Unit - III

Laplace Transform

Def: Integral transform → mathematical technique to solve the given integral in the complicated original domain by transforming it into the transformed domain.

Integral in original domain $\xrightarrow{\text{using}}$ Integral transform → Integral in the transformed domain → Calculation will be easier in transformed domain

using Inverse Integral transform → the resultant can be converted back to the original variables

$$I[f(x)] = \int_{x=a}^{b} f(x)\, K(s,x)\, dx$$

parameter.

$f(x)$ → continuous in (a,b)

$K(s,x)$ → Kernel

$x \longrightarrow s$

$x \longleftarrow s$

Def: Laplace Transform → one kind of integral transform to solve the complicated integrals

$$L[f(t)] = \int_{t=0}^{\infty} f(t)\, \underbrace{e^{-st}}\, dt = F(s)$$

Kernel, s → parameter $(0, \infty)$.

$f(t)$ is continuous in $(0, \infty)$

$$\boxed{L[f(t)] = F(s) = \int_{t=0}^{\infty} e^{-st} f(t)\, dt}$$

Application of the Laplace transform:-

(i) is used to solve the Differential equations, especially ODE with initial conditions.

(ii) complicated indefinite integrals (limits can be $(0,\infty)$, $(-\infty,\infty)$) can also be solved by Laplace transform.

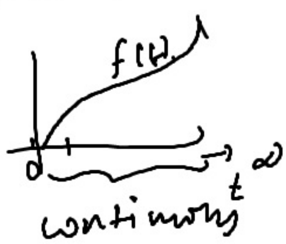
Engineering problems
↓
mathematical modelling containing DE.
↓
necessary to solve DE. → By some technique (one kind is Laplace)
↓
Results
↓
Going back with the inference to the engg. problems.

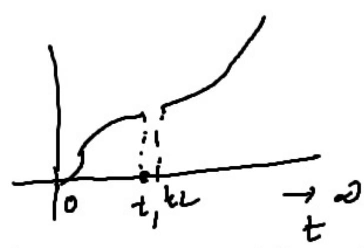
Question:-

* If we have any $f(t)$ with some $t \in (0,\infty)$, can it be possible to find Laplace transform.?

Ans:- No, it is not possible in general. But if $f(t)$ satisfies the following necessary conditions:-

(i) $f(t)$ is continuous in $(0,\infty)$ if not then atleast $f(t) \rightarrow$ piecewise continuous in $(0,\infty)$





(after correcting the discontinuous point, cheking whether that group of points is bounded or not. If it is bounded then $f(t)$ - piecewise)

(ii) $f(t)$ should have exponential order $\alpha > 0$.

i.e, $\lim_{t\to\infty} |f(t)| = Me^{\alpha t}$, (or) $|f(t)e^{-\alpha t}| \le M$

(exponentially bounded).

By satisfying these two conditions, we can able to find the Laplace transform.

Elementary Formulas for the Laplace transforms:-

(i) Find $L[1]$ $\quad L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt.$

Sol:

$$L\{1\} = \int_0^\infty e^{-st} \cdot 1 \, dt$$

$$= \left\{ \frac{e^{-st}}{-s} \right\}_0^\infty$$

$$= \frac{-1}{s}\left\{ e^{-\infty} - e^{0} \right\}$$

$$\boxed{L\{1\} = \frac{1}{s}}, \quad s > 0.$$

$e^{-\infty} = 0$

$e^{\infty} = \infty$

$e^{0} = 1$

(ii) $L\{t\} = \int_0^\infty e^{-st}\, t \, dt$

$st = y \Rightarrow t = \frac{y}{s}$

$dt = \frac{dy}{s}$, $s > 0$.

$t \to 0 \Rightarrow y \to 0$

$t \to \infty \Rightarrow y \to \infty$.

$$= \int_{y=0}^\infty e^{-y} \cdot \left(\frac{y}{s}\right)^1 \cdot \frac{dy}{s}$$

$$= \frac{1}{s^2} \int_{y=0}^\infty e^{-y} y^1 \, dy$$

$$= \frac{1}{s^2} \int_0^\infty e^{-y} y^{2-1} \, dy$$

$$= \frac{1}{s^2} \Gamma(2) = \frac{1}{s^2}(1) = \boxed{\frac{1}{s^2}}, \quad s > 0$$

Rough:

Gamma function

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} \, dx$$

$$\boxed{\Gamma(n) = (n-1)!}$$

$\Gamma(2) = 1!$

$= 1$

(iii) $L\{t^n\} = \int_0^\infty e^{-st}\, t^n\, dt$; $st = y \Rightarrow t = \frac{y}{s}$.

$dt = \frac{dy}{s}$

$t \to 0 \Rightarrow y \to 0$

$t \to \infty \Rightarrow y \to \infty$.

$$= \int_{y=0}^\infty e^{-y} \left(\frac{y}{s}\right)^n \frac{dy}{s}$$

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

$$= \frac{1}{s^{n+1}} \int_{y=0}^\infty e^{-y}\, y^n\, dy$$

$$\Gamma(n+1) = n!$$

$$= \frac{1}{s^{n+1}} \underbrace{\int_{y=0}^\infty e^{-y}\, y^{(n+1)-1}\, dy}$$

$$= \frac{\Gamma(n+1)}{s^{n+1}} \quad (\text{or}) \quad \frac{n!}{s^{n+1}}$$

$$\boxed{L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}} \quad (\text{or}) \quad \frac{n!}{s^{n+1}}, \quad s > 0}$$

From this result, $n=0 \Rightarrow L\{1\} = \frac{1}{s}$

$\boxed{0! = 1}$

$n=1 \Rightarrow L\{t\} = \frac{1!}{s^2} = \frac{1}{s^2}$

$n=2 \Rightarrow L\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3}$

⋮

(iv) $L\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at}\, dt$

$$= \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left\{ \frac{e^{-(s-a)t}}{-(s-a)} \right\}_0^\infty = \left\{ 0 - \frac{1}{-(s-a)} \right\}$$

$$L\{e^{at}\} = \frac{1}{(s-a)}, \quad \underline{s > a}$$

(v) $L\{e^{-at}\} = \int_0^\infty e^{-st} e^{-at} dt$

$$= \int_0^\infty e^{-(s+a)t} dt$$

$$= \left\{ \cancel{\frac{e^{-(s+a)t}}{-(s+a)}} \right\}_0^\infty = \left\{ 0 - \frac{1}{-(s+a)} \right\}$$

$$L\{e^{-at}\} = \frac{1}{s+a}, \quad \underline{\underline{s > -a}}.$$

(vi) $L\{\sin at\} = \int_0^\infty e^{-st} \sin(at)\, dt$

Rough.

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b \cos bx\right]$$

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}\left[a \cos bx + b \sin bx\right]$$

$$= \left\{ \cancel{\frac{e^{-st}}{(-s)^2+a^2}} \left(-s \sin at - a \cos at\right) \right\}_0^\infty$$

$$= \left\{ 0 - \frac{1}{(s^2+a^2)}(0-a) \right\}$$

$$= \frac{a}{s^2+a^2}, \quad s > 0.$$

$$\boxed{L\{\sin at\} = \frac{a}{s^2+a^2}}, \quad s > 0.$$

(vii) $L\{\cos at\} = \int_0^\infty e^{-st} \cos at\, dt$

$$= \left\{ \cancel{\frac{e^{-st}}{(-s)^2+a^2}} \left(-s \cos at + a \sin at\right) \right\}_0^\infty$$

$$= \left\{ (0) - \left(\frac{1}{s^2+a^2}\left(-s.1. + 0\right) \right) \right\} = \boxed{\frac{s}{s^2+a^2}, \quad s > 0.}$$

(viii) $L\{\sinh(at)\} = \frac{a}{s^2-a^2}, \quad s>|a|$

(ix) $L\{\cosh(at)\} = \frac{s}{s^2-a^2}, \quad s>|a|$

Properties of the Laplace Transforms :-

(i) Linearity Property :-

$$L\{a f(t) + b g(t)\} = a.L\{f(t)\} + b.L\{g(t)\}$$

where $a, b \to$ constants, $f(t), g(t) \to$ two functions.

