

C^* -ALGEBRAS GENERATED BY PROJECTIVE REPRESENTATIONS OF FREE NILPOTENT GROUPS

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ABSTRACT. We compute the two-cocycles (or multipliers) of the free nilpotent groups of class 2 and rank n and give conditions for simplicity of the corresponding twisted group C^* -algebras. These groups are representation groups for \mathbb{Z}^n and can be considered as a family of generalized Heisenberg groups with higher-dimensional center. Their group C^* -algebras are in a natural way isomorphic to continuous fields over $\mathbb{T}^{\frac{1}{2}n(n-1)}$ with the noncommutative n -tori as fibers. In this way, the twisted group C^* -algebras associated with the free nilpotent groups of class 2 and rank n may be thought of as “second order” noncommutative n -tori.

INTRODUCTION

The discrete Heisenberg group may be described as the group generated by three elements u_1 , u_2 , and v_{12} satisfying the commutation relations

$$[u_1, v_{12}] = [u_2, v_{12}] = 1 \quad \text{and} \quad [u_1, u_2] = v_{12}.$$

The group has received much attention in the literature, partly because it is one of the easiest examples of a nonabelian torsion-free group. Moreover, the continuous Heisenberg group (see below) is a connected nilpotent Lie group that arises in certain quantum mechanical systems.

As a natural consequence of this attention, several classes of generalized Heisenberg groups have been investigated. For example, in [14, 15] Milnes and Walters describe the four- and five-dimensional nilpotent groups, and in [11, 12] Lee and Packer study the finitely generated torsion-free two-step nilpotent groups with one-dimensional center.

In this paper, on the other hand, we will consider a family of generalized Heisenberg groups, denoted by $G(n)$ for $n \geq 2$, with larger center. The groups $G(n)$ are the so-called free nilpotent groups of class 2 and rank n and will be defined properly in Section 1. Here we also provide further motivation for our investigation of these groups. Inspired by the work of Packer [21] we compute the second cohomology group $H^2(G(n), \mathbb{T})$ of $G(n)$ and study the structure of the twisted group C^* -algebras $C^*(G(n), \sigma)$ associated with two-cocycles σ of $G(n)$.

Section 2 is devoted to two-cocycle calculations, where we decompose $G(n)$ into a semidirect product and apply techniques introduced by Mackey [13]. In particular, we will see that

$$H^2(G(n), \mathbb{T}) \cong \mathbb{T}^{\frac{1}{2}(n+1)n(n-1)},$$

and in Theorem 2.6 we give explicit formulas for the two-cocycles of $G(n)$ up to similarity.

Next, in Section 3 we describe $C^*(G(n), \sigma)$ as a universal C^* -algebra of a set of generators and relations. Then we construct the algebra that in a natural way appear as a continuous field over the compact space $H^2(G(n), \mathbb{T})$ with $C^*(G(n), \sigma)$ as fibers. We also explain that for $n = 2$, this algebra is the group C^* -algebra of the free nilpotent group of class 3 and rank 2.

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In Section 4 we investigate the center of $C^*(G(n), \sigma)$ and give conditions for simplicity of these twisted group C^* -algebras in Theorem 4.4 and Corollary 4.6.

Finally, in Section 5 we study the automorphism group of $G(n)$ and discuss isomorphism invariants of $C^*(G(n), \sigma)$ coming from $\text{Aut } G(n)$.

1. THE FREE NILPOTENT GROUPS $G(n)$ OF CLASS 2 AND RANK n

For each natural number $n \geq 2$, let $G(n)$ be the group generated by elements $\{u_i\}_{1 \leq i \leq n}$ and $\{v_{jk}\}_{1 \leq j < k \leq n}$ subject to the relations

$$[v_{jk}, v_{lm}] = [u_i, v_{jk}] = 1 \quad \text{and} \quad [u_j, u_k] = v_{jk} \quad (1)$$

for $1 \leq i \leq n$, $1 \leq j < k \leq n$, and $1 \leq l < m \leq n$. Clearly, $G(2)$ is the usual discrete Heisenberg group. For some purposes, it can be useful to set $G(1) = \langle u_1 \rangle \cong \mathbb{Z}$. Note that $G(n)$ is generated by n elements, while its dimension equals $n + \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1)$.

The group $G(n)$ is called the free nilpotent group of class 2 and rank n . Indeed, $G(n)$ is a free object on n generators in the category of nilpotent groups of step at most two. To see this, note first that $G(n)$ is the group generated by $\{u_i\}_{i=1}^n$ subject to the relations that all commutators of order greater than two involving the generators are trivial. Let $G'(n)$ be any other nilpotent group of step at most two and let $\{u'_i\}_{i=1}^n$ be any set of n elements in $G'(n)$. Then there is a unique homomorphism from $G(n)$ to $G'(n)$ that maps u_i to u'_i for $1 \leq i \leq n$. Of course, every free object on n generators in this category is isomorphic to $G(n)$. For a more extensive treatment of free nilpotent groups, see the article on Terence Tao's website [27] (see also 2. in the list below).

Furthermore, we will need the following concrete realization, say $\tilde{G}(n)$, of $G(n)$. For $n \geq 2$, we denote the elements of $\tilde{G}(n)$ by

$$r = (r_1, \dots, r_n, r_{12}, r_{13}, \dots, r_{n-1,n})^1,$$

where all entries are integers, and define multiplication by

$$r \cdot s = (r_1 + s_1, \dots, r_n + s_n, r_{12} + s_{12} + r_1 s_2, r_{13} + s_{13} + r_1 s_3, \dots, r_{n-1,n} + s_{n-1,n} + r_{n-1} s_n).$$

By letting u_i have 1 in the i 'th spot and 0 else and v_{jk} have 1 in the jk 'th spot and 0 else, the relations (1) are satisfied for these elements. Next, we define the map

$$\tilde{G}(n) \longrightarrow G(n), \quad r \longmapsto v_{12}^{r_{12}} \cdots v_{n-1,n}^{r_{n-1,n}} \cdot u_n^{r_n} \cdots u_1^{r_1},$$

and then it is not difficult to see that $\tilde{G}(n)$ is isomorphic to $G(n)$. Henceforth, we will not distinguish between $G(n)$ and the realization $\tilde{G}(n)$ just described, but this should cause no confusion.

Denote by $V(n)$ the subgroup of $G(n)$ generated by the v_{jk} 's. Then $V(n)$ coincides with the center $Z(G(n))$ of $G(n)$ and

$$V(n) = Z(G(n)) \cong \mathbb{Z}^{\frac{1}{2}n(n-1)}.$$

Indeed, both this and the next observations follow after noticing that

$$r \cdot s \cdot r^{-1} = (s_1, \dots, s_n, s_{12} + r_1 s_2 - s_1 r_2, \dots, s_{n-1,n} + r_{n-1} s_n - s_{n-1} r_n).$$

Moreover, consider the subgroups $G(n-1)$ and $H(n)$ of $G(n)$ defined by

$$G(n-1) = \langle u_i, v_{jk} : 1 \leq i \leq n-1, 1 \leq j < k \leq n-1 \rangle, \\ H(n) = \langle u_n, v_{jn} : 1 \leq j < n \rangle.$$

¹To be absolutely precise, the entries with double index are colexicographically ordered, that is, $(i, j) < (k, l)$ if $j < l$ or if $j = l$ and $i < k$.

Note that $G(n-1)$ sits inside $G(n)$ as a subgroup and that $H(n) \cong \mathbb{Z}^n$ is a normal subgroup of $G(n)$. Clearly, we have $G(n)/V(n) \cong \mathbb{Z}^n$ and $G(n)/H(n) \cong G(n-1)$. Therefore, there are short exact sequences

$$1 \longrightarrow V(n) \longrightarrow G(n) \longrightarrow \mathbb{Z}^n \longrightarrow 1$$

and

$$1 \longrightarrow H(n) \longrightarrow G(n) \longrightarrow G(n-1) \longrightarrow 1,$$

where the second one splits and the first does not. In particular, $G(n)$ is a central extension of \mathbb{Z}^n by $\mathbb{Z}^{\frac{1}{2}n(n-1)}$, showing that $G(n)$ is indeed a two-step nilpotent group.

To motivate our investigation of $G(n)$, we present a few aspects about these groups and some appearances in the literature.

1. Consider in the first place the *continuous* Heisenberg group. We will represent this group in two different ways, G_{matrix} and G_{wedge} , both with elements $(x, x') = (x_1, x_2, x') \in \mathbb{R}^3$, i.e. $x = (x_1, x_2) \in \mathbb{R}^2$, and with multiplication as follows. For G_{matrix} we define

$$(x_1, x_2, x')(y_1, y_2, y') = (x_1 + y_1, x_2 + y_2, x' + y' + x_1 y_2),$$

and for G_{wedge} we set

$$(x_1, x_2, x')(y_1, y_2, y') = (x_1 + y_1, x_2 + y_2, x' + y' + \frac{1}{2}(x_1 y_2 - x_2 y_1)).$$

One can deduce that $G_{\text{matrix}} \cong G_{\text{wedge}}$. To motivate the notation, note that G_{matrix} can be represented as matrix multiplication in $M_3(\mathbb{R})$ if one identifies

$$(x_1, x_2, x') \longleftrightarrow \begin{bmatrix} 1 & x_1 & x' \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{bmatrix},$$

and that the multiplication in G_{wedge} may be written as

$$(x, x')(y, y') = (x + y, x' + y' + \frac{1}{2}(x \wedge y)).$$

In general, the wedge product on \mathbb{R}^n is defined as a certain bilinear map (see e.g. [25, p. 79])

$$\mathbb{R}^n \times \mathbb{R}^n \rightarrow \bigwedge^2(\mathbb{R}^n),$$

where $\bigwedge^2(\mathbb{R}^n)$ is a $\frac{1}{2}n(n-1)$ -dimensional real vector space. The elements of $\bigwedge^2(\mathbb{R}^n)$ are called bivectors and if $\{e_i\}_{i=1}^n$ is a basis for \mathbb{R}^n , then $\{e_i \wedge e_j\}_{i < j}$ is a basis for $\bigwedge^2(\mathbb{R}^n)$. For every $n \geq 2$, define the group $\widehat{G}(n, \mathbb{R})$ with elements

$$(x, x') \in \mathbb{R}^n \oplus \bigwedge^2(\mathbb{R}^n), \quad \text{where } x = (x_1, \dots, x_n), \quad x' = (x'_{12}, x'_{13}, \dots, x'_{n-1, n}),$$

and where multiplication is given by

$$(x, x')(y, y') = (x + y, x' + y' + \frac{1}{2}(x \wedge y)).$$

This group is of dimension $n + \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1)$. Remark especially that if $n = 3$, the wedge product can be identified with the vector cross product on \mathbb{R}^3 . That is, the product in $\widehat{G}(3, \mathbb{R})$ is given by

$$(x, x')(y, y') = (x + y, x' + y' + \frac{1}{2}(x \times y)).$$

It is not hard to see that $\widehat{G}(n, \mathbb{R})$ is isomorphic to the group consisting of the same elements, but with multiplication given by

$$(x, x')(y, y') = (x + y, x' + y' + (x_1 y_2, x_1 y_3, \dots, x_{n-1} y_n)). \quad (2)$$

Let $G(n, \mathbb{R})$ denote the group defined by (2). Then $G(n)$ is the integer version of $G(n, \mathbb{R})$.

