ON TWISTED GROUP C^*-ALGEBRAS ASSOCIATED WITH FC-HYPERCENTRAL GROUPS AND OTHER RELATED GROUPS

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Abstract. We show that the twisted group C^*-algebra associated with a discrete FC-hypercentral group is simple (resp. has a unique tracial state) if and only if Kleppner's condition is satisfied. This generalizes a result of Packer for countable nilpotent groups. We also consider a larger class of groups, for which we can show that the corresponding reduced twisted group C^*-algebras have a unique tracial state if and only if Kleppner's condition holds.

1. Introduction

In this article, all groups will be considered as discrete groups. Letting $\sigma\colon G \times G \to \mathbb{T}$ denote a normalized two-cocycle on a group G, in other words, $\sigma \in Z^2(G, \mathbb{T})$, we will say that (G, σ) is *C^*-simple* (resp. has *the unique trace property*) whenever the reduced twisted group C^*-algebra $C_r^*(G, \sigma)$ is simple (resp. has a unique tracial state). If G is amenable, then we can equally consider the (full) twisted group C^*-algebra $C^*(G, \sigma)$, since $C^*(G, \sigma)$ is canonically isomorphic to $C_r^*(G, \sigma)$ in this case (cf. [27]). It is well known that a necessary condition for (G, σ) to be C^*-simple (resp. have the unique trace property) is *Kleppner's condition* [16], which says that every nontrivial σ-regular conjugacy class in G is infinite. In general, Kleppner's condition is not sufficient for (G, σ) to be C^*-simple (resp. have the unique trace property). However, for certain classes of groups, Kleppner's condition is sufficient for both these properties to hold. We will therefore say that a group G belongs to the class $\mathcal{K}_{C^*S}$ (resp. $\mathcal{K}_{UT}$) if, for every $\sigma \in Z^2(G, \mathbb{T})$, we have that (G, σ) is C^*-simple (resp. has the unique trace property) if and only if (G, σ) satisfies Kleppner's condition. Moreover, we will let $\mathcal{K}$ denote the intersection of $\mathcal{K}_{C^*S}$ and $\mathcal{K}_{UT}$, while $\mathcal{K}^{am}$ will denote the subclass of $\mathcal{K}$ consisting of amenable groups.

It is a classical fact that the class of finite groups is contained in $\mathcal{K}^{am}$; see for example [14] or [16]. Packer [22] has shown that all countable nilpotent groups belong to $\mathcal{K}^{am}$ (see also [26] for the case of abelian groups). Large families of non-amenable groups in $\mathcal{K}$ are described in [1, 2, 4]. We mention explicitly the class $\mathcal{P}$ introduced in [4], as we will refer to it later: it consists of all PH groups [24] and of all groups with property (P_{com}) [5]. In particular, all weak Powers groups [13] belong to $\mathcal{P}$, and the class $\mathcal{P}$ contains many amalgamated free products, HNN-extensions, hyperbolic groups, Coxeter groups, and lattices in semisimple Lie groups. Any group belonging to $\mathcal{P}$ is ICC (meaning that all of its nontrivial conjugacy classes are infinite), so Kleppner's condition is trivially satisfied for any $\sigma \in Z^2(G, \mathbb{T})$ when G lies in $\mathcal{P}$.

Our first aim in this paper is to show that the class $\mathcal{K}^{am}$ contains all *FC-hypercentral* groups (cf. Theorem 3.1). The class of FC-hypercentral groups [25] is quite large: it contains for instance all groups that have only finite conjugacy classes (usually called FC-groups); moreover, it contains all virtually nilpotent groups and all FC-nilpotent groups. A simple way

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to describe that a group is FC-hypercentral is to say that it has no nontrivial ICC quotient group. We will recall the equivalent definitions in Section 2.3. We just mention here that if a group G is finitely generated, then G is FC-hypercentral if and only if G is virtually nilpotent, if and only if G has polynomial growth. An interesting open problem raised by our result is whether the class $\mathcal{K}^{am}$ coincides with the class of FC-hypercentral groups.

In our proof of Theorem 3.1, we will use the observation that if a group G is amenable and (G, σ) has the unique trace property for some $\sigma \in Z^2(G, \mathbb{T})$, then (G, σ) is C^*-simple. This follows easily from the fact that $C_r^*(G, \sigma)$ has the QTS property introduced by Murphy [19] whenever G is amenable. Thus, the main burden of our proof will be to show that for an FC-hypercentral group, the unique trace property follows from Kleppner's condition. This will be achieved by streamlining and generalizing the proof of the same implication given by Packer [22] in the case of a countable nilpotent group. For completeness, we will also give another proof that for an FC-hypercentral group, the C^*-simplicity can be deduced from Kleppner's condition, by making use of a deep result of Echterhoff in [9].

A consequence of Theorem 3.1 is that if G is a countable FC-hypercentral group, $\sigma \in Z^2(G, \mathbb{T})$ and (G, σ) satisfies Kleppner's condition, then $C^*(G, \sigma) \simeq C_r^*(G, \sigma)$ belongs to the class of separable simple nuclear C^*-algebras with a unique tracial state, a class that is of particular interest in the classification program for C^*-algebras.

In the second part of this paper, we consider a larger class of groups and show that it is contained in $\mathcal{K}_{UT}$. To describe this class, we first recall [25] that any group G has a canonical normal FC-hypercentral subgroup, $FCH(G)$, called its *FC-hypercenter*. The quotient group $G/FCH(G)$ is an ICC group, that we will denote by $ICC(G)$. We will say that G belongs to the class $\mathcal{ICCP}$ when $ICC(G)$ belongs to the class $\mathcal{P}$ mentioned above. As the trivial group is the only amenable group belonging to $\mathcal{P}$, we have that the class of FC-hypercentral groups is the intersection of $\mathcal{ICCP}$ with the class of amenable groups. Our result is that the class $\mathcal{ICCP}$ is contained in $\mathcal{K}_{UT}$ (cf. Theorem 4.1). We believe that $\mathcal{ICCP}$ is also contained in $\mathcal{K}_{C^*S}$, and include a result supporting this conjecture.