proof:

$$LHS = L\{a f(t) + b g(t)\} = \int_0^\infty e^{-st}(a f(t) + b g(t))\,dt$$

$$= \int_0^\infty e^{-st} a f(t)\,dt + \int_0^\infty e^{-st} b g(t)\,dt$$

$$= a \underbrace{\int_0^\infty e^{-st} f(t)\,dt} + b \underbrace{\int_0^\infty e^{-st} g(t)\,dt}$$

$$= a\, L\{f(t)\} + b\, L\{g(t)\} = RHS.$$

(ii) First shifting property :-

If $L\{f(t)\} = F(s)$, then

(i) $L\{e^{at} f(t)\} = F(s-a)$

(ii) $L\{e^{-at} f(t)\} = F(s+a)$

Rough

$L\{f(t)\} = F(s)$

$\int_0^\infty e^{-st} f(t)\,dt$

proof:

$$LHS \Rightarrow L\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt = F(s-a) = RHS.$$

(iii) second shifting property :-

If $L\{f(t)\} = F(s)$, $g(t) = \begin{cases} f(t-a), & t>a \\ 0, & t<a \end{cases}$

then,

$$L\{g(t)\} = e^{-as} F(s).$$

(iv) Change of Scale property :-

If $L\{f(t)\} = F(s)$, then

$$L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right), \quad a>0.$$

Proof :-

$$LHS = L\{f(at)\} = \int_0^\infty e^{-st} f(at)\, dt$$

$at = x$

$t = \frac{x}{a}$

$dt = \frac{dx}{a}$

$t \to 0 \Rightarrow x \to 0$

$t \to \infty \Rightarrow x \to \infty.$

$$= \int_{x=0}^{\infty} e^{-s\left(\frac{x}{a}\right)} f(x) \frac{dx}{a}$$

$$= \frac{1}{a} \int_{x=0}^{\infty} e^{-\left(\frac{s}{a}\right)x} f(x)\, dx$$

changing $x \to t$.

$$= \frac{1}{a} \underbrace{\int_0^\infty e^{-\left(\frac{s}{a}\right)t} f(t)}\, dt$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right) = RHS$$

(v) Laplace transform on derivatives :-

$f^{(n)}(t) = \frac{d^n}{dt^n}(f(t))$

If $L\{f(t)\} = F(s)$, then

(1) $L\{f'(t)\} = s.F(s) - f(0)$

(2) $L\{f''(t)\} = s^2 F(s) - s f(0) - f'(0).$

(3) $L\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - \dots - f^{(n-1)}(0).$

(vi) Laplace transforms on Integrals.

deri $\Leftrightarrow$ $\times$ly
Int $\Leftrightarrow$ $\div$

If $L\{f(t)\} = F(s)$, then

$$L\left\{\int_0^t f(t)\,dt\right\} = \frac{F(s)}{s}$$

(vii) If $L\{f(t)\} = F(s)$, then

(i) $L\{t\,f(t)\} = -\frac{d}{ds}F(s)$.

(ii) $L\{t^2 f(t)\} = \frac{d^2}{ds^2}F(s)$

(iii) $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}(F(s))$.

(iv) $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$, if $\lim_{t\to 0}\frac{f(t)}{t}$ exist

① Find $L\{2e^{-3t}+3t^2-4\sin 2t+2\cos 3t\}$

sol:

$$= 2L\{e^{-3t}\}+3L\{t^2\}-4L\{\sin 2t\}+2L\{\cos 3t\}$$

$$= 2\left(\frac{1}{s+3}\right)+3\left(\frac{2!}{s^3}\right)-4\left(\frac{2}{s^2+4}\right)+2\left(\frac{s}{s^2+9}\right)$$

$$= \frac{2}{s+3}+\frac{6}{s^3}-\frac{8}{s^2+4}+\frac{2s}{s^2+9} \;//$$

Rough

Linearity Property

$L\{e^{-at}\}=\frac{1}{s+a}$

$L\{t^n\}=\frac{n!}{s^{n+1}}$

$L\{\sin at\}=\frac{a}{s^2+a^2}$

$L\{\cos at\}=\frac{s}{s^2+a^2}$

② $L\{e^{-2t}\sin 5t\}$, $L\{e^{3t}\cosh(4t)\}$

sol:

By first shifting,

(i) $$L\{e^{-2t}\sin 5t\} = \{L\{\sin 5t\}\}_{s\to s+2}$$

$$= \{F(s)\}_{s\to(s+2)}$$

$$= \left(\frac{5}{s^2+25}\right)_{s\to(s+2)}$$

$$= \frac{5}{(s+2)^2+25} \;//$$

Rough

$L\{e^{-at}f(t)\}=F(s+a)$

(ii) $$L\{e^{3t}\cosh(4t)\} = \left[L\{\cosh(4t)\}\right]_{s\to(s-3)}$$

$$= \left(\frac{s}{s^2-16}\right)_{s\to(s-3)}$$

$$= \frac{(s-3)}{(s-3)^2-16} \;//.$$

Rough

$L^{-1}\left\{\frac{2}{(s-2)^2+4}\right\}$

$= e^{2t}\sin 2t.$

(13) $L\{\sin 3t \cos 4t + \cos^2(2t) + 3\}$

Sol:

$= L\{\sin 3t \cos 4t\} + L\{\cos^2(2t)\} + L\{3\}$

$= L\left\{\dfrac{\sin(7t)+\sin(-t)}{2}\right\} + L\left\{\dfrac{1+\cos 2(2t)}{2}\right\} + 3L\{1\}$

$= \dfrac{1}{2}\left(\dfrac{7}{s^2+49} + \dfrac{(-1)}{s^2+1}\right) + \dfrac{1}{2}\left\{\dfrac{1}{s} + \dfrac{s}{s^2+16}\right\} + 3\left(\dfrac{1}{s}\right)$

$= \dfrac{1}{2}\left\{\dfrac{7}{s^2+49} - \dfrac{1}{s^2+1} + \dfrac{1}{s} + \dfrac{s}{s^2+16}\right\} + \dfrac{3}{s}$ //

Rough

$\sin(A+B) \longrightarrow ①$

$= \sin A \cos B + \cos A \sin B$

$\sin(A-B) \longrightarrow ②$

$= \sin A \cos B - \cos A \sin B$

$\cos(A-B) \longrightarrow ③$

$= \cos A \cos B + \sin A \sin B$

$\cos(A+B) \longrightarrow ④$

$= \cos A \cos B - \sin A \sin B$

$\cos 2A = 2\cos^2 A - 1$

$= 1 - 2\sin^2 A$

$= \cos^2 A - \sin^2 A$

$\cos^2 A = \left(\dfrac{1+\cos 2A}{2}\right)$

(4) $L\{t^3 e^{-4t}\}$

Sol:

$= \left[L\{t^3\}\right]_{s\to(s+4)}$

$= \left\{\dfrac{3!}{s^4}\right\}_{s\to(s+4)} = \dfrac{6}{(s+4)^4}$ //.

$L\{t^n\} = \dfrac{n!}{s^{n+1}}$

(5) $L\{t \sin 2t\}$, $L\{t^2 \cos 3t\}$

Sol:

(i) $L\{t^1 \sin 2t\} = (-1)^1 \dfrac{d}{ds}\{L\{\sin 2t\}\}$

$= (-1)\dfrac{d}{ds}\left\{\dfrac{2}{s^2+4}\right\}$

$= (-1)\left\{\dfrac{(s^2+4)(0) - 2(2s)}{(s^2+4)^2}\right\}$

$= -\left\{\dfrac{-4s}{(s^2+4)^2}\right\} = \dfrac{4s}{(s^2+4)^2}$ //.

Rough.