We also mention that Nielsen [17] has classified all the six-dimensional connected, simply connected, nilpotent Lie groups. In this setting, $G(3, \mathbb{R})$ is the group denoted by $G_{6,15}$.

2. One may define the free nilpotent group $G(m, n)$ of class m and rank n for every $m \geq 1$. Indeed, $G(m, n)$ is the group generated by $\{u_i\}_{i=1}^n$ subject to the relations that all commutators of order greater than m involving the generators are trivial. More precisely, for $m = 1, 2, 3$ and $n \geq 2$, we have that $G(m, n)$ can be described as the groups with presentations

$$\begin{aligned} G(1, n) &= \langle \{u_i\}_{i=1}^n : [u_i, u_j] = 1 \rangle \cong \mathbb{Z}^n, \\ G(2, n) &= \langle \{u_i\}_{i=1}^n : [[u_i, u_j], u_k] = 1 \rangle = G(n), \\ G(3, n) &= \langle \{u_i\}_{i=1}^n : [[[u_i, u_j], u_k], u_l] = 1 \rangle, \end{aligned} \quad (3)$$

and it should now be clear how to define $G(m, n)$ for all $m \geq 1$ and $n \geq 2$. Finally, we set $G(m, 1) = \langle u_1 \rangle \cong \mathbb{Z}$ for each $m \geq 1$. One can also formally define $G(m, n)$ as follows: For $n \geq 1$, let \mathbb{F}_n denote the free group on n generators, set $\gamma_1(\mathbb{F}_n) = \mathbb{F}_n$ and for $m \geq 1$, set $\gamma_{m+1}(\mathbb{F}_n) = [\mathbb{F}_n, \gamma_m(\mathbb{F}_n)]$. Then $G(m, n) = \mathbb{F}_n / \gamma_{m+1}(\mathbb{F}_n)$.

Moreover, for all $m \geq 1$ and $n \geq 2$, the group $G(m, n)$ is the free object on n generators in the category of nilpotent groups of step at most m . In particular, notice that $G(m, n)$ is m -step nilpotent and that

$$G(m, n) \cong G(m+1, n) / Z(G(m+1, n)). \quad (4)$$

Again, we refer to [27] for additional details.

In [15, Section 4] Milnes and Walters describe the simple quotients of the C^* -algebra associated with a five-dimensional group denoted by $H_{5,4}$. One can check that $H_{5,4}$ is isomorphic to the group $G(3, 2)$. See Remark 3.2 for more about this group.

3. The group $G(3)$ is briefly discussed by Baggett and Packer [4, Example 4.3]. The purpose of that paper is to describe the primitive ideal space of group C^* -algebras of some two-step nilpotent groups. However, $G(3)$ only serves as an example of a group the authors could not handle.
4. Let $n \geq 2$. It is well-known that the group C^* -algebra $A = C^*(G(n))$ may be described as the universal C^* -algebra generated by unitaries $\{U_i\}_{1 \leq i \leq n}$ and $\{V_{jk}\}_{1 \leq j < k \leq n}$ satisfying the relations

$$[V_{jk}, V_{lm}] = [U_i, V_{jk}] = I \quad \text{and} \quad [U_j, U_k] = V_{jk}$$

for all $1 \leq i \leq n$, $1 \leq j < k \leq n$, and $1 \leq l < m \leq n$.

For $\lambda = (\lambda_{12}, \lambda_{13}, \dots, \lambda_{n-1,n}) \in \mathbb{T}^{\frac{1}{2}n(n-1)}$, let \mathcal{A}_λ be the noncommutative n -torus. It is the universal C^* -algebra generated by unitaries $\{W_i\}_{i=1}^n$ and relations $[W_i, W_j] = \lambda_{ij}I$ for $1 \leq i < j \leq n$. The universal property of A gives that for each λ in $\mathbb{T}^{\frac{1}{2}n(n-1)}$ there is a surjective $*$ -homomorphism

$$\pi_\lambda : A \rightarrow \mathcal{A}_\lambda$$

satisfying $\pi_\lambda(U_i) = W_i$ for $1 \leq i \leq n$ and $\pi_\lambda(V_{jk}) = \lambda_{jk}I$ for $1 \leq j < k \leq n$.

Furthermore, A has center $Z(A) = C^*(\{V_{jk}\}_{1 \leq j < k \leq n}) \cong C^*(V(n))$. Indeed, this is the case since $G(n)$ is amenable and its finite conjugacy classes are precisely the one-point sets of central elements (see Lemma 4.1 below). Therefore, we set

$$T = \text{Prim } Z(A) \cong \widehat{Z(A)} = \mathbb{T}^{\frac{1}{2}n(n-1)}.$$

Let λ be a primitive ideal of $Z(A)$ identified with an element of $\mathbb{T}^{\frac{1}{2}n(n-1)}$. Let \mathcal{I}_λ be the ideal of A generated by λ , that is, the ideal generated by $\{V_{jk} - \lambda_{jk}I : 1 \leq j < k \leq n\}$. It is clear that $\mathcal{I}_\lambda \subseteq \ker \pi_\lambda$. By the universal property of \mathcal{A}_λ , there is a $*$ -homomorphism

$$\rho : \mathcal{A}_\lambda \rightarrow A / \mathcal{I}_\lambda$$

such that $\rho(W_i) = U_i + \mathcal{I}_\lambda$ for $1 \leq i \leq n$. Hence, $\rho \circ \pi_\lambda$ coincides with the quotient map $A \rightarrow A/\mathcal{I}_\lambda$ and consequently, $\ker \pi_\lambda \subseteq \mathcal{I}_\lambda$. Therefore, $\mathcal{A}_\lambda \cong A/\mathcal{I}_\lambda$ and π_λ may be regarded as the quotient map $A \rightarrow A/\mathcal{I}_\lambda$.

For an element a of A , let \tilde{a} be the section $T \rightarrow \bigsqcup_T \mathcal{A}_\lambda$ given by $\tilde{a}(\lambda) = \pi_\lambda(a)$ and let $\tilde{A} = \{\tilde{a} \mid a \in A\}$ be the set of all such sections. Then the following can be deduced from the Dauns-Hofmann Theorem [5].

Theorem 1.1. *The triple $(T, \{\mathcal{A}_\lambda\}, \tilde{A})$ consisting of the base space T , C^* -algebras \mathcal{A}_λ for each λ in T , and the set of sections \tilde{A} , is a full continuous field of C^* -algebras. Moreover, the C^* -algebra associated with this continuous field is naturally isomorphic to A .*

This result may be obtained as a corollary to [24, Theorem 1.2] which employs tools of Williams [29] related to Fell bundle theory, by taking $G = G(n)$ and $\sigma = 1$ in that theorem. It is also a special case of [3, Corollary 2.3]. Our proof is more direct and partly inspired by [1, Theorem 1.1] which covers the case where $n = 2$.

From the above discussion it now follows that $G(n)$ is a *representation group* for \mathbb{Z}^n in the sense of Moore [16]. In this case, that means $G(n)$ is (up to isomorphism) the unique central extension of \mathbb{Z}^n by $H^2(\mathbb{Z}^n, \mathbb{T})$ such that the ordinary irreducible representation theory of $G(n)$ coincides with the projective irreducible representation theory of \mathbb{Z}^n .

This fact plays an important role in [6], where the noncommutative principal torus bundles over locally compact spaces are classified up to equivariant Morita equivalence. As explained in [6, Section 2], the group C^* -algebra of $G(n)$ serves as a “universal” bundle in this classification.

We refer to [7, Section 4] for more information on representation groups, where the groups $G(n, \mathbb{R})$ and $G(n)$ are treated particularly in [7, Example 4.7].

2. THE TWO-COCYCLES OF THE FREE NILPOTENT GROUPS $G(n)$

Let G be *any* discrete group with identity e . A function $\sigma: G \times G \rightarrow \mathbb{T}$ satisfying

$$\begin{aligned}\sigma(r, s)\sigma(rs, t) &= \sigma(r, st)\sigma(s, t) \\ \sigma(r, e) &= \sigma(e, r) = 1\end{aligned}$$

for all elements $r, s, t \in G$ is called a *two-cocycle of G with values in \mathbb{T}* (or a *multiplier of G*). Moreover, two two-cocycles σ and τ are said to be *similar*, written $\sigma \sim \tau$, if

$$\tau(r, s) = \beta(r)\beta(s)\overline{\beta(rs)}\sigma(r, s)$$

for all $r, s \in G$ and some function $\beta: G \rightarrow \mathbb{T}$. The set of similarity classes of two-cocycles of G is an abelian group under pointwise multiplication. This group is the second cohomology group $H^2(G, \mathbb{T})$.

Let G be a semidirect product of a normal subgroup H and a subgroup K . By properties of the semidirect product, the elements of G can be uniquely written as products ab , where a belongs to H and b belongs to K . Define the action α of K on H by $\alpha_b(a) = bab^{-1}$. One often writes $G = H \rtimes_\alpha K$, but to simplify the notation, we will still denote the elements of G by ab instead (a, b) and write the group product in G as $(ab)(a'b') = a\alpha_b(a')bb'$ for $a, a' \in H$ and $b, b' \in K$. Hopefully, the reader is familiar with semidirect products so that this does not cause any confusion.

Next, we apply Mackey’s theorem [13, Theorem 9.4] and obtain the following result.

Theorem 2.1. *Every two-cocycle of G is similar to a two-cocycle σ of G of the form*

$$\sigma(a'b, ab') = \sigma_H(a', \alpha_b(a))g(a, b)\sigma_K(b, b'), \quad (5)$$

where σ_H and σ_K are two-cocycles of H and K , respectively,

$$g: H \times K \rightarrow \mathbb{T}$$

is a function such that $g(a, e) = g(e, b) = 1$ for all $a \in H$, $b \in K$, and σ_H and g satisfy

$$\begin{aligned} g(aa', b) &= \sigma_H(\alpha_b(a), \alpha_b(a')) \overline{\sigma_H(a, a')} \cdot g(a, b)g(a', b), \\ g(a, bb') &= g(\alpha_{b'}(a), b)g(a, b'). \end{aligned} \quad (6)$$

Moreover, for every choice of σ_H , g , and σ_K satisfying the conditions above, σ is a two-cocycle of G .

