C^*-SIMPLICITY OF HNN EXTENSIONS AND GROUPS ACTING ON TREES

RASMUS SYLVESTER BRYDER, NIKOLAY A. IVANOV, AND TRON OMLAND

ABSTRACT. We study non-ascending HNN extensions acting on their Bass-Serre trees, and characterize C^*-simplicity and the unique trace property by means of the kernel and quasi-kernels of the HNN extension in question. We also present a concrete example of an HNN extension that is a new example of a group that is not C^*-simple but does have the unique trace property. Additionally, we include certain more general results, mostly based on previous work of various authors, concerning C^*-simplicity of groups admitting extreme boundary actions, and in particular, groups acting on trees.

1. INTRODUCTION

A discrete group is said to be C^*-*simple* when its reduced C^*-algebra, i.e., the C^*-algebra associated to its left regular representation is simple. This property for discrete groups found its primus motor in a paper by Powers, who proved in 1975 that the non-abelian free group on two generators is C^*-simple [31]. Since then, many other examples of C^*-simple groups have been found (see Pierre de la Harpe's survey [17]). What these groups have in common is that they are all proven to be C^*-simple by using variations of Powers' technique. A common denominator for C^*-simple groups is that they are highly non-amenable, in the sense that they have *trivial amenable radical*, i.e., they admit no normal non-trivial amenable subgroups. Another property related to C^*-simplicity of a discrete group is *the unique trace property*, meaning that its reduced group C^*-algebra admits a unique tracial state. The unique trace property also implies triviality of the amenable radical. An early question of de la Harpe [16] was whether there was any connection between the aforementioned properties. Until recently, no characterizations of C^*-simplicity nor the unique trace property were known, nor did there exist examples of groups that only satisfied one of these two properties. Only in 2014 did Kalantar and Kennedy obtain the first known characterization of C^*-simplicity [22], and later that year, Breuillard, Kalantar, Kennedy, and Ozawa gave a characterization of the unique trace property in terms of its amenable radical [4]. By means of the result of Kalantar and Kennedy, de la Harpe's question was finally completely settled in 2015, when Le Boudec found examples of non-C^*-simple groups with the unique trace property, by examining actions of countable groups on trees [26].

In the 1960's, Furstenberg considered what came to be known as boundary actions [11], in order to investigate the irreducible unitary representations of a group. A boundary action is a minimal action of a group on a compact Hausdorff space that is strongly proximal, i.e., the weak*-closure of each orbit in the space of probability measures on the space contains a point mass. Furstenberg proved that any locally compact group G always admits a universal boundary action, meaning a compact Hausdorff space $\partial_F G$ that maps uniquely G-equivariantly

onto any other compact Hausdorff space with a boundary action of the group G. This was the technology used in [22] to obtain a characterization of C^*-simplicity, where the authors showed that a discrete group Γ is C^*-simple if and only if the action of Γ on the Furstenberg boundary $\partial_F \Gamma$ is topologically free. The theory of boundary actions was also the main force used in [4] to show that the unique trace property of a discrete group is equivalent to the group having trivial amenable radical. Because of the above characterizations, as well as their close affiliation with non-amenability, C^*-simplicity and the unique trace property have since witnessed a spike in interest.

In this article, we mainly study HNN extensions and obtain necessary and sufficient conditions for C^*-simplicity, in which we take both a purely algebraic point of view, and a geometric point of view, giving a partial answer to [17, Problem 28].

First, in Sections 2 and 3 we include a number of results that hold in more general situations, for boundary actions and groups acting on trees, mostly inspired from the work of Le Boudec and Matte Bon [27]. Especially, we investigate the case of when a group acts on a tree via an action that is minimal and of general type (cf. [27, Section 4.3]). These are fairly weak assumptions, and together they give rise to a boundary action of the group on a natural boundary of the tree. With this set-up, Proposition 3.8 provides a connection to conditions implying C^*-simplicity earlier studied by de la Harpe, and Theorem 3.9 gives necessary and sufficient conditions for C^*-simplicity.