$L\{t f(t)\} = -\dfrac{d}{ds}\left[L\{f(t)\}\right]$

$L\{t^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n}\left[L\{f(t)\}\right]$

$d\left(\dfrac{u}{v}\right) = \dfrac{v\,du - u\,dv}{v^2}$

(ii) $L\{t^2 \cos 3t\}$

$$= \frac{d^2}{ds^2}\left\{L\{\cos 3t\}\right\}$$

$$= \frac{d^2}{ds^2}\left\{\frac{s}{s^2+9}\right\}$$

$$= \frac{d}{ds}\left\{\frac{(s^2+9).1 - s(2s)}{(s^2+9)^2}\right\}$$

$$= \frac{d}{ds}\left\{\frac{s^2+9-2s^2}{(s^2+9)^2}\right\}$$

$$= \frac{d}{ds}\left\{\frac{-s^2+9}{(s^2+9)^2}\right\}$$

$$= \frac{(s^2+9)^2(-2s) - (-s^2+9)(2(s^2+9).2s)}{(s^2+9)^4}$$

$$= \frac{(s^2+9)(-2s) - 2s(-s^2+9)\ 2(s^2+9)}{(s^2+9)^4}$$

$$= \frac{-2s - 4s(-s^2+9)}{(s^2+9)^3} //.$$

$$= \frac{-2s+4s^3-36s}{(s^2+9)^3}$$

$$= \frac{4s^3-38s}{(s^2+9)^3} //.$$

① Find $L\left\{\dfrac{e^{-t}-e^{-3t}}{t}\right\}$

sol:

$$= \int_s^\infty L\{e^{-t}-e^{-3t}\}\,ds$$

$$= \int_s^\infty \left(\frac{1}{s+1} - \frac{1}{s+3}\right) ds$$

$$= \left\{\cancelto{0}{\log(s+1)} - \cancelto{0}{\log(s+3)}\right\}_s^\infty$$

$$= \{(0) - (\log(s+1) - \log(s+3))\}$$

$$= -\log\left(\frac{s+1}{s+3}\right) = \log\left(\frac{s+3}{s+1}\right)$$

Rough

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty L\{f(t)\}\,ds$$

$$L\{e^{-at}\} = \frac{1}{(s+a)}$$

$$\int \frac{f'(t)}{f(t)}\,dt = \log[f(t)]$$

$$\int \frac{1}{t}\,dt = \log t$$

$$-\log(x) = \log x^{-1} = \log\left(\frac{1}{x}\right)$$

② $L\left\{\dfrac{\cos at - \cos bt}{t}\right\}$

sol:

$$= \int_s^\infty L\{\cos at - \cos bt\}\,ds$$

$$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right) ds$$

$$= \frac{1}{2}\left\{\log(s^2+a^2) - \log(s^2+b^2)\right\}_s^\infty$$

$$= \frac{1}{2}\left\{0 - (\log(s^2+a^2) - \log(s^2+b^2))\right\}$$

$$= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

Initial value Theorem (IVT)

If $L\{f(t)\} = F(s)$, then $\lim_{t \to 0} f(t) = \lim_{s \to \infty} s.F(s)$

Final value Theorem (FVT)

If $L\{f(t)\} = F(s)$, then $\lim_{t \to \infty} f(t) = \lim_{s \to 0} s.F(s)$.

① Verify initial value theorem for the function $(3-2\cos t)$

sol: Let $f(t) = (3-2\cos t)$

$$L\{f(t)\} = F(s) = L\{(3-2\cos t)\} = 3.L\{1\} - 2L\{\cos t\}$$

$$F(s) = \frac{3}{s} - 2\left(\frac{s}{s^2+1}\right)$$

LHS:-

$$\lim_{t \to 0} f(t) = \lim_{t \to 0} (3-2\cos t) = 3 - 2\cos 0 = 3-2 = 1$$

RHS:-

$$\lim_{s \to \infty} s.F(s) = \lim_{s \to \infty} s\left(\frac{3}{s} - \frac{2s}{s^2+1}\right) = \lim_{s \to \infty} \cancel{s}\frac{3}{\cancel{s}} - \lim_{s \to \infty} \frac{2s^2}{s^2+1}$$

$$= 3 - \lim_{s \to \infty} \left(\frac{2}{(1+\frac{1}{s^2})}\right) = 3 - \frac{2}{(1+0)} = 3-2 = 1$$

LHS = RHS, Verified IVT.

② verify Final value theorem.

$1 + e^{-t}(\sin t + \cos t)$

sol: Let $f(t) = 1 + e^{-t}(\sin t) + e^{-t}(\cos t)$

$$F(s) = \frac{1}{s} + \{L(\sin t)\}_{s \to s+1} + \{L(\cos t)\}_{s \to (s+1)}$$

$$= \frac{1}{s} + \left(\frac{1}{s^2+1}\right)_{s \to s+1} + \left(\frac{s}{s^2+1}\right)_{s \to s+1}$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{(s+1)}{(s+1)^2+1}$$

Rough

If $F(s) = \boxed{\frac{1}{s+2}}$

find $f(0)$

sol:

Using IVT,

$$f(0) = \lim_{s \to \infty} s.F(s)$$

$$= \lim_{s \to \infty} s.\left(\frac{1}{s+2}\right)$$

$$= \lim_{s \to \infty} \frac{\cancel{s}}{\cancel{s}}\left(\frac{1}{1+2/s}\right) = 1$$

LHS:

$$\lim_{t \to \infty} f(t) = \lim_{t \to \infty} \left(1 + \underbrace{e^{-t} \sin t}_{\to 0} + \underbrace{e^{-t} \cos t}_{\to 0}\right) = \underline{\underline{1}}.$$

$e^{-\infty} = 0$

RHS:

$$\lim_{s \to 0} s \cdot F(s) = \lim_{s \to 0} s \left\{ \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{(s+1)}{(s+1)^2+1} \right\}$$

$$= \lim_{s \to 0} \cancel{s}\left(\frac{1}{\cancel{s}}\right) + \lim_{s \to 0} \frac{s}{(s+1)^2+1} + \lim_{s \to 0} \frac{s(s+1)}{(s+1)^2+1}$$

$$= 1 + 0 + 0 = 1$$

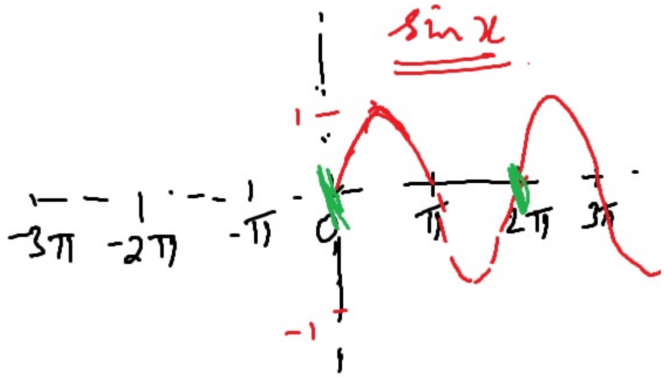
$\frac{0}{(\ )} = 0$

LHS = RHS, Verified FVT.

Periodic Functions :-

A function $f(t)$ is said to be periodic with period 'T', then for all 't', there exist a smaller number T such that

$$\boxed{f(t+T) = f(t)}$$



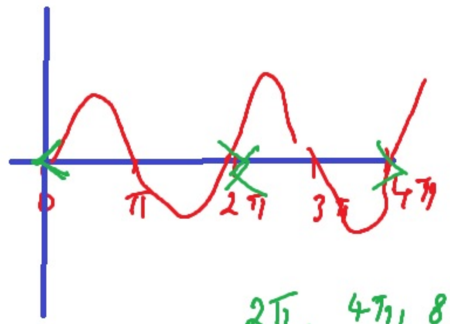
Sin x is a periodic function with period $\underline{\underline{(2\pi)}}$.

Defn: periodic function

A function $f(t)$ is said to be periodic function with period T, then for all t, there exist some smaller $T>0$ such that $f(t+T) = f(t)$.