Proposition 2.2. *Let (σ_H, g, σ_K) and $(\sigma'_H, g', \sigma'_K)$ be triples satisfying the conditions of Theorem 2.1 and let σ and σ' be the corresponding two-cocycles of G . Then $\sigma \sim \sigma'$ if and only if the following conditions hold:*

- (i) $\sigma_K \sim \sigma'_K$,
- (ii) there exists a function $\beta: H \rightarrow \mathbb{T}$ such that

$$\begin{aligned} \sigma'_H(a, a') &= \overline{\beta(a)\beta(a')}\beta(aa')\sigma_H(a, a'), \\ g'(a, b) &= \beta(\alpha_b(a))\overline{\beta(a)}g(a, b). \end{aligned}$$

Remark 2.3. If (ii) holds, then $\sigma_H \sim \sigma'_H$. If $\sigma_H \sim \sigma'_H$ and β and β' are two functions implementing the similarity, then $\beta' = f \cdot \beta$ for some homomorphism $f: H \rightarrow \mathbb{T}$.

Proof of Proposition 2.2. Suppose $\sigma \sim \sigma'$. Then there exists some $\gamma: G \rightarrow \mathbb{T}$ such that

$$\sigma(a'b, ab') = \gamma(a'b)\gamma(ab')\overline{\gamma(a'bab')} \sigma'(a'b, ab') \quad (7)$$

for all $a, a' \in H$ and $b, b' \in K$. In particular, if $a = a' = e$, then

$$\sigma_K(b, b') = \gamma(b)\gamma(b')\overline{\gamma(bb')}\sigma'_K(b, b')$$

for all $b, b' \in K$, so $\sigma_K \sim \sigma'_K$. Moreover, the formula (5) from Theorem 2.1 with $a = e$ and $b = e$ gives that

$$\sigma(a', b') = 1 = \sigma'(a', b')$$

for all $a' \in H$ and $b' \in K$. Applying this fact to (7) shows that $\gamma(a'b') = \gamma(a')\gamma(b')$ for all $a' \in H$ and $b' \in K$. Define β on H by $\beta(a) = \gamma(a)$. Then, by letting $b = b' = e$ in (5) and (7), we get

$$\sigma'_H(a', a) = \overline{\beta(a')\beta(a)}\beta(a'a)\sigma_H(a', a)$$

for all $a', a \in H$. Furthermore, by letting $a' = e$ and $b' = e$ in (5) and (7), we compute

$$\begin{aligned} g(a, b) &= \gamma(b)\gamma(a)\overline{\gamma(ba)}g'(a, b) \\ &= \gamma(b)\gamma(a)\overline{\gamma(\alpha_b(a)b)}g'(a, b) \\ &= \gamma(b)\gamma(a)\overline{\gamma(\alpha_b(a))}\gamma(b)g'(a, b) \\ &= \gamma(a)\overline{\gamma(\alpha_b(a))}g'(a, b) \\ &= \beta(a)\overline{\beta(\alpha_b(a))}g'(a, b) \end{aligned}$$

for all $a \in H$ and $b \in K$.

Assume next that β is such that (ii) holds, and that (i) holds through δ , that is,

$$\sigma_K(b, b') = \delta(b)\delta(b')\overline{\delta(bb')}\sigma'_K(b, b').$$

Define γ on G by $\gamma(ab) = \beta(a)\delta(b)$. Then

$$\begin{aligned}\sigma(a'b, ab') &= \sigma_H(a', \alpha_b(a))g(a, b)\sigma_K(b, b') \\ &= \beta(a')\beta(\alpha_b(a))\overline{\beta(a'\alpha_b(a))}\sigma'_H(a', \alpha_b(a)) \\ &\quad \cdot \beta(a)\overline{\beta(\alpha_b(a))}g'(a, b) \cdot \delta(b)\delta(b')\overline{\delta(bb')}\sigma'_K(b, b') \\ &= \beta(a')\delta(b) \cdot \beta(a)\delta(b') \cdot \overline{\beta(a'\alpha_b(a))\delta(bb')}\sigma'(a'b, ab') \\ &= \gamma(a'b)\gamma(ab')\overline{\gamma(a'bab')}\sigma'(a'b, ab').\end{aligned}$$

□

Fix $n \geq 2$. To compute the two-cocycles of $G(n)$ up to similarity, we will proceed in the following way. Consider $G(n)$ as the split extension of $G(n-1)$ by $H(n)$ as described in Section 1. We will identify the elements

$$\begin{aligned}a &= (0, \dots, 0, a_n, 0, \dots, 0, a_{1n}, \dots, a_{n-1, n}), \\ b &= (b_1, \dots, b_{n-1}, 0, b_{12}, \dots, b_{n-2, n-1}, 0, \dots, 0),\end{aligned}$$

of $H(n)$ and $G(n-1)$, respectively, with ones of the form

$$\begin{aligned}a &\longleftrightarrow (a_n, a_{1n}, \dots, a_{n-1, n}), \\ b &\longleftrightarrow (b_1, \dots, b_{n-1}, b_{12}, \dots, b_{n-2, n-1}).\end{aligned}$$

The elements of $G(n)$ will be written as products ab , where a belongs to $H(n)$ and b belongs to $G(n-1)$, and the action α of $G(n-1)$ on $H(n)$ is then given by

$$\alpha_b(a) = bab^{-1} = (a_n, a_{1n} + b_1a_n, \dots, a_{n-1, n} + b_{n-1}a_n).$$

Remark 2.4. In the published version, Theorem 2.1 and Proposition 2.2 were only shown to hold for $G(n)$, not for any semidirect product.

Proposition 2.2 can be deduced from [24, Appendix 2], but in any case it may be useful to give a proof by a direct computation.

Let τ_n be a two-cocycle of $G(n)$ coming from a pair $(\sigma_{H(n)}, g_n)$, that is,

$$\tau_n(a'b, ab') = \sigma_{H(n)}(a', \alpha_b(a))g_n(a, b), \quad (8)$$

where $(\sigma_{H(n)}, g_n)$ satisfies (6). By Theorem 2.1 and Proposition 2.2, every two-cocycle of $G(n)$ that is trivial on $G(n-1)$ is similar to one of this form. Denote the abelian group of similarity classes of two-cocycles of this type by $\tilde{H}^2(G(n), \mathbb{T})$.

Corollary 2.5. *The second cohomology group of $G(n)$ may be decomposed as*

$$H^2(G(n), \mathbb{T}) = \tilde{H}^2(G(n), \mathbb{T}) \oplus H^2(G(n-1), \mathbb{T}) = \bigoplus_{k=2}^n \tilde{H}^2(G(k), \mathbb{T}).$$

Proof. It follows from Theorem 2.1 and Proposition 2.2 (see our comment above) that

$$H^2(G(n), \mathbb{T}) = \tilde{H}^2(G(n), \mathbb{T}) \oplus H^2(G(n-1), \mathbb{T}).$$

Thus, the second inequality is proven by induction after noticing that

$$\{1\} = H^2(\mathbb{Z}, \mathbb{T}) = H^2(G(1), \mathbb{T}) = \tilde{H}^2(G(1), \mathbb{T}). \quad \square$$

Theorem 2.6. *We have*

$$H^2(G(n), \mathbb{T}) \cong \mathbb{T}^{\frac{1}{3}(n+1)n(n-1)},$$

and for each set of $\frac{1}{3}(n+1)n(n-1)$ parameters

$$\{\lambda_{i,jk} : 1 \leq i \leq k, 1 \leq j < k \leq n\} \subseteq \mathbb{T},$$

the associated $[\sigma]$ in $H^2(G(n), \mathbb{T})$ may be represented by

$$\begin{aligned} \sigma(r, s) = & \prod_{i < j < k} \lambda_{i,jk}^{s_{jk}r_i + s_k r_{ij}} \lambda_{j,ik}^{s_{ik}r_j + s_k(r_i r_j - r_{ij})} \\ & \cdot \prod_{j < k} \lambda_{j,jk}^{s_{jk}r_j + \frac{1}{2}s_k r_j(r_j - 1)} \lambda_{k,jk}^{r_k(s_{jk} + r_j s_k) + \frac{1}{2}r_j s_k(s_k - 1)}. \end{aligned} \quad (9)$$

The proof of this theorem will be given in Section 2.1.

See the paragraph following Theorem 3.1 for an explanation of why $\lambda_{i,jk}$ for $i > k$ is not involved in the above.

Example 2.7. For $G(1) \cong \mathbb{Z}$ there are no nontrivial two-cocycles. The two-cocycles of the usual Heisenberg group $G(2)$ are, up to similarity, given by two parameters (as computed in [21, Proposition 1.1]):

$$\sigma(r, s) = \lambda_{1,12}^{s_{12}r_1 + \frac{1}{2}s_2 r_1(r_1 - 1)} \lambda_{2,12}^{r_2(s_{12} + r_1 s_2) + \frac{1}{2}r_1 s_2(s_2 - 1)} \quad (10)$$

The two-cocycles of $G(3)$ are, up to similarity, given by eight parameters:

$$\begin{aligned} \sigma(r, s) = & \lambda_{1,23}^{s_{23}r_1 + s_3 r_{12}} \lambda_{2,13}^{s_{13}r_2 + s_3(r_1 r_2 - r_{12})} \\ & \cdot \lambda_{1,12}^{s_{12}r_1 + \frac{1}{2}s_2 r_1(r_1 - 1)} \lambda_{2,12}^{r_2(s_{12} + r_1 s_2) + \frac{1}{2}r_1 s_2(s_2 - 1)} \\ & \cdot \lambda_{1,13}^{s_{13}r_1 + \frac{1}{2}s_3 r_1(r_1 - 1)} \lambda_{3,13}^{r_3(s_{13} + r_1 s_3) + \frac{1}{2}r_1 s_3(s_3 - 1)} \\ & \cdot \lambda_{2,23}^{s_{23}r_2 + \frac{1}{2}s_3 r_2(r_2 - 1)} \lambda_{3,23}^{r_3(s_{23} + r_2 s_3) + \frac{1}{2}r_2 s_3(s_3 - 1)} \end{aligned}$$

Remark 2.8. One may associate a Lyndon-Hochschild-Serre spectral sequence with the extension (see e.g. [28, 6.8.2]):

$$1 \longrightarrow V(n) \longrightarrow G(n) \longrightarrow \mathbb{Z}^n \longrightarrow 1$$

By applying [10, Theorem 4] to this sequence, one can compute the second homology group of $G(n)$ (which is recently also done more generally for $G(m, n)$ in [26, Proposition 2.1]), and deduce that

$$H_2(G(n), \mathbb{Z}) \cong \mathbb{Z}^{\frac{1}{3}(n+1)n(n-1)},$$

which gives that $H^2(G(n), \mathbb{T}) \cong \mathbb{T}^{\frac{1}{3}(n+1)n(n-1)}$ after dualizing, using the universal coefficient theorem for cohomology. However, this does not give an explicit description of $H^2(G(n), \mathbb{T})$.

