On the structure of certain C^*-algebras arising from groups

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Outline of thesis

The dissertation consists of a collection of five papers and an introductory section that explains the connection between these papers and puts them into a context. The following five papers are included:

Paper I
: The full group C^*-algebra of the modular group is primitive. Joint work with Erik Bédos. Published in *Proc. Amer. Math. Soc.*, 140(4):1403-1411, 2012.

Paper II
: Primitivity of some full group C^*-algebras. Joint work with Erik Bédos. Published in *Banach J. Math. Anal.*, 5(2):44-58, 2011.

Paper III
: Primeness and primitivity conditions for twisted group C^*-algebras. Accepted for publication in *Math. Scand.*, 2012.

Paper IV
: C^*-algebras generated by projective representations of free nilpotent groups. Submitted for publication, 2013.

Paper V
: Cuntz-Li algebras from a-adic numbers. Joint work with Steve Kaliszewski and John Quigg. Submitted for publication, 2012.

1 C^*-algebras

This section is an attempt to very briefly motivate the study of C^*-algebras and explain some aspects of the theory related to the thesis, without going too much into detail on the historical background.

The theory of C^*-algebras can be developed in two different ways, either as certain algebras of bounded operators on Hilbert spaces or as special cases of Banach algebras.

1.1 Concrete approach

The motivation for studying operator algebras originally comes from quantum mechanics, and almost every survey on the topic starts with the Heisenberg commutation relation for a free particle,

$$PQ - QP = -iI, \tag{1}$$

where P and Q are self-adjoint operators on a Hilbert space $\mathcal{H}$ representing momentum and position, respectively. It turns out that (1) has nontrivial solutions only if $\mathcal{H}$ is infinite-dimensional and at least one of P or Q is unbounded. However, a theorem by Stone describes a bijective correspondence via "exponentiation" between possibly unbounded self-adjoint operators on $\mathcal{H}$ and one-parameter unitary subgroups of $B(\mathcal{H})$. As a consequence, the Weyl form of (1) is introduced, that is, for every real number t one defines the bounded unitary operators $U(t) = e^{itP}$ and $V(t) = e^{itQ}$ on $\mathcal{H}$ and observes that

$$U(s)V(t) = e^{ist}V(t)U(s). \tag{2}$$

In this way, U and V become unitary representations of $\mathbb{R}$ on $\mathcal{H}$. Moreover, for (s,t) in $\mathbb{R}^2$, set $W(s,t) = U(s)V(t)$, and define $\sigma\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{T}$ by

$$\sigma((s,t),(s',t')) = e^{its'}. \tag{3}$$

It is then evident that σ is a multiplier (or 2-cocycle) of $\mathbb{R}^2$ and that W is a σ-projective unitary representation of $\mathbb{R}^2$ on $\mathcal{H}$.

Equivalently, the pair (U,V) determines a unitary representation of the real Heisenberg group. Indeed, set $\widetilde{W}(r,s,t) = e^{ir}U(s)V(t) = e^{ir}W(s,t)$, and then

$$\widetilde{W}(r,s,t)\widetilde{W}(r',s',t') = e^{i(r+r')}e^{its'}W(s+s',t+t') = \widetilde{W}(r+r'+ts',s+s',t+t').$$

Schrödinger's solution of (2) are the bounded operators U and V on $L^2(\mathbb{R})$ given by

$$U(s)\psi(t) = \psi(t-s) \quad \text{and} \quad V(s)\psi(t) = e^{-ist}\psi(t), \tag{4}$$

and every pair of irreducible unitary representations of $\mathbb{R}$ satisfying (2) is unitarily equivalent to this pair. Also, let σ be defined by (3). Then every irreducible σ-projective representation of $\mathbb{R}^2$ is unitarily equivalent with the one coming from (4).

This uniqueness result is generalized by Mackey to hold for all locally compact second countable abelian groups G and is called the "Stone-von Neumann theorem". That is, there is, up to unitary equivalence, only one pair of irreducible unitary representations U of G and V of $\widehat{G}$, such that

$$U(a)V(b) = \langle b, a\rangle V(b)U(a)$$

for all $a \in G$ and $b \in \widehat{G}$ (see also Example 3.1 below).

In general, states of a quantum mechanical system may be considered as elements ψ of a Hilbert space $\mathcal{H}$ and observables as self-adjoint operators T on $\mathcal{H}$, such that the result of a measurement of T is given by the expected value $\langle T\psi, \psi\rangle$. The dynamical evolution of a system is determined by a self-adjoint operator H through $T(t) = e^{itH}Te^{-itH}$ or $\psi(t) = e^{-itH}\psi$ for t in $\mathbb{R}$, so that the expected value at t is

$$\langle e^{itH}Te^{-itH}\psi, \psi\rangle = \langle Te^{-itH}\psi, e^{-itH}\psi\rangle. \tag{5}$$

Moreover, there is a need for a study of families of operators, for example for the consideration of spectral decomposition of a single operator. The "rings of operators" that Murray and von Neumann considered in the first place are the weakly closed and self-adjoint subalgebras of $B(\mathcal{H})$ containing the identity operator. This class of operator algebras is now called von Neumann algebras. In this framework, the quantum observables are identified with the self-adjoint elements of such operator algebras.

1.2 Abstract approach

One can argue that it is sufficient to consider uniformly closed and self-adjoint subalgebras of $B(\mathcal{H})$, and therefore the notion C^* appears, for closed *-subalgebra.

Gelfand and Naimark (and Segal) then discover that C^*-algebras may also be studied abstractly, without any reference to operators on Hilbert spaces. That is, a C^*-algebra A can be defined axiomatically as a Banach algebra together with an involution $A \to A$, $x \mapsto x^*$ such that

$$\|x^*x\| = \|x\|^2.$$

Then, for every such (abstract) C^*-algebra A there exists a Hilbert space $\mathcal{H}$ and an injective *-homomorphism $\pi\colon A \to B(\mathcal{H})$. That is, $A \cong \pi(A) \subset B(\mathcal{H})$, as every *-homomorphism is norm-decreasing and thus continuous.

