

# INTRODUCTION TO C\*-ALGEBRAS

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ABSTRACT. These are lectures notes for an introductory course on  $C^*$ -algebras held in Autumn 2016 at the University of Oslo.

## INTRODUCTION

Good references for the contents (and way beyond) are [Mur90, Ped79].

### 1. A BRIEF REVIEW OF COMMUTATIVE BANACH ALGEBRAS

We will mainly be interested in algebras over the field  $\mathbb{C}$ , so this will implicitly be assumed, unless stated otherwise. Recall the following facts:

- (1) For an element  $a$  in a unital algebra  $A$ , the spectrum is defined as  $\text{Sp}a = \{\lambda \in \mathbb{C} \mid \lambda - a \text{ is not invertible}\}$ .
- (2) A Banach algebra is an algebra  $A$  with a submultiplicative norm  $\|\cdot\|$ , i.e.  $\|ab\| \leq \|a\|\|b\|$ , for which  $A$  is complete.
- (3) For every  $a$  in a unital Banach algebra  $A$ , the spectrum  $\text{Sp}a$  is a nonempty, compact set with spectral radius  $r(a) := \lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}} \leq \|a\|$ .

Now suppose  $A$  is a commutative, unital Banach algebra. Then an ideal  $I$  in  $A$  is maximal if and only if  $A/I \cong \mathbb{C}$ . This yields a correspondence between maximal ideals and nonzero homomorphisms  $\varphi: A \rightarrow \mathbb{C}$ . For every such homomorphism  $\varphi$  and  $a \in A$

$$\varphi(\varphi(a) - a) = 0 \Rightarrow \varphi(a) \in \text{Sp } a \Rightarrow |\varphi(a)| \leq \|a\|$$

holds true. Thus every nonzero homomorphism  $\varphi: A \rightarrow \mathbb{C}$  is continuous, and  $\text{Spec } A := \{\varphi: A \rightarrow \mathbb{C} \text{ nonzero homomorphism}\}$  becomes a compact Hausdorff space under the topology of pointwise convergence. This leads to  $\chi: A \rightarrow C(\text{Spec } A), a \mapsto \hat{a}$  with  $\hat{a}(\varphi) := \varphi(a)$ , the homomorphism known as the *Gelfand transformation*.

*Remark 1.1.* If  $X$  is a compact Hausdorff space and  $A = C(X)$ , then  $\text{Spec } A$  is homeomorphic to  $X$  with  $\chi$  being the identity. Hence the range of  $\hat{a}$  equals  $\text{Sp } a$  for all  $a \in A$  here, and consequently  $\|\hat{a}\| = r(a)$ .

### 2. C\*-ALGEBRAS: UNIQUENESS OF THE C\*-NORM, UNITIZATION, AND THE FUNCTIONAL CALCULUS FOR NORMAL ELEMENTS

Recall that an *involution*  $*$ :  $A \rightarrow A, a \mapsto a^*$  on a normed algebra  $(A, \|\cdot\|)$  is an conjugate linear, anti-multiplicative symmetry, i.e.  $(\lambda a + bc)^* = \overline{\lambda}a^* + c^*b^*$ , and  $(a^*)^* = a$  for  $\lambda \in \mathbb{C}, a, b, c \in A$ . Involutions are norm-decreasing, and they are often isometric.

**Definition 2.1.** A  $C^*$ -algebra is a Banach algebra  $(A, \|\cdot\|)$  with an involution  $*$  satisfying the  $C^*$ -norm condition  $\|aa^*\| = \|a\|^2$  for all  $a \in A$ .

*Remark 2.2.* As  $\|aa^*\| \leq \|a\|\|a^*\| = \|a\|^2$ , it suffices to request  $\|aa^*\| \geq \|a\|^2$ .

*Examples 2.3.* Here are some basic examples to get started:

- (a) The algebra of bounded, linear operators on a Hilbert space  $H$ , denoted by  $\mathcal{L}(H)$ , forms a  $C^*$ -algebra with the operator norm and involution given by taking the adjoint.
- (b) For every  $C^*$ -algebra  $A$ , every subalgebra  $B \subset A$  that is closed and invariant under the involution yields a  $C^*$ -algebra.
- (c) Given two Banach algebras  $A, B$ , their direct sum  $A \oplus B$  becomes a Banach algebra under component-wise operations and  $\|(a, b)\| := \max(\|a\|, \|b\|)$ . If  $A$  and  $B$  are  $C^*$ -algebras, then  $A \oplus B$  becomes a  $C^*$ -algebra with  $(a, b)^* := (a^*, b^*)$ .
- (d) Let  $A$  be a  $C^*$ -algebra, and  $M \subset A$  an arbitrary subset. Then there exists a smallest  $C^*$ -subalgebra of  $A$  containing  $M$  which is given by  $C^*(M) := \bigcap_{\substack{B \subset AC^*\text{-algebra} \\ M \subset B}} B$ . Equivalently,  $C^*(M)$  can be described as the closure of all polynomials in  $M \cup M^*$ .

If  $A$  is a Banach algebra, then it can be embedded into a unital Banach algebra  $\tilde{A}$  in an elementary way: On the  $\mathbb{C}$ -vector space  $\tilde{A} := A \oplus \mathbb{C}$ , define multiplication by  $(a, \lambda)(b, \mu) := (ab + \mu a + \lambda b, \lambda\mu)$ . Then  $\tilde{A}$  is an algebra with unit  $1 = (0, 1)$ . It becomes a Banach algebra for the norm  $\|(a, \lambda)\| := \|a\| + \|\lambda\|$ , and  $A$  embeds into  $\tilde{A}$  isometrically via  $a \mapsto (a, 0)$ .

For  $C^*$ -algebras, the story is a little more subtle, but has a happy ending.

**Proposition 2.4.** *If  $a$  is a normal element in a  $C^*$ -algebra  $A$ , then  $r(a) = \|a\|$ . Therefore, every unital  $*$ -algebra  $A$  admits at most one norm satisfying the  $C^*$ -norm condition.*

*Proof.* Suppose first that  $a$  is self-adjoint. Then the  $C^*$ -norm condition gives  $\|a^2\| = \|a\|^2$ , and thus  $r(a) = \lim_{n \rightarrow \infty} \|a^{2^n}\|^{2^{-n}} = \|a\|$ . For  $a$  normal, we get

$$\|a\|^{2^n} = \|aa^*\|^{2^{n-1}} = \|(aa^*)^{2^{n-1}}\|^{\frac{1}{2}} = \|a^{2^n} a^{*2^n}\|^{\frac{1}{2}} = \|a^{2^n}\|,$$

and thus  $r(a) = \|a\|$ . For the second claim, let  $\|\cdot\|_1$  and  $\|\cdot\|_2$  be  $C^*$ -norms on  $A$ . By the first part, we get

$$\|a\|_1^2 = \|aa^*\|_1 = r(a) = \|aa^*\|_2 = \|a\|_2^2 \text{ for all } a \in A.$$

□

A remarkable aspect of Proposition 2.4 is that the spectral radius, originally defined algebraically without reference to a norm, actually coincides with the unique  $C^*$ -norm for every  $C^*$ -algebra.

**Theorem 2.5.** *Let  $A$  be a  $C^*$ -algebra with norm  $\|\cdot\|_A$  and  $\tilde{A}$  the unital algebra described above. Then there exists a unique  $C^*$ -norm  $\|\cdot\|$  on  $\tilde{A}$ .*

*Proof.* By Proposition 2.4, we know that there can only be one such norm, so we need to prove the existence of this norm. Suppose first that  $A$  already has a unit  $e$ . As  $1 - e$  is a self-adjoint idempotent, i.e. a projection, with  $(1 - e)a = 0$  for all  $a \in A$ , we get a

decomposition  $\tilde{A} = A \oplus \mathbb{C}(1 - e) \cong A \oplus \mathbb{C}$  as a direct sum of involutive Banach algebras. In that case, we get a  $C^*$ -algebra-structure on  $\tilde{A}$  via Example 2.3 (c).

Now suppose that  $A$  does not have a unit. For  $x = (a, \lambda) \in \tilde{A}$ , define a linear operator  $L_x: A \rightarrow A$  by  $L_x(b) = xb = ab + \lambda b$ . Then  $L_x$  is bounded with  $\|L_x\| \leq \|a\|_A + |\lambda|$ , and we define  $\|x\| := \|L_x\| = \sup\{\|L_x(b)\| \mid b \in A, \|b\| \leq 1\}$ . To see that this defines a norm on  $\tilde{A}$ , note that

$$\|x\| = \|L_x\| = 0 \Rightarrow L_x = 0 \Rightarrow xb = 0 \text{ for all } b \in A.$$

In this case, we must have  $\lambda \neq 0$  (for otherwise  $x = 0$ ). Assume  $\lambda \neq 0$ . Then  $b = -\lambda^{-1}ab$  for all  $b \in A$  would show that  $A$  admits a unit on the left. Taking involution, we would also get a unit on the right. In that case, these two units would necessarily coincide and  $A$  would be unital - a contradiction. Hence  $\|\cdot\|$  defines a norm on  $\tilde{A}$ .

Next, we show that it satisfies the  $C^*$ -norm condition, that is  $\|x^*x\| = \|x\|^2$ : Let  $\varepsilon > 0$ , and  $b \in A, \|b\|_A \leq 1$  with  $\|xb\| \geq \|xb\| - \varepsilon$ . As  $\|b^*\|_A \leq 1$ , we get

$$\|x^*x\| \geq \|b^*\|_A \|x^*xb\| \geq \|b^*x^*xb\| = \|(xb)^*xb\| = \|(xb)^*xb\|_A = \|xb\|^2 \geq (\|x\| - \varepsilon)^2,$$

which suffices, see Remark 2.2. Finally,  $\tilde{A}$  is complete with respect to  $\|\cdot\|$  as a sum of a finite dimensional space and a Banach space inside the  $C^*$ -algebra of bounded linear operators on  $A$  denoted by  $\mathcal{L}(A)$ . Another way to see this is to note that  $\tilde{A}$  is a closed  $*$ -invariant subalgebra of the  $C^*$ -algebra  $\mathcal{L}(A)$ .  $\square$

*Remark 2.6.* Let  $A, B$  be algebras, and assume that  $B$  is unital. Then every homomorphism  $\varphi: A \rightarrow B$  admits a unique extension  $\tilde{\varphi}: \tilde{A} \rightarrow B$  with  $\tilde{\varphi}(1_{\tilde{A}}) = 1_B$ .

**Theorem 2.7.** *Suppose  $A$  is a Banach algebra with isometric involution and  $B$  is a  $C^*$ -algebra. Then every  $*$ -homomorphism  $\varphi: A \rightarrow B$  is contractive, i.e.  $\|\varphi(a)\| \leq \|a\|$  for all  $a \in A$ .*

*Proof.* Consider the unique extension  $\tilde{\varphi}: \tilde{A} \rightarrow \tilde{B}$ . As  $\text{Sp}_{\tilde{B}}\tilde{\varphi}(a) \subset \text{Sp}_{\tilde{A}}a$ , we get

$$\|a\|^2 = \|a^*\| \|a\| \geq \|aa^*\| \geq r(aa^*) \geq r(\varphi(aa^*)) = \|\varphi(aa^*)\| = \|\varphi(a)\|^2,$$

so that  $\varphi$  is contractive.  $\square$

We now revisit and extend some spectral considerations for elements in general  $C^*$ -algebras, starting with a straightforward observation.

**Proposition 2.8.** *Suppose  $X$  is a compact Hausdorff space and  $x_0 \in X$ . Then  $Y := X \setminus \{x_0\}$  is locally compact,  $C_0(Y) \cong \{f \in C(X) \mid f(x_0) = 0\}$ , and  $C(X) = C_0(Y) \oplus \mathbb{C}$  as vector spaces.*

*Remark 2.9.* For a locally compact Hausdorff space  $Y$ , define a topology on  $X := Y \sqcup \{\infty\}$  (extending the original one on  $Y$ ) by declaring  $N \subset X$  to be a neighbourhood of  $\infty$  if there exists  $K \subset Y$  compact such that  $X \setminus K \subset N$ . With this topology,  $X$  becomes a compact Hausdorff space, and  $Y = X \setminus \{\infty\}$  with the subspace topology.  $X$  is called the *Alexandroff compactification* of  $Y$ . In this context, Proposition 2.8 states that  $C_0(Y) \cong \{f \in C(X) \mid f(\infty) = 0\}$ .

*Example 2.10.* The Alexandroff compactification of a nonempty half-open interval in  $\mathbb{R}$  gives a space that is homeomorphic to  $[0, 1]$ , whereas any nonempty open interval yields the torus.

Recall that a *character* on an algebra  $A$  over a field  $K$  is an algebra homomorphism  $\varphi: A \rightarrow K$ .

**Definition 2.11.** For a commutative  $C^*$ -algebra  $A$ , the *spectrum* of  $A$  is  $\text{Spec } A := \{\varphi \text{ nonzero character on } A\}$ .

Recall that the Gelfand transformation  $\chi: A \rightarrow C(\text{Spec } A)$  defines an isometric  $*$ -isomorphism for every unital commutative  $C^*$ -algebra  $A$ .

Note that  $\text{Spec } A$  sits inside  $\text{Spec } \tilde{A}$ , which corresponds to the space of all characters on  $A$ . Thus we have  $\text{Spec } A = \text{Spec } \tilde{A} \setminus \{\tilde{0}\}$ , and we conclude that  $\text{Spec } A$  is locally compact and Hausdorff, see Proposition 2.8.

**Proposition 2.12.** *Let  $A$  be a commutative  $C^*$ -algebra. Then the restriction of the Gelfand transformation for  $\tilde{A}$  defines an isometric  $*$ -isomorphism*

$$A \rightarrow C_0(\text{Spec } A) \cong \{f \in C(\text{Spec } \tilde{A}) \mid f(0) = 0\}.$$

*Proof.* Using the previous results, this is an easy exercise.  $\square$

We will see that Proposition 2.12 is particularly interesting in connection with Example 2.3 (d) applied to  $M = \{a\}$  or  $\tilde{M} = \{a, 1\}$  for a normal element  $a \in A$ . This yields commutative  $C^*$ -subalgebras  $C^*(a) \subset A$  and  $C^*(a, 1) \subset \tilde{A}$ .

We want to define a notion of a spectrum for elements inside a not necessarily unital  $C^*$ -algebra. To this end, we make the following preliminary definition: For an element  $a$  in a  $C^*$ -algebra  $A$ , we define  $\text{Sp}' a := \text{Sp}_{\tilde{A}} a$ .

*Remark 2.13.* If  $A$  is a  $C^*$ -algebra with unit  $e$ , then we know that  $\tilde{A} \cong A \oplus \mathbb{C}(1 - e)$ . Using the canonical embedding  $A \rightarrow \tilde{A}, a \mapsto (a, 0)$ , we see that  $\lambda 1 - a = (\lambda e - a, \lambda)$  is invertible in  $\tilde{A}$  if and only if  $\lambda \neq 0$  and  $\lambda \notin \text{Sp}' a$ . This means that  $\text{Sp}' a = \text{Sp } a \cup \{0\}$ .

**Theorem 2.14.** *Let  $a$  be a normal element in a  $C^*$ -algebra  $A$ .*

- (1) *If  $A$  is unital, then the map  $\text{Spec } C^*(a, 1) \rightarrow \text{Sp}_A a, \varphi \mapsto \varphi(a)$  is a homeomorphism and the Gelfand transformation defines an isomorphism  $C^*(a, 1) \cong C(\text{Sp } a)$ .*
- (2) *The map  $\text{Spec } C^*(a) \rightarrow \text{Sp}' a \setminus \{0\}, \varphi \mapsto \varphi(a)$  is a homeomorphism and Gelfand transformation defines an isomorphism  $C^*(a) \cong \{f \in C(\text{Sp}' a) \mid f(0) = 0\} \cong C_0(\text{Sp}' a \setminus \{0\})$ .*

*Proof.* (1) The map is injective: If  $\varphi$  and  $\psi$  are characters on  $C^*(a, 1)$  with  $\varphi(a) = \psi(a)$ , then they agree on all polynomials in  $a$  and  $a^*$ . Hence we get  $\varphi = \psi$  by Example 2.3 (d). Surjectivity is easy as every  $\lambda \in \text{Sp } a$  defines a nonzero character  $\varphi \in \text{Spec } C^*(a, 1)$  via  $\varphi(a) := \lambda$ . As  $\text{Spec } C^*(a, 1)$  carries the topology of pointwise convergence, the map is also continuous, and hence a homeomorphism because domain and range are compact.

- (2) According to (1), the map  $\text{Spec } \widetilde{C^*(a)} \rightarrow \text{Sp}' a$  is a homeomorphism with  $\tilde{0} \mapsto 0$  and  $\widetilde{C^*(a)} \cong C(\text{Sp}' a)$ . Therefore, we have  $\text{Spec } C^*(a) = \text{Spec } \widetilde{C^*(a)} \setminus \{\tilde{0}\} \cong \text{Sp}' a \setminus \{0\}$  and  $C^*(a) \cong C_0(\text{Sp}' a \setminus \{0\})$  by Proposition 2.12 and the preceding discussion.  $\square$

*Remark 2.15.* If  $A$  is a unital  $C^*$ -algebra, then  $\text{Sp}' a \setminus \{0\} = \text{Sp } a \setminus \{0\}$  so that  $\text{Spec } C^*(a)$  is homeomorphic to  $\text{Sp } a \setminus \{0\}$ .

**Definition 2.16.** For an element  $a$  in a  $C^*$ -algebra  $A$ , we define its *spectrum*  $\text{Sp } a$  as

$$\text{Sp } a = \begin{cases} \text{Sp}_A a & , \text{ if } A \text{ is unital, and} \\ \text{Sp}_{\bar{A}} a & , \text{ otherwise.} \end{cases}$$

*Remark 2.17* (The functional calculus for normal elements in  $C^*$ -algebra). Let  $a$  be a normal element in a  $C^*$ -algebra  $A$ . Then the following hold by invoking Theorem 2.14:

- (i) Every  $f \in C(\text{Sp } a)$  defines an element  $f(a) \in C^*(a, 1)$ . This gives rise to a functional calculus with the properties that

$$\lambda f + \mu g(a) = \lambda f(a) + \mu g(a), (fg)(a) = f(a)g(a), \bar{f}(a) = f(a^*), \text{ and } f \circ g(a) = f(g(a)).$$

- (ii) Every element  $f \in C(\text{Sp } a)$  with  $f(0) = 0$  then defines an element  $f(a) \in C^*(a)$ .
- (iii) Due to the isomorphisms in Theorem 2.14, every element in  $C(a, 1)$  and  $C^*(a)$  admits such a description.

**Proposition 2.18.** *Let  $A$  be a unital  $C^*$ -algebra and  $a \in A$  normal. Then  $a$  is invertible in  $A$  if and only if  $1 \in C^*(a)$ .*

*Proof.* Recall that  $a$  is invertible in  $A$  if and only if  $0$  does not belong to  $\text{Sp } a$ . If  $0 \notin \text{Sp } a$ , then  $f(x) := x^{-1}$  defines an element of  $C(\text{Sp } a)$  for which  $1 = f(a)a \in C^*(a)$  by the properties listed under Remark 2.17 (i). Conversely, let  $1 \in C^*(a)$ . Under the functional calculus, the unit corresponds to the characteristic function  $\chi_{\text{Sp } a}$ , and hence  $\chi_{\text{Sp } a}(0) = 0$  according to Remark 2.17 (ii), which implies  $0 \notin \text{Sp } a$ .  $\square$

**Proposition 2.19.** *Let  $a$  and  $b$  be elements in a unital  $C^*$ -algebra  $A$ . Then  $\text{Sp } ab \cup \{0\} = \text{Sp } ba \cup \{0\}$  holds.*

*Proof.* Suppose  $\lambda \notin \text{Sp } ab \cup \{0\}$ . Without loss of generality, we can assume that  $\lambda = 1$  (by replacing  $a$  with  $\lambda^{-1}x$ , if necessary). Set  $u := (1 - ab)^{-1} \in A$ . Then we get

$$\begin{aligned} (1 - ba)(1 + bua) &= 1 - ba + bua - babua = 1 - ba + b(1 - ab)ua = 1 \text{ and} \\ (1 + bua)(1 - ba) &= 1 - ba + bua - buaba = 1 - ba + bu(1 - ab)a = 1. \end{aligned}$$

Thus  $1 = \lambda \notin \text{Sp } ba$ .  $\square$

*Remark 2.20.* If  $A$  is a  $C^*$ -algebra with unit  $1$  and  $B$  is a  $C^*$ -subalgebra of  $A$  with  $1 \in B$ , then  $\text{Sp}_A b = \text{Sp}_B b$  for every normal element  $b \in B$ . Thus it suffices to calculate the spectrum of  $b$  in  $C^*(b, 1)$ . This is also true for not necessarily normal elements, but the proof involves spectral measures, see [Mur90, Section 2.5].

### 3. POSITIVE ELEMENTS IN A $C^*$ -ALGEBRA

For the whole section,  $A$  is assumed to be a  $C^*$ -algebra.

**Definition 3.1.** An element  $a \in A$  is called *positive* (denoted  $a \geq 0$ ) if  $a$  is self-adjoint and  $\text{Sp } a \subset [0, \infty)$ .

*Remark 3.2.* Due to the functional calculus from Remark 2.17, every positive element  $a$  has a (uniquely determined) square root  $\sqrt{a} = a^{\frac{1}{2}} \in A$ .

**Lemma 3.3.** *Let  $A$  be unital,  $a \in A$  self-adjoint, and  $\lambda \geq \|a\|$ . Then  $a$  is positive if and only if  $\|\lambda - a\| \leq \lambda$ .*

*Proof.* This follows easily from the functional calculus in Remark 2.17.  $\square$

**Proposition 3.4.** *The sum of two positive elements in  $A$  is positive.*

*Proof.* Let  $a$  and  $b$  be positive elements. For  $\lambda := \|a\| + \|b\|$ , we get

$$\|\lambda - (a + b)\| \leq \| \|a\| - a \| + \| \|b\| - b \| \leq \lambda$$

by applying Lemma 3.3 twice. Thus, Lemma 3.3 now implies that  $a + b \geq 0$ .  $\square$

*Remark 3.5.* Elements in  $A$  can be expressed as linear combinations of positive elements:

- (i) If  $a \in A$  is self-adjoint, then there are  $a_+, a_- \geq 0$  with  $a_+ a_- = 0$  such that  $a = a_+ - a_-$ . To see this, use the functional calculus from Remark 2.17 (ii) for  $f_+ := \max(\text{id}, 0)$  and  $f_- := \min(\text{id}, 0)$ .
- (ii) If  $a \in A$  is an arbitrary element, we can decompose as  $a = a_1 + ia_2$  with  $a_1 = \text{Re } a = \frac{a+a^*}{2}$  and  $a_2 = \text{Im } a = \frac{a-a^*}{2i}$ . As  $a_1, a_2$  are self-adjoint, we end up with

$$a = a_{1,+} - a_{1,-} + ia_{2,+} - ia_{2,-}.$$

**Proposition 3.6.** *For  $a \in A$ , the following conditions are equivalent:*

- (1)  $a \geq 0$ .
- (2)  $\exists h \in A, h^* = h$  with  $h^2 = a$ .
- (3)  $\exists b \in A$  with  $b^*b = a$ .