2. Preliminaries

2.1. **On twisted group C^*-algebras.** Let G denote a group with identity e and let $\sigma\colon G \times G \to \mathbb{T}$ denote a normalized two-cocycle (sometimes called a multiplier) on G with values in the circle group $\mathbb{T}$. We recall that σ satisfies

$$\sigma(g,h)\sigma(gh,k) = \sigma(h,k)\sigma(g,hk)\,,$$
$$\sigma(g,e) = \sigma(e,g) = 1$$

for all $g, h, k \in G$.

The set of all such two-cocycles will be denoted by $Z^2(G, \mathbb{T})$, as in [27]. The trivial two-cocycle is simply written as 1. We will use the convention that when $\sigma = 1$, we just drop σ from all our notation.

The *left regular σ-projective unitary representation* λ_σ *of* G *on* $B(\ell^2(G))$ is given by

$$(\lambda_\sigma(g)\xi)(h) = \sigma(g, g^{-1}h)\,\xi(g^{-1}h)$$

for $\xi \in \ell^2(G)$ and $g, h \in G$. Note that

$$\lambda_\sigma(g)\,\delta_h = \sigma(g,h)\,\delta_{gh}$$

for all $g, h \in G$ (where $\delta_g(h) = 1$ if $g = h$ and $\delta_g(h) = 0$ otherwise). The *reduced twisted group C^*-algebra* and the *twisted group von Neumann algebra of* (G, σ), $C_r^*(G, \sigma)$ and $W^*(G, \sigma)$ are, respectively, the C^*-algebra and the von Neumann algebra generated by $\lambda_\sigma(G)$. The *(full) twisted group C^*-algebra of* (G, σ), $C^*(G, \sigma)$, is the enveloping C^*-algebra of the Banach *-algebra $\ell^1(G, \sigma)$, equipped with the twisted convolution and involution (see [27]).

The canonical tracial state on $C_r^*(G,\sigma)$ will be denoted by τ; it is simply given as the restriction to $C_r^*(G,\sigma)$ of the vector state associated with δ_e. As is well-known, τ is faithful and satisfies $\tau(\lambda_\sigma(g)) = 0$ for every $g \neq e$.

We recall [16, 22, 20] that $g \in G$ is called *σ-regular* if

$$\sigma(g,h) = \sigma(h,g) \text{ for every } h \in G \text{ that commutes with } g\,.$$

If g is σ-regular, then hgh^{-1} is σ-regular for all h in G, so the notion of σ-regularity makes sense for conjugacy classes in G.

As mentioned in the Introduction, the pair (G,σ) will be said to satisfy *Kleppner's condition* if every nontrivial σ-regular conjugacy class of G is infinite. It is known [16, 22, 20] that (G,σ) satisfies Kleppner's condition if and only if $W^*(G,\sigma)$ is a factor, if and only if $C_r^*(G,\sigma)$ has trivial center, if and only if $C_r^*(G,\sigma)$ is prime. It follows easily from these equivalences that Kleppner's condition is necessary for (G,σ) to be C^*-simple (resp. to have the unique trace property). On the other hand, if G is amenable, then $C_r^*(G) \simeq C^*(G)$ has a 1-dimensional *-representation [23]. Hence, if G is a nontrivial amenable ICC group, then $(G,1)$ satisfies Kleppner's condition, but neither is $(G,1)$ C^*-simple, nor does it have the unique trace property.

The following lemma, which is a slight adaptation of a technical result due to Carey and Moran ([7, Lemma 4.1]), will be important in the proof of Theorem 3.1.

Lemma 2.1. *Let G be a group and assume ψ is a tracial state on $C^*(G)$. Let $g \to u(g) \in C^*(G)$ denote the canonical embedding of G into $C^*(G)$ and let ψ_G be the function on G given by $\psi_G = \psi \circ u$. Assume that there exist $h \in G$ and a sequence $\{g_i\}_{i\in\mathbb{N}}$ in G such that*

$$\psi_G\big(g_j h^{-1} g_j^{-1} g_i h g_i^{-1}\big) = 0 \quad \textit{for every } i \neq j \textit{ in } \mathbb{N}\,. \tag{2.1}$$

Then $\psi_G(h) = 0$.

Proof. For each $N \in \mathbb{N}$, let $a_N \in C^*(G)$ be defined by $a_N = I - \overline{\psi_G(h)}\sum_{i=1}^N u\big(g_i h g_i^{-1}\big)\,.$ Then we have

$$(a_N)^*\, a_N = \Big[I - \psi_G(h) \sum_{j=1}^N u\big(g_j h^{-1} g_j^{-1}\big)\Big]\, \Big[I - \overline{\psi_G(h)} \sum_{i=1}^N u\big(g_i h g_i^{-1}\big)\Big]$$

$$= I - \overline{\psi_G(h)} \sum_{i=1}^N u\big(g_i h g_i^{-1}\big) - \psi_G(h) \sum_{j=1}^N u\big(g_j h^{-1} g_j^{-1}\big) + |\psi_G(h)|^2 \sum_{i,j=1}^N u\big(g_j h^{-1} g_j^{-1} g_i h g_i^{-1}\big)\,.$$

Using that ψ is a tracial state, we get

$$\psi\big((a_N)^*\, a_N\big) = 1 - 2N\,|\psi_G(h)|^2 + |\psi_G(h)|^2 \sum_{i,j=1}^N \psi_G\big(g_j h^{-1} g_j^{-1} g_i h g_i^{-1}\big)$$

$$= 1 - N\,|\psi_G(h)|^2 + |\psi_G(h)|^2 \sum_{i,j=1,\, i\neq j}^N \psi_G\big(g_j h^{-1} g_j^{-1} g_i h g_i^{-1}\big)\,.$$

Using (2.1), we get

$$0 \leq \psi\big((a_N)^*\, a_N\big) = 1 - N\,|\psi_G(h)|^2\,,$$

hence $|\psi_G(h)| \leq \sqrt{1/N}$. Letting $N \to \infty$, we obtain the desired conclusion. $\square$

2.2. **On the QTS property.** Let A denote a unital C^*-algebra. Following Murphy [19], A is said to have the *QTS property* if, for each proper (closed two-sided) ideal J of A, the quotient A/J admits a tracial state. As observed by Murphy, if A has the QTS property, then A is simple if and only if all its tracial states are faithful. As an immediate consequence, we get:

Theorem 2.2. *Assume that A has the QTS property and a unique tracial state, which is* faithful. *Then A is simple.*

We recall (cf. [12]) that a unital C^*-algebra is simple with at most one tracial state if and only if it has the Dixmier property. Hence, the assumptions of Theorem 2.2 imply that A has the Dixmier property. In this connection, we remark that Ozawa has recently shown [21] that the QTS property may be characterized by a weaker Dixmier type property.

