GROUPS IN QUANTUM INFORMATION THEORY

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Quantum information theory is a very active and rapidly expanding interdisciplinary field of research, that have recently attracted a lot of interest from physicists, mathematicians and computer scientists. The power of quantum computers comes from their ability to use quantum mechanical principles such as superposition, measurement, and entanglement. Arguably, one of the most attractive features of quantum computing is that quantum algorithms are conjectured to solve a certain computational problems exponentially faster than any classical algorithm [25, 12, 6]. Therefore, many companies and research institutes are spending a lot on this futuristic and potentially game-changing technology. The promise of new technologies like safe cryptography and new "super-computer", capable of handling otherwise untractable problems, has excited not only researchers from many different fields like physics, mathematics and computer science, but also a large public audience. Some even built toy models for a quantum computer in the lab, for instance, IBM's 53-qubit quantum computer [8]. On a practical level all these new visions are based on the ability to control the quantum states of (a small number of) microsystems individually and to use them for information transmission and processing. From a more fundamental point of view the crucial point is a reconsideration of the foundations of quantum mechanics in an information theoretical context, formulated in the mathematical language of linear operators and vector spaces.

Mathematics has been crucial to the technological development of the modern society, and pure mathematics is the foundation of almost every science. In the future, there will probably be an even greater demand for more advanced mathematics, that will be used in new and unexpected ways. Any breakthrough usually comes as a consequence of a joint effort by researchers, working over a long time-span on improving the current status of knowledge. Operator algebras contribute to this development by studying methods that has its origin in quantum phenomena, such as noncommutativity, and is essential to progress in quantum information theory. Symmetries have been observed in nature and studied for thousands of years, and the mathematical model of a group is what embodies symmetries into a theoretical framework. The study of groups and dynamics in mathematics has applications and interactions across several fields of science and technology. There exist many symmetries behind several quantum information processes based on group representation, and these symmetries have not yet been sufficiently studied. Moreover, random walks and Markov chains provide important tools to describe behavior of systems, and their quantum counterparts are essential to designing new quantum algorithms.

Even though concrete implementations require finite-dimensionality, the success of quantum mechanics is due to the discovery that nature is described in infinite-dimensional Hilbert spaces, so it is desirable to demonstrate aspects of certain quantum processes for observables and states living in infinite dimensions.

Mathematical basis: operator algebras and group representations

The theory of operator algebras is the mathematical foundation for quantum information theory. This field, where certain algebras of bounded operators on Hilbert spaces are being studied, has its origin in quantum mechanics and much of its motivation comes from unitary

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representations of locally compact groups. In this framework, quantum states of qubits are represented on Hilbert spaces and quantum gates correspond to unitary operators.

Henceforth, G will always denote a discrete group with identity element e. We will mostly be concerned with the cases where G is countable, finitely-generated, or finite. Let σ be a 2-cocycle on G, that is, a map $G \times G \to \mathbb{T}$ such that $\sigma(g,e) = \sigma(e,g) = 1$ and

$$\sigma(g,h)\sigma(gh,k) = \sigma(g,hk)\sigma(h,k)$$

for all $g,h,k \in G$. If there exists a function $\beta\colon G \to \mathbb{T}$, with $\beta(e) = 1$ such that $\sigma(g,h) = \beta(g)\beta(h)\overline{\beta(gh)}$, then σ is called a coboundary. We let $Z^2(G,\mathbb{T})$ and $B^2(G,\mathbb{T})$ denote the group of 2-cocycles and coboundaries on G, respectively. Then the second cohomology group $H^2(G,\mathbb{T})$ is the quotient $Z^2(G,\mathbb{T})/B^2(G,\mathbb{T})$.