*Proof.* For (1) implies (2), take  $h := \sqrt{a}$ , see Remark 3.2. (2) implies (3) is obvious, so it remains to prove that (3) implies (1): As a first step, we show that  $-c^*c \geq 0$  forces  $c = 0$  for any  $c \in A$ . Note that  $-c^*c \geq 0$  is equivalent to  $-cc^* \geq 0$ , see Proposition 2.19. If we now invoke Remark 3.5 (ii) to write  $c = c_1 + ic_2$  with  $c_i$  self-adjoint, then we get  $c^*c + cc^* = 2c_1^2 + 2c_2^2$ . Thus  $-cc^* \geq 0$  would imply that  $c^*c = 2c_1^2 + 2c_2^2 + (-cc^*)$  is positive by Proposition 3.4. As we also have  $-c^*c \geq 0$ , we get  $\text{Sp } c^*c = \{0\}$ , and hence

$$\|c\|^2 = \|c^*c\| = r(c^*c) = 0.$$

Now let  $a = b^*b = u - v$  with  $u, v \geq 0, uv = 0$ , see Remark 3.5 (i), and set  $c := bv$ . Then  $-c^*c = -vb^*bv = v^3 \geq 0$  as  $v \geq 0$ . According to the first step, this forces  $c = 0$  so that  $v^3 = 0$ . Hence  $v = 0$  and thus  $a \geq 0$ .  $\square$

*Remark 3.7.* We shall denote the set of self-adjoint elements by  $A_{sa}$ , and the set of positive elements by  $A_+$ . Then Remark 3.5, Proposition 3.4, and Proposition 3.6 show that  $A_+$  is a cone in  $A_{sa}$ , i.e.

- (i)  $A_{sa} = A_+ - A_+$ ,
- (ii)  $A_+ + A_+ \subset A_+$ ,
- (iii)  $\lambda A_+ \subset A_+$  for all  $\lambda \in [0, \infty)$ , and
- (iv)  $A_+ \cap -A_+ = \{0\}$ .

Thus  $a \geq b : \Leftrightarrow a - b \geq 0$  defines a partial order on  $A_{sa}$ .

**Theorem 3.8.** *Let  $A$  be a  $C^*$ -algebra. Then the following statement hold:*

- (1)  $A_+ = \{a^*a \mid a \in A\}$ .

- (2) For  $a, b \in A_{sa}, c \in A$ ,  $a \leq b$  implies  $c^*ac \leq c^*bc$ .
- (3) For  $a, b \in A$ ,  $0 \leq a \leq b$  implies  $\|a\| \leq \|b\|$ .
- (4) If  $A$  is unital, then  $0 \leq a \leq b$  for invertible  $a$  and  $b$  implies  $0 \leq b^{-1} \leq a^{-1}$ .

*Proof.* We address the items in the same order:

- (1) This has been proven in Proposition 3.6.
- (2) We have  $c^*bc - c^*ac = c^*(b - a)c = (\sqrt{b - a})^*\sqrt{b - a}c \geq 0$  using Remark 3.2 and Proposition 3.6.
- (3) By passing to the unitization of  $A$  if necessary, see Theorem 2.5, we can assume that  $A$  is unital. Then  $\|b\| = \inf\{\lambda \in (0, \infty) \mid \lambda \geq b(\geq a)\} \geq \|a\|$ .
- (4) The assumption  $0 \leq a \leq b$  implies that  $a^{-1}$  and  $b^{-1}$  are positive, for instance by using Definition 3.1 and functional calculus. As  $\sqrt{a^{-1}}a\sqrt{a^{-1}} = 1$ , we can use (2) for  $c = \sqrt{a^{-1}}$  to get  $1 \leq \sqrt{a^{-1}}b\sqrt{a^{-1}}$ . Then functional calculus shows  $1 \geq (\sqrt{a^{-1}}b\sqrt{a^{-1}})^{-1} = \sqrt{a}b^{-1}\sqrt{a}$ . Now we apply (2) for  $c = \sqrt{a^{-1}}$  again.

□

*Remark 3.9.* Suppose  $a, b \in A$  satisfy  $0 \leq a \leq b$ . Then we have  $0 \leq a^\alpha \leq b^\alpha$  for every  $\alpha \in (0, 1]$ . If  $a$  and  $b$  commute, then  $0 \leq a^2 \leq b^2$  holds as well. However, this is not true in general. Indeed, take  $A = M_2(\mathbb{C})$ . Then

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad q = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

are projections, i.e.  $p^* = p = p^2$ . Then  $p \leq p + q$  by Proposition 3.4, but

$$(p + q)^2 - p = pq + qp + q = \frac{1}{2} \left[ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right] = \frac{1}{2} \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$$

has eigenvalues  $\frac{2 \pm \sqrt{5}}{2}$  and thus  $\text{Sp}(p + q)^2 - p^2$  is not contained in  $[0, \infty)$ . So we have  $0 \leq p \leq p + q$ , but  $p^2 \not\leq (p + q)^2$ . In fact, if  $0 \leq a^\alpha \leq b^\alpha$  for all  $\alpha \geq 1$ , then  $a$  and  $b$  necessarily commute.

#### 4. APPROXIMATE UNITS, IDEALS, AND QUOTIENTS

**Definition 4.1.** Let  $A$  be a  $C^*$ -algebra. An *approximate unit* in  $A$  is a monotone increasing net  $(u_\lambda)_{\lambda \in \Lambda} \subset A$  with  $0 \leq u_\lambda \leq 1$  (in  $\tilde{A}$ ) and  $u_\lambda a \rightarrow a \leftarrow au_\lambda$  for all  $a \in A$ .

*Example 4.2.* Of course, the unit in a unital  $C^*$ -algebra yields an approximate unit, but here are two natural, non-tautological examples:

- (a) For  $A = C_0(\mathbb{R})$ , an approximate unit is given by  $u_n := (n + 1 - t)\chi_{[-(n+1), -n]} + \chi_{[-n, n]} + (n + 1 - t)\chi_{(n, n+1]}$ .
- (b) Let  $H$  be a Hilbert space, e.g.  $H = \ell^2(\mathbb{N})$ , with orthonormal basis  $(e_i)_{i \in I}$ , and  $A = \mathcal{K}(H)$ , the  $C^*$ -algebra of compact operators on  $H$ . Note that the family  $\Lambda$  of finite subsets  $F$  of  $I$  form a directed set with respect to inclusion. For  $F \subset I$  finite, let  $P_F$  denote the orthogonal projection onto the space  $\langle \{e_i \mid i \in F\} \rangle \subset H$ . Then  $0 \leq P_F \leq 1$ , and  $F \subset G$  corresponds to  $0 \leq p_F \leq p_G$ . Thus  $(p_F)_{F \in \Lambda}$  is a monotone increasing net of positive contractions. Moreover, for every compact operator  $k \in \mathcal{K}(H)$ , we have  $p_F k, k p_F \rightarrow k$  as the finite subset  $F$  gets larger.

The next theorem shows that an approximate unit is not available in contexts that we arguably understand rather well, but a general tool that is always at our disposal.

**Theorem 4.3.** *Let  $A$  be a  $C^*$ -algebra (or a norm-closed ideal in a  $C^*$ -algebra). Then  $(u_h)_{h \in \Lambda}$  given by*

$$\Lambda = \{h \in A_+ \mid \|h\| < 1\} \quad \text{and} \quad u_h = h$$

*constitutes an approximate unit for  $A$ .*

*Proof.* We need to prove that

a)  $\Lambda$  is direct, and

b)  $ha \rightarrow a \leftarrow ah$  for all  $a \in A$  with the limit taken over  $h \in \Lambda$ .

For  $\lambda \in (0, \infty)$  and  $a, b \in A$  with  $0 \leq a \leq b$ , the elements  $\lambda^{-1} + a$  and  $\lambda^{-1} + b$  are invertible (in  $\tilde{A}$ ), and  $\lambda^{-1} + a \leq \lambda^{-1} + b$ . Thus Theorem 3.8 (4) gives

$$(\lambda^{-1} + a)^{-1} \geq (\lambda^{-1} + b)^{-1}.$$

Therefore we get  $\lambda - (\lambda^{-1} + a)^{-1} \leq \lambda - (\lambda^{-1} + b)^{-1}$ . Since  $\lambda = (1 + \lambda c)(\lambda^{-1} + c)^{-1}$  for  $c = a, b$ , this leads to

$$\lambda a(\lambda^{-1} + a)^{-1} \leq \lambda b(\lambda^{-1} + b)^{-1}$$

and then

$$(4.1) \quad a(\lambda^{-1} + a)^{-1} \leq b(\lambda^{-1} + b)^{-1}.$$

Let us denote by  $h'$  the positive contraction  $h(1 + h)^{-1}$  for  $h \in \Lambda$ . Then functional calculus shows that  $h = h'(1 + h')^{-1}$ , so the process is self-inverse. Therefore, (4.1) states that we have  $g \leq h$  if and only if  $g' \leq h'$  for  $g, h \in \Lambda$ . Now let  $g, h \in \Lambda$ . As  $\|c(1 + c)^{-1}\| < 1$  for any  $c \in A_+$ , we have  $(g' + h')(1 + g' + h')^{-1} \in \Lambda$ . By (4.1),  $g', h' \leq g' + h'$  gives  $g, h \leq (g' + h')(1 + g' + h')^{-1}$ , which proves a).

For b), let  $a \in A$  and consider  $b := a^*a$ . For  $n \in \mathbb{N}^\times$ , define  $h_n := b(\frac{1}{n} + b)^{-1} \in \Lambda$ . Then  $b - bh_n \leq \frac{1}{n}$ . So if  $h \in \Lambda$  satisfies  $h_n \leq h$ , we get

$$\begin{aligned} \|b - hb\|^2 &= \|b(1 - h)^2b\| \\ &\stackrel{3.8 (2),(3)}{\leq} \|b(1 - h)b\| \\ &\stackrel{3.8 (2),(3)}{\leq} \|b(1 - h_n)b\| \\ &\leq \frac{1}{n} \|b\| \rightarrow 0. \end{aligned}$$

Thus we have  $bu_h, u_hb \rightarrow b$  for all  $b \in A_+$  (by taking adjoints). For  $a \in A$ , we now use

$$\|a - ah\|^2 = \|(1 - h)b(1 - h)\| \leq \|1 - h\| \|b(1 - h)\| \rightarrow 0.$$

□

**Definition 4.4.** A  $C^*$ -algebra  $A$  is called  $\sigma$ -unital if it admits a countable approximate unit.

**Proposition 4.5.** *Every separable  $C^*$ -algebra  $A$  is  $\sigma$ -unital.*

*Proof.* Let  $(a_n)_{n \in \mathbb{N}} \subset A$  be dense, and  $(\nu_\lambda)_{\lambda \in \Lambda}$  an approximate unit for  $A$ , which exists by Theorem 4.3. Choose  $(\lambda_n)_{n \in \mathbb{N}}$  such that

$$\|\nu_\lambda a_k - a_k\| \leq \frac{1}{n} \geq \|a_k \nu_\lambda - a_k\| \quad \text{for all } k \leq n \text{ and } \lambda \geq \lambda_n,$$



and set  $u_n := \nu_{\lambda_n}$ . Then  $(u_n)_{n \in \mathbb{N}}$  is an approximate unit for  $A$ : For  $a \in A$  and  $\varepsilon > 0$ , there is  $k \in \mathbb{N}$  such that  $\|a - a_k\| < \varepsilon$ . Then we can also choose  $n \geq k$  such that  $\varepsilon > \frac{1}{n}$ . This allows us to compute for  $m \geq n$ :

$$\|u_m a - a\| \leq \|u_m a - u_m a_k\| + \|u_m a_k - a_k\| + \|a_k - a\| < 3\varepsilon.$$

□

Unless stated otherwise, an ideal in a  $C^*$ -algebra means a two-sided ideal, and is denoted by  $I \triangleleft A$ .

**Corollary 4.6.** *Let  $A$  be a  $C^*$ -algebra. If  $I \triangleleft A$  is closed, then  $I^* = I$ . Moreover, if  $J \triangleleft I$  is closed, then  $J$  is a closed ideal in  $A$ .*

*Proof.* Fix an approximate unit  $(u_\lambda)$  for  $I$ , which exists by Theorem 4.3. For  $a \in I$ , we have  $a^* u_\lambda \in I$  as  $I$  is an ideal in  $A$ . Since  $*$  is continuous on  $A$ , we get  $a^* = \lim (u_\lambda a)^* = \lim a^* u_\lambda \in I$ .

For the second claim, it is clear that  $J$  is closed as the  $C^*$ -norm on  $I$  is the restriction of the one on  $A$ , see Proposition 2.4. If  $b \in J, a \in A$  then  $b u_\lambda a \in J$  as  $u_\lambda a \in I$ , and  $b u_\lambda a \rightarrow ba$  as  $b u_\lambda \rightarrow b$ . Thus  $ba \in J$  since  $J$  is a closed ideal in  $I$ . □

It follows from Corollary 4.6 that every closed ideal of a  $C^*$ -algebra is again a  $C^*$ -algebra. Next, we will show that  $C^*$ -algebras are also well-behaved under taking quotients with respect to closed ideals. To do so, we need to combine algebraic and analytic considerations:

If  $A$  is an algebra and  $I \triangleleft A$ , then  $A/I$  is an algebra with the natural multiplication. If an algebra  $A$  has an involution  $*$ , and  $I^* = I \triangleleft A$ , then  $A/I$  has an involution given by  $\overline{a^*} := \overline{a}^*$ .

If  $A$  is a Banach space and  $I$  is a closed subspace, then  $A/I$  is a Banach space for  $\|\overline{a}\| = \inf_{i \in I} \|a + i\|$ . Now if  $A$  is a Banach algebra and  $I \triangleleft A$  is a closed ideal, then  $A/I$  is a Banach algebra with the quotient norm described above. Indeed, the norm is submultiplicative because for  $a, b \in A$  and  $\varepsilon > 0$ , we can find  $i, j \in I$  with

$$\|\overline{a}\| \|\overline{b}\| \sim^\varepsilon \|a + i\| \|b + j\| \geq \|(a + i)(b + j)\| \geq \|\overline{ab}\|.$$

**Theorem 4.7.** *For every closed ideal  $I \triangleleft A$  in a  $C^*$ -algebra  $A$ , the quotient  $A/I$  is a  $C^*$ -algebra.*

*Proof.* By the preceding remarks we know that  $A/I$  is an involutive Banach algebra, so we need to show that the quotient norm satisfies the  $C^*$ -norm condition. For this purpose, we invoke an approximate unit  $(u_\lambda)$  for  $I$ , whose existence is guaranteed by Theorem 4.3. We claim that  $\|\overline{a}\| = \lim \|a - a u_\lambda\|$  for every  $a \in A$ . Indeed, for  $\varepsilon > 0$ , choose  $b \in I$  with  $\|\overline{a}\| + \varepsilon \geq \|a + b\|$ , and  $\lambda_0$  such that  $\|b - b u_\lambda\| < \varepsilon$  for all  $\lambda \geq \lambda_0$ . Then

$$\|\overline{a}\| \leq \|a - a u_\lambda\| \leq \|(a + b)(1 - u_\lambda)\| + \|b(1 - u_\lambda)\| \leq \|\overline{a}\| + 2\varepsilon$$

for  $\lambda \geq \lambda_0$ , so  $\|\overline{a}\| = \lim \|a - a u_\lambda\|$ . This in turn gives

$$\begin{aligned} \|\overline{a}\|^2 &= \lim \|a - a u_\lambda\|^2 = \lim \|(a(1 - u_\lambda))^* a(1 - u_\lambda)\| \\ &\leq \lim \|(1 - u_\lambda)\| \|a^* a(1 - u_\lambda)\| \\ &\leq \lim \|a^* a(1 - u_\lambda)\| = \|\overline{a^* a}\| \end{aligned}$$

□

*Remark 4.8.* If  $\varphi: A \rightarrow B$  is a  $*$ -homomorphism between  $C^*$ -algebras  $A$  and  $B$ , then  $\text{Sp } \varphi(a) \subset \text{Sp } a$  for every  $a \in A$ . Moreover, for every normal  $a \in A$  and  $f \in C_0(\text{Sp } a)$ , we have  $\varphi(f(a)) = f(\varphi(a))$ .

**Corollary 4.9.** *Let  $A$  and  $B$  be  $C^*$ -algebras, and  $\varphi: A \rightarrow B$  a  $*$ -homomorphism.*

- (1) *If  $\varphi$  is injective, then it is isometric.*
- (2) *The image  $\varphi(A)$  is a  $C^*$ -algebra, which is isomorphic to  $A/\ker \varphi$ .*

*Proof.* By Theorem 2.7, we know that  $\|\varphi\| \leq 1$ . Now suppose there was  $a \in A$  such that  $\|\varphi(a)\| < \|a\|$ . This is equivalent to  $\|\varphi(a^*a)\| < \|a^*a\|$ . Then there would exist a continuous function  $f$  on  $\text{Sp } a^*a \subset [0, \|a^*a\|]$  with the properties

$$f = 0 \text{ on } [0, \|\varphi(a^*a)\|] \quad \text{while} \quad f(\|a^*a\|) \neq 0.$$

It would follow that  $f(a^*a) \neq 0$ , but  $\varphi(f(a^*a)) = f(\varphi(a^*a)) = 0$ , see Remark 4.8, a contradiction to the hypothesis that  $\varphi$  is injective. Thus  $\varphi$  is isometric.

The natural map  $\bar{\varphi}$  given by  $\bar{a} \mapsto \varphi(a)$  is a well-defined  $*$ -homomorphism.

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ & \searrow & \nearrow \bar{\varphi} \\ & A/\ker \varphi & \end{array}$$

By Theorem 4.7  $A/\ker \varphi$  is a  $C^*$ -algebra as  $\ker \varphi \triangleleft A$  is closed. The map  $\bar{\varphi}$  is clearly injective, hence also isometric by (1). Therefore,  $\varphi(A)$  is complete, and thus a  $C^*$ -algebra as a closed involutive subalgebra of  $B$ . Surjectivity of  $\bar{\varphi}$  then shows that  $\bar{\varphi}$  defines an isomorphism  $A/\ker \varphi \cong \varphi(A)$ .  $\square$

## 5. POSITIVE LINEAR FUNCTIONALS

**Definition 5.1.** Let  $A$  be a  $C^*$ -algebra. A linear functional  $\varphi: A \rightarrow \mathbb{C}$  is called *positive* (denoted  $\varphi \geq 0$ ) if  $\varphi(a) \geq 0$  for all  $a \geq 0$ .

Note that  $A_+ = \{a \in A \mid a \geq 0\} = \{a^*a \mid a \in A\}$ , see Proposition 3.6.

Whenever there is no confusion, we will suppress linearity and speak of (positive) functionals to improve the readability.

*Examples 5.2.* Here are three canonical examples of positive functionals on  $C^*$ -algebras:

- (a) For  $A = C([0, 1])$ , the functionals  $\varphi_\lambda(f) := \int_{[0, 1]} f(x) dx$  and  $\varphi_{1/2}(f) := f(\frac{1}{2})$  for  $f \in A$  are positive. More generally, if  $X$  is a compact Hausdorff space, then Riesz's Theorem asserts that the positive functionals on  $C(X)$  are in 1 : 1-correspondence with Borel measures on  $X$  via  $\mu \mapsto \varphi_\mu$  with  $\varphi_\mu(f) := \int_X f d\mu$ .
- (b) For a Hilbert space  $H$  and  $A = \mathcal{L}(H)$ , pick  $\xi \in H$  and define  $\varphi_\xi(a) := \langle a\xi, \xi \rangle$ . Then  $\varphi_\xi$  is positive because  $\varphi_\xi(a^*a) = \|a\xi\|^2 \geq 0$ .
- (c) If  $A = M_n(\mathbb{C}) = \mathcal{L}(\mathbb{C}^n)$  for some  $n \in \mathbb{N}$ , then the trace  $\tau(a) := \sum_{1 \leq k \leq n} a_{kk}$  is a positive functional on  $A$ , where  $a = (a_{k\ell})_{1 \leq k, \ell \leq n}$ . This follows from (b) and  $\tau = \sum_{1 \leq k \leq n} \varphi_{\xi_k}$ , where  $\xi_1, \dots, \xi_n$  are the canonical orthonormal basis for  $\mathbb{C}^n$ .

**Theorem 5.3.** *If  $\varphi$  is a positive functional on a C\*-algebra  $A$ , then  $\varphi$  is bounded.*

*Proof.* According to Remark 3.5 (ii), every element in  $A$  can be displayed as a linear combination of at most four positive elements. Thus it suffices to show that  $\varphi$  is bounded on  $A_+^1 := \{a \in A_+ \mid \|a\| \leq 1\}$ . If  $\varphi$  was not bounded on  $A_+^1$ , there would exist a sequence  $(a_n)_{n \geq 1} \subset A_+^1$  with  $\varphi(a_n) \geq 2^n$  for all  $n$ . Define  $a := \sum_{n \geq 1} 2^{-n} a_n \in A$  using completeness of  $A$ . Then  $a \geq \sum_{1 \leq n \leq N} 2^{-n} a_n \geq 0$  for every  $N \geq 1$ . Hence  $\varphi(a)$  would need to satisfy  $\varphi(a) \geq \varphi(\sum_{1 \leq n \leq N} 2^{-n} a_n) \geq N$  for all  $N \geq 1$ , which is impossible.  $\square$

**Proposition 5.4.** *If  $\varphi$  is a positive functional on a C\*-algebra  $A$ , then*

- (1)  $\varphi(a^*) = \overline{\varphi(a)}$ , and
- (2)  $|\varphi(a)|^2 \leq \|\varphi\| \varphi(a^*a)$

*hold for all  $a \in A$ .*

*Proof.* Consider  $\langle \cdot, \cdot \rangle_\varphi: A \times A \rightarrow \mathbb{C}$  given by  $\langle a, b \rangle_\varphi := \varphi(b^*a)$ . This defines a positive semidefinite sesquilinear form on  $A$  as  $\langle a, a \rangle_\varphi = \varphi(a^*a) \geq 0$ . Hence, the polarization identity for such maps gives  $\overline{\langle a, b \rangle_\varphi} = \langle b, a \rangle_\varphi$ . Now take an approximate unit  $(u_\lambda)$  in  $A$  and let  $a \in A$ . Then

$$\overline{\varphi(a)} \leftarrow \overline{\varphi(u_\lambda a)} = \overline{\langle a, u_\lambda \rangle_\varphi} = \langle u_\lambda, a \rangle_\varphi = \varphi(a^* u_\lambda) \rightarrow \varphi(a^*)$$

because  $\varphi$  is bounded, see Theorem 5.3.

