C^*-ALGEBRAS GENERATED BY PROJECTIVE REPRESENTATIONS OF FREE NILPOTENT GROUPS

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ABSTRACT. We compute the two-cocycles (or multipliers) of the free nilpotent groups of class 2 and rank n and give conditions for simplicity of the corresponding twisted group C^*-algebras. These groups are representation groups for $\mathbb{Z}^n$ and can be considered as a family of generalized Heisenberg groups with higher-dimensional center. Their group C^*-algebras are in a natural way isomorphic to continuous fields over $\mathbb{T}^{\frac{1}{2}n(n-1)}$ with the noncommutative n-tori as fibers. In this way, the twisted group C^*-algebras associated with the free nilpotent groups of class 2 and rank n may be thought of as "second order" noncommutative n-tori.

INTRODUCTION

The discrete Heisenberg group may be described as the group generated by three elements u_1, u_2, and v_{12} satisfying the commutation relations

$$[u_1, v_{12}] = [u_2, v_{12}] = 1 \quad \text{and} \quad [u_1, u_2] = v_{12}.$$

The group has received much attention in the literature, partly because it is one of the easiest examples of a nonabelian torsion-free group. Moreover, the continuous Heisenberg group (see below) is a connected nilpotent Lie group that arises in certain quantum mechanical systems.

As a natural consequence of this attention, several classes of generalized Heisenberg groups have been investigated. For example, in [14, 15] Milnes and Walters describe the four- and five-dimensional nilpotent groups, and in [11, 12] Lee and Packer study the finitely generated torsion-free two-step nilpotent groups with one-dimensional center.

In this paper, on the other hand, we will consider a family of generalized Heisenberg groups, denoted by $G(n)$ for $n \geq 2$, with larger center. The groups $G(n)$ are the so-called free nilpotent groups of class 2 and rank n and will be defined properly in Section 1. Here we also provide further motivation for our investigation of these groups. Inspired by the work of Packer [21] we compute the second cohomology group $H^2(G(n), \mathbb{T})$ of $G(n)$ and study the structure of the twisted group C^*-algebras $C^*(G(n), \sigma)$ associated with two-cocycles σ of $G(n)$.

Section 2 is devoted to two-cocycle calculations, where we decompose $G(n)$ into a semidirect product and apply techniques introduced by Mackey [13]. In particular, we will see that

$$H^2(G(n), \mathbb{T}) \cong \mathbb{T}^{\frac{1}{3}(n+1)n(n-1)},$$

and in Theorem 2.6 we give explicit formulas for the two-cocycles of $G(n)$ up to similarity.

Next, in Section 3 we describe $C^*(G(n), \sigma)$ as a universal C^*-algebra of a set of generators and relations. Then we construct the algebra that in a natural way appear as a continuous field over the compact space $H^2(G(n), \mathbb{T})$ with $C^*(G(n), \sigma)$ as fibers. We also explain that for $n = 2$, this algebra is the group C^*-algebra of the free nilpotent group of class 3 and rank 2.

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In Section 4 we investigate the center of $C^*(G(n),\sigma)$ and give conditions for simplicity of these twisted group C^*-algebras in Theorem 4.4 and Corollary 4.6.

Finally, in Section 5 we study the automorphism group of $G(n)$ and discuss isomorphism invariants of $C^*(G(n),\sigma)$ coming from $\operatorname{Aut} G(n)$.

1. The free nilpotent groups $G(n)$ of class 2 and rank n

For each natural number $n \geq 2$, let $G(n)$ be the group generated by elements $\{u_i\}_{1\leq i\leq n}$ and $\{v_{jk}\}_{1\leq j<k\leq n}$ subject to the relations

$$[v_{jk}, v_{lm}] = [u_i, v_{jk}] = 1 \quad \text{and} \quad [u_j, u_k] = v_{jk} \tag{1}$$

for $1 \leq i \leq n$, $1 \leq j < k \leq n$, and $1 \leq l < m \leq n$. Clearly, $G(2)$ is the usual discrete Heisenberg group. For some purposes, it can be useful to set $G(1) = \langle u_1 \rangle \cong \mathbb{Z}$. Note that $G(n)$ is generated by n elements, while its dimension equals $n + \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1)$.

The group $G(n)$ is called the free nilpotent group of class 2 and rank n. Indeed, $G(n)$ is a free object on n generators in the category of nilpotent groups of step at most two. To see this, note first that $G(n)$ is the group generated by $\{u_i\}_{i=1}^n$ subject to the relations that all commutators of order greater than two involving the generators are trivial. Let $G'(n)$ be any other nilpotent group of step at most two and let $\{u_i'\}_{i=1}^n$ be any set of n elements in $G'(n)$. Then there is a unique homomorphism from $G(n)$ to $G'(n)$ that maps u_i to u_i' for $1 \leq i \leq n$. Of course, every free object on n generators in this category is isomorphic to $G(n)$. For a more extensive treatment of free nilpotent groups, see the article on Terence Tao's website [27] (see also 2. in the list below).

Furthermore, we will need the following concrete realization, say $\widetilde{G}(n)$, of $G(n)$. For $n \geq 2$, we denote the elements of $\widetilde{G}(n)$ by

$$r = (r_1, \ldots, r_n, r_{12}, r_{13}, \ldots, r_{n-1,n})^1,$$

where all entries are integers, and define multiplication by

$$\begin{aligned} r \cdot s = (&r_1 + s_1, \ldots, r_n + s_n, \\ &r_{12} + s_{12} + r_1 s_2, r_{13} + s_{13} + r_1 s_3, \ldots, r_{n-1,n} + s_{n-1,n} + r_{n-1} s_n). \end{aligned}$$

By letting u_i have 1 in the i'th spot and 0 else and v_{jk} have 1 in the jk'th spot and 0 else, the relations (1) are satisfied for these elements. Next, we define the map

$$\widetilde{G}(n) \longrightarrow G(n), \quad r \longmapsto v_{12}^{r_{12}} \cdots v_{n-1,n}^{r_{n-1,n}} \cdot u_n^{r_n} \cdots u_1^{r_1},$$

and then it is not difficult to see that $\widetilde{G}(n)$ is isomorphic to $G(n)$. Henceforth, we will not distinguish between $G(n)$ and the realization $\widetilde{G}(n)$ just described, but this should cause no confusion.

Denote by $V(n)$ the subgroup of $G(n)$ generated by the v_{jk}'s. Then $V(n)$ coincides with the center $Z(G(n))$ of $G(n)$ and

$$V(n) = Z(G(n)) \cong \mathbb{Z}^{\frac{1}{2}n(n-1)}.$$

Indeed, both this and the next observations follow after noticing that

$$r \cdot s \cdot r^{-1} = (s_1, \ldots, s_n, s_{12} + r_1 s_2 - s_1 r_2, \ldots, s_{n-1,n} + r_{n-1} s_n - s_{n-1} r_n).$$

Moreover, consider the subgroups $G(n-1)$ and $H(n)$ of $G(n)$ defined by

$$\begin{aligned} G(n-1) &= \langle u_i, v_{jk} : 1 \leq i \leq n-1, 1 \leq j < k \leq n-1 \rangle, \\ H(n) &= \langle u_n, v_{jn} : 1 \leq j < n \rangle. \end{aligned}$$

[1] To be absolutely precise, the entries with double index are colexicographically ordered, that is, $(i,j) < (k,l)$ if $j < l$ or if $j = l$ and $i < k$.

Note that $G(n-1)$ sits inside $G(n)$ as a subgroup and that $H(n) \cong \mathbb{Z}^n$ is a normal subgroup of $G(n)$. Clearly, we have $G(n)/V(n) \cong \mathbb{Z}^n$ and $G(n)/H(n) \cong G(n-1)$. Therefore, there are short exact sequences

$$1 \longrightarrow V(n) \longrightarrow G(n) \longrightarrow \mathbb{Z}^n \longrightarrow 1$$

and

$$1 \longrightarrow H(n) \longrightarrow G(n) \longrightarrow G(n-1) \longrightarrow 1,$$

where the second one splits and the first does not. In particular, $G(n)$ is a central extension of $\mathbb{Z}^n$ by $\mathbb{Z}^{\frac{1}{2}n(n-1)}$, showing that $G(n)$ is indeed a two-step nilpotent group.

To motivate our investigation of $G(n)$, we present a few aspects about these groups and some appearances in the literature.

1. Consider in the first place the *continuous* Heisenberg group. We will represent this group in two different ways, G_{matrix} and G_{wedge}, both with elements $(x, x') = (x_1, x_2, x') \in \mathbb{R}^3$, i.e. $x = (x_1, x_2) \in \mathbb{R}^2$, and with multiplication as follows. For G_{matrix} we define

$$(x_1, x_2, x')(y_1, y_2, y') = (x_1 + y_1, x_2 + y_2, x' + y' + x_1 y_2)\,,$$

and for G_{wedge} we set

$$(x_1, x_2, x')(y_1, y_2, y') = \big(x_1 + y_1, x_2 + y_2, x' + y' + \tfrac{1}{2}(x_1 y_2 - x_2 y_1)\big)\,.$$

One can deduce that $G_{\text{matrix}} \cong G_{\text{wedge}}$. To motivate the notation, note that G_{matrix} can be represented as matrix multiplication in $M_3(\mathbb{R})$ if one identifies

$$(x_1, x_2, x') \longleftrightarrow \begin{bmatrix} 1 & x_1 & x' \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{bmatrix},$$

and that the multiplication in G_{wedge} may be written as

$$(x, x')(y, y') = \big(x + y, x' + y' + \tfrac{1}{2}(x \wedge y)\big)\,.$$

In general, the wedge product on $\mathbb{R}^n$ is defined as a certain bilinear map (see e.g. [25, p. 79])

$$\mathbb{R}^n \times \mathbb{R}^n \to \textstyle\bigwedge^2(\mathbb{R}^n),$$

where $\bigwedge^2(\mathbb{R}^n)$ is a $\frac{1}{2}n(n-1)$-dimensional real vector space. The elements of $\bigwedge^2(\mathbb{R}^n)$ are called bivectors and if $\{e_i\}_{i=1}^n$ is a basis for $\mathbb{R}^n$, then $\{e_i \wedge e_j\}_{i<j}$ is a basis for $\bigwedge^2(\mathbb{R}^n)$. For every $n \geq 2$, define the group $\widehat{G}(n, \mathbb{R})$ with elements

$$(x, x') \in \mathbb{R}^n \oplus \textstyle\bigwedge^2(\mathbb{R}^n), \quad \text{where } x = (x_1, \ldots, x_n), \quad x' = (x'_{12}, x'_{13}, \ldots, x'_{n-1,n}),$$

and where multiplication is given by

$$(x, x')(y, y') = \big(x + y, x' + y' + \tfrac{1}{2}(x \wedge y)\big)\,.$$

This group is of dimension $n + \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1)$. Remark especially that if $n = 3$, the wedge product can be identified with the vector cross product on $\mathbb{R}^3$. That is, the product in $\widehat{G}(3, \mathbb{R})$ is given by

$$(x, x')(y, y') = \big(x + y, x' + y' + \tfrac{1}{2}(x \times y)\big)\,.$$

It is not hard to see that $\widehat{G}(n, \mathbb{R})$ is isomorphic to the group consisting of the same elements, but with multiplication given by

$$(x, x')(y, y') = (x + y, x' + y' + (x_1 y_2, x_1 y_3, \ldots, x_{n-1} y_n))\,. \tag{2}$$

Let $G(n, \mathbb{R})$ denote the group defined by (2). Then $G(n)$ is the integer version of $G(n, \mathbb{R})$.

