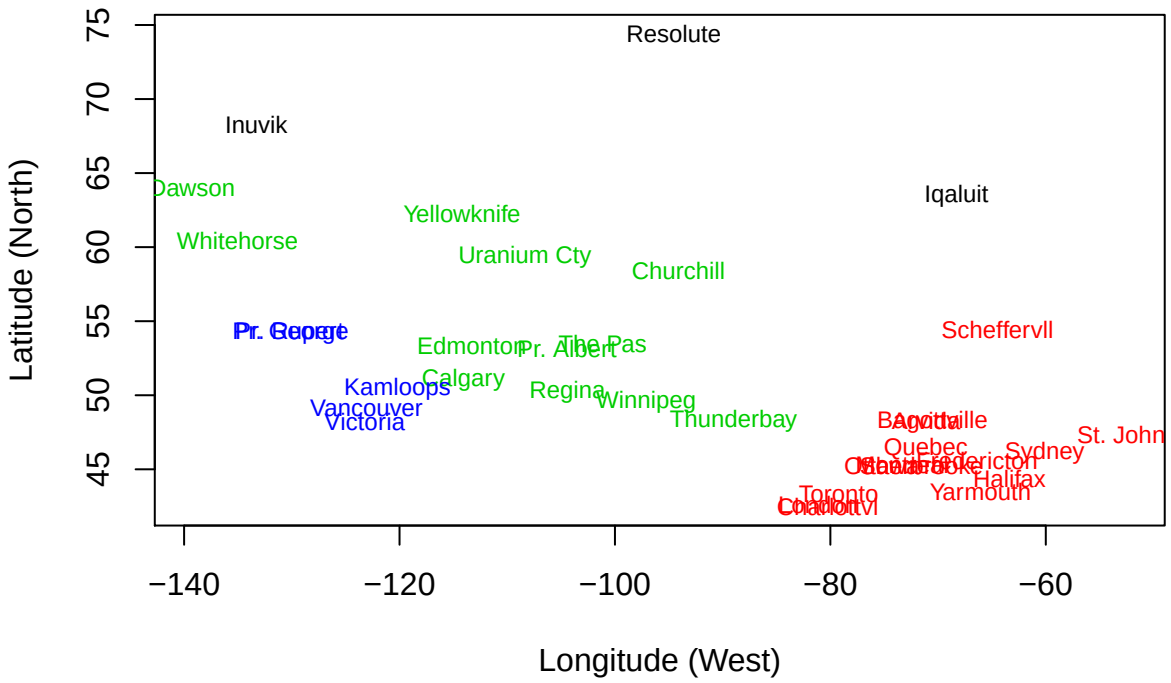
Note that except for a sign change in Z2, the result (in terms of the truncated SVD with k=2) is almost identical to the one based on all 20 θ's. This seems to indicate that almost all trend information is captured by the first 10 polynomial basis functions.

3.1. Interpretation of Z1

From the plots above, it is clear that Z1 corresponds largely to the division in regions, with the central, Continental region to the right and the coastal regions (Atlantic and Pacific) to the left. We observe that the Arctic region is somewhat in the middle, but also that the Pacific region is very heterogenous in these plots.

Before we continue, let us consider the geographical location of the 35 cities under consideration. From the included data, we can construct a simple map of Canada (compare it to a real map!):

```
cities <- cbind(-MetaData$coord.W.longitude,
                MetaData$coord.N.latitude)
rownames(cities) <- rownames(MetaData)
plot(cities, type="n", xlab="Longitude (West)", ylab="Latitude (North)")
text(cities, rownames(cities), cex=0.75, col=as.numeric(MetaData$region))
```



We look in more detail at the cities of the Pacific region. It seems strange that Pr.George and Pr.Rupert are geographically very close to each other, whereas they are very different in the Z1/Z2-space. Requesting their coordinates:

```
cities[rownames(cities)=="Pr. George"]
```