Specializing further, for a graph of groups one can study its fundamental group and its action on the Bass-Serre tree (see Section 4). Graphs of groups with only one edge are the most studied examples, and their fundamental groups are of two types, either an amalgamated free product or an HNN extension. The former case was investigated in [21] and the latter case is studied in detail in the remaining sections of this paper, where we define two "quasi-kernels" of an HNN extension in order to determine C^*-simplicity. Theorem 4.10 gives combinatorial properties analog to Proposition 3.8, and Proposition 4.12 shows that in certain cases, C^*-simplicity is equivalent to the group being icc. A characterization of C^*-simplicity in terms of quasi-kernels is then given in Theorem 4.19.

Finally, in Section 5 we produce a concrete HNN extension construced from locally finite groups, that is not C^*-simple, but has the unique trace property.

2. Preliminaries on boundary actions and C^*-simplicity

In this section, we first recall the theory of boundary actions and extreme boundary actions, and their relation to C^*-simplicity and the unique trace property. We employ the terminology of [21, Section 5] to formalize the results.

Let Γ be a discrete group with identity element 1. Consider the Hilbert space $\ell^2(\Gamma)$ with the standard orthonormal basis $\{\delta_g\}_{g \in \Gamma}$, and define the left regular representation λ of Γ on $\ell^2(\Gamma)$ by $\lambda(g)\delta_h = \delta_{gh}$. The reduced group C^*-algebra of Γ, denoted by $C^*_r(\Gamma)$, is the C^*-subalgebra of $B(\ell^2(\Gamma))$ generated by $\lambda(\Gamma)$. A group Γ is called C^*-simple if $C^*_r(\Gamma)$ is simple, that is, if it has no non-trivial proper two-sided closed ideals.

A state on a unital C^*-algebra B is a linear functional $\phi \colon B \to \mathbb{C}$ that is positive, i.e., $\phi(a) \geq 0$ whenever $a \in B$ and $a \geq 0$, and unital, i.e., $\phi(1) = 1$. A state ϕ is called tracial if it satisfies the additional property that $\phi(ab) = \phi(ba)$ for all $a, b \in B$. There is a canonical faithful tracial state τ on $C^*_r(\Gamma)$, namely the vector state associated with δ_1, that is, $\tau(a) = \langle a\delta_1, \delta_1 \rangle$ for all $a \in C^*_r(\Gamma)$. The group Γ is said to have the *unique trace property* if τ is the only tracial state on $C^*_r(\Gamma)$.

The first large class of groups that were shown to be C^*-simple with the unique trace property, consisted of the *Powers groups* (see [16] for a definition).

Further, recall that a group Γ is *amenable* if there exists a state on $\ell^\infty(\Gamma)$ which is invariant under the left translation action by Γ. It is explained in [9] that every group Γ has a unique

maximal normal amenable subgroup $R(\Gamma)$, called the *amenable radical* of Γ. It was shown in [4, Theorem 1.3] that Γ has the unique trace property if and only if the amenable radical of Γ is trivial. As a consequence, C^*-simplicity is stronger than the unique trace property by [30], and strictly stronger by [26, Theorem A].

The following study of C^*-simplicity is based on the dynamical approach to and characterization of C^*-simplicity given in [22]. Let Γ be a group acting continuously (i.e., by homeomorphisms) on a topological space X. The fixed-point set of an element $g \in \Gamma$ is denoted by $X^g = \{x \in X \mid gx = x\}$. Let $\Gamma_x = \{g \in \Gamma \mid gx = x\}$ be the stabilizer subgroup of $x \in X$ and let Γ_x° denote the subgroup of Γ_x consisting of all elements that fix a neighborhood of x pointwise. We then define $\ker(\Gamma \curvearrowright X) = \{g \in \Gamma \mid X^g = X\}$ and

$$\mathrm{int}(\Gamma \curvearrowright X) = \langle \Gamma_x^\circ \mid x \in X \rangle = \langle \{g \in \Gamma \mid X^g \text{ has non-empty interior}\} \rangle.$$

If $\ker(\Gamma \curvearrowright X) = \{1\}$, the action is *faithful*, and when $\mathrm{int}(\Gamma \curvearrowright X) = \{1\}$, the action is *topologically free*. If X is a Hausdorff space, then X^g is always closed.