Let H be a Hilbert space and σ a 2-cocycle on G. A σ-projective unitary representation of G on H is a map U from G into the unitary group of H such that $U(g)U(h) = \sigma(g,h)U(gh)$ for all $g,h \in G$. The left regular representation λ_σ of G on $\ell^2(G)$ is defined by $\lambda_\sigma(g)\delta_h = \sigma(g,h)\delta_{gh}$ for $g,h \in G$, where $\{\delta_g\}_{g\in G}$ denotes the standard orthonormal basis on $\ell^2(G)$. We now define the reduced group C^*-algebra $C^*_r(G,\sigma)$ as the C^*-subalgebra of $B(\ell^2(G))$ generated by the image of λ_σ, and the group von Neumann algebra $W^*(G,\sigma)$ as $\lambda_\sigma(G)''$. Both these algebras comes with a canonical tracial state τ given by $\tau(x) = \langle x\delta_e, \delta_e\rangle$. Two 2-cocycles that belong to the same class in $H^2(G,\mathbb{T})$ produce algebras that are canonically isomorphic. See [5] for more on projective group representations.

Let $G \times_\sigma \mathbb{T}$ be the group consisting of elements $(g,z) \in G \times \mathbb{T}$, and with multiplication given by

$$(g,z)(h,w) = (gh, zw\sigma(g,h)).$$

Then we define G^σ as the subgroup of $G \times_\sigma \mathbb{T}$ generated by $G \times \{1\}$. Clearly, G^σ is countable, finitely generated, or finite if and only if the same holds for G. Moreover, since $\mathbb{T}$ is a central subgroup of $G \times_\sigma \mathbb{T}$, it follows that G^σ is FC-hypercentral (see [4]), virtually nilpotent, or nilpotent if and only if the same holds for G. There is a one-to-one correspondence between σ-projective unitary representations U of G on H, and (ordinary) unitary representations V of G^σ on H, given by $V(g,z) = zU(g)$. Moreover, U is irreducible if and only V is irreducible, and in this case the images of U and V generate the same subalgebras of $B(H)$.

An element $g \in G$ is called σ-regular if $\sigma(g,h) = \sigma(h,g)$ whenever $h \in G$ commutes with g. We say that (G,σ) satisfies Kleppner's condition if it does not have any σ-regular elements with a finite conjugacy class. In [16, 20] it is shown that Kleppner's condition is equivalent to the group von Neumann algebra $W^*(G,\sigma)$ being a factor, and therefore generalizes the icc property from the untwisted case, and is also equivalent to triviality of the center of $C^*_r(G,\sigma)$.

It is of particular interest to describe the class of groups for which, for any 2-cocycle σ, (G,σ) satisfies Kleppner's condition if and only if $C^*_r(G,\sigma)$ is simple and has a unique tracial state. In [5], we showed that the so-called FC-hypercentral groups belongs to this class. Recently, the class of FC-hypercentral groups has received attention in other situations [10].

What characterizes the research in the area of group representations is that it often requires a wide range of techniques, so we would need to draw both on classical and modern results, both on general theory and on very specific computations.

1. Unitary error bases

Both classical and quantum error correction are concerned with the fundamental problem of communications in the presence of noise. Quantum information is represented by the states of a quantum system, which is affected by decoherence effects due to inevitable interaction between the quantum system and its environment. The challenge of performing quantum error-correction consists two of parts: first, the physics of error processes and their reversal, and secondly the construction of a good quantum error-correcting code.

Unitary error bases are the fundamental primitives in the construction of quantum error control codes, and this technology is essentially based on group symmetry.

It is well known from the theory of quantum codes that if a code can correct a set of error operators, then it can correct their linear span. For this reason, it is sufficient to focus our attention to errors that form a unitary error basis of the vector space of linear operators. Knill (see e.g. [17]) introduced a special type of basis of unitary operators called a nice error basis and investigated its properties and applications to quantum codes. Originally, the motivation comes from the Pauli basis

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

which is a unitary error basis for $\mathbb{C}^{2\times 2}$. Bases of $\mathbb{C}^{d\times d}$ play a fundamental role for quantum error-correcting codes and other typical applications of quantum information processing, like e.g. super-dense coding and teleportation schemes.