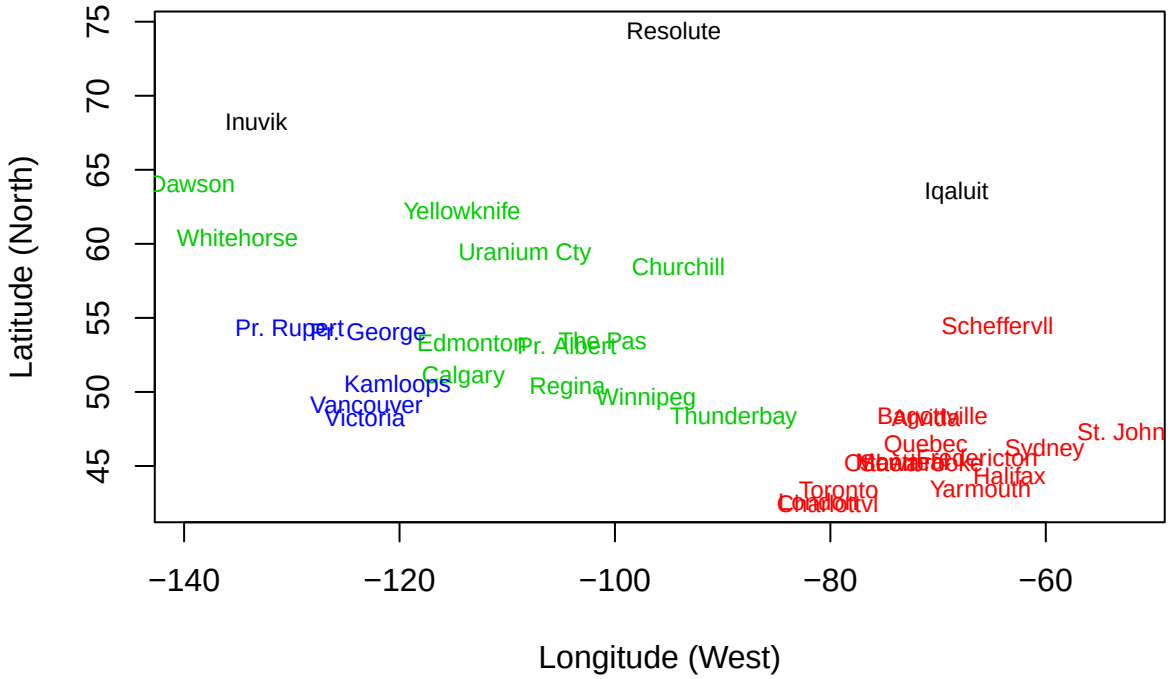
```
## [1] -130.11   54.20
```

```
cities[rownames(cities)=="Pr. Rupert"]
```

```
## [1] -130.19   54.19
```

and comparing them to the coordinates from e.g. Google Maps, shows that the included coordinates for Pr.George are incorrect. We rectify this, and redraw the map:

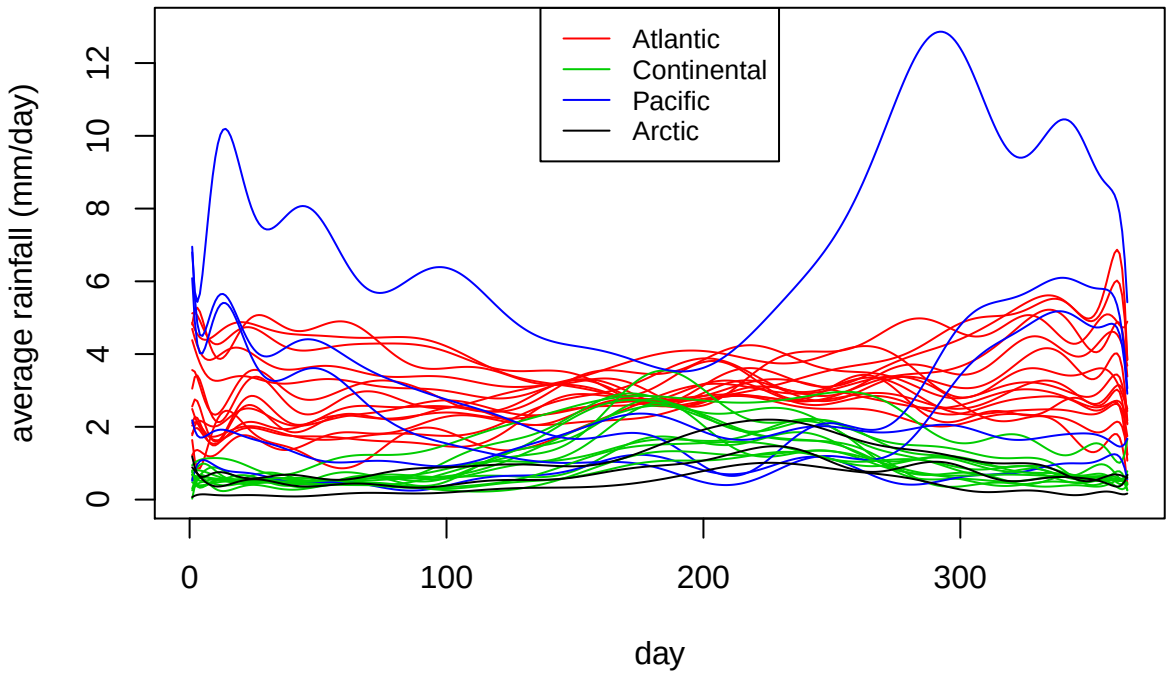
```
cities[rownames(cities)=="Pr. George"] <- c(-122.89, 53.93)
# corrected map
plot(cities, type="n", xlab="Longitude (West)", ylab="Latitude (North)")
text(cities, rownames(cities), cex=0.75, col=as.numeric(MetaData$region))
```



It is now apparent that the 2 locations in the Rocky Mountains (Kamloops and Pr.George) have a rainfall pattern more resembling that of the cities in the Continental region than that of the cities on the Pacific coastline (Vancouver, Victoria and Pr.Rupert). Note that the region variable has not necessarily been defined based on rainfall patterns!

Now that it is clear that the cities on (or close to) the Atlantic and Pacific coastlines are to the left of Z1, and cities in the Continental centre are to the right of Z1, a logical explanation might be that Z1 corresponds negatively with the total amount of rainfall (i.e. low Z1 equals high total rainfall). We will check this by redrawing a previous plot, this time grouped by region:

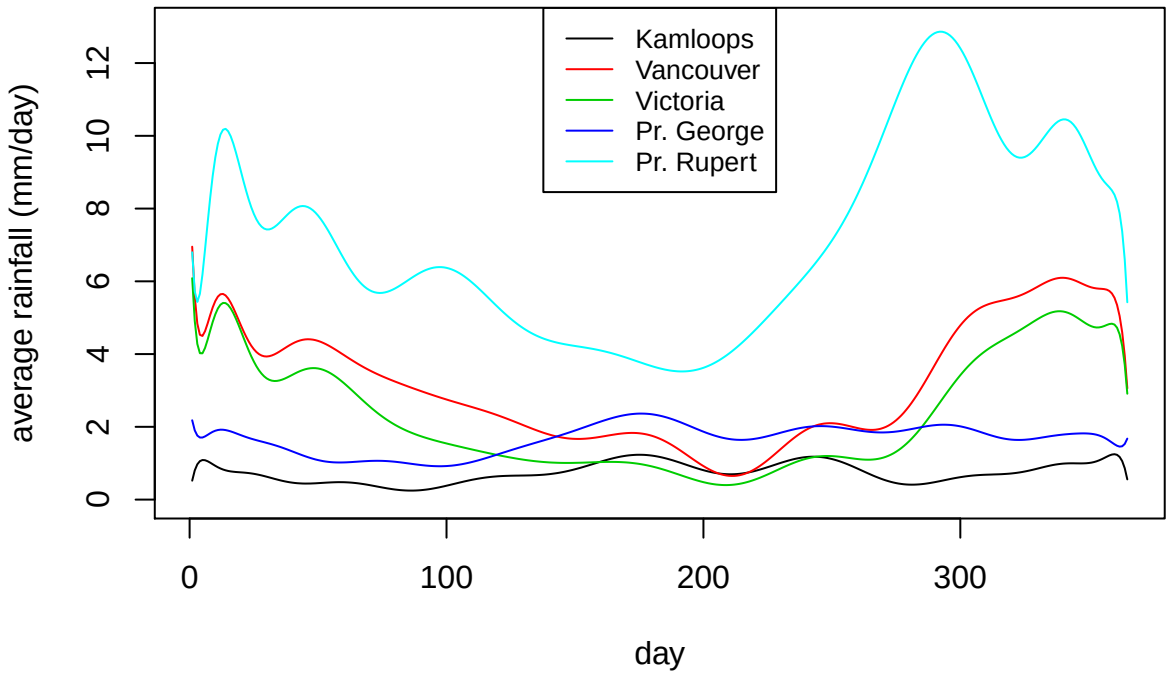
```
plot(days*364+1,models[[1]]$fitted.values,type="n", ylim=c(0,13),
     xlab="day", ylab="average rainfall (mm/day)")
for(i in 1:dim(da)[2]) {
  lines(days*364+1,models[[i]]$fitted.values, col=as.numeric(MetaData$region[i]))
}
legend("top",legend=unique(MetaData$region), col=as.numeric(unique(MetaData$region)),
       cex=0.8, lty=1)
```