We also mention that Nielsen [17] has classified all the six-dimensional connected, simply connected, nilpotent Lie groups. In this setting, $G(3,\mathbb{R})$ is the group denoted by $G_{6,15}$.

2. One may define the free nilpotent group $G(m,n)$ of class m and rank n for every $m \geq 1$. Indeed, $G(m,n)$ is the group generated by $\{u_i\}_{i=1}^n$ subject to the relations that all commutators of order greater than m involving the generators are trivial. More precisely, for $m = 1,2,3$ and $n \geq 2$, we have that $G(m,n)$ can be described as the groups with presentations
$$\begin{aligned} G(1,n) &= \langle \{u_i\}_{i=1}^n : [u_i, u_j] = 1\rangle \cong \mathbb{Z}^n, \\ G(2,n) &= \langle \{u_i\}_{i=1}^n : [[u_i, u_j], u_k] = 1\rangle = G(n), \\ G(3,n) &= \langle \{u_i\}_{i=1}^n : [[[u_i, u_j], u_k], u_l] = 1\rangle, \end{aligned} \tag{3}$$
and it should now be clear how to define $G(m,n)$ for all $m \geq 1$ and $n \geq 2$. Finally, we set $G(m,1) = \langle u_1\rangle \cong \mathbb{Z}$ for each $m \geq 1$. One can also formally define $G(m,n)$ as follows: For $n \geq 1$, let $\mathbb{F}_n$ denote the free group on n generators, set $\gamma_1(\mathbb{F}_n) = \mathbb{F}_n$ and for $m \geq 1$, set $\gamma_{m+1}(\mathbb{F}_n) = [\mathbb{F}_n, \gamma_m(\mathbb{F}_n)]$. Then $G(m,n) = \mathbb{F}_n/\gamma_{m+1}(\mathbb{F}_n)$.
   Moreover, for all $m \geq 1$ and $n \geq 2$, the group $G(m,n)$ is the free object on n generators in the category of nilpotent groups of step at most m. In particular, notice that $G(m,n)$ is m-step nilpotent and that
$$G(m,n) \cong G(m+1,n)/Z(G(m+1,n)). \tag{4}$$
Again, we refer to [27] for additional details.
   In [15, Section 4] Milnes and Walters describe the simple quotients of the C^*-algebra associated with a five-dimensional group denoted by $H_{5,4}$. One can check that $H_{5,4}$ is isomorphic to the group $G(3,2)$. See Remark 3.2 for more about this group.
3. The group $G(3)$ is briefly discussed by Baggett and Packer [4, Example 4.3]. The purpose of that paper is to describe the primitive ideal space of group C^*-algebras of some two-step nilpotent groups. However, $G(3)$ only serves as an example of a group the authors could not handle.
4. Let $n \geq 2$. It is well-known that the group C^*-algebra $A = C^*(G(n))$ may be described as the universal C^*-algebra generated by unitaries $\{U_i\}_{1\leq i\leq n}$ and $\{V_{jk}\}_{1\leq j<k\leq n}$ satisfying the relations
$$[V_{jk}, V_{lm}] = [U_i, V_{jk}] = I \quad \text{and} \quad [U_j, U_k] = V_{jk}$$
for all $1 \leq i \leq n$, $1 \leq j < k \leq n$, and $1 \leq l < m \leq n$.
   For $\lambda = (\lambda_{12}, \lambda_{13}, \ldots, \lambda_{n-1,n}) \in \mathbb{T}^{\frac{1}{2}n(n-1)}$, let $\mathcal{A}_\lambda$ be the noncommutative n-torus. It is the universal C^*-algebra generated by unitaries $\{W_i\}_{i=1}^n$ and relations $[W_i, W_j] = \lambda_{ij} I$ for $1 \leq i < j \leq n$. The universal property of A gives that for each λ in $\mathbb{T}^{\frac{1}{2}n(n-1)}$ there is a surjective *-homomorphism
$$\pi_\lambda \colon A \to \mathcal{A}_\lambda$$
satisfying $\pi_\lambda(U_i) = W_i$ for $1 \leq i \leq n$ and $\pi_\lambda(V_{jk}) = \lambda_{jk} I$ for $1 \leq j < k \leq n$.
   Furthermore, A has center $Z(A) = C^*(\{V_{jk}\}_{1\leq j<k\leq n}) \cong C^*(V(n))$. Indeed, this is the case since $G(n)$ is amenable and its finite conjugacy classes are precisely the one-point sets of central elements (see Lemma 4.1 below). Therefore, we set
$$T = \operatorname{Prim} Z(A) \cong \widehat{Z(A)} = \mathbb{T}^{\frac{1}{2}n(n-1)}.$$
Let λ be a primitive ideal of $Z(A)$ identified with an element of $\mathbb{T}^{\frac{1}{2}n(n-1)}$. Let $\mathcal{I}_\lambda$ be the ideal of A generated by λ, that is, the ideal generated by $\{V_{jk} - \lambda_{jk} I : 1 \leq j < k \leq n\}$. It is clear that $\mathcal{I}_\lambda \subseteq \ker \pi_\lambda$. By the universal property of $\mathcal{A}_\lambda$, there is a *-homomorphism
$$\rho \colon \mathcal{A}_\lambda \to A/\mathcal{I}_\lambda$$

such that $\rho(W_i) = U_i + \mathcal{I}_\lambda$ for $1 \leq i \leq n$. Hence, $\rho \circ \pi_\lambda$ coincides with the quotient map $A \to A/\mathcal{I}_\lambda$ and consequently, $\ker \pi_\lambda \subseteq \mathcal{I}_\lambda$. Therefore, $\mathcal{A}_\lambda \cong A/\mathcal{I}_\lambda$ and π_λ may be regarded as the quotient map $A \to A/\mathcal{I}_\lambda$.

For an element a of A, let $\tilde{a}$ be the section $T \to \bigsqcup_T \mathcal{A}_\lambda$ given by $\tilde{a}(\lambda) = \pi_\lambda(a)$ and let $\widetilde{A} = \{\tilde{a} \mid a \in A\}$ be the set of all such sections. Then the following can be deduced from the Dauns-Hofmann Theorem [5].

Theorem 1.1. *The triple $(T, \{\mathcal{A}_\lambda\}, \widetilde{A})$ consisting of the base space T, C^*-algebras $\mathcal{A}_\lambda$ for each λ in T, and the set of sections $\widetilde{A}$, is a full continuous field of C^*-algebras. Moreover, the C^*-algebra associated with this continuous field is naturally isomorphic to A.*

This result may be obtained as a corollary to [24, Theorem 1.2] which employs tools of Williams [29] related to Fell bundle theory, by taking $G = G(n)$ and $\sigma = 1$ in that theorem. It is also a special case of [3, Corollary 2.3]. Our proof is more direct and partly inspired by [1, Theorem 1.1] which covers the case where $n = 2$.

From the above discussion it now follows that $G(n)$ is a *representation group* for $\mathbb{Z}^n$ in the sense of Moore [16]. In this case, that means $G(n)$ is (up to isomorphism) the unique central extension of $\mathbb{Z}^n$ by $\widehat{H^2(\mathbb{Z}^n, \mathbb{T})}$ such that the ordinary irreducible representation theory of $G(n)$ coincides with the projective irreducible representation theory of $\mathbb{Z}^n$.

This fact plays an important role in [6], where the noncommutative principal torus bundles over locally compact spaces are classified up to equivariant Morita equivalence. As explained in [6, Section 2], the group C^*-algebra of $G(n)$ serves as a "universal" bundle in this classification.

We refer to [7, Section 4] for more information on representation groups, where the groups $G(n, \mathbb{R})$ and $G(n)$ are treated particularly in [7, Example 4.7].

2. The two-cocycles of the free nilpotent groups $G(n)$

Let G be *any* discrete group with identity e. A function $\sigma\colon G \times G \to \mathbb{T}$ satisfying

$$\sigma(r, s)\sigma(rs, t) = \sigma(r, st)\sigma(s, t)$$
$$\sigma(r, e) = \sigma(e, r) = 1$$

for all elements $r, s, t \in G$ is called a *two-cocycle of G with values in* $\mathbb{T}$ (or a *multiplier of G*). Moreover, two two-cocycles σ and τ are said to be *similar*, written $\sigma \sim \tau$, if

$$\tau(r, s) = \beta(r)\beta(s)\overline{\beta(rs)}\sigma(r, s)$$

for all $r, s \in G$ and some function $\beta\colon G \to \mathbb{T}$. The set of similarity classes of two-cocycles of G is an abelian group under pointwise multiplication. This group is the second cohomology group $H^2(G, \mathbb{T})$.

Let G be a semidirect product of a normal subgroup H and a subgroup K. By properties of the semidirect product, the elements of G can be uniquely written as products ab, where a belongs to H and b belongs to K. Define the action α of K on H by $\alpha_b(a) = bab^{-1}$. One often writes $G = H \rtimes_\alpha K$, but to simplify the notation, we will still denote the elements of G by ab instead (a, b) and write the group product in G as $(ab)(a'b') = a\alpha_b(a')bb'$ for $a, a' \in H$ and $b, b' \in K$. Hopefully, the reader is familiar with semidirect products so that this does not cause any confusion.

Next, we apply Mackey's theorem [13, Theorem 9.4] and obtain the following result.

Theorem 2.1. *Every two-cocycle of G is similar to a two-cocycle σ of G of the form*

$$\sigma(a'b, ab') = \sigma_H(a', \alpha_b(a))g(a, b)\sigma_K(b, b'), \tag{5}$$

where σ_H and σ_K are two-cocycles of H and K, respectively,

$$g\colon H \times K \to \mathbb{T}$$

is a function such that $g(a,e) = g(e,b) = 1$ for all $a \in H$, $b \in K$, and σ_H and g satisfy

$$\begin{aligned} g(aa', b) &= \sigma_H(\alpha_b(a), \alpha_b(a'))\overline{\sigma_H(a,a')} \cdot g(a,b)g(a',b), \\ g(a, bb') &= g(\alpha_{b'}(a), b)g(a,b'). \end{aligned} \tag{6}$$

Moreover, for every choice of σ_H, g, and σ_K satisfying the conditions above, σ is a two-cocycle of G.