The following result will be useful to us:

Corollary 2.3. *Assume that G is amenable and let $\sigma \in Z^2(G, \mathbb{T})$. Then (G, σ) is C^*-simple whenever it has the unique trace property.*

Proof. It is known (cf. [19]) that a unital C^*-algebra A has the QTS property whenever A is hypertracial (as defined in [3]). Since G is amenable if and only if $C_r^*(G, \sigma)$ is hypertracial (cf. [3]), the assertion follows from Theorem 2.2. For the ease of the reader, we sketch a direct proof that $C_r^*(G, \sigma)$ has the QTS property when G is assumed to be amenable. Let J be a proper ideal of $C_r^*(G, \sigma)$, let π denote the canonical quotient map from $C_r^*(G, \sigma)$ onto $B = C_r^*(G, \sigma)/J$, let φ denote a state on B, and set $v(g) = \pi(\lambda_\sigma(g))$ for each $g \in G$. For each $x \in B$, define $x_\varphi \in \ell^\infty(G)$ by

$$x_\varphi(g) = \varphi\big(v(g)\, x\, v(g)^*\big) \quad \text{for each } g \in G\,.$$

Now, let m be a right-invariant mean on $\ell^\infty(G)$ and define $\psi \colon B \to \mathbb{C}$ by

$$\psi(x) = m(x_\varphi) \quad \text{for each } x \in B\,.$$

Then ψ is a state on B. Moreover, as $\big(v(h)x\, v(h)^*\big)_\varphi$ is the right-translate of x_φ by h for each $h \in G$, the invariance of m gives that $\psi\big(v(h)\, xv\,(h)^*\big) = \psi(x)$ for all $h \in G$ and $x \in B$. As $\{v(h) \mid h \in G\}$ generates B as a C^*-algebra, it follows readily that ψ is tracial. □

Murphy also shows that a unital C^*-algebra A has the QTS property whenever A is exact [6] and has stable rank one (i.e., the invertible elements of A are dense in A). Now, it follows from [10] that $C_r^*(G, \sigma)$ is exact whenever G is exact. Hence, another corollary of Theorem 2.2 is:

Corollary 2.4. *Let $\sigma \in Z^2(G, \mathbb{T})$. Assume that G is exact and $C_r^*(G, \sigma)$ has stable rank one. Then (G, σ) is C^*-simple whenever it has the unique trace property.*

Not much seems to be known about conditions ensuring that $C_r^*(G, \sigma)$ has stable rank one. We plan to investigate this in a separate paper.

2.3. **On FC-hypercentral groups.** Let G be a group. We recall that the *FC-center of* G is given by

$$FC(G) = \{g \in G \mid \text{the conjugacy class of } g \text{ is finite}\}\,.$$

The FC-center of G is a normal subgroup of G, which is trivial if and only if G is ICC. The group G is said to be an *FC-group* when $FC(G) = G$.

The *upper (ascending) FC-central series* $\{F_\alpha\}_\alpha$ of G is a normal series of subgroups of G indexed by the ordinal numbers. It is defined as follows (cf. [25, Section 4.3]):

We set $F_0 = \{e\}$, $F_\alpha/F_\beta = FC(G/F_\beta)$ if $\alpha = \beta + 1$, and $F_\alpha = \bigcup_{\beta<\alpha} F_\beta$ when α is a limit ordinal number. This series eventually stabilizes and $FCH(G) = \lim_\alpha F_\alpha = \bigcup_\alpha F_\alpha$ is called the *FC-hypercenter of* G. Note that $FCH(G)$ is a normal subgroup of G since it is a union of normal subgroups.

Since $F_1 = FC(G)$, we have that $FCH(G)$ is trivial if and only if $FC(G)$ is trivial if and only if G is ICC.

A group G is called *FC-hypercentral* [25] when $FCH(G) = G$. If the upper FC-central series stabilizes to G after a finite number of steps, then G is called *FC-nilpotent.* For example, G is FC-nilpotent whenever G is virtually nilpotent (i.e., it contains a nilpotent subgroup of finite index). If G is finitely generated and FC-hypercentral, then G is FC-nilpotent; further, if G is finitely generated and FC-nilpotent, then G is a finite extension of finitely generated nilpotent subgroup. (See [17, Theorem 2 and its proof]).

As observed by Echterhoff in [9], it follows that FC-hypercentral groups have polynomial growth and are therefore amenable. Moreover, a deep result proved by Echterhoff is that G is FC-hypercentral if and only if G is amenable and every prime ideal of $C^*(G)$ is maximal. This generalizes an earlier result of Moore and Rosenberg [18], which says that any countable amenable T_1-group is FC-hypercentral. (We recall that G is called a *T_1-group* when every primitive ideal of $C^*(G)$ is maximal).

To sum up, consider the following conditions for a group G:

(i) G is virtually nilpotent.
(ii) G is FC-nilpotent.
(iii) G is FC-hypercentral.
(iv) G is an amenable T_1-group.
(v) G has polynomial growth.

In general, we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv), and (iii) $\Rightarrow$ (v). If G is countable, then (iii) $\Leftrightarrow$ (iv). (To our knowledge, it is open whether (iv) implies (iii) in general; it seems also to be unknown whether all T_1-groups have polynomial growth.) Finally, for a finitely generated group, all the five conditions are equivalent, the implication (v) $\Rightarrow$ (i) being a famous result due to Gromov [11].

