Assembly map and the Baum-Connes conjecture

Trial lecture

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Conjecture (Baum-Connes)

The assembly map

$$\mu_i^G \colon RK_i^G(\underline{E}G) \to K_i(C_r^*(G))$$

is an isomorphism for $i = 0, 1$.

We restrict to countable discrete groups in this talk.
The structure of $C_r^*(G)$ can be highly nontrivial, so its K-theory can be hard to compute.
The left hand side is computable up to torsion.
Therefore "the left hand side explains the right hand side".
The conjecture has a lot of implications (e.g. the Novikov conjecture and the Kadison-Kaplansky conjecture).

Conjecture (Baum-Connes)

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The K_0-group

Let A be a unital C^*-algebra.

- $p \in A$ is called a projection if $p^2 = p = p^*$.
- $p \sim q$ if there is partial isometry $v \in A$ s.t. $v^*v = p$ and $vv^* = q$.

For $n \in \mathbb{N}$, let $P_n(A)$ denote the set of projections in $M_n(A)$.
If $k < n$, then $M_k(A)$ sits inside $M_n(A)$ as the upper left corner. Let

$$M_\infty(A) = \bigcup_{n\in\mathbb{N}} M_n(A) \quad \text{and} \quad P_\infty(A) = \bigcup_{n\in\mathbb{N}} P_n(A).$$

Let $p \in P_n(A)$ and $q \in P_m(A)$, and define $p \oplus q = \left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right) \in M_{n+m}(A)$.
Moreover, $p \sim_0 q$ in $M_\infty(A)$ if there is k such that $p \oplus 0_{k-n} \sim q \oplus 0_{k-m}$ in $M_k(A)$.
Set $V(A) = P_\infty(A)/\sim_0$. Then $V(A)$ is an abelian semigroup and

$$\begin{aligned} K_0(A) &= G(V(A)), \text{ i.e. the Grothendick group of } V(A) \\ &= \text{“the group of formal differences of projections over } A\text{”} \end{aligned}$$

The K_1-group

Let A be a unital C^*-algebra.

- $u \in A$ is a unitary if $uu^* = u^*u = 1$.

For $n \in \mathbb{N}$, let $\mathcal{U}_n(A)$ denote the group of unitaries of $M_n(A)$ and set

$$\mathcal{U}_\infty(A) = \bigcup_{n \in \mathbb{N}} \mathcal{U}_n(A).$$

Let $u \in \mathcal{U}_n(A)$ and $v \in \mathcal{U}_m(A)$, and define $u \oplus v = \left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right) \in \mathcal{U}_{n+m}(A)$. Moreover, $u \sim_1 v$ in $\mathcal{U}_\infty(A)$ if there is k such that $u \oplus 1_{k-n} \sim_h v \oplus 1_{k-m}$ in $M_k(A)$. Set $[u]_1 + [v]_1 = [u \oplus v]_1$ and note that $[u]_1 + [u^*]_1 = 0$, so we define the abelian group

$$\begin{aligned} K_1(A) &= \mathcal{U}_\infty(A)/\sim_1 \\ &= \text{“homotopy classes of unitary elements over } A\text{”} \end{aligned}$$

Example

If A is finite-dimensional, then $K_1(A) = 0$.

Group C^*-algebras

Let G be a discrete group.
Define multiplication in the group algebra $\mathbb{C}G$ by

$$(f * g)(s) = \sum_{t \in G} f(t) g(t^{-1} s)$$

and for $f \in \mathbb{C}G$ define an operator T_f on $\ell^2(G)$ by $T_f(g) = f * g$. Then

$$\mathbb{C}G \to B(\ell^2(G)), \qquad f \mapsto T_f$$

is an injection. The reduced group C^*-algebra of G, $C_r^*(G)$ is then the norm closure of $\mathbb{C}G$ in $B(\ell^2(G))$.

Remark

There are also other group C^*-algebras associated with G.

Examples - right hand side

Let G be a discrete group.
Let $\widehat{G}$ denote the classes of irreducible unitary representations of G.

Example

Let G be abelian.
Then the dual group of G is a compact space and coincides with $\widehat{G}$.
Moreover, $C_r^*(G) \cong C(\widehat{G})$ via the Fourier transform, and for $i = 0, 1$

$$K_i(C_r^*(G)) \cong K_i(C(\widehat{G})) \cong K_{\text{top}}^i(\widehat{G}).$$

In particular, $K_i(C_r^*(\mathbb{Z}^n)) \cong K_{\text{top}}^i(\mathbb{T}^n) \cong \mathbb{Z}^{2^{n-1}}$ for $i = 0, 1$ and the generators for $K_0(C_r^*(\mathbb{Z}))$ and $K_1(C_r^*(\mathbb{Z}))$ are $[1]_0$ and $[u]_1$, respectively.