Remark 2.1. Let Γ act on a Hausdorff space X. If D is a Γ-invariant subset of X, then so is $\overline{D}$, and we have

$$\ker(\Gamma \curvearrowright D) = \ker(G \curvearrowright \overline{D}), \quad \mathrm{int}(\Gamma \curvearrowright D) = \mathrm{int}(\Gamma \curvearrowright \overline{D}).$$

Here the first equality follows from continuity of the action. If D^g has non-empty interior for some $g \in \Gamma$, there is a non-empty open subset V of $\overline{D}$ such that $V \cap D \subseteq D^g$ which ensures that $V \subseteq \overline{V \cap D} \subseteq \overline{D^g} \subseteq \overline{D}^g$. Conversely, if $\overline{D}^g$ has non-empty interior, then there is a non-empty open subset V of $\overline{D}$ such that $V \subseteq \overline{D}^g$, meaning that $V \cap D \subseteq D^g$. As V is non-empty, so is $V \cap D$.

The action of Γ on the topological space X is said to be *minimal* if the Γ-orbit Γx is dense in X for all $x \in X$. Now assume that X is compact Hausdorff, so that X always admits a minimal and closed Γ-invariant subset by Zorn's lemma. By the Riesz representation theorem we may identify the space $\mathcal{P}(X)$ of Radon probability measures on X with the state space of the unital C^*-algebra $C(X)$. We then say that the action of Γ on X is *strongly proximal* if the weak*-closure of every Γ-orbit in $\mathcal{P}(X)$ contains a Dirac measure.

The action is a *boundary action* (and X is said to be a Γ-*boundary*) if the action of Γ on X is minimal and strongly proximal. Any Γ-boundary has an isolated point if and only if it is a one-point space. The major result [22, Theorem 6.2] states (among other things) that if $\Gamma \curvearrowright X$ is a topologically free boundary action, then Γ is C^*-simple.

A result due to Furstenberg [11] states that any discrete group Γ admits a universal Γ-boundary $\partial_F \Gamma$, known as the *Furstenberg boundary*, in the sense that for any Γ-boundary X there exists a unique Γ-equivariant continuous surjection $\partial_F \Gamma \to X$. Then [22, Theorem 6.2] says that Γ is C^*-simple if and only if the action of Γ on the Furstenberg boundary $\partial_F \Gamma$ is topologically free, i.e., if and only if $\mathrm{int}(\Gamma \curvearrowright \partial_F \Gamma)$ is trivial. Moreover, in [10, Corollary 8], it is proved that $R(\Gamma) = \ker(\Gamma \curvearrowright \partial_F \Gamma)$. Thus, minimality of the Γ-action on $\partial_F \Gamma$ implies that Γ is amenable if and only if $\partial_F \Gamma$ is a one-point space.

Another, intrinsic, description of C^*-simplicity is obtained in [23]. A subgroup H of a group Γ is called *residually normal*[1] if there exists a finite subset $F \subseteq G \setminus \{1\}$ such that $F \cap gHg^{-1} \neq \emptyset$ for all $g \in G$. Then [23, Theorem 5.3] says that Γ is C^*-simple if it has no amenable residually normal subgroups.

In [21, Section 7] the *amenablish radical* $AH(\Gamma)$ of a group Γ is defined. It is the smallest normal subgroup of Γ producing a C^*-simple quotient. In particular, a group is C^*-simple if and only if $AH(\Gamma) = \{1\}$ and is called *amenablish* if $AH(\Gamma) = \Gamma$. For any group Γ, the normal subgroup $\mathrm{int}(\Gamma \curvearrowright \partial_F \Gamma)$ is always amenablish. In [21, Section 6 and 7], it is explained that the class of amenablish groups is radical and is "dual" to the class of C^*-simple groups, which

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[1]In an earlier preprint such subgroups were called *recurrent*.

is residual, analogously to the duality between the radical class of amenable groups and the residual class of groups with the unique trace property.

The following result is an adaptation of [22, Theorem 6.2] and [13, Theorem 5.2].