For (2), an application of the Cauchy-Schwartz inequality gives

$$|\varphi(a)|^2 \leftarrow |\langle a, u_\lambda \rangle|^2 \leq |\langle u_\lambda, u_\lambda \rangle| |\langle a, a \rangle| = \varphi(u_\lambda^2) \varphi(a^*a) \leq \|\varphi\| \varphi(a^*a).$$

$\square$

The relevance of approximate units in the context of positive functionals goes far beyond Proposition 5.4, as we shall now see.

**Theorem 5.5.** *For every bounded functional  $\varphi$  on a C\*-algebra  $A$ , the following are equivalent:*

- (1)  $\varphi$  is positive.
- (2)  $\|\varphi\| = \lim \varphi(u_\lambda)$  for every approximate unit  $(u_\lambda)$  in  $A$ .
- (3)  $\|\varphi\| = \lim \varphi(u_\lambda)$  for some approximate unit  $(u_\lambda)$  in  $A$ .

*Proof.* Suppose (1) holds. We can assume that  $\|\varphi\| = 1$ . Let  $(u_\lambda)$  be an approximate unit in  $A$ . Then  $(\varphi(u_\lambda))$  is a monotone increasing net in  $\mathbb{C}$  with an upper bound 1, due to Proposition 5.4. Thus  $\varphi(u_\lambda) \rightarrow \alpha$  for a suitable  $\alpha \in [0, 1]$ . We need to show that  $\alpha = 1$ . To this end, let  $a \in A$  with  $\|a\| \leq 1$ . Then

$$|\varphi(a)|^2 \leftarrow |\varphi(u_\lambda a)|^2 \leq \varphi(u_\lambda^2) \varphi(a^*a) \leq \varphi(u_\lambda) \varphi(a^*a) \leq \lim \varphi(u_\lambda) = \alpha$$

shows that  $1 = \|\varphi\| \leq \alpha$ .

The implication from (2) to (3) is trivial.

Now suppose (3) holds, and for convenience assume  $\|\varphi\| = 1$ . First, we show that  $\varphi(A_{sa}) \subset \mathbb{R}$ . So let  $a \in A_{sa}$  and  $\varphi(a) = \alpha + i\beta$ ,  $\alpha, \beta \in \mathbb{R}$ . We can assume that  $\beta \leq 0$  by  $a \mapsto -a$  if necessary. For every  $n \geq 1$ , the C\*-norm condition gives

$$\|a - inu_\lambda\|^2 = \|a\|^2 + n^2 u_\lambda^2 - in(au_\lambda - u_\lambda a) \leq 1 + n^2 + n\|au_\lambda - u_\lambda a\|.$$

While  $\|\varphi\| = 1$  gives

$$|\varphi(a - inu_\lambda)|^2 \leq \|(a - inu_\lambda)^*(a - inu_\lambda)\| \stackrel{\|a\|, \|u_\lambda\| \leq 1}{\leq} 1 + n^2 + n\|au_\lambda - u_\lambda a\| \rightarrow 1 + n^2,$$

(3) implies

$$|\varphi(a - inu_\lambda)|^2 \rightarrow |\varphi(a) - in|^2 = \alpha^2 + (\beta - n)^2.$$

This forces  $\beta = 0$ . Now let  $a \in A_+^1$ . Then  $-1 \leq -a \leq u_\lambda - a \leq u_\lambda \leq 1$  so that  $\|u_\lambda - a\| \leq 1$  and hence

$$1 \geq \varphi(u_\lambda - a) \rightarrow 1 - \varphi(a) \implies \varphi(a) \geq 0.$$

□

Note that the conditions in Theorem 5.5 are non-void by Theorem 4.3.

**Corollary 5.6.** *If  $A$  is a unital  $C^*$ -algebra, then a functional  $\varphi$  on  $A$  is positive if and only if  $\varphi(1) = \|\varphi\|$ .*

*Proof.* This follows immediately from (1) $\Leftrightarrow$ (3) in Theorem 5.5 applied to the unit of  $A$ . □

**Corollary 5.7.** *If  $\varphi$  and  $\psi$  are positive functionals on a  $C^*$ -algebra  $A$ , then  $\|\varphi + \psi\| = \|\varphi\| + \|\psi\|$ .*

*Proof.* This follows immediately from linearity and Theorem 5.5. □

**Definition 5.8.** A positive functional  $\varphi$  on a  $C^*$ -algebra  $A$  is called a *state* if  $\|\varphi\| = 1$ .

**Theorem 5.9.** *For every normal element  $a$  in a  $C^*$ -algebra  $A$ , there exists a state  $\varphi$  on  $A$  with  $|\varphi(a)| = \|a\|$ .*

*Proof.* Note that  $|\psi(a)| \leq \|a\|$  for every state  $\psi$  on  $A$ . As  $a$  is normal, there exists  $\lambda \in \text{Sp } a$  with  $|\lambda| = r(a) = \|a\|$ , see Proposition 2.4. This gives rise to a character  $\varphi_\lambda: C(\text{Sp } a \cup \{0\}) \rightarrow \mathbb{C}, f \mapsto f(\lambda)$ . As  $C(\text{Sp } a \cup \{0\}) \cong C^*(a, 1)$ , see Theorem 2.14, this map defines a state on the subalgebra  $C^*(a, 1)$  of  $\tilde{A}$ . Then we can use the Hahn-Banach extension theorem to get a linear extension  $\tilde{\varphi}: \tilde{A} \rightarrow \mathbb{C}$  with  $\|\tilde{\varphi}\| = \|\varphi_\lambda\| = 1$ . By Theorem 5.5,  $\tilde{\varphi} \geq 0$  and hence  $\tilde{\varphi}$  is a state on  $\tilde{A}$ . The desired  $\varphi$  is now given by the restriction of  $\tilde{\varphi}$  to  $A \subset \tilde{A}$ . Here is a visualization:

$$\begin{array}{ccc} A & \hookrightarrow & \tilde{A} \\ & & \uparrow \\ & & C^*(a, 1) \xrightarrow{\cong} C(\text{Sp } a \cup \{0\}) \xrightarrow[f \mapsto f(\lambda)]{\varphi_\lambda} \mathbb{C} \end{array}$$

$\tilde{\varphi}$  (curved arrow from  $\tilde{A}$  to  $\mathbb{C}$ )

□

**Proposition 5.10.** *Let  $\varphi$  be a positive functional on a  $C^*$ -algebra  $A$ .*

- (1) *For  $a \in A$ , we have  $\varphi(a^*a) = 0$  if and only if  $\varphi(ba) = 0$  for all  $b \in A$ .*
- (2) *For  $a, b \in A$ , we have  $\varphi(b^*a^*ab) \leq \|a^*a\|\varphi(b^*b)$ .*

*Proof.* For (1), the Cauchy-Schwartz inequality yields  $|\varphi(ba)|^2 \leq \varphi(b^*b)\varphi(a^*a)$  for  $a, b \in A$ . To get (2), we just note that  $b^*a^*ab \leq \|a^*a\|b^*b$ , see Theorem 3.8 (2). □

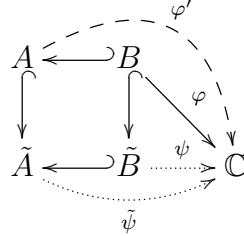
**Theorem 5.11.** *Let  $\varphi$  be a positive functional on a  $C^*$ -subalgebra  $B$  of a  $C^*$ -algebra  $A$ . Then  $\varphi$  extends to a positive functional  $\varphi'$  on  $A$  with  $\|\varphi'\| = \|\varphi\|$ .*

*Proof.* First, suppose  $A = \tilde{B}$ . Define  $\varphi'(\lambda + b) := \lambda\|\varphi\| + \varphi(b)$  for all  $b \in B$  and  $\lambda \in \mathbb{C}$ . Then  $\|\varphi'\| = \|\varphi\| < \infty$  because for  $b \in B, \lambda \in \mathbb{C}$ , and an approximate unit  $(u_\nu)$  in  $B$ , we have

$$\begin{aligned} |\varphi'(\lambda + b)| &= |\lambda\|\varphi\| + \varphi(b)| \\ &= \lim |\lambda\varphi(u_\nu) + \varphi(bu_\nu)| \\ &= \lim |\varphi((\lambda + b)u_\nu)| \\ &\leq \lim \|\varphi\| \|\lambda + b\| \|u_\nu\| \leq \|\varphi\| \|\lambda + b\|. \end{aligned}$$

Thus Corollary 5.6 shows that  $\varphi' \geq 0$  as  $\varphi'(1) = \|\varphi\| = \|\varphi'\|$ .

For the general case, the strategy is displayed by the following diagram:



We use the first part of the proof to extend  $\varphi$  to  $\psi \geq 0$  with  $\|\psi\| = \|\varphi\|$ . Applying the Hahn-Banach theorem yields an extension of  $\psi$  to a bounded linear functional  $\tilde{\psi}$  on  $\tilde{A}$  satisfying  $\tilde{\psi}(1) = \|\tilde{\psi}\| = \|\psi\| = \|\varphi\|$ . Due to Corollary 5.6, we can thus conclude that  $\tilde{\psi}$  is positive. Its restriction to  $A \subset \tilde{A}$  gives the desired positive functional  $\varphi'$ .  $\square$

## 6. THE GNS-REPRESENTATION

The aim of this section is to prove that every  $C^*$ -algebra is isomorphic to a subalgebra of the bounded linear operators on some Hilbert space. This celebrated result is attributed to Gelfand, Naimark, and Segal.

**Definition 6.1.** A *representation* of a  $C^*$ -algebra  $A$  is a pair  $(\pi, H)$  consisting of a Hilbert space  $H$  and a  $*$ -homomorphism  $\pi: A \rightarrow \mathcal{L}(H)$ . A representation  $(\pi, H)$  is called

- (i) *nondegenerate* if  $\overline{\pi(A)H} = H$ , and
- (ii) *cyclic* if there is  $\xi \in H$  such that  $\overline{\pi(A)\xi} = H$ . In this case, we write  $(\pi, H, \xi)$ .

*Remark 6.2.* We note the following observations:

- (i) If  $(\pi_i, H_i)_{i \in I}$  is a family of representations of a  $C^*$ -algebra  $A$ , then their direct sum  $(\oplus_{i \in I} \pi_i, \oplus_{i \in I} H_i)$  is a representation of  $A$ .
- (ii) If  $(\pi, H)$  is a representation of  $A$  and  $K \subset H$  is a closed invariant subspace, that is,  $\pi(A)K \subset K$ , then its orthogonal complement  $K^\perp$  is also a closed invariant subspace. Thus,  $\pi$  decomposes into a direct sum  $\pi = \pi_K \oplus \pi_{K^\perp}$  of two representations  $(\pi_K, K)$  and  $(\pi_{K^\perp}, K^\perp)$ .

A natural question is whether representations can be described as direct sums of representations that form irreducible components. Note that every cyclic representation is necessarily irreducible in this informal sense. For this purpose, we need a notion of equivalence that identifies representations that are morally the same.

**Definition 6.3.** Two representations  $(\pi, H)$  and  $(\pi', H')$  of a  $C^*$ -algebra  $A$  are *unitarily equivalent*, denoted  $\pi \sim \pi'$ , if there exists a unitary  $U: H \rightarrow H'$  such that  $\pi'(a) = U\pi(a)U^*$  for all  $a \in A$ .

**Proposition 6.4.** *Let  $A$  be a  $C^*$ -algebra. Then the following statements hold:*

- (1) *Every representation of  $A$  is unitarily equivalent to a direct sum of nondegenerate representations and a 0-representation.*
- (2) *Every nondegenerate representation of  $A$  is unitarily equivalent to a direct sum of cyclic representations.*

*Proof.* Let  $(\pi, H)$  be a representation. For part (1), note that  $K := \overline{\pi(A)H}$  is a closed invariant subspace of  $H$ . Hence  $\pi \sim \pi_K \oplus \pi_{K^\perp}$ , see Remark 6.2 (ii). Then  $\pi_K$  is nondegenerate because

$$\overline{\pi_K(A)K} = \overline{\pi_K(A)\pi_K(A)H} \stackrel{A^2=A}{=} \overline{\pi_K(A)H} = K.$$

Moreover,  $\pi_{K^\perp} = 0$  as  $\pi(a)h \in K$  for every  $h \in K^\perp$ .

For (2), let  $(\xi_i)_{i \in I} \subset H \setminus \{0\}$  be a maximal family with the property that the closed invariant subspaces  $K_i := \overline{\pi(A)\xi_i}$  are mutually orthogonal. Assume that  $\bigoplus_{i \in I} K_i$  is a proper subspace of  $H$ . Then there exists  $\xi \in (\bigoplus_{i \in I} K_i)^\perp \setminus \{0\}$ , and  $K := \overline{\pi(A)\xi}$  is orthogonal to every  $K_i$  as

$$\langle K_i, K \rangle = \langle \pi(A)K_i, \xi \rangle = \langle K_i, \xi \rangle = 0.$$

We arrive at a contradiction and thus deduce that  $H = \bigoplus_{i \in I} K_i$ , so that  $\pi \sim \bigoplus_{i \in I} \pi_i$  with  $\pi_i = \pi_{K_i}$ .  $\square$

Cyclic representations also have the advantage that unitary equivalence is witnessed by a uniquely determined unitary.

**Proposition 6.5.** *Let  $(\pi_1, H_1, \xi_1)$  and  $(\pi_2, H_2, \xi_2)$  be cyclic representations of a  $C^*$ -algebra  $A$ . If the positive functionals  $\varphi_i$  on  $A$  given by  $\varphi_i(a) := \langle \pi_i(a)\xi_i, \xi_i \rangle$  coincide, then there exists a unique unitary  $U: H_1 \rightarrow H_2$  with  $U\xi_1 = \xi_2$  and  $\pi_2(a) = U\pi_1(a)U^*$ . In particular, we have  $\pi_1 \sim \pi_2$ .*

*Proof.* The map  $V: \pi_1(A)H_1 \rightarrow \pi_2(A)H_2$  defined by  $\pi_1(a)\xi_1 \mapsto \pi_2(a)\xi_2$  is well-defined and isometric because

$$\langle \pi_2(a)\xi_2, \pi_2(a)\xi_2 \rangle = \langle \pi_2(a^*a)\xi_2, \xi_2 \rangle = \varphi_2(a^*a) = \varphi_1(a^*a) = \langle \pi_1(a)\xi_1, \pi_1(a)\xi_1 \rangle.$$

As  $\xi_1$  and  $\xi_2$  are both cyclic vectors, domain and range of  $V$  are dense in  $H_1$  and  $H_2$ , respectively, so  $V$  admits a unique extension  $U: H_1 \rightarrow H_2$ . Then  $U$  is necessarily a unitary, and

$$U\pi_1(a)U^*\pi_2(b)\xi_2 = U\pi_1(ab)\xi_1 = \pi_2(a)\pi_2(b)\xi_2$$

shows that  $U\pi_1(a)U^* = \pi_2(a)$  for all  $a \in A$  on  $H_2$ .  $\square$

Note that  $\xi_1$  and  $\xi_2$  in Proposition 6.5 necessarily need to have the same norm. By scaling, we can restrict our attention to normalized vectors, i.e.  $\|\xi_i\| = 1$ . In this case, the map  $\varphi_i$  is a state on  $A$ .

**Theorem 6.6** (GNS-representation for a state). *If  $\varphi$  is a state on a  $C^*$ -algebra  $A$ , then there exists a cyclic representation  $(\pi_\varphi, H_\varphi, \xi_\varphi)$  with  $\langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi \rangle = \varphi(a)$ . The representation  $(\pi_\varphi, H_\varphi, \xi_\varphi)$  is unique up to unitary equivalence.*

*Proof.* Uniqueness up to unitary equivalence will be a consequence of Proposition 6.5, once we prove the existence of such a cyclic representation. The basic idea is to use  $\varphi$  to produce a Hilbert space out of  $A$ . To do so, we start by noting that  $\langle a, b \rangle_\varphi := \varphi(b^*a)$  defines a positive semidefinite Hermitian form on  $A$ . Let  $N_\varphi := \{a \in A \mid \langle a, a \rangle_\varphi = 0\}$  denote the null space. For  $a, b \in N_\varphi$ , we have

$$|\varphi((a+b)^*(a+b))| = |\varphi(a^*b) + \varphi(b^*a)| \leq |\varphi(a^*b)| + |\varphi(b^*a)| \stackrel{5.10(1)}{=} 0.$$

Thus  $N_\varphi$  is a (necessarily closed) subspace of  $A$ . Therefore,  $K_\varphi := A/N_\varphi$  is a Hermitian vector space for the scalar product induced by  $\langle \cdot, \cdot \rangle$ , and its completion  $H_\varphi$  with respect to the norm induced by this scalar product is a Hilbert space. Let  $\gamma: A \rightarrow H_\varphi$  denote the map  $a \mapsto a + N_\varphi$ , and define  $\pi(a): \gamma(A) \rightarrow \gamma(A)$  by  $\gamma(b) \mapsto \gamma(ab)$  for all  $a \in A$ . As

$$\|\gamma(ab)\|_\varphi^2 = \varphi(b^*a^*ab) \stackrel{5.4}{\leq} \|a^*a\| \varphi(b^*b) = \|a\|^2 \|\gamma(b)\|_\varphi^2,$$

for all  $a, b \in A$ , each map  $\pi(a)$  is well-defined and  $\|\pi(a)\| \leq \|a\|$ . Since  $\gamma(A)$  is dense in  $H_\varphi$ , we can thus extend  $\pi(a)$  to a bounded linear operator  $\pi_\varphi(a)$  on  $H_\varphi$ . This defines a  $*$ -homomorphism  $\pi_\varphi$  because  $\pi$  is multiplicative on elements in  $\gamma(A)$ , and, similarly,

$$\langle \pi(a)\gamma(b), \gamma(c) \rangle_\varphi = \varphi(c^*ab) = \varphi((a^*c)^*b) = \langle \gamma(b), \pi(a^*)\gamma(c) \rangle_\varphi$$

implies  $\pi_\varphi(a)^* = \pi_\varphi(a^*)$ .

The map  $f: K_\varphi \rightarrow \mathbb{C}$  determined by  $a + N_\varphi \mapsto \varphi(a)$  is continuous because  $|f(a)| \leq \varphi(a^*a) = \|a\|_\varphi^2$  for all  $a \in A$ . Thus Riesz's theorem asserts that the continuous extension  $f'$  of  $f$  to  $H_\varphi$  admits a unique  $\xi_\varphi \in H_\varphi$  such that

$$\varphi(a) = f(a + N_\varphi) = f'(\gamma(a)) = \langle \gamma(a), \xi_\varphi \rangle_\varphi \quad \text{for all } a \in A.$$

Then

$$\langle \gamma(b), \pi(a)\xi_\varphi \rangle_\varphi = \langle \gamma(a^*b), \xi_\varphi \rangle_\varphi = \varphi(a^*b) = \langle \gamma(b), \gamma(a) \rangle_\varphi$$

for  $a, b \in A$  implies  $\pi_\varphi(a)\xi_\varphi = \gamma(a)$ . Therefore,  $\xi_\varphi$  is a cyclic vector for  $\pi_\varphi$ , and  $\langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi \rangle_\varphi = \langle \gamma(a), \xi_\varphi \rangle_\varphi = \varphi(a)$ . Note that since  $\varphi$  is a state,  $\xi_\varphi$  has norm 1.  $\square$

*Remark 6.7.* The unique cyclic  $\xi_\varphi \in H_\varphi$  can be obtained explicitly by observing that it is the norm limit of  $(\gamma(u_\lambda))$  for any approximate unit  $(u_\lambda)$  in  $A$ .

*Remark 6.8.* Theorem 6.6 yields a one-to-one correspondence

$$\{(\pi, H, \xi) \text{ cyclic representation of } A\} / \sim \longleftrightarrow \{\varphi \text{ state on } A\}.$$

For a state  $\varphi$  on  $A$ , the triple  $(\pi_\varphi, H_\varphi, \xi_\varphi)$  is also called the GNS-representation for  $\varphi$ .

Combining the GNS-representation for states with Theorem 5.9, which guarantees a sufficient supply of states on  $C^*$ -algebras, we arrive at:

**Theorem 6.9** (The universal GNS-representation). *Every  $C^*$ -algebra  $A$  admits a faithful representation  $(\pi, H)$ . In particular, every  $C^*$ -algebra is (isometrically) isomorphic to a subalgebra of bounded linear operators on a Hilbert space.*

*Proof.* Define  $(\pi, H)$  as the direct sum representation over  $(\pi_\varphi, H_\varphi)_{\varphi \text{ state on } A}$ , see Remark 6.2 (i) and Theorem 6.6. For  $a \in A \setminus \{0\}$ , there exists a state  $\varphi$  on  $A$  with  $\varphi(a^*a) = \|a^*a\|$ , see Theorem 5.9. Thus we get

$$\|\pi(a)\|^2 \geq \|\pi_\varphi(a)\|^2 \geq \|\langle \pi_\varphi(a^*a)\xi_\varphi, \xi_\varphi \rangle_\varphi = \varphi(a^*a) = \|a^*a\| = \|a\|^2,$$

so that  $\pi$  is faithful. As every injective  $*$ -homomorphism is isometric, see Corollary 4.9,  $\pi$  is isometric.  $\square$

The pair  $(\pi, H)$  is commonly referred to as the *universal GNS-representation* of the  $C^*$ -algebra  $A$ .

*Examples 6.10.* Let us look at two explicit examples:

- (a) Consider  $A = C([0, 1])$  with the state given by integration with respect to the Lebesgue measure  $\lambda$ , i.e.  $\varphi(f) := \int_{[0, 1]} f(t) \lambda(dt)$ . Then we have  $\langle f, g \rangle_\varphi = \int_{[0, 1]} \overline{g(t)} f(t) \lambda(dt)$ , which coincides with the usual scalar product on  $L^2([0, 1], \lambda)$ . As  $C([0, 1])$  is dense in  $L^2([0, 1], \lambda)$  with respect to the  $L^2$ -norm, we get  $H_\varphi \cong L^2([0, 1])$ . Moreover,  $\pi_\varphi$  is given by multiplication, and  $\xi_\varphi = 1$ .
- (b) For  $n \geq 1$ , let  $A = M_n(\mathbb{C})$  and  $\varphi((a_{ij})_{1 \leq i, j \leq n}) = a_{11}$ . Then  $\langle a, b \rangle_\varphi = \varphi(b^*a) = \sum_{1 \leq i \leq n} \overline{b_{i1}} a_{i1}$  is the euclidean scalar product on  $\mathbb{C}^n$  (on the first column of the matrices). Thus  $H_\varphi \cong \mathbb{C}^n$ ,  $\gamma(a) = (a_{11}, a_{21}, \dots, a_{n1})^t$ , and  $\pi_\varphi$  is given by left multiplication of a vector by a matrix.

*Remark 6.11.* If  $A$  is a  $C^*$ -algebra, then there exists a unique  $C^*$ -norm on  $M_n(A) := \{(a_{ij})_{1 \leq i, j \leq n} \mid a_{ij} \in A\}$  for all  $n \geq 1$ . Indeed, the universal GNS-representation from Theorem 6.9 allows us to view  $A$  as a subalgebra of  $\mathcal{L}(H)$ . Therefore  $M_n(A)$  is a subalgebra of  $\mathcal{L}(\oplus_{1 \leq i \leq n} H)$ , and the latter has a  $C^*$ -norm. Uniqueness is an outcome of Proposition 2.4.