Proposition 2.2. *Let (σ_H, g, σ_K) and $(\sigma'_H, g', \sigma'_K)$ be triples satisfying the conditions of Theorem 2.1 and let σ and σ' be the corresponding two-cocycles of G. Then $\sigma \sim \sigma'$ if and only if the following conditions hold:*

(i) $\sigma_K \sim \sigma'_K$,
(ii) there exists a function $\beta\colon H \to \mathbb{T}$ such that

$$\begin{aligned} \sigma'_H(a,a') &= \overline{\beta(a)\beta(a')}\beta(aa')\sigma_H(a,a'), \\ g'(a,b) &= \beta(\alpha_b(a))\overline{\beta(a)}g(a,b). \end{aligned}$$

Remark 2.3. If (ii) holds, then $\sigma_H \sim \sigma'_H$. If $\sigma_H \sim \sigma'_H$ and β and β' are two functions implementing the similarity, then $\beta' = f \cdot \beta$ for some homomorphism $f\colon H \to \mathbb{T}$.

Proof of Proposition 2.2. Suppose $\sigma \sim \sigma'$. Then there exists some $\gamma\colon G \to \mathbb{T}$ such that

$$\sigma(a'b, ab') = \gamma(a'b)\gamma(ab')\overline{\gamma(a'bab')}\sigma'(a'b, ab') \tag{7}$$

for all $a, a' \in H$ and $b, b' \in K$. In particular, if $a = a' = e$, then

$$\sigma_K(b,b') = \gamma(b)\gamma(b')\overline{\gamma(bb')}\sigma'_K(b,b')$$

for all $b, b' \in K$, so $\sigma_K \sim \sigma'_K$. Moreover, the formula (5) from Theorem 2.1 with $a = e$ and $b = e$ gives that

$$\sigma(a', b') = 1 = \sigma'(a', b')$$

for all $a' \in H$ and $b' \in K$. Applying this fact to (7) shows that $\gamma(a'b') = \gamma(a')\gamma(b')$ for all $a' \in H$ and $b' \in K$. Define β on H by $\beta(a) = \gamma(a)$. Then, by letting $b = b' = e$ in (5) and (7), we get

$$\sigma'_H(a', a) = \overline{\beta(a')\beta(a)}\beta(a'a)\sigma_H(a', a)$$

for all $a', a \in H$. Furthermore, by letting $a' = e$ and $b' = e$ in (5) and (7), we compute

$$\begin{aligned} g(a,b) &= \gamma(b)\gamma(a)\overline{\gamma(ba)}g'(a,b) \\ &= \gamma(b)\gamma(a)\overline{\gamma(\alpha_b(a)b)}g'(a,b) \\ &= \gamma(b)\gamma(a)\overline{\gamma(\alpha_b(a))\gamma(b)}g'(a,b) \\ &= \gamma(a)\overline{\gamma(\alpha_b(a))}g'(a,b) \\ &= \beta(a)\overline{\beta(\alpha_b(a))}g'(a,b) \end{aligned}$$

for all $a \in H$ and $b \in K$.

Assume next that β is such that (ii) holds, and that (i) holds through δ, that is,

$$\sigma_K(b,b') = \delta(b)\delta(b')\overline{\delta(bb')}\sigma'_K(b,b').$$

Define γ on G by $\gamma(ab) = \beta(a)\delta(b)$. Then

$$\begin{aligned}\sigma(a'b, ab') &= \sigma_H(a', \alpha_b(a))g(a,b)\sigma_K(b,b')\\ &= \beta(a')\beta(\alpha_b(a))\overline{\beta(a'\alpha_b(a))}\sigma'_H(a', \alpha_b(a))\\ &\quad \cdot \beta(a)\overline{\beta(\alpha_b(a))}g'(a,b) \cdot \delta(b)\delta(b')\overline{\delta(bb')}\sigma'_K(b,b')\\ &= \beta(a')\delta(b) \cdot \beta(a)\delta(b') \cdot \overline{\beta(a'\alpha_b(a))\delta(bb')}\sigma'(a'b, ab')\\ &= \gamma(a'b)\gamma(ab')\overline{\gamma(a'bab')}\sigma'(a'b, ab').\end{aligned}$$

□

Fix $n \geq 2$. To compute the two-cocycles of $G(n)$ up to similarity, we will proceed in the following way. Consider $G(n)$ as the split extension of $G(n-1)$ by $H(n)$ as described in Section 1. We will identify the elements

$$\begin{aligned} a &= (0, \dots, 0, a_n, 0, \dots, 0, a_{1n}, \dots, a_{n-1,n}),\\ b &= (b_1, \dots, b_{n-1}, 0, b_{12}, \dots, b_{n-2,n-1}, 0, \dots, 0),\end{aligned}$$

of $H(n)$ and $G(n-1)$, respectively, with ones of the form

$$\begin{aligned} a &\longleftrightarrow (a_n, a_{1n}, \dots, a_{n-1,n}),\\ b &\longleftrightarrow (b_1, \dots, b_{n-1}, b_{12}, \dots, b_{n-2,n-1}).\end{aligned}$$

The elements of $G(n)$ will be written as products ab, where a belongs to $H(n)$ and b belongs to $G(n-1)$, and the action α of $G(n-1)$ on $H(n)$ is then given by

$$\alpha_b(a) = bab^{-1} = (a_n, a_{1n} + b_1 a_n, \dots, a_{n-1,n} + b_{n-1}a_n).$$

Remark 2.4. In the published version, Theorem 2.1 and Proposition 2.2 were only shown to hold for $G(n)$, not for any semidirect product.

Proposition 2.2 can be deduced from [24, Appendix 2], but in any case it may be useful to give a proof by a direct computation.

Let τ_n be a two-cocycle of $G(n)$ coming from a pair $(\sigma_{H(n)}, g_n)$, that is,

$$\tau_n(a'b, ab') = \sigma_{H(n)}(a', \alpha_b(a))g_n(a,b), \tag{8}$$

where $(\sigma_{H(n)}, g_n)$ satisfies (6). By Theorem 2.1 and Proposition 2.2, every two-cocycle of $G(n)$ that is trivial on $G(n-1)$ is similar to one of this form. Denote the abelian group of similarity classes of two-cocycles of this type by $\widetilde{H}^2(G(n), \mathbb{T})$.

Corollary 2.5. *The second cohomology group of $G(n)$ may be decomposed as*

$$H^2(G(n), \mathbb{T}) = \widetilde{H}^2(G(n), \mathbb{T}) \oplus H^2(G(n-1), \mathbb{T}) = \bigoplus_{k=2}^{n} \widetilde{H}^2(G(k), \mathbb{T}).$$

Proof. It follows from Theorem 2.1 and Proposition 2.2 (see our comment above) that

$$H^2(G(n), \mathbb{T}) = \widetilde{H}^2(G(n), \mathbb{T}) \oplus H^2(G(n-1), \mathbb{T}).$$

Thus, the second inequality is proven by induction after noticing that

$$\{1\} = H^2(\mathbb{Z}, \mathbb{T}) = H^2(G(1), \mathbb{T}) = \widetilde{H}^2(G(1), \mathbb{T}).$$ □

Theorem 2.6. *We have*

$$H^2(G(n), \mathbb{T}) \cong \mathbb{T}^{\frac{1}{3}(n+1)n(n-1)},$$

and for each set of $\frac{1}{3}(n+1)n(n-1)$ parameters

$$\{\lambda_{i,jk} : 1 \leq i \leq k, 1 \leq j < k \leq n\} \subseteq \mathbb{T},$$

the associated $[\sigma]$ *in* $H^2(G(n),\mathbb{T})$ *may be represented by*

$$\begin{aligned}\sigma(r,s) = &\prod_{i<j<k} \lambda_{i,jk}^{s_{jk}r_i+s_kr_{ij}} \lambda_{j,ik}^{s_{ik}r_j+s_k(r_ir_j-r_{ij})} \\ &\cdot \prod_{j<k} \lambda_{j,jk}^{s_{jk}r_j+\frac{1}{2}s_kr_j(r_j-1)} \lambda_{k,jk}^{r_k(s_{jk}+r_js_k)+\frac{1}{2}r_js_k(s_k-1)}.\end{aligned} \tag{9}$$

The proof of this theorem will be given in Section 2.1.

See the paragraph following Theorem 3.1 for an explanation of why $\lambda_{i,jk}$ for $i>k$ is not involved in the above.

Example 2.7. For $G(1)\cong\mathbb{Z}$ there are no nontrivial two-cocycles. The two-cocycles of the usual Heisenberg group $G(2)$ are, up to similarity, given by two parameters (as computed in [21, Proposition 1.1]):

$$\sigma(r,s) = \lambda_{1,12}^{s_{12}r_1+\frac{1}{2}s_2r_1(r_1-1)} \lambda_{2,12}^{r_2(s_{12}+r_1s_2)+\frac{1}{2}r_1s_2(s_2-1)} \tag{10}$$

The two-cocycles of $G(3)$ are, up to similarity, given by eight parameters:

$$\begin{aligned}\sigma(r,s) = &\lambda_{1,23}^{s_{23}r_1+s_3r_{12}} \lambda_{2,13}^{s_{13}r_2+s_3(r_1r_2-r_{12})} \\ &\cdot \lambda_{1,12}^{s_{12}r_1+\frac{1}{2}s_2r_1(r_1-1)} \lambda_{2,12}^{r_2(s_{12}+r_1s_2)+\frac{1}{2}r_1s_2(s_2-1)} \\ &\cdot \lambda_{1,13}^{s_{13}r_1+\frac{1}{2}s_3r_1(r_1-1)} \lambda_{3,13}^{r_3(s_{13}+r_1s_3)+\frac{1}{2}r_1s_3(s_3-1)} \\ &\cdot \lambda_{2,23}^{s_{23}r_2+\frac{1}{2}s_3r_2(r_2-1)} \lambda_{3,23}^{r_3(s_{23}+r_2s_3)+\frac{1}{2}r_2s_3(s_3-1)}\end{aligned}$$

Remark 2.8. One may associate a Lyndon-Hochschild-Serre spectral sequence with the extension (see e.g. [28, 6.8.2]):

$$1 \longrightarrow V(n) \longrightarrow G(n) \longrightarrow \mathbb{Z}^n \longrightarrow 1$$

By applying [10, Theorem 4] to this sequence, one can compute the second homology group of $G(n)$ (which is recently also done more generally for $G(m,n)$ in [26, Proposition 2.1]), and deduce that

$$H_2(G(n),\mathbb{Z}) \cong \mathbb{Z}^{\frac{1}{3}(n+1)n(n-1)},$$

which gives that $H^2(G(n),\mathbb{T})\cong\mathbb{T}^{\frac{1}{3}(n+1)n(n-1)}$ after dualizing, using the universal coefficient theorem for cohomology. However, this does not give an explicit description of $H^2(G(n),\mathbb{T})$.