Example,

(i) $f(t) = \sin t$.



$$f(t+2\pi) = \sin(t+2\pi) = \sin t = f(t)$$

$$f(t+4\pi) = \sin(t+4\pi) = \sin t = f(t)$$

$$f(t+6\pi) = \sin(t+6\pi) = \sin t = f(t)$$

Range 2π (i) $(0, 2\pi) = (2\pi, 4\pi) = (4\pi, 8\pi)$

Range 4π (ii) $(0, 4\pi) = (4\pi, 8\pi) = (8\pi, 12\pi)$

$2\pi, 4\pi, 8\pi$

Smaller $= 2\pi$

Hence,

$f(t) = \sin t$ satisfies

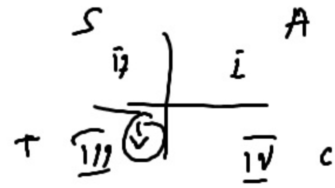
for $T = \underbrace{2\pi, 4\pi, 6\pi, 8\pi, \ldots}$

out of that, smaller is $= 2\pi$

$\therefore$ the period of $\sin(t) = 2\pi$

$\sin(360° + \theta) = \sin\theta$

$\cos(360° + \theta) = \cos\theta$



(ii) $f(t) = \tan(t)$.

$$f(t+\pi) = \tan(t+\pi) = \tan t = f(t)$$

$\tan(t+180°)$
$= \tan t$
$= \tan t$.

$$f(t+2\pi) = \tan(t+2\pi) = \tan t = f(t)$$

$$f(t+3\pi) = \tan(t+3\pi) = \tan(180° + \pi)$$
$$= f(t).$$

$3 \times 180°$
$= 540$.
$= 360° + 180°$

$f(t) = \tan\pi$ satisfies

for $t = \pi, 2\pi, 3\pi, 4\pi, \ldots$

$\therefore$ period $= \pi$.

(i) $\sin x$ $\cos x$ are periodic with period $T = 2\pi$

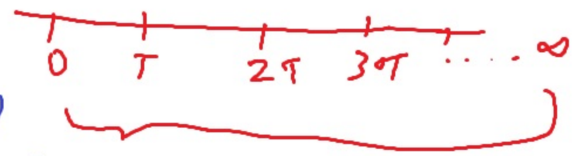
(ii) $\tan x$ is periodic with period $T = \pi$.

Laplace transform on periodic functions

If $f(t)$ is a periodic function with period $T>0$, then

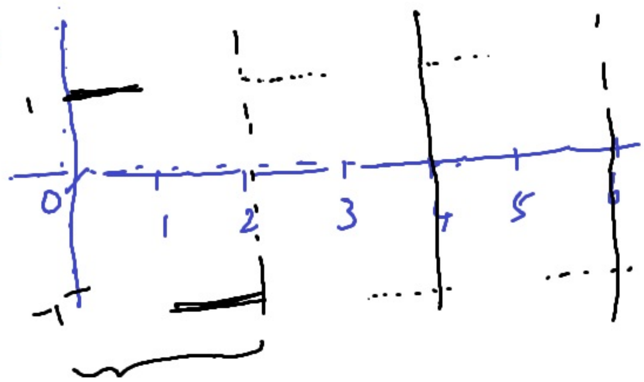
$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

(Instead of calculating in $(0,\infty)$ enough to calculate only $(0,T)$ since the similar behaviours in $(T,2T), (2T,3T), \ldots$)



① Find the Laplace transform of a periodic function $f(t)$ with period 2, given by

$$f(t) = \begin{cases} 1 & ; \ 0<t<1 \\ -1 & ; \ 1<t<2 \end{cases}$$



Sol: $T = \text{Period} = 2$

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2s}} \left\{ \int_0^1 e^{-st} \cdot 1\, dt + \int_1^2 e^{-st} (-1)\, dt \right\}$$

$$= \frac{1}{1-e^{-2s}} \left\{ \left(\frac{e^{-st}}{-s}\right)_{t=0}^{1} - \left(\frac{e^{-st}}{-s}\right)_1^2 \right\}$$

$$= \frac{1}{1-e^{-2s}} \left\{ \left(\frac{e^{-s}}{-s} - \frac{1}{-s}\right) - \left(\frac{e^{-2s}}{-s} - \frac{e^{-s}}{-s}\right) \right\}$$

$$= \frac{1}{1-e^{-2s}} \left\{ -\frac{2e^{-s}}{s} + \frac{1}{s} + \frac{e^{-2s}}{s} \right\} = \frac{1}{s} \frac{1}{(1-e^{-2s})} (1-2e^{-s}+e^{-2s})$$

$$= \frac{1}{s(1-e^{-2s})}\left(1-2e^{-s}+(e^{-s})^2\right)$$

$$= \frac{1}{s\cancel{(1-e^{-s})}(1+e^{-s})}(1-e^{-s})^{\cancel{2}}$$

$$= \frac{(1-e^{-s})}{s(1+e^{-s})}$$

$$= \frac{\cancel{e^{-s/2}}(e^{s/2}-e^{-s/2})}{s\,\cancel{e^{-s/2}}(e^{s/2}+e^{-s/2})}$$

$$= \frac{1}{s}\cdot\tanh\left(\frac{s}{2}\right)$$

$a^2-2ab+b^2 = (a-b)^2$

$a=1$

$b=e^{-s}$

$(1-e^{-2s}) = (1)^2-(e^{-s})^2$

$= (1-e^{-s})(1+e^{-s})$

$(1-e^{-s}) = (e^{-s/2}e^{s/2} - e^{-s/2}e^{-s/2})$

$= e^{-s/2}(e^{s/2}-e^{-s/2})$

$(1+e^{-s}) = (e^{-s/2}e^{s/2} + e^{-s/2}e^{-s/2})$

$= e^{-s/2}(e^{+s/2}+e^{-s/2})$

$\tan\theta = \frac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}$

$\tanh\theta = \frac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}}$

(2) Find the laplace transform of the periodic function

$$f(t) = \begin{cases} t & ; 0<t<1 \\ 2-t & ; 1<t<2 \end{cases}$$

sol:

Here, T = period = 2

$$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2s}}\int_0^2 e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2s}}\left\{\int_0^1 e^{-st}\,t\,dt + \int_1^2 e^{-st}\cdot(2-t)\,dt\right\}$$

$$= \frac{1}{1-e^{-2s}} \left\{ \int_0^1 e^{-st}\, t\, dt + \int_1^2 e^{-st}(2-t)\, dt \right\}$$

$$= \frac{1}{1-e^{-2s}} \left\{ \left[(t)\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right) \right]_0^1 + \left[(2-t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right) \right]_1^2 \right\}$$

$$= \frac{1}{1-e^{-2s}} \left\{ \left\{ \left[\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} \right] - \left[0 - \frac{1}{s^2} \right] \right\} + \left\{ \left[0 + \frac{e^{-2s}}{s^2} \right] - \left[\frac{e^{-s}}{-s} + \frac{e^{-s}}{s^2} \right] \right\} \right\}$$

$t=1$

$$= \frac{1}{1-e^{-2s}} \left\{ \frac{1}{s}\left(\cancel{-e^{-s}} + \cancel{e^{-s}} \right) + \frac{1}{s^2}\left(-e^{-s} + 1 + e^{-2s} - e^{-s} \right) \right\}$$

$$= \frac{1}{(1-e^{-2s})} \left\{ \frac{1}{s^2}\left(1 - 2e^{-s} + e^{-2s} \right) \right\}$$

$$= \frac{1}{s^2} \frac{1}{(1)^2 - (e^{-s})^2} \left\{ (1-e^{-s})^2 \right\}$$

$$= \frac{1}{s^2} \frac{\cancel{(1-e^{-s})}(1-e^{-s})}{\cancel{(1-e^{-s})}(1+e^{-s})}$$

$$= \frac{1}{s^2} \frac{(e^{s/2} - e^{-s/2})}{(e^{s/2} + e^{-s/2})}$$

$$= \frac{1}{s^2} \tanh\left(\frac{s}{2}\right) \quad //.$$

Rough

$\int u\, dv = uv - u' v_1 + u'' v_2 - \cdots$

$\int e^{-st}\, t\, dt$

$u = t, \quad \int dv = \int e^{-st} dt$

$u' = 1, \quad v = \frac{e^{-st}}{-s}$

$v_1 = \frac{e^{-st}}{s^2}$

$u'' = 0$, no need v_2.