Another condition one might consider is:

(vi) G is elementary amenable with subexponential growth.

Then we have (v) $\Rightarrow$ (vi). Indeed, Gromov's result gives that any finitely generated subgroup of a group with polynomial growth must be virtually nilpotent, and thus elementary amenable. Since a group is a direct limit of its finitely generated subgroups, this means that a group with polynomial growth is elementary amenable.

If G is a finitely generated group, we also have (vi) $\Rightarrow$ (v), hence conditions (i)-(vi) are all equivalent in this case. This assertion is an immediate consequence of a result due to Chou [8], which says that a finitely generated elementary amenable group have either polynomial growth or exponential growth.

In [15], Jaworski defines a group G to be *identity excluding* if the only irreducible unitary representation of G which weakly contains the 1-dimensional identity representation is the 1-dimensional identity representation itself. An interesting result in our context is [15, Theorem 4.5], which says that a countable group is FC-hypercentral if and only if it is amenable and identity excluding.

The FC-hypercenter of a group G may be described as the smallest normal subgroup of G that produces an ICC quotient group. This fact is mentioned without proof in [15, Remark 4.1]. For completeness, we give a proof of this useful characterization.

Proposition 2.5. *Let G be a group. Then the quotient group $G/FCH(G)$ is ICC. Moreover, if N is a normal subgroup of G such that G/N is ICC, then $FCH(G) \subset N$.*

Proof. Since the upper FC-central series of G stabilizes at $FCH(G)$, the first assertion is clear. The second assertion also follows from the construction. Assume that N is a normal subgroup of G such that G/N is ICC. Then the quotient map $G \to G/N$ sends $FC(G)$ to a subgroup of $FC(G/N) = \{e\}$, so one has $F_1 = FC(G) \subset N$.

Next, suppose that α and β are ordinals such that $\alpha = \beta + 1$ and $F_\beta \subset N$. Then the quotient map $G/F_\beta \to (G/F_\beta)/(N/F_\beta) = G/N$ sends $FC(G/F_\beta) = F_\alpha/F_\beta$ to the identity, that is, $F_\alpha \subset N$.

Finally, if α is a limit ordinal and $F_\beta \subset N$ for all $\beta < \alpha$, then $F_\alpha = \bigcup_{\beta<\alpha} F_\beta \subset N$. Hence, $F_\alpha \subset N$ for all ordinals α, so $FCH(G) = \bigcup_\alpha F_\alpha \subset N$. □

Corollary 2.6. *A group G is FC-hypercentral if and only if G has no nontrivial ICC quotients, that is, if and only if $FC(G/N)$ is nontrivial for every proper normal subgroup N of G.*

We also mention some permanence properties of FC-hypercentrality.

We will say that G is an *FC-central extension* of a group K if there is a short exact sequence

$$e \longrightarrow H \longrightarrow G \longrightarrow K \longrightarrow e$$

such that $H \subset FC(G)$. In particular, H must be an FC-group. Note that FC-central extensions generalize both central and finite extensions. The class of FC-nilpotent groups forms the smallest class that is closed under FC-central extensions.

Similarly, we will say that G is an *FC-hypercentral extension* of K if $H \subset FCH(G)$. The class of FC-hypercentral groups is closed under FC-hypercentral extensions:

Proposition 2.7. *Suppose a group G is an FC-hypercentral extension of a group K. Then G is FC-hypercentral if and only if K is FC-hypercentral.*

Proof. Let $q\colon G \to K$ denote the canonical surjection. According to Lemma 2.5, we have to show that G has a nontrivial ICC quotient if and only if K has a nontrivial ICC quotient.

First, let N be a proper normal subgroup of K such that K/N is ICC. Then $q^{-1}(N)$ is a proper normal subgroup of G that contains H and $G/q^{-1}(N) \cong (G/H)/(q^{-1}(N)/H) \cong K/N$ is ICC.

For the converse implication, let N be a proper normal subgroup of G such that G/N is ICC. Then N must contain $FCH(G)$, in particular, N must contain H. Hence, $q(N) \cong N/H$ is a proper normal subgroup of K and $G/N \cong (G/H)/(N/H) \cong K/q(N)$, which is ICC. □

It follows from Proposition 2.7 that the class of FC-hypercentral groups is closed under quotients and direct products. We also have:

Proposition 2.8. *Suppose H is a subgroup of G.*

a) *If G is FC-hypercentral, then H is FC-hypercentral.*
b) *If H is FC-hypercentral and of finite index in G, then G is FC-hypercentral.*

Proof. For *a*) we have that $FC(G)\cap H \subset FC(H)$ for any G, and a routine induction argument shows that $FC_\alpha(G) \cap H \subset FC_\alpha(H)$ for all ordinals. Hence, $FCH(H) \cap H \subset FCH(H)$, so if G is FC-hypercentral, then so is H.

To prove *b*), assume that G is not FC-hypercentral. Then there is a proper normal subgroup N of G such that G/N is ICC. Another routine argument gives that $H \cap N$ is a proper normal subgroup of H and $H/(H \cap N)$ is ICC. Hence, H is not FC-hypercentral. As suggested by the referee, one may also argue as follows: if H is FC-hypercentral and of finite index in G, then the core $\cap_{g\in G}\, gHg^{-1} \subseteq H$ is a normal subgroup of G with finite index; it is FC-hypercentral by *a*) and then G is FC-hypercentral by Proposition 2.7. □

3. On C^*-simplicity and the unique trace property for FC-hypercentral groups

This section is mainly devoted to the proof of the following result:

Theorem 3.1. *Assume that G is an FC-hypercentral group and let $\sigma \in Z^2(G, \mathbb{T})$. Then the following properties are equivalent:*

(i) *(G, σ) satisfies Kleppner's condition;*
(ii) *(G, σ) is C^*-simple;*
(iii) *(G, σ) has the unique trace property,*

This means the class of FC-hypercentral groups is contained in the class $\mathcal{K}^{am}$.