2.1. **Proof of Theorem 2.6.** We will in this proof first compute $\widetilde{H}^2(G(n),\mathbb{T})$ through several lemmas and then use Corollary 2.5 to conclude the argument.

Lemma 2.1.1. *Every element of* $\widetilde{H}^2(G(n),\mathbb{T})$ *may be represented by a pair* $(\sigma_{H(n)}, g_n)$, *where* $\sigma_{H(n)}$ *is a two-cocycle of* $H(n)$ *given by*

$$\sigma_{H(n)}(a',a) = \prod_{i=1}^{n-1} \lambda_i^{a'_n a_{in}} \tag{11}$$

for some $\lambda_1,\dots,\lambda_{n-1}\in\mathbb{T}$, *and* g_n *satisfies*

$$g_n(a+a',b) = \Big(\prod_{i=1}^{n-1} \lambda_i^{b_i a_n a'_n}\Big) g_n(a,b) g_n(a',b) \tag{12}$$

for all $a,a'\in H(n)$ *and* $b\in G(n-1)$.

Proof. Every element of $\widetilde{H}^2(G(n),\mathbb{T})$ may be represented by a two-cocycle of the form (8), that is, by a pair $(\sigma_{H(n)}, g_n)$ satisfying (6).

Moreover, it is well-known (see e.g. [2]) that every two-cocycle of $H(n) \cong \mathbb{Z}^n$ is similar to one of the form

$$\sigma_{H(n)}(a',a) = \prod_{1\le i\le n-1} \lambda_i^{a'_n a_{in}} \cdot \prod_{1\le j<k\le n-1} \mu_{jk}^{a'_{jn} a_{kn}}$$

for some sets of scalars $\{\lambda_i\}_{1\le i\le n-1}, \{\mu_{jk}\}_{1\le j<k\le n-1} \subseteq \mathbb{T}$. Since $H(n)$ is abelian, (6) gives that

$$\begin{aligned}\sigma_{H(n)}(\alpha_b(a),\alpha_b(a'))\overline{\sigma_{H(n)}(a,a')} &= g_n(a+a',b)\overline{g_n(a,b)g_n(a',b)}\\ &= g_n(a'+a,b)\overline{g_n(a',b)g_n(a,b)}\\ &= \sigma_{H(n)}(\alpha_b(a'),\alpha_b(a))\overline{\sigma_{H(n)}(a',a)}\end{aligned}$$

for all $a, a' \in H(n)$ and $b \in G(n-1)$. Furthermore, we have

$$\begin{aligned}&\sigma_H(\alpha_b(a),\alpha_b(a'))\overline{\sigma_H(a,a')}\\ &\quad= \prod_{1\le i\le n-1} \lambda_i^{a_n(a'_{in}+b_i a'_n)-a_n a'_{in}} \cdot \prod_{1\le j<k\le n-1} \mu_{jk}^{(a_{jn}+b_j a_n)(a'_{kn}+b_k a'_n)-a_{jn}a'_{kn}}\\ &\quad= \prod_{1\le i\le n-1} \lambda_i^{b_i a_n a'_n} \cdot \prod_{1\le j<k\le n-1} \mu_{jk}^{b_j a'_{kn} a_n + b_k a_{jn} a'_n + b_j b_k a_n a'_n}.\end{aligned}$$

This is equal to $\sigma_H(\alpha_b(a'),\alpha_b(a))\overline{\sigma_H(a',a)}$ for all $a,a' \in H(n)$ and $b \in G(n-1)$ if and only if the expression remains unchanged under the substitution $a \longleftrightarrow a'$, that is, if and only if all the μ_{jk}'s are 1. ☐

Lemma 2.1.2. *For every element of* $\widetilde{H}^2(G(n),\mathbb{T})$ *there is a* unique *associated pair* $(\sigma_{H(n)}, g_n)$ *satisfying the conditions of Lemma 2.1.1 such that*

$$g_n(u_n,u_i) = 1 \quad \textit{for all} \quad 1\le i\le n-1. \tag{13}$$

Proof. Suppose that $(\sigma_{H(n)}, g_n)$ satisfies (11) and (12). Let $f\colon H(n)\to\mathbb{T}$ be the homomorphism determined by $f(u_n) = 1$ and $f(v_{in}) = \overline{g_n(u_n,u_i)}$ for all $1\le i\le n-1$ and define g'_n by $g'_n(a,b) = f(\alpha_b(a))\overline{f(a)}g_n(a,b)$. Then, $g'_n(u_n,u_i) = 1$ for all $1\le i\le n-1$ and by Proposition 2.2, $(\sigma_{H(n)}, g'_n)$ determines a two-cocycle of $H(n)$ in the same similarity class as the one coming from $(\sigma_{H(n)}, g_n)$.

Suppose now that there are two pairs $(\sigma_{H(n)}, g_n)$ and $(\sigma'_{H(n)}, g'_n)$ both satisfying the conditions of Lemma 2.1.1. Then $\sigma'_{H(n)} = \sigma_{H(n)}$, so by Proposition 2.2 and the succeeding remark, there is a homomorphism $f\colon H(n)\to\mathbb{T}$ such that

$$g'_n(a,b) = f(\alpha_b(a))\overline{f(a)}g_n(a,b) = \Big(\prod_{i=1}^{n-1} f(v_{in})^{a_n b_i}\Big) g_n(a,b)$$

for all $a\in H(n)$ and $b\in G(n-1)$. In particular,

$$g'_n(u_n,u_i) = f(v_{in})g_n(u_n,u_i) \quad \text{for all} \quad 1\le i\le n-1,$$

so that $g'_n = g_n$ if $g'_n(u_n,u_i) = g_n(u_n,u_i)$ for all $1\le i\le n-1$. ☐

In the forthcoming lemmas we fix an element of $\widetilde{H}^2(G(n),\mathbb{T})$, and let $(\sigma_{H(n)}, g)$ be the unique associated pair satisfying (11), (12), and (13) for some set of scalars $\{\lambda_i\}_{i=1}^{n-1} \subseteq \mathbb{T}$.

For computational reasons, we introduce the following notation. For $a = (a_n, a_{1n}, \dots, a_{n-1,n})$ in $H(n)$, we write $a = w(a) + z(a)$, where $w(a) = (a_n, 0, \dots, 0)$, and $z(a)$ is the "central part", i.e. $z(a) = (0, a_{1n}, \dots, a_{n-1,n})$. Similarly, for $b = (b_1, \dots, b_{n-1}, b_{12}, \dots, b_{n-2,n-1})$

in $G(n-1)$, we write $b = w(b)z(b)$, where $w(b) = (b_1, \ldots, b_{n-1}, 0, \ldots, 0)$ and $z(b) = (0, \ldots, 0, b_{12}, \ldots, b_{n-2,n-1})$. Note that $\alpha_b(a) = a$ if either $w(a)$ or $w(b)$ is trivial, i.e. if either a or b is central.

Lemma 2.1.3. *For all $a \in H(n)$ and $b \in G(n-1)$ we have*

$$g(a, b) = g(w(a), w(b))g(w(a), z(b))g(z(a), w(b)).$$

Proof. It follows immediately from Lemma 2.1.1 that if $a, a' \in H(n)$ and $w(a)$ or $w(a')$ is 0, then

$$g(a + a', b) = g(a, b)g(a', b), \tag{14}$$

hence,

$$g(a, b) = g(w(a) + z(a), b) = g(w(a), b)g(z(a), b)$$

for all $a \in H(n)$ and $b \in G(n-1)$. If $b' \in G(n-1)$ and $w(b') = e$, then b' is central and $\alpha_{b'}(a) = a$ for all $a \in H(n)$. Therefore,

$$g(a, b)g(a, b') = g(a, bb') = g(a, b'b) = g(\alpha_b(a), b')g(a, b) \tag{15}$$

for all $a \in H(n)$, $b \in G(n-1)$. By (14), we then get

$$\begin{aligned} 1 &= g(\alpha_b(a), b')\overline{g(a, b')} \\ &= g(a + (0, b_1 a_n, \ldots, b_{n-1}a_n), b')\overline{g(a, b')} \\ &= g(a, b')g((0, b_1 a_n, \ldots, b_{n-1}a_n), b')\overline{g(a, b')} \\ &= g((0, b_1 a_n, \ldots, b_{n-1}a_n), b') \end{aligned}$$

for all $a \in H(n)$ and $b \in G(n-1)$. Consequently, since this holds for all $a \in H(n)$ and $b \in G(n-1)$, and central $b' \in G(n-1)$, we get that if $\tilde{a}$ and $\tilde{b}$ are *any* elements in $H(n)$ and $G(n-1)$, respectively, then $g(z(\tilde{a}), z(\tilde{b})) = 1$. Moreover, (15) also imply that if $b, b' \in G(n-1)$ and *either* $w(b)$ *or* $w(b')$ is equal to e, that is, either b or b' is central, then

$$g(a, bb') = g(a, b)g(a, b'). \tag{16}$$

Hence, by (16) and (14),

$$\begin{aligned} g(a, b) &= g(a, w(b)z(b)) = g(a, w(b))g(a, z(b)) \\ &= g(w(a), w(b))g(z(a), w(b))g(w(a), z(b)) \cdot 1 \end{aligned}$$

for all $a \in H(n)$ and $b \in G(n-1)$. $\square$

Lemma 2.1.4. *For all $a \in H(n)$ and $b, b' \in G(n-1)$ we have*

$$g(z(a), w(b)) = \prod_{i,j=1}^{n-1} g(v_{in}, u_j)^{a_{in} b_j},$$

$$g(w(a), z(b)) = \prod_{1 \le i < j \le n} g(u_n, v_{ij})^{a_n b_{ij}} = \prod_{1 \le i < j \le n} \left(\overline{g(v_{in}, u_j)} g(v_{jn}, u_i)\right)^{a_n b_{ij}},$$

and

$$g(a, bb') = \Big(\prod_{i,j=1}^{n-1} g(v_{in}, u_j)^{b'_i b_j a_n}\Big) g(a, b)g(a, b'). \tag{17}$$

Proof. Let $z(H(n)) = \{z(a) \mid a \in H(n)\}$ and $z(G(n-1)) = \{z(b) \mid b \in G(n-1)\}$. Then g is a bihomomorphism when restricted to $z(H(n)) \times G(n-1)$ or $H(n) \times z(G(n-1))$. Therefore, the first two identities hold. Indeed, this follows directly from (6) after noticing that since $z(a)$ and $z(b)$ are central,

$$\alpha_{w(b)}(z(a)) = z(a) \quad \text{and} \quad \alpha_{z(b)}(w(a)) = w(a).$$

