THE PEDERSEN RIGIDITY PROBLEM

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ABSTRACT. If α is an action of a locally compact abelian group G on a C^*-algebra A, Takesaki-Takai duality recovers (A, α) up to Morita equivalence from the dual action of $\widehat{G}$ on the crossed product $A \rtimes_\alpha G$. Given a bit more information, Landstad duality recovers (A, α) up to isomorphism. In between these, by modifying a theorem of Pedersen, (A, α) is recovered up to outer conjugacy from the dual action and the position of A in $M(A \rtimes_\alpha G)$. Our search (still unsuccessful, somehow irritating) for examples showing the necessity of this latter condition has led us to formulate the "Pedersen rigidity problem". We present numerous situations where the condition is redundant, including G discrete or A stable or commutative. The most interesting of these "no-go theorems" is for locally unitary actions on continuous-trace algebras.

1. INTRODUCTION

Given an action α of a locally compact group G on a C^*-algebra A, our first reaction is to form the crossed product C^*-algebra $A \rtimes_\alpha G$.

Question 1.1. How do we recover the action from the crossed product?

Short answer: we can't.

Example 1.2. If G (is infinite and) acts on $C_0(G)$ by translation, then $C_0(G) \rtimes G \simeq \mathcal{K}(L^2(G))$. But also $(C_0(G) \otimes \mathcal{K}(L^2(G))) \rtimes G \simeq \mathcal{K}(L^2(G)) \otimes \mathcal{K}(L^2(G)) \simeq \mathcal{K}(L^2(G))$, where G acts trivially on $\mathcal{K}$.

Now, $C_0(G) \otimes \mathcal{K}$ is not isomorphic to $C_0(G)$, but they are at least Morita equivalent.

Example 1.3. Examples arising from number theory (and which began with Cuntz) give *nonisomorphic* commutative C^*-algebras, carrying actions of a discrete abelian group G that have isomorphic crossed products (see [KOQ14, Remark 4.3]). Note that by commutativity, A and B are not Morita equivalent.

So, there's no hope of recovering (A, α) just from $A \rtimes G$, even up to Morita equivalence.

We assume throughout that G is abelian (a lot of what we say here can be done with nonabelian groups, see [KOQ19], but that requires the technical machinery of coactions, which would tend to obscure the fundamental issues). Then there is a dual action $\widehat{\alpha}$ of the dual group $\widehat{G}$ on $A \rtimes_\alpha G$.

We assume that we know the group G, the crossed product $A \rtimes G$, and the dual action $\widehat{\alpha}$.

We want to know what other information we need to recover the C^*-algebra A and the action $\alpha \colon G \curvearrowright A$, at least in some sense.

With only the dual action, Takesaki-Takai duality tells us that the following is the best we can hope for in general:

Theorem 1.4 (Takesaki-Takai). *$A \rtimes_\alpha G \rtimes_{\widehat{\alpha}} \widehat{G} \simeq A \otimes \mathcal{K}(L^2(G))$. Moreover, $\widehat{\widehat{\alpha}}$ corresponds to $\alpha \otimes \operatorname{Ad} \rho$.*

Here ρ is the right regular representation of G. Thus, Takesaki-Takai duality implies that we can recover the action (A, α) from the dual action $(A \rtimes_\alpha G, \widehat{\alpha})$ up to Morita equivalence.

To recover more, we need to keep track of more information about the the crossed product.

The crossed product $A \rtimes_\alpha G$ is generated by a universal covariant representation

$$(i_A, i_G) \colon (A, G) \to M(A \rtimes G).$$

The extra data we consider involves the maps i_A, i_G.

At the opposite extreme from Takesaki-Takai duality, Landstad duality gives the action up to isomorphism (and we could not reasonably ask for more than that), given one more piece of information: i_G.

Landstad constructs (what we now call) the *generalized fixed-point algebra* $\mathrm{Fix}(A \rtimes_\alpha G, \widehat{\alpha}, i_G)$, which is a C^*-subalgebra of $M(A \rtimes_\alpha G)$.

Theorem 1.5 (Landstad). *The original action (A, α) is isomorphic to $(\mathrm{Fix}(A \rtimes_\alpha G, \widehat{\alpha}, i_G), \mathrm{Ad}\, i_G)$.*

Indeed, $i_A \colon A \to M(A \rtimes_\alpha G)$ gives the isomorphism.

Question 1.6. What's the structure underlying Landstad duality?

Definition 1.7. An *equivariant action* of $\widehat{G}$ is a triple (C, γ, v), where (C, γ) is an action of $\widehat{G}$ and $v \colon G \to M(C)$ is a strictly continuous unitary homomorphism such that $\gamma_\chi(v_s) = \chi(s)v_s$ for all $\chi \in \widehat{G}$ and $s \in G$.

Note that from the given action (A, α) the crossed product gives an equivariant action $(A \rtimes_\alpha G, \widehat{\alpha}, i_G)$.