We aim to study error bases for higher dimensional quantum systems from a group representations perspective, and develop techniques for constructing quantum error-correcting codes.

Nice error bases. Following Knill (and others, see e.g., [17]), in the n-dimensional case, a *nice error basis* is a set of the form $\{U(g) : g \in G\}$ where G is a finite group of order n^2, U is a σ-projective unitary representation of G into the algebra $M_n(\mathbb{C})$ of $n \times n$-matrices satisfying that $\mathrm{tr}(U(g)) = 0$ for all $g \neq e$, where tr is denotes the normalized canonical trace on $M_n(\mathbb{C})$. Then G^σ is called the *abstract error group* and G is the *index group* for the basis of the basis. A nice error basis is the main example of a unitary error basis for $M_n(\mathbb{C})$, i.e., an orthonormal basis for $M_n(\mathbb{C})$, considered as a Hilbert space with respect to the inner product associated with tr, which consists of unitary matrices.

A finite group G is said to be of *central type* if there exists σ on G such that (G, σ) satisfies Kleppner's condition, also called "nondegenerate" in the abelian case. This is only possible when the number of elements of G is a square. If n is square-free, then the only group of central type of order n^2 is $\mathbb{Z}_n \times \mathbb{Z}_n$ [11]. The following open problems are natural to consider:

(1) For a given non-square-free n, classify the groups of central type of order n^2.
(2) Given a group G of central type, classify the $\mathrm{Aut}(G)$-orbits of the cohomology classes in $H^2(G, \mathbb{T})$.

One strategy is exploiting the connections between groups of central type and their Sylow subgroups, as suggested in [11]. Decomposition techniques, either decomposing G or $H^2(G, \mathbb{T})$, and looking at extensions, should be useful, combined with the "relative" Kleppner condition (see [5]) for a pair (G, N), where N is a normal subgroup of G, taking the normal subgroup structure of G into account.

In the infinite-dimensional case we first consider the (unique) hyperfinite II_1 factor M (having separable predual) and let τ denote its canonical tracial state. A (tentative) definition of a nice error basis for M could be a set of the form $\{U(g) : g \in G\}$ where G is a countably infinite group, U is a σ-projective unitary representation of G in M satisfying that $M = U(G)''$ and $\tau(U(g)) = 0$ for all $g \neq e$. We note that such a set will give an orthonormal basis for the Hilbert space $L^2(M, \tau)$.

Moreover, using Connes' famous result [7], examples of nice error bases for M are obtained by letting the index group G be a countably infinite amenable group and σ be a 2-cocycle on G such that the pair (G, σ) satisfies Kleppner's condition, and choosing U to be the left (or right) regular σ-projective unitary representation of G on $\ell^2(G)$.

Since $M = U(G)''$ is injective as a von Neumann algebra, the index group G has to be amenable. Indeed, assume that $\{U(g) : g \in G\}$ is a nice error basis for the hyperfinite II_1 factor M, with associated $\sigma \in Z^2(G,T)$. Then using that $M = U(G)''$ and $\tau(U(g)) = 0$ for all $g \neq e$, we get that

$$\langle U(g)1_M, 1_M\rangle = \langle \lambda_\sigma(g)\delta_e, \delta_e\rangle$$

for all $g \in G$, and that 1_M and δ_e are cyclic for U and λ_σ, respectively. Hence, U is unitary equivalent to λ_σ. Since M is injective, this implies that $\lambda_\sigma(G)''$ is injective, so it follows that G is amenable. It also follows that (G,σ) must satisfy Kleppner's condition.

In particular, it follows that any nontrivial countable amenable icc group is an index group for a nice error basis for M (with trivial 2-cocycle), as well as $G = \mathbb{Z}^n$ for any $n \geq 2$ (with a nondegenerate 2-cocycle). An interesting open problem is to determine which (necessarily countably infinite amenable) groups are index groups for nice error bases for M. Another natural problem arising when one fixes an "index pair" (G,σ) for M is to investigate what can be said (up to unitary equivalence) about the family of σ-projective representations of G which produce nice error bases for M.