From this plot, it is clear that the rainfall year-round is lowest in the Continental and Arctic regions, higher in the Atlantic region and highest in one Pacific region (Pr.Rupert, see next plot). This corresponds mostly, but not completely, with the ordering according to Z1. The main difference is that the Pacific coastal towns are clearly separated from the Atlantic region according to Z1, while no clear distinction is present in the plot above. An alternative explanation is that Z1 corresponds more with rainfall in deep winter (days <30 and >335) than with rainfall year-round.

We can also zoom in on the cities of the Pacific region:

```
plot(days*364+1,models[[1]]$fitted.values,type="n", ylim=c(0,13),
     xlab="day", ylab="average rainfall (mm/day)")
for(i in 25:29) {
  lines(days*364+1,models[[i]]$fitted.values, col=i)
}
legend("top",legend=MetaData$city[25:29], col=25:29, cex=0.8, lty=1)
```



We observe that the rainfall is indeed much higher in the 3 coastal cities than in the 2 cities in the Rocky Mountains.

3.2. Interpretation of Z2

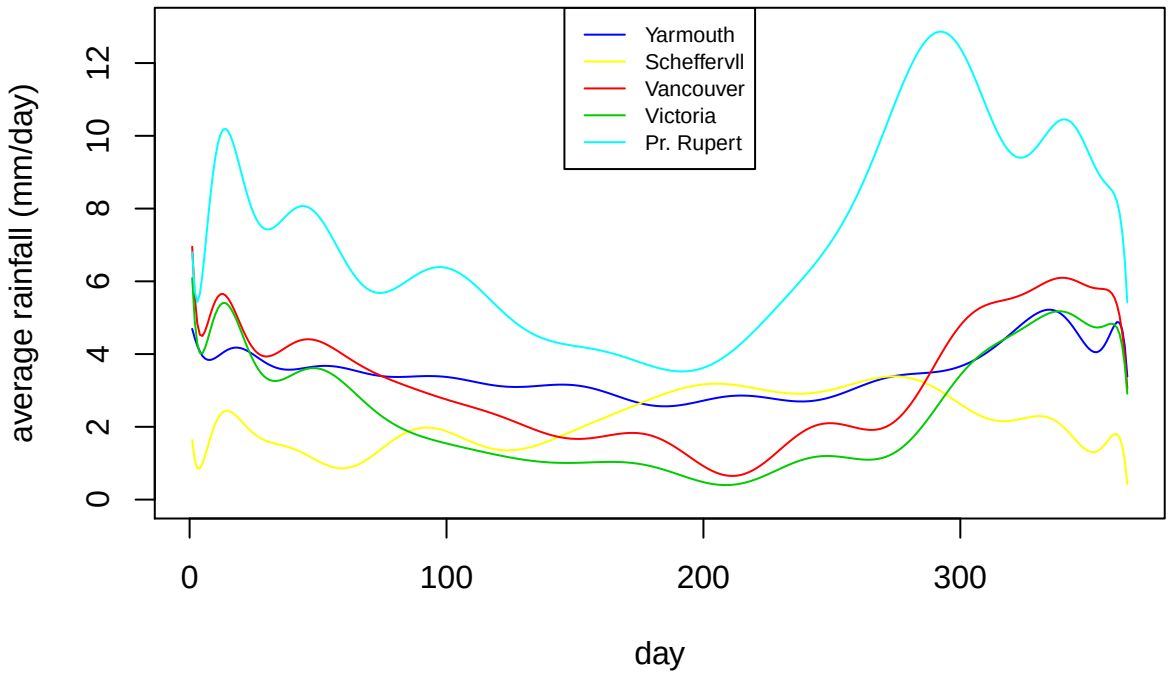
The interpretation of Z2 is even less obvious. The interpretation is complicated by the fact that the city which is most distinct in terms of Z2 (Pr.Rupert) was also most distinct in terms of Z1. In other words: in addition to having the highest rainfall, what makes Pr.Rupert so special?