Proposition 2.2. *Let X be a Γ-boundary. The following hold:*

(i) *Γ has the unique trace property if and only if $\ker(\Gamma \curvearrowright X)$ has the unique trace property.*
(ii) *Γ is C^*-simple if and only if $\mathrm{int}(\Gamma \curvearrowright X)$ is C^*-simple.*

Proof. Every $g \in R(\Gamma)$ fixes every point in X by [10, Corollary 8]. Hence $R(\Gamma) \subseteq \ker(\Gamma \curvearrowright X)$, so by e.g., [21, Lemma 6.7], we have that $R(\Gamma) = R(\ker(\Gamma \curvearrowright X))$. Consequently, (i) holds.

For (ii), we set $N = \mathrm{int}(\Gamma \curvearrowright X)$. For any Γ-boundary Y, there exists a Γ-equivariant continuous surjective map $\partial_F \Gamma \to Y$. If $g \in \Gamma$ is such that $(\partial_F \Gamma)^g$ has non-empty interior, then the set Y^g has non-empty interior by [4, Lemma 3.2]. Thus, $\mathrm{int}(\Gamma \curvearrowright \partial_F \Gamma) \subseteq \mathrm{int}(\Gamma \curvearrowright Y)$. Since N is a normal subgroup of Γ, the boundary action $N \curvearrowright \partial_F N$ extends to a boundary action $\Gamma \curvearrowright \partial_F N$ by [4, Lemma 5.2]. It follows that $\mathrm{int}(\Gamma \curvearrowright \partial_F \Gamma) \subseteq \mathrm{int}(\Gamma \curvearrowright \partial_F N) \cap N = \mathrm{int}(N \curvearrowright \partial_F N)$. Hence, if N is C^*-simple, then Γ is C^*-simple.

The converse holds by [4, Theorem 1.4] because $\mathrm{int}(\Gamma \curvearrowright X)$ is a normal subgroup of Γ. $\square$

For Γ and X as above and a non-empty subset $U \subseteq X$, define the fixator subgroup $\Gamma_U = \{g \in \Gamma \mid gx = x \text{ for all } x \in U\}$. Since group elements act continuously, it should be clear that $\Gamma_U = \Gamma_{\overline{U}}$. In particular, if V is dense in X, then $\ker(\Gamma \curvearrowright X) = \Gamma_V \subseteq \Gamma_U$ for all non-empty $U \subseteq X$.

Remark that the notation Γ_C for rigid stabilizers used in [27] coincides with our Γ_U for $C = X \setminus U$.

Recall that the *normal closure* in a group Γ of a subset $S \subseteq \Gamma$ is the smallest normal subgroup, denoted here by $\langle\!\langle S \rangle\!\rangle$, of Γ containing S.

Lemma 2.3. *For any minimal action of a group Γ on a Hausdorff space X and any $x \in X$, the interior of $\Gamma \curvearrowright X$ equals the normal closure of Γ_x°.*

Proof. The normal closure of Γ_x° is always contained in $\mathrm{int}(\Gamma \curvearrowright X)$. Now let $y \in X$ and let $g \in \Gamma_y^\circ$. Then there exists a neighbourhood U of y such that g fixes U pointwise. Since the action is minimal, we can find an $h \in \Gamma$ such that $hx \in U$. Then $h^{-1}gh$ fixes $h^{-1}U$ pointwise, so that $h^{-1}gh \in \Gamma_x^\circ$, i.e., $g \in h\Gamma_x^\circ h^{-1}$. Hence g is contained in the normal closure of Γ_x°, completing the proof. $\square$

Lemma 2.4. *Let $\Gamma \curvearrowright X$ be a minimal action on a Hausdorff space X. The following are equivalent:*

(i) *$\Gamma \curvearrowright X$ is topologically free;*
(ii) *Γ_x° is trivial for some $x \in X$;*
(iii) *Γ_U is trivial for all non-empty open $U \subseteq X$.*

Proof. The implication (i) $\implies$ (ii) is obvious.

If Γ_x° is trivial for some $x \in X$, then Γ_x° is trivial for all $x \in X$ by Lemma 2.3. Let U be an arbitrary non-empty open set in X and pick any $x \in U$. Clearly, Γ_U is a subgroup of Γ_x°, which is trivial. Thus Γ_U is trivial, and we have shown (ii) $\implies$ (iii).