To simplify notation, we set $A = C^*_r(G, \sigma)$ and let A_{FC} denote the C^*-subalgebra of A generated by $\{\lambda_\sigma(g) \mid g \in FC(G)\}$.

A simple computation gives that for all $g, h \in G$, we have

$$\lambda_\sigma(g)\,\lambda_\sigma(h)\,\lambda_\sigma(g)^* = \widetilde{\sigma}(g,h)\,\lambda_\sigma(ghg^{-1})\,,$$

where

$$\widetilde{\sigma}(g,h) = \sigma(g,h)\,\overline{\sigma(ghg^{-1}, g)}\,.$$

Hence, we get an action γ of G on A_{FC}, given by

$$\gamma_g(a) = \lambda_\sigma(g)\,a\,\lambda_\sigma(g)^* \quad \text{for all } g \in G \text{ and } a \in A_{FC}\,.$$

We will let τ_{FC} denote the tracial state on A_{FC} obtained by restricting the canonical tracial state τ on A to A_{FC}. Clearly, τ_{FC} is γ-invariant.

Proposition 3.2. *The following conditions are equivalent:*

(i) *(G, σ) satisfies Kleppner's condition,*
(ii) *τ_{FC} is the unique γ-invariant tracial state of A_{FC}.*

Proof. $(i) \Rightarrow (ii)$: Assume that (i) holds. If G is ICC, then $A_{FC} \simeq \mathbb{C}$, and (ii) is trivially satisfied in this case. We may therefore assume that G is not ICC, so $FC(G) \neq \{e\}$. Let φ be a γ-invariant state of A_{FC}.

Consider $h \in FC(G)$, $h \neq e$. As h is not σ-regular, there exists $g \in G$ such that $gh = hg$ and $\sigma(g,h) \neq \sigma(h,g)$. It clearly follows that $\widetilde{\sigma}(g,h) \neq 1$ and

$$\gamma_g\big(\lambda_\sigma(h)\big) = \widetilde{\sigma}(g,h)\,\lambda_\sigma(ghg^{-1}) = \widetilde{\sigma}(g,h)\,\lambda_\sigma(h)\,.$$

Thus, we get

$$\varphi\big(\lambda_\sigma(h)\big) = \varphi\big(\gamma_g(\lambda_\sigma(h))\big) = \widetilde{\sigma}(g,h)\,\varphi\big(\lambda_\sigma(h)\big)\,,$$

so $\varphi\big(\lambda_\sigma(h)\big) = 0$. This implies that φ agrees with τ_{FC}.

$(ii) \Rightarrow (i)$: Assume that (i) does not hold. It is then known that the center Z of A is nontrivial. In fact, $Z_{FC} := A_{FC} \cap Z$ is nontrivial in this case (cf. [20, Proof of Theorem 2.7]). So we may pick a non-scalar positive element $z_0 \in Z_{FC}$ and define a tracial state on A_{FC} by

$$\varphi(a) = \frac{1}{\tau_{FC}(z_0)}\,\tau_{FC}(az_0)\,, \quad \text{for } a \in A_{FC}\,.$$

As Z_{FC} is fixed by γ and τ_{FC} is γ-invariant, φ is also γ-invariant.

Now, observe that $\varphi(z_0) \neq \tau_{FC}(z_0)$. Indeed, assume (for contradiction) that this is not true. This means that $\tau_{FC}({z_0}^2) = \tau_{FC}(z_0)^2$, hence

$$\tau_{FC}\big((z_0 - \tau_{FC}(z_0)I)^2\big) = 0\,.$$

Since τ_{FC} is faithful, we get that $z_0 = \tau_{FC}(z_0)I$, i.e., z_0 is a scalar, which gives a contradiction.

This shows that $\varphi \neq \tau_{FC}$, so τ_{FC} is not the only γ-invariant tracial state of A_{FC}. Thus, (ii) does not hold. ∎

An immediate consequence of Proposition 3.2 is the following:

Corollary 3.3. *Assume that (G,σ) satisfies Kleppner's condition and let φ be a tracial state of A. Then φ agrees with τ on A_{FC}.*

Let now D^σ denote the subgroup of $\mathbb{T}$ generated by the image of σ, i.e., by the set $\sigma(G\times G)$. We consider D^σ as a discrete group and define the extension G^σ of G by D^σ as the set $G\times D^\sigma$ equipped with the product given by $(g,z)(h,w) = (gh, \sigma(g,h)zw)$. Moreover, we let $(g,z)\to u_\sigma(g,z)\in C^*(G^\sigma)$ denote the canonical embedding of G^σ into $C^*(G^\sigma)$.

Lemma 3.4. *Let ψ be a tracial state on $C^*(G^\sigma)$ and let $\widetilde{\psi}$ be the function on G^σ given by $\widetilde{\psi} = \psi\circ u_\sigma$. Assume that*

$$\widetilde{\psi}(g,z) = z\,\widetilde{\psi}(g,1)$$

for all $g\in G$ and $z\in D^\sigma$, and

$$\widetilde{\psi}(h,z) = 0$$

for all $h\in FC(G)\setminus\{e\}$ and $z\in D^\sigma$.

Then $\widetilde{\psi}(h,z) = 0$ for all $h\in FCH(G)\setminus\{e\}$ and $z\in D^\sigma$.

Proof. If G is ICC, then $FCH(G) = FC(G) = \{e\}$, so there is nothing to show. Thus, suppose G is not ICC.

Let $\{F_\alpha\}_\alpha$ denote the upper FC-central series of G and let α be an ordinal. It suffices to show that $\widetilde{\psi}(h,z) = 0$ for all $h\in F_\alpha\setminus\{e\}$ and all $z\in D^\sigma$.

For $\alpha = 1$ the result holds by assumption since $F_1 = FC(G)$. So let $\alpha > 1$ be an ordinal and suppose that $\widetilde{\psi} = 0$ on $(F_\beta\setminus\{e\})\times D^\sigma\subset G^\sigma$ for all $\beta<\alpha$.