The algebraic structure in a C^*-algebra is strong. In fact, $\|x\|^2$ coincides with the spectral radius of x^*x so that there is only one norm on a *-algebra making it a C^*-algebra.

Furthermore, Gelfand and Naimark show that for every commutative C^*-algebra A there is a locally compact Hausdorff space X such that $A \cong C_0(X)$, the set of complex-valued continuous functions on X vanishing at infinity with pointwise operations and sup-norm. Moreover, two commutative C^*-algebras are isomorphic if and only if their associated topological spaces are homeomorphic. There is a (contravariant) category equivalence between the category of unital commutative C^*-algebras with *-homomorphism and the category of compact Hausdorff spaces with continuous maps. There is also a version of this result relating nonunital commutative C^*-algebras with locally compact noncompact Hausdorff spaces. Thus, topological properties of X can be translated into algebraic properties of $C_0(X)$, and vice versa, and the theory of C^*-algebras is often referred to as noncommutative topology in the modern language.

For example, let X be a compact Hausdorff space. If f is a projection in $C(X)$, that is, $\overline{f(x)} = f(x) = f(x)^2$ for all $x \in X$, then f can only take the values 0 and 1. Hence, X is connected if and only if $C(X)$ is projectionless.

Open and closed sets of X correspond to ideals and quotients of $C_0(X)$, respectively. Clearly, $C_0(X)$ is simple only if $X = \{*\}$. In the noncommutative case, on the other hand, the theory is much more intriguing, and highly nontrivial C^*-algebras can still be simple.

2 Projective unitary representations

The importance of (projective) unitary representations in the theory of C^*-algebras should now be obvious from the previous section. In particular, the way W and $\widetilde{W}$ are obtained above indicate a connection between projective representations of a group and ordinary representations of an extension of that group. In addition, since two states of a quantum mechanical system are equivalent if they are scalar multiples of each other, states are really elements of a projective Hilbert space $P\mathcal{H} = \mathcal{H}/\mathbb{C}1$.

The original approach concerns representations of the group $\mathbb{R}$ and then generalizations to locally compact second countable abelian groups. However, we will delay the discussion of locally compact groups until Section 2.3, and first focus on (arbitrary) discrete groups.

All of the five included papers deal with unitary representations of groups. In Paper III and IV, we study C^*-algebras associated with projective unitary representations of discrete groups in detail, and the consideration of locally compact groups is needed for Paper V.

2.1 Twisted group C^*-algebras

Let G be a discrete group and $\mathcal{H}$ a nontrivial Hilbert space. The automorphism group of $P\mathcal{H}$ is the projective unitary group $PU(\mathcal{H})$, that is, the quotient of $U(\mathcal{H})$ by its center, i.e.

$$PU(\mathcal{H}) = U(\mathcal{H})/\mathbb{T}1_{\mathcal{H}}.$$

A projective unitary representation of G is a homomorphism $G \to PU(\mathcal{H})$. Every lift of a projective representation to a map $U\colon G \to U(\mathcal{H})$ satisfies

$$U(a)U(b) = \sigma(a,b)U(ab) \tag{6}$$

for all $a, b \in G$ and some function $\sigma\colon G \times G \to \mathbb{T}$. From the associativity of U and by requiring that $U(e) = 1_{\mathcal{H}}$, the identities

$$\begin{aligned} \sigma(a,b)\sigma(ab,c) &= \sigma(a,bc)\sigma(b,c) \\ \sigma(a,e) = \sigma(e,b) &= 1 \end{aligned} \tag{7}$$

must hold for all elements $a, b, c \in G$. Motivated by these observations, any function $\sigma\colon G \times G \to \mathbb{T}$ satisfying (7) and is called a *multiplier of* G, and any map $U\colon G \to U(\mathcal{H})$ satisfying (6) is called a *σ-projective unitary representation of G on $\mathcal{H}$*.

The lift of a homomorphism $G \to PU(\mathcal{H})$ up to U is not unique, but any other lift is of the form βU for some function $\beta\colon G \to \mathbb{T}$. Consequently, one says that two multipliers σ and τ are *similar* and writes $\sigma \sim \tau$ if

$$\tau(a,b) = \beta(a)\beta(b)\overline{\beta(ab)}\sigma(a,b) \tag{8}$$

for all $a, b \in G$ and some $\beta\colon G \to \mathbb{T}$. The set of similarity classes of multipliers of G is an abelian group under pointwise multiplication. This group was originally called the *Schur multiplier* of G, but it coincides with the second cohomology group $H^2(G, \mathbb{T})$ consisting of 2-*cocycles on G with values in* $\mathbb{T}$.

Let σ be a multiplier of G. The Banach *-algebra $\ell^1(G,\sigma)$ is defined as the Banach space $\ell^1(G)$ together with twisted convolution and involution given by

$$\begin{aligned} (f * g)(a) &= \sum_{b \in G} f(b)\sigma(b, b^{-1}a)g(b^{-1}a) \\ f^*(a) &= \overline{\sigma(a, a^{-1})f(a^{-1})} \end{aligned}$$

for elements f, g in $\ell^1(G)$.

The full twisted group C^*-algebra $C^*(G,\sigma)$ is the universal enveloping algebra of $\ell^1(G,\sigma)$, that is, the completion of $\ell^1(G,\sigma)$ with respect to the norm $\|\cdot\|_{\max}$ given by

$$\|f\|_{\max} = \sup\{\|\pi(f)\| : \pi \text{ is representation of } \ell^1(G,\sigma)\}.$$

Let i_σ denote the canonical injection of G into $C^*(G,\sigma)$. Then $C^*(G,\sigma)$ satisfies the following universal property. Every σ-projective unitary representation of G on some Hilbert space $\mathcal{H}$ (or in some C^*-algebra A) factors uniquely through i_σ.

The *left regular σ-projective unitary representation λ_σ of G on $B(\ell^2(G))$* is given by

$$(\lambda_\sigma(a)\xi)(b) = \sigma(a, a^{-1}b)\xi(a^{-1}b).$$

The integrated form of λ_σ on $\ell^1(G,\sigma)$ is defined by

$$\lambda_\sigma(f) = \sum_{a\in G} f(a)\lambda_\sigma(a).$$

The reduced twisted group C^*-algebra and the twisted group von Neumann algebra of (G,σ), $C_r^*(G,\sigma)$ and $W^*(G,\sigma)$ are, respectively, the C^*-algebra and the von Neumann algebra generated by $\lambda_\sigma(\ell^1(G,\sigma))$, or equivalently by $\lambda_\sigma(G)$.