2.1. Proof of Theorem 2.6. We will in this proof first compute $\tilde{H}^2(G(n), \mathbb{T})$ through several lemmas and then use Corollary 2.5 to conclude the argument.

Lemma 2.1.1. *Every element of $\tilde{H}^2(G(n), \mathbb{T})$ may be represented by a pair $(\sigma_{H(n)}, g_n)$, where $\sigma_{H(n)}$ is a two-cocycle of $H(n)$ given by*

$$\sigma_{H(n)}(a', a) = \prod_{i=1}^{n-1} \lambda_i^{a'_i a_{in}} \quad (11)$$

for some $\lambda_1, \dots, \lambda_{n-1} \in \mathbb{T}$, and g_n satisfies

$$g_n(a + a', b) = \left(\prod_{i=1}^{n-1} \lambda_i^{b_i a_n a'_i} \right) g_n(a, b) g_n(a', b) \quad (12)$$

for all $a, a' \in H(n)$ and $b \in G(n-1)$.

Proof. Every element of $\tilde{H}^2(G(n), \mathbb{T})$ may be represented by a two-cocycle of the form (8), that is, by a pair $(\sigma_{H(n)}, g_n)$ satisfying (6).

Moreover, it is well-known (see e.g. [2]) that every two-cocycle of $H(n) \cong \mathbb{Z}^n$ is similar to one of the form

$$\sigma_{H(n)}(a', a) = \prod_{1 \leq i \leq n-1} \lambda_i^{a'_n a_{in}} \cdot \prod_{1 \leq j < k \leq n-1} \mu_{jk}^{a'_{jn} a_{kn}}$$

for some sets of scalars $\{\lambda_i\}_{1 \leq i \leq n-1}, \{\mu_{jk}\}_{1 \leq j < k \leq n-1} \subseteq \mathbb{T}$. Since $H(n)$ is abelian, (6) gives that

$$\begin{aligned} \sigma_{H(n)}(\alpha_b(a), \alpha_b(a')) \overline{\sigma_{H(n)}(a, a')} &= g_n(a + a', b) \overline{g_n(a, b) g_n(a', b)} \\ &= g_n(a' + a, b) \overline{g_n(a', b) g_n(a, b)} \\ &= \sigma_{H(n)}(\alpha_b(a'), \alpha_b(a)) \overline{\sigma_{H(n)}(a', a)} \end{aligned}$$

for all $a, a' \in H(n)$ and $b \in G(n-1)$. Furthermore, we have

$$\begin{aligned} \sigma_H(\alpha_b(a), \alpha_b(a')) \overline{\sigma_H(a, a')} &= \prod_{1 \leq i \leq n-1} \lambda_i^{a_n(a'_{in} + b_i a'_n) - a_n a'_{in}} \cdot \prod_{1 \leq j < k \leq n-1} \mu_{jk}^{(a_{jn} + b_j a_n)(a'_{kn} + b_k a'_n) - a_{jn} a'_{kn}} \\ &= \prod_{1 \leq i \leq n-1} \lambda_i^{b_i a_n a'_n} \cdot \prod_{1 \leq j < k \leq n-1} \mu_{jk}^{b_j a'_{kn} a_n + b_k a_{jn} a'_n + b_j b_k a_n a'_n}. \end{aligned}$$

This is equal to $\sigma_H(\alpha_b(a'), \alpha_b(a)) \overline{\sigma_H(a', a)}$ for all $a, a' \in H(n)$ and $b \in G(n-1)$ if and only if the expression remains unchanged under the substitution $a \longleftrightarrow a'$, that is, if and only if all the μ_{jk} 's are 1. \square

Lemma 2.1.2. *For every element of $\tilde{H}^2(G(n), \mathbb{T})$ there is a unique associated pair $(\sigma_{H(n)}, g_n)$ satisfying the conditions of Lemma 2.1.1 such that*

$$g_n(u_n, u_i) = 1 \quad \text{for all } 1 \leq i \leq n-1. \quad (13)$$

Proof. Suppose that $(\sigma_{H(n)}, g_n)$ satisfies (11) and (12). Let $f: H(n) \rightarrow \mathbb{T}$ be the homomorphism determined by $f(u_n) = 1$ and $f(v_{in}) = \overline{g_n(u_n, u_i)}$ for all $1 \leq i \leq n-1$ and define g'_n by $g'_n(a, b) = f(\alpha_b(a)) \overline{f(a)} g_n(a, b)$. Then, $g'_n(u_n, u_i) = 1$ for all $1 \leq i \leq n-1$ and by Proposition 2.2, $(\sigma_{H(n)}, g'_n)$ determines a two-cocycle of $H(n)$ in the same similarity class as the one coming from $(\sigma_{H(n)}, g_n)$.

Suppose now that there are two pairs $(\sigma_{H(n)}, g_n)$ and $(\sigma'_{H(n)}, g'_n)$ both satisfying the conditions of Lemma 2.1.1. Then $\sigma'_{H(n)} = \sigma_{H(n)}$, so by Proposition 2.2 and the succeeding remark, there is a homomorphism $f: H(n) \rightarrow \mathbb{T}$ such that

$$g'_n(a, b) = f(\alpha_b(a)) \overline{f(a)} g_n(a, b) = \left(\prod_{i=1}^{n-1} f(v_{in})^{a_n b_i} \right) g_n(a, b)$$

for all $a \in H(n)$ and $b \in G(n-1)$. In particular,

$$g'_n(u_n, u_i) = f(v_{in}) g_n(u_n, u_i) \quad \text{for all } 1 \leq i \leq n-1,$$

so that $g'_n = g_n$ if $g'_n(u_n, u_i) = g_n(u_n, u_i)$ for all $1 \leq i \leq n-1$. \square

In the forthcoming lemmas we fix an element of $\tilde{H}^2(G(n), \mathbb{T})$, and let $(\sigma_{H(n)}, g)$ be the unique associated pair satisfying (11), (12), and (13) for some set of scalars $\{\lambda_i\}_{i=1}^{n-1} \subseteq \mathbb{T}$.

For computational reasons, we introduce the following notation. For $a = (a_n, a_{1n}, \dots, a_{n-1,n})$ in $H(n)$, we write $a = w(a) + z(a)$, where $w(a) = (a_n, 0, \dots, 0)$, and $z(a)$ is the “central part”, i.e. $z(a) = (0, a_{1n}, \dots, a_{n-1,n})$. Similarly, for $b = (b_1, \dots, b_{n-1}, b_{12}, \dots, b_{n-2,n-1})$

in $G(n-1)$, we write $b = w(b)z(b)$, where $w(b) = (b_1, \dots, b_{n-1}, 0, \dots, 0)$ and $z(b) = (0, \dots, 0, b_{12}, \dots, b_{n-2, n-1})$. Note that $\alpha_b(a) = a$ if either $w(a)$ or $w(b)$ is trivial, i.e. if either a or b is central.

Lemma 2.1.3. *For all $a \in H(n)$ and $b \in G(n-1)$ we have*

$$g(a, b) = g(w(a), w(b))g(w(a), z(b))g(z(a), w(b)).$$

Proof. It follows immediately from Lemma 2.1.1 that if $a, a' \in H(n)$ and $w(a)$ or $w(a')$ is 0, then

$$g(a + a', b) = g(a, b)g(a', b), \quad (14)$$

hence,

$$g(a, b) = g(w(a) + z(a), b) = g(w(a), b)g(z(a), b)$$

for all $a \in H(n)$ and $b \in G(n-1)$. If $b' \in G(n-1)$ and $w(b') = e$, then b' is central and $\alpha_{b'}(a) = a$ for all $a \in H(n)$. Therefore,

$$g(a, b)g(a, b') = g(a, bb') = g(a, b'b) = g(\alpha_b(a), b')g(a, b) \quad (15)$$

for all $a \in H(n)$, $b \in G(n-1)$. By (14), we then get

$$\begin{aligned} 1 &= g(\alpha_b(a), b')\overline{g(a, b')} \\ &= g(a + (0, b_1 a_n, \dots, b_{n-1} a_n), b')\overline{g(a, b')} \\ &= g(a, b')g((0, b_1 a_n, \dots, b_{n-1} a_n), b')\overline{g(a, b')} \\ &= g((0, b_1 a_n, \dots, b_{n-1} a_n), b') \end{aligned}$$

for all $a \in H(n)$ and $b \in G(n-1)$. Consequently, since this holds for all $a \in H(n)$ and $b \in G(n-1)$, and central $b' \in G(n-1)$, we get that if \tilde{a} and \tilde{b} are *any* elements in $H(n)$ and $G(n-1)$, respectively, then $g(z(\tilde{a}), z(\tilde{b})) = 1$. Moreover, (15) also imply that if $b, b' \in G(n-1)$ and *either* $w(b)$ or $w(b')$ is equal to e , that is, either b or b' is central, then

$$g(a, bb') = g(a, b)g(a, b'). \quad (16)$$

Hence, by (16) and (14),

$$\begin{aligned} g(a, b) &= g(a, w(b)z(b)) = g(a, w(b))g(a, z(b)) \\ &= g(w(a), w(b))g(z(a), w(b))g(w(a), z(b)) \cdot 1 \end{aligned}$$

for all $a \in H(n)$ and $b \in G(n-1)$. □

Lemma 2.1.4. *For all $a \in H(n)$ and $b, b' \in G(n-1)$ we have*

$$\begin{aligned} g(z(a), w(b)) &= \prod_{i,j=1}^{n-1} g(v_{in}, u_j)^{a_{in}b_j}, \\ g(w(a), z(b)) &= \prod_{1 \leq i < j \leq n} g(u_n, v_{ij})^{a_n b_{ij}} = \prod_{1 \leq i < j \leq n} \left(\overline{g(v_{in}, u_j)} g(v_{jn}, u_i) \right)^{a_n b_{ij}}, \end{aligned}$$

and

$$g(a, bb') = \left(\prod_{i,j=1}^{n-1} g(v_{in}, u_j)^{b'_i b_j a_n} \right) g(a, b)g(a, b'). \quad (17)$$

Proof. Let $z(H(n)) = \{z(a) \mid a \in H(n)\}$ and $z(G(n-1)) = \{z(b) \mid b \in G(n-1)\}$. Then g is a bihomomorphism when restricted to $z(H(n)) \times G(n-1)$ or $H(n) \times z(G(n-1))$. Therefore, the first two identities hold. Indeed, this follows directly from (6) after noticing that since $z(a)$ and $z(b)$ are central,

$$\alpha_{w(b)}(z(a)) = z(a) \quad \text{and} \quad \alpha_{z(b)}(w(a)) = w(a).$$

