FREE NILPOTENT GROUPS ARE C^*-SUPERRIGID

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Abstract. The free nilpotent group $G_{m,n}$ of class m and rank n is the free object on n generators in the category of nilpotent groups of class at most m. We show that $G_{m,n}$ can be recovered from its reduced group C^*-algebra, in the sense that if H is any group such that $C_r^*(H)$ is isomorphic to $C_r^*(G_{m,n})$, then H must be isomorphic to $G_{m,n}$.

Introduction

Group C^*-algebras play an important role in the theory of operator algebras. A natural question to ask, yet not much studied, is to what extent a group can be recovered from its (reduced) group C^*-algebra. The analog problem for group von Neumann algebras has received some attention in the last few years, but to this day there are less than a handful of results available, the first one presented in [4]. A group G is called W^*-superrigid if it can be recovered from its group von Neumann algebra $L(G)$, that is, if H is any group such that $L(H) \cong L(G)$, then $H \cong G$. The group von Neumann algebra of any nontrivial countable amenable group with infinite conjugacy classes is isomorphic to the hyperfinite II_1 factor, so in general, much of the group structure is lost in the construction. However, examples of W^*-superrigid groups are known to exist, in particular, some classes of generalized wreath products [4, 1] and amalgamated free products [2].

Inspired by this terminology, a group G is said to be C^*-superrigid if $C_r^*(H) \cong C_r^*(G)$ implies that $H \cong G$. It has been known for some time that torsion-free abelian groups are C^*-superrigid [7], and only very recently, it was shown that certain torsion-free virtually abelian groups, so-called Bieberbach groups, are C^*-superrigid [5], providing the first result for nonabelian groups. In a somewhat different direction, specific examples of amalgamated free products were proven to be C^*-superrigid in [2], including a continuum of groups that can contain torsion. Returning to the amenable situation, it is conjectured that all finitely generated torsion-free nilpotent groups are C^*-superrigid, and important progress towards solving this problem was made in [3], where the authors gave a positive answer in the case of nilpotency class 2.

We remark that there is no known example of a torsion-free group that is not C^*-superrigid. For more background on the topic, see [5, 3] and references therein.

In this short note, we show that also the free nilpotent groups are C^*-superrigid.

1. Preliminaries and various results

Let G be a discrete group. As usual, $C^*(G)$ denotes the full group C^*-algebra of G, and we let $g \mapsto u_g$ be the canonical inclusion of G into $C^*(G)$. The left regular representation λ of G on $\ell^2(G)$ is given by $\lambda_g \delta_h = \delta_{gh}$ for all $g, h \in G$, and the reduced group C^*-algebra $C_r^*(G)$ of G is the C^*-subalgebra of $B(\ell^2(G))$ generated by the image of λ. It follows that λ induces a homomorphism of $C^*(G)$ onto $C_r^*(G)$, mapping u_g to λ_g for all $g \in G$. Moreover,

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it is well-known that if $C^*(G) \cong C^*_r(G)$, then λ must be faithful, and in this case, G is called amenable, and we use λ to identify $C^*(G)$ with $C^*_r(G)$.

The subgroup G' of G generated by all the elements $ghg^{-1}h^{-1}$ for $g, h \in G$ is called the commutator (or derived) subgroup of G. It is normal in G, and the quotient $G_{\mathrm{ab}} = G/G'$ is an abelian group, called the abelianization of G. The group G_{ab} is the largest abelian quotient of G, that is, whenever N is a normal subgroup of G and G/N is abelian, $G' \subseteq N$.

Let $\widetilde{\pi}_{\mathrm{ab}} \colon C^*(G) \to C^*(G_{\mathrm{ab}})$ denote the homomorphism induced by the quotient map $\pi_{\mathrm{ab}} \colon G \to G_{\mathrm{ab}}$. Note that π_{ab} induces a map $C^*_r(G) \to C^*_r(G_{\mathrm{ab}}) = C^*(G_{\mathrm{ab}})$ if and only if G', or equivalently, G is amenable.