## 7. IRREDUCIBLE REPRESENTATIONS

**Definition 7.1.** A representation  $(\pi, H)$  of a  $C^*$ -algebra  $A$  with  $\pi \neq 0$  is *irreducible* if the only  $\pi(A)$ -invariant closed subspaces of  $H$  are 0 and  $H$ .

We will simply talk about invariant closed subspaces whenever there is no ambiguity concerning the representation. Moreover, we mostly suppress  $H$  by saying that  $\pi$  is irreducible.

**Proposition 7.2.** For a representation  $(\pi, H)$  of a  $C^*$ -algebra  $A$  with  $\pi \neq 0$ , the following statements are equivalent:

- (1)  $(\pi, H)$  is irreducible.
- (2) The relative commutant  $\pi(A)' := \{T \in \mathcal{L}(H) \mid T\pi(a) = \pi(a)T \text{ for all } a \in A\}$  equals  $\mathbb{C}1$ .
- (3) Every nonzero vector in  $H$  is cyclic for  $\pi$ .

*Proof.* We prove (1)  $\Leftrightarrow$  (2) and (1)  $\Leftrightarrow$  (3). To show that (1)  $\Rightarrow$  (2), we observe that  $\pi(A)'$  is a  $C^*$ -subalgebra of  $\mathcal{L}(H)$ . Let  $T \in \pi(A)'_{sa}$ . If  $T \notin \mathbb{R}1$ , then  $\text{Sp } T$  contains at least two distinct real numbers, say  $\lambda_1 < \lambda_2$ . This allows us to choose (nonnegative) bump functions  $f_1, f_2 \in C(\text{Sp } T)$  with disjoint support (so that  $f_1 f_2 = 0$ ) and  $f_i(\lambda_i) > 0$ . Then  $H_i := \overline{f_i(T)H}$  defines two nontrivial, closed subspaces of  $H$  that are  $\pi(A)$ -invariant as

$$\overline{\pi(A)H_i} \stackrel{T \in \pi(A)'}{=} \overline{f_i(T)\pi(A)H} \subset H_i.$$

Moreover,  $f_1 f_2$  implies that  $H_1$  and  $H_2$  are orthogonal as

$$\langle H_1, H_2 \rangle = \langle \overline{f_1(T)H}, \overline{f_2(T)H} \rangle = \langle H, \overline{f_1(T)f_2(T)H} \rangle \stackrel{f_1 f_2 = 0}{=} 0.$$



This contradicts irreducibility so that we must have  $T \in \mathbb{R}1$ , which amounts to  $\pi(A)' = \mathbb{C}1$ . For the converse direction, Let  $K$  be a closed invariant subspace of  $H$ , and denote by  $P_K \in \mathcal{L}(H)$  the orthogonal projection onto  $K$ . As  $\pi(a)P_K\xi \in K$  for all  $\xi \in H$  by  $\pi(A)$ -invariance of  $K$ , we have  $P_K\pi(a)P_K = \pi(a)P_K$  for all  $a \in A$ . So if  $a \in A_{sa}$ , then

$$P_K\pi(a) = (\pi(a)P_K)^* = (P_K\pi(a)P_K)^* = P_K\pi(a)P_K = \pi(a)P_K.$$

Thus we have  $P_K \in \pi(A)' = \mathbb{C}1$ , which forces  $K \in \{0, H\}$ .

For (1)  $\Rightarrow$  (3), let  $\xi \in H \setminus \{0\}$  and consider  $K := \overline{\pi(A)\xi}$ . This is a closed, invariant subspace of  $H$ . If we had  $K = 0$ , then  $\mathbb{C}\xi$  would be a nonzero, closed, invariant subspace and (1) would imply  $H = \mathbb{C}\xi$ . However, then we would have  $\pi = 0$ , which is ruled out by the standing assumptions. Thus  $K \neq 0$  and (1) implies  $K = H$  so that  $\xi$  is cyclic for  $\pi$ .

Conversely, every nonzero, closed, invariant subspace  $K$  of  $H$  contains some nonzero vector  $\xi$ , and (3) implies that  $H = \pi(A)\xi \subset \overline{\pi(A)K} \subset K$ .  $\square$

*Examples 7.3.* Let us look at some examples:

- (a) If  $A$  is a  $C^*$ -algebra and  $\pi: A \rightarrow \mathbb{C}$  is a character, then  $(\pi, \mathbb{C})$  is an irreducible representation.
- (b) If  $A$  is a commutative  $C^*$ -algebra and  $(\pi, H)$  is irreducible, then Proposition 7.2 tells us that  $\pi(A) \subset \pi(A)' = \mathbb{C}1$ , so that  $\pi(A) = \mathbb{C}1$ . Hence every closed subspace of  $H$  is invariant, and thus  $H \cong \mathbb{C}$  so that  $\pi: A \rightarrow \mathbb{C}$  is in fact a character. So we see that for commutative  $C^*$ -algebras, irreducible representations are in one-to-one correspondence with characters.
- (c) If  $(\pi, H)$  is a finite-dimensional, nondegenerate representation of a  $C^*$ -algebra  $A$ , then  $(\pi, H)$  is unitary equivalent to a finite direct sum of irreducible representations (using an induction argument).

We have seen in Theorem 6.6 that every state on a  $C^*$ -algebra  $A$  gives rise to a cyclic representation. Our next goal is to characterize the irreducible representations among these. This requires some preparation.

**Lemma 7.4.** *Let  $H$  be a Hilbert space and  $(\cdot, \cdot)$  a continuous sesquilinear form on  $H$ , i.e.  $|(\xi, \eta)| \leq C\|\xi\|\|\eta\|$  for all  $\xi, \eta \in H$  for some  $C < \infty$ . Then there exists  $T \in \mathcal{L}(H)$  such that  $(\xi, \eta) = \langle B\xi, \eta \rangle$  for all  $\xi, \eta \in H$ .*

*Proof.* For every fixed  $\eta \in H$ , the map  $\xi \mapsto (\xi, \eta)$  is a continuous functional on  $H$ . Thus the Riesz representation theorem asserts that there is  $\tilde{\eta} \in H$  such that the functional is given by  $\xi \mapsto \langle \xi, \tilde{\eta} \rangle$ . The map  $\eta \mapsto \tilde{\eta}$  is apparently linear, bounded. Then its adjoint  $T \in \mathcal{L}(H)$  satisfies

$$\langle T\xi, \eta \rangle = \langle \xi, \tilde{\eta} \rangle = (\xi, \eta).$$

$\square$

Note that an operator  $T \in \mathcal{L}(H)$  on some Hilbert space  $H$  is positive if and only if  $\langle T\xi, \xi \rangle \geq 0$ . The next lemma is crucial.

**Lemma 7.5.** *Let  $(\pi, H)$  be a representation of a  $C^*$ -algebra  $A$ ,  $\xi \in H$  and  $\varphi(a) := \langle \pi(a)\xi, \xi \rangle$  for  $a \in A$ . Then the following statements hold:*

- 1) *For every  $T \in \pi(A)', 0 \leq T \leq 1$ , the positive functional  $\varphi_T$  on  $A$  given by  $\varphi_T(a) := \langle \pi(a)T\xi, T\xi \rangle$  for  $a \in A$  satisfies  $\varphi_T \leq \varphi$ , i.e.  $\varphi - \varphi_T \geq 0$  on  $A_+$ .*

- 2) If  $\xi$  is cyclic for  $\pi$ , then the mapping  $T \mapsto \varphi_T$  from 1) is injective.  
 3) If  $\psi$  is a positive functional on  $A$  such that  $\psi \leq \varphi$ , then there exists  $T \in \pi(A)', 0 \leq T \leq 1$  with  $\varphi_T = \psi$ .

*Proof.* For every  $a \in A_+$ , we have

$$\begin{aligned}
 \varphi_T(a) &= \langle \pi(a)T\xi, T\xi \rangle \\
 &= \langle T\pi(\sqrt{a})\pi(\sqrt{a})T\xi, \xi \rangle \quad (T^* = T, a \geq 0) \\
 &= \langle \pi(\sqrt{a})T^2\pi(\sqrt{a})\xi, \xi \rangle \quad (T \in \pi(A)') \\
 &= \langle \pi(a)\xi, \xi \rangle \quad (3.8 (2), (3) \text{ and } \|T\| \leq 1) \\
 &= \varphi(a),
 \end{aligned}$$

which proves 1). For 2), let  $S$  and  $T$  as in 1) with  $\varphi_S = \varphi_T$ . Then we get

$$\langle S^2\pi(a)\xi, \pi(b)\xi \rangle = \langle \pi(b^*a)S\xi, S\xi \rangle = \varphi_S(b^*a) = \varphi_T(b^*a) = \langle T^2\pi(a)\xi, \pi(b)\xi \rangle,$$

as  $S, T \in \pi(A)'_{sa}$ . If  $\xi$  is cyclic for  $\pi$ , this implies  $S^2 = T^2$ , and hence  $S = T$  by uniqueness of the square root for positive operators, see Remark 3.2.

Part 3) will be more involved than 1) and 2), and uses Lemma 7.4. Given a positive functional  $\psi$  on  $A$  with  $\psi \leq \varphi$ , we define a sesquilinear form  $(\cdot, \cdot)_\psi$  on  $K := \overline{\pi(A)\xi}$  by  $(\pi(a)\xi, \pi(b)\xi)_\psi := \psi(b^*a)$  for  $a, b \in A$ . Note that  $(\cdot, \cdot)_\psi$  is well-defined and  $\|(\cdot, \cdot)_\psi\| \leq 1$  because

$$|(\pi(a)\xi, \pi(b)\xi)_\psi| = |\psi(b^*a)| \leq \psi(b^*b)^{\frac{1}{2}}\psi(a^*a)^{\frac{1}{2}} \stackrel{\psi \leq \varphi}{\leq} \varphi(b^*b)^{\frac{1}{2}}\varphi(a^*a)^{\frac{1}{2}} = \|\pi(b)\xi\| \cdot \|\pi(a)\xi\|.$$

Thus Lemma 7.4 yields  $T_0 \in \mathcal{L}(K)$  with

$$\psi(b^*a) = (\pi(a)\xi, \pi(b)\xi)_\psi = \langle T_0\pi(a)\xi, \pi(b)\xi \rangle \quad \text{for all } a, b \in A.$$

Note that  $T_0$  is a positive contraction as

$$\langle T_0\pi(a)\xi, \pi(a)\xi \rangle = \psi(a^*a) \geq 0 \quad \text{for all } a \in A \quad \text{and } \|T_0\| \leq \|(\cdot, \cdot)_\psi\| \leq 1.$$

We can therefore consider an operator  $T \in \mathcal{L}(H)$  that agrees with  $\sqrt{T_0}$  on  $K$ , and vanishes on  $K^\perp$ . We claim that  $\psi = \varphi_T$ . Clearly,  $0 \leq T \leq 1$ , so we need to show that

- a)  $T \in \pi(A)'$ , and
- b)  $\psi = \varphi_T$ .

Note that  $K = \overline{\pi(A)\xi}$  and its orthogonal complement  $K^\perp$  are  $\pi$ -invariant subspaces of  $H$ . Thus  $T\pi(a) = 0 = \pi(a)T$  on  $K^\perp$ . So, it suffices to show that  $\pi(a)T_0\pi(b)\xi = T_0\pi(ab)\xi$  for all  $a, b \in A$ . This follows from

$$\langle T_0\pi(ab)\xi, \pi(c)\xi \rangle = \psi(c^*ab) = \psi((a^*c)^*b) = \langle T_0\pi(b)\xi, \pi(a^*c)\xi \rangle = \langle \pi(a)T_0\pi(b)\xi, \pi(c)\xi \rangle,$$

so  $T \in \pi(A)'$ . But then

$$\psi(b^*a) = \langle T_0\pi(a)\xi, \pi(b)\xi \rangle = \langle T^2\pi(a)\xi, \pi(b)\xi \rangle \stackrel{a)}{=} \langle \pi(a)T\xi, \pi(b)T\xi \rangle = \varphi_T(b^*a).$$

If we use an approximate unit  $(u_\lambda)$  in  $A$  in place of  $b$ , which exists by Theorem 4.3, we conclude that  $\psi = \varphi_T$ , because both functionals are positive, hence continuous, see Theorem 5.3.  $\square$

**Definition 7.6.** A state  $\varphi$  on a  $C^*$ -algebra  $A$  is called *pure* if  $0 \leq \psi \leq \varphi$  implies that  $\psi = \lambda\varphi$  for some  $\lambda \in [0, 1]$ .

*Example 7.7.* If  $A$  is a commutative  $C^*$ -algebra, i.e.  $A = C_0(X)$  for some locally compact Hausdorff space  $X$ , then the pure states on  $A$  are the Dirac measures on  $X$ .

**Theorem 7.8.** *A state  $\varphi$  on a  $C^*$ -algebra  $A$  is pure if and only if its GNS-representation  $(\pi_\varphi, H_\varphi, \xi_\varphi)$  is irreducible.*

*Proof.* Suppose  $(\pi_\varphi, H_\varphi, \xi_\varphi)$  is irreducible, and let  $0 \leq \psi \leq \varphi$ . Due to Lemma 7.5 3), there exists  $T \in \pi_\varphi(A)'$ ,  $0 \leq T \leq 1$  such that  $\psi = \varphi_T$ . As  $\pi_\varphi$  is irreducible, Proposition 7.2 implies  $\pi_\varphi(A)' = \mathbb{C}1$ , so that  $T = \lambda 1$  with  $\lambda \in [0, 1]$ . In this case, we get  $\psi = \varphi_T = \lambda^2 \varphi$ , so  $\varphi$  is pure.

Conversely, assume that  $\varphi$  is pure, and let  $K$  be a closed  $\pi_\varphi$ -invariant subspace of  $H_\varphi$ . Then the orthogonal projection  $P$  onto  $K$  is a positive contraction in  $\pi(A)'$ . Therefore, Lemma 7.5 1) implies that the positive functional  $\varphi_P$  satisfies  $0 \leq \varphi_P \leq \varphi$ . Since  $\varphi$  is pure, there is  $\lambda \in [0, 1]$  with  $\varphi_P = \lambda \varphi$ . We use this to obtain

$$\begin{aligned} \langle P\pi_\varphi(a)\xi_\varphi, \pi_\varphi(b)\xi_\varphi \rangle &= \langle \pi_\varphi(b^*a)P\xi_\varphi, P\xi_\varphi \rangle && (P \text{ is a projection in } \pi(A)'.) \\ &= \varphi_P(b^*a) \\ &= \lambda \varphi(b^*a) \\ &= \langle \pi_\varphi(a)\xi_\varphi, \lambda \pi_\varphi(b)\xi_\varphi \rangle. \end{aligned}$$

for arbitrary  $a, b \in A$ . Since  $\xi_\varphi$  is cyclic for  $\pi_\varphi$ , this shows that  $P\eta = \lambda\eta$  for all  $\eta \in H_\varphi$ . Since  $P$  is the orthogonal projection onto  $K$ , we conclude that  $P \in \{0, 1\}$  corresponding to  $K = 0$  and  $K = H_\varphi$ , respectively. Thus  $\pi_\varphi$  is irreducible.  $\square$

We now show that every positive functional on a  $C^*$ -algebra with norm at most 1 is a convex combination of pure states. This essentially reduces the study of positive functionals to the study of pure states. In combination with Theorem 7.8, this provides us with an ample supply of irreducible representations for arbitrary  $C^*$ -algebras.

**Definition 7.9.** Let  $V$  be a vector space.

- (i) A subset  $C$  of  $V$  is *convex* if for all  $x, y \in C$  and  $t \in [0, 1]$ , the element  $tx + (1-t)y$  belongs to  $C$ .
- (ii) A subset  $M$  of a convex set  $C$  is *extremal* if it has the following property: If  $x, y \in C$  and  $t \in (0, 1)$  satisfy  $tx + (1-t)y \in M$ , then  $x, y \in M$ .
- (iii) A point  $x$  in a convex set  $C$  is called an *extreme point* if  $\{x\}$  is extremal.

**Proposition 7.10.** *Let  $A$  be a  $C^*$ -algebra and  $B := \{\varphi: A \rightarrow \mathbb{C} \mid \varphi \geq 0, \|\varphi\| \leq 1\}$ . Then the following statements hold:*

- (1)  *$B$  is convex and compact with respect to the weak topology.*
- (2) *The extreme points of  $B$  are the pure states and 0.*

*Proof.* For (1), note that the set of continuous functionals  $\varphi: A \rightarrow \mathbb{C}$  with  $\|\varphi\| \leq 1$  is compact with respect to the weak topology.  $B$  forms a closed subset of this former set as  $\varphi \geq 0$  is a closed condition. Convexity is clear.

For (2), we first note that 0 is an extreme point as  $0 = \varphi + \psi$  for positive functionals  $\varphi, \psi$  implies  $\varphi = 0 = \psi$ . Next, let  $\varphi$  be a pure state, and suppose we have  $\varphi = t\psi_1 + (1-t)\psi_2$  for some  $\psi_i \geq 0$  with  $\|\psi_i\| \leq 1$  and  $t \in (0, 1)$ . Since  $\varphi$  is pure, there are  $\lambda_1, \lambda_2 \in [0, 1]$  such that  $\psi_i = \lambda_i \varphi$  for  $i = 1, 2$ . Thus

$$\varphi = t\psi_1 + (1-t)\psi_2 = (t\lambda_1 + (1-t)\lambda_2)\varphi$$

implies  $t\lambda_1 + (1-t)\lambda_2 = 1$ , and therefore  $\lambda_i = 1$ , i.e.  $\psi_i = \varphi$ . Conversely, let  $\varphi \in B$  be an extreme point with  $\varphi \neq 0$ . Then we necessarily have  $\|\varphi\| = 1$ . Suppose there is  $0 \leq \psi \leq \varphi$ . Then  $\varphi = \psi + \psi'$  for  $\psi' := \varphi - \psi \in B$ . Consider

$$\varphi = \|\psi\| \frac{\psi}{\|\psi\|} + \|\psi'\| \frac{\psi'}{\|\psi'\|}.$$

As  $\varphi$  is an extreme point and  $\|\psi\| + \|\psi'\| = \|\psi + \psi'\| = \|\varphi\| = 1$  by Corollary 5.7, we get  $\psi = \|\psi\|\varphi$ , so  $\varphi$  is pure.  $\square$

Let us recall the celebrated theorem of Krein and Milman:

**Theorem 7.11** (Krein–Milman). *Let  $V$  be a locally convex Hausdorff space. Then every nonempty, compact, convex set  $C \subset V$  has an extreme point.*

**Definition 7.12.** Let  $M$  be a subset of a vector space  $V$ . The *convex hull* of  $M$ , denoted by  $\text{conv}(M)$ , is the smallest convex subset of  $V$  containing  $M$ .

*Remark 7.13.* The convex hull can be described explicitly by  $\text{conv}(M) = \{\sum_{1 \leq k \leq n} c_k x_k \mid n \in \mathbb{N}, x_k \in M, \text{ and } \sum_{1 \leq k \leq n} c_k = 1\}$ .

**Corollary 7.14.** *Let  $V$  be a locally convex  $\mathbb{R}$ -vector space and  $C \subset V$  convex and compact. Then  $C = \overline{\text{conv}(\text{Ext } C)}$ , where  $\text{Ext } C$  is the set of extreme points in  $C$ .*

*Proof.* Let us set  $A := \overline{\text{conv}(\text{Ext } C)}$  for convenience. As  $C$  is convex and closed,  $A$  is a subset of  $C$  because  $\text{Ext } C \subset C$ . So suppose that there exists  $c \in C \setminus A$ . We can assume that  $0 \in C$  so that  $0 \in \text{Ext } C \subset A$ . Then there is a convex neighbourhood of 0  $U'$  such that  $c + U' \cap A = \emptyset$ . Then  $U := \frac{1}{2}U'$  is a convex neighbourhood of 0 with  $c + U \cap A + U = \emptyset$ . Consider the Minkowski functional  $p_B(v) := \inf\{t > 0 \mid v \in t(B)\}$  for  $v \in V$  for a nonempty convex  $B \subset C$  containing 0. For the convex sets  $A + U$  and  $U$ , we have  $p_{A+U} \leq p_U$ . As  $p_B$  is always sublinear and  $p_U$  is continuous because  $U$  is a neighbourhood of 0, the map  $p_{A+U}$  is continuous and thus bounded on  $C$  since the latter is compact. By the Hahn-Banach theorem for the locally convex case, there exists a continuous map  $f: V \rightarrow \mathbb{R}$  with  $f(c) = p_{A+U}(c) > 1$  and  $f(b) \leq p_{A+U}(b)$  for all  $b \in V$ . Since  $p_{A+U}$  is bounded on  $C$ , so is  $f$ . Therefore, the set  $K := \{c' \in C \mid f(c') = \sup_{b \in C} f(b)\}$  is non-empty and compact. It is also extremal in  $C$ , so Theorem 7.11 implies that  $K$  contains an extreme point  $c' \in \text{Ext } C \subset A$ . However,  $f(c') = \sup_{b \in C} f(b) \geq f(c) > 1$  contradicts  $f|_A \leq 1$ . Thus we conclude that  $A = C$ .  $\square$

Corollary 7.14 has a great application to the study of representations of  $C^*$ -algebras:

**Corollary 7.15.** *Let  $A$  be a  $C^*$ -algebra and  $a \in A \setminus \{0\}$ . Then there exists an irreducible representation  $(\pi, H)$  such that  $\|\pi(a)\| = \|a\|$ .*

*Proof.* With the notation of Proposition 7.10, consider  $K := \{\varphi \in B \mid \varphi(a^*a) = \|a\|^2\}$ . Then  $K$  is non-empty and extremal, and compact as a closed subset of  $B$ , see Proposition 7.10 (1). By Theorem 7.11,  $K$  contains an element  $\varphi \in \text{Ext } B$ . According to Proposition 7.10 (2),  $\varphi$  is a pure state. Thus its associated GNS-representation  $(\pi_\varphi, H_\varphi, \xi_\varphi)$  is irreducible by Theorem 7.8. In addition,

$$\|\pi_\varphi(a)\xi_\varphi\|^2 = \varphi(a^*a) = \|a\|^2$$

implies  $\|\pi(a)\| = \|a\|$  as  $\|\xi_\varphi\| = 1$ .  $\square$

## 8. VON NEUMANN ALGEBRAS

In the previous Section, we saw that every  $C^*$ -algebra  $A$  can be thought of as a concrete norm-closed,  $*$ -invariant subalgebra of  $\mathcal{L}(H)$  for some Hilbert space, see Theorem 6.9. But there are more interesting topologies on  $\mathcal{L}(H)$  that are weaker than the norm topology, namely the *strong operator topology* (SOT) and the *weak operator topology* (WOT).

**Norm:** It suffices to consider sequences instead of nets. A sequence  $(a_n)$  converges to  $a$  if  $\|a_n - a\| \rightarrow 0$ . Multiplication, addition, and similar operations are continuous, the involution is isometric.