If τ is similar with σ, then in all three cases, the operator algebras associated with (G,τ) and (G,σ) are isomorphic.

Moreover, there is a natural one-to-one correspondence between the representations of $C^*(G,\sigma)$ and the σ-projective unitary representations of G. In particular, λ_σ extends to a *-homomorphism of $C^*(G,\sigma)$ onto $C_r^*(G,\sigma)$. If G is amenable, then λ_σ is faithful. Also, if $\sigma = 1$, faithfulness of λ_σ implies that G is amenable, but it is not known whether this holds for nontrivial σ. The reason why the argument does not carry over to the twisted case is that there is in general no trivial 1-dimensional representation of $C^*(G,\sigma)$.

2.2 Cohomology of groups

Let G be a discrete group. Denote the group of all multipliers of G by $Z^2(G,\mathbb{T})$ and the group of all trivial multipliers of G by $B^2(G,\mathbb{T})$, so that

$$H^2(G,\mathbb{T}) = Z^2(G,\mathbb{T})/B^2(G,\mathbb{T}),$$

where $\mathbb{T}$ is regarded as a $\mathbb{Z}G$-module for which G acts trivially. To analyze the projective representations of G and its associated twisted group C^*-algebras, it is useful to compute its cohomology group $H^2(G,\mathbb{T})$, and there are several approaches. In particular, it is interesting to study how the structure of $C^*(G,\sigma)$ varies with σ and to determine precisely when $C^*(G,\sigma)$ and $C^*(G,\tau)$ are isomorphic also when $\tau \not\sim \sigma$.

For the results mentioned below, we refer to the books by Brown [3] and Weibel [25].

Central extensions Recall that two extensions E_1 and E_2 of $\mathbb{T}$ by G are equivalent if there exists a homomorphism $f\colon E_1 \to E_2$ such that the diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{T} \arrow[r] \arrow[d, "="'] & E_1 \arrow[r] \arrow[d, "f"] & G \arrow[r] \arrow[d, "="] & e \\
1 \arrow[r] & \mathbb{T} \arrow[r] & E_2 \arrow[r] & G \arrow[r] & e
\end{tikzcd}$$

commutes. Then f must be an isomorphism by the "five-lemma". Denote by $\mathrm{Ext}(G, \mathbb{T})$ the set of equivalence classes of (algebraic) central extensions of $\mathbb{T}$ by G. Then there is a bijection

$$\mathrm{Ext}(G, \mathbb{T}) \cong H^2(G, \mathbb{T}).$$

In particular, if σ is a multiplier of G, then the corresponding extension is given as the group G^σ defined by the product

$$(z, a)(w, b) = (zw\sigma(a, b), ab) \tag{9}$$

on $\mathbb{T} \times G$. The trivial element in $\mathrm{Ext}(G, \mathbb{T})$ corresponds to the direct product $\mathbb{T} \times G$ and is the only split extension. Since every extension of a free group splits, $H^2(G, \mathbb{T})$ is trivial for all free groups G. Moreover, if G is abelian, then every abelian central extension of $\mathbb{T}$ by G is trivial in $\mathrm{Ext}(G, \mathbb{T})$. Hence, a multiplier of an abelian group is trivial if and only if it is symmetric.

Homology of groups The universal coefficient theorem gives an isomorphism

$$H^2(G, \mathbb{T}) \cong \mathrm{Hom}(H_2(G, \mathbb{Z}), \mathbb{T}),$$

that is, a "duality" between homology and cohomology of groups. Here is a few examples from Brown's book:

- Let $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ be the cyclic group of order n. Then $H_2(\mathbb{Z}_n, \mathbb{Z}) = \{0\}$.
- Let $F(S)$ be a free group on a set S. Then $H_2(F(S), \mathbb{Z}) = \{0\}$.
- More generally, let $G = \langle S \mid R \rangle$, that is, $G = F/N$ where $F = F(S)$ is the free group on the set S and N is the normal subgroup of F generated by the relations R. Then Hopf's formula gives that

$$H_2(G, \mathbb{Z}) = (N \cap [F, F])/[F, N].$$

One can apply a Mayer-Vietoris sequence to compute the homology of a free product of groups and obtain that

$$H_2(G_1 * G_2, \mathbb{Z}) \cong H_2(G_1, \mathbb{Z}) \oplus H_2(G_2, \mathbb{Z}).$$

By dualizing, we get that

$$H^2(G_1 * G_2, \mathbb{T}) \cong H^2(G_1, \mathbb{T}) \oplus H^2(G_2, \mathbb{T}),$$

and an explicit description of the multipliers is given in [15, Section 5] and [19, Section 4]. The facts stated above illustrates that there are several ways to see that $H^2(\mathbb{F}_n, \mathbb{T})$ is trivial for all $n \geq 1$. Moreover, the Künneth formula gives that

$$H_2(G_1 \times G_2, \mathbb{Z}) \cong H_2(G_1, \mathbb{Z}) \oplus H_2(G_2, \mathbb{Z}) \oplus (H_1(G_1, \mathbb{Z}) \otimes_{\mathbb{Z}} H_1(G_2, \mathbb{Z})),$$

where $H_1(G, \mathbb{Z})$ is the abelianization of G, and thus,

$$H^2(G_1 \times G_2, \mathbb{T}) \cong H^2(G_1, \mathbb{T}) \oplus H^2(G_2, \mathbb{T}) \oplus \mathrm{Hom}(G_1, \mathrm{Hom}(G_2, \mathbb{T})).$$

By applying Mackey's theorem [14, Theorem 9.4], one can compute the multipliers up to similarity as explained in [19, Section 3].