For a C^*-algebra A, the commutator ideal $\mathcal{J}$ of A is the ideal generated by all elements $xy - yx$ for $x, y \in A$. Let $\phi \colon A \to A/\mathcal{J}$ denote the quotient map. The Gelfand spectrum Γ_A of A is given by

$$\Gamma_A = \{\text{nonzero algebra homomorphisms } \gamma \colon A \to \mathbb{C}\}.$$

For every $\gamma \in \Gamma_A$, we clearly have $\gamma(xy - yx) = 0$ for all $x, y \in A$, and thus $\mathcal{J} \subseteq \ker \gamma$. If $\rho \in \Gamma_{A/\mathcal{J}}$, then $\rho \circ \phi$ belongs to Γ_A, and every $\gamma \in \Gamma_A$ defines an element $\rho \in \Gamma_{A/\mathcal{J}}$ given by $\rho(x + \mathcal{J}) = \gamma(x)$. Together, this gives that $\Gamma_{A/\mathcal{J}} = \Gamma_A$. Moreover, if $x \notin \mathcal{J}$, then $0 \neq \phi(x) \in A/\mathcal{J}$, which is a commutative C^*-algebra, so there exists $\rho \in \Gamma_{A/\mathcal{J}}$ such that $\rho(\phi(x)) \neq 0$. That is, $x \notin \ker \rho \circ \phi$, and as explained above, $\rho \circ \phi \in \Gamma_A$. We conclude that

$$\mathcal{J} = \bigcap_{\gamma \in \Gamma_A} \ker \gamma. \tag{1}$$

Lemma 1.1. *The commutator ideal $\mathcal{J}$ of $C^*(G)$ coincides with the kernel of $\widetilde{\pi}_{\mathrm{ab}}$.*

Proof. First, since $C^*(G_{\mathrm{ab}})$ is commutative, $\ker \widetilde{\pi}_{\mathrm{ab}}$ must contain all commutators in $C^*(G)$, and thus $\mathcal{J} \subseteq \ker \widetilde{\pi}_{\mathrm{ab}}$. Next, we note that

$$\Gamma_{C^*(G_{\mathrm{ab}})} = \mathrm{Hom}(G_{\mathrm{ab}}, \mathbb{T}) = \mathrm{Hom}(G, \mathbb{T}) = \Gamma_{C^*(G)}.$$

The second identification is given by $\chi' \mapsto \chi' \circ \pi_{\mathrm{ab}}$ for $\chi' \in \mathrm{Hom}(G_{\mathrm{ab}}, \mathbb{T})$, and the inverse by $\chi \mapsto \chi'$ for $\chi \in \mathrm{Hom}(G, \mathbb{T})$, where $\chi'(g + G') = \chi(g)$. The last identification is the usual integrated form, with inverse $\gamma \mapsto \chi$ for $\gamma \in \Gamma_{C^*(G)}$, where $\chi(g) = \gamma(u_g)$; and the first equality is similar. Combined, the first and last space is identified via $\gamma' \mapsto \gamma' \circ \widetilde{\pi}_{\mathrm{ab}}$ for $\gamma' \in \Gamma_{C^*(G_{\mathrm{ab}})}$.

Thus, if $x \notin \mathcal{J}$, then by (1) there is $\gamma \in \Gamma_{C^*(G)}$ such that $\gamma(x) \neq 0$. Since $\gamma = \gamma' \circ \widetilde{\pi}_{\mathrm{ab}}$ for some $\gamma' \in \Gamma_{C^*(G_{\mathrm{ab}})}$, we have $\gamma'(\widetilde{\pi}_{\mathrm{ab}}(x)) \neq 0$, and hence $x \notin \ker \widetilde{\pi}_{\mathrm{ab}}$. □

The following result is due to [7].