Moreover, for $i < j$ we have $u_i u_j = v_{ij} u_j u_i$. By (6) and the previous lemma, one calculates

$$\begin{aligned} g(u_n, u_i u_j) &= g(\alpha_{u_j}(u_n), u_i) g(u_n, u_j) \\ &= g(u_n v_{jn}, u_i) g(u_n, u_j) \\ &= g(u_n, u_i) g(v_{jn}, u_i) g(u_n, u_j) \end{aligned}$$

and

$$\begin{aligned} g(u_n, v_{ij} u_j u_i) &= g(u_n, v_{ij}) g(u_n, u_j u_i) \\ &= g(u_n, v_{ij}) g(\alpha_{u_i}(u_n), u_j) g(u_n, u_i) \\ &= g(u_n, v_{ij}) g(u_n v_{in}, u_j) g(u_n, u_i) \\ &= g(u_n, v_{ij}) g(u_n, u_j) g(v_{in}, u_j) g(u_n, u_i), \end{aligned}$$

so that

$$g(v_{jn}, u_i) = g(u_n, v_{ij}) g(v_{in}, u_j), \tag{18}$$

which gives the last identity in the second line of the statement. Finally, we compute

$$\begin{aligned} g(a, bb') &= g(\alpha_{b'}(a), b) g(a, b') = g(a + (0, b'_1 a_n, \ldots, b'_{n-1} a_n), b) g(a, b') \\ &= g((0, b'_1 a_n, \ldots, b'_{n-1} a_n), w(b)) g(a, b) g(a, b') \\ &= \Big(\prod_{i=1}^{n-1} g(v_{in}, w(b))^{b'_i a_n} \Big) g(a, b) g(a, b') \\ &= \Big(\prod_{i=1}^{n-1} \Big(\prod_{j=1}^{n-1} g(v_{in}, u_j)^{b_j} \Big)^{b'_i a_n} \Big) g(a, b) g(a, b'). \end{aligned}$$

□

Lemma 2.1.5. *For all $a \in H(n)$ and $b \in G(n-1)$ we have*

$$\begin{aligned} g(w(a), w(b)) = \Big(&\prod_{i=1}^{n-1} \lambda_i^{\frac{1}{2} b_i a_n (a_n - 1)} g(v_{in}, u_i)^{\frac{1}{2} a_n b_i (b_i - 1)} \Big) \\ &\cdot \prod_{1 \le i < j \le n-1} g(v_{in}, u_j)^{b_i b_j a_n}. \end{aligned}$$

Proof. First we see from (17) that if $b_j \geq 1$, then

$$\begin{aligned} g(u_n, u_j^{b_j}) &= g(u_n, u_j^{b_j - 1} u_j) \\ &= g(v_{jn}, u_j)^{b_j - 1} g(u_n, u_j^{b_j - 1}) g(u_n, u_j) \\ &= \cdots = g(v_{jn}, u_j)^{\frac{1}{2} b_j (b_j - 1)} g(u_n, u_j)^{b_j} \end{aligned}$$

and then it is not hard to see that

$$g(u_n, u_j^{b_j}) = g(v_{jn}, u_j)^{\frac{1}{2} b_j (b_j - 1)} g(u_n, u_j)^{b_j}$$

for negative b_j as well, for example by applying (17) again.

Moreover, note that $w(b) = u_{n-1}^{b_{n-1}} \cdots u_1^{b_1}$, so that by (17),

$$\begin{aligned} g(u_n, w(b)) &= g(u_n, u_{n-1}^{b_{n-1}} \cdots u_1^{b_1}) \\ &= \big(\prod_{j=2}^{n-1} g(v_{1n}, u_j)^{b_1 b_j}\big) g(u_n, u_{n-1}^{b_{n-1}} \cdots u_2^{b_2}) g(u_n, u_1^{b_1}) \\ &= \cdots = \big(\prod_{1 \le i < j \le n-1} g(v_{in}, u_j)^{b_i b_j}\big)\big(\prod_{j=1}^{n-1} g(u_n, u_j^{b_j})\big). \end{aligned}$$

Then by (12) for $a_n \ge 1$,

$$\begin{aligned} g(w(a), w(b)) &= g(a_n u_n, u_{n-1}^{b_{n-1}} \cdots u_1^{b_1}) \\ &= \Big(\prod_{i=1}^{n-1} \lambda_i^{b_i(a_n-1)}\Big) \cdot g((a_n-1)u_n, u_{n-1}^{b_{n-1}} \cdots u_1^{b_1}) g(u_n, u_{n-1}^{b_{n-1}} \cdots u_1^{b_1}) \\ &= \cdots = \Big(\prod_{i=1}^{n-1} \lambda_i^{b_i \cdot \frac{1}{2} a_n(a_n-1)}\Big) \cdot g(u_n, u_{n-1}^{b_{n-1}} \cdots u_1^{b_1})^{a_n} \\ &= \Big(\prod_{i=1}^{n-1} \lambda_i^{\frac{1}{2} b_i a_n(a_n-1)}\Big) \cdot \Big(\prod_{1 \le i < j \le n-1} g(v_{in}, u_j)^{b_i b_j a_n}\Big) \\ &\cdot \Big(\prod_{j=1}^{n-1} g(v_{jn}, u_j)^{\frac{1}{2} a_n b_j (b_j-1)} g(u_n, u_j)^{a_n b_j}\Big). \end{aligned}$$

Again, it is not hard to see that a similar argument also works for negative a_n. Finally, recall that we have chosen g so that $g(u_n, u_j) = 1$ by (13). ☐

Lemma 2.1.6. *We have*

$$\widetilde{H}^2(G(n), \mathbb{T}) \cong \mathbb{T}^{n(n-1)},$$

and for each set of $n(n-1)$ parameters

$$\{\lambda_{i,jn} : 1 \le i \le n, 1 \le j \le n-1\} \subseteq \mathbb{T},$$

the associated $[\tau]$ in $\widetilde{H}^2(G(n), \mathbb{T})$ may be represented by

$$\begin{aligned} \tau(a'b, ab') = &\prod_{1 \le i < j \le n-1} \lambda_{i,jn}^{a_{jn} b_i + a_n b_{ij}} \lambda_{j,in}^{a_{in} b_j + a_n(b_i b_j - b_{ij})} \prod_{j=1}^{n-1} \lambda_{j,jn}^{a_{jn} b_j + \frac{1}{2} a_n b_j (b_j-1)} \\ &\cdot \prod_{j=1}^{n-1} \lambda_{n,jn}^{a_n'(a_{jn} + b_j a_n) + \frac{1}{2} b_j a_n (a_n-1)}. \end{aligned}$$

Proof. If one puts $\lambda_{i,jn} = g(v_{jn}, u_i)$ for $i, j < n$ and $\lambda_{n,jn} = \lambda_j$ for $j < n$, then this is a consequence of the preceding lemmas. Indeed, by (8) we can represent τ as a pair $(\sigma_{H(n)}, g)$. Here $\sigma_{H(n)}$ is of the form (11) and g can decomposed as in Lemma 2.1.3 with factors computed in Lemma 2.1.4 and Lemma 2.1.5. ☐

To complete the proof of Theorem 2.6, we set $r = a'b$ and $s = ab'$ and recall that by Corollary 2.5 we can compute σ_n inductively as $[\sigma_n] = \prod_{k=2}^{n} [\tau_n]$.

Finally, we can also check that $\sum_{k=2}^{n} k(k-1) = \frac{1}{3}(n+1)n(n-1)$.

3. The twisted group C^*-algebras $C^*(G(n),\sigma)$ of $G(n)$

Again, let G be *any* discrete group, σ a two-cocycle of G and $\mathcal{H}$ a nontrivial Hilbert space. A map U from G into the unitary group of $\mathcal{H}$ satisfying

$$U(r)U(s) = \sigma(r,s)U(rs)$$

for all $r,s \in G$ is called a *σ-projective unitary representation of G on $\mathcal{H}$*.

We recall the following facts about twisted group C^*-algebras and refer to Zeller-Meier [30] for further details of the construction.

To each pair (G,σ), we may associate the full twisted group C^*-algebra $C^*(G,\sigma)$. Denote the canonical injection of G into $C^*(G,\sigma)$ by i_σ. Then $C^*(G,\sigma)$ satisfies the following universal property. Every σ-projective unitary representation of G on some Hilbert space $\mathcal{H}$ (or in some unital C^*-algebra A) factors uniquely through i_σ.

The reduced twisted group C^*-algebra $C_r^*(G,\sigma)$ is generated by the left regular σ-projective unitary representation λ_σ of G on $B(\ell^2(G))$. Consequently, λ_σ extends to a *-homomorphism of $C^*(G,\sigma)$ onto $C_r^*(G,\sigma)$. If G is amenable, then λ_σ is faithful. Note especially that every nilpotent group is amenable, so that $C^*(G(n),\sigma) \cong C_r^*(G(n),\sigma)$ through λ_σ for every $n \geq 1$ and all two-cocycles σ of $G(n)$.

Finally, we remark that if $\tau \sim \sigma$ through some function $\beta\colon G \to \mathbb{T}$, then the assignment $i_\tau(r) \mapsto \beta(r)i_\sigma(r)$ induces an isomorphism $C^*(G,\tau) \to C^*(G,\sigma)$.

Theorem 3.1 (Remark 3.1 in the published version)**.** *Fix $n \geq 2$, let σ be a two-cocycle of $G(n)$ of the form* (9)*, that is, determined by the $\frac{1}{3}(n+1)n(n-1)$ parameters*

$$\{\lambda_{i,jk} : 1 \leq i \leq k, 1 \leq j < k \leq n\} \subseteq \mathbb{T},$$

and set

$$\lambda_{k,ij} = \overline{\lambda_{i,jk}}\lambda_{j,ik} \tag{19}$$

when $1 \leq i < j < k \leq n$.

Then the twisted group C^*-algebra $C^*(G(n),\sigma)$ is the universal C^*-algebra generated by unitaries $\{U_i\}_{1\leq i\leq n}$ and $\{V_{jk}\}_{1\leq j<k\leq n}$ satisfying the relations

$$[V_{jk}, V_{lm}] = I, \quad [U_i, V_{jk}] = \lambda_{i,jk}I, \quad [U_j, U_k] = V_{jk} \tag{20}$$

for $1 \leq i \leq n$, $1 \leq j < k \leq n$, and $1 \leq l < m \leq n$.

Proof. Set $U_i = i_\sigma(u_i)$ and $V_{jk} = i_\sigma(v_{jk})$ and note that (9) gives that $\sigma(u_i, v_{jk}) = \lambda_{i,jk}$ and $\sigma(v_{jk}, u_i) = 1$ for all $1 \leq i \leq n$ and $1 \leq j < k \leq n$. Thus,

$$[U_i, V_{jk}] = \sigma(u_i, v_{jk})\overline{\sigma(v_{jk}, u_i)}I = \lambda_{i,jk}I \text{ for all } 1 \leq i \leq n,\ 1 \leq j < k \leq n.$$

Moreover, note that $\sigma(u_i, u_j) = 1$ for all $1 \leq i,j \leq n$ and $\sigma(v_{jk}, v_{lm}) = 1$ for all $1 \leq j < k \leq n$ and $1 \leq l < m \leq n$. Hence, it is clear that $C^*(G(n),\sigma)$ is generated as a C^*-algebra by unitaries satisfying (20).