Finally, to prove (iii) $\implies$ (i), assume that $\Gamma \curvearrowright X$ is not topologically free, and choose a non-trivial $g \in \Gamma$ such that X^g has non-empty interior U. Then $g \in \Gamma_U$, which is therefore non-trivial. $\square$

We now say that an action $\Gamma \curvearrowright X$ is *amenably free* if Γ_x° is amenable for all $x \in X$.

Lemma 2.5. *Let $\Gamma \curvearrowright X$ be a minimal action on a Hausdorff space X. The following are equivalent:*

(i) *$\Gamma \curvearrowright X$ is amenably free;*

    (ii) Γ_x° *is amenable for some $x \in X$;*
    (iii) Γ_U *is amenable for all non-empty open $U \subseteq X$.*

Proof. The implication (i) $\implies$ (ii) is obvious.

    Let $U \subseteq X$ be non-empty and open. By minimality, there exists $h \in \Gamma$ such that $hx \in U$, and thus $\Gamma_U \subseteq \Gamma_{hx}^\circ = h\Gamma_x^\circ h^{-1}$. Hence, it follows that (ii) $\implies$ (iii).

    Finally, to show (iii) $\implies$ (i), we assume that Γ_U is amenable for all non-empty open $U \subseteq X$. Choose an arbitrary $x \in X$ and let $\mathcal{V}$ be the collection of all open neighbourhoods of x. The collection is clearly closed under finite intersections, so it is downward directed, and therefore Γ_x° equals the direct limit $\bigcup_\mathcal{V} \Gamma_U$, which is amenable. $\qquad\square$

Remark 2.6. (i) If $\Gamma \curvearrowright X$ is an amenable action on a compact Hausdorff space, then it is amenably free. Indeed, it follows from the definition of amenability of $\Gamma \curvearrowright X$ that for all $x \in X$, the action $\Gamma_x \curvearrowright \{x\}$ is amenable, and therefore Γ_x is amenable (see e.g. [5, Sections 4.3-4.4]). Hence, Γ_x° is amenable for all $x \in X$.

    (ii) Let $\Gamma \curvearrowright X$ be a minimal action of a *countable* group Γ on a Hausdorff space X. Then there exists $x \in X$ such that $\Gamma_x^\circ = \Gamma_x$ (see [27, Proposition 2.4]). Therefore, in this case, $\Gamma \curvearrowright X$ is topologically free if and only if Γ_x is trivial for some $x \in X$, and amenably free if and only if Γ_x is amenable for some $x \in X$.

Proposition 2.7. *Let $\Gamma \curvearrowright X$ be a minimal action on a compact Hausdorff space X. If Γ is C^*-simple, then the action is either topologically free or not amenably free.*

Proof. This follows from [4, Theorem 7.1] and [29, Theorem 14 (2)]. $\qquad\square$

Corollary 2.8. *If $\Gamma \curvearrowright X$ is an amenably free boundary action, then Γ is C^*-simple if and only if $\Gamma \curvearrowright X$ is topologically free.*

Proposition 2.9. *Let $\Gamma \curvearrowright X$ be a minimal action on a compact Hausdorff space such that $N = \ker(\Gamma \curvearrowright X)$ is C^*-simple. Suppose that Γ_U/N is non-amenable for all non-dense open $U \subseteq X$. Then Γ is C^*-simple.*

Proof. The action of Γ on X factors to a faithful action of Γ/N on X, and for any subset $U \subseteq X$, the fixator subgroup Γ_U in Γ satisfies $\Gamma_U/N \cong (\Gamma/N)_U$. Hence $(\Gamma/N)_U$ is non-amenable for all non-dense open $U \subseteq X$, and it now follows from [27, Corollary 3.6] and [23, Theorem 4.1] that Γ/N is C^*-simple (note also that the countability assumption used in [27, Lemma 2.1] is not necessary, as shown in [23, Proposition 5.2]). Hence, since N and Γ/N are both C^*-simple, Γ is C^*-simple by [4, Theorem 1.4]. $\qquad\square$

Corollary 2.10. *Let $\Gamma \curvearrowright X$ be a faithful boundary action that is* not *topologically free. Consider the following properties:*

    (i) Γ_U *is non-amenable for all non-empty open $U \subseteq X$;*
    (ii) Γ *is C^*-simple;*
    (iii) Γ_U *is non-amenable for some non-empty open $U \subseteq X$.*

Then (i) $\implies$ (ii) $\implies$ (iii).