To aid us in finding an explanation, we again zoom in on five cities: 3 with a value of Z2 close to zero (Vancouver, Victoria and Yarmouth) and 2 on the other side of Z2 (Pr.Rupert and Schefferville):

```
selected.cities <- c(4,7,26,27,29)
MetaData$city[selected.cities]
```

```
## [1] Yarmouth    Scheffervll Vancouver   Victoria    Pr. Rupert
## 35 Levels: Arvida Bagottville Calgary Charlottvl Churchill ... Yellowknife
```

```
plot(days*364+1,models[[1]]$fitted.values,type="n", ylim=c(0,13),
     xlab="day", ylab="average rainfall (mm/day)")
for(i in selected.cities) {
  lines(days*364+1,models[[i]]$fitted.values,col=i)
}
legend(x="top", legend=MetaData$city[selected.cities],
       col=selected.cities, cex=0.7, lty=1)
```



However, no clear interpretation follows from this plot. A possible explanation could be that in Pr.Rupert and Schefferville, the majority of days can be regarded as "wet" and a minority as "dry" (relative to the total rainfall of each city). Whereas in the other 3 cities, the majority of days can be regarded as "dry" and a minority as "wet" (again in relative terms). However, this explanation is tentative at best - other explanations are certainly possible. It's also possible that there simply does not exist an easy to describe interpretation to Z2 in the current example.

When doing data analysis in a consulting capacity, it is highly recommended to request feedback from the subject matter (in this case: climate) scientists soon enough. They may offer possible explanations and/or have insights which are not readily available to the consulting statistician.

4. Functional Biplot

A biplot directly constructed from the SVD of $\boldsymbol{\Theta}$ is hard to interprete. A better interpretation arises from backtransforming the SVD to the original function space.

From Section 1.3.2 we have the statistical model for city i in matrix notation.

$$\boldsymbol{Y}_i = \boldsymbol{X}_i\boldsymbol{\theta}_i + \boldsymbol{\varepsilon}_i$$

with $\boldsymbol{Y}_i$ the vector with the outcomes of observation i (one for each day t_{ij}, $j = 1, \ldots, p$), $\boldsymbol{\theta}_i$ the vector with the θ_{ik} (one for each basis function k), $\boldsymbol{X}_i$ the matrix with the x_{ijk} (days j in the rows, basis function index k in columns), and $\boldsymbol{\varepsilon}_i$ the vector with the i.i.d. error terms.