Evaluation of the indefinite integral using Laplace transf.

problem :-

① Using Laplace Transform, evaluate $\int_0^\infty \frac{e^{-t}\sin\sqrt{3}t}{t}\,dt$.

Sol:

$$\int_0^\infty e^{-t}\,\frac{\sin\sqrt{3}t}{t}\,dt = \left[\int_0^\infty e^{-st}\underbrace{\left(\frac{\sin\sqrt{3}t}{t}\right)}_{f(t)}dt\right]_{s\to 1}$$

$$= \left[L\left\{\frac{\sin\sqrt{3}t}{t}\right\}\right]_{s\to 1}$$

$$= \left\{\int_s^\infty L\{\sin\sqrt{3}t\}\,ds\right\}_{s\to 1}$$

$$= \left\{\int_s^\infty \frac{\sqrt{3}}{s^2+3}\,ds\right\}_{s\to 1}$$

$$= \left\{\cancel{\sqrt{3}}\left[\frac{1}{\cancel{\sqrt{3}}}\tan^{-1}\left(\frac{s}{\sqrt{3}}\right)\right]_s^\infty\right\}_{s\to 1}$$

Rough

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$= \left\{\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{\sqrt{3}}\right)\right]\right\}_{s\to 1}$$

$\tan^{-1}\infty = \pi/2$

$$= \left\{\cot^{-1}\left(\frac{s}{\sqrt{3}}\right)\right\}_{s\to 1}$$

$$= \cot^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\left\{\begin{array}{l}\tan\frac{\pi}{3} = \frac{\sqrt{3}}{1}\\ \cot\pi/3 = \frac{1}{\sqrt{3}}\end{array}\right.$$

$$= \pi/3 \;//.$$

① Evaluate $\int_0^\infty t e^{-3t} \sin t \, dt$, using L.T.

Sol:

$$= \int_0^\infty e^{-3t} (t \sin t) \, dt$$

$$= \left\{ \int_0^\infty e^{-st} (t \sin t) \, dt \right\}_{s \to 3}$$

$$= \left\{ L(t \sin t) \right\}_{s \to 3}$$

$$= \left\{ -\frac{d}{ds} \left(L(\sin t) \right) \right\}_{s \to 3}$$

$$= \left\{ -\frac{d}{ds} \left\{ \frac{1}{s^2+1} \right\} \right\}_{s \to 3}$$

$$= -\left\{ \frac{(s^2+1)0 - 1.(2s)}{(s^2+1)^2} \right\}_{s \to 3}$$

$$= \left(\frac{2s}{(s^2+1)^2} \right)_{s \to 3}$$

$$= \frac{6}{100}$$

$$= \frac{3}{50} //.$$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt.$$

$$L\{t' f(t)\} = -\frac{d}{ds} (L(f(t)))$$

$$d\left(\frac{u}{v}\right) = \frac{v.u' - u.v'}{v^2}$$

Rough (objective)

$$\int_0^\infty e^{-2t} t \, dt$$

$$= (L(t))_{s \to +2}$$

$$= (1/s^2)_{s \to 2}$$

$$= 1/4 //.$$

Home work :-

① Evaluate $\int_0^\infty \frac{\cos 6t - \cos 4t}{t} dt$.

Ans: $\ln\left(\frac{3}{2}\right)$

Inverse Laplace transforms :- (L^{-1})

(I) Elementary formulas :-

(i) $L\{1\} = \frac{1}{s} \Rightarrow L^{-1}\{L\{1\}\} = L^{-1}\{\frac{1}{s}\} \Rightarrow L^{-1}\{\frac{1}{s}\} = 1$

(ii) $L\{t\} = \frac{1}{s^2} \Rightarrow L^{-1}\{\frac{1}{s^2}\} = t$

(iii) $L\{t^n\} = \frac{n!}{s^{n+1}} \Rightarrow L^{-1}\{\frac{1}{s^{n+1}}\} = \frac{t^n}{n!}$

(iv) $L\{e^{-at}\} = \frac{1}{s+a} \Rightarrow L^{-1}\{\frac{1}{s+a}\} = e^{-at}$

(v) $L\{\sin at\} = \frac{a}{s^2+a^2} \Rightarrow L^{-1}\{\frac{1}{s^2+a^2}\} = \frac{\sin at}{a}$

(vi) $L\{\cos at\} = \frac{s}{s^2+a^2} \Rightarrow L^{-1}\{\frac{s}{s^2+a^2}\} = \cos at$.

Rough

$L^{-1}\{\frac{3}{s^2}\}$

$= 3\, L^{-1}\{\frac{1}{s^2}\}$

$= 3.t.$

$\Rightarrow L\{3t\} = 3.L\{t\}$

$= 3/s^2$

Properties of L^{-1} :-

(i) If $L^{-1}\{F(s)\} = f(t)$, $L^{-1}\{G(s)\} = g(t)$,

$$L^{-1}\{aF(s) + bG(s)\} = a\,L^{-1}\{F(s)\} + b\,L^{-1}\{G(s)\}$$

(ii) If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\{F(s-a)\} = e^{at}f(t)$.

(iii) if $L^{-1}\{F(s)\} = f(t)$ then

$$L^{-1}\{e^{-as}F(s)\} = \begin{cases} f(t-a) & ; t > a \\ 0 & ; t < a \end{cases}$$

(iv) If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\{F(as)\} = \frac{1}{a} f(\frac{t}{a})$

(v) L^{-1} on derivatives :-

If $L^{-1}\{F(s)\} = f(t)$, then

$$L^{-1}\{F^{(n)}(s)\} = (-1)^n t^n f(t).$$

$F^{(n)}(s) = \frac{d^n}{ds^n}(F(s))$

(vi) L^{-1} on Integrals :-

If $L^{-1}\{F(s)\} = f(t)$, then

$$L^{-1}\left\{\int_s^\infty F(s)\,ds\right\} = \frac{f(t)}{t}$$

$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$

$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$

(vii) L^{-1} (which is x'lyed by s) and $\boxed{\text{if } f(0)=0}$

$$L^{-1}\{s.F(s)\} = f'(t).$$

$L\{f'(t)\} = s.F(s) - f(0)$

(viii) L^{-1} on division by s

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)\,dt.$$

$L\left\{\int_0^t f(t)\,dt\right\} = \frac{F(s)}{s}.$

Convolution theorem for Laplace :-

If $L\{f(t)\} = F(s)$, $L\{g(t)\} = G(s)$

$$L\{f(t) * g(t)\} = L\{F(s)\}.L\{G(s)\}$$

where, the convolution of $f(t)$ and $g(t)$ is defined as

$$f(t) * g(t) = \int_0^t f(u)\,g(t-u)\,du$$

$* \rightarrow$ convolution.

Convolution theorem for Inverse Laplace :-

If $L^{-1}\{F(s)\} = f(t)$, $L^{-1}\{G(s)\} = g(t)$.

$$L^{-1}\{F(s).G(s)\} = L^{-1}\{F(s)\} * L^{-1}\{G(s)\} = f(t) * g(t)$$

Problems:-

① Find $L^{-1}\left\{\frac{1}{s^2+9}\right\}$

$L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$

$L\{\sin at\} = \frac{a}{s^2+a^2}$

Sol:

$$= \frac{1}{3}L^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$= \frac{1}{3}\sin 3t = \frac{\sin 3t}{3}$$

② Find $L^{-1}\left\{\frac{4}{2s-4} + \frac{4+2s}{16s^2-4}\right\}$

Rough

$L\{e^{at}\} = \frac{1}{s-a}$

Sol:

$$= L^{-1}\left\{\frac{4}{2s-4}\right\} + L^{-1}\left\{\frac{4+2s}{16s^2-4}\right\}$$

$$= L^{-1}\left\{\frac{4}{2s-4}\right\} + 4L^{-1}\left\{\frac{1}{16s^2-4}\right\} + L^{-1}\left\{\frac{2s}{16s^2-4}\right\}$$

$$= L^{-1}\left\{\frac{2}{s-2}\right\} + 4L^{-1}\left\{\frac{1}{16\left(s^2-\frac{1}{4}\right)}\right\} + L^{-1}\left\{\frac{2s}{16\left(s^2-\frac{1}{4}\right)}\right\}$$

$$= 2L^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{4}L^{-1}\left\{\frac{1}{s^2-\left(\frac{1}{2}\right)^2}\right\} + \frac{1}{8}L^{-1}\left\{\frac{s}{s^2-(1/2)^2}\right\}$$

$$= 2e^{2t} + \frac{1}{4}\frac{\sinh\left(\frac{1}{2}t\right)}{(1/2)} + \frac{1}{8}\cosh\left(\frac{1}{2}t\right)$$