If α is a limit ordinal, then by construction

$$\big(F_\alpha\setminus\{e\}\big)\times D^\sigma = \Big(\big(\bigcup_{\beta<\alpha} F_\beta\big)\setminus\{e\}\Big)\times D^\sigma = \Big(\bigcup_{\beta<\alpha}\big(F_\beta\setminus\{e\}\big)\Big)\times D^\sigma.$$

By hypothesis, $\widetilde{\psi} = 0$ on the right-hand side, hence also on the left-hand side.

If α is a successor ordinal, then $\alpha = \beta+1$ for some ordinal β. Pick an element $h\in F_\alpha\setminus F_\beta$. Then the set $\{ghg^{-1}\mid g\in G\}$ is infinite since $FC(G)\subset F_\beta$, while the set $\{ghg^{-1}F_\beta\mid g\in G\}$ is finite in G/F_β since $F_\alpha/F_\beta = FC(G/F_\beta)$. Hence, there exists an infinite sequence $(g_i)_{i\in\mathbb{N}}$ in G such that $g_ihg_i^{-1}F_\beta = g_jhg_j^{-1}F_\beta$ for all $i,j\in\mathbb{N}$, and $g_ihg_i^{-1}\neq g_jhg_j^{-1}$ whenever $i\neq j$. This means that $g_jh^{-1}g_j^{-1}g_ihg_i^{-1}\in F_\beta\setminus\{e\}$ whenever $i\neq j$ in $\mathbb{N}$. Since

$$(g_j,1)(h,1)^{-1}(g_j,1)^{-1}(g_i,1)(h,1)(g_i,1)^{-1} = (g_jh^{-1}g_j^{-1}g_ihg_i^{-1}g_j, w)$$

for some $w\in D^\sigma$, and $\widetilde{\psi} = 0$ on $(F_\beta\setminus\{e\})\times D^\sigma$, we get

$$\widetilde{\psi}\Big((g_j,1)(h,1)^{-1}(g_j,1)^{-1}(g_i,1)(h,1)(g_i,1)^{-1}\Big) = \widetilde{\psi}(g_jh^{-1}g_j^{-1}g_ihg_i^{-1}, w) = 0$$

whenever $i\neq j$ in $\mathbb{N}$. We may now apply Lemma 2.1 and conclude that $\widetilde{\psi}(h,1) = 0$. Thus, $\widetilde{\psi}(h,z) = z\,\widetilde{\psi}(h,1) = 0$ for all $h\in F_\alpha\setminus\{e\}$ and $z\in D^\sigma$.

□

Let A_{FCH} denote the C^*-subalgebra of A generated by $\{\lambda_\sigma(h)\mid h\in FCH(G)\}$.

Lemma 3.5. *Let φ be a tracial state on A which agrees with τ on A_{FC}. Then φ agrees with τ on A_{FCH}.*

Proof. As the map $(g,z)\mapsto z\lambda_\sigma(g)$ is a unitary representation of G^σ on $\ell^2(G)$, there exists a surjective *-homomorphism $\pi\colon C^*(G^\sigma)\to A$ satisfying $\pi(u_\sigma(g,z)) = z\,\lambda_\sigma(g)$ for each

$(g, z) \in G^\sigma$. Thus φ lifts to a tracial state $\psi = \varphi \circ \pi$ on $C^*(G^\sigma)$. For all $g \in G$ and $z \in D^\sigma$, we have

$$\psi(u_\sigma(g, z)) = \varphi(z\,\lambda_\sigma(g)) = z\,\varphi(\lambda_\sigma(g)) = z\,\psi(u_\sigma(g, 1)).$$

Since $\varphi(\lambda_\sigma(g)) = 0$ for all $g \in FC(G) \setminus \{e\}$ (by assumption), it follows that $\psi(u_\sigma(g, z)) = 0$ for all $g \in FC(G) \setminus \{e\}$ and $z \in D^\sigma$. Hence, using Lemma 3.4, we get that $\psi(u_\sigma(g, z)) = 0$ for all $g \in FCH(G) \setminus \{e\}$ and $z \in D^\sigma$. In particular, we get

$$\varphi(\lambda_\sigma(g)) = \psi(u_\sigma(g, 1)) = 0$$

for all $g \in FCH(G) \setminus \{e\}$. Thus, φ agrees with τ on A_{FCH}, as desired. □

Proof of Theorem 3.1. Assume that (i) holds and let φ be a tracial state on A. Corollary 3.3 tells us that φ agrees with τ on A_{FC}. Since G is FC-hypercentral, we have that $A = A_{FCH}$. Applying Lemma 3.5, we conclude that φ coincides with τ. Hence, (iii) holds. Since FC-hypercentral groups are amenable, we get from Corollary 2.3 that (ii) follows from (iii). Finally, we know that (ii) implies (i) in general. □

We will also show how Echterhoff's characterization of FC-hypercentrality mentioned in Section 2.3 may be used to give a different proof of the implication $(i) \Rightarrow (ii)$ in Theorem 3.1. We will need the following result:

Proposition 3.6. *Assume that every prime ideal of $C^*(G^\sigma)$ is maximal. Then (G, σ) is C^*-simple whenever (G, σ) satisfies Kleppner's condition.*

Proof. Assume that (G, σ) satisfies Kleppner's condition. Then $C^*_r(G, \sigma)$ is prime [20, Theorem 2.7]. Let π denote the surjective *-homomorphism $C^*(G^\sigma) \to C^*_r(G, \sigma)$ obtained in the proof of Lemma 3.5. The kernel $\mathcal{J}$ of π is then a prime ideal of $C^*(G^\sigma)$. Hence, the assumption gives that $\mathcal{J}$ is maximal, so $C^*(G^\sigma)/\mathcal{J} \simeq C^*_r(G, \sigma)$ is simple. □

Another proof of $(i) \Rightarrow (ii)$ in Theorem 3.1. Using Proposition 2.7, we get that G^σ is also FC-hypercentral. Hence, [9, Corollary 3.2] tells us that every prime ideal of $C^*_r(G^\sigma)$ is maximal, so the assertion follows from Proposition 3.6. □

We also record another consequence of Proposition 3.6.