If one replaces M by a general II_1 factor (with separable predual), any group can potentially be a candidate for an index group. As above, there is a canonical trace, a cyclic and separating vector, and we must have that (G,σ) satisfies Kleppner's condition. The advantage with the hyperfinite II_1 factor is the existence of an abundance of countably infinite amenable groups that can be index groups for nice error bases for M, and the class of FC-hypercentral groups is a good starting point for constructing such bases.

Moving to the C^*-algebra situation, suppose that G is countable FC-hypercentral, and that U is an irreducible σ-projective unitary representation of G on a Hilbert space H. Let A be the C^*-subalgebra of $B(H)$ generated by $U(G)$. Then the kernel of the induced map $V\colon C_r^*(G^\sigma) \to A$ (where $V(g,z) = zU(g)$ is well-defined since G is amenable), is a primitive ideal, and therefore maximal, meaning that A is simple (see [5]). It then follows that (G,σ) satisfies Kleppner's condition, which again means that $C_r^*(G,\sigma)$ is simple with a unique faithful tracial state. Thus the induced surjective map $U\colon C_r^*(G,\sigma) \to A$ must be an isomorphism.

For example, if $G = \mathbb{Z}_n \times \mathbb{Z}_n$ for $2 \leq n < \infty$ or $G = \mathbb{Z} \times \mathbb{Z}$ and σ is such that (G,σ) satisfies Kleppner's condition, then the abstract error group G^σ is a Heisenberg group. If $G = \mathbb{Z}_n \times \mathbb{Z}_n$, then $A = M_n(\mathbb{C})$, and for $G = \mathbb{Z} \times \mathbb{Z}$, then A is a simple noncommutative torus. More general versions of these groups, of larger dimension and "higher order", so-called free nilpotent groups, are discussed in [21].

We remark that there is an ongoing classification program for separable, simple, nuclear C^*-algebras, and the A above is precisely of this type, and therefore one can in certain situations tell when different groups and 2-cocycles give the same algebra, and construct families of nice error bases in these cases.

2. Open quantum walks

The concept of quantum walks play an important role in quantum information science, especially as a tool to design quantum algorithms, applied to search and sampling problems [1]. It is based on the simulated quantum evolution of a particle moving randomly within some underlying graph structure. A quantum system is said to be open if it interacts with its environment, i.e., the evolution loses information and is therefore nonreversible. Open quantum walks were defined in [2] and are considered as good quantum analogues of Markov chains. These random walks are typically stepwise quantum in their behavior, but they seem to show up a rather classical asymptotic behavior. The process is implemented by Markovian dynamics influenced by internal degrees of freedom, and can be useful to model a variety of phenomena, in particular quantum algorithms, e.g. [26].

The non-commutative nature of the objects under study, and specifically the fact that the transition probabilities are replaced by operators acting on a Hilbert space, are the cause of higher mathematical complexity.

General setup. Consider a directed graph with vertices V and edges $E \subseteq V \times V$. To every vertex $i \in V$ we associate a C^*-algebra or a von Neumann algebra A_i, representing the internal degrees of freedom of the walker. We assume that V is finite or countably infinite, that the graph is connected, and that each A_i is separable and unital. For every edge $(i, j) \in E$ we define a transition map, which is a completely positive map

$$\Psi_{i,j} \colon A_i \to A_j$$

such that for every j we have

$$\sum_{i:(i,j)\in E} \Psi_{i,j}(I_i) \leq I_j. \tag{1}$$

The $\Psi_{i,j}$'s describe the transformation of the internal degrees of freedom as the walker shifts from i to j.