$\frac{1}{a}\, L\{\sinh(at)\} = \frac{a}{s^2-a^2}$

$$= 2e^{2t} + \frac{1}{2}\sinh\left(\frac{t}{2}\right) + \frac{1}{8}\cosh\left(\frac{t}{2}\right).$$

$L(\cosh at) = \frac{s}{s^2-a^2}$

③ Find $L^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\}$

Sol:

$$= L^{-1}\left\{\frac{3s+7}{(s-1)^2-4}\right\}$$

s^2-2s-3

$= (s^2-2s+1)-1-3$

$= (s-1)^2-4.$

$$= L^{-1}\left\{\frac{3s+7}{(s-1)^2-2^2}\right\}$$

$$= L^{-1}\left\{\frac{3s-3+3+7}{(s-1)^2-2^2}\right\}$$

$$= L^{-1}\left\{\frac{3(s-1)+10}{(s-1)^2-2^2}\right\}$$

$$= 3L^{-1}\left\{\frac{(s-1)}{(s-1)^2-2^2}\right\} + 10\,L^{-1}\left\{\frac{1}{(s-1)^2-2^2}\right\}$$

$$= 3.e^{1t}\cosh(2t) + 10\,e^{t}\frac{\sinh(2t)}{2}$$

$$= 3e^{t}\cosh(2t) + 5\,e^{t}\sinh(2t).$$

Rough

$$L^{-1}\left\{\frac{s}{s^2-a^2}\right\}$$

$$= \cosh(at)$$

$$L^{-1}\left\{\frac{(s-a)}{(s-a)^2-b^2}\right\}$$

$$= e^{at}\cosh(bt)$$

Rough

$$L\{e^{at}\cosh bt\}$$

$$= \left(\frac{s}{s^2-b^2}\right)_{s\to(s-a)}$$

$$= \frac{(s-a)}{(s-a)^2-b^2}$$

④ Find $L^{-1}\left\{\frac{s}{(s+2)^2}\right\}$

Sol:

$$= L^{-1}\left\{s\cdot\left(\frac{1}{(s+2)^2}\right)\right\}$$

$$= \frac{d}{dt}\left\{L^{-1}\left\{\frac{1}{(s+2)^2}\right\}\right\}$$

$$= \frac{d}{dt}\left\{e^{-2t}\cdot t\right\}$$

$$= e^{-2t}(1) + t\,(e^{-2t}\cdot(-2))$$

$$= e^{-2t} - 2t\,e^{-2t}$$

Rough

simple

(i) $L^{-1}\left\{\frac{1}{(s+2)^2}\right\}$

$$= e^{-2t}\cdot t$$

$$L^{-1}\{s.F(s)\}$$

$$= \frac{d}{dt}\left\{L^{-1}\{F(s)\}\right\}$$

① $L^{-1}\left\{\dfrac{1}{s(s^2+2s+2)}\right\}$

Sol:

$$= L^{-1}\left\{\dfrac{1/s^2+2s+2}{s}\right\}$$

$$= \int_0^t L^{-1}\left\{\dfrac{1}{s^2+2s+2}\right\} dt$$

$$= \int_0^t L^{-1}\left\{\dfrac{1}{(s+1)^2+1^2}\right\} dt$$

$$= \int_0^t \left(e^{-t}\sin t\right) dt$$

$$= \left\{\dfrac{e^{-t}}{(-1)^2+1^2}\left((-1)\sin t - 1.\cos t\right)\right\}_0^t$$

$$= \left[\dfrac{e^{-t}}{2}\left(-\sin t-\cos t\right) - \left[\dfrac{1}{2}(-1)\right]\right]$$

$$= -\dfrac{e^{-t}}{2}(\sin t+\cos t)+\dfrac{1}{2} \;//.$$

$L^{-1}\left\{\dfrac{F(s)}{s}\right\}$

$$= \int_0^t f(t)\,dt$$

$$= \int_0^t L^{-1}[F(s)]\,dt$$

$\underbrace{s^2+2s}+2$

$= (s^2+2s+1^2)-1^2+2$

$= (s+1)^2+1$

$= (s+1)^2+1^2$

$L(e^{at}f(t)) = F(s-a)$

$L(e^{-t}\sin t)$

$= \left(\dfrac{1}{s^2+1^2}\right)$

$s\to s+1$

$= \dfrac{1}{(s+1)^2+1^2}$ /

② Evaluate $L^{-1}\left[\dfrac{s}{(s^2+a^2)^2}\right]$, by convolution theorem.

$\int e^{ax}\sin bx\,dx$

$= \dfrac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$

Sol:

convolution theorem for L^{-1} :-

$$\boxed{\begin{aligned} L^{-1}\{F(s).G(s)\} &= L^{-1}\{F(s)\} * L^{-1}\{G(s)\} \\ &= f(t) * g(t) \\ &= \int_0^t f(u)\,g(t-u)\,du \end{aligned}}$$

$$L^{-1}\left\{ \frac{s}{(s^2+a^2)} \cdot \frac{1}{(s^2+a^2)} \right\} = L^{-1}\{F(s).G(s)\} = f(t) * g(t)$$

Here, $F(s) = \frac{s}{s^2+a^2} \Rightarrow f(t) = L^{-1}\left\{ \frac{s}{s^2+a^2} \right\} = \cos at$

$G(s) = \frac{1}{s^2+a^2} \Rightarrow g(t) = L^{-1}\left\{ \frac{1}{s^2+a^2} \right\} = \frac{\sin at}{a}$

$$L^{-1}\{F(s) \cdot G(s)\} = f(t) * g(t)$$

$$= \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t \cos au \frac{\sin a(t-u)}{a}\, du$$

$$= \frac{1}{a} \int_0^t \underbrace{\cos au \cdot \sin(at-au)}\, du$$

$$= \frac{1}{a} \int_0^t \frac{1}{2} \left\{ \sin(\cancel{au}+at-\cancel{au}) - \sin(au-(at-au)) \right\} du$$

Rough

$\cos A \sin B$

$= \frac{1}{2}\left\{ \sin(A+B) - \sin(A-B) \right\}$

$$= \frac{1}{2a} \int_0^t \left(\sin(at) - \sin(2au-at) \right) du$$

$$= \frac{1}{2a} \int_0^t \sin(at)\, du - \frac{1}{2a} \int_0^t \sin(2au-at)\, du$$

$$= \frac{\sin(at)}{2a} (u)_0^t - \frac{1}{2a} \left\{ -\frac{\cos(2au-at)}{2a} \right\}_0^t$$

$$= \frac{t \sin at}{2a} + \frac{1}{2a} \left\{ \left(\frac{\cos at}{2a} \right) - \left(\frac{\cos(0-at)}{2a} \right) \right\}$$

$$= \frac{t \sin at}{2a} + \frac{1}{2a^2} \left\{ \frac{\cancel{\cos at}}{\cancel{2a}} - \frac{\cancel{\cos at}}{\cancel{2a}} \right\}$$

Ans:- $\frac{t \sin at}{2a}$ //.