Landstad duality says that if we are given an equivariant action (C, γ, v) of $\widehat{G}$, the generalized fixed-point algebra $A := \mathrm{Fix}(C, \gamma, v)$ is a C^*-subalgebra of $M(C)$, $\mathrm{Ad}\, v$ gives an action α of G on A, and

$$(A \rtimes_\alpha G, \widehat{\alpha}, i_G) \simeq (C, \gamma, v).$$

2. Explore the middle

We've seen that the "classical" crossed-product duality theorems — Takesaki-Takai and Landstad dualities — give two extremes: the first is the weakest, giving the original action up to Morita equivalence, and the second is the strongest, giving the original action up to isomorphism. This leads us to the following:

Question 2.1. What lies between these extremes?

The foundation of our contribution is the following, which focuses on one part of Landstad duality:

Question 2.2. Given an equivariant action (C, γ, v), how much does the generalized fixed-point algebra $\mathrm{Fix}(C, \gamma, v)$ depend upon v?

In other words, could we recover v from the action (C, γ) if we also knew the generalized fixed-point algebra $\mathrm{Fix}(C, \gamma, v)$?

In our study of this question, we were guided by a theorem of Pedersen (recorded as Theorem 2.3 below). First, recall that a *cocycle* for an action (A, α) is a strictly continuous unitary map $u \colon G \to M(A)$ such that $u_{st} = u_s \alpha_s(u_t)$. Then $\mathrm{Ad}\, u \circ \alpha$ is another action on A, said to be *exterior equivalent* to α. The following two theorems (modulo the mention of generalized fixed-point algebras in the second) first appeared in [Ped82, Theorem 35], and in more precise form in [RR88, Theorem 0.10].

Theorem 2.3 (Pedersen)**.** *Two actions α and β of G on A are exterior equivalent if and only if there is an isomorphism*

$$\theta \colon (A \rtimes_\alpha G, \widehat{\alpha}) \xrightarrow{\simeq} (A \rtimes_\beta G, \widehat{\beta})$$

such that $\theta \circ i_A^\alpha = i_A^\beta$.

We can escape from A using outer conjugacy. Recall that actions (A, α) and (B, β) of G are *outer conjugate* if there is an action γ exterior equivalent to β such that $(A, \alpha) \simeq (B, \gamma)$. We call the following result "outer duality".

Theorem 2.4 (Pedersen, KOQ)**.** *Two actions (A, α) and (B, β) of G are outer conjugate if and only if there is an isomorphism*

$$(A \rtimes_\alpha G, \widehat{\alpha}) \simeq (B \rtimes_\beta G, \widehat{\beta})$$

taking $\mathrm{Fix}(A \rtimes_\alpha G, \widehat{\alpha}, i_G^\alpha)$ to $\mathrm{Fix}(B \rtimes_\beta G, \widehat{\beta}, i_G^\beta)$.

So, Pedersen's theorem says we can recover the action (A, α) of G up to outer conjugacy if we know the dual action $(A \rtimes_\alpha G, \widehat{\alpha})$ of $\widehat{G}$ and the generalized fixed-point algebra $\mathrm{Fix}(A \rtimes_\alpha G, \widehat{\alpha}, i_G)$, but perhaps not the homomorphism $i_G \colon G \to M(A \rtimes_\alpha G)$ itself.

Remark 2.5. All of the crossed-product dualities, including our outer duality, can be promoted to category equivalences (see [KOQ16]), but the Pedersen rigidity problem we discuss here does not depend upon that.

3. Pedersen rigidity

After we developed the outer duality in [KOQ18], we wanted to give some examples showing that Pedersen's condition about the generalized fixed-point algebras is necessary. That is, we wanted to exhibit examples of the following: actions (A, α) and (B, β) of G such that

   (i) $(A \rtimes_\alpha G, \widehat{\alpha}) \simeq (B \rtimes_\beta G, \widehat{\beta})$, but
   (ii) α and β are not outer conjugate.

Equivalently, by Pedersen's theorem we want there to exist an isomorphism between the dual actions $\widehat{\alpha}$ and $\widehat{\beta}$, but not one that preserves the generalized fixed-point algebras.

Somehow irritating, we are frustrated by our inability to find any examples of this phenomenon.

As we searched for this behavior among various special types of actions, we continually discovered that it cannot happen. More precisely, we proved a string of "no-go theorems", which together tell us that there are no examples in any of the following cases:

   (i) G is discrete;
   (ii) A and B are stable;
   (iii) A and B are commutative;
   (iv) α or β is inner;
   (v) G is compact, and α and β are faithful and ergodic;
   (vi) A and B are continuous trace and α and β are locally unitary.

Let's see why:

Theorem 3.1. *If G is discrete, then two actions (A, α) and (B, β) of G are outer conjugate if and only if the dual actions $(A \rtimes_\alpha G, \widehat{\alpha})$ and $(B \rtimes_\beta G, \widehat{\beta})$ are conjugate.*

The reason is that $\widehat{G}$ is compact, so the dual action has a genuine fixed-point algebra, and hence all generalized fixed-point algebras coincide.