Moreover, for $i < j$ we have $u_i u_j = v_{ij} u_j u_i$. By (6) and the previous lemma, one calculates

$$\begin{aligned} g(u_n, u_i u_j) &= g(\alpha_{u_j}(u_n), u_i) g(u_n, u_j) \\ &= g(u_n v_{jn}, u_i) g(u_n, u_j) \\ &= g(u_n, u_i) g(v_{jn}, u_i) g(u_n, u_j) \end{aligned}$$

and

$$\begin{aligned} g(u_n, v_{ij} u_j u_i) &= g(u_n, v_{ij}) g(u_n, u_j u_i) \\ &= g(u_n, v_{ij}) g(\alpha_{u_i}(u_n), u_j) g(u_n, u_i) \\ &= g(u_n, v_{ij}) g(u_n v_{in}, u_j) g(u_n, u_i) \\ &= g(u_n, v_{ij}) g(u_n, u_j) g(v_{in}, u_j) g(u_n, u_i), \end{aligned}$$

so that

$$g(v_{jn}, u_i) = g(u_n, v_{ij}) g(v_{in}, u_j), \quad (18)$$

which gives the last identity in the second line of the statement. Finally, we compute

$$\begin{aligned} g(a, bb') &= g(\alpha_{b'}(a), b) g(a, b') = g(a + (0, b'_1 a_n, \dots, b'_{n-1} a_n), b) g(a, b') \\ &= g((0, b'_1 a_n, \dots, b'_{n-1} a_n), w(b)) g(a, b) g(a, b') \\ &= \left(\prod_{i=1}^{n-1} g(v_{in}, w(b))^{b'_i a_n} \right) g(a, b) g(a, b') \\ &= \left(\prod_{i=1}^{n-1} \left(\prod_{j=1}^{n-1} g(v_{in}, u_j)^{b_j} \right)^{b'_i a_n} \right) g(a, b) g(a, b'). \end{aligned}$$

□

Lemma 2.1.5. *For all $a \in H(n)$ and $b \in G(n-1)$ we have*

$$\begin{aligned} g(w(a), w(b)) &= \left(\prod_{i=1}^{n-1} \lambda_i^{\frac{1}{2} b_i a_n (a_n - 1)} g(v_{in}, u_i)^{\frac{1}{2} a_n b_i (b_i - 1)} \right) \\ &\quad \cdot \prod_{1 \leq i < j \leq n-1} g(v_{in}, u_j)^{b_i b_j a_n}. \end{aligned}$$

Proof. First we see from (17) that if $b_j \geq 1$, then

$$\begin{aligned} g(u_n, u_j^{b_j}) &= g(u_n, u_j^{b_j-1} u_j) \\ &= g(v_{jn}, u_j)^{b_j-1} g(u_n, u_j^{b_j-1}) g(u_n, u_j) \\ &= \dots = g(v_{jn}, u_j)^{\frac{1}{2} b_j (b_j-1)} g(u_n, u_j)^{b_j} \end{aligned}$$

and then it is not hard to see that

$$g(u_n, u_j^{b_j}) = g(v_{jn}, u_j)^{\frac{1}{2} b_j (b_j-1)} g(u_n, u_j)^{b_j}$$

for negative b_j as well, for example by applying (17) again.

Moreover, note that $w(b) = u_{n-1}^{b_{n-1}} \cdots u_1^{b_1}$, so that by (17),

$$\begin{aligned} g(u_n, w(b)) &= g(u_n, u_{n-1}^{b_{n-1}} \cdots u_1^{b_1}) \\ &= \left(\prod_{j=2}^{n-1} g(v_{1n}, u_j)^{b_1 b_j} \right) g(u_n, u_{n-1}^{b_{n-1}} \cdots u_2^{b_2}) g(u_n, u_1^{b_1}) \\ &= \cdots = \left(\prod_{1 \leq i < j \leq n-1} g(v_{in}, u_j)^{b_i b_j} \right) \left(\prod_{j=1}^{n-1} g(u_n, u_j^{b_j}) \right). \end{aligned}$$

Then by (12) for $a_n \geq 1$,

$$\begin{aligned} g(w(a), w(b)) &= g(a_n u_n, u_{n-1}^{b_{n-1}} \cdots u_1^{b_1}) \\ &= \left(\prod_{i=1}^{n-1} \lambda_i^{b_i(a_n-1)} \right) \cdot g((a_n-1)u_n, u_{n-1}^{b_{n-1}} \cdots u_1^{b_1}) g(u_n, u_{n-1}^{b_{n-1}} \cdots u_1^{b_1}) \\ &= \cdots = \left(\prod_{i=1}^{n-1} \lambda_i^{b_i \cdot \frac{1}{2} a_n(a_n-1)} \right) \cdot g(u_n, u_{n-1}^{b_{n-1}} \cdots u_1^{b_1})^{a_n} \\ &= \left(\prod_{i=1}^{n-1} \lambda_i^{\frac{1}{2} b_i a_n(a_n-1)} \right) \cdot \left(\prod_{1 \leq i < j \leq n-1} g(v_{in}, u_j)^{b_i b_j a_n} \right) \\ &\quad \cdot \left(\prod_{j=1}^{n-1} g(v_{jn}, u_j)^{\frac{1}{2} a_n b_j(b_j-1)} g(u_n, u_j)^{a_n b_j} \right). \end{aligned}$$

Again, it is not hard to see that a similar argument also works for negative a_n . Finally, recall that we have chosen g so that $g(u_n, u_j) = 1$ by (13). \square

Lemma 2.1.6. *We have*

$$\tilde{H}^2(G(n), \mathbb{T}) \cong \mathbb{T}^{n(n-1)},$$

and for each set of $n(n-1)$ parameters

$$\{\lambda_{i,jn} : 1 \leq i \leq n, 1 \leq j \leq n-1\} \subseteq \mathbb{T},$$

the associated $[\tau]$ in $\tilde{H}^2(G(n), \mathbb{T})$ may be represented by

$$\begin{aligned} \tau(a'b, ab') &= \prod_{1 \leq i < j \leq n-1} \lambda_{i,jn}^{a_{jn} b_i + a_n b_{ij}} \lambda_{j,in}^{a_{in} b_j + a_n(b_i b_j - b_{ij})} \prod_{j=1}^{n-1} \lambda_{j,jn}^{a_{jn} b_j + \frac{1}{2} a_n b_j(b_j-1)} \\ &\quad \cdot \prod_{j=1}^{n-1} \lambda_{n,jn}^{a'_n(a_{jn} + b_j a_n) + \frac{1}{2} b_j a_n(a_n-1)}. \end{aligned}$$

Proof. If one puts $\lambda_{i,jn} = g(v_{jn}, u_i)$ for $i, j < n$ and $\lambda_{n,jn} = \lambda_j$ for $j < n$, then this is a consequence of the preceding lemmas. Indeed, by (8) we can represent τ as a pair $(\sigma_{H(n)}, g)$. Here $\sigma_{H(n)}$ is of the form (11) and g can be decomposed as in Lemma 2.1.3 with factors computed in Lemma 2.1.4 and Lemma 2.1.5. \square

To complete the proof of Theorem 2.6, we set $r = a'b$ and $s = ab'$ and recall that by Corollary 2.5 we can compute σ_n inductively as $[\sigma_n] = \prod_{k=2}^n [\tau_k]$.

Finally, we can also check that $\sum_{k=2}^n k(k-1) = \frac{1}{3}(n+1)n(n-1)$.

3. THE TWISTED GROUP C*-ALGEBRAS $C^*(G(n), \sigma)$ OF $G(n)$

Again, let G be *any* discrete group, σ a two-cocycle of G and \mathcal{H} a nontrivial Hilbert space. A map U from G into the unitary group of \mathcal{H} satisfying

$$U(r)U(s) = \sigma(r, s)U(rs)$$

for all $r, s \in G$ is called a σ -projective unitary representation of G on \mathcal{H} .

We recall the following facts about twisted group C*-algebras and refer to Zeller-Meier [30] for further details of the construction.

To each pair (G, σ) , we may associate the full twisted group C*-algebra $C^*(G, \sigma)$. Denote the canonical injection of G into $C^*(G, \sigma)$ by i_σ . Then $C^*(G, \sigma)$ satisfies the following universal property. Every σ -projective unitary representation of G on some Hilbert space \mathcal{H} (or in some unital C*-algebra A) factors uniquely through i_σ .

The reduced twisted group C*-algebra $C_r^*(G, \sigma)$ is generated by the left regular σ -projective unitary representation λ_σ of G on $B(\ell^2(G))$. Consequently, λ_σ extends to a *-homomorphism of $C^*(G, \sigma)$ onto $C_r^*(G, \sigma)$. If G is amenable, then λ_σ is faithful. Note especially that every nilpotent group is amenable, so that $C^*(G(n), \sigma) \cong C_r^*(G(n), \sigma)$ through λ_σ for every $n \geq 1$ and all two-cocycles σ of $G(n)$.

Finally, we remark that if $\tau \sim \sigma$ through some function $\beta: G \rightarrow \mathbb{T}$, then the assignment $i_\tau(r) \mapsto \beta(r)i_\sigma(r)$ induces an isomorphism $C^*(G, \tau) \rightarrow C^*(G, \sigma)$.

Theorem 3.1 (Remark 3.1 in the published version). *Fix $n \geq 2$, let σ be a two-cocycle of $G(n)$ of the form (9), that is, determined by the $\frac{1}{3}(n+1)n(n-1)$ parameters*

$$\{\lambda_{i,jk} : 1 \leq i \leq k, 1 \leq j < k \leq n\} \subseteq \mathbb{T},$$

and set

$$\lambda_{k,ij} = \overline{\lambda_{i,jk}} \lambda_{j,ik} \quad (19)$$

when $1 \leq i < j < k \leq n$.

Then the twisted group C-algebra $C^*(G(n), \sigma)$ is the universal C*-algebra generated by unitaries $\{U_i\}_{1 \leq i \leq n}$ and $\{V_{jk}\}_{1 \leq j < k \leq n}$ satisfying the relations*

$$[V_{jk}, V_{lm}] = I, \quad [U_i, V_{jk}] = \lambda_{i,jk} I, \quad [U_j, U_k] = V_{jk} \quad (20)$$

for $1 \leq i \leq n$, $1 \leq j < k \leq n$, and $1 \leq l < m \leq n$.

Proof. Set $U_i = i_\sigma(u_i)$ and $V_{jk} = i_\sigma(v_{jk})$ and note that (9) gives that $\sigma(u_i, v_{jk}) = \lambda_{i,jk}$ and $\sigma(v_{jk}, u_i) = 1$ for all $1 \leq i \leq n$ and $1 \leq j < k \leq n$. Thus,

$$[U_i, V_{jk}] = \sigma(u_i, v_{jk}) \overline{\sigma(v_{jk}, u_i)} I = \lambda_{i,jk} I \text{ for all } 1 \leq i \leq n, 1 \leq j < k \leq n.$$

Moreover, note that $\sigma(u_i, u_j) = 1$ for all $1 \leq i, j \leq n$ and $\sigma(v_{jk}, v_{lm}) = 1$ for all $1 \leq j < k \leq n$ and $1 \leq l < m \leq n$. Hence, it is clear that $C^*(G(n), \sigma)$ is generated as a C*-algebra by unitaries satisfying (20).