The $\boldsymbol{\theta}_i$ parameters estimates are denoted by $\hat{\boldsymbol{\theta}}_i$, and as before the ith row of $\boldsymbol{\Theta}$ is $\hat{\boldsymbol{\theta}}_i^t$. The fitted model for all n cities may then be simultaneously written as

$$\hat{\boldsymbol{Y}} = \boldsymbol{\Theta}\boldsymbol{X}^t$$

with $\hat{\boldsymbol{Y}}$ the $n \times p$ matrix with ith row $\boldsymbol{Y}_i^t$, and $\boldsymbol{X}$ the $p \times (m+1)$ matrix $\boldsymbol{X}_i$ as defined before (note that all $\boldsymbol{X}_i$ are equal because for all cities i the measurements were obtained at the same p time points (days)).

After substituting $\boldsymbol{\Theta}$ with its truncated SVD (after k terms) we get the model fit

$$\hat{\boldsymbol{Y}}_k = \boldsymbol{\Theta}_k\boldsymbol{X}^t = \boldsymbol{U}_k\boldsymbol{D}_k\boldsymbol{V}_k^t\boldsymbol{X}^t = \boldsymbol{Z}_k\boldsymbol{V}_k^t\boldsymbol{X}^t.$$

The latter expression relates an approximate model fit ($\hat{\boldsymbol{Y}}_k$) for the complete data set of n cities to the scores of the SVD ($\boldsymbol{Z}_k$), which have been plotted as part of HW1, and the loadings ($\boldsymbol{V}_k$). The relation is established through the matrix $\boldsymbol{X}$ which was used to relate the basis functions to the θ-parameters. In particular, the (j,k) element of $\boldsymbol{X}$ was given by $\phi_k(t_j)$ (before we wrote t_{ij} but since for all cities i the measurements were taken at the same times, we simplified the notation from t_{ij} to t_j). The model fit of the previous equation may thus also be written in functional form

$$\hat{Y}_{ki}(t) = \sum_{j=1}^{k} \sum_{r=0}^{m} z_{kij} v_{rj} \phi_r(t)$$

where z_{kij} is the (i,j)th element of $\boldsymbol{Z}_k$ and v_{rj} is the (r,j)th element of $\boldsymbol{V}_k$ (or, equivalently, of $\boldsymbol{V}$).

For a given set of basis functions $\phi_r(t)$ and the SVD of the matrix with parameter estimates $\boldsymbol{\Theta}$, the expression for $\hat{Y}_{ki}(t)$ may be used for plotting informative graphs to express the interpretation of the SVD of $\boldsymbol{\Theta}$ in terms of the original functions.

Some motivation/suggestions for informative plots for $k = 2$:

- $\hat{Y}_{2i}(t)$ is written as a function of the scores z_{2ij}, and these scores are plotted as points (z_{2i1}, z_{2i2}) in a two-dimensional graph (e.g. MDS plot)
- to understand the meaning of the first dimension, $\hat{Y}_{2i}(t)$ can be plotted as a function of t with z_{2i1} varying between a small and a large score, and with z_{2i2} fixed at a meaningful constant
- to understand the meaning of the second dimension, $\hat{Y}_{2i}(t)$ can be plotted as a function of t with z_{2i2} varying between a small and a large score, and with z_{2i1} fixed at a meaningful constant

Many other variants are possible.

5. Assignment of HW2

You may start from the SVD of $\boldsymbol{\Theta}$ and the MDS plot from HW1. It is up to you to choose an appropriate basis (either you use the basis you used for your HW1, or you choose a basis used in the feedback to HW1). Your choice will not strongly affect the evaluation of your HW.

You are asked to interpret the SVD as a MDS and as a PCA, but you need to express you conclusions in terms of the original precipitation functions (i.e. rainfall as a function of time) and thus not in terms of the estimated θ parameters. This will require that you find good interpretations of the two dimensions of the MDS plot (or biplot). You can do this by showing informative graphs based on the method outlined in Section 4 of this document.

You are asked to present your results as an R markdown file. Briefly comment your R code, motivate your choices and provide interpretation of your final results.

There is an R package that contains functions for an FDA. However, you are explicitly asked NOT to use this package. Using the package will not give you deep insight into how an FDA works.