Next, suppose that A is any C^*-algebra generated by a set of unitaries satisfying the relations (20). For each r in $G(n)$ we define the unitary W_r in A by

$$W_r = V_{12}^{r_{12}} \cdots V_{n-1,n}^{r_{n-1,n}} \cdot U_n^{r_n} \cdots U_1^{r_1}.$$

Then a computation using (20) repeatedly gives that[2] $W_rW_s = \tau(r,s)W_{rs}$, where $\tau(r,s)$ is a scalar in $\mathbb{T}$ for all $r,s \in G(n)$. Now, the associativity of A immediately implies that τ is a two-cocycle of $G(n)$, so that W is a τ-projective unitary representation of $G(n)$ in A. Furthermore, note that τ satisfies

$$\tau(u_i, v_{jk})\overline{\tau(v_{jk}, u_i)} = \lambda_{i,jk} \text{ for } 1 \leq i \leq n,\ 1 \leq j < k \leq n.$$

[2]In general, it will require much work to compute the formula for τ and it is not needed for this argument. However, for $n = 2$, the expression for τ is precisely of the form (10).

By the universal property of the full twisted group C^*-algebra, there exists a unique *-homomorphism φ of $C^*(G(n),\tau)$ onto A such that $\varphi(i_\tau(r)) = W(r)$ for all $r \in G(n)$.

Therefore, it is sufficient to show that $\tau \sim \sigma$, because then $C^*(G(n),\tau)$ is canonically isomorphic to $C^*(G(n),\sigma)$. By Theorem 2.6, there is some $\beta\colon G(n) \to \mathbb{T}$ such that σ', given by

$$\sigma'(r,s) = \beta(r)\beta(s)\overline{\beta(rs)}\tau(r,s),$$

is of the form (9). We calculate that

$$\begin{aligned}
&\sigma'(u_i,v_{jk})\overline{\sigma'(v_{jk},u_i)} \\
&= \beta(u_i)\beta(v_{jk})\overline{\beta(u_iv_{jk})}\tau(u_i,v_{jk})\overline{\beta(v_{jk})\beta(u_i)}\beta(v_{jk}u_i)\overline{\tau(v_{jk},u_i)} \\
&= \tau(u_i,v_{jk})\overline{\tau(v_{jk},u_i)} = \lambda_{i,jk}
\end{aligned}$$

for all $1 \le i \le n$ and $1 \le j < k \le n$. Hence, $\sigma' = \sigma$, so $\tau \sim \sigma$. □

The above relation (19) is a consequence of (18) in the proof of Theorem 2.6 and is the reason why $\lambda_{i,jk}$ for $i > k$ is not involved in the expression (9). To illustrate this further, consider the case of rank 3. Let U_1, U_2, U_3 and V_{12}, V_{13}, V_{23} be unitaries in a C^*-algebra B satisfying

$$[V_{jk}, V_{lm}] = I, \quad [U_i, V_{jk}] = \mu_{i,jk} I, \quad [U_j, U_k] = V_{jk}$$

for $1 \le i \le 3$, $1 \le j < k \le 3$, and $1 \le l < m \le 3$ where $\{\mu_{i,jk}\}$ is *any* set of nine scalars in $\mathbb{T}$. Then we can compute that

$$\begin{aligned}
U_1U_2U_3 &= V_{12}U_2U_1U_3 = \cdots = \mu_{2,13}V_{12}V_{13}V_{23}U_3U_2U_1, \\
U_1U_2U_3 &= U_1V_{23}U_3U_2 = \cdots = \mu_{1,23}\mu_{3,12}V_{12}V_{13}V_{23}U_3U_2U_1,
\end{aligned}$$

that is, we must have $\mu_{2,13} = \mu_{1,23}\mu_{3,12}$.

For rank $n > 3$, any choice of a triple of unitaries from the family $\{U\}_{i=1}^n$ gives a similar dependence. In the $n \cdot \frac{1}{2}n(n-1)$ commutation relations, these $\binom{n}{3}$ dependencies are the only possible ones since

$$n \cdot \tfrac{1}{2}n(n-1) - \tbinom{n}{3} = \tfrac{1}{2}n(n-1)\left(n - \tfrac{1}{3}(n-2)\right) = \tfrac{1}{3}(n+1)n(n-1).$$

Remark 3.2. For $n \ge 2$, let ω be the dual two-cocycle of $G(n)$, that is,

$$\omega\colon G(n) \times G(n) \to H^2(\widehat{G(n), \mathbb{T}}) \cong \mathbb{Z}^{\frac{1}{3}(n+1)n(n-1)}$$

is determined by $\omega(r,s)(\sigma) = \sigma(r,s)$ for a two-cocycle σ of $G(n)$. Let the group $R(G(n))$ be defined as the set $\mathbb{Z}^{\frac{1}{3}(n+1)n(n-1)} \times G(n)$ with product

$$(j,r)(k,s) = (j + k + \omega(r,s), rs).$$

It is not entirely obvious that ω and $R(G(n))$ are well-defined and we refer to [24, p. 689–690] and [7, Section 4] for details on this and the fact that $R(G(n))$ is a representation group for $G(n)$. Moreover, according to [24, Corollary 1.3] we may construct a continuous field A over $H^2(G(n),\mathbb{T})$ with fibers $A_\lambda \cong C^*(G(n),\sigma_\lambda)$ for each $\lambda \in H^2(G(n),\mathbb{T})$. Then the C^*-algebra associated with this continuous field will be naturally isomorphic to the group C^*-algebra of the group $R(G(n))$.

Next, we briefly consider the group $G(3,2)$ generated by $u_1, u_2, v_{12}, w_1, w_2$ satisfying

$$[u_1,u_2] = v_{12}, \quad [u_1,v_{12}] = w_1, \quad [u_2,v_{12}] = w_2, \quad w_1, w_2 \text{ central}.$$

Then we have $Z(G(3,2)) \cong \mathbb{Z}^2$ and $Z(C^*(G(3,2))) \cong C(\mathbb{T}^2)$.

The following statement can also be deduced from [24, Theorem 1.2 and Examples 1.4 (3)], but we include the analysis that follows, because it is similar to that used in Theorem 1.1.

Let i denote the canonical injection of $G(3,2)$ into $C^*(G(3,2))$. For each $\lambda = (\lambda_1, \lambda_2) \in \mathbb{T}^2$, let $C^*(G(2), \sigma_\lambda)$ be generated by unitaries satisfying (20). By a similar argument as in Theorem 1.1, there is a surjective *-homomorphism

$$\pi_\lambda \colon C^*(G(3,2)) \to C^*(G(2), \sigma_\lambda)$$

such that $i(u_i) = U_i$, $i(v_{12}) = V_{12}$, and $i(w_i) = \lambda_i I$ for $i = 1, 2$. Moreover, the kernel of π_λ coincides with the ideal of $C^*(G(3,2))$ generated by

$$\lambda \in \operatorname{Prim} Z(C^*(G(3,2))) \cong Z(\widehat{C^*(G(3},2))) = \mathbb{T}^2 \cong H^2(G(2), \mathbb{T}).$$

Again, similarly as in Theorem 1.1, we define a set of sections and apply the Dauns-Hofmann Theorem. In this way, the triple

$$\Big(H^2(G(2), \mathbb{T}), \{C^*(G(2), \sigma_\lambda)\}_\lambda, C^*\widetilde{(G(3,2))}\Big)$$

is a full continuous field of C^*-algebras, and the C^*-algebra associated with this continuous field is naturally isomorphic to $C^*(G(3,2))$.

It is not difficult to see that $R(G(2))$ is isomorphic to $G(3,2)$. We conjecture that $R(G(n)) \cong G(3,n)$ also for $n \geq 3$, where $G(3,n)$ is the free nilpotent group of class 3 and rank n as described in (3), so that A is isomorphic to $C^*(G(3,n))$. For $n \geq 3$, the complicated part is to construct a homomorphism $R(G(n)) \to G(3,n)$, find an isomorphism $\mathbb{Z}^{\frac{1}{3}(n+1)n(n-1)} \cong Z(G(3,n))$, and then use (4) to produce a commuting diagram:

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}^{\frac{1}{3}(n+1)n(n-1)} \arrow[r] \arrow[d, "\cong"'] & R(G(n)) \arrow[r] \arrow[d] & G(n) \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & Z(G(3,n)) \arrow[r] & G(3,n) \arrow[r] & G(n) \arrow[r] & 1
\end{tikzcd}$$

In fact, [26, Proposition 2.2 and Remark 2.3] indicate that the representation group $R(G(m,n))$ for $G(m,n)$ defined similarly as above is isomorphic to $G(m+1,n)$ for all $m, n \geq 1$.

4. Simplicity of the twisted group C^*-algebras $C^*(G(n), \sigma)$

Let σ be a two-cocycle of any group G. An element r of G is called *σ-regular* if $\sigma(r,s) = \sigma(s,r)$ whenever s in G commutes with r. If r is σ-regular, then every conjugate of r is also σ-regular. Therefore, we say that a conjugacy class of G is σ-regular if it contains a σ-regular element.

Let $n \geq 2$. The conjugacy class C_r of $r \in G(n)$ is infinite if $r \notin V(n) = Z(G(n))$. Indeed, for any $s \in G(n)$ we have

$$(srs^{-1})_i = r_i \text{ and } (srs^{-1})_{jk} = r_{jk} + s_j r_k - r_j s_k.$$

Hence, $|C_r| = \infty$ if $r_i \neq 0$ for some i. Of course, $C_r = \{r\}$ if $r \in V(n)$.

More generally, it is explained in [9, Section 2.1.1] that if G is any torsion-free nilpotent group and $r \in G \setminus Z(G)$, then $|C_r| = \infty$.

In the published version, the following was only shown to hold for $G(n)$.

Lemma 4.1. *Let σ be a two-cocycle of a group G. Suppose that $|C_r| \in \{1, \infty\}$ whenever $r \in G$ is σ-regular. Define $S(G)$ to be the set of σ-regular central elements of G, that is,*

$$S(G) = \{r \in Z(G) \mid \sigma(r,s) = \sigma(s,r) \textit{ for all } s \in G\}.$$

Then $S(G)$ is a subgroup of G and $Z(C_r^*(G), \sigma) \cong C(\widehat{S(G)})$.

Proof. It is not hard to check that $S(G)$ is a subgroup of $Z(G)$.