**SOT:** A net  $(a_\lambda)$  converges to  $a$  if  $a_\lambda \xi \rightarrow a\xi$  for all  $\xi \in H$ . This can be described by the collection of seminorms  $p_\xi, \xi \in H$ , where  $p_\xi(a) := \|a\xi\|$ . The multiplication  $\mathcal{L}(H) \times \mathcal{L}(H) \rightarrow \mathcal{L}(H), (a, b) \mapsto ab$  is not continuous. A counterexample can be given using the unilateral shift on  $\ell^2(\mathbb{N})$ . However, for fixed  $b \in \mathcal{L}(H)$ , left and right multiplication by  $b$ , i.e.  $a \mapsto ba$  and  $a \mapsto ab$ , are continuous. Moreover, multiplication is continuous when restricted to bounded domains in  $\mathcal{L}(H)$ . That is to say, if  $a_\lambda \rightarrow a$  and  $b_\mu \rightarrow b$  strongly with  $\|a_\lambda\|, \|b_\mu\| \leq C$  for some  $C < \infty$ , then  $a_\lambda b_\mu \rightarrow ab$  as

$$\|a_\lambda b_\mu \xi - ab\xi\| \leq C\|b_\mu \xi - b\xi\| + \|a_\lambda b\xi - ab\xi\| \rightarrow 0.$$

Also, the involution is not strongly continuous: Let  $H = \ell^2(\mathbb{N})$  with ONB  $(\xi_n)_{n \in \mathbb{N}}$  and define operators  $a_n \in \mathcal{L}(H)$  by  $a_n \eta := \langle \eta, \xi_n \rangle \xi_1$ . Then  $(a_n)$  tends to 0 strongly, but  $a_n^* \xi_1 = \xi_n$  for all  $n \in \mathbb{N}$ .

**WOT:** A net  $(a_\lambda)$  converges to  $a$  if  $\langle a_\lambda \xi, \eta \rangle \rightarrow \langle a\xi, \eta \rangle$  for all  $\xi, \eta \in H$ . As for SOT, left and right multiplication by a fixed element in  $\mathcal{L}(H)$  is weakly continuous, but multiplication is not jointly continuous. Indeed, let  $V$  denote the unilateral shift on  $\ell^2(\mathbb{N})$ , i.e.  $V\xi_n = \xi_{n+1}$  for the canonical ONB  $(\xi_n)$ . Then  $V^n \rightarrow 0$  and  $V^{*n} \rightarrow 0$  weakly, but  $V^{*n}V^n = 1$  for all  $n \in \mathbb{N}$ . However, the involution is weakly continuous because  $\langle a_\lambda^* \xi, \eta \rangle = \overline{\langle a_\lambda \eta, \xi \rangle}$ .

*Remark 8.1.* For every Hilbert space  $H$ , the unit ball  $\mathcal{L}(H)_1 = \{a \in \mathcal{L}(H) \mid \|a\| \leq 1\}$  is weakly compact. To see this, note that  $\mathbb{C}_1$  is the closed unit disc, which is compact. Due to Tychonov's theorem,  $\mathbb{C}_1^I$  is compact for every index set  $I$ . Now the embedding  $a \mapsto \langle a\xi, \eta \rangle$  of  $\mathcal{L}(H)_1$  into  $\mathbb{C}_1^{\{(\xi, \eta) \in H \times H \mid \|\xi\|, \|\eta\| \leq 1\}}$  is a homeomorphism onto its range, which is closed, hence compact.

If  $H$  is separable, then bounded nets can be replaced by sequences when concerned with questions around SOT and WOT.

*Remark 8.2.* Note that convergence with respect to the norm topology implies convergence in SOT, which in turn implies convergence in WOT. Hence, every weakly closed (open) set is strongly closed (open), and every strongly closed (open) set is norm-closed (open).

Even though it already appeared in Proposition 7.2, let us now define a commutant for subsets in  $\mathcal{L}(H)$  properly.

**Definition 8.3.** For a Hilbert space  $H$  and  $X \subset \mathcal{L}(H)$ , the *commutant* of  $X$  in  $\mathcal{L}(H)$  is  $X' := \{T \in \mathcal{L}(H) \mid Tx = xT \text{ for all } x \in X\}$ .

*Remark 8.4.* The commutant  $X'$  is always weakly closed as multiplication by a fixed element is weakly continuous. In addition, we have  $(X^*)' = (X')^*$ . So if  $X^* = X$ , then  $X'$  is a  $C^*$ -algebra, see Remark 8.2.

Observe that  $X \subset X''$  and  $X' \subset X'''$ . In principle, if  $X^* = X$ , then these could be the starting bits of two increasing chains of  $C^*$ -algebras (ignoring the fact that  $X$  may not be a  $C^*$ -algebra). But the next lemma shows that these sequences stabilize after the first step already, if  $X$  is suitably chosen.

**Lemma 8.5.** *If  $A \subset \mathcal{L}(H)$  is a  $*$ -subalgebra with  $1 \in A$ , then  $A$  is strongly dense in  $A''$ .*

*Proof.* Let  $b \in A''$  and  $\varepsilon > 0$ . We need to show that for all  $n \in \mathbb{N}$  and  $\xi_1, \dots, \xi_n \in H$ , there exists  $a \in A$  such that  $p_{\xi_k}(b - a) = \|(b - a)\xi_k\| < \varepsilon$  for  $1 \leq k \leq n$ .

For  $n = 1$ , let  $\xi \in H$  and  $K := \overline{A\xi}$ . Then  $K$  is a closed  $A$ -invariant subspace of  $H$  that contains  $\xi$  (as  $1 \in A$ ). Thus the orthogonal projection  $P_K$  belongs to  $A'$ , and hence commutes with  $b$  so that  $b\xi \in K$ . Therefore, there exists  $a \in A$  with  $\|b\xi - a\xi\| < \varepsilon$ .

Now let  $n > 1$  and  $\xi_1, \dots, \xi_n \in H$ . Consider the map  $\varphi: \mathcal{L}(H) \rightarrow M_n(\mathcal{L}(H))$  given by  $T \mapsto \text{diag}(T, \dots, T)$ . This is an isometric homomorphism of  $*$ -algebras, so  $\varphi(A)^* = \varphi(A)$  is a  $*$ -algebra with  $1_{M_n(\mathcal{L}(H))} \in \varphi(A)$ . As

$$\varphi(A)' = \{(T_{ij})_{1 \leq i, j \leq n} \mid T_{ij} \in A' \text{ for all } i, j\},$$

we know that  $b \in A''$  implies  $\varphi(b) \in \varphi(A)''$ . Now, applying the case  $n = 1$  to  $\varphi(A)$ ,  $\xi := (\xi_1, \dots, \xi_n)^t$  and  $\varphi(b)$  yields an element  $a \in A$  with

$$\varepsilon > \|\varphi(b - a)\xi\| = \max_{1 \leq k \leq n} \|(b - a)\xi_k\|.$$

□

**Theorem 8.6.** *If  $A \subset \mathcal{L}(H)$  is a  $*$ -subalgebra with  $1 \in A$ , then the following are equivalent:*

- (1)  $A = A''$ .
- (2)  $A$  is weakly closed.
- (3)  $A$  is strongly closed.

*Proof.* Clearly, we have (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3), and by Lemma 8.5, (3) forces  $A = \overline{A}^{SOT} = A''$ . □

**Definition 8.7.** A *von Neumann algebra* is a weakly closed  $*$ -subalgebra  $A \subset \mathcal{L}(H)$  that contains the identity.

We will now collect a number of relatively straightforward, but nonetheless useful observations. For the rest of the section, let  $M$  be a von Neumann algebra.

*Remark 8.8.* For every normal element  $a \in M$ , we can perform functional calculus with Borel functions on  $\text{Sp } a$  inside  $M$ . The reason is that the algebra of bounded Borel functions  $\mathcal{B}_b(\text{Sp } a)$  is the smallest algebra of functions on  $\text{Sp } a$  that contains  $C(\text{Sp } a)$  and is closed under pointwise limits of sequences.

*Remark 8.9.* If  $K$  is a closed subspace of a Hilbert space  $H$  and  $a \in \mathcal{L}(H)$ , then  $aK \subset K$  if and only if  $aP_K = P_K a P_K$  for the orthogonal projection  $P_K$  onto  $K$ . Thus  $P_K$  commutes with  $a$  if and only if  $K$  is invariant under  $a$  and  $a^*$ . This allows us to draw the following consequence:

For  $a \in M$ , let  $K := \overline{aH}$  and  $L := \ker a$ . Then  $P_K$  and  $P_L$  belong to  $M$  as well. Indeed,  $K$  and  $L$  are invariant under  $b$  and  $b^*$  for every  $b \in M' = (M')^*$  because  $ab = ba$ . Thus  $P_K, P_L \in M''$  which equals  $M$ , by Theorem 8.6.

*Remark 8.10.* Every element in  $M$  can be approximated in norm by linear combinations of projections. Since  $M$  is a  $*$ -algebra, it suffices to show this for  $a \in M_{sa}$ . Via the Gelfand transformation,  $a$  corresponds to the identity function on  $\text{Sp } a$ . This function can be approximated by linear combinations of characteristic functions coming from finer and finer finite partitions of  $\text{Sp } a$  into Borel sets. By Remark 8.8, each such characteristic function give rise to projections with the desired property.

*Remark 8.11.* Recall from a previous exercise that in every unital  $C^*$ -algebra, every element is a linear combination of four unitaries. Thus  $M = \{a \in \mathcal{L}(H) \mid uau^* = a \text{ for all } u \in \mathcal{U}(M')\}$ , that is,  $M$  is the set of all elements in  $\mathcal{L}(H)$  that are fixed under conjugation by all unitaries from the von Neumann algebra  $M'$ .

*Remark 8.12.* For  $a \in M$ , let  $a = v|a|$  be its polar decomposition in  $\mathcal{L}(H)$ , and recall that  $v$  is a partial isometry and  $|a| = \sqrt{a^*a}$ . Recall that it is unique if we request  $v|_{\ker a} = 0$ , which we shall do. Then  $|a|$  belongs to  $M$  by functional calculus, but Remark 8.11 also implies  $a = uau^* = uvu^*|a|$  for every  $u \in \mathcal{U}(M')$ . Notice that  $uvu^*$  is a partial isometry that vanishes on  $\ker a$ . So by uniqueness of the polar decomposition, we get  $uvu^* = v$ , and hence  $v \in M$  by Remark 8.11.

**Definition 8.13.** For  $X \subset \mathcal{L}(H)$ , the *von Neumann algebra generated by  $X$*   $W^*(X)$  is the smallest von Neumann subalgebra of  $\mathcal{L}(H)$  that contains  $X$ .

**Proposition 8.14.** For every  $X \subset \mathcal{L}(H)$ , the von Neumann algebra  $W^*(X)$  is given by  $(X \cup X^*)''$ .

*Proof.* By Remark 8.4,  $(X \cup X^*)''$  is a von Neumann algebra, and contains  $X$ . So suppose  $M$  is a von Neumann algebra with  $X \subset M$ . Then

$$X \cup X^* \subset M \implies (X \cup X^*)' \supset M' \implies (X \cup X^*)'' \subset M'' = M,$$

so that  $W^*(X) = (X \cup X^*)''$ . □

**Definition 8.15.** For a von Neumann algebra  $M$ , the set  $Z(M) := M \cap M'$  is called the *center* of  $M$ .

Note that  $Z(M') = Z(M)$ .

**Proposition 8.16.** For every von Neumann algebra  $M$ , its center  $Z(M)$  is a von Neumann algebra, and  $Z(M)' = W^*(M \cup M')$ .

*Proof.* It is clear that  $Z(M)$  is a von Neumann algebra as unitality, WOT, and  $*$ -subalgebra structures are well-behaved with respect to finite intersections. The second claim follows from

$$(M \cup M')' = M' \cap M'' = Z(M)$$

as this leads to

$$W^*(M \cup M') = (M \cup M')'' = Z(M)'. \quad \square$$

*Example 8.17.* For  $M = \mathcal{L}(H)$ , we get  $M' = \mathbb{C}1$ , so that  $Z(M) = Z(M') = \mathbb{C}1$ .

There are more von Neumann algebras with trivial center, which motivates the next definition.

**Definition 8.18.** A von Neumann algebra  $M$  is called a *factor* if  $Z(M) = \mathbb{C}1$ .

**Proposition 8.19.** For a von Neumann algebra  $M$ , the following are equivalent:

- (1)  $M$  is a factor.
- (2)  $M'$  is a factor.
- (3)  $W^*(M \cup M') = \mathcal{L}(H)$ .

*Proof.* As  $Z(M) = Z(M')$ , (1) and (2) are equivalent. Using Proposition 8.16, we have  $Z(M)' = W^*(M \cup M')$  and  $Z(M) = Z(M)''$ , so the equivalence between (1) and (3) follows from Example 8.17.  $\square$

## 9. TENSOR PRODUCTS

In this section we discuss the notion of a tensor product of Hilbert spaces, which leads to an elementary class of factors.

Let  $V$  and  $W$  be vector spaces over some field  $K$ . We want to define an object  $V \otimes W$  such that all bases  $A$  for  $V$  and  $B$  for  $W$  yield a basis  $A \times B$  for  $V \otimes W$ . If we took this as a definition for  $V \otimes W$ , we would have to show that it does not depend on the particular choice of bases. Therefore, we make the following alternative definition.

**Definition 9.1.** For vector spaces  $V$  and  $W$  over some field  $K$ , the (*algebraic*) *tensor product*  $V \otimes W$  is defined to be the quotient space of the  $K$ -vector space with basis given by the symbols  $\xi \otimes \eta$ ,  $\xi \in V, \eta \in W$ , by the linear subspace generated by

- (a)  $(\xi_1 + \xi_2) \otimes \eta - (\xi_1 \otimes \eta + \xi_2 \otimes \eta)$ ,
- (b)  $\xi \otimes (\eta_1 + \eta_2) - (\xi \otimes \eta_1 + \xi \otimes \eta_2)$ ,
- (c)  $\lambda(\xi \otimes \eta) - (\lambda\xi) \otimes \eta$ , and  $\lambda(\xi \otimes \eta) - \xi \otimes \lambda\eta$

for all  $\lambda \in K, \xi, \xi_1, \xi_2 \in V, \eta, \eta_1, \eta_2 \in W$ .

*Remark 9.2.* The algebraic tensor product  $V \otimes W$  is characterized by the following universal property: The assignment  $(\xi, \eta) \mapsto \xi \otimes \eta$  defines a bilinear map  $\alpha: V \times W \rightarrow V \otimes W$ , and for every bilinear map  $\beta: V \times W \rightarrow Z$  there exists a unique linear map  $\beta': V \otimes W \rightarrow Z$  such that  $\beta = \beta' \circ \alpha$ . This results in a bijection between bilinear maps on  $V \times W$  and linear maps on  $V \otimes W$ .

Moreover, one can show that the Cartesian product  $A \times B$  of bases  $A$  for  $V$  and  $B$  for  $W$  defines a basis for  $V \otimes W$ , as desired.

*Example 9.3.* The tensor product  $K^m \otimes K^n$  is isomorphic to  $K^{mn}$ , where  $m, n \in \mathbb{N}$ . Now let  $m = n$  and consider the bilinear map  $\langle \cdot, \cdot \rangle: K^n \times K^n \rightarrow K, (\xi, \eta) \mapsto \sum_{i=1}^n \xi_i \eta_i$ . Then  $\xi \otimes \eta \mapsto |\xi\rangle\langle\eta|$ , where  $|\xi\rangle\langle\eta|(\zeta) := \xi\langle\eta, \zeta\rangle$  for  $\xi, \eta, \zeta \in K^n$  defines an element in  $M_n(K)$ . As the image of  $\{\xi \otimes \eta \mid \xi, \eta \in B\}$  for a basis  $B$  of  $K^n$  is linearly independent, the map defines an isomorphism  $K^n \otimes K^n \cong M_n(K)$ . Under this isomorphism, the universal property of  $K^n \otimes K^n$  from Remark 9.2 applied to  $\langle \cdot, \cdot \rangle: K^n \times K^n \rightarrow K$  yields the trace on  $M_n(K)$ .



*Remark 9.4.* Let  $H$  be a  $\mathbb{C}$ -vector space. We define a new  $\mathbb{C}$ -vector space  $\overline{H}$  that is given by  $H$  as a set and additive group, but with scalar multiplication  $\lambda \cdot \xi := \overline{\lambda}\xi$  for  $\lambda \in \mathbb{C}, \xi \in \overline{H}$ . The identity defines a linear map between the spaces  $H$  and  $\overline{H}$  under which antilinear maps  $\varphi: H \rightarrow X$ , i.e. additive maps with  $\varphi(\lambda\xi) = \overline{\lambda}\varphi(\xi)$  for  $\lambda \in \mathbb{C}, \xi \in H$ , correspond to linear maps  $\overline{H} \rightarrow X$ . This is actually a universal property characterizing  $\overline{H}$ . In particular, we get a bijection between antilinear maps  $H \rightarrow X$  and linear maps  $\overline{H} \rightarrow X$  (for every  $X$ ).

We also observe that  $\overline{H_1 \otimes H_2} \cong \overline{H_1} \otimes \overline{H_2}$  due to their universal properties.

**Proposition 9.5.** *Suppose  $H_1$  and  $H_2$  are Hilbert spaces (over  $\mathbb{C}$ ). Then*

$$\langle \cdot, \cdot \rangle: H_1 \otimes H_2 \times H_1 \otimes H_2 \rightarrow \mathbb{C} \text{ given by } (\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2) \mapsto \langle \xi_1, \eta_1 \rangle_{H_1} \langle \xi_2, \eta_2 \rangle_{H_2}$$

*defines an inner product on  $H_1 \otimes H_2$ .*

*Proof.* Recall that the inner product on  $H_i$  is a (particular) sesquilinear form  $\langle \cdot, \cdot \rangle: H_i \times H_i \rightarrow \mathbb{C}$ . Using Remark 9.4, we can pass to a bilinear map  $H_i \times \overline{H_i} \rightarrow \mathbb{C}$ . Thus

$$H_1 \times H_2 \times \overline{H_1} \times \overline{H_2} \rightarrow \mathbb{C} \text{ given by } (\xi_1, \xi_2, \eta_1, \eta_2) \mapsto \langle \xi_1, \eta_1 \rangle_{H_1} \langle \xi_2, \eta_2 \rangle_{H_2}$$

is a 4-linear map corresponding to a bilinear map  $H_1 \otimes H_2 \times \overline{H_1} \otimes \overline{H_2} \rightarrow \mathbb{C}$ , see Remark 9.2. By the last line of Remark 9.4, we thus see that  $H_1 \otimes H_2 \times \overline{H_1} \otimes \overline{H_2} \rightarrow \mathbb{C}, (\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2) \mapsto \langle \xi_1, \eta_1 \rangle_{H_1} \langle \xi_2, \eta_2 \rangle_{H_2}$  is a bilinear map. Hence we end up with a sesquilinear map

$$\langle \cdot, \cdot \rangle: H_1 \otimes H_2 \times H_1 \otimes H_2 \rightarrow \mathbb{C} \text{ given by } (\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2) \mapsto \langle \xi_1, \eta_1 \rangle_{H_1} \langle \xi_2, \eta_2 \rangle_{H_2}.$$

It remains to show that  $\langle \cdot, \cdot \rangle$  is positive definite. To this end, let  $\xi \in H_1 \otimes H_2$ . Then  $\xi = \sum_{i=1}^n \xi_1^{(i)} \otimes \xi_2^{(i)}$  for some  $n \in \mathbb{N}$  and suitable  $\xi_j^{(i)} \in H_j, j = 1, 2$ . Without loss of generality, we can assume that  $(\xi_2^{(i)})_{1 \leq i \leq n}$  is an orthonormal family in  $H_2$ , as the finite linear combination for  $\xi$  can be adjusted accordingly by choosing an orthonormal basis for the subspace of  $H_2$  spanned by  $\{\xi_2^{(1)}, \dots, \xi_2^{(n)}\}$ . Then

$$\langle \xi, \xi \rangle = \sum_{1 \leq i, j \leq n} \langle \xi_1^{(i)} \otimes \xi_2^{(i)}, \xi_1^{(j)} \otimes \xi_2^{(j)} \rangle = \sum_{1 \leq i \leq n} \|\xi_1^{(i)}\|_{H_1}^2$$

shows that  $\langle \cdot, \cdot \rangle$  is indeed positive definite.  $\square$

**Definition 9.6.** For two complex Hilbert spaces  $H_1$  and  $H_2$ , the *Hilbert space tensor product*  $H_1 \widehat{\otimes} H_2$  is defined as the completion of  $H_1 \otimes H_2$  with respect to the norm induced by the inner product  $\langle \cdot, \cdot \rangle$  from Proposition 9.5.

**Proposition 9.7.** *If  $(\xi_i)_{i \in I}$  and  $(\eta_j)_{j \in J}$  are orthonormal bases for (complex) Hilbert spaces  $H_1$  and  $H_2$ , respectively, then  $(\xi_i \otimes \eta_j)_{(i,j) \in I \times J}$  is an orthonormal basis for  $H_1 \widehat{\otimes} H_2$ .*

*Proof.* It is clear that  $(\xi_i \otimes \eta_j)_{(i,j) \in I \times J}$  is an orthonormal family in  $H_1 \widehat{\otimes} H_2$ , so we need to show that its span is dense. First, let  $\alpha \in H_1$ , so  $\alpha = \sum_{i \in I} c_i \xi_i$  for suitable  $c_i \in \mathbb{C}$ . Then  $\sum_{i \in F} c_i \xi_i \otimes \eta_j \rightarrow \alpha \otimes \eta_j$ , where  $F$  ranges over finite subsets of  $I$  (directed with respect to inclusion). Hence  $\alpha \otimes \eta_j \in \overline{\text{span}\{\xi_i \otimes \eta_j \mid (i, j) \in I \times J\}}$  for every  $j \in J$ . Displaying  $\beta \in H_2$  similarly as  $\beta = \sum_{j \in J} d_j \eta_j$  with suitable  $d_j \in \mathbb{C}$ , we get

$$\alpha \otimes \beta = \lim_{F \subset J \text{ finite}} \sum_{j \in F} \alpha \otimes d_j \eta_j \in \overline{\text{span}\{\xi_i \otimes \eta_j \mid (i, j) \in I \times J\}}$$

for all  $\alpha \in H_1, \beta \in H_2$ , so  $H_1 \otimes H_2 \subset \overline{\text{span}}\{\xi_i \otimes \eta_j \mid (i, j) \in I \times J\}$ . As  $H_1 \widehat{\otimes} H_2$  is the completion of the algebraic tensor product  $H_1 \otimes H_2$ , the subspace  $\overline{\text{span}}\{\xi_i \otimes \eta_j \mid (i, j) \in I \times J\}$  is all of  $H_1 \widehat{\otimes} H_2$ .  $\square$

The following is immediate from Proposition 9.7.