Next, suppose that A is any C*-algebra generated by a set of unitaries satisfying the relations (20). For each r in $G(n)$ we define the unitary W_r in A by

$$W_r = V_{12}^{r_{12}} \cdots V_{n-1,n}^{r_{n-1,n}} \cdot U_n^{r_n} \cdots U_1^{r_1}.$$

Then a computation using (20) repeatedly gives that² $W_r W_s = \tau(r, s) W_{rs}$, where $\tau(r, s)$ is a scalar in \mathbb{T} for all $r, s \in G(n)$. Now, the associativity of A immediately implies that τ is a two-cocycle of $G(n)$, so that W is a τ -projective unitary representation of $G(n)$ in A . Furthermore, note that τ satisfies

$$\tau(u_i, v_{jk}) \overline{\tau(v_{jk}, u_i)} = \lambda_{i,jk} \text{ for } 1 \leq i \leq n, 1 \leq j < k \leq n.$$

²In general, it will require much work to compute the formula for τ and it is not needed for this argument. However, for $n = 2$, the expression for τ is precisely of the form (10).

By the universal property of the full twisted group C^* -algebra, there exists a unique $*$ -homomorphism φ of $C^*(G(n), \tau)$ onto A such that $\varphi(i_\tau(r)) = W(r)$ for all $r \in G(n)$.

Therefore, it is sufficient to show that $\tau \sim \sigma$, because then $C^*(G(n), \tau)$ is canonically isomorphic to $C^*(G(n), \sigma)$. By Theorem 2.6, there is some $\beta: G(n) \rightarrow \mathbb{T}$ such that σ' , given by

$$\sigma'(r, s) = \beta(r)\beta(s)\overline{\beta(rs)}\tau(r, s),$$

is of the form (9). We calculate that

$$\begin{aligned} & \sigma'(u_i, v_{jk})\overline{\sigma'(v_{jk}, u_i)} \\ &= \beta(u_i)\beta(v_{jk})\overline{\beta(u_i v_{jk})}\tau(u_i, v_{jk})\overline{\beta(v_{jk})\beta(u_i)}\beta(v_{jk}u_i)\overline{\tau(v_{jk}, u_i)} \\ &= \tau(u_i, v_{jk})\overline{\tau(v_{jk}, u_i)} = \lambda_{i,jk} \end{aligned}$$

for all $1 \leq i \leq n$ and $1 \leq j < k \leq n$. Hence, $\sigma' = \sigma$, so $\tau \sim \sigma$. \square

The above relation (19) is a consequence of (18) in the proof of Theorem 2.6 and is the reason why $\lambda_{i,jk}$ for $i > k$ is not involved in the expression (9). To illustrate this further, consider the case of rank 3. Let U_1, U_2, U_3 and V_{12}, V_{13}, V_{23} be unitaries in a C^* -algebra B satisfying

$$[V_{jk}, V_{lm}] = I, \quad [U_i, V_{jk}] = \mu_{i,jk}I, \quad [U_j, U_k] = V_{jk}$$

for $1 \leq i \leq 3$, $1 \leq j < k \leq 3$, and $1 \leq l < m \leq 3$ where $\{\mu_{i,jk}\}$ is any set of nine scalars in \mathbb{T} . Then we can compute that

$$\begin{aligned} U_1 U_2 U_3 &= V_{12} U_2 U_1 U_3 = \cdots = \mu_{2,13} V_{12} V_{13} V_{23} U_3 U_2 U_1, \\ U_1 U_2 U_3 &= U_1 V_{23} U_3 U_2 = \cdots = \mu_{1,23} \mu_{3,12} V_{12} V_{13} V_{23} U_3 U_2 U_1, \end{aligned}$$

that is, we must have $\mu_{2,13} = \mu_{1,23} \mu_{3,12}$.

For rank $n > 3$, any choice of a triple of unitaries from the family $\{U\}_{i=1}^n$ gives a similar dependence. In the $n \cdot \frac{1}{2}n(n-1)$ commutation relations, these $\binom{n}{3}$ dependencies are the only possible ones since

$$n \cdot \frac{1}{2}n(n-1) - \binom{n}{3} = \frac{1}{2}n(n-1)(n - \frac{1}{3}(n-2)) = \frac{1}{3}(n+1)n(n-1).$$

Remark 3.2. For $n \geq 2$, let ω be the dual two-cocycle of $G(n)$, that is,

$$\omega: G(n) \times G(n) \rightarrow H^2(\widehat{G(n)}, \mathbb{T}) \cong \mathbb{Z}^{\frac{1}{3}(n+1)n(n-1)}$$

is determined by $\omega(r, s)(\sigma) = \sigma(r, s)$ for a two-cocycle σ of $G(n)$. Let the group $R(G(n))$ be defined as the set $\mathbb{Z}^{\frac{1}{3}(n+1)n(n-1)} \times G(n)$ with product

$$(j, r)(k, s) = (j + k + \omega(r, s), rs).$$

It is not entirely obvious that ω and $R(G(n))$ are well-defined and we refer to [24, p. 689–690] and [7, Section 4] for details on this and the fact that $R(G(n))$ is a representation group for $G(n)$. Moreover, according to [24, Corollary 1.3] we may construct a continuous field A over $H^2(G(n), \mathbb{T})$ with fibers $A_\lambda \cong C^*(G(n), \sigma_\lambda)$ for each $\lambda \in H^2(G(n), \mathbb{T})$. Then the C^* -algebra associated with this continuous field will be naturally isomorphic to the group C^* -algebra of the group $R(G(n))$.

Next, we briefly consider the group $G(3, 2)$ generated by $u_1, u_2, v_{12}, w_1, w_2$ satisfying

$$[u_1, u_2] = v_{12}, \quad [u_1, v_{12}] = w_1, \quad [u_2, v_{12}] = w_2, \quad w_1, w_2 \text{ central}.$$

Then we have $Z(G(3, 2)) \cong \mathbb{Z}^2$ and $Z(C^*(G(3, 2))) \cong C(\mathbb{T}^2)$.

The following statement can also be deduced from [24, Theorem 1.2 and Examples 1.4 (3)], but we include the analysis that follows, because it is similar to that used in Theorem 1.1.

Let i denote the canonical injection of $G(3, 2)$ into $C^*(G(3, 2))$. For each $\lambda = (\lambda_1, \lambda_2) \in \mathbb{T}^2$, let $C^*(G(2), \sigma_\lambda)$ be generated by unitaries satisfying (20). By a similar argument as in Theorem 1.1, there is a surjective $*$ -homomorphism

$$\pi_\lambda: C^*(G(3, 2)) \rightarrow C^*(G(2), \sigma_\lambda)$$

such that $i(u_i) = U_i$, $i(v_{12}) = V_{12}$, and $i(w_i) = \lambda_i I$ for $i = 1, 2$. Moreover, the kernel of π_λ coincides with the ideal of $C^*(G(3, 2))$ generated by

$$\lambda \in \text{Prim } Z(C^*(G(3, 2))) \cong Z(C^*(\widehat{G(3, 2)})) = \mathbb{T}^2 \cong H^2(G(2), \mathbb{T}).$$

Again, similarly as in Theorem 1.1, we define a set of sections and apply the Dauns-Hofmann Theorem. In this way, the triple

$$\left(H^2(G(2), \mathbb{T}), \{C^*(G(2), \sigma_\lambda)\}_\lambda, C^*(\widehat{G(3, 2)}) \right)$$

is a full continuous field of C^* -algebras, and the C^* -algebra associated with this continuous field is naturally isomorphic to $C^*(G(3, 2))$.

It is not difficult to see that $R(G(2))$ is isomorphic to $G(3, 2)$. We conjecture that $R(G(n)) \cong G(3, n)$ also for $n \geq 3$, where $G(3, n)$ is the free nilpotent group of class 3 and rank n as described in (3), so that A is isomorphic to $C^*(G(3, n))$. For $n \geq 3$, the complicated part is to construct a homomorphism $R(G(n)) \rightarrow G(3, n)$, find an isomorphism $\mathbb{Z}^{\frac{1}{3}(n+1)n(n-1)} \cong Z(G(3, n))$, and then use (4) to produce a commuting diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z}^{\frac{1}{3}(n+1)n(n-1)} & \longrightarrow & R(G(n)) & \longrightarrow & G(n) \longrightarrow 1 \\ & & \downarrow \cong & & \downarrow & & \downarrow = \\ 1 & \longrightarrow & Z(G(3, n)) & \longrightarrow & G(3, n) & \longrightarrow & G(n) \longrightarrow 1 \end{array}$$

In fact, [26, Proposition 2.2 and Remark 2.3] indicate that the representation group $R(G(m, n))$ for $G(m, n)$ defined similarly as above is isomorphic to $G(m+1, n)$ for all $m, n \geq 1$.

4. SIMPLICITY OF THE TWISTED GROUP C^* -ALGEBRAS $C^*(G(n), \sigma)$

Let σ be a two-cocycle of any group G . An element r of G is called σ -regular if $\sigma(r, s) = \sigma(s, r)$ whenever s in G commutes with r . If r is σ -regular, then every conjugate of r is also σ -regular. Therefore, we say that a conjugacy class of G is σ -regular if it contains a σ -regular element.

Let $n \geq 2$. The conjugacy class C_r of $r \in G(n)$ is infinite if $r \notin V(n) = Z(G(n))$. Indeed, for any $s \in G(n)$ we have

$$(srs^{-1})_i = r_i \text{ and } (srs^{-1})_{jk} = r_{jk} + s_j r_k - r_j s_k.$$

Hence, $|C_r| = \infty$ if $r_i \neq 0$ for some i . Of course, $C_r = \{r\}$ if $r \in V(n)$.

More generally, it is explained in [9, Section 2.1.1] that if G is any torsion-free nilpotent group and $r \in G \setminus Z(G)$, then $|C_r| = \infty$.

In the published version, the following was only shown to hold for $G(n)$.

Lemma 4.1. *Let σ be a two-cocycle of a group G . Suppose that $|C_r| \in \{1, \infty\}$ whenever $r \in G$ is σ -regular. Define $S(G)$ to be the set of σ -regular central elements of G , that is,*

$$S(G) = \{r \in Z(G) \mid \sigma(r, s) = \sigma(s, r) \text{ for all } s \in G\}.$$

Then $S(G)$ is a subgroup of G and $Z(C_r^(G), \sigma) \cong C(\widehat{S(G)})$.*

Proof. It is not hard to check that $S(G)$ is a subgroup of $Z(G)$.