Theorem 3.2. *If A and B are stable and have strictly positive elements* (*which is satisfied if they are separable, for example*) *actions (A, α) and (B, β) of G are outer conjugate if and only if the dual actions are conjugate.*

The reason is that if $\widehat{\alpha} \simeq \widehat{\beta}$ then by Takesaki-Takai duality the actions α and β must at least be Morita equivalent, and then by stability must be outer conjugate, by a result of Combes [Com84, Section 8 Proposition].

For commutative algebras, we can say even more:

Theorem 3.3. *If A and B are commutative, then actions (A, α) and (B, β) of G are conjugate if and only if the dual actions are conjugate.*

The reason is that, if $\widehat{\alpha} \simeq \widehat{\beta}$, then α and β are Morita equivalent, and then the associated Rieffel homeomorphism $\widehat{B} \simeq \widehat{A}$ [Rie74, Corollary 6.27] is G-equivariant, and this gives an $\alpha - \beta$ equivariant isomorphism $A \simeq B$.

In the inner case, we only require one of the actions to be inner:

Theorem 3.4. *An inner action (A, α) and any other action (B, β) of G are outer conjugate if and only if the dual actions are conjugate.*

The reason is that every inner action (A, α) is exterior equivalent to the trivial action ι, and hence we can take the dual actions to be the same:

$$(A \rtimes_\alpha G, \widehat{\alpha}) = (A \rtimes_\iota G, \widehat{\iota}).$$

Since G is abelian, the homomorphism $i_G^\iota : G \to M(A \rtimes_\iota G)$ maps into the center, and hence it commutes with $i_G^\alpha : G \to M(A \rtimes_\iota G)$. Therefore, by [QR95, Lemma 1.6] (see also [BE15, Proposition 3.12] for a slightly more general result) the generalized fixed-point algebras coincide:

$$(A \rtimes_\alpha G)^{\widehat{\alpha}, i_G^\alpha} = (A \rtimes_\iota G)^{\widehat{\iota}, i_G^\iota}.$$

Then transitivity gives the no-go theorem. Note that this argument required G to be abelian.

Theorem 3.5. *If G is compact, then faithful ergodic actions (A, α) and (B, β) of G are conjugate if and only if the dual actions are conjugate.*

Note that, as in the commutative case, we get more than we expect: actual, rather than outer, conjugacy.

The reason is that every spectral subspace

$$A_\gamma = \{a \in A : \alpha_s(a) = \gamma(s)a \text{ for all } s \in G\}$$

is nonzero, and hence is linearly spanned by a unitary u_γ. The resulting map $u : \widehat{G} \to A$ is twisted by a 2-cocycle $\omega : \widehat{G} \times \widehat{G} \to \mathbb{T}$. One shows that the cocycles for the actions (A, α) and (B, β) are cohomologous, and hence, by the results of [OPT80], (A, α) and (B, β) are conjugate.

Theorem 3.6. *If A is continuous-trace and an action α of G on A is locally unitary, then (A, α) and another action (B, β) of G are outer conjugate if and only if the dual actions are conjugate.*

The reason is that A is the cross-sectional algebra of a continuous C^*-bundle $\mathcal{U}$ over the spectrum $\widehat{A}$, and by local unitarity $\widehat{A}$ is covered by open sets N for which the action on the cross-sectional algebra of the restricted bundle over N is inner. From this we construct matching families of G-invariant ideals of A and B such that for each pair the desired property holds, namely that conjugacy of dual actions implies outer conjugacy of the original actions, and then standard techniques of C^*-dynamical systems allow us to combine the ideals to deduce the property for (A, α) and (B, β).

Question 3.7. Can Theorem 3.6 be extended to pointwise unitary actions?

The above no-go theorems lead us to formulate the

Pedersen rigidity problem. If (A, α) and (B, β) are actions of G such that $(A \rtimes_\alpha G, \widehat{\alpha}) \simeq (B \rtimes_\beta G, \widehat{\beta})$, are α and β outer conjugate?

We call this the "Pedersen rigidity problem", because an affirmative answer would mean that Pedersen's condition, namely that the isomorphism preserves the generalized fixed-point algebras, is superfluous. The no-go theorems are evidence hinting at an affirmative answer.

4. Conclusion

We have proved versions of almost all our no-go theorems for nonabelian G, using coactions of G instead of actions of (the non-existent) dual group $\widehat{G}$. The exception is for inner actions, and consequently for locally unitary actions, because our proof of Theorem 3.4 depended upon G being abelian. We suspect that the theorem is in fact true for nonabelian G.

That being said, we actually believe that the Pedersen rigidity problem will have a negative answer — that is, we think that there do exist pairs of non-outer-conjugate actions with conjugate dual actions. We were surprised to discover not only a complete absence of such examples in the literature, but even the lack of apparent interest in the issue. This is striking, since it is tempting to conjecture that one of the first questions researchers must have asked about crossed products is, how much information do we get from only knowing the dual actions? It seems to us that this investigation is long overdue.

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