Corollary 3.7. *Assume that G is a countable discrete group such that G^σ is a T_1-group. Then (G, σ) is C^*-simple whenever (G, σ) satisfies Kleppner's condition.*

Proof. Since G is countable, G^σ is also countable. Hence, $C^*(G^\sigma)$ is separable, so its prime ideals are primitive ideals, and the assumption in Proposition 3.6 is therefore satisfied. □

We note that if G is amenable and it satisfies the assumptions in Corollary 3.7, then it is not difficult to deduce from the Moore-Rosenberg result cited in Section 2.3 that G is FC-hypercentral. Hence, Corollary 3.7 is covered by Theorem 3.1 in this case.

We do not know of any group in $\mathcal{K}^{am}$ that is not FC-hypercentral. Thus, a natural question is:

Question 3.8. Does the class of FC-hypercentral groups coincide with $\mathcal{K}^{am}$?

This question may be reformulated as follows. Assume that G is neither ICC, nor FC-hypercentral, but amenable. Can one always find $\sigma \in Z^2(G, \sigma)$ such that (G, σ) satisfies Kleppner's condition, but (G, σ) is not C^*-simple or does not have the unique trace property ? It does not seem easy to answer this question positively.

4. On the unique trace property for a larger class of groups

In this section, we let G be a group and $\sigma \in Z^2(G, \mathbb{T})$. Motivated by Proposition 2.5, we define $ICC(G) = G/FCH(G)$. Moreover, we let $\mathcal{P}$ denote the class of groups considered in [4], that consists of all PH groups [24] and of all groups satisfying property (P_{com}) [5]. The class $\mathcal{P}$ is a large subclass of the class of ICC groups, and the only amenable group belonging to $\mathcal{P}$ is the trivial group. The main purpose of this section is to show the following result:

Theorem 4.1. *Assume that $K = ICC(G)$ belongs to $\mathcal{P}$. Then we have:*

a) *(G, σ) satisfies Kleppner's condition if and only if (G, σ) has the unique trace property.*
b) *Set $H = FCH(G)$ and let σ_H denote the restriction of σ to $H \times H$. If (H, σ_H) satisfies Kleppner's condition, then (G, σ) is C^*-simple and has the unique trace property.*

We set $A = C_r^*(G, \sigma)$ and let A_{FC} and A_{FCH} be defined as in the previous section. Since $FCH(G)$ is normal in G, the action γ of G on A_{FC} extends to an action $\widetilde{\gamma}$ of G on A_{FCH} given by

$$\widetilde{\gamma}_g(a) = \lambda_\sigma(g)\, a\, \lambda_\sigma(g)^*$$

for all $g \in G$ and $a \in A_{FCH}$. Let τ_{FCH} denote the restriction of the canonical tracial state τ on A to A_{FCH}. Clearly, τ_{FCH} is $\widetilde{\gamma}$-invariant. Analogously to Proposition 3.2, we have:

Proposition 4.2. *The following conditions are equivalent:*

(i) *(G, σ) satisfies Kleppner's condition,*
(ii) *τ_{FCH} is the unique $\widetilde{\gamma}$-invariant tracial state of A_{FCH}.*

Proof. $(i) \Rightarrow (ii)$: Assume that (i) holds. If G is ICC, then $A_{FCH} \cong \mathbb{C}$, and (ii) is trivially satisfied in this case. We may therefore assume that G is not ICC, so $FC(G) \neq \{e\}$. Let ϕ be a $\widetilde{\gamma}$-invariant tracial state of A_{FCH}. As in Proposition 3.2, we compute that $\phi(\lambda_\sigma(h)) = 0$ for all $h \in FC(G) \setminus \{e\}$. Now, let E denote the canonical conditional expectation from A onto A_{FCH}. Then $\varphi := \phi \circ E$ is a tracial state on A such that $\varphi(\lambda_\sigma(h)) = 0$ for all $h \in FC(G) \setminus \{e\}$. Applying Lemma 3.5, we get that $\varphi = \phi_{|A_{FCH}} = \tau_{FCH}$.

Since $A_{FC} \subset A_{FCH}$, the proof of $(ii) \Rightarrow (i)$ goes along the same lines as the one given for this implication in Proposition 3.2. □

Set $H = FCH(G)$ and $K = ICC(G) = G/H$, and let q denote the canonical homomorphism from G onto K. Further, let $n\colon K \to G$ be a section for q satisfying $n(e) = e$, and define $m\colon K \times K \to H$ by $m(k, l) = n(k)n(l)n(kl)^{-1}$. Finally, let σ_H denote the restriction of σ to $H \times H$.

Moreover, define $\beta\colon K \to \mathrm{Aut}(A_{FCH})$ by $\beta = \widetilde{\gamma} \circ n$ and $\omega\colon K \times K \to \mathcal{U}(A_{FCH})$ by

$$\omega(k, l) = \sigma\big(n(k), n(l)\big)\, \sigma\big(m(k, l), n(kl)\big)^*\, \lambda_\sigma(m(k, l))\,.$$

Then (β, ω) is a twisted action of K on A_{FCH} such that $C_r^*(G, \sigma)$ is *-isomorphic to the twisted crossed product $C_r^*(A_{FCH}, K, \beta, \omega)$. This follows from [1] after noticing that A_{FCH} may be identified with $C_r^*(H, \sigma_H)$ via the *-isomorphism sending $\lambda_\sigma(h)$ to $\lambda_{\sigma_H}(h)$ for each $h \in H$.