Group extensions and semidirect products If one has a short exact sequence of discrete groups,

$$e \longrightarrow H \longrightarrow G \longrightarrow K \longrightarrow e,$$

one may try to apply a Lyndon-Hochschild-Serre spectral sequence on this to compute the group homology or cohomology. Of course, in general there might be complicated to compute the boundary maps.

The drawback with applying purely homological techniques is that one does not get an explicit description of the multipliers up to similarity. To study twisted group C^*-algebras, we want a concrete family of multipliers in $Z^2(G, \mathbb{T})$, that represents the similarity classes in $H^2(G, \mathbb{T})$.

In some cases, for example, $G = \mathbb{Z}^n$ or $\mathbb{Z}_n$, the group $Z^2(G, \mathbb{T})$ is known as well. Apart from this, one of the most important techniques used for explicit computations is given by Mackey in [14, Theorem 9.6] for semidirect products. That is, this applies if the short exact sequence above splits, but it is easier to handle the calculations if H is abelian.

In Paper IV [21, Section 2], we apply this technique to give an explicit description of a family of representatives of $H^2(G, \mathbb{T})$ when G is a free nilpotent group of class 2.

2.3 Locally compact groups

We now give a brief explanation of projective unitary representations of locally compact groups and twisted group C^*-algebras associated with these.

First, if G is a topological group that is T_0 (i.e. points are topologically distinguishable), then it is also completely regular and Hausdorff, so T_0 is therefore often part of the definition of a topological group (see [10, p. 83]). In particular, when we consider a locally compact group, we will always mean a locally compact T_0 group.

Every locally compact group G will be equipped with a left Haar measure μ and the other spaces in question with the obvious Borel measures. Then a multiplier σ of G is a measurable function $G \times G \to \mathbb{T}$ such that (7) hold, and a σ-projective unitary representation of G on a Hilbert space $\mathcal{H}$ is a measurable function $G \to \mathcal{U}(\mathcal{H})$ satisfying (6). As above, we say that two multipliers σ and τ are similar if there is a measurable function $G \to \mathbb{T}$ such that (8) holds. The topological structure of $H^2(G,\mathbb{T}) = Z^2(G,\mathbb{T})/B^2(G,\mathbb{T})$ is handled by Moore in [16].

The Banach *-algebra $L^1(G,\sigma)$ is defined as the Banach space $L^1(G)$ together with twisted convolution and involution given by

$$(f * g)(a) = \int_G f(b)\sigma(b, b^{-1}a)g(b^{-1}a)\, d\mu(b),$$
$$f^*(a) = \Delta(a)^{-1}\overline{\sigma(a, a^{-1})f(a^{-1})},$$

for elements f, g in $L^1(G)$ and the modular function Δ of G.

Similarly as for the discrete case, one defines the left regular σ-projective unitary representation λ_σ of G on $B(L^2(G))$, as well as the full and reduced twisted group C^*-algebras and the group von Neumann algebra. Also, λ_σ is faithful on $C^*(G,\sigma)$ whenever G is amenable. However, $C^*(G,\sigma)$ and $C^*_r(G,\sigma)$ are unital only if G is discrete and the canonical map i_σ is in general a map from G into the multiplier algebra of $C^*(G,\sigma)$. The twisted group C^*-algebras are separable precisely when G is second countable.

To avoid topological issues, we now assume that G is a second countable locally compact group (or an arbitrary discrete group, see [26, Section D.3]). For a multiplier σ of G, the algebraic central extension G^σ of $\mathbb{T}$ by G defined by (9) has a unique second countable locally compact topology such that the Borel structures of G^σ and $\mathbb{T} \times G$ coincide and $\mu_{G^\sigma} = \mu_{\mathbb{T}} \times \mu_G$ is a left Haar measure on G^σ, by [13]. If $\mathrm{Ext}(G,\mathbb{T})$ denotes the set of locally compact second countable central extensions of $\mathbb{T}$ by G, then $\sigma \mapsto G^\sigma$ gives a bijection $H^2(G,\mathbb{T}) \to \mathrm{Ext}(G,\mathbb{T})$, by [16].

There is a 1-1-correspondence between σ-projective unitary representations of G and unitary representations of G^σ satisfying $(z, e) \mapsto z1_{\mathcal{H}}$, and $C^*(G,\sigma)$ is a quotient of $C^*(G^\sigma)$.

Compact groups and the twisted Peter-Weyl theorem Let G be a compact group and σ a multiplier of G. If U and U' are two σ-projective unitary representations of G on $\mathcal{H}$ and $\mathcal{H}'$, respectively, we can in the usual way form the σ-projective unitary representation $U \oplus U'$ on $\mathcal{H} \oplus \mathcal{H}'$.

Let $\widehat{(G, \sigma)}$ denote the set of all equivalence classes of irreducible σ-projective representations of G. We reserve the symbol d_U for the dimension of the representation space for $[U] \in \widehat{(G, \sigma)}$. Then the following hold:

- Each irreducible σ-projective unitary representation of a compact group G is finite-dimensional. The left regular σ-projective unitary representation λ of G is unitarily equivalent to the direct sum of irreducible ones, that is,

$$\lambda \simeq \bigoplus_{[U]\in\widehat{(G,\sigma)}} \bigoplus_{j=1}^{d_U} U.$$

- The twisted group C^*-algebra decomposes into a direct sum of matrix algebras, that is,

$$C^*(G, \sigma) \cong \bigoplus_{[U]\in\widehat{(G,c)}} M_{d_U}(\mathbb{C}),$$

and for $f \in L^1(G, \sigma)$, the isomorphism is given by

$$f \mapsto (U(f))_{[U]\in\widehat{(G,\sigma)}}.$$

- For every nontrivial $a \in G$, there exists an irreducible σ-projective unitary representation U of G such that $U(a) \neq I$.

  In particular, $C^*(G, \sigma)$ is residually finite-dimensional, that is, has a separating family of finite-dimensional representations.

3 Dynamical systems and crossed products

In this section we consider (twisted) group actions on C^*-algebras. Again, this is motivated by topological dynamics in the commutative case. To see this, let X be a compact metric space and φ a homeomorphism $X \to X$. Then φ induces an action of $\mathbb{Z}$ on $C(X)$ by $n \cdot f(x) = f(\varphi^{-n}(x))$.