Let δ_e in $\ell^2(G)$ be the characteristic function on $\{e\}$ and for an operator T in $B(\ell^2(G))$, set $f_T = T\delta_e \in \ell^2(G)$. If T belongs to the center of $C_r^*(G, \sigma)$, then f_T can be nonzero only on the finite σ -regular conjugacy classes of G , that is, on $S(G)$ (see e.g. [19, Lemmas 2.3 and 2.4]).

Next, let $C^*(S(G), \sigma)$ be identified with $\{\lambda_\sigma(s) \mid s \in S(G)\} \subseteq B(\ell^2(G))$. This means that $Z(C_r^*(G, \sigma)) \subseteq C^*(S(G), \sigma)$. As the reverse inclusion obviously holds, we have $Z(C_r^*(G, \sigma)) = C^*(S(G), \sigma)$.

Now, it is not difficult to see that $C^*(S(G)) \cong C^*(S(G), \sigma)$. Indeed, as $s \mapsto \lambda_\sigma(s)$ is a unitary representation of $S(G)$ into $C^*(S(G), \sigma)$ and the canonical tracial state τ on $C^*(S(G), \sigma)$ is faithful and satisfies $\tau(\lambda_\sigma(s)) = 0$ for each nonzero $s \in S(G)$, this is just a consequence of [30, Théorème 4.22]. Altogether, we get

$$Z(C_r^*(G), \sigma) = C^*(S(G), \sigma) \cong C^*(S(G)) \cong C(\widehat{S(G)}). \quad \square$$

Now, we fix a two-cocycle σ of $G(n)$ of the form (9).

Remark 4.2. If $S(G(n))$ is nontrivial, we can describe $C^*(G(n), \sigma)$ as a continuous field of C^* -algebras over the base space $\widehat{S(G(n))}$. The fibers will be isomorphic to $C^*(G(n)/S(G(n)), \omega)$ for some two-cocycle ω of $G(n)/S(G(n))$ (see [11, Theorem 1.1] and [24, Theorem 1.2] for further details).

Example 4.3 ([11, Lemma 3.8 and Theorem 3.9]). Fix a two-cocycle σ of $G(2)$ of the form (10) such that both $\lambda_{1,12}$ and $\lambda_{2,12}$ are torsion elements. Let p and q be the smallest natural numbers such that $\lambda_{1,12}^p = \lambda_{2,12}^q = 1$ and set $k = \text{lcm}(p, q)$. Clearly, $V(2) = \mathbb{Z}$ and $S(G(2)) = k\mathbb{Z}$. Moreover, $G(2)/S(G(2))$ can be identified with the group with product

$$(r_1, r_2, r_{12})(s_1, s_2, s_{12}) = (r_1 + s_1, r_2 + s_2, r_{12} + s_{12} + r_1 s_2 \bmod k\mathbb{Z})$$

for $r_1, r_2, s_1, s_2 \in \mathbb{Z}$ and $r_{12}, s_{12} \in \{0, 1, \dots, k-1\}$.

Then $C^*(G(2), \sigma)$ is a continuous field of C^* -algebras over the base space $\widehat{S(G(2))} \cong \mathbb{T}$. The fibers will be isomorphic to $C^*(G(n)/S(G(n)), \omega_\lambda)$, where $\lambda \in \mathbb{T}$ and

$$\omega_\lambda(r, s) = \sigma(r, s)\mu^{r_1 s_2}$$

for some $\mu \in \mathbb{T}$ with $\mu^k = \lambda$.

Characterizations for simplicity of twisted group C^* -algebras of two-step nilpotent groups have been given in [11, Corollary 1.4] and [24, Corollary 1.6]. For the groups $G(n)$, the necessary and sufficient conditions for simplicity are somewhat easier to provide.

Theorem 4.4. *The following are equivalent:*

- (i) $C^*(G(n), \sigma)$ is simple.
- (ii) $C^*(G(n), \sigma)$ has trivial center.
- (iii) There are no nontrivial central σ -regular elements in $G(n)$.

Proof. By [22, Theorem 1.7] $C^*(G(n), \sigma)$ is simple if and only if every nontrivial σ -regular conjugacy class of $G(n)$ is infinite. Since every finite conjugacy class of $G(n)$ is a one-point set of a central element, then (i) is equivalent with (iii).

Moreover, (iii) is the same as saying that $S(G(n))$ is trivial, so therefore, (ii) is equivalent with (iii) by Lemma 4.1. This also follows from [19, Theorem 2.7]. \square

Lemma 4.5. *A central element $s = (0, \dots, 0, s_{12}, s_{13}, \dots, s_{n-1,n})$ of $G(n)$ is σ -regular if and only if*

$$\prod_{1 \leq j < k \leq n} \lambda_{i,jk}^{s_{jk}} = 1$$

for all $1 \leq i \leq n$.

Proof. Clearly, an element $s = (0, \dots, 0, s_{12}, s_{13}, \dots, s_{n-1,n}) \in V(n)$ is σ -regular if and only if $\sigma(s, r) = \sigma(r, s)$ for all $r \in G(n)$. By a direct calculation from the cocycle formula (9), we get that

$$\begin{aligned} \sigma(r, s) \overline{\sigma(s, r)} &= \left(\prod_{i < j < k} \lambda_{i,jk}^{s_{jk} r_i} \lambda_{j,ik}^{s_{ik} r_j} \right) \left(\prod_{j < k} \lambda_{j,jk}^{s_{jk} r_j} \lambda_{k,jk}^{r_k s_{jk}} \right) \left(\prod_{i < j < k} \lambda_{i,jk}^{-r_k s_{ij}} \lambda_{j,ik}^{r_k s_{ij}} \right) \\ &= \prod_{i=1}^n \left(\prod_{1 \leq j < k \leq n} \lambda_{i,jk}^{s_{jk}} \right)^{r_i} \end{aligned}$$

is equal to 1 for all $r \in G(n)$ if and only if the inner parenthesis is 1 for each $1 \leq i \leq n$. \square

Corollary 4.6. *We have that $C^*(G(n), \sigma)$ is simple if and only if for each nontrivial central element $s = (0, \dots, 0, s_{12}, s_{13}, \dots, s_{n-1,n})$ there is some $1 \leq i \leq n$ such that*

$$\prod_{1 \leq j < k \leq n} \lambda_{i,jk}^{s_{jk}} \neq 1.$$

Example 4.7. In particular, $C^*(G(3), \sigma)$ is simple if and only if for each nontrivial central element $s = (0, 0, 0, s_{12}, s_{13}, s_{23})$ at least one of the following hold:

$$\begin{aligned} \lambda_{1,12}^{s_{12}} \lambda_{1,13}^{s_{13}} \lambda_{1,23}^{s_{23}} &\neq 1, \\ \lambda_{2,12}^{s_{12}} \lambda_{2,13}^{s_{13}} \lambda_{2,23}^{s_{23}} &\neq 1, \\ \lambda_{3,12}^{s_{12}} \lambda_{3,13}^{s_{13}} \lambda_{3,23}^{s_{23}} &\neq 1. \end{aligned}$$

Next, set $\lambda_{i,jk} = e^{2\pi i t_{i,jk}}$ for $t_{i,jk} \in [0, 1)$ and consider the $n \times \frac{1}{2}n(n-1)$ -matrix T with entries $t_{i,jk}$ in the corresponding spots. Then T induces a linear map

$$\mathbb{R}^{\frac{1}{2}n(n-1)} \rightarrow \mathbb{R}^n.$$

Corollary 4.8. *Let T be the matrix described above. Then following are equivalent:*

- (i) $C^*(G(n), \sigma)$ is simple
- (ii) $T^{-1}(\mathbb{Z}^n) \cap \mathbb{Z}^{\frac{1}{2}n(n-1)} = \{0\}$
- (iii) $T(\mathbb{Z}^{\frac{1}{2}n(n-1)} \setminus \{0\}) \cap \mathbb{Z}^n = \emptyset$

Remark 4.9. Clearly, condition (ii) above is equivalent to that T restricts to an injective map

$$\mathbb{Z}^{\frac{1}{2}n(n-1)} \rightarrow \mathbb{R}^n / \mathbb{Z}^n \cong \mathbb{T}^n.$$

Furthermore, for $1 \leq j < k \leq n$, define

$$\Lambda_{jk} = \{t_{i,jk} \in [0, 1), 1 \leq i \leq n \mid e^{2\pi i t_{i,jk}} = \lambda_{i,jk}\}$$

and for $1 \leq i \leq n$, define

$$\Lambda_i = \{t_{i,jk} \in [0, 1), 1 \leq j < k \leq n \mid e^{2\pi i t_{i,jk}} = \lambda_{i,jk}\}.$$

Proposition 4.10. *If there exists i such that all the elements of Λ_i are irrational and linearly independent over \mathbb{Q} , then $C^*(G(n), \sigma)$ is simple.*

Proof. It follows immediately from Lemma 4.5, that “equation i ” cannot be satisfied unless $s = 0$. Hence, no nontrivial σ -regular central elements exists. \square

Proposition 4.11. *If there exists $j < k$ such that Λ_{jk} consists of only rational elements, then $C^*(G(n), \sigma)$ is not simple.*

Proof. Let q be the least common two-cocycle of the denominators of the elements of Λ_{jk} . Then qv_{jk} is central and σ -regular. Indeed,

$$\sigma(r, qv_{jk})\overline{\sigma(qv_{jk}, r)} = \prod_{i=1}^{n-1} \lambda_{i,jk}^{qr_i} = 1$$

for all $r \in G(n)$. □

Remark 4.12. In the case where $C^*(G(n), \sigma)$ is not simple, some more information about the primitive ideal space can be deduced from [11, Proposition 1.3].

5. ON ISOMORPHISMS INVARIANTS OF $C^*(G(n), \sigma)$

Fix $n \geq 2$ and let σ be a two-cocycle of $G(n)$. If φ is an automorphism of $G(n)$, define the two-cocycle σ_φ of $G(n)$ by

$$\sigma_\varphi(r, s) = \sigma(\varphi(r), \varphi(s)). \quad (21)$$

Then it is well-known that the associated twisted group C^* -algebras $C^*(G(n), \sigma)$ and $C^*(G(n), \sigma_\varphi)$ are isomorphic. Indeed, the map

$$i_{(G, \sigma)}(r) \mapsto i_{(G, \sigma_\varphi)}(\varphi^{-1}(r))$$

extends to an isomorphism $C^*(G(n), \sigma) \rightarrow C^*(G(n), \sigma_\varphi)$. Moreover, for any automorphism φ of $G(n)$, it is easily seen that two two-cocycles σ and τ of $G(n)$ are similar if and only if σ_φ and τ_φ are similar. Hence, there is a well-defined group action of the automorphism group $\text{Aut } G(n)$ on $H^2(G(n), \mathbb{T})$ defined by $\varphi \cdot [\sigma] = [\sigma_\varphi]$.