Proposition 1.2. *Suppose that G is torsion-free and abelian and let H be any group such that $C^*(H) \cong C^*(G)$. Then $H \cong G$.*

Corollary 1.3. *If H is any group such that $C^*(H) \cong C^*(G)$, then $C^*(H_{\mathrm{ab}}) \cong C^*(G_{\mathrm{ab}})$. In particular, if G_{ab} is torsion-free, then $H_{\mathrm{ab}} \cong G_{\mathrm{ab}}$.*

Proof. Any isomorphism $C^*(H) \cong C^*(G)$ takes the commutator ideal of $C^*(H)$ to the commutator ideal of $C^*(G)$, and thus, the quotients $C^*(H_{\mathrm{ab}})$ and $C^*(G_{\mathrm{ab}})$ must be isomorphic. The last statement now follows from Proposition 1.2. □

The upper central sequence of G, denoted $Z_0 \subset Z_1 \subset Z_2 \subset \cdots$, is defined by $Z_0 = \{e\}$, $Z_1 = Z(G)$, and for all $i \geq 0$,

$$Z_{i+1} = \{g \in G : [g, h] \in Z_i \text{ for all } h \in G\}.$$

In particular, we remark that Z_i is a normal subgroup of Z_{i+1} and $Z_{i+1}/Z_i = Z(G/Z_i)$ for all $i \geq 0$. If there exists an m such that $G = Z_m$, then G is called a nilpotent group, and the smallest such m is said to be the class of G.

Lemma 1.4. *Suppose that G is a nilpotent group and let $S \subseteq G$ be a set such that $\pi_{\mathrm{ab}}(S)$ generates G_{ab}. Then S generates G.*

Proof. Let m be the nilpotency class of G, and let $\{e\} = Z_0 \subset Z_1 \subset \cdots \subset Z_{m-1} \subset Z_m = G$ be the upper central series of G. Denote by H the subgroup of G generated by S. For $0 \le i \le m$, set $H_i = HZ_i$. Then H_i is a subgroup of G and a normal subgroup of H_{i+1} for all i. Indeed, for $h, h' \in H$, $z_i, z_i' \in Z_i$,

$$(h'z_i)(hz_i') = h'hz_i[z_i^{-1}, h^{-1}]z_i' \in HZ_iZ_{i-1}Z_i = HZ_i = H_i$$

since $[z_i^{-1}, h^{-1}] \in Z_{i-1}$. Moreover, for $h, h' \in H$, $z_i, z_i' \in Z_i$, and $z_{i+1} \in Z_{i+1}$,

$$(hz_{i+1})(h'z_i)(hz_{i+1})^{-1} = h[z_{i+1}, h']h'z_i[z_i^{-1}, z_{i+1}]h^{-1} \in HZ_iHZ_iZ_{i-1}H = H_i.$$

If $H \neq G$, there would exist some $0 \le k < m$ such that $H_k \neq G$ and $H_{k+1} = G$. Then

$$G/H_k = H_{k+1}/H_k = HZ_{k+1}/HZ_k \cong Z_{k+1}/Z_k,$$

where the last identification is the second isomorphism theorem, and the last quotient is abelian. Thus, H_k contains the commutator subgroup G', and therefore also HG'. Since $\pi_{\mathrm{ab}}(H) = G_{\mathrm{ab}}$, then HG' equals G, so

$$G = HG' \subseteq H_k \subsetneq G,$$

which is a contradiction. Hence, we conclude that $H = G$. □

Note that the abelianization of a torsion-free nilpotent group is not necessarily torsion-free, so in general, we do not know if it can be recovered from its group C^*-algebra. E.g., consider the index 2 subgroup of the Heisenberg group given by

$$G = \begin{pmatrix} 1 & 2\mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}.$$

The abelianization of G is $\mathbb{Z}^2 \oplus (\mathbb{Z}/2\mathbb{Z})$, and all generating sets of G have at least three elements. This is in constrast to the Heisenberg group itself, whose abelianization is $\mathbb{Z}^2$, and which can be generated by two elements.

2. C^*-superrigidity of free nilpotent groups

The free nilpotent group $G_{m,n}$ of class m and rank n is the free object on n generators in the category of nilpotent groups of class at most m. It is defined by a set of generators $\{g_i\}_{i=1}^n$ subject to the relations that all commutators of length $m+1$ involving the generators are trivial, i.e., $[\cdots[[g_{i_1}, g_{i_2}], g_{i_3}]\cdots], g_{i_{m+1}}]$ is trivial for any choice of sequence of generators. For all $m \ge 1$, we have $G_{m,1} \cong \mathbb{Z}$, while $G_{m,n}$ is an m-step nilpotent group for every $n \ge 2$. As an easy example, we mention that $G_{2,2}$ is isomorphic to the Heisenberg group, and refer to [8, 6] for further details about free nilpotent groups.