Set $A = \bigoplus_{i\in V} A_i$, where we use c_0-sum in the C^*-case and ℓ^∞-sum in the von Neumann case. Then the association

$$\Psi\left((a_i)_{i\in V}\right) = \Big(\sum_{i:(i,j)\in E} \Psi_{i,j}(a_i)\Big)_{j\in V} \in A \tag{2}$$

defines a completely positive map $\Psi \colon A \to A$. Note that (1) suffices for convergence in von Neumann case, while in the C^*-algebra setting additional convergence requirements are needed, often (3).

The map Ψ is unital if and only if (1) is replaced by the normalization condition

$$\sum_{i:(i,j)\in E} \Psi_{i,j}(I_i) = I_j. \tag{3}$$

Unital quantum channels. In the general setup above, assuming the normalization condition (3), then Ψ is called a quantum channel. The dual map Ψ^* is then trace-preserving (equivalent to Ψ being unital). For simplicity, following [2, 26], assume that for each i, j, there is an operator $B_{i,j}$ such that

$$\Psi_{i,j}(a) = B_{i,j}^* a B_{i,j}, \quad \Psi_{i,j}^*(x) = B_{i,j} x B_{i,j}^*,$$

where $\Psi_{i,j}^*$ is from A_j to A_i, and

$$\sum_{i:(j,i)\in E} B_{i,j}^* B_{i,j} = I_j.$$

Consider the direct sum of the underlying Hilbert spaces $H = \bigoplus_{i\in V} H_i$, set $M_{i,j} = B_{i,j} \otimes |i\rangle\langle j|$ and note that then

$$\sum_{(j,i)\in E} M_{i,j}^* M_{i,j} = I_H.$$

The completely positive maps Ψ and Ψ^* are then defined by

$$\Psi(a) = \sum_{(j,i)\in E} M_{i,j}^* \rho M_{i,j}, \quad \Psi^*(a) = \sum_{(j,i)\in E} M_{i,j} \rho M_{i,j}^*$$

One can compute that

$$\Psi^*\left(\sum_{(i,j)\in E} \rho_{i,j} \otimes |i\rangle\langle j|\right) = \sum_{i\in V}\left(\sum_{j:(i,j)\in E} B_{i,j}\rho_{j,j}B_{i,j}^*\right) \otimes |i\rangle\langle i|,$$

that is, the image takes a certain diagonal form, so one may assume the initial state is of this form.

Problems that are essential to study are analogues of the ones from classical Markov chains; existence of steady states, hitting times, recurrence, periodicity, ergodicity and mixing. In particular, can we describe to what extent these properties depend on the underlying graph? We will analyze quantum trajectories and compute distribution probabilities.

Cayley graphs of monoids. The above questions will be studied for Cayley graphs of monoids. For example, let $\lambda \in \mathbb{C}$ such that $3|\lambda|^2 = 1$ and define matrices

$$A = \lambda \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \lambda \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Then one can compute that $ABA = BAB$ and $A^*A + B^*B = I$. Note also that $ABABAB = -\lambda^6 I$. Consider the graph whose vertices are elements the braid monoid $B_3^+ = \langle a, b : aba = bab \rangle^+$, and where A and B are maps corresponding to the edges (x, ax) and (x, bx), respectively, for every $x \in B_3^+$, i.e., the transition maps are defined by $\rho \mapsto A\rho A^*$ and $\rho \mapsto B\rho B^*$ for density matrices ρ.

Braid group representations are realized as certain particles called nonabelian anyons, which may be applied in quantum computing, see [15].

Moreover, some authors have already studied certain cases where $\mathbb{Z}^n$ is the set of vertices and there is an edge between two nodes if their difference is a unit vector. One may associate $\mathbb{C}^2$ to represent the internal degrees of freedom of each node, i.e., representing the lamp being on/off in a lamplighter setup, or more generally, finite-dimensional spaces $\mathbb{C}^{n_i}$ on the nodes i, whose dimension may vary in $\mathbb{Z}^n$ in some structural way. Open quantum walks based on this setup have applications for quantum circuits realized as sequences of quantum gates.