**Corollary 9.8.** *For all index sets  $I$  and  $J$ , the Hilbert space  $\ell^2(I) \widehat{\otimes} \ell^2(J)$  is isomorphic to  $\ell^2(I \times J)$ .*

Proposition 9.7 also implies that  $L^2(\mathbb{R}^m) \widehat{\otimes} L^2(\mathbb{R}^n)$  is isomorphic to  $L^2(\mathbb{R}^{m+n})$  for all  $m, n \in \mathbb{N}$ : The map

$$\begin{aligned} L^2(\mathbb{R}^m) \times L^2(\mathbb{R}^n) &\rightarrow L^2(\mathbb{R}^{m+n}) \\ (f, g) &\mapsto [(x, y) \mapsto f(x)g(y)] \end{aligned}$$

is bilinear and thus induces a linear map  $\varphi: L^2(\mathbb{R}^m) \otimes L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^{m+n})$ . The map  $\varphi$  is isometric because the inner product on  $L^2(\mathbb{R}^m) \otimes L^2(\mathbb{R}^n)$  from Proposition 9.5 is equal to the pullback of the inner product on  $L^2(\mathbb{R}^{m+n})$  under  $\varphi$ . However, proving that the unique extension of  $\varphi$  to an isometric linear map  $\varphi': L^2(\mathbb{R}^m) \widehat{\otimes} L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^{m+n})$  is a surjective map is more difficult.

**Proposition 9.9.** *Let  $H_i$  be a Hilbert space and  $x_i \in \mathcal{L}(H_i)$  for  $i = 1, 2$ . Then there exists  $x_1 \otimes x_2 \in \mathcal{L}(H_1 \widehat{\otimes} H_2)$  uniquely determined by the property  $x_1 \otimes x_2(\xi_1 \otimes \xi_2) = x_1 \xi_1 \otimes x_2 \xi_2$  for all  $\xi_i \in H_i$ . The operator  $x_1 \otimes x_2$  has the following properties:*

- (i)  $\|x_1 \otimes x_2\| = \|x_1\| \cdot \|x_2\|$ .
- (ii)  $(x_1 + y_1) \otimes x_2 = x_1 \otimes x_2 + y_1 \otimes x_2$  and  $x_1 \otimes (x_2 + y_2) = x_1 \otimes x_2 + x_1 \otimes y_2$ .
- (iii)  $(x_1 \otimes x_2)(y_1 \otimes y_2) = x_1 y_1 \otimes x_2 y_2$ .
- (iv)  $(x_1 \otimes x_2)^* = x_1^* \otimes x_2^*$ .

*Proof.* Note that if  $T \in \mathcal{L}(H_1 \widehat{\otimes} H_2)$  has the property that  $T(\xi_1 \otimes \xi_2) = x_1 \xi_1 \otimes x_2 \xi_2$  for all  $\xi_i \in H_i$ , then  $T$  agrees with  $x_1 \otimes x_2$  on  $H_1 \otimes H_2$ , and hence  $T = x_1 \otimes x_2$  by denseness.

Suppose first that  $x_2 = 1$ . A generic element of  $H_1 \otimes H_2$  is of the form  $\sum_{i=1}^n \xi_1^{(i)} \otimes \xi_2^{(i)}$  with  $n \in \mathbb{N}$ . As in the proof of Proposition 9.7, we can arrange for  $(\xi_2^{(i)})_{1 \leq i \leq n}$  to be orthonormal. Then we get

$$\|(x_1 \otimes 1)(\sum_{i=1}^n \xi_1^{(i)} \otimes \xi_2^{(i)})\|^2 = \sum_{i=1}^n \|x_1 \xi_1^{(i)}\|^2 \leq \|x_1\|^2 \sum_{i=1}^n \|\xi_1^{(i)}\|^2 = \|x_1\|^2 \|\sum_{i=1}^n \xi_1^{(i)} \otimes \xi_2^{(i)}\|.$$

Therefore, the linear map  $x_1 \otimes 1: H_1 \otimes H_2$  admits a unique extension to an element of  $\mathcal{L}(H_1 \widehat{\otimes} H_2)$  (again denoted by  $x_1 \otimes 1$ ) with  $\|x_1 \otimes 1\| \leq \|x_1\|$ . The case of  $1 \otimes x_2$  is analogous, and we define  $x_1 \otimes x_2 := (x_1 \otimes 1)(1 \otimes x_2)$ . Clearly,  $\|x_1 \otimes x_2\| \leq \|x_1\| \cdot \|x_2\|$  by construction, but  $\|(x_1 \otimes x_2)(\xi_1 \otimes \xi_2)\| = \|x_1 \xi_1\| \cdot \|x_2 \xi_2\|$  yields the reverse inequality. This shows (i). Using denseness of  $H_1 \otimes H_2 \subset H_1 \widehat{\otimes} H_2$  and linearity for the vectors, it suffices to check (ii)–(iv) on elementary tensors  $\xi_1 \otimes \xi_2$ . But for these the claims are trivial by the defining property of  $x_1 \otimes x_2$ .  $\square$

*Remark 9.10.* If  $(\xi_i)_{i \in I}$  and  $(\eta_j)_{j \in J}$  are orthonormal bases for Hilbert spaces  $H_1$  and  $H_2$ , respectively, then we know from Proposition 9.7 that  $(\xi_i \otimes \eta_j)_{(i,j) \in I \times J}$  is an orthonormal basis for  $H_1 \widehat{\otimes} H_2$ . This leads to

$$H_1 \widehat{\otimes} H_2 \cong \bigoplus_{i \in I} \mathbb{C} \xi_i \widehat{\otimes} H_2 \cong \bigoplus_{j \in J} H_1 \widehat{\otimes} \mathbb{C} \eta_j,$$

where we note that there is no need to take the completion for each of summands appearing in the direct sum decompositions. But we use the symbol  $\widehat{\otimes}$  to stress that we are considering Hilbert spaces. For  $j \in J$ , the map

$$\begin{aligned} u_j: H_1 &\rightarrow H_1 \widehat{\otimes} \mathbb{C}\eta_j \\ \xi &\mapsto \xi \otimes \eta_j \end{aligned}$$

is an isometric isomorphism with  $u_j^* u_j = 1_{H_1}$  and  $u_j u_j^* = P_{H_1 \widehat{\otimes} \mathbb{C}\eta_j}$ . We observe that for  $i, j \in J$ , we have  $u_i u_j^*(\xi \otimes \eta_k) = \delta_{j,k} \xi \otimes \eta_i$  for all  $k \in J$ . Thus  $e_{ij} := u_i u_j^* = 1 \otimes e'_{ij}$  for  $e'_{ij} \in \mathcal{L}(H_2)$  determined by  $e'_{ij}(\eta_k) = \delta_{j,k} \eta_i$  for  $k \in J$ . In particular,  $(e_{jj})_{j \in J}$  is a family of mutually orthogonal projections (whose sum converges strongly to the identity on  $H_1 \widehat{\otimes} H_2$ ).

Given  $x \in \mathcal{L}(H_1 \widehat{\otimes} H_2)$ , we use the maps  $u_j, j \in J$  from Remark 9.10 to define  $x_{ij} := u_i^* x u_j$  for  $i, j \in J$ .

**Proposition 9.11.** *Let  $H_1$  and  $H_2$  be Hilbert spaces. For  $x \in \mathcal{L}(H_1 \widehat{\otimes} H_2)$ , the following conditions are equivalent:*

- (1)  $x = x_1 \otimes 1$  for some  $x_1 \in \mathcal{L}(H_1)$ .
- (2)  $x \in (1 \otimes \mathcal{L}(H_2))'$ , where  $1 \otimes \mathcal{L}(H_2) = \{1 \otimes x_2 \mid x_2 \in \mathcal{L}(H_2)\}$ .
- (3)  $x_{ij} = \delta_{i,j} x_1$  for some  $x_1 \in \mathcal{L}(H_1)$ .

*Proof.* Clearly, (1) implies (2). Suppose (2) holds and let  $i, j \in J, i \neq j$ . Then we get

$$x_{ij} = u_i^* x u_j = u_i^* e_{ii} x e_{jj} u_j \stackrel{(2)}{=} u_i^* x e_{ii} e_{jj} u_j \stackrel{9.10}{=} 0.$$

Moreover, we have

$$x_{ii} = u_i^* x u_i = u_i^* u_i u_j^* \underbrace{u_j u_i}_{=e_{ji}=1 \otimes e'_{ji}} x u_i \stackrel{(2)}{=} u_i^* u_i u_j^* x u_j u_i^* u_i = x_{jj},$$

which together with the first computation gives (3). Given (3), we use the fact that  $\sum_{j \in F} e_{jj} \rightarrow 1_{H_1 \widehat{\otimes} H_2}$  strongly as  $F$  ranges over finite subsets of  $J$  to compute

$$x(\xi \otimes \eta_i) = x u_i(\xi) = \sum_{j \in J} u_j \underbrace{u_j^* x u_i}_{=x_{ji}}(\xi) \stackrel{(3)}{=} u_i x_1(\xi) = x_1(\xi) \otimes \eta_i.$$

According to Proposition 9.9, this shows  $x = x_1 \otimes 1$ , so we arrive at (1).  $\square$

**Proposition 9.12.** *Let  $H_1$  and  $H_2$  be Hilbert spaces. Then the following statements hold:*

- (1)  $(\mathcal{L}(H_1) \otimes 1)'$  equals  $1 \otimes \mathcal{L}(H_2)$  and  $(1 \otimes \mathcal{L}(H_2))' = \mathcal{L}(H_1) \otimes 1$ . In particular, both algebras are factors.
- (2)  $\mathcal{L}(H_1) \rightarrow \mathcal{L}(H_1) \otimes 1, x \mapsto x \otimes 1$  is a  $*$ -isomorphism.
- (3)  $\mathcal{L}(H_2) \rightarrow 1 \otimes \mathcal{L}(H_2), y \mapsto 1 \otimes y$  is a  $*$ -isomorphism.

*Proof.* Part (1) is essentially due to Proposition 9.11 as  $Z(\mathcal{L}(H_1) \otimes 1) = \mathcal{L}(H_1) \otimes 1 \cap 1 \otimes \mathcal{L}(H_2) = \mathbb{C}1_{H_1 \widehat{\otimes} H_2}$ . For parts (2) and (3), it is clear that the map defines a surjective  $*$ -homomorphism by Proposition 9.9. But injectivity follows from  $x \otimes 1 = 0 \Rightarrow x = 0$ , and similarly for  $1 \otimes y$ .  $\square$

## 10. MURRAY-VON NEUMANN (SUB-)EQUIVALENCE FOR FACTORS

In this section, we introduce the notion of Murray-von Neumann (sub-)equivalence of projections and prove that projections in factors are comparable. We assume from now on that  $H$  is a separable Hilbert space and that  $M \subset \mathcal{L}(H)$  is a von Neumann algebra.

**Definition 10.1.** Two projections  $e, f \in M$  are *Murray-von Neumann equivalent*, denoted  $e \sim f$ , if there exists  $v \in M$  such that  $e = v^*v$  and  $vv^* = f$ .

It is sometimes convenient to write  $e \sim_v f$  if  $v \in M$  satisfies the condition from Definition 10.2. We note that any such  $v$  is necessarily a partial isometry with source projection  $e$  and range projection  $f$ . The relation  $\sim$  is an equivalence relation. Indeed, reflexivity and symmetry are apparent. For transitivity, let  $e \sim_v f \sim_w g$ . Then  $wv$  satisfies

$$(wv)^*wv = v^*fv = v^*vv^*v = e \quad \text{and} \quad wv(wv)^* = wfw^* = ww^*ww^* = g.$$

Next, we introduce a weakening of Murray-von Neumann equivalence.

**Definition 10.2.** A projection  $e \in M$  is *Murray-von Neumann subequivalent* to another projection  $f \in M$ , denoted  $e \prec f$ , if there exists a partial isometry  $v \in M$  such that  $e = v^*v$  and  $vv^* \leq f$ .

Note that  $\prec$  is also transitive. The proof is analogous to the one for Murray-von Neumann equivalence.

*Remark 10.3.* Note that for projections  $e, f \in M$  the following conditions are equivalent:

- (a)  $eH \subset fH$ .
- (b)  $fe = e$ .
- (c)  $e \leq f$ , i.e.  $f - e \geq 0$ .

*Example 10.4.* Let  $M = \mathcal{L}(H)$  and  $e = P_K, f = P_L$  for closed subspaces  $K, L \subset H$ . Then  $e \prec f$  holds if and only if there is a closed subspace  $\tilde{K} \subset L$  such that  $K \cong \tilde{K}$ , which is equivalent to  $\dim K \leq \dim L$ .

**Proposition 10.5.** *If  $(e_i)_{i \in I}, (f_i)_{i \in I} \subset M$  are two families of mutually orthogonal projections. If  $e_i \sim f_i$  for all  $i \in I$ , then  $\sum_{i \in I} e_i \sim \sum_{i \in I} f_i$ . Similarly, if  $e_i \prec f_i$  for all  $i \in I$ , then  $\sum_{i \in I} e_i \prec \sum_{i \in I} f_i$ .*

*Proof.* By assumption, we have  $e_i \sim_{v_i} f_i$  for suitable partial isometries  $v_i \in M$  for all  $i \in I$ . Then  $w \in \mathcal{L}(H)$  defined by  $w|_{e_i H} := v_i$  and  $w|_{(\bigoplus_{i \in I} e_i H)^\perp} = 0$  satisfies  $\sum_{i \in I} e_i = w^*w$  and  $ww^* = \sum_{i \in I} f_i$ . We need to show that  $w$  belongs to  $M$ , for instance by showing that it commutes with  $M'$ . Every element in  $M'$  is a linear combination of unitaries, so it suffices to prove this for all unitaries  $u \in M'$ , see Remark 8.11. This follows from

$$uw|_{e_i H} = uv_i|_{e_i H} \stackrel{v_i \in M}{=} v_i u|_{e_i H} = wu|_{e_i H} \quad \text{and} \quad uw|_{(\bigoplus_{i \in I} e_i H)^\perp} = 0 = wu|_{(\bigoplus_{i \in I} e_i H)^\perp}$$

where we used that  $e_i H$  and  $(\bigoplus_{i \in I} e_i H)^\perp$  are  $M'$ -invariant subspaces.

If the families only satisfy  $e_i \prec f_i$  for all  $i \in I$ , say via partial isometries  $v_i$ , then the previous argument applied to  $(e_i)_{i \in I}$  and  $(v_i v_i^*)_{i \in I}$  yields a partial isometry  $w$  with  $\sum_{i \in I} e_i \sim_w \sum_{i \in I} v_i v_i^* \leq \sum_{i \in I} f_i$ , which proves the claim.  $\square$

**Proposition 10.6.** *Two projections  $e, f \in M$  satisfy  $e \sim f$  if and only if  $e \prec f$  and  $e \succ f$ .*

*Proof.* It is clear that  $e \sim f$  implies  $e \prec f$  and  $e \succ f$ . The proof for the converse direction is analogous to the proof of the Schröder-Bernstein theorem from set theory: Let  $e, f \in M$  be projections with  $e \prec f$  and  $e \succ f$ . By replacing  $f$  by the range projection from  $f \prec e$  we can assume  $f \leq e$ . Due to  $e \prec f$ , there is a partial isometry  $v \in M$  such that  $e = v^*v$  and  $vv^* \leq f$ . Then  $e_{2n} := v^n e v^{*n}$  and  $e_{2n+1} := v^n f v^{*n}$  for  $n \geq 0$  defines a family  $(e_n)_{n \in \mathbb{N}}$  of projections in  $M$  with

$$e_{2n} \geq e_{2n+1}, \quad e_{2n} \sim e_{2n+2}, \quad \text{and } e_{2n+1} \sim e_{2n+3} \quad \text{for all } n \geq 0.$$

Observe that  $e_{2n} - e_{2n+1} \sim e_{2n+2} - e_{2n+3}$  via  $v_n := v(e_{2n} - e_{2n+1})$  as

$$(v(e_{2n} - e_{2n+1}))^* v(e_{2n} - e_{2n+1}) = (e_{2n} - e_{2n+1})e(e_{2n} - e_{2n+1}) = e_{2n} - e_{2n+1}$$

and  $v(e_{2n} - e_{2n+1})(v(e_{2n} - e_{2n+1}))^* = e_{2n+2} - e_{2n+3}$ . Noting that  $e = \sum_{n \geq 0} (e_n - e_{n+1})$  and  $f = \sum_{n \geq 1} (e_n - e_{n+1})$  (both as strong limits), we get a partial isometry  $w \in M$  with  $e \sim_w f$  by setting  $w|_{(1-e)H} = 0$ ,  $w|_{(e_{2n} - e_{2n+1})H} = v_n$ , and  $w|_{(e_{2n+1} - e_{2n+2})H} = \text{id}$ .  $\square$

For an element  $a$  in a von Neumann algebra  $M$ , we denote by  $\text{supp } a$  the orthogonal projection onto  $(\ker a)^\perp$ . According to Remark 8.9,  $\text{supp } a$  also belongs to  $M$ .

**Proposition 10.7.** *For every element  $a$  in a von Neumann algebra  $M$ ,  $\text{supp } a^* \sim \text{supp } a$  holds.*

*Proof.* If  $a = v|a|$  is the polar decomposition for  $a$ , then  $v \in M$  by Remark 8.12, and we have  $\text{supp } a = v^*v$  and  $\text{supp } a^* = vv^*$ .  $\square$

**Lemma 10.8.** *Let  $M \subset \mathcal{L}(H)$  be a factor. If  $e, f \in M \setminus \{0\}$  are projections, then there exist projections  $e_1, f_1 \in M \setminus \{0\}$  such that  $e \geq e_1 \sim f_1 \leq f$ .*

*Proof.* Since  $e \neq 0$ ,  $K := \overline{MeH}$  is a non-trivial, closed  $M$ -invariant subspace of  $H$ . Thus we have  $0 \neq P_K \in M'$ , and also  $P_K \in M''$ . Due to  $M$  being a factor, we get  $P_K \in M' \cap M'' = Z(M) = \mathbb{C}1$ . Since  $P_K$  is a nonzero projection, this amounts to  $P_K = 1$ . Therefore, there exists  $z \in M$  such that  $fze \neq 0$  because  $f \neq 0$ . Now  $e \geq \text{supp } fze \sim \text{supp } ez^*f \leq f$  by Proposition 10.7, which proves the claim as  $\text{supp } fze$  and  $\text{supp } (fze)^* = \text{supp } ez^*f$  vanish if and only if  $fze = 0$ .  $\square$

**Theorem 10.9.** *If  $e$  and  $f$  are projections in a factor  $M$ , then  $e \prec f$  or  $e \succ f$  holds.*

*Proof.* Consider families of pairs of non-zero projections  $(e_i, f_i)_{i \in I} \subset M \times M$  such that  $(e_i)_{i \in I}$  and  $(f_i)_{i \in I}$  are families of mutually orthogonal projections with  $e \geq e_i \sim f_i$  for all  $i \in I$ . Such families are partially ordered by inclusion, and every totally ordered collection of such families admits a maximal element. By Lemma 10.8, there exist such families, at least with  $|I| = 1$ . So by Zorn's lemma, there exists a maximal family  $(e_i, f_i)_{i \in I}$ . Then  $e' := \sum_{i \in I} e_i$  and  $f' := \sum_{i \in I} f_i$  are projections in  $M$  (as strong limits of the respective partial sums) satisfying  $e \geq e' \sim f' \leq f$  by Proposition 10.5. Assume  $e - e' \neq 0 \neq f - f'$ . Then Lemma 10.8 implies that there are projections  $e'', f'' \in M \setminus \{0\}$  with  $e - e' \geq e'' \sim f'' \leq f - f'$ . As this would contradict maximality of  $(e_i, f_i)_{i \in I}$ , we must have  $e = e'$  or  $f = f'$ , and hence  $e \prec f$  or  $e \succ f$ , respectively.  $\square$

**Theorem 10.10.** *Let  $M$  be a factor. If  $e$  and  $f$  are projections in  $M$  with  $e \neq 0$ , then there exists a family  $(e_i)_{i \in I} \subset M$  of mutually orthogonal projections and a projection  $r \in M$  with  $e \sim e_i \perp r$  for all  $i \in I$  such that*

$$f = \sum_{i \in I} e_i + r \quad \text{and} \quad e \not\sim r \prec e.$$

*If  $I$  is infinite, then one can arrange for  $r = 0$ .*

*Proof.* Similar to the proof of Theorem 10.9, Zorn's lemma yields a maximal family of projections  $(e_i)_{i \in I} \subset M$  with respect to the conditions that  $e_i e_j = 0$  for  $i \neq j$  and  $e_i \sim e$  for all  $i, j \in I$ . Then  $r := f - \sum_{i \in I} e_i$  is a projection in  $M$ . According to Theorem 10.9, we have  $r \prec e$  or  $r \succ e$ . But then maximality of  $(e_i)_{i \in I}$  forces  $e \not\sim r \prec e$ .

Now let  $I$  be infinite, and pick  $k \in I$ . Then

$$\sum_{i \in I} e_i = \sum_{i \in I \setminus \{k\}} e_i + e_k \stackrel{e_k \sim e \succ r}{\succ} \sum_{i \in I \setminus \{k\}} e_i + r \stackrel{10.5 \text{ for } I \setminus \{k\} \& I}{\sim} f,$$

yields  $f \prec \sum_{i \in I} e_i$ , and hence  $f \sim \sum_{i \in I} e_i$  by Proposition 10.6. If  $v \in M$  is a partial isometry with  $f = v^* v$  and  $vv^* = \sum_{i \in I} e_i$ , then  $e_i \leq f$  leads to

$$f = v^* v v^* v = \sum_{i \in I} v^* e_i v \quad \text{with } v^* e_i v \sim e, v^* e_i v v^* e_j v = \delta_{i,j} v^* e_i v \text{ for all } i, j \in I.$$

□

## 11. DISCRETE FACTORS

In this short section we characterize *discrete factors*  $M \subset \mathcal{L}(H)$ , which are factors that are isomorphic to  $\mathcal{L}(\tilde{H})$  (as  $C^*$ -algebras for some Hilbert space  $\tilde{H}$ ).

**Definition 11.1.** Let  $M$  be a factor. A nonzero projection  $e \in M$  is *minimal*, if the only projections in  $M$  below  $e$  are 0 and  $e$ .

**Proposition 11.2.** *If  $e$  and  $f$  are minimal projections in a factor  $M$ , then  $e \sim f$ .*

*Proof.* By definition,  $e \neq 0 \neq f$ . Due to Theorem 10.9 we have  $e \prec f$  or  $e \succ f$ , but either option gives  $e \sim f$  by minimality of  $f$  or  $e$ , respectively. □

**Proposition 11.3.** *A nonzero projection  $e$  in a factor  $M$  is minimal if and only if  $eMe = \mathbb{C}e$ .*

*Proof.* If  $eMe = \mathbb{C}e$ , then  $0 \leq f \leq e$  for a projection  $f \in M$  implies  $f = efe \in \mathbb{C}e$ , which amounts to  $f \in \{0, e\}$  since it is a projection. For the converse, we note that  $eMe \subset \mathcal{L}(eH)$  is a von Neumann algebra. Therefore,  $eMe$  contains all spectral projections of its self-adjoint elements, see Remark 8.10. Since  $e$  is minimal in  $M$ , this forces  $eMe = \mathbb{C}e$ . □

**Theorem 11.4.** *For a factor  $M \subset \mathcal{L}(H)$ , the following conditions are equivalent:*

- (1)  *$M$  is discrete, i.e.  $M \cong \mathcal{L}(H_1)$  for some Hilbert space  $H_1$ .*
- (2) *Every nonzero projection in  $M$  dominates a minimal projection in  $M$ .*
- (3)  *$M$  has a minimal projection.*
- (4)  *$M'$  is discrete, i.e.  $M' \cong \mathcal{L}(H_2)$  for some Hilbert space  $H_2$ .*

- (5) Every nonzero projection in  $M'$  dominates a minimal projection in  $M'$ .  
 (6)  $M'$  has a minimal projection.