Find $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+a^2)}\right\} = L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\}$

Sol:

$$L^{-1}\left\{\frac{s}{(s^2+a^2)} \cdot \frac{s}{(s^2+a^2)}\right\} = L^{-1}\{F(s) \cdot G(s)\} = f(t) * g(t) \longrightarrow (1)$$

Here,

$$F(s) = \frac{s}{s^2+a^2} \Rightarrow f(t) = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$$

$$G(s) = \frac{s}{s^2+a^2} \Rightarrow g(t) = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$$

$$(1) \Rightarrow L^{-1}\{F(s)\, G(s)\} = f(t) * g(t)$$

$$= \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t \underbrace{\cos au \cdot \cos a(t-u)}\, du$$

$$= \int_0^t \cos \underbrace{au}_{A} \cdot \cos \underbrace{(at-au)}_{B}\, du$$

$$= \int_0^t \frac{1}{2}\left\{\cos(au-at+au) + \cos(\cancel{au}+at-\cancel{au})\right\} du$$

$$= \frac{1}{2}\left\{\int_0^t \cos(2au-at)\, du + \int_0^t \cos at\, du\right\}$$

$$= \frac{1}{2}\left\{\left(\frac{\sin(2au-at)}{2a}\right)_0^t + \cos at\, (u)_0^t\right\}$$

$$= \frac{1}{2}\left\{\left[\left(\frac{\sin at}{2a}\right) - \left(\frac{\sin(-at)}{2a}\right)\right] + t\cos at\right\}$$

$$= \frac{1}{2}\left\{\frac{\cancel{2}\sin(at)}{\cancel{2}a} + t\cos at\right\} = \frac{1}{2}\left(\frac{\sin at}{a} + t\cos at\right) //.$$

HW:-

① $L^{-1}\left\{\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \dfrac{a\sin at - b\sin bt}{a^2-b^2}$ //.

Evaluating the Inverse Laplace transform by the method of partial Fractions :-

Basic Formulas for partial Fraction :- (the degree of the numerator < the degree of the denominator)

① $\dfrac{1}{(x+a)(x+b)} = \dfrac{A}{(x+a)} + \dfrac{B}{(x+b)}$

② $\dfrac{1}{(x^2+a^2)(x+b)} = \dfrac{Ax+B}{(x^2+a^2)} + \dfrac{C}{(x+b)}$

③ $\dfrac{1}{(x+a)^2(x+b)} = \dfrac{A}{(x+a)} + \dfrac{B}{(x+a)^2} + \dfrac{C}{(x+b)}$

problems using partial Fraction method :-

① Find $L^{-1}\left\{\dfrac{3s+16}{s^2-s-6}\right\}$

$$\begin{array}{c} -6 \\ \wedge \\ -3 \quad +2 \end{array}$$

$$L^{-1}\left\{\frac{3s+16}{\underbrace{s^2-3s}+2s-6}\right\} = L^{-1}\left\{\frac{3s+16}{(s-3)(s+2)}\right\}$$

consider,

$$\frac{3s+16}{(s-3)(s+2)} = \frac{A}{(s-3)} + \frac{B}{(s+2)} \longrightarrow ①$$

$$= \frac{A(s+2)+B(s-3)}{(s-3)(s+2)}$$

$$3s+16 = A(s+2)+B(s-3) \longrightarrow ②$$

$s=3 \Rightarrow \quad 25 = 5A \Rightarrow \quad A = \frac{25}{5} = 5$

$s=-2 \Rightarrow \quad 10 = -5B \Rightarrow \quad B = \frac{10}{-5} = -2$

$$\frac{3s+16}{(s-3)(s+2)} = \frac{5}{(s-3)} + \frac{(-2)}{(s+2)}$$

Now, $L^{-1}\left\{\frac{3s+16}{(s-3)(s+2)}\right\} = L^{-1}\left\{\frac{5}{(s-3)} + \frac{-2}{(s+2)}\right\}$

$$= L^{-1}\left\{\frac{5}{s-3}\right\} - 2L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= 5 \cdot e^{3t} - 2e^{-2t}$$

Find $L^{-1}\left\{\frac{s^2-2s+3}{(s-1)^2(s+1)}\right\}$ by partial fraction method.

$L\{e^{at}\} = \frac{1}{s-a}$

$L\{e^{-at}\} = \frac{1}{s+a}$

Sol:

consider, $\frac{s^2-2s+3}{(s-1)^2(s+1)} = \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C}{(s+1)} \to (*)$

$$s^2-2s+3 = A(s-1)(s+1) + B(s+1) + C(s-1)^2 \to (1)$$

$s=1 \Rightarrow \quad 1-2+3 = B(2) \Rightarrow 2B = 2 \Rightarrow B=1$

$s=-1 \Rightarrow \quad 1-2(-1)+3 = C(-2)^2 \Rightarrow 4C = 6 \Rightarrow C = 3/2$

Equating the coeff of s^2:- $\quad 1 = A + C$

$A = 1 - C \Rightarrow 1 - 3/2 = -1/2$

$A = -1/2, \quad B = 1, \quad C = 3/2$

$$L^{-1}\left\{\frac{s^2-2s+3}{(s-1)^2(s+1)}\right\} = L^{-1}\left\{\frac{-1/2}{(s-1)} + \frac{1}{(s-1)^2} + \frac{3/2}{(s+1)}\right\}$$

$$= -\frac{1}{2}L^{-1}\left(\frac{1}{s-1}\right) + L^{-1}\left(\frac{1}{(s-1)^2}\right) + \frac{3}{2}L^{-1}\left(\frac{1}{s+1}\right)$$

$$= -\frac{1}{2}e^{t} + e^{t}t + \frac{3}{2}e^{-t}$$

Rough

$L(e^{at}f(t)) = F(s-a)$

$L(t) = \frac{1}{s^2}$

Find $L^{-1}\left\{\frac{10}{(s^2+1)(s^2+4s+3)}\right\} = L^{-1}\left\{\frac{10}{(s^2+1)(s+1)(s+3)}\right\}$

sol: consider $\frac{10}{(s^2+1)(s+1)(s+3)} = \frac{As+B}{(s^2+1)} + \frac{C}{(s+1)} + \frac{D}{(s+3)} \rightarrow (*)$

$$10 = (As+B)(s+1)(s+3) + C(s^2+1)(s+3) + D(s^2+1)(s+1)$$

$s=-1 \Rightarrow \quad 10 = C(1+1)(2) \Rightarrow \quad 10 = 4C \Rightarrow C = 5/2$

$s=-3 \Rightarrow \quad 10 = D(9+1)(-2) \Rightarrow \quad 10 = -20D \Rightarrow D = -1/2$

Eq the coeff: s^3 :- $\quad 0 = A + C + D \Rightarrow A = -C - D$

$$= -5/2 + 1/2 = -2$$

the coeff: s^0 :- $\quad 10 = 3B + 3C + D \qquad (As+B)(s^2+4s+3)$

$3B$

$$10 = 3B + \frac{15}{2} - 1/2$$

$$10 = \frac{6B + 15 - 1}{2}$$

$$6B = 20 - 14 \Rightarrow \boxed{B=1}$$

$A = -2, \; B = 1, \; C = 5/2, \; D = -1/2$

$$L^{-1}\left\{\frac{As+B}{(s^2+1)} + \frac{C}{(s+1)} + \frac{D}{(s+3)}\right\}$$

$$= L^{-1}\left\{\frac{-2s+1}{s^2+1} + \frac{5/2}{(s+1)} + \frac{-1/2}{(s+3)}\right\}$$

$$= L^{-1}\left\{\frac{-2s+1}{s^2+1}\right\} + 5/2\, L^{-1}\left\{\frac{1}{s+1}\right\} - 1/2\, L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= -2\cos t + \sin t + \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t}$$

Application of Laplace transforms
(Solving ODE with initial conditions using Laplace Transforms)

eg: ODE

$$\frac{dy}{dx}+3y=x$$

$$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+y=x^2$$

$$y''+3y'+y=x^2$$

soln is to find $y(x)$

Formulas :- Rule :-

(i) Taking Laplace Transform on both sides

(ii) Applying initial conditions in the above expression and make simplified expression.