Graphs from second cohomology groups. Let G be a discrete group and σ, ω two 2-cocycles of G. For a function $\varphi\colon G \to \mathbb{C}$ define $m_\varphi\colon \mathbb{C}[G,\omega] \to \mathbb{C}[G,\sigma]$ by $m_\varphi(\lambda_\sigma(g)) = \varphi(g)\lambda_\omega(g)$. If m_φ is bounded in operator norm, then it induces bounded linear operators

$$M_\varphi\colon C_r^*(G,\omega) \to C_r^*(G,\sigma) \quad \text{and} \quad \widetilde{M}_\varphi\colon W_r^*(G,\omega) \to W_r^*(G,\sigma).$$

Moreover, φ is positive definite with respect to (ω,σ) if there exists a $\overline{\omega}\sigma$-projective unitary representation U of G on some Hilbert space H and some vector $\xi \in H$ such that

$$\varphi(g) = \langle U(g)\xi, \xi \rangle.$$

Then M_φ and $\widetilde{M}_\varphi$ are completely positive maps, and unital when $\varphi(e) = 1$, see e.g., [3, Section 2.2].

Assume that G is a group such that its second cohomology group $H^2(G,\mathbb{T})$ is finite. This is always the case if G is finite, and one particular example of interest is $G = \mathbb{Z}_n \times \mathbb{Z}_n$. Let the graph be determined by (the Cayley graph of) $H^2(G,\mathbb{T})$. For every node in $V = H^2(G,\mathbb{T})$, the associated Hilbert space is $\ell^2(G) = \mathbb{C}^{|G|}$, and the C^*-algebra is $C_r^*(G,\sigma)$. For $(\omega,\sigma) \in E$, we associate a positive definite function φ with respect to (ω,σ) and consider transition maps M_φ.

Commutative cases and resistance networks. Let X denote the vertex set and set $A_x = \mathbb{C}$ for every $x \in X$. Then $A = \bigoplus_{x\in X} \mathbb{C} = C_0(X)$ (or $A = \ell^\infty(X)$ in the von Neumann case). For each pair x, y, the transition map c_{xy} is an element of $[0,\infty)$. The condition $c_{xy} = 0$ means that there is no edge between x and y. We also require that $c_x = \sum_{y\in X} c_{xy}$ is ≤ 1 for every $x \in X$ (this is not done in [14]).

In the framework of resistance networks, see [14], the c is called the conductor function, and the additional assumptions $c_{xy} = c_{yx}$ and $c_{xx} = 0$ are being made. The function $x \mapsto c_x$ induces a multiplication operator M_c on A, i.e., Define two operators M_c and T_c on A by

$$(M_c f)(x) = c_x f(x) \quad \text{and} \quad (T_c f)(x) = \sum_{y\in X} c_{xy} f(y),$$

and note that T_c coincides with $\Psi\colon A \to A$ as defined in (2).

These networks also contain additional structure, e.g., the Laplacian $\Delta\colon A \to A$ given by

$$(\Delta f)(x) = \sum_{y\in X} c_{xy}(f(x) - f(y)),$$

that is, $\Delta = M_c - T_c$. If we let τ be the trace on A defined by $\tau(\delta_x) = 1$ for all $x \in X$, then one can compute that the energy function $\mathcal{E}\colon A \times A \to \mathbb{C}$ is given by

$$\mathcal{E}(f, g) = \langle f, (M_c - T_c)g\rangle = \tau(f^*(M_c - T_c)g).$$

Parts of this is also described by Rieffel [24], in particular in Example 2.3 and Section 8 when X is finite. We hope that this approach will answer Rieffel's question at the end of Example 3.3.