Since τ_{FCH} is $\widetilde{\gamma}$-invariant, τ_{FCH} is also β-invariant. Moreover, we have:

Proposition 4.3. *The following conditions are equivalent:*

(i) *(G, σ) satisfies Kleppner's condition,*
(ii) *τ_{FCH} is the unique β-invariant tracial state of A_{FCH}.*

Proof. Assume (i) holds. Proposition 4.2 gives that τ_{FCH} is the unique $\widetilde{\gamma}$-invariant tracial state on A_{FCH}. Consider now a β-invariant tracial state ω of A_{FCH} and let $g \in G$. Write $g = h\,n(k)$ where $k = q(g) \in K$ and $h = g\,n(k)^{-1} \in H$. Then, for each $s \in H$, we have

$$\omega\big(\widetilde{\gamma}_g(\lambda_\sigma(s))\big) = \omega\big(\widetilde{\gamma}_h\beta_k(\lambda_\sigma(s))\big)$$
$$= \omega\big(\lambda_\sigma(h)\,\beta_k(\lambda_\sigma(s))\,\lambda_\sigma(h)^*\big) = \omega\big(\beta_k(\lambda_\sigma(s))\big) = \omega(\lambda_\sigma(s))\,.$$

It follows that ω is $\widetilde{\gamma}$-invariant. Hence, $\omega = \tau_{FCH}$. This shows that (ii) holds.

Conversely, if (ii) holds, then, as $\beta = \widetilde{\gamma} \circ n$, it is clear that τ_{FCH} is the unique $\widetilde{\gamma}$-invariant tracial state of A_{FCH}, so (i) holds by using Proposition 4.2. □

Proof of Theorem 4.1. a) From [4, Corollary 3.9], we know that when K belongs to the class $\mathcal{P}$, the tracial states of $C_r^*(A_{FCH}, K, \beta, \omega)$ are in a one-to-one correspondence with the β-invariant tracial states of A_{FCH}. Hence, it follows from Proposition 4.3 that $C_r^*(G,\sigma) \cong C_r^*(A_{FCH}, K, \beta, \omega)$ has a unique tracial state if and only if (G,σ) satisfies Kleppner's condition.

b) Assume that (H, σ_H) satisfies Kleppner's condition. Since H is FC-hypercentral, we get from Theorem 3.1 that $A_{FCH} \simeq C_r^*(H, \sigma_H)$ is simple with a unique tracial state. This implies that A_{FCH} has a unique β-invariant tracial state and that the system $(A_{FCH}, K, \beta, \omega)$ is minimal. Hence, it follows from [4, Corollary 3.11] that $C_r^*(G,\sigma) \cong C_r^*(A_{FCH}, K, \beta, \omega)$ is simple with a unique tracial state. □

Let $\mathcal{ICCP}$ denote the class of groups satisfying the assumption in Theorem 4.1. Part a) of this theorem shows that $\mathcal{ICCP}$ is contained in the class $\mathcal{K}_{UT}$. We believe that $\mathcal{ICCP} \subset \mathcal{K}$, i.e., that we also have $\mathcal{ICCP} \subset \mathcal{K}_{C^*S}$, but we have not been able to prove this. Part b) of Theorem 4.1 is a somewhat weaker statement; its proof shows that we would have $\mathcal{ICCP} \subset \mathcal{K}_{C^*S}$ if one could answer positively the following:

Question 4.4. Assume that (G,σ) satisfies Kleppner's condition. Is the system $(A_{FCH}, K, \beta, \omega)$ always minimal ? That is, is $\{0\}$ the only proper β-invariant ideal of A_{FCH} ?

In this regard, we also remark that if G belongs to $\mathcal{ICCP}$, G is exact, and $C_r^*(G,\sigma)$ has stable rank one whenever (G,σ) satisfies Kleppner's condition, then Corollary 2.4 and Theorem 4.1 a) together give that G belongs to $\mathcal{K}$.

Note that if $G/FC(G)$ belongs to $\mathcal{P}$, then $G/FC(G)$ is ICC, so the upper FC-central series of G stops at F_1, i.e., $FCH(G) = FC(G)$. Hence, Theorem 4.1 gives:

Corollary 4.5. *Assume that $G/FC(G)$ belongs to $\mathcal{P}$. Then we have:*

a) *(G,σ) satisfies Kleppner's condition if and only if (G,σ) has the unique trace property.*
b) *Set $H = FC(G)$ and let σ_H denote the restriction of σ to $H \times H$. If (H, σ_H) satisfies Kleppner's condition, then (G,σ) is C^*-simple and has the unique trace property.*

Example 4.6. Let $n \in \mathbb{N}$, $n \geq 2$ and set $G = \langle a, b \mid ab^n = b^n a\rangle$. Then G is a so-called Baumslag-Solitar group, often denoted by $BS(n,n)$. We have

$$FCH(G) = FC(G) = Z(G) = \langle b^n \rangle \simeq \mathbb{Z}$$

and $ICC(G) \simeq \mathbb{Z} * \mathbb{Z}_n \in \mathcal{P}$ (since $\mathbb{Z} * \mathbb{Z}_n$ is a Powers group [13]), so $G \in \mathcal{ICCP}$.

Let f denote the surjective homomorphism $f\colon G \to \mathbb{Z}^2$ satisfying $f(a) = (1,0)$ and $f(b) = (0,1)$. For $\theta \in \mathbb{T}$, let $\omega_\theta \in Z^2(\mathbb{Z}^2, \mathbb{T})$ be given by $\omega_\theta(m,n) = e^{2\pi i \theta m_2 n_1}$, and define $\sigma_\theta \in Z^2(G, \mathbb{T})$ by $\sigma_\theta(x,y) = \omega_\theta(f(x), f(y))$. It can be shown that every two-cocycle on G is cohomologous to one of this form.

Then one easily verifies that (G, σ_θ) satisfies Kleppner's condition if and only if θ is irrational. Hence, Theorem 4.1 a) gives that (G, σ_θ) has the unique trace property if and only if θ is irrational. Theorem 4.1 b) is not useful in this example since σ_θ restricts to 1 on $Z(G)$. However, it can be shown that (G, σ_θ) is C^*-simple if and only if θ is irrational. Hence, it follows that $G = BS(n,n)$ belongs to $\mathcal{K}$.

We will come back to this example and also discuss other conditions ensuring simplicity and/or uniqueness of the tracial state for reduced twisted group C^*-algebras in a subsequent paper.

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