The group $G_{m,n}$ satisfies the following universal property: If H is any nilpotent group of class at most m and $h_1, \ldots, h_n$ are elements in H, there exists a unique homomorphism $G_{m,n} \to H$ mapping g_i to h_i for all i.

The abelianization of $G_{m,n}$ is isomorphic to $\mathbb{Z}^n$ and π_{ab} maps g_i to the generator e_i of the i'th summand of $\mathbb{Z}^n$.

The center $Z(G_{m,n})$ of $G_{m,n}$ is a free abelian group (its rank can be computed, but it is not relevant here), and for $m, n \ge 2$ we have

$$G_{m,n}/Z(G_{m,n}) \cong G_{m-1,n}, \tag{2}$$

as seen by mapping generators to generators.

Lemma 2.1. *Let $m, n \geq 2$, and let H be a nilpotent group of class at most m that can be generated by n elements. Suppose that $H/Z(H) \cong G_{m-1,n}$. Then $H \cong G_{m,n}$.*

Proof. The universal property of $G_{m,n}$ means that there exists a surjective map $\varphi\colon G_{m,n} \to H$. Clearly, $\varphi(Z(G_{m,n})) \subseteq Z(H)$, and we set $K = \varphi^{-1}(Z(H))$. Consider the maps

$$G_{m,n}/Z(G_{m,n}) \to G_{m,n}/K \to H/Z(H),$$

given by $aZ(G_{m,n}) \mapsto aK$ and $aK \mapsto \varphi(a)Z(H)$. The composition map ψ is surjective since φ is surjective. Since finitely generated nilpotent groups are Hopfian, $G_{m-1,n} \cong G_{m,n}/Z(G_{m,n})$ does not have any proper quotient isomorphic to itself. Hence, the composition map ψ must be an isomorphism, and $K = Z(G_{m,n})$. We get the following commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & Z(G_{m,n}) \arrow[r, "i"] \arrow[d, "\varphi|_Z"', "\cong"] & G_{m,n} \arrow[r, "q"] \arrow[d, "\varphi"'] & G_{m,n}/Z(G_{m,n}) \arrow[r] \arrow[d, "\psi"', "\cong"] & 1 \\
1 \arrow[r] & Z(H) \arrow[r, "i"] & H \arrow[r, "q"] & H/Z(H) \arrow[r] & 1
\end{tikzcd}$$

By the five lemma, φ is an isomorphism. □

Theorem 2.2. *For every pair of natural numbers m and n, the free nilpotent group $G_{m,n}$ of class m and rank n is C^*-superrigid.*

Proof. The case $n = 1$ is obvious, so let $n \geq 2$. We do this by induction on m. Note first that $G_{1,n} \cong \mathbb{Z}^n$, which is C^*-superrigid (see Proposition 1.2). Let $m \geq 2$, and suppose that $G_{m-1,n}$ is C^*-superrigid. Let H be any group and assume that $C^*(H) \cong C^*(G_{m,n})$. It follows from [3, Theorem B] that H is a torsion-free nilpotent group of class m.

Moreover, $C^*(H/Z(H)) \cong C^*(G_{m,n}/Z(G_{m,n}))$ by [3, Proof of Lemma 4.2], and (2) implies that the latter is isomorphic to $C^*(G_{m-1,n})$. By the induction hypothesis, the group $G_{m-1,n}$ is C^*-superrigid, so $H/Z(H) \cong G_{m-1,n}$.

The abelianization of $G_{m,n}$ is isomorphic to $\mathbb{Z}^n$ and thus $H_{\mathrm{ab}} \cong \mathbb{Z}^n$ by Corollary 1.3. For each $1 \leq i \leq n$, choose an element s_i of H that is mapped to the generator e_i of $\mathbb{Z}^n \cong H_{\mathrm{ab}}$. If $S = \{s_i : 1 \leq i \leq n\}$, then $\pi_{\mathrm{ab}}(S)$ generates H_{ab}, so S generates H by Lemma 1.4, i.e., H can be generated by n elements.

Therefore, we apply Lemma 2.1 to conclude that $H \cong G_{m,n}$. □

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