Example

If G is finite, then $K_0(C_r^*(G)) \cong \mathbb{Z}\widehat{G}$ and $K_1(C_r^*(G)) = 0$.

Proper actions

Let G be discrete and X a G-space.
An action $G \curvearrowright X$ is called proper if for all $x, y \in X$, there are neighborhoods U_x, U_y around x, y such that $|\{t \in G : t \cdot U_x \cap U_y\}| < \infty$.
Alternatively, $G \curvearrowright X$ is proper if $G \times X \to X \times X$, $(t, x) \mapsto (t \cdot x, x)$ is a proper map, i.e. inverse images of compact sets are compact.
Henceforth, assume that both X and the orbit space X/G are metrizable.

Definition

A proper G-space $\underline{E}G$ is called a "universal space for proper G-actions" if for every other proper G-space X, there is a G-map $X \to \underline{E}G$ unique up to G-homotopy.

Two G-maps f_0, f_1 are G-homotopic if there is a homotopy $\{f_t\}_{t\in[0,1]}$ of G-maps connecting them.
Clearly, $\underline{E}G$ is unique up to G-homotopy.

Classifying spaces

Let BG denote a classifying space for G (i.e. a $K(G,1)$-space), and EG its universal covering space.
That is, $EG \to BG$ is a principal G-bundle and $BG \cong EG/G$.

Example

If G is torsion-free, every proper action is free.
Hence every proper G-space is a locally trivial principal G-bundle over X/G, so $\underline{E}G$ coincides with EG.
In particular, $\underline{E}\mathbb{Z}^n = E\mathbb{Z}^n = \mathbb{R}^n$ and $B\mathbb{Z}^n = \mathbb{T}^n$.

Example

If G is finite, then all G-spaces are proper, so $\underline{E}G = \{*\}$.

Elliptic operators

Setting: G discrete countable, X, X/G metrizable, $G \curvearrowright X$ proper.

Definition

A "generalized elliptic G-operator" or "G-equivariant abstract elliptic operator" or "cycle over X" is a triple (U, π, F), where

- $U\colon G \to U(\mathcal{H})$ is a unitary representation of G on $\mathcal{H}$.
- $\pi\colon C_0(X) \to B(\mathcal{H})$ is a representation of $C_0(X)$ on $\mathcal{H}$, so that $\pi(t \cdot f) = U(t)\pi(f)U(t)^*$ for all $t \in G$ and $f \in C_0(X)$.
- $F^* = F \in B(\mathcal{H})$ so that $[F, U(t)] = 0$ for all $t \in G$, while $[F, \pi(f)]$ and $\pi(f)(F^2 - I)$ are compact for all $f \in C_0(X)$.

(U, π, F) is called even if it is $\mathbb{Z}/2\mathbb{Z}$-graded, and odd otherwise. (even: $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, $U = \left(\begin{smallmatrix} U_0 & 0 \\ 0 & U_1 \end{smallmatrix}\right)$, $\pi = \left(\begin{smallmatrix} \pi_0 & 0 \\ 0 & \pi_1 \end{smallmatrix}\right)$, and $F = \left(\begin{smallmatrix} 0 & P^* \\ P & 0 \end{smallmatrix}\right)$).

Equivariant K-homology

- $(U, \pi, F) \oplus (U', \pi', F') = (U \oplus U', \pi \oplus \pi', F \oplus F')$.
- (U, π, F) is called degenerate if $[F, U(t)] = [F, \pi(f)] = \pi(f)(F^2 - I) = 0$.
- $(U, \pi, F) \sim_h (U', \pi', F')$ if $U = U'$, $\pi = \pi'$, and there is a homotopy $\{F_t\}$ connecting F and F'.
- $(U, \pi, F) \sim (U', \pi', F')$ if $(U, \pi, F) \oplus \text{deg.} \sim_h (U', \pi', F') \oplus \text{deg.}$

Definition

$K_0^G(X)$ is the abelian group of equivalence classes of even cycles over X.
$K_1^G(X)$ is the abelian group of equivalence classes of odd cycles over X.

If we view an even cycle as an odd one by forgetting about the $\mathbb{Z}/2\mathbb{Z}$-graduation, then it is homotopic to a degenerate cycle.