*Proof.* It is apparent that (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3) and (4)  $\Rightarrow$  (5)  $\Rightarrow$  (6). We will show that (3) implies (4), which will then also prove (6)  $\Rightarrow$  (1).

For (3) implies (4), we let  $e$  be a minimal projection in  $M$ . We claim that

- (a)  $M' \rightarrow M'e, a \mapsto ea (= ae = eae)$  is an injective  $*$ -homomorphism, and  
 (b)  $M'e \subset \mathcal{L}(eH)$  is a von Neumann algebra with  $(M'e)' \cap e\mathcal{L}(H)e = eMe = \mathbb{C}e$ .

For (a), the closed subspace  $\overline{M'aH}$  for every  $a \in M' \setminus \{0\}$  is  $M$ - and  $M'$ -invariant. As  $M$  is a factor,  $a \neq 0$  gives  $P_K = 1$ , that is,  $K = H$ . Thus  $e \neq 0$  yields  $eK \neq 0$ , and thus  $ea \neq 0$ .

For (b), we note that  $M'e \subset \mathcal{L}(eH)$  is a von Neumann algebra as the property of being SOT-closed is preserved under compression by any projection in  $\mathcal{L}(H)$ . The second part of the equality is due to Proposition 11.3. For the first one, let  $a \in M', b \in M$ . Then  $(ebe)ae = eabe = ae(ebe)$  shows  $eMe \subset (M'e)' \cap e\mathcal{L}(H)e$ . The reverse inclusion is obtained from

$$\begin{aligned}
 (M'e)' \cap e\mathcal{L}(H)e &= \{b \in \mathcal{L}(H) \mid b = be = eb, aeb = bae \text{ for all } a \in M'\} \\
 &\stackrel{ae=ea}{=} \{b \in \mathcal{L}(H) \mid b = be = eb, ab = ba \text{ for all } a \in M'\} \\
 &= \{b \in \mathcal{L}(H) \mid b = be = eb, b \in M''\} \\
 &\stackrel{M''=M}{=} \{b \in M \mid b = be = eb\} \\
 &= eMe.
 \end{aligned}$$

We thus conclude by (b) that for  $H_2 := eH$ , we have  $M' \cong M'e = (M'e)'' \cap \mathcal{L}(H_2) = (\mathbb{C}e)' \cap \mathcal{L}(H_2) = \mathcal{L}(H_2)$ , which completes (4).

Now suppose the equivalent conditions (1)–(6) hold, and let  $e$  be a minimal projection in  $M$ . Then  $1 = \sum_{i \in I} e_i$  for some index set  $I$  with projections  $e_i \in M$  such that  $e \sim e_i \perp e_j$  for all  $i, j \in I, i \neq j$  by Theorem 10.10. Note that the remainder  $r$  vanishes as  $e$  is minimal, and that  $I \neq \emptyset$ . Choose  $k \in I$  to define  $H_2 := e_k H$  and  $H_1 := \ell^2(I)$ .  $\square$

*Remark 11.5.* The conditions (1)–(6) from Theorem 11.4 are also equivalent to:

- (7) There are Hilbert spaces  $H_1$  and  $H_2$  such that there is an isomorphism  $H \cong H_1 \widehat{\otimes} H_2$  that takes  $M$  to  $\mathcal{L}(H_1) \otimes 1$  and  $M'$  to  $1 \otimes \mathcal{L}(H_2)$ .

However, we do not provide a proof.

## 12. GROUP C\*-ALGEBRAS

Recall that a topological group is a group together with a topology such that the map  $G \times G \rightarrow G$  given by  $(g, h) \mapsto gh$ , and the map  $G \rightarrow G$  given by  $g \mapsto g^{-1}$ , are both continuous (the former with respect to the product topology).

In operator algebras one generally considers locally compact Hausdorff groups.

However, in this course, we will only consider *discrete* groups (although some exceptions could briefly be mentioned). Recall that  $G$  is discrete if all of its subsets are open. In particular, this means that every map from  $G$  into any space is automatically continuous.

*Remark 12.1.* If  $G$  is a locally compact Hausdorff group and the cardinality of  $G$  is countable, then  $G$  must be discrete.

*Examples 12.2.* Some locally compact Hausdorff groups:

- (i) Finite groups are discrete (but not so interesting from the viewpoint of operator algebras, as they only give rise to finite-dimensional  $C^*$ -algebras).
- (ii)  $\mathbb{Z}^n$  is discrete and abelian ( $\mathbb{Z}$  is probably the most important group).
- (iii)  $\mathbb{F}_n$  is discrete, but not abelian for  $n \geq 2$  (also a very important group for operator algebras).
- (iv)  $\mathbb{T}$ , the circle group under multiplication and topology inherited from the complex plane; this group is non-discrete, compact, and abelian.
- (v)  $\mathbb{R}$ , the real numbers under addition and the usual topology; this group is non-discrete, non-compact, and abelian.
- (vi) For any Hilbert space  $H$ , define the unitary group  $U(H)$  as the set  $\{U \in B(H) : UU^* = U^*U = I\}$ ; then  $U(H)$  is a group under multiplication (and we usually give it the strong operator topology).
- (vii) the above are just the most standard examples – even for countable discrete groups, the class of examples is vast!

Let  $G$  be a discrete group. For any  $g \in G$ , define a function  $\delta_g: G \rightarrow \mathbb{C}$  by

$$\delta_g(h) = \begin{cases} 1 & \text{if } g = h, \\ 0 & \text{else.} \end{cases}$$

Define the group ring  $\mathbb{C}[G]$  as the vector space of functions  $G \rightarrow \mathbb{C}$  with finite support. Since  $G$  is discrete, this coincides with  $C_c(G)$ , the vector space of functions  $G \rightarrow \mathbb{C}$  with compact support. For a function  $\varphi \in \mathbb{C}[G]$ , define

$$\|\varphi\|_1 = \sum_{g \in G} |\varphi(g)|,$$

and denote the completion of  $\mathbb{C}[G]$  with respect to  $\|\cdot\|_1$  by  $\ell^1(G)$ .

**Proposition 12.3.**  $\ell^1(G)$  is a Banach  $*$ -algebra when equipped with convolution, involution, and norm given by

$$\begin{aligned} (\varphi * \psi)(s) &= \sum_{g \in G} \varphi(g) \psi(g^{-1}s), \\ \varphi^*(s) &= \overline{\varphi(s^{-1})}, \\ \|\varphi\|_1 &= \sum_{g \in G} |\varphi(g)|. \end{aligned}$$

*Proof.* Exercise; for example, you need to show that

$$\|\varphi * \psi\|_1 \leq \|\varphi\|_1 \cdot \|\psi\|_1.$$

□

**Definition 12.4.** A unitary representation of  $G$  on a Hilbert space  $H$  is a homomorphism  $\pi: G \rightarrow U(H)$ .



This induces a (nondegenerate) representation  $\tilde{\pi}: \ell^1(G) \rightarrow B(H)$  given by

$$\tilde{\pi}(\varphi) = \sum_{g \in G} \varphi(g) \pi(g).$$

Check that  $\|\tilde{\pi}(\varphi)\| \leq \|\varphi\|_1$ . In fact, every nondegenerate representation of  $\ell^1(G)$  arises this way.

**Definition 12.5.** For any  $G$ , we define the Hilbert space  $\ell^2(G)$  as the space of all functions  $\xi: G \rightarrow \mathbb{C}$  such that

$$\sum_{g \in G} |\xi(g)|^2 < \infty,$$

together with inner product

$$\langle \xi, \zeta \rangle = \sum_{g \in G} \xi(g) \overline{\zeta(g)}.$$

We also note that  $\{\delta_h\}_{h \in G}$  forms an orthonormal basis for  $\ell^2(G)$ .

**Definition 12.6.** For any  $G$ , we define the unitary representation  $\lambda: G \rightarrow U(\ell^2(G))$  by

$$(\lambda(g)\xi)(s) = (\delta_g * \xi)(s) = \xi(g^{-1}s).$$

Then  $\lambda$  is called the left regular representation of  $G$ .

Its extension to  $\ell^1(G)$  satisfies  $\lambda(\varphi)\xi = \varphi * \xi$ .

**Definition 12.7.** The reduced group  $C^*$ -algebra of  $G$  is denoted  $C_r^*(G)$  (or sometimes  $C_\lambda^*(G)$ ), and is the norm-closure of  $\lambda(\ell^1(G))$  inside  $B(\ell^2(G))$ .

It coincides with the  $C^*$ -subalgebra of  $B(\ell^2(G))$  generated by the set  $\{\lambda(g)\}_{g \in G}$ .

**Definition 12.8.** The group von Neumann algebra of  $G$  is denoted  $W^*(G)$  or sometimes  $L(G)$ , and is the closure in the weak operator topology of  $\lambda(\ell^1(G))$  inside  $B(\ell^2(G))$ . That is,  $W^*(G) = C_r^*(G)''$ .

It coincides with the von Neumann subalgebra of  $B(\ell^2(G))$  generated by the set  $\{\lambda(g)\}_{g \in G}$ .

For every  $\varphi \in \ell^1(G)$ , define

$$\|\varphi\|_{\max} = \sup\{\|\pi(\varphi)\| : \pi \text{ is a representation of } \ell^1(G)\}.$$

Since  $\|\pi(\varphi)\| \leq \|\varphi\|_1$  for all  $\varphi \in \ell^1(G)$  and all representations  $\pi$ , the subset of  $\mathbb{R}$  on the right-hand side is bounded for every  $\varphi \in \ell^1(G)$ . Moreover, it is nonempty (contains the left regular representation). Hence, the supremum always exists. Moreover,  $\|\cdot\|_{\max}$  is a  $C^*$ -norm on  $\ell^1(G)$ .

**Definition 12.9.** The full group  $C^*$ -algebra of  $G$  is denoted  $C^*(G)$  and it is the completion of  $\ell^1(G)$  with respect to the  $\|\cdot\|_{\max}$ -norm.

Next, for a discrete group  $G$  and an element  $g \in G$ , we define its conjugacy class  $C_g = \{hgh^{-1} : h \in G\}$ . We say that  $G$  is ICC if  $C_g$  is infinite for every  $g \in G$ ,  $g \neq e$ . Note that if  $G$  is abelian, then  $C_g = \{g\}$  for all  $g \in G$ , so ICC groups are in some sense far away from being abelian. For example, the free group on two generators  $\mathbb{F}_2$  is ICC.

**Theorem 12.10.** *The group von Neumann algebra  $W^*(G)$  is a factor if and only if  $G$  is ICC.*

*Proof.* Coming soon. □

*Remark 12.11.* If  $G$  is nontrivial, countable, and ICC, then  $W^*(G)$  is a  $\text{II}_1$  factor.

Indeed,  $W^*(G)$  has a tracial state, namely the vector state associated with  $\delta_e$ , so  $W^*(G)$  is finite. Moreover, since  $G$  is countably infinite,  $W^*(G)$  is infinite-dimensional and separable, so  $W^*(G)$  is a  $\text{II}_1$  factor.

*Example 12.12.*  $W^*(\mathbb{Z}_2 * \mathbb{Z}_2)$  is not a factor, but  $W^*(\mathbb{Z}_2 * \mathbb{Z}_3)$  is a factor (exercise).

*Remark 12.13.* For every  $G$ , the left regular representation  $\lambda$  extends to a surjective  $*$ -homomorphism  $C^*(G) \rightarrow C_r^*(G)$ .

**Definition 12.14.** A group  $G$  is called *amenable* if  $\lambda$  is faithful on  $C^*(G)$ .

For example, all finite and abelian groups are amenable, while  $\mathbb{F}_2$  is not amenable.

*Remark 12.15.* For every  $G$ , we can define the unitary representation  $\iota: G \rightarrow \mathbb{C}$  by  $\iota(g) = 1$  for all  $g \in G$ . This gives rise to a surjective  $*$ -homomorphism  $C^*(G) \rightarrow \mathbb{C}$ , and  $\ker \iota$  is an ideal of  $C^*(G)$  of codimension one. Thus  $C^*(G)$  is never simple, unless  $G$  is trivial.

On the other hand,  $C_r^*(G)$  can be simple, which is the case, for example for  $G = \mathbb{F}_2$ . Determining when  $C_r^*(G)$  is in general hard, and is an active research area.

**Proposition 12.16.**  $C_r^*(\mathbb{Z}) \cong C(\mathbb{T})$ .

*Proof.* Coming soon. □

*Remark 12.17.* For every locally compact Hausdorff abelian group  $G$ , one can define its Pontryagin dual  $\widehat{G} = \text{Hom}(G, \mathbb{T})$  with pointwise operations and the compact-open topology. Then  $C_r^*(G) \cong C_0(\widehat{G})$  by essentially the same argument as above. In particular,  $\widehat{\mathbb{Z}} \cong \mathbb{T}$  via  $z \mapsto (n \mapsto z^n)$ . Moreover, note that for all  $G$  we have  $G \cong \widehat{\widehat{G}}$  via  $x \mapsto (\chi \mapsto \chi(x))$  for  $x \in G$  and  $\chi \in \widehat{G}$ .

### 13. TENSOR PRODUCTS

Given Hilbert spaces  $H_1, H_2$ , there is a unique inner product on  $H_1 \otimes H_2$  such that

$$\langle \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \rangle = \langle \xi_1, \eta_1 \rangle \cdot \langle \xi_2, \eta_2 \rangle.$$

Let  $H_1 \widehat{\otimes} H_2$  denote its completion (see Section 9 and [Mur90, p. 184–186]).

Given  $x_1 \in B(H_1)$  and  $x_2 \in B(H_2)$ , there exists unique element in  $B(H_1 \widehat{\otimes} H_2)$ , denoted  $x_1 \otimes x_2$  such that

$$(x_1 \otimes x_2)(\xi_1 \otimes \xi_2) = x_1 \xi_1 \otimes x_2 \xi_2$$

for all  $\xi_i \in H_i$ . Moreover,  $\|x_1 \otimes x_2\| = \|x_1\| \cdot \|x_2\|$  (see Section 9 and [Mur90, Lemma 6.3.2]).

Let  $A_1$  and  $A_2$  be  $*$ -algebras. There exists unique multiplication and involution on  $A_1 \otimes A_2$  such that (see [Mur90, p. 188])

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = a_1 b_1 \otimes a_2 b_2 \quad (a_1 \otimes a_2)^* = a_1^* \otimes a_2^*.$$

Let  $\varphi: A_1 \rightarrow C$  and  $\psi: A_2 \rightarrow C$  be  $*$ -homomorphisms such that  $\varphi(a_1)\psi(a_2) = \psi(a_2)\varphi(a_1)$  for all  $a_1 \in A_1$  and  $a_2 \in A_2$ . Then there is a unique  $*$ -homomorphism

$\pi: A_1 \otimes A_2 \rightarrow C$  such that  $\pi(a_1 \otimes a_2) = \varphi(a_1)\psi(a_2)$ . Indeed, the map  $(a_1, a_2) \mapsto \varphi(a_1)\psi(a_2)$  is bilinear, so it induces a map  $A_1 \otimes A_2 \rightarrow C$  (see [Mur90, p. 189]).

Henceforth  $A_1 \otimes A_2$  will always denote the *algebraic* tensor product, even when  $A_1$  and  $A_2$  are  $C^*$ -algebras.

**Theorem 13.1.** *Suppose that  $\varphi: A_1 \rightarrow B(H_1)$  and  $\psi: A_2 \rightarrow B(H_2)$  are representations of  $C^*$ -algebras  $A_1$  and  $A_2$ . There exists a unique  $*$ -homomorphism  $\pi: A_1 \otimes A_2 \rightarrow B(H_1 \widehat{\otimes} H_2)$  such that  $\pi(a_1 \otimes a_2) = \varphi(a_1) \otimes \psi(a_2)$ . If  $\varphi$  and  $\psi$  are both injective, then  $\pi$  is injective. We write  $\pi = \varphi \otimes \psi$ .*

*Proof.* See [Mur90, Theorem 6.3.3]. □

**Definition 13.2.** Let  $A_1$  and  $A_2$  be  $C^*$ -algebras, and let  $\varphi$  and  $\psi$  denote the universal GNS-representations of  $A_1$  and  $A_2$ , respectively, and set  $\pi = \varphi \otimes \psi$ . For  $x \in A_1 \otimes A_2$ , set  $\|x\|_* = \|\pi(x)\|$ . This norm is called the spatial  $C^*$ -norm on  $A_1 \otimes A_2$ . The completion of  $A_1 \otimes A_2$  with respect to  $\|\cdot\|_*$  is denoted  $A_1 \otimes_* A_2$ , and called the spatial tensor product.

We note that  $\|a_1 \otimes a_2\|_* = \|a_1\| \cdot \|a_2\|$ .

*Remark 13.3.* In general, there are more than one  $C^*$ -norm on  $A_1 \otimes A_2$ . If  $\gamma$  is a  $C^*$ -norm on  $A_1 \otimes A_2$ , we denote the completion with respect to  $\gamma$  by  $A_1 \otimes_\gamma A_2$ .

**Theorem 13.4.** *Let  $A_1$  and  $A_2$  be nonzero  $C^*$ -algebras, and let  $\gamma$  be a  $C^*$ -norm on  $A_1 \otimes_\gamma A_2$ . Let  $\pi: A_1 \otimes_\gamma A_2 \rightarrow B(H)$  be a nondegenerate representation. There exist unique  $\varphi: A_1 \rightarrow B(H)$  and  $\psi: A_2 \rightarrow B(H)$  such that  $\pi(a_1 \otimes a_2) = \varphi(a_1)\psi(a_2) = \psi(a_2)\varphi(a_1)$ . We write  $\varphi = \pi|_{A_1}$  and  $\psi = \pi|_{A_2}$ .*

*Proof.* See [Mur90, Theorem 6.3.5]. □

**Corollary 13.5.** *Let  $A_1$  and  $A_2$  be  $C^*$ -algebras and  $\gamma$  a  $C^*$ -seminorm on  $A_1 \otimes A_2$ . Then  $\gamma(a_1 \otimes a_2) \leq \|a_1\| \cdot \|a_2\|$ .*

*Proof.* See [Mur90, Corollary 6.3.6]. □

Let  $\Gamma$  be the set of  $C^*$ -norms and for each  $x \in A_1 \otimes A_2$ , set

$$\|x\|_{\max} = \sup_{\gamma \in \Gamma} \gamma(x).$$

Define the maximal tensor product  $A_1 \otimes_{\max} A_2$  as the completion of  $A_1 \otimes A_2$  with respect to  $\|\cdot\|_{\max}$ .

**Definition 13.6.** A  $C^*$ -algebra  $A$  is called *nuclear* if for all  $C^*$ -algebras  $B$ , there is a unique  $C^*$ -norm on  $A \otimes B$ .

*Remark 13.7.* The spatial norm is the smallest  $C^*$ -norm on  $A \otimes B$ , so  $A$  is nuclear if and only if  $A \otimes_* B = A \otimes_{\max} B$  for all  $C^*$ -algebras  $B$ .

Moreover, since the spatial norm is minimal, if  $\gamma$  is any  $C^*$ -norm on  $A_1 \otimes A_2$ , then

$$\|a_1\| \cdot \|a_2\| = \|a_1 \otimes a_2\|_* \leq \gamma(a_1 \otimes a_2)$$

for all  $a_1 \in A_1$  and  $a_2 \in A_2$ , that is, combined with the above corollary, we have that  $\gamma(a_1 \otimes a_2) = \|a_1\| \cdot \|a_2\|$  for all  $a_1 \in A_1$  and  $a_2 \in A_2$ .

*Examples 13.8.* Some tensor products of familiar algebras:

- (i) Every finite-dimensional  $C^*$ -algebra is nuclear and  $M_n(\mathbb{C}) \otimes_\gamma A = M_n(A)$  for all  $\gamma$ .
- (ii) Every commutative  $C^*$ -algebra is nuclear and  $C_0(X) \otimes_\gamma A = C_0(X, A)$  for all  $\gamma$ .
- (iii)  $C^*(G)$  is nuclear if and only if  $C_r^*(G)$  is nuclear if and only if  $G$  is amenable.
- (iv) The compact operators  $K(H)$  on a Hilbert space  $H$  is a nuclear  $C^*$ -algebra.

#### 14. $C^*$ -DYNAMICAL SYSTEMS AND CROSSED PRODUCTS

Let  $A$  be a  $C^*$ -algebra and define the automorphism group of  $A$  as the set

$$\text{Aut}(A) = \{\varphi: A \rightarrow A \mid \varphi \text{ is a }^*\text{-isomorphism}\},$$

which is a group under composition (and is often given the point-norm topology). An action of a discrete group  $G$  on a  $C^*$ -algebra  $A$  is a homomorphism  $\alpha: G \rightarrow \text{Aut}(A)$ , i.e.,  $\alpha_g$  is an automorphism of  $A$  and  $\alpha_{gh} = \alpha_g \circ \alpha_h$  for all  $g, h \in G$ .

**Definition 14.1.** A  $C^*$ -dynamical system is a triple  $(A, G, \alpha)$ , where  $A$  is a  $C^*$ -algebra,  $G$  is a discrete group, and  $\alpha$  is an action of  $G$  on  $A$ .

**Definition 14.2.** Let  $(A, G, \alpha)$  be a  $C^*$ -dynamical system. A covariant representation of  $(A, G, \alpha)$  on a Hilbert space  $H$  is a pair  $(\pi, U)$ , where  $\pi: A \rightarrow B(H)$  is a representation of  $A$ ,  $U: G \rightarrow U(H)$  is a unitary representation of  $G$ , and

$$U(s)\pi(a)U(s)^* = \pi(\alpha_s(a))$$

for all  $s \in G$  and  $a \in A$ .

Define  $\ell^1(G, A)$  as the space of all functions  $\varphi: G \rightarrow A$  such that  $\sum_{g \in G} \|\varphi(g)\| < \infty$ . This is a Banach  $^*$ -algebra with convolution and involution given by

$$\begin{aligned} (\varphi * \psi)(t) &= \sum_{s \in G} \varphi(s) \alpha_s(\psi(s^{-1}t)), \\ \varphi^*(t) &= \alpha_t(\varphi(t^{-1})^*), \end{aligned}$$

and norm  $\|\varphi\|_1 = \sum_{g \in G} \|\varphi(g)\|$ . (Checking that this is indeed a Banach  $^*$ -algebra is a technical exercise with summation formulas.)