More generally, let G be a locally compact group acting on a locally compact Hausdorff space X, i.e. (X, G) is a transformation group. Define the induced action α of G on $C_0(X)$ by $\alpha_g(f)(x) = f(g^{-1} \cdot x)$. Then $(C_0(X), G, \alpha)$ is a so-called C^*-dynamical system.

Dynamical systems are dealt with in the enclosed Paper I and II (the unital twisted case) and Paper V (the separable case).

3.1 C^*-dynamical systems

In general, a C^*-dynamical system is a triple (A, G, α) consisting of a C^*-algebra A, a locally compact group G, and a continuous homomorphism $\alpha\colon G \to \operatorname{Aut} A$ (i.e. $g \mapsto \alpha_g(a)$ is continuous for all $a \in A$). There are two cases that are of particular interest:

- A is separable and G is second countable,
- A is unital and G is discrete.

Let (A, G, α) be a C^*-dynamical system and let $C_c(G, A)$ be the set of continuous functions $G \to A$ with compact support. For $f, g \in C_c(G, A)$, define $f * g$, f^*, and $\|f\|_1$ by

$$\begin{aligned} f * g(a) &= \int_G f(b)\alpha_b(g(b^{-1}a))\, d\mu(b), \\ f^*(a) &= \Delta(a^{-1})\alpha_a(f(a^{-1})^*), \\ \|f\|_1 &= \int_G \|f(a)\|\, d\mu(a). \end{aligned}$$

Banach space valued integration and the Bochner integral are treated in Williams' book [26, Section B.1]. The completion of $C_c(G, A)$ with respect to $\|\cdot\|_1$ is a Banach *-algebra denoted by $L^1(A, G, \alpha)$. A covariant representation of (A, G, α) is a pair (π, U) consisting of a representation π of A on a Hilbert space $\mathcal{H}$ and a unitary representation U of G on $\mathcal{H}$ satisfying

$$\pi \circ \alpha_a = \operatorname{Ad}(U(a))\pi$$

for all $a \in G$. There is 1-1 correspondence between covariant representations of (A, G, α) and representations of $L^1(A, G, \alpha)$. In particular, a covariant representation (π, U) of (A, G, α) induces a representation $\pi \times U$ of $L^1(A, G, \alpha)$ given by

$$(\pi \times U)(f) = \int_G \pi(f(a))U(a)\, d\mu(a).$$

For $f \in C_c(G, A)$, define

$$\begin{aligned} \|f\|_{\max} &= \sup\{\|\pi(f)\| : \pi \text{ is representation of } L^1(A, G, \alpha)\} \\ &= \sup\{\|\pi \times U(f)\| : (\pi, U) \text{ is a covariant representation of } (A, G, \alpha)\}. \end{aligned}$$

The completion of $C_c(G, A)$ with respect to $\|\cdot\|_{\max}$ is the crossed product of A by G and is denoted by $A \rtimes_\alpha G$.

An interesting example of a transformation group C^*-algebra is when G is a locally compact group and H is a subgroup of G acting on G by left translation. A special case of this example is when $H = G$.

Example 3.1 (The Stone-von Neumann theorem, part II)**.** Let G be a locally compact group. Then

$$C_0(G) \rtimes_{\mathrm{lt}} G \cong \mathcal{K}(L^2(G)),$$

where $\mathcal{K}(L^2(G))$ is the compact operators on $L^2(G)$, which is a simple C^*-algebra.

3.2 Coactions and duality theory

If G is a locally compact abelian group, then $C^*(G) \cong C^*_r(G) \cong C_0(\widehat{G})$ via the Fourier transform. Moreover, if (A, G, α) is a C^*-dynamical system and G is abelian, then there is an action $\hat{\alpha}$ of $\widehat{G}$ on $A \rtimes_\alpha G$ such that

$$(A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \widehat{G} \cong A \otimes \mathcal{K}(L^2(G)).$$

Motivated by the goal of extending this result to nonabelian groups, one introduces coactions, so that if G is abelian, then a coaction of G on a C^*-algebra A is an action of $\widehat{G}$ on A.

Moreover, an action α of G on A may be identified with a map

$$\widetilde{\alpha}\colon A \to M(A \otimes C_0(G)) \cong C_b(G, M(A)), \quad \widetilde{\alpha}(x)(a) = \alpha_a(x), \quad x \in A, a \in G.$$

Inspired by this fact, one says that a coaction of G on A is an injective nondegenerate homomorphism $\delta\colon A \to M(A \otimes C^*(G))$ satisfying

$$\begin{gathered} \overline{\operatorname{span}}\{\delta(A)(1 \otimes C^*(G))\} = A \otimes C^*(G) \\ (\delta \otimes i) \circ \delta = (i \otimes \delta_G) \circ \delta, \end{gathered}$$

where the coaction δ_G of G on $C^*(G)$ is given by $C^*(G) \to M(C^*(G) \otimes C^*(G))$, $a \mapsto a \otimes a$. The associated (co-)crossed product of (A, G, δ) is a C^*-algebra $A \rtimes_\delta G$ whose representations are the same as the covariant representations of (A, G, δ) (see [9, Appendix A.5]).

In Paper V [11, Appendix], we consider an injective homomorphism $\varphi\colon H \to G$, its integrated form $\pi_\varphi\colon C^*(H) \to M(C^*(G))$, and the coaction of G on $C^*(H)$ defined by $\delta = (\overline{\mathrm{id} \otimes \pi_\varphi}) \circ \delta_H$.

3.3 Twisted C^*-dynamical systems

To unify the constructions of twisted group C^*-algebras and crossed products, we now consider twisted C^*-dynamical systems. Since we only need this construction in the unital case, we will assume that the C^*-algebras are unital and that the groups are discrete. The more general construction of separable twisted C^*-dynamical systems is nicely treated by Packer and Raeburn [23].