Let δ_e in $\ell^2(G)$ be the characteristic function on $\{e\}$ and for an operator T in $B(\ell^2(G))$, set $f_T = T\delta_e \in \ell^2(G)$. If T belongs to the center of $C_r^*(G,\sigma)$, then f_T can be nonzero only on the finite σ-regular conjugacy classes of G, that is, on $S(G)$ (see e.g. [19, Lemmas 2.3 and 2.4]).

Next, let $C^*(S(G),\sigma)$ be identified with $\{\lambda_\sigma(s) \mid s \in S(G)\} \subseteq B(\ell^2(G))$. This means that $Z(C_r^*(G,\sigma)) \subseteq C^*(S(G),\sigma)$. As the reverse inclusion obviously holds, we have $Z(C_r^*(G),\sigma) = C^*(S(G),\sigma)$.

Now, it is not difficult to see that $C^*(S(G)) \cong C^*(S(G),\sigma)$. Indeed, as $s \mapsto \lambda_\sigma(s)$ is a unitary representation of $S(G)$ into $C^*(S(G),\sigma)$ and the canonical tracial state τ on $C^*(S(G),\sigma)$ is faithful and satisfies $\tau(\lambda_\sigma(s)) = 0$ for each nonzero $s \in S(G)$, this is just a consequence of [30, Théorème 4.22]. Altogether, we get

$$Z(C_r^*(G),\sigma) = C^*(S(G),\sigma) \cong C^*(S(G)) \cong C(\widehat{S(G)}). \qquad \square$$

Now, we fix a two-cocycle σ of $G(n)$ of the form (9).

Remark 4.2. If $S(G(n))$ is nontrivial, we can describe $C^*(G(n),\sigma)$ as a continuous field of C^*-algebras over the base space $\widehat{S(G(n))}$. The fibers will be isomorphic to $C^*(G(n)/S(G(n)),\omega)$ for some two-cocycle ω of $G(n)/S(G(n))$ (see [11, Theorem 1.1] and [24, Theorem 1.2] for further details).

Example 4.3 ([11, Lemma 3.8 and Theorem 3.9])**.** Fix a two-cocycle σ of $G(2)$ of the form (10) such that both $\lambda_{1,12}$ and $\lambda_{2,12}$ are torsion elements. Let p and q be the smallest natural numbers such that $\lambda_{1,12}^p = \lambda_{2,12}^q = 1$ and set $k = \mathrm{lcm}(p,q)$. Clearly, $V(2) = \mathbb{Z}$ and $S(G(2)) = k\mathbb{Z}$. Moreover, $G(2)/S(G(2))$ can be identified with the group with product

$$(r_1, r_2, r_{12})(s_1, s_2, s_{12}) = (r_1 + s_1, r_2 + s_2, r_{12} + s_{12} + r_1 s_2 \bmod k\mathbb{Z})$$

for $r_1, r_2, s_1, s_2 \in \mathbb{Z}$ and $r_{12}, s_{12} \in \{0, 1, \ldots, k-1\}$.

Then $C^*(G(2),\sigma)$ is a continuous field of C^*-algebras over the base space $\widehat{S(G(2))} \cong \mathbb{T}$. The fibers will be isomorphic to $C^*(G(n)/S(G(n)),\omega_\lambda)$, where $\lambda \in \mathbb{T}$ and

$$\omega_\lambda(r,s) = \sigma(r,s)\mu^{r_1 s_2}$$

for some $\mu \in \mathbb{T}$ with $\mu^k = \lambda$.

Characterizations for simplicity of of twisted group C^*-algebras of two-step nilpotent groups have been given in [11, Corollary 1.4] and [24, Corollary 1.6]. For the groups $G(n)$, the necessary and sufficient conditions for simplicity are somewhat easier to provide.

Theorem 4.4. *The following are equivalent:*

(i) $C^*(G(n),\sigma)$ *is simple.*
(ii) $C^*(G(n),\sigma)$ *has trivial center.*
(iii) There are no nontrivial central σ*-regular elements in* $G(n)$.

Proof. By [22, Theorem 1.7] $C^*(G(n),\sigma)$ is simple if and only if every nontrivial σ-regular conjugacy class of $G(n)$ is infinite. Since every finite conjugacy class of $G(n)$ is a one-point set of a central element, then (i) is equivalent with (iii).

Moreover, (iii) is the same as saying that $S(G(n))$ is trivial, so therefore, (ii) is equivalent with (iii) by Lemma 4.1. This also follows from [19, Theorem 2.7]. $\square$

Lemma 4.5. *A central element* $s = (0, \ldots, 0, s_{12}, s_{13}, \ldots, s_{n-1,n})$ *of* $G(n)$ *is* σ*-regular if and only if*

$$\prod_{1 \le j < k \le n} \lambda_{i,jk}^{s_{jk}} = 1$$

for all $1 \le i \le n$.

Proof. Clearly, an element $s = (0, \dots, 0, s_{12}, s_{13}, \dots, s_{n-1,n}) \in V(n)$ is σ-regular if and only if $\sigma(s, r) = \sigma(r, s)$ for all $r \in G(n)$. By a direct calculation from the cocycle formula (9), we get that

$$\begin{aligned}\sigma(r,s)\overline{\sigma(s,r)} &= \Big(\prod_{i<j<k} \lambda_{i,jk}^{s_{jk}r_i}\lambda_{j,ik}^{s_{ik}r_j}\Big)\Big(\prod_{j<k}\lambda_{j,jk}^{s_{jk}r_j}\lambda_{k,jk}^{r_k s_{jk}}\Big)\Big(\prod_{i<j<k}\lambda_{i,jk}^{-r_k s_{ij}}\lambda_{j,ik}^{r_k s_{ij}}\Big)\\ &= \prod_{i=1}^{n}\Big(\prod_{1\le j<k\le n}\lambda_{i,jk}^{s_{jk}}\Big)^{r_i}\end{aligned}$$

is equal to 1 for all $r \in G(n)$ if and only if the inner parenthesis is 1 for each $1 \le i \le n$. ☐

Corollary 4.6. *We have that* $C^*(G(n), \sigma)$ *is simple if and only if for each nontrivial central element* $s = (0, \dots, 0, s_{12}, s_{13}, \dots, s_{n-1,n})$ *there is some* $1 \le i \le n$ *such that*

$$\prod_{1\le j<k\le n} \lambda_{i,jk}^{s_{jk}} \neq 1.$$

Example 4.7. In particular, $C^*(G(3), \sigma)$ is simple if and only if for each nontrivial central element $s = (0, 0, 0, s_{12}, s_{13}, s_{23})$ at least one of the following hold:

$$\begin{aligned}\lambda_{1,12}^{s_{12}}\lambda_{1,13}^{s_{13}}\lambda_{1,23}^{s_{23}} &\neq 1,\\ \lambda_{2,12}^{s_{12}}\lambda_{2,13}^{s_{13}}\lambda_{2,23}^{s_{23}} &\neq 1,\\ \lambda_{3,12}^{s_{12}}\lambda_{3,13}^{s_{13}}\lambda_{3,23}^{s_{23}} &\neq 1.\end{aligned}$$

Next, set $\lambda_{i,jk} = e^{2\pi i t_{i,jk}}$ for $t_{i,jk} \in [0, 1)$ and consider the $n \times \frac{1}{2}n(n-1)$-matrix T with entries $t_{i,jk}$ in the corresponding spots. Then T induces a linear map

$$\mathbb{R}^{\frac{1}{2}n(n-1)} \to \mathbb{R}^n.$$

Corollary 4.8. *Let* T *be the matrix described above. Then following are equivalent:*

(i) $C^*(G(n), \sigma)$ *is simple*
(ii) $T^{-1}(\mathbb{Z}^n) \cap \mathbb{Z}^{\frac{1}{2}n(n-1)} = \{0\}$
(iii) $T(\mathbb{Z}^{\frac{1}{2}n(n-1)} \setminus \{0\}) \cap \mathbb{Z}^n = \varnothing$

Remark 4.9. Clearly, condition (ii) above is equivalent to that T restricts to an injective map

$$\mathbb{Z}^{\frac{1}{2}n(n-1)} \to \mathbb{R}^n/\mathbb{Z}^n \cong \mathbb{T}^n.$$

Furthermore, for $1 \le j < k \le n$, define

$$\Lambda_{jk} = \{t_{i,jk} \in [0,1),\ 1 \le i \le n \mid e^{2\pi i t_{i,jk}} = \lambda_{i,jk}\}$$

and for $1 \le i \le n$, define

$$\Lambda_i = \{t_{i,jk} \in [0,1),\ 1 \le j < k \le n \mid e^{2\pi i t_{i,jk}} = \lambda_{i,jk}\}.$$

Proposition 4.10. *If there exists* i *such that all the elements of* Λ_i *are irrational and linearly independent over* $\mathbb{Q}$*, then* $C^*(G(n), \sigma)$ *is simple.*

Proof. It follows immediately from Lemma 4.5, that "equation i" cannot be satisfied unless $s = 0$. Hence, no nontrivial σ-regular central elements exists. ☐

Proposition 4.11. *If there exists* $j < k$ *such that* Λ_{jk} *consists of only rational elements, then* $C^*(G(n), \sigma)$ *is not simple.*

Proof. Let q be the least common two-cocycle of the denominators of the elements of Λ_{jk}. Then qv_{jk} is central and σ-regular. Indeed,

$$\sigma(r, qv_{jk})\overline{\sigma(qv_{jk}, r)} = \prod_{i=1}^{n-1} \lambda_{i,jk}^{qr_i} = 1$$

for all $r \in G(n)$. □

Remark 4.12. In the case where $C^*(G(n), \sigma)$ is not simple, some more information about the primitive ideal space can be deduced from [11, Proposition 1.3].

5. On isomorphisms invariants of $C^*(G(n), \sigma)$

Fix $n \geq 2$ and let σ be a two-cocycle of $G(n)$. If φ is an automorphism of $G(n)$, define the two-cocycle σ_φ of $G(n)$ by

$$\sigma_\varphi(r, s) = \sigma(\varphi(r), \varphi(s)). \tag{21}$$

Then it is well-known that the associated twisted group C^*-algebras $C^*(G(n), \sigma)$ and $C^*(G(n), \sigma_\varphi)$ are isomorphic. Indeed, the map

$$i_{(G,\sigma)}(r) \mapsto i_{(G,\sigma_\varphi)}(\varphi^{-1}(r))$$

extends to an isomorphism $C^*(G(n), \sigma) \to C^*(G(n), \sigma_\varphi)$. Moreover, for any automorphism φ of $G(n)$, it is easily seen that two two-cocycles σ and τ of $G(n)$ are similar if and only if σ_φ and τ_φ are similar. Hence, there is a well-defined group action of the automorphism group $\operatorname{Aut} G(n)$ on $H^2(G(n), \mathbb{T})$ defined by $\varphi \cdot [\sigma] = [\sigma_\varphi]$.