Therefore, we will now briefly investigate $\text{Aut } G(n)$. Let $V(n)^n$ be the subgroup of $\text{Aut } G(n)$ consisting of the automorphisms $G(n) \rightarrow G(n)$ of the form $u_i \mapsto z_i u_i$ for $1 \leq i \leq n$ and elements $z_i \in V(n) = Z(G(n))$. Clearly, $V(n)^n$ contains $\text{Inn } G(n)$. In fact, in the case $n = 2$, we have $V(2)^2 = \text{Inn } G(2)$.

Proposition 5.1. *There is a split short exact sequence:*

$$1 \longrightarrow V(n)^n \longrightarrow \text{Aut } G(n) \longrightarrow \text{GL}(n, \mathbb{Z}) \longrightarrow 1$$

Proof. Assume that φ is any endomorphism $G(n) \rightarrow G(n)$. The image of a central element under φ must be central, so φ restricts to an endomorphism $\varphi_1: V(n) \rightarrow V(n)$. Therefore, φ also induces an endomorphism $\varphi_2: G(n)/V(n) \rightarrow G(n)/V(n)$ determined by $\varphi_2(q(r)) = q(\varphi(r))$. Consider now the following commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & V(n) & \xrightarrow{i} & G(n) & \xrightarrow{q} & \mathbb{Z}^n \longrightarrow 1 \\ & & \varphi_1 \downarrow & & \varphi \downarrow & & \downarrow \varphi_2 \\ 1 & \longrightarrow & V(n) & \xrightarrow{i} & G(n) & \xrightarrow{q} & \mathbb{Z}^n \longrightarrow 1 \end{array}$$

Assume that φ_2 is an automorphism. Since φ_2 is surjective, then for all u_i there is some $s_i \in G(n)$ such that $\varphi(s_i) = z_i u_i$ for some $z_i \in V(n)$. Hence, for all $j < k$, we have $\varphi_1(s_j s_k s_j^{-1} s_k^{-1}) = v_{jk}$ and therefore, φ_1 is surjective. Every surjective endomorphism of \mathbb{Z}^n is also injective, so φ_1 is an automorphism as well. Thus, by the “short five lemma”, φ is an automorphism.

The converse obviously holds and hence, φ is an automorphism if and only if φ_2 is an automorphism.

Furthermore, the construction of $G(n)$ in terms of generators and relations means that every endomorphism $G(n) \rightarrow G(n)$ is uniquely determined by its values at $\{u_i\}_{i=1}^n$. In particular, we let $\varphi: G(n) \rightarrow G(n)$ be determined by the pair of matrices given by its entries

$$(\varphi(u_i)_j), (\varphi(u_i)_{jk}) \in M_n(\mathbb{Z}) \times M_{n, \frac{1}{2}n(n-1)}(\mathbb{Z})$$

so that the induced endomorphism φ_2 is coming from a matrix in $M_n(\mathbb{Z})$.

By the above argument, the map between endomorphism groups defined by

$$\text{End } G(n) \rightarrow \text{End } \mathbb{Z}^n, \quad (\varphi(u_i)_j), (\varphi(u_i)_{jk}) \mapsto (\varphi(u_i)_j) \quad (22)$$

restricts to a surjective map $\text{Aut } G(n) \rightarrow \text{Aut } \mathbb{Z}^n = \text{GL}(n, \mathbb{Z})$.

Before concluding the argument, we need the following.

Lemma 5.2 (nested inside the proof). *If φ and φ' are two endomorphisms of $G(n)$, then*

$$(\varphi \circ \varphi')(u_i)_j = \sum_{k=1}^n \varphi'(u_i)_k \varphi(u_k)_j.$$

If φ and φ' are two endomorphisms of $G(n)$ that both induce the trivial map on $G(n)/V(n)$, then

$$(\varphi \circ \varphi')(u_i)_{jk} = \varphi'(u_i)_{jk} + \varphi(u_i)_{jk}.$$

Proof. For the moment, set $\varphi(u_i)_j = r_{ij}$ and $\varphi'(u_i)_j = s_{ij}$. Then

$$(\varphi \circ \varphi')(u_i) = \varphi(u_n^{s_{in}} \cdots u_1^{s_{i1}} z) = (u_n^{r_{nn}} \cdots u_1^{r_{n1}})^{s_{in}} \cdots (u_n^{r_{1n}} \cdots u_1^{r_{11}})^{s_{i1}} z'$$

for some elements $z, z' \in V(n)$. Moreover, we can change the order of the u_i 's in the expression just by replacing z' by another central element z'' and thus,

$$(\varphi \circ \varphi')(u_i)_j = r_{nj}s_{in} + r_{n-1,j}s_{i,n-1} + \cdots + r_{1j}s_{i1} = \sum_{k=1}^n s_{ik}r_{kj}.$$

If both φ_2 and φ'_2 are trivial, then $\varphi(u_i) = z_i u_i$ and $\varphi'(u_i) = z'_i u_i$ for all $1 \leq i \leq n$ and some elements $z_i, z'_i \in V(n)$. Hence, $\varphi(v_{jk}) = \varphi'(v_{jk}) = v_{jk}$ for all $j < k$ and thus,

$$(\varphi \circ \varphi')(u_i) = \varphi(z'_i u_i) = z'_i z_i u_i. \quad \square$$

Therefore, (22) restricts to a surjective *homomorphism* $\text{Aut } G(n) \rightarrow \text{GL}(n, \mathbb{Z})$ with kernel isomorphic to the group $M_{n, \frac{1}{2}n(n-1)}(\mathbb{Z})$ under addition, that is, to $V(n)^n$.

Moreover, if A is a matrix in $\text{GL}(n, \mathbb{Z})$ with entries a_{ij} , then one can define an automorphism φ_A of $G(n)$ by $\varphi_A(u_i)_j = a_{ij}$. Thus, it should be clear that $\text{GL}(n, \mathbb{Z})$ sits inside $\text{Aut } G(n)$ as a subgroup so that the sequence splits. \square

Proposition 5.3. *If φ belongs to $V(n)^n$, then σ_φ is similar to σ . Thus, the action of $V(n)^n$ on $H^2(G(n), \mathbb{T})$ given by (21) is trivial.*

Proof. It is not hard to see that

$$\sigma(u_i, v_{jk}) \overline{\sigma(v_{jk}, u_i)} = \sigma_\varphi(u_i, v_{jk}) \overline{\sigma_\varphi(v_{jk}, u_i)},$$

that is,

$$[i_{(G, \sigma)}(u_i), i_{(G, \sigma)}(v_{jk})] = [i_{(G, \sigma_\varphi)}(u_i), i_{(G, \sigma_\varphi)}(v_{jk})]$$

for all $1 \leq i \leq n$ and $1 \leq j < k \leq n$. It then follows from the universal property of $C^*(G(n), \sigma)$ described in Theorem 3.1 that $\sigma_\varphi \sim \sigma$. \square

To describe the $\mathrm{GL}(n, \mathbb{Z})$ -action on $H^2(G(n), \mathbb{T})$ requires more work. To any A in $\mathrm{GL}(n, \mathbb{Z})$ we may associate a square matrix \tilde{A} of dimension $\frac{1}{2}n(n-1)$, with entries coming from the determinant of all 2×2 -matrices inside A . More precisely, if $A = (a_{ij})$, \tilde{A} is given by entries $\tilde{a}_{ij,kl}$ for $i < j, k < l$ such that $\tilde{a}_{ij,kl} = a_{ik}a_{jl} - a_{il}a_{jk}$. Then A acts on the matrix T defined prior to Corollary 4.8 by $A \cdot T = AT\tilde{A}$. Tedious computations of commutation relations and use of the universal property of $C^*(G(n), \sigma)$ from Theorem 3.1 now lead to the following result.

Proposition 5.4. *Let σ and σ' be two-cocycles of $G(n)$ of the form (9) and let T and T' be the associated matrices of Corollary 4.8. If there exists a matrix A in $\mathrm{GL}(n, \mathbb{Z})$ such that $A \cdot T = T'$, then $C^*(G(n), \sigma)$ and $C^*(G(n), \sigma')$ are isomorphic.*

For $n = 2$, it is shown by Packer [21, Theorem 2.9] that $C^*(G(2), \sigma)$ and $C^*(G(2), \sigma')$, where σ and σ' are of the form (9), are isomorphic if and only if there is a $\mathrm{GL}(2, \mathbb{Z})$ -matrix A taking σ to σ' . Note in this case that $\det \tilde{A} = \det A = \pm 1$.

For $n \geq 3$, it is at the moment not clear whether the $\mathrm{GL}(n, \mathbb{Z})$ -action on $H^2(G(n), \mathbb{T})$ described above is such that the orbits represent different isomorphism classes of twisted group C^* -algebras. Therefore, the problem of determining the isomorphism classes of $C^*(G(n), \sigma)$ remains open for future investigation.

Remark 5.5. Every two-cocycle σ_λ of $G(n)$ is of the form $e^{2\pi i \tilde{\sigma}_\lambda}$ for some two-cocycle $\tilde{\sigma}_\lambda$ on $G(n, \mathbb{R})$. Moreover, any pair of two-cocycles σ_λ and σ_μ of the form described above are homotopic in the sense of Packer and Raeburn [23, Section 4]. Hence, one may use [23, Theorem 4.2 and Corollary 4.5] to deduce that

$$K_i(C^*(G(n)), \sigma_\lambda) \cong K_i(C^*(G(n)), \sigma_\mu) \cong K_{\mathrm{top}}^{i+\frac{1}{2}n(n-1)}(G(n, \mathbb{R})/G(n)).$$

Remark 5.6 (new). The groups $G(n)$, or more generally all the free nilpotent groups $G(m, n)$ of class m and rank n are finitely generated (torsion-free) nilpotent groups. Let σ be a two-cocycle of $G(m, n)$ and suppose that there are no nontrivial central σ -regular elements in $G(m, n)$, that is, $C^*(G(m, n), \sigma)$ is simple with a unique trace by [22] (see Theorem 4.4 and the subsequent results above for the case of $G(n)$). Then there exists an irreducible representation π_σ of $G(m+1, n)$ such that $C^*(G(m, n), \sigma) \cong C^*(\pi_\sigma(G(m+1, n)))$ by using Theorem 1.1, Remark 3.2, and (4). Thus it follows from [9] and [8] that $C^*(G(m, n), \sigma)$ is classified by its ordered K -theory within the class of simple, unital, nuclear C^* -algebras with finite nuclear dimension that satisfy the universal coefficient theorem.

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UPDATE JUNE 2017

This document includes several computations and details that were left out in the published version of the article from 2015, in addition to a reformulation of Proposition 2.2 in terms of arbitrary semidirect products, a more general statement in Lemma 4.1, and a new Remark 5.6 that makes a connection to recent work. The latest arxiv version of the paper [20] includes all these updates, except Lemma 4.1 and a few other minor details.

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