A (unital) twisted C^*-dynamical system is a quadruple (A, G, α, ω) consisting of a unital C^*-algebra A, a discrete group G, and maps $\alpha\colon G \to \operatorname{Aut} A$ and $\omega\colon G \times G \to \mathcal{U}(A)$ satisfying

$$\begin{gathered}\alpha_a\alpha_b = \operatorname{Ad}(\omega(a,b))\alpha_{ab}\\ \omega(a,b)\omega(ab,c) = \alpha_a(\omega(b,c))\omega(a,bc)\\ \omega(e,e) = 1_A\end{gathered}$$

for all $a, b, c \in G$, and from this it is easily deduced that

$$\omega(a,e) = \omega(e,b) = 1_A\,, \quad \alpha_e = \mathrm{id}_A\,, \quad \omega(a,a^{-1}) = \alpha_a(\omega(a^{-1},a)).$$

Twisted C^*-dynamical systems coming from discrete groups were introduced by Zeller-Meier [27] in the case where ω is central-valued, and then in more generality by Busby and Smith [4].

A covariant representation of a twisted C^*-dynamical system (A, G, α, ω) is a pair (π, U) consisting of a representation π of A on a Hilbert space $\mathcal{H}$ and a map $U\colon G \to \mathcal{U}(\mathcal{H})$ satisfying

$$\begin{gathered}U(a)U(b) = \pi(\omega(a,b))U(ab)\\ \pi\circ\alpha_a = \operatorname{Ad}(U(a))\pi\end{gathered}$$

for all $a, b \in G$.

We equip the Banach space $\ell^1(G, A)$ with twisted convolution and involution given by

$$\begin{gathered}(f * g)(a) = \sum_{b\in G} f(b)\alpha_b(g(b^{-1}a))\omega(b,b^{-1}a)\\ f^*(a) = \omega(a,a^{-1})^*\alpha_a(f(s^{-1}))^*\end{gathered}$$

and denote the resulting Banach *-algebra by $\ell^1(A, G, \alpha, \omega)$. There is 1-1 correspondence between covariant representations of (A, G, α, ω) and representations of $\ell^1(A, G, \alpha, \omega)$. In particular, every covariant representation (π, U) of a twisted C^*-dynamical system (A, G, α, ω) induces a representation, denoted by $\pi \times U$, of $\ell^1(A, G, \alpha, \omega)$ defined by

$$(\pi \times U(f)\xi)(a) = \sum_{b\in G} \pi(\alpha_{a^{-1}}(f(b)))(U(b)\xi)(a).$$

Define now a C^*-norm on $\ell^1(A, G, \gamma, \omega)$ by

$$\begin{aligned}\|f\|_{\max} &= \sup\{\|\pi(f)\| : \pi \text{ is a representation of } \ell^1(A, G, \alpha, \omega)\} \\ &= \sup\{\|\pi \times U(f)\| : (\pi, U) \text{ is a covariant representation of } (A, G, \alpha, \omega)\}.\end{aligned}$$

The full twisted crossed product $A \rtimes_{(\alpha,\omega)} G$ is the completion of $\ell^1(A, G, \gamma, \omega)$ with respect to $\|\cdot\|_{\max}$, that is, the enveloping C^*-algebra of $\ell^1(A, G, \gamma, \omega)$.

The following example was one of the motivations for studying twisted crossed products in the first place and is essential when constructing induced representations.

Example 3.2. Let H be a normal subgroup of a group G with quotient group $K = G/H$, that is, we have a short exact sequence of groups

$$e \longrightarrow H \longrightarrow G \longrightarrow K \longrightarrow e.$$

Let σ be a multiplier of G and σ' the restriction to H. Then we may decompose $C^*(G, \sigma)$ into a twisted crossed product ([20] based on [23, Theorem 4.1])

$$C^*(G, \sigma) \cong C^*(H, \sigma') \rtimes_{(\alpha,\omega)} K,$$

where

$$\begin{aligned}\alpha_a(i_H(b)) &= i_G(n(a)) i_G(b) i_G(n(a))^* \\ \omega(a, b) &= i_G(n(a)) i_G(n(b)) i_G(n(ab))^*\end{aligned}$$

for a normalized section n for the quotient map $G \to K$.

This decomposition becomes an ordinary crossed product if and only if the sequence splits and $\sigma = 1$ when restricted to K.

Let π be a representation of A on a Hilbert space $\mathcal{H}$ and define the covariant representation $(\widetilde{\pi}, \lambda)$ on $\ell^2(G, \mathcal{H})$ by

$$\begin{aligned}(\widetilde{\pi}(x)\xi)(a) &= \pi(\alpha_{a^{-1}}(x))\xi(a) \\ (\lambda(b)\xi)(a) &= \pi(\omega(a^{-1}, b))\xi(b^{-1}a)\end{aligned}$$

for $a, b \in G$, $\xi \in \ell^2(G, \mathcal{H})$, and $x \in A$, and set $\operatorname{Ind} \pi = \widetilde{\pi} \times \lambda$. For $f \in \ell^1(A, G, \alpha, \omega)$, define

$$\|f\|_{\mathrm{red}} = \sup\{\|\operatorname{Ind} \pi(f)\| : \pi \text{ is a representation of } A\} = \|\operatorname{Ind} \rho(f)\|$$

for some faithful representation ρ of A. The completion of $\ell^1(A, G, \alpha, \omega)$ with respect to $\|\cdot\|_{\mathrm{red}}$ is the reduced twisted crossed product of A by G and is denoted by $A \rtimes_{(\alpha,\omega),r} G$. If G is amenable, then it is well known that the full and reduced twisted crossed products are isomorphic.

Moreover, $\mathrm{Ind}\,\pi$ is faithful on $A \rtimes_{(\alpha,\omega),r} G$ if and only if $\{\pi \circ \alpha_a\}_{a \in G}$ is separating for A, as explained in [20] based on [27, Théorèm 4.11]. The next results can be deduced from Mackey's work [13], but is shown in [20] and [2, Appendix] as well. First, the following are equivalent:

(i) $\mathrm{Ind}\,\pi$ is irreducible,

(ii) π is irreducible and the stabilizer group $\{a \in G \mid \pi \circ \alpha_a \simeq \pi\}$ is trivial,

(iii) $[\pi] \in \widehat{A}$ is a free point for the natural action of G on $\widehat{A}$, that is, $[\pi] \neq [\pi \circ \alpha_a]$ for all $a \neq e$.