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Definition

Suppose $G \curvearrowright Y$ is proper. Then $X \subseteq Y$ is G-compact if the action of G preserves X and X/G is compact.
Define the "G-equivariant K-homology of Y with compact supports" by

$$RK_i^G(Y) = \lim_{\substack{X \subseteq Y \\ X \ G\text{-compact}}} K_i^G(X).$$

Examples - left hand side

Theorem

If $G \curvearrowright X$ is free, then $K_i^G(X) = K_i^{\text{top}}(X/G)$ $(= K_i^e(X/G))$.
In particular, if G is torsion-free, then $K_i^G(\underline{E}G) = K_i^{\text{top}}(BG)$.

Example

For $i = 0, 1$, $(R)K_i^{\mathbb{Z}^n}(\underline{E}\mathbb{Z}^n) = K_i^{\text{top}}(B\mathbb{Z}^n) = K_i^{\text{top}}(\mathbb{T}^n) = \mathbb{Z}^{2^{n-1}}$.
$K_0(\mathbb{T})$ is generated by $(1, f \mapsto f(1), 0)$, and
$K_1(\mathbb{T})$ by $(1, M, \text{diag}(\text{sign}(n))_n)$.

Example

If G is finite, then $\underline{E}G = \{*\}$, which means that $K_0^G(\underline{E}G) \cong \mathbb{Z}\widehat{G}$,
$(U_0 \oplus U_1, \iota, 0) \mapsto [U_0] - [U_1]$ and $K_1^G(\underline{E}G) = 0$.

"$K_i^G(\underline{E}G) = K$-homology of BG + repr. theory of finite subgroups of G"

A Hilbert module over $C_r^*(G)$

Let (U, π, F) be a cycle for $G \curvearrowright X = \underline{E}G$ (assume G-compact).
We may (up to homotopy) assume that

- $\overline{\pi(C_0(X))\mathcal{H}} = \mathcal{H}$.
- for all $f \in C_c(X)$ there is $g \in C_c(X)$ such that $F\pi(f) = \pi(g)F\pi(f)$.

Set $H = \pi(C_c(X))\mathcal{H}$ and define a (right) Hilbert C^*-module $\mathcal{E} = \overline{H}$ over $C_r^*(G) = \overline{\mathbb{C}G}$ as follows (for $t \in G \subset \mathbb{C}G$ and $\xi \in H$):

$$\xi \cdot t = U(t^{-1})\xi, \qquad \langle \xi_1, \xi_2 \rangle(t) = \langle \xi_1, U(t)\xi_2 \rangle_{\mathcal{H}}$$

$F(H) \subseteq H$ and F extends continuously to a bounded operator $\mathcal{F}$ on $\mathcal{E}$.

Theorem (Kasparov)

$\mathcal{F}^2 - I$ is a compact operator on $\mathcal{E}$.
Moreover, $[\mathcal{F}] = \mathrm{Index}(\mathcal{F}) \in K_i(C_r^*(G)) \quad (= KK_i(\mathbb{C}, C_r^*(G)))$.
For $i = 0$, $\mathcal{F} = \begin{pmatrix} 0 & P^* \\ P & 0 \end{pmatrix}$ *and* "$\mathrm{Index}(\mathcal{F}) = [\ker(P)] - [\ker(P^*)]$".

The assembly map

Definition

The map $(U, \pi, F) \mapsto [\mathcal{F}]$ is a group homomorphism and extends to the direct limit and defines the assembly map

$$\mu_i^G \colon RK_i^G(\underline{E}G) \to K_i(C_r^*(G))$$

Conjecture (Baum-Connes)

μ_i^G is an isomorphism for $i = 0, 1$.

The conjecture is known to be true for finite groups, abelian groups, amenable groups, and a-T-menable groups (Higson-Kasparov).
It is not known whether it is true for $G = \mathrm{SL}(3, \mathbb{Z})$.
The only "counter-example" is the "Gromov group".

Examples

Example

Let G be finite and then $K_0^G(\{*\}) \cong \mathbb{Z}\widehat{G} \cong K_0(C_r^*(G))$.
The map can be obtained as
$(U, \iota, F = \left(\begin{smallmatrix} 0 & P^* \\ P & 0 \end{smallmatrix}\right)) \mapsto \mathrm{Index}(F) = [\ker(P)] - [\ker(P^*)]$.
For $i = 1$, it is 0 on both sides.

Example

Let $G = \mathbb{Z}$.
Then $K_0^{\mathbb{Z}}(\mathbb{R}) \cong K_0(\mathbb{T})$, and $\mu_0^G(1, f \mapsto f(1), 0) = [1] \in K_0(C_r^*(\mathbb{Z}))$.
More work to get $\mu_1^G((1, M, \mathrm{diag}(\mathrm{sign}(n))_n)) = [u] \in K_1(C_r^*(\mathbb{Z}))$.