The other ingredients of Rieffel's setup is related to the above by the identities $\Gamma(b, c) = \langle \partial b, \partial c\rangle_A$ and $\mathcal{E}(b, c) = \tau(\Gamma(b, c))$, where ∂ is a derivation from the C^*-algebra A into an A-bimodule Ω. Finally, this gives rise to seminorms $L_{\mathcal{E}}$ and Monge-Kantorovich-type metrics $\rho_{\mathcal{E}}$, as described below. Can we find a generalization of Rieffel's setup to graphs with an infinite vertex set?

In many examples the condition that $c_x = 1$ for all $x \in X$ is being made, i.e., the (3) above. Then the Laplace operator Δ coincides with $I - \Psi$.

3. Quantum channels and metrics

A central role in quantum information theory is played by the notion of a quantum channel, which transmits both quantum and classical information, and is a noncommutative analog of a transition probability matrix in classical theory.

In algebraic quantum mechanics, states are linearly dual to the *-algebra of observables, and (completely) positive maps provide a good model for transformations of states that are more general than just unitary time evolution.

The purpose of this work package is to study quantum channels as communication channels between infinite-dimensional spaces, and use this new approach to gain insight in the connection between group symmetries and quantum information.

Let A be a unital C^*-algebra with a faithful tracial positive linear functional τ. Following Farenick and Rahaman [9], a quantum channel on (A, τ) is a positive linear map $Q\colon A \to A$ which preserves τ. If A is a von Neumann algebra, τ is also assumed to be normal, and Q has to be normal. The usual definition in the case where $A = M_n(\mathbb{C})$ and τ is the usual trace on $M_n(\mathbb{C})$ is to require in addition that Q is *completely* positive as defined above, but we will not do that here. We also note that if $A = \mathbb{C}^n$ and $\tau(z_1, \ldots, z_n) = z_1 + \cdots + z_n$, then a quantum channel may be identified with a stochastic $n \times n$-matrix, giving rise to Markov chains. Motivated by the fact that the state space $S(M_n(\mathbb{C}))$ of $M_n(\mathbb{C})$ may be identified with the density matrices in $M_n(\mathbb{C})$, while $S(\mathbb{C}^n)$ corresponds bijectively to stochastic vectors, Farenick and Rahaman introduce the density space of a pair (A, τ)

$$D_\tau(A) = \{a \in A_+ : \tau(a) = 1\},$$

as an analog in the infinite-dimensional case.

Clearly, a quantum channel Q maps $D_\tau(A)$ into itself. On the other hand, if Q is unital, its adjoint map $\varphi \to \varphi \circ Q$ maps $S(A)$ into itself. When $A = M_n(\mathbb{C})$ or $A = \mathbb{C}^n$, there has been a lot of interest in studying when quantum channels are strict contractions with respect to the metric on $D_\tau(A)$ associated with the 1-norm on A induced by τ, i.e., $\|a\| = \tau(\sqrt{a^*a})$. This metric also makes sense in the infinite-dimensional case. Farenick and Rahaman consider another metric on $D_\tau(A)$, called the Bures metric (see also [19]), defined for any two $a, b \in D_\tau(A)$ by

$$d_B^\tau(a, b) = \sqrt{1 - \tau(||a^{1/2}b^{1/2}||)}.$$

It is shown in [9] that this is indeed a metric and that it induces the same topology on $D_\tau(A)$ as the one coming from the 1-norm.

Their motivation is to initiate a study of Bures contractive channels, but several questions are left untouched. For instance, to be able to deduce the existence of a unique fixed-point in $D_\tau(A)$ for a strictly contractive quantum channel Q, it would be nice to find conditions (other than finite-dimensionality of A) ensuring that $D_\tau(A)$ is compact in the topology from the Bures metric.

In another direction, following Connes, Rieffel, and others (see e.g., the survey [18]), several constructions of metrics on $S(A)$ are known, generalizing the Monge-Kantorovich metric on the regular Borel probability measures on a compact Hausdorff space. In some cases the associated topology is known to coincide with the relative weak*-topology on $S(A)$.