Furthermore, suppose that π_1 and π_2 are irreducible representations such that $\pi_1 \circ \alpha_a \not\simeq \pi_2$ for all $a \in G$. Then $\mathrm{Ind}\,\pi_1 \not\simeq \mathrm{Ind}\,\pi_2$.

4 Some aspects of structure and classification

The structure and classification theory for C^*-algebras are vast subjects, and we will only mention a few aspects here. One of the most interesting topics is investigation of the ideal structure (and especially simplicity) which is central in all of the included papers.

Moreover, K-theory plays a major role in the classification program for C^*-algebras. Even though K-theory is not dealt with directly in any of the included papers, we consider the properties of being nuclear (Paper III and V) and purely infinite (Paper V), since these notions are very useful for classification by K-theory.

Representation theory is also central in all of the included papers, and this motivates the study of Morita equivalence of C^*-algebras.

4.1 Ideals

By an ideal of a C^*-algebra we will always mean a closed (and thus self-adjoint) two-sided ideal. As usual, a C^*-algebra is *simple* if it contains no proper nontrivial ideals, and *prime* if any pair of nonzero ideals has nonzero intersection. A C^*-algebra with a faithful irreducible representation is called *primitive*. In general, primitivity is a property between simplicity and primeness. Obviously, every simple C^*-algebra is primitive, and it is not difficult to see that every primitive C^*-algebra is prime. Conversely, every prime and separable C^*-algebra is primitive by a result of Dixmier [6], that is, the notions of primeness and primitivity are equivalent for separable C^*-algebras. There are rather few examples of prime nonprimitive C^*-algebras (the first was presented by Weaver [24]). It is also well known that every prime C^*-algebra has trivial center, so that we have the following:

$$\text{simplicity} \Longrightarrow \text{primitivity} \Longrightarrow \text{primeness} \Longrightarrow \text{trivial center}$$

Moreover, a von Neumann algebra is a *factor* if it has trivial center, or equivalently, if it contains no proper nontrivial weakly closed ideals. If A is a concrete unital C^*-algebra, and A'' is a factor, then A is prime. Hence, a von Neumann algebra is a factor if and only if it is prime (as a C^*-algebra).

Following Dixmier [7], a C^*-algebra A is *antiliminary* if $\pi(A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$ for all, or equivalently, some faithful irreducible representations π of A.

Let A be a separable unital C^*-algebra. Then, by [7], A is primitive and antiliminary if and only if the pure state space of A is weak*-dense in the state space of A.

4.2 Kirchberg algebras

A C^*-algebra is called *nuclear* if the identity map, as a completely positive map, approximately factors through matrix algebras. Equivalently, A is nuclear if $A \otimes_{\min} B \cong A \otimes_{\max} B$ for all C^*-algebras B, or yet equivalently, if A'' is an injective von Neumann algebra.

A simple C^*-algebra A is *purely infinite* if and only if every hereditary C^*-subalgebra of A contains an infinite projection. In the separable case, this is the same as saying that every corner $\overline{xAx^*}$ of A contains an infinite projection.

A *Kirchberg algebra* is a separable, simple, nuclear, purely infinite C^*-algebra in the UCT class (meaning KK-equivalent with a commutative C^*-algebra). Moreover, Kirchberg algebras are classifiable by K-theory, and it is therefore of interest to show that C^*-algebras coming from for example C^*-dynamical systems belong to this class.

4.3 Morita equivalence

Let A and B be C^*-algebras. Then A and B are *Morita equivalent* if there exists an $A-B$-imprimitivity bimodule. That is, if there is an $A-B$-bimodule E which is simultaneously a full left Hilbert A-module under an A-valued inner product ${}_A\langle \cdot, \cdot \rangle$ and a full right Hilbert B-module under a B-valued inner product $\langle \cdot, \cdot \rangle_B$ such that

$${}_A\langle \xi, \eta \rangle \zeta = \xi \langle \eta, \zeta \rangle_B$$

for all $\xi, \eta, \zeta \in E$.

One important feature of a Morita equivalence is that it gives a functorial correspondence between the representations of the algebras. In particular, the spectrum and primitive ideal spaces are homeomorphic.

If A and B are separable, they are Morita equivalent if and only if they are stably isomorphic.

4.4 Group C^*-algebras and crossed products

The trivial representation ι of any locally compact group G on $\mathbb{C}$ given by $\iota(g) = 1$ for all $g \in G$, induces a representation of $C^*(G)$ on $\mathbb{C}$. Hence, $C^*(G)$ will always have an ideal of codimension 1, called the augmented ideal. That is, $C^*(G)$ is never simple, unless G is trivial. Therefore, primitive and prime group C^*-algebras may be considered as the building blocks for the class of group C^*-algebras.

The problem of determining whether a group C^*-algebra is primitive seems hard in general. For example, primitivity of the group C^*-algebra of the group $\mathbb{F}_2 \times \mathbb{F}_2$ can be related to Connes' embedding problem [2, Remark 2.2]. In [17], Murphy gives some conditions and examples of primitive group C^*-algebras.

On the other hand, the reduced group C^*-algebra $C_r^*(G)$ can be simple for nontrivial G. Much work is done in the area of determining the class of C^*-simple groups, see e.g. de la Harpe [5]. Simplicity of $C_r^*(G)$ is in general unrelated to primitivity of $C^*(G)$.

Also, the full twisted group C^*-algebra $C^*(G, \sigma)$ may be simple when G is amenable. For example, by the work of Kleppner, it is known that if G is abelian, then $\operatorname{Prim} C^*(G, \sigma)$ is homeomorphic with $\widehat{S_\sigma}$, where

$$S_\sigma = \{a \in G \mid \sigma(a, b) = \sigma(b, a) \text{ for all } b \in G\},$$

and $C^*(G, \sigma)$ is simple if S_σ is trivial.

Moreover, if G is discrete, then $C^*(G, \sigma)$ is simple and nuclear if and only if $C_r^*(G, \sigma)$ is simple and nuclear, and for this to hold, we must have that G is amenable and σ is nontrivial. This gives another motivation for considering the twisted case. However, our main focus is to study primeness and primitivity of the full and reduced twisted group C^*-algebras corresponding to discrete groups.