Exact groups

A group G is called exact if whenever

$$1 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 1$$

is a short exact sequence of G-maps, then

$$1 \longrightarrow C_r^*(G,A) \longrightarrow C_r^*(G,A) \longrightarrow C_r^*(G,C) \longrightarrow 1$$

is a short exact sequence of C^*-algebras.
Do nonexact groups exist?
Probably yes; the "Gromov group".

Conjecture (Possible happy ending?)

If G is exact, then the Baum-Connes conjecture holds.

Intermediate crossed products

Consider a crossed product as a functor $A \mapsto C^*_\tau(G, A)$,
$\{G\text{-}C^*\text{-algebras}\} \to \{C^*\text{-algebras}\}$.
C^*_τ is exact if for all short exact sequences of G-C^*-algebras, then the resulting sequence of C^*-algebras is exact.
In particular, C^* is exact, while C^*_r is (probably) not exact.
For an intermediate crossed product $C^*_\tau(G, A)$ there are surjections:

$$C^*(G, A) \to C^*_\tau(G, A) \to C^*_r(G, A)$$

An intermediate crossed product is then a function τ that assigns to each G-C^*-algebras a norm-closed ideal $\tau(G, A)$ of $C^*(G, A)$, such that

- $\tau(G, A) \subseteq \ker \lambda \colon C^*(G, A) \to C^*_r(G, A)$ for all A
- for each *-homomorphism $A \to B$, the resulting *-homomorphism $C^*(G, A) \to C^*(G, B)$ maps $\tau(G, A)$ to $\tau(G, B)$

Then $C^*_\tau(G, A) = C^*(G, A)/\tau(G, A)$.

Reformulated Baum-Connes

Theorem (Kirchberg)

There exists a unique, minimal intermediate crossed product which is exact. Denote this crossed by C^*_{exact}.

Conjecture (Reformulated Baum-Connes)

The "corrected" assembly map

$$\mu^G_{\text{exact},i}\colon RK^G_i(\underline{E}G) \to K_i(C^*_{\text{exact}}(G))$$

is an isomorphism for $i = 0, 1$.

Assembly map via KK-theory (sketch)

Let A and B be separable G-C^*-algebras (with B unital).
Let $\mathcal{E}$ be a right Hilbert C^*-module over B.

Definition

A cycle over (A, B) is a triple $(U, \pi, \mathcal{F})$, where

- $U\colon G \to L_B(\mathcal{E})$, for some right Hilbert C^*-module $\mathcal{E}$ over B, s.t. $\langle U(t)\xi, U(t)\nu\rangle_B = t \cdot \langle \xi, \nu\rangle_B$.
- $\pi\colon A \to L_B(\mathcal{E})$ s.t. $\pi(t \cdot a) = U(t)\pi(a)U(t^{-1})$.
- $\mathcal{F}^* = \mathcal{F} \in L_B(\mathcal{E})$, and $\pi(a)(\mathcal{F}^2 - I), [\pi(a), \mathcal{F}], [U(t), \mathcal{F}]$ are compact for all $a \in A$ and $t \in G$.

A cycle is even if it is $\mathbb{Z}/2\mathbb{Z}$-graded and odd if not.

Assembly map via KK-theory (sketch) cont.

Definition

Let $KK_0^G(A,B)$ be the group of (equivalence classes) of even cycles and $KK_1^G(A,B)$ of odd cycles.

- $K_i^G(X) = KK_i^G(C_0(X),\mathbb{C})$
- $K_i(B) \cong KK_i^e(\mathbb{C},B)$;
  $i=0$: $x=[e_1]-[e_0]$ is associated with $(1,\pi_x,0)$,
  $\pi_x\colon \mathbb{C}\to L(\mathcal{H}_B\oplus\mathcal{H}_B)$, $\lambda\to\left(\begin{smallmatrix}\lambda e_0 & 0\\ 0 & \lambda e_1\end{smallmatrix}\right)$, where $\mathcal{H}_B=\ell^2\otimes B$.
  $i=1$: more technical, but not difficult

$$KK_i^G(C_0(\underline{E}G),\mathbb{C}) \xrightarrow{p_{\underline{E}G}\times j_G(\cdot)} KK_0(\mathbb{C},C_r^*(G,C_0(\underline{E}G)))\times KK_i(C_r^*(G,C_0(\underline{E}G)),C_r^*(G))$$
$$\longrightarrow KK_i(\mathbb{C},C_r^*(G))$$

where last map is the Kasparov product, and j_G is the descent homomorphism

$$j_G\colon KK_i^G(A,B)\to KK_i(C_r^*(G,A),C_r^*(G,B))$$