In particular, for a unital C^*-algebra A equipped with a lower semicontinuous seminorm defined on its self-adjoints, $L\colon A_{\mathrm{sa}} \to [0,\infty]$, such that $\mathrm{dom}(L) = \{a \in A_{\mathrm{sa}} : L(a) < \infty\}$ is dense in A_{sa} and $\{a \in A_{\mathrm{sa}} : L(a) = 0\} = \mathbb{R}1_A$. We call (A, L) a *compact quantum metric space* if the associated *Monge-Kantorovich* metric defined $S(A)$, for any two states μ, ν by

$$\mathrm{mk}_L(\mu,\nu) = \sup\{|\mu(a) - \nu(a)| : a \in A_{\mathrm{sa}}, L(a) \leq 1\}$$

metrizes the weak* topology. This was introduced by Rieffel in [22].

Now, to each $a \in D_\tau(A)$, one may associate $\varphi_a \in S(A)$ by setting $\varphi_a(b) = \tau(ab)$ for all $b \in A$. Since τ is faithful, the map $a \mapsto \varphi_a$ is injective. Hence, if d is a metric on $S(A)$, then we get a metric d_τ on $D_\tau(A)$ given by $d_\tau(a,b) = d(\varphi_a, \varphi_b)$.

Following the above, we may define a metric on $D_\tau(A)$ for any two $a, b \in D_\tau(A)$ by

$$\begin{aligned} d^\tau_{\mathrm{mk}_L}(a,b) = \mathrm{mk}_L(\varphi_a,\varphi_b) &= \sup\{|\varphi_a(c) - \varphi_b(c)| : c \in A_{\mathrm{sa}}, L(c) \leq 1\} \\ &= \sup\{|\tau(ac) - \tau(bc)| : c \in A_{\mathrm{sa}}, L(c) \leq 1\} \end{aligned}$$

Do these metrics induce the same topology as the Bures on the density space?

Whether this approach can be used to show that $D_\tau(A)$ becomes compact (or complete) with respect to d_τ in certain cases is not clear. If Q is a unital quantum channel, it seems that Q acts on $D_\tau(A)$ and on $S(A)$ in a non-compatible way under the embedding $a \mapsto \varphi_a$, so this may provide a challenge. Finally, if Q is a unital quantum channel on (A, τ), and there is a metric d on $S(A)$ giving the relative weak*-topology on $S(A)$, then it would be interesting to study when Q acts contractively on $S(A)$, because Q would then have a unique fixed-point, namely τ, and we would also have that $d(\phi \circ Q^n, \tau) \to 0$ as $n \to \infty$ for every $\phi \in S(A)$.

In general, what can we learn about a quantum channel from its ergodicity and mixing properties?

Group algebras. A typical situation we will study is when A is a twisted group C^*-algebra or a group von Neumann algebra, that is, when $A = C^*_r(G,\sigma)$ or $A = W^*(G,\sigma)$, where G is a countably infinite discrete group, σ is a 2-cocycle on G, and τ is the canonical tracial state on A (see e.g., [5]).

When $A = C^*_r(G,\sigma)$, a length function ℓ on G will induce a metric on $S(A)$ (via the Dirac operator associated with ℓ), as explained by Rieffel's theory on compact quantum metric spaces [23].

In [13, Section 3.3], the authors discuss a generalization of frames that have been used in quantum computing. Let U be a σ-projective unitary representation of G on a Hilbert space H, and suppose that $a \in B(H)$ is such that $\sum_{g \in G} U_g^* a^* a U_g = I$. Then the map

$$\Phi_{U,a}(x) = \sum_{g \in G} U_g^* a^* x a U_g$$

defines a unital completely positive map $B(H) \to B(H)$ with range contained in $U(G)'$. In which cases does $\Phi_{U,a}$ restrict to a map between von Neumann algebras? And when is it a strict contraction on the density space?

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