(iii) Taking inverse Laplace on both sides.

$$L\{y''(t)\}=L\{y''\}=s^2L\{y\}-sy(0)-y'(0)$$

$$L\{y'(t)\}=L\{y'\}=sL\{y\}-y(0)$$

Rough

$$L\{f'(t)\}=s.F(s)-f(0)$$

$$L\{f''(t)\}=s^2F(s)-sf(0)-f'(0)$$

$$L\{f'''(t)\}=s^3F(s)-s^2f(0)-sf'(0)-f''(0)$$

① Solve $(D^2+4)y=\cos 2t$, $y(0)=3$, $y'(0)=4$. using Laplace Transform.

Sol:

$$D\equiv\frac{d}{dt},\quad D^2\equiv\frac{d^2}{dt^2}$$

$$\frac{d^2y}{dt^2}+4y=\cos 2t$$

$$y''+4y=\cos 2t \longrightarrow ①$$

$$\left(\frac{d^2}{dt^2}y\right)$$

$$\left(\frac{d^2y}{dt^2}\right)$$

taking Laplace on both sides,

$$L\{y''\}+4L\{y\}=L\{\cos 2t\}$$

$$\{s^2L(y)-sy(0)-y'(0)\}+4L\{y\}=\frac{s}{s^2+4}$$

$$\{s^2L\{y\}-s(3)-4\}+4L\{y\}=\frac{s}{s^2+4}$$

$$(s^2+4)L\{y\}-3s-4=\frac{s}{s^2+4}$$

$$(s^2+4)\,L(y) = \frac{s}{s^2+4} + 3s + 4 \qquad y = y(t).$$

$$L\{y\} = \frac{s}{(s^2+4)^2} + \frac{3s+4}{(s^2+4)}$$

Taking L^{-1} on both sides,

$$y(t) = L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} + L^{-1}\left\{\frac{3s+4}{s^2+4}\right\}$$

$$= \quad (\mathrm{I}) \quad + \quad (\mathrm{II})$$

To calculate :- (II)

$$L^{-1}\left\{\frac{3s+4}{(s^2+4)}\right\} = L^{-1}\left\{\frac{3s}{s^2+4}\right\} + L^{-1}\left\{\frac{4}{s^2+4}\right\}$$

$$= 3.\cos 2t + 4\,\frac{\sin 2t}{2}$$

$$= 3\cos 2t + 2\sin 2t$$

To calculate :- (I) (by convolution only)

$$L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = L^{-1}\left\{\frac{s}{(s^2+4)}\cdot\frac{1}{(s^2+4)}\right\}$$

Here,

$$F(s) = \frac{s}{s^2+4} \Rightarrow f(t) = L^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t$$

$$G(s) = \frac{1}{s^2+4} \Rightarrow g(t) = L^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{\sin 2t}{2}$$

By convolution theorem,

$$L^{-1}\{F(s)\,G(s)\} = f(t) * g(t)$$

$$= \int_0^t f(u)\,g(t-u)\,du$$

$$= \frac{1}{2}\int_0^t \cos(2u)\cdot\sin(2t-2u)\,du = \frac{t}{2\times 2}\sin 2t$$

$$= \frac{t}{4}\sin 2t$$

Rough

$$L^{-1}\left\{\frac{s}{(s+4)^2}\right\}$$

$$= \left[\frac{d}{dt}\,L^{-1}\left\{\frac{1}{(s+4)^2}\right\}\right]$$

$$= \left[\frac{d}{dt}\left(t.e^{-4t}\right)\right]$$

(Refer convolution section)

$\therefore\ y(t) = \frac{t}{4}\sin 2t + 3\cos 2t + 2\sin 2t$ //.

② solve $y'' - 4y' + 8y = e^{2t}$, $y(0) = 2$, $y'(0) = -2$

sol: Taking Laplace

$$L\{y''\} - 4L\{y'\} + 8L\{y\} = L\{e^{2t}\}$$

$$\{s^2 L(y) - sy(0) - y'(0)\} - 4\{sL\{y) - y(0)\} + 8L\{y\} = L(e^{2t})$$

$$\{s^2 L(y) - 2s + 2\} - 4\{sL(y) - 2\} + 8L\{y\} = \frac{1}{(s-2)}$$

$$(s^2 - 4s + 8)L(y) - 2s + 10 = \frac{1}{(s-2)}$$

$$(s^2 - 4s + 8)L\{y\} = \frac{1}{(s-2)} + 2s - 10$$

$$L\{y\} = \frac{1}{(s-2)(s^2-4s+8)} + \frac{2s-10}{(s^2-4s+8)}$$

Taking L^{-1} on both sides,

$$y = y(t) = \underbrace{L^{-1}\left\{\frac{1}{(s-2)(s^2-4s+8)}\right\}}_{(I)} + \underbrace{L^{-1}\left\{\frac{2s-10}{s^2-4s+8}\right\}}_{(II)}$$

to calculate :- (I), partial method

(I) : Consider, $\frac{1}{(s-2)(s^2-4s+8)} = \frac{A}{(s-2)} + \frac{Bs+C}{(s^2-4s+8)}$

Final Ans :- $\frac{1}{4}e^{2t}\{1 + 7\cos 2t - 12\sin 2t\}$

Solve $y'' + 2y' - 3y = \sin t$, $y(0) = 0$, $y'(0) = 0$, using Laplace Transform

$L\{\sin at\} = \frac{a}{s^2+a^2}$

Sol: Taking Laplace on both sides,

$$L\{y''\} + 2L\{y'\} - 3L\{y\} = L\{\sin t\}$$

$$\{s^2 L\{y\} - s\,y(0) - y'(0)\} + 2\{s L\{y\} - y(0)\} - 3L\{y\} = \frac{1}{(s^2+1)}$$

$$(s^2 + 2s - 3)L\{y\} = \frac{1}{(s^2+1)}$$

Rough

$y = L^{-1}\left\{\frac{1}{(s^2+2s-3)}\right.$

$$L\{y\} = \frac{1}{(s^2+1)(s^2+2s-3)}$$

$$L\{y\} = \frac{1}{(s^2+1)(s-1)(s+3)}$$

Taking L^{-1} on bothsides,

$$y = L^{-1}\left\{\frac{1}{(s^2+1)(s-1)(s+3)}\right\} \rightarrow (1)$$

By partial Fraction method,

consider, $\frac{1}{(s^2+1)(s-1)(s+3)} = \frac{As+B}{(s^2+1)} + \frac{C}{(s-1)} + \frac{D}{(s+3)}$ $\rightarrow (2)$

$$0s^3 + 0s^2 + 0s^1 + 1s^0 = (As+B)(s-1)(s+3) + C(s^2+1)(s+3) + D(s^2+1)(s-1)$$

$s=1 \Rightarrow 1 = C(2)(4) \Rightarrow 8C = 1 \Rightarrow C = 1/8$

$s=-3 \Rightarrow 1 = D(10)(-4) \Rightarrow -40D = 1 \Rightarrow D = -1/40$

Equating the coeff of s^3: $0 = A + C + D \Rightarrow A + \frac{1}{8} - \frac{1}{40} = 0 \Rightarrow A = -\frac{1}{8} + \frac{1}{40} = \frac{-5+1}{40} = \frac{-1}{10}$

Coeff of s^0: $1 = 3B + 3C - D \Rightarrow 3B = 1 - 3C + D$

$$3B = 1 - 3/8 - 1/40$$

$$3B = \frac{40-15-1}{40} \Rightarrow B = \frac{1}{5} \;//.$$

$A = -\frac{1}{10}$, $B = \frac{1}{5}$, $C = \frac{1}{8}$, $D = -\frac{1}{40}$

using ①, ②,

$$y = L^{-1}\left\{\frac{As+B}{(s^2+1)} + \frac{C}{(s-1)} + \frac{D}{(s+3)}\right\}$$

$$= -\frac{1}{10} L^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{5} L^{-1}\left\{\frac{1}{s^2+1}\right\} + \frac{1}{8} L^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{40} L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= -\frac{1}{10}\cos t + \frac{1}{5}\sin t + \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t}$$

$$y(t) = \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}\cos t + \frac{1}{5}\sin t \quad //.$$

Unit-step function (or) Heaviside's Function

$$H(t-a) = \begin{cases} 1; & \text{if } t \geq a \\ 0; & \text{if } t < a \end{cases}$$