Furthermore, we have the following (references given in [11]):

- If (A, G, α, ω) is a twisted C^*-dynamical system with A nuclear and G amenable, then $A \rtimes_{(\alpha,\omega)} G$ is nuclear.

- Let (A, G, α) be a C^*-dynamical system with $A = C_0(X)$ commutative and G amenable and discrete so that the action of G on X is topologically free. Then $A \rtimes_\alpha G$ is simple if the action of G on X is minimal and $A \rtimes_\alpha G$ is purely infinite if the action of G on X is locally contractive.

- If (A, G, α) is a C^*-dynamical system with A commutative and G amenable and discrete, then $A \rtimes_\alpha G$ belongs to the UCT class.

Finally, as an application of "Green's symmetric imprimitivity theorem", we get the following (see e.g. [26, p. 126]). Suppose that K and H are closed subgroups of a locally compact group G. Let K act by left multiplication on G and let H act by right multiplication on G. Then $C_0(K\backslash G) \rtimes_{\mathrm{rt}} H$ is Morita equivalent to $C_0(G/H) \rtimes_{\mathrm{lt}} K$.

5 Overview of the thesis

Paper I and II

In [2], we study the projective special linear groups $\mathrm{PSL}(n, \mathbb{Z})$ for $n \geq 2$. The main result is [2, Theorem 2.3], which says that $C^*(\mathrm{PSL}(2, \mathbb{Z}))$ is primitive, and also antiliminary. The proof of this result uses the techniques mentioned in the end of Section 3.3 (and Example 3.2), namely we construct a faithful irreducible representation through an inducing process [2, Theorem 2.1 and Appendix]. In [17] Murphy mentions that he knows no example of an icc group whose full group C^*-algebra is nonprimitive. When $n \geq 3$, we show that $C^*(\mathrm{PSL}(n, \mathbb{Z}))$ is nonprimitive so that for example $\mathrm{SL}(3, \mathbb{Z})$ provides such an example.

The main result of [1] is [1, Theorem 1.2], where we show that $C^*(G_1 * G_2)$ is primitive whenever G_1 and G_2 are countable discrete amenable groups such that $|G_1 - 1| \cdot |G_2 - 1| \geq 2$. Since $\mathrm{PSL}(2, \mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$, this is a generalization of the previous paper, and the technique of the proof is again an application of [2, Theorem 2.1 and Appendix]. However, the argument turns out to be combinatorially harder in this case. Moreover, in [1, Lemma 3.2, Corollary 3.3, and Corollary 3.4] we give conditions to ensure that $C^*(G_1 * G_2)$ is antiliminary whenever G_1 and G_2 satisfy the conditions of [1, Theorem 1.2].

Finally, we remark that in a recent preprint by Dykema and Torres-Ayala [8], related results are shown with a different approach.

Paper III

Let G be an arbitrary discrete group and σ a multiplier of G. The aim of [19] is to generalize results of Murphy [17] and Kleppner [12] and give precise conditions for primeness of $C_r^*(G, \sigma)$. Following [12], an element a of G is called *σ-regular* if $\sigma(a, b) = \sigma(b, a)$ whenever b commutes with a. Moreover, σ-regularity is a property of conjugacy classes, and we will say that (G, σ) satisfies *condition K* if every nontrivial σ-regular conjugacy class of G is infinite. The main result is [19, Theorem 2.7], which says that condition K on (G, σ) is equivalent with primeness of $C_r^*(G, \sigma)$. Also, [19, Corollary 2.8] gives that condition K on (G, σ) is necessary for primeness of $C^*(G, \sigma)$.

In the final sections, we consider the cases where $G = G_1 \times G_2$ and where $G = G_1 * G_2$. The direct product is in general harder to handle since a multiplier σ of G does not necessarily decompose nicely in this case, but has a "cross-term" as discussed in Section 2.2. In the free product case, we obtain with [1, Theorem 1.2] a generalization of [19, Theorem 4.1] to the twisted case.

Remark. Significant parts of [19, Section 2], especially [19, Lemma 2.2 and 2.4], were already obtained in [20], although rewritten here.

Paper IV

In [21], we study the free nilpotent groups of class 2 and rank n, denoted by $G(n)$. These groups may be considered as generalized Heisenberg groups with higher-dimensional center. Motivated by Packer [22], we compute the second cohomology group of $G(n)$ and give explicit formulas for the multipliers in [21, Theorem 2.7], by applying techniques of Mackey [14, Section 9]. Then we give conditions for simplicity of the twisted group C^*-algebras $C^*(G(n), \sigma)$ in [21, Section 4]. We also describe $C^*(G(n), \sigma)$ in terms of generators and relations in [21, Theorem 3.1], and as a continuous field over $\mathbb{T}^{\frac{1}{2}n(n-1)}$ with the noncommutative n-tori as fibers in [21, Theorem 1.1].

Paper V

Inspired by the work of Cuntz and Li on ring C^*-algebras, we give a crossed product construction of a family of C^*-algebras $\overline{\mathcal{Q}}$ associated with the a-adic numbers. We show that these algebras are nonunital Kirchberg algebras in the UCT class [11, Corollary 2.8].

The a-adic numbers are locally compact abelian groups that appear as Hausdorff completions of additive subgroups of $\overline{\mathbb{Q}}$, and the most commonly studied examples are the p-adic numbers $\mathbb{Q}_p$.

The main result is [11, Theorem 4.1] which says that $\overline{\mathcal{Q}}$ is Morita equivalent with a crossed product C^*-algebra coming from an $ax + b$-action on $\mathbb{R}$ of a certain subgroup of $\mathbb{Q} \rtimes \mathbb{Q}_+^\times$. The proof uses "Green's symmetric imprimitivity theorem" and relies especially on two additional results, a duality result for groups [11, Theorem 3.3], and a "subgroup of dual group theorem" that we prove in a more general setting, for coactions, in [11, Appendix].

Remark. The main results of [11] are also summarized in a preprint for a conference proceedings paper [18].

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