Therefore, we will now briefly investigate $\operatorname{Aut} G(n)$. Let $V(n)^n$ be the subgroup of $\operatorname{Aut} G(n)$ consisting of the automorphisms $G(n) \to G(n)$ of the form $u_i \mapsto z_i u_i$ for $1 \leq i \leq n$ and elements $z_i \in V(n) = Z(G(n))$. Clearly, $V(n)^n$ contains $\operatorname{Inn} G(n)$. In fact, in the case $n = 2$, we have $V(2)^2 = \operatorname{Inn} G(2)$.

Proposition 5.1. *There is a split short exact sequence:*

$$1 \longrightarrow V(n)^n \longrightarrow \operatorname{Aut} G(n) \longrightarrow \mathrm{GL}(n, \mathbb{Z}) \longrightarrow 1$$

Proof. Assume that φ is any endomorphism $G(n) \to G(n)$. The image of a central element under φ must be central, so φ restricts to an endomorphism $\varphi_1 \colon V(n) \to V(n)$. Therefore, φ also induces an endomorphism $\varphi_2 \colon G(n)/V(n) \to G(n)/V(n)$ determined by $\varphi_2(q(r)) = q(\varphi(r))$. Consider now the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & V(n) \arrow[r, "i"] \arrow[d, "\varphi_1"'] & G(n) \arrow[r, "q"] \arrow[d, "\varphi"'] & \mathbb{Z}^n \arrow[r] \arrow[d, "\varphi_2"] & 1 \\
1 \arrow[r] & V(n) \arrow[r, "i"] & G(n) \arrow[r, "q"] & \mathbb{Z}^n \arrow[r] & 1
\end{tikzcd}$$

Assume that φ_2 is an automorphism. Since φ_2 is surjective, then for all u_i there is some $s_i \in G(n)$ such that $\varphi(s_i) = z_i u_i$ for some $z_i \in V(n)$. Hence, for all $j < k$, we have $\varphi_1(s_j s_k s_j^{-1} s_k^{-1}) = v_{jk}$ and therefore, φ_1 is surjective. Every surjective endomorphism of $\mathbb{Z}^n$ is also injective, so φ_1 is an automorphism as well. Thus, by the "short five lemma", φ is an automorphism.

The converse obviously holds and hence, φ is an automorphism if and only if φ_2 is an automorphism.

Furthermore, the construction of $G(n)$ in terms of generators and relations means that every endomorphism $G(n) \to G(n)$ is uniquely determined by its values at $\{u_i\}_{i=1}^n$. In particular, we let $\varphi\colon G(n) \to G(n)$ be determined by the pair of matrices given by its entries

$$(\varphi(u_i)_j), (\varphi(u_i)_{jk}) \in M_n(\mathbb{Z}) \times M_{n,\frac{1}{2}n(n-1)}(\mathbb{Z})$$

so that the induced endomorphism φ_2 is coming from a matrix in $M_n(\mathbb{Z})$.

By the above argument, the map between endomorphism groups defined by

$$\operatorname{End} G(n) \to \operatorname{End} \mathbb{Z}^n, \quad (\varphi(u_i)_j), (\varphi(u_i)_{jk}) \mapsto (\varphi(u_i)_j) \tag{22}$$

restricts to a surjective map $\operatorname{Aut} G(n) \to \operatorname{Aut} \mathbb{Z}^n = \mathrm{GL}(n, \mathbb{Z})$.

Before concluding the argument, we need the following.

Lemma 5.2 (nested inside the proof)**.** *If φ and φ' are two endomorphisms of $G(n)$, then*

$$(\varphi \circ \varphi')(u_i)_j = \sum_{k=1}^n \varphi'(u_i)_k \varphi(u_k)_j.$$

If φ and φ' are two endomorphisms of $G(n)$ that both induce the trivial map on $G(n)/V(n)$, then

$$(\varphi \circ \varphi')(u_i)_{jk} = \varphi'(u_i)_{jk} + \varphi(u_i)_{jk}.$$

Proof. For the moment, set $\varphi(u_i)_j = r_{ij}$ and $\varphi'(u_i)_j = s_{ij}$. Then

$$(\varphi \circ \varphi')(u_i) = \varphi(u_n^{s_{in}} \cdots u_1^{s_{i1}} z) = (u_n^{r_{nn}} \cdots u_1^{r_{n1}})^{s_{in}} \cdots (u_n^{r_{1n}} \cdots u_1^{r_{11}})^{s_{i1}} z'$$

for some elements $z, z' \in V(n)$. Moreover, we can change the order of the u_i's in the expression just by replacing z' by another central element z'' and thus,

$$(\varphi \circ \varphi')(u_i)_j = r_{nj} s_{in} + r_{n-1,j} s_{i,n-1} + \cdots + r_{1j} s_{i1} = \sum_{k=1}^n s_{ik} r_{kj}.$$

If both φ_2 and φ_2' are trivial, then $\varphi(u_i) = z_i u_i$ and $\varphi'(u_i) = z_i' u_i$ for all $1 \le i \le n$ and some elements $z_i, z_i' \in V(n)$. Hence, $\varphi(v_{jk}) = \varphi'(v_{jk}) = v_{jk}$ for all $j < k$ and thus,

$$(\varphi \circ \varphi')(u_i) = \varphi(z_i' u_i) = z_i' z_i u_i.$$ ☐

Therefore, (22) restricts to a surjective *homomorphism* $\operatorname{Aut} G(n) \to \mathrm{GL}(n, \mathbb{Z})$ with kernel isomorphic to the group $M_{n,\frac{1}{2}n(n-1)}(\mathbb{Z})$ under addition, that is, to $V(n)^n$.

Moreover, if A is a matrix in $\mathrm{GL}(n, \mathbb{Z})$ with entries a_{ij}, then one can define an automorphism φ_A of $G(n)$ by $\varphi_A(u_i)_j = a_{ij}$. Thus, it should be clear that $\mathrm{GL}(n, \mathbb{Z})$ sits inside $\operatorname{Aut} G(n)$ as a subgroup so that the sequence splits. ☐

Proposition 5.3. *If φ belongs to $V(n)^n$, then σ_φ is similar to σ. Thus, the action of $V(n)^n$ on $H^2(G(n), \mathbb{T})$ given by* (21) *is trivial.*

Proof. It is not hard to see that

$$\sigma(u_i, v_{jk})\overline{\sigma(v_{jk}, u_i)} = \sigma_\varphi(u_i, v_{jk})\overline{\sigma_\varphi(v_{jk}, u_i)},$$

that is,

$$[i_{(G,\sigma)}(u_i), i_{(G,\sigma)}(v_{jk})] = [i_{(G,\sigma_\varphi)}(u_i), i_{(G,\sigma_\varphi)}(v_{jk})]$$

for all $1 \le i \le n$ and $1 \le j < k \le n$. It then follows from the universal property of $C^*(G(n), \sigma)$ described in Theorem 3.1 that $\sigma_\varphi \sim \sigma$. ☐

To describe the $\mathrm{GL}(n,\mathbb{Z})$-action on $H^2(G(n),\mathbb{T})$ requires more work. To any A in $\mathrm{GL}(n,\mathbb{Z})$ we may associate a square matrix $\tilde{A}$ of dimension $\frac{1}{2}n(n-1)$, with entries coming from the determinant of all 2×2-matrices inside A. More precisely, if $A=(a_{ij})$, $\tilde{A}$ is given by entries $\tilde{a}_{ij,kl}$ for $i<j, k<l$ such that $\tilde{a}_{ij,kl}=a_{ik}a_{jl}-a_{il}a_{jk}$. Then A acts on the matrix T defined prior to Corollary 4.8 by $A\cdot T=AT\tilde{A}$. Tedious computations of commutation relations and use of the universal property of $C^*(G(n),\sigma)$ from Theorem 3.1 now lead to the following result.

Proposition 5.4. *Let σ and σ' be two-cocycles of $G(n)$ of the form* (9) *and let T and T' be the associated matrices of Corollary 4.8. If there exists a matrix A in $\mathrm{GL}(n,\mathbb{Z})$ such that $A\cdot T=T'$, then $C^*(G(n),\sigma)$ and $C^*(G(n),\sigma')$ are isomorphic.*

For $n=2$, it is shown by Packer [21, Theorem 2.9] that $C^*(G(2),\sigma)$ and $C^*(G(2),\sigma')$, where σ and σ' are of the form (9), are isomorphic if and only if there is a $\mathrm{GL}(2,\mathbb{Z})$-matrix A taking σ to σ'. Note in this case that $\det\tilde{A}=\det A=\pm 1$.

For $n\geq 3$, it is at the moment not clear whether the $\mathrm{GL}(n,\mathbb{Z})$-action on $H^2(G(n),\mathbb{T})$ described above is such that the orbits represent different isomorphism classes of twisted group C^*-algebras. Therefore, the problem of determining the isomorphism classes of $C^*(G(n),\sigma)$ remains open for future investigation.

Remark 5.5. Every two-cocycle σ_λ of $G(n)$ is of the form $e^{2\pi i\widetilde{\sigma}_\lambda}$ for some two-cocycle $\widetilde{\sigma}_\lambda$ on $G(n,\mathbb{R})$. Moreover, any pair of two-cocycles σ_λ and σ_μ of the form described above are homotopic in the sense of Packer and Raeburn [23, Section 4]. Hence, one may use [23, Theorem 4.2 and Corollary 4.5] to deduce that

$$K_i(C^*(G(n)),\sigma_\lambda)\cong K_i(C^*(G(n)),\sigma_\mu)\cong K_{\mathrm{top}}^{i+\frac{1}{2}n(n-1)}(G(n,\mathbb{R})/G(n)).$$

Remark 5.6 (new)**.** The groups $G(n)$, or more generally all the free nilpotent groups $G(m,n)$ of class m and rank n are finitely generated (torsion-free) nilpotent groups. Let σ be a two-cocycle of $G(m,n)$ and suppose that there are no nontrivial central σ-regular elements in $G(m,n)$, that is, $C^*(G(m,n),\sigma)$ is simple with a unique trace by [22] (see Theorem 4.4 and the subsequent results above for the case of $G(n)$). Then there exists an irreducible representation π_σ of $G(m+1,n)$ such that $C^*(G(m,n),\sigma)\cong C^*(\pi_\sigma(G(m+1,n)))$ by using Theorem 1.1, Remark 3.2, and (4). Thus it follows from [9] and [8] that $C^*(G(m,n),\sigma)$ is classified by its ordered K-theory within the class of simple, unital, nuclear C^*-algebras with finite nuclear dimension that satisfy the universal coefficient theorem.

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This document includes several computations and details that were left out in the published version of the article from 2015, in addition to a reformulation of Proposition 2.2 in terms of arbitrary semidirect products, a more general statement in Lemma 4.1, and a new Remark 5.6 that makes a connection to recent work. The latest arxiv version of the paper [20] includes all these updates, except Lemma 4.1 and a few other minor details.

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