Let  $(\pi, U)$  be a covariant representation of a  $C^*$ -dynamical system  $(A, G, \alpha)$  on  $H$ . Define its integrated form  $\pi \times U$  as the representation of  $\ell^1(G, A)$  on  $H$  given by

$$(\pi \times U)(\varphi) = \sum_{t \in G} \pi(\varphi(t))U(t).$$

Note that  $\pi \times U$  is norm-decreasing (check!).

Let  $(A, G, \alpha)$  be a  $C^*$ -dynamical system. Let  $\pi: A \rightarrow B(H)$  be a faithful (nondegenerate) representation (e.g. the universal GNS-representation). Define a covariant representation  $(\tilde{\pi}, \lambda)$  of  $(A, G, \alpha)$  on  $\ell^2(G, H)$  by

$$\begin{aligned} (\tilde{\pi}(a)\xi)(s) &= \pi(\alpha_{s^{-1}}(a))\xi(s), \\ (\lambda(t)\xi)(s) &= \xi(t^{-1}s). \end{aligned}$$

For  $\varphi \in \ell^1(G, A)$ , define  $\|\varphi\|_r = \|(\tilde{\pi} \times \lambda)(\varphi)\|$ . This norm is independent of the choice of faithful  $\pi$ .

**Definition 14.3.** Let  $(A, G, \alpha)$  be a  $C^*$ -dynamical system. The reduced crossed product, denoted  $A \rtimes_{\alpha, r} G$  is the completion of  $\ell^1(G, A)$  with respect to  $\|\cdot\|_r$ .

For  $\varphi \in \ell^1(G, A)$ , define

$$\|\varphi\|_{\max} = \sup\{\|(\pi \times U)(\varphi)\| : (\pi, U) \text{ is a covariant representation}\}.$$

The right hand side is always bounded and nonempty, so the supremum exists.

**Definition 14.4.** Let  $(A, G, \alpha)$  be a  $C^*$ -dynamical system. The full crossed product, denoted  $A \rtimes_{\alpha} G$  is the completion of  $\ell^1(G, A)$  with respect to  $\|\cdot\|_{\max}$ .

If  $A$  is represented on  $H$ , one may think of  $A \rtimes_{\alpha} G$  as the  $C^*$ -subalgebra of  $B(\ell^2(G, H))$  generated by all the products  $\tilde{\pi}(a)\lambda(g)$  for  $a \in A$ ,  $g \in G$ .

There is always a surjection  $A \rtimes_{\alpha} G \rightarrow A \rtimes_{\alpha, r} G$ . If  $G$  is amenable, then the map is also injective. But in contrast to the group  $C^*$ -algebra case it is also possible that the full and reduced crossed products are isomorphic when  $G$  is nonamenable.

*Example 14.5.* If  $\alpha$  is trivial, i.e.,  $\alpha_g$  is the identity for all  $g \in G$ , then  $A \rtimes_{\alpha, r} G \cong A \otimes_* C_r^*(G)$  and  $A \rtimes_{\alpha} G \cong A \otimes_{\max} C^*(G)$ . In some sense, one can think of crossed products as “skew” tensor products

*Example 14.6.* If  $A = \mathbb{C}$ , then  $\alpha$  must be the identity, so  $\mathbb{C} \rtimes_{\alpha, r} G \cong C_r^*(G)$  and  $\mathbb{C} \rtimes_{\alpha} G \cong C^*(G)$ .

**Proposition 14.7.** Define the left translation action  $\text{lt}$  of  $G$  on  $C_0(G)$  by  $(\text{lt}_s f)(t) = f(s^{-1}t)$ . Let  $G$  be a finite group with  $|G| = n$ . Then  $C_0(G) \rtimes_{\text{lt}} G \cong M_n(\mathbb{C})$ .

*Proof.* Later. □

In fact, for any group  $G$ , one has that  $C_0(G) \rtimes_{\text{lt}} G \cong K(\ell^2(G))$ .

*Example 14.8.* Let  $X$  be a locally compact Hausdorff space and  $\varphi: X \rightarrow X$  a homeomorphism. Then there is an action of  $\mathbb{Z}$  on  $C_0(X)$  given by  $\alpha_n(f) = \varphi \circ f^{-n}$ , and  $(C_0(X), \mathbb{Z}, \alpha)$  is a  $C^*$ -dynamical system.

*Example 14.9.* Let  $\theta \in \mathbb{R}$  and let  $\alpha^\theta$  be the action of  $\mathbb{Z}$  on  $C(\mathbb{T})$  given by  $(\alpha_n^\theta f)(z) = f(e^{-2\pi i n \theta} z)$ . Then  $(C(\mathbb{T}), \mathbb{Z}, \alpha^\theta)$  is a  $C^*$ -dynamical system and its crossed product, denoted  $A_\theta$ , is called the rotation algebra. If  $\theta \notin \mathbb{Z}$ , it is also often called a noncommutative two-torus.

Define  $M: C(\mathbb{T}) \rightarrow B(L^2(\mathbb{T}))$  by  $(M(h)f)(z) = h(z)f(z)$  and  $U: \mathbb{Z} \rightarrow B(L^2(\mathbb{T}))$  by  $(U_n f)(z) = f(e^{-2\pi i n \theta} z)$ . Then  $(M, U)$  is a covariant representation of  $(C(\mathbb{T}), \mathbb{Z}, \alpha^\theta)$  on  $L^2(\mathbb{T})$ .

**Theorem 14.10.** Let  $\theta \in \mathbb{R} \setminus \mathbb{Q}$ , consider the  $C^*$ -dynamical system  $(C(\mathbb{T}), \mathbb{Z}, \alpha^\theta)$ , and its crossed product algebra  $A_\theta$ .

- (i)  $A_\theta$  is generated by two unitaries  $u, v$  satisfying  $uv = e^{2\pi i \theta} vu$ .
- (ii) If  $H$  is any Hilbert space and  $U, V$  are unitaries in  $B(H)$  such that  $UV = e^{2\pi i \theta} VU$ , then there exists a faithful representation  $L: A_\theta \rightarrow B(H)$  such that  $L(u) = U$  and  $L(v) = V$ .
- (iii)  $A_\theta$  is simple.

*Proof.* Later. □

*Remark 14.11.*  $A_\theta \cong A_{\theta'}$  if and only if  $\theta + \theta' \in \mathbb{Z}$  or  $\theta - \theta' \in \mathbb{Z}$ .

15. UNIVERSAL  $C^*$ -ALGEBRAS BY GENERATORS AND RELATIONS

*Remark 15.1.* In this course, we restrict to unital algebras and algebraic relations.

Let  $G = \{x_i\}_{i \in I}$  be a set of generators. By a set of relations  $R$  for  $G$ , we mean a set of polynomials in  $\{x_i, x_i^*\}_{i \in I}$  (with complex coefficients). A representation of  $(G, R)$  is a set of operators  $\{T_i, T_i^*\}_{i \in I} \subseteq B(H)$  for some Hilbert space  $H$ , satisfying all the relations in  $R$ . Let  $A$  denote the (unital, complex) free  $*$ -algebra on  $G$  (i.e., consisting of all polynomials in  $\{x_i, x_i^*\}_{i \in I}$ ). For every  $x \in A$ , set

$$\rho(x) = \sup\{\|\pi(x)\| : \pi \text{ is a representation of } (G, R)\}.$$

If this supremum exists for every  $x \in A$ , then it defines a  $C^*$ -seminorm on  $A$ . We then divide out by all elements of norm zero to get a pre- $C^*$ -algebra, and then complete it to get a  $C^*$ -algebra (see Remark 15.4). This resulting  $C^*$ -algebra is denoted  $C^*(G, R)$  and called the universal  $C^*$ -algebra of  $(G, R)$ .

*Remark 15.2.* Note that:

- Universal  $C^*$ -algebras do not always exist.
- If it exists, it will in many cases just be  $\mathbb{C}$ .
- Every unital  $C^*$ -algebra is isomorphic to  $C^*(G, R)$  for some  $(G, R)$ .
- Universal  $C^*$ -algebras are especially interesting when they have a simple description.

Suppose that  $(G, R)$  are generators and relations such that  $C^*(G, R)$  exists, and suppose that  $\{T_i, T_i^*\}_{i \in I}$  are operators on a Hilbert space  $H$  satisfying the relations from  $R$ . Then there exists a  $*$ -homomorphism  $\pi: C^*(G, R) \rightarrow B(H)$  such that  $\pi(x_i) = T_i$  for all  $i \in I$ . Moreover,  $C^*(G, R)$  satisfies the following universal property: whenever  $B$  is any other  $C^*$ -algebra generated by set  $\{x'_i\}_{i \in I}$  satisfying the relations from  $R$ , then there exists a unique surjective  $*$ -homomorphism  $\pi: C^*(G, R) \rightarrow B$  such that  $\pi(x_i) = x'_i$  for all  $i \in I$ .

*Examples 15.3.* Consider the following examples:

- (i)  $C^*(\mathbb{Z}) \cong C_r^*(\mathbb{Z}) \cong C(\mathbb{T})$  is the universal  $C^*$ -algebra  $C^*(G, R)$  with  $G = \{u\}$  and  $R = \{uu^* = u^*u = 1\}$ .
- (ii) Let  $H$  be a discrete group. Then the full group  $C^*$ -algebra  $C^*(H)$  is the universal  $C^*$ -algebra  $C^*(G, R)$  with  $G = \{U_h : h \in H\}$  and

$$R = \{U_g U_h = U_{gh}, U_h^* = U_{h^{-1}}, U(e) = 1, U_h U_h^* = U_h^* U_h = 1 \text{ for all } g, h \in H\}.$$

(It is possible to describe the above with fewer relations.)

- (iii) Let  $\theta \in \mathbb{R}$ , and let  $A_\theta$  denote the rotation algebra (the (non-)commutative two-torus). Then  $A_\theta$  is the universal  $C^*$ -algebra  $C^*(G, R)$  with  $G = \{u, v\}$  and  $R = \{uu^* = u^*u = vv^* = v^*v = 1, uv = e^{2\pi i \theta} vu\}$ . Note that  $A_\theta$  is commutative if and only if  $\theta \in \mathbb{Z}$ .
- (iv) The Toeplitz algebra  $\mathcal{T}$  is the universal  $C^*$ -algebra  $C^*(G, R)$  with  $G = \{v\}$  and  $R = \{v^*v = 1\}$ .

- (v) For  $n \geq 2$ , the Cuntz algebras  $\mathcal{O}_n$  is the universal  $C^*$ -algebra  $C^*(G, R)$  with  $G = \{s_i\}_{i=1}^n$

$$R = \{s_i^* s_i = 1 \text{ for all } i \text{ and } \sum_{i=1}^n s_i s_i^* = 1\}.$$

- (vi) Let  $G = \{v, w\}$  and  $R = \{vv^* = w, w^2 = v, w^*w = 1\}$ . Then  $C^*(G, R) = \mathbb{C}1$ .  
 (vii) Let  $G = \{v\}$  and  $R = \emptyset$ . Then the universal  $C^*$ -algebra of  $(G, R)$  does not exist.  
 (viii) Crossed product by  $\mathbb{Z}$ .  
 (ix) Tensor products.

*Remark 15.4.* Let  $A$  be a  $*$ -algebra. We say that a seminorm  $\rho$  on  $A$  is a  $C^*$ -seminorm if it satisfies

$$\rho(xy) \leq \rho(x)\rho(y) \quad \rho(x) = \rho(x^*) \quad \rho(x^*x) = \rho(x)^2.$$

Let  $N := p^{-1}(0)$ . Then  $N$  is a self-adjoint ideal of  $A$ , so the quotient  $A/N$  is a pre- $C^*$ -algebra (i.e., a  $*$ -algebra with a  $C^*$ -norm). The usual Banach space completion of  $A/N$  is then a  $C^*$ -algebra, sometimes called the enveloping algebra of  $(A, \rho)$ .

## 16. DIRECT LIMITS OF $C^*$ -ALGEBRAS

Let  $(A_n)_{n=1}^\infty$  be a sequence of  $*$ -algebras, and  $(\varphi_n)_{n=1}^\infty$  a sequence of  $*$ -homomorphisms  $\varphi_n: A_n \rightarrow A_{n+1}$ . For  $n \leq m$ , define  $\varphi_{n,m}: A_n \rightarrow A_m$  by composition, and set  $\varphi_{n,n+1} = \varphi_n$  and  $\varphi_{n,n} = \text{id}$ . Thus, if  $n \leq m \leq k$ , we have  $\varphi_{n,k} = \varphi_{m,k} \varphi_{n,m}$ .

The product  $\prod_{n=1}^\infty A_n$  is a  $*$ -algebra under pointwise operations. Define  $A \subseteq \prod_{n=1}^\infty A_n$  as all elements  $x = (x_n)_{n=1}^\infty \in A$  for which there exists an  $N$  such that  $x_{n+1} = \varphi_n(x_n)$  for all  $n \geq N$ . Then one can check that  $A$  is a  $*$ -subalgebra of  $\prod_{n=1}^\infty A_n$ .

Moreover, for  $x, y \in A$ , we write  $x \sim y$  if there exists some  $N$  such that  $x_n = y_n$  for all  $n \geq N$ , i.e., if their sequences eventually become equal. Then  $\sim$  is an equivalence relation on  $A$  and  $A/\sim$  is a  $*$ -algebra, which is the *algebraic* direct limit of the system  $(A_n, \varphi_n)_{n=1}^\infty$ .

Now, let us turn to  $C^*$ -algebras. Let  $(A_n, \varphi_n)_{n=1}^\infty$  be a system of  $C^*$ -algebras and  $*$ -homomorphisms. As above, we define the  $*$ -subalgebra  $A$  of  $\prod_{n=1}^\infty A_n$ . Let  $x \in A$ , and let  $N$  be such that  $x_{n+1} = \varphi_n(x_n)$  for all  $n \geq N$ . Since each  $\varphi_n$  is a  $*$ -homomorphism, it is norm-decreasing for all  $n$ , so we have that

$$\|x_{n+1}\| = \|\varphi_n(x_n)\| \leq \|x_n\|$$

for all  $n \geq N$ . That is,  $(\|x_n\|)_{n=1}^\infty$  is eventually decreasing and bounded below, so it converges. We set

$$\rho(x) = \lim_{n \rightarrow \infty} \|x_n\|,$$

and then  $x \mapsto \rho(x)$  is a  $C^*$ -seminorm on  $A$ .

**Definition 16.1.** The direct limit (sometimes called the inductive limit) of the system  $(A_n, \varphi_n)_{n=1}^\infty$  is the enveloping algebra (see Remark 15.4) of  $(A, \rho)$ , where  $A \subseteq \prod_{n=1}^\infty A_n$  is defined as above. We denote the direct limit by  $\varinjlim (A_n, \varphi_n)$  or just  $\varinjlim A_n$  if the maps are understood.

Further, define maps  $\Phi_n: A_n \rightarrow A$  by

$$(\Phi_n(x))_k = \begin{cases} 0 & \text{if } k < n, \\ \varphi_{n,k} & \text{if } k \geq n, \end{cases}$$

and let  $\tilde{\Phi}_n: A_n \rightarrow \varinjlim A_n$  be the composition, i.e.,  $A_n \rightarrow A \rightarrow A/N \rightarrow \overline{A/N} = \varinjlim A_n$ .

Then  $\tilde{\Phi}_n(A_n) \subseteq \tilde{\Phi}_{n+1}(A_{n+1})$ , and

$$\bigcup_{n=1}^{\infty} \tilde{\Phi}_n(A_n)$$

is dense in  $\varinjlim A_n$ . Henceforth, we omit the tilde, and write  $\Phi_n$  for the map  $A_n \rightarrow \varinjlim A_n$ . We have that  $\Phi_n = \Phi_{n+1} \circ \varphi_n$  for all  $n$ .

**Theorem 16.2.** *Given  $(A_n, \varphi_n)_{n=1}^{\infty}$ ,  $\varinjlim(A_n, \varphi_n)$ , and  $(\Phi_n)_{n=1}^{\infty}$  as above.*

(i) *Let  $x \in A_n$ ,  $y \in A_m$  such that  $\Phi_n(x) = \Phi_m(y)$ . For all  $\varepsilon > 0$ , there exists  $k \geq n, m$  such that  $\|\varphi_{n,k}(x) - \varphi_{m,k}(y)\| < \varepsilon$ .*

(ii) *If  $A'$  and  $(\Phi'_n: A_n \rightarrow A')_{n=1}^{\infty}$  are such that  $\Phi'_n = \Phi'_{n+1} \circ \varphi_n$  for all  $n$ , then there exists a unique  $*$ -homomorphism  $\Phi: \varinjlim(A_n, \varphi_n) \rightarrow A'$  such that  $\Phi'_n = \Phi \circ \Phi_n$  for all  $n$ .*

*Proof.* See [Mur90, Theorem 6.1.1 and 6.1.2].  $\square$

**Example 16.3.** If  $A$  is a  $C^*$ -algebra and  $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq A_{n+1} \cdots$  is an increasing sequence of  $C^*$ -subalgebras, such that the union  $\bigcup_{n=1}^{\infty} A_n$  is dense in  $A$ , then  $\varinjlim(A_n, \iota_n)$ , where  $\iota_n$  is the inclusion map.

If in addition all the  $A_n$ 's are finite-dimensional, then  $A$  is called an approximately finite-dimensional algebra, or more commonly, just an AF-algebra (these are discussed in the next section).

**Theorem 16.4.** *Suppose that  $A$  is an increasing union of simple  $C^*$ -subalgebras  $A_n$ . Then  $A$  is simple.*

*Proof.* See [Mur90, Theorem 6.1.3 and 6.1.4], but the proof given in class is maybe a little bit more direct.  $\square$

## 17. APPROXIMATELY FINITE-DIMENSIONAL $C^*$ -ALGEBRAS

First, we discuss finite-dimensional  $C^*$ -algebras.

**Lemma 17.1.** *Let  $\pi$  is an irreducible representation of a (nonzero)  $C^*$ -algebra  $A$  on a finite-dimensional Hilbert space  $H$ . Then  $\pi(A) = M_{\dim H}(\mathbb{C})$ .*

*Proof.* Let  $n = \dim H$ . Since  $\pi$  is irreducible, we have  $\pi(A)' = \mathbb{C}1$ , and thus  $\pi(A)'' = \mathbb{C}1' = B(H) \cong B(\mathbb{C}^n) = M_n(\mathbb{C})$ . On a finite-dimensional Hilbert space, the strong and operator topologies all coincide, so by invoking the bicommutant theorem, we get

$$\pi(A) = \overline{\pi(A)}^{\text{op}} = \overline{\pi(A)}^{\text{SOT}} = \pi(A)'' = M_n(\mathbb{C}). \quad \square$$

**Proposition 17.2.** *Let  $A$  be a (nonzero) finite-dimensional  $C^*$ -algebra. Then  $A$  is simple if and only if  $A \cong M_n(\mathbb{C})$  for some  $n \geq 1$ .*



*Proof.* If  $A \cong M_n(\mathbb{C}) = B(\mathbb{C}^n) = K(\mathbb{C}^n)$ , then  $A$  is simple, since the compact operators on any Hilbert space is a simple  $C^*$ -algebra.

If  $A$  is simple, then pick an irreducible representation  $\pi$  of  $A$  on a Hilbert space  $H$ . Since  $\pi$  is irreducible, it is cyclic, so there exists  $\xi \in H$  such that  $\overline{\pi(A)\xi} = H$ . Thus, since  $A$  is finite-dimensional, we must have that  $H$  is finite-dimensional. It follows from Lemma 17.1 that  $\pi(A) = M_n(\mathbb{C})$  for some  $n \geq 1$ . Moreover, as  $A$  is assumed to be simple, then  $\ker \pi = \{0\}$ , so  $\pi$  is also faithful, and hence  $\pi$  gives an isomorphism  $A \rightarrow M_n(\mathbb{C})$ .  $\square$

**Theorem 17.3.** *Every (nonzero) finite-dimensional  $C^*$ -algebra is a direct sum of matrix algebras.*

*Proof.* There exist irreducible representations  $\pi_1, \pi_2, \dots, \pi_N$  of  $A$  such that  $\pi = \pi_1 \oplus \pi_2 \oplus \dots \oplus \pi_N$  is faithful. Indeed, ... (to be continued)  $\square$

**Corollary 17.4.** *Every (nonzero) finite-dimensional  $C^*$ -algebra is unital (whether  $\{0\}$  is unital or not is a somewhat open to opinion).*

**Definition 17.5.** AF-algebras

*Remark 17.6.* Let  $A$  be an AF-algebra, and  $J$  an ideal of  $A$ . Then both  $J$  and  $A/J$  are AF-algebras. AF-algebras are always nuclear.

*Remark 17.7.* AF-algebras has a lot of projections. For example, a commutative  $C^*$ -algebra  $C_0(X)$  is an AF-algebra if and only if  $X$  is totally disconnected.

*Example 17.8.* Let  $H$  be an infinite-dimensional separable Hilbert space. Then the compact operators  $K(H)$  is a two-sided closed ideal of  $B(H)$ . Let  $F(H)$  be the finite-rank operators of  $B(H)$ . Then  $F(H)$  is a two-sided selfadjoint (nonclosed) ideal of  $B(H)$ . If  $J$  is any nonzero, proper, two-sided, selfadjoint, (not necessarily closed) ideal of  $B(H)$ , then  $F(H) \subseteq J \subseteq K(H)$ . Thus,  $K(H)$  is the *only* nonzero, proper two-sided closed ideal of  $B(H)$ .

Let  $P_n$  be a sequence of rank  $n$  operators converging strongly to the identity in  $B(H)$ . Set  $A_n = P_n K(H) P_n$ . Then  $A_n \simeq M_n(\mathbb{C})$ , and  $K(H)$  is the direct limit of the system  $(M_n(\mathbb{C}), \varphi_n)$ , where  $\varphi_n$  maps  $M_n(\mathbb{C})$  into the upper left corner of  $M_{n+1}(\mathbb{C})$ . Thus  $K(H)$  is an AF-algebra.

Moreover,  $K(H)$  is simple (to be continued...).

**Definition 17.9.** A  $C^*$ -algebra  $A$  is called a uniformly hyperfinite algebra (UHF-algebra), if it is unital and has an increasing sequence of simple finite-dimensional  $C^*$ -subalgebras containing the unit of  $A$ , such that its union is dense in  $A$ .

Let  $(k_n)_{n=1}^\infty$  be a sequence of natural numbers  $\geq 2$  such that  $k_n$  divides  $k_{n+1}$  for all  $n$ . Let  $A_n = M_{k_n}$ , and let  $\varphi_n: A_n \rightarrow A_{n+1}$  be the embedding  $k_{n+1}/k_n$  copies of  $M_{k_n}$  into the diagonal of  $M_{k_{n+1}}$ . Let  $A = \varinjlim A_n$ . Then  $A$  is an UHF-algebra, and every UHF-algebra is isomorphic to one of this form.

*Remark 17.10.* Supernatural numbers form a complete invariant for the UHF-algebras. They also form a complete invariant for the Bunce-Deddens algebras. The Bunce-Deddens algebras also have a direct limit description of algebras  $M_n(C(\mathbb{T}))$ .

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