Proof. (i) $\implies$ (ii) is just Proposition 2.9 with trivial kernel, while (ii) $\implies$ (iii) follows from Proposition 2.7 and Lemma 2.5. $\qquad\square$

    The following definition is based on the notion of an extremely proximal flow from [12].

Definition 2.11. An action of a group Γ on a compact Hausdorff space X (with more than two points) is called an *extreme boundary action* (called a *strong boundary action* in [24]) if for every closed $K \subsetneq X$ and non-empty open $U \subseteq X$ there exists $g \in \Gamma$ such that $gK \subseteq U$.

Example 2.12. (a) Let $\Gamma = G \times H$, where $G \curvearrowright X$ is a topologically free extreme boundary action, and H is non-amenable and non-C^*-simple. Then $H = \ker(\Gamma \curvearrowright X) = \operatorname{int}(\Gamma \curvearrowright X) = \Gamma_x^\circ = \Gamma_U$ for all $x \in X$ and non-empty open $U \subseteq X$. In particular, since H is not C^*-simple, Γ is not C^*-simple. In other words, when the kernel is not C^*-simple, non-amenability of all Γ_x° and Γ_U is in general not sufficient for C^*-simplicity.

(b) Let H be a non-C^*-simple group with the unique trace property admitting a faithful extreme boundary action on some compact Hausdorff space X, for example the group from Section 5. Let $\mathbb{F}_H$ be the free group on the set H. We get an induced action of $\mathbb{F}_H$ on X whose kernel coincides with the kernel of the canoncal surjection $\mathbb{F}_H \to H$ which is C^*-simple. In other words, it is possible that a group Γ is C^*-simple and $\Gamma/\ker(\Gamma \curvearrowright X)$ is not C^*-simple, i.e., the kernel of the action $\Gamma \curvearrowright X$ is responsible for the C^*-simplicity of Γ.

Lemma 2.13 ([12, 24])**.** *Let $\Gamma \curvearrowright X$ be an extreme boundary action. Then the action is minimal and strongly proximal, and X is a Γ-boundary in the sense of Furstenberg.*

Lemma 2.14. *Let $\Gamma \curvearrowright X$ be an extreme boundary action. Let U and V be two non-empty open sets that are not dense in X. Then the normal closures of Γ_U and Γ_V coincide, and equal $\operatorname{int}(\Gamma \curvearrowright X)$. Moreover, Γ_U is amenable (resp. trivial) if and only if Γ_V is amenable (resp. trivial).*

Proof. By definition, there exists $g \in \Gamma$ such that $g\overline{U} \subseteq V$. Let $h \in \Gamma_V$, and let $v \in \overline{U}$. Then $g \cdot v \in V$, so $g^{-1}hg \cdot v = g^{-1}g \cdot v = v$. Since v was arbitrarily chosen, this means that $g^{-1}hg \in \Gamma_{\overline{U}}$. Thus $\Gamma_V \subseteq g\Gamma_{\overline{U}}g^{-1}$, and the normal closure of Γ_V is contained in the normal closure of $\Gamma_{\overline{U}} = \Gamma_U$. The other inclusion is similar. Hence, the normal closures of Γ_U and Γ_V coincide. If $g \in \Gamma_U$, then $U \subseteq X^g$, so $g \in \operatorname{int}(\Gamma \curvearrowright X)$, which implies that the normal closure of Γ_U is contained in $\operatorname{int}(\Gamma \curvearrowright X)$. Conversely, if $g \in \Gamma$ and X^g has non-empty interior $W \subseteq X$, then $g \in \Gamma_W$. Therefore, the conclusion follows from the containments

$$\{g \in \Gamma \mid X^g \text{ has non-empty interior}\} \subseteq \bigcup \Gamma_W \subseteq \langle\!\langle \Gamma_U \rangle\!\rangle \subseteq \operatorname{int}(\Gamma \curvearrowright X),$$

where the union is taken over all non-empty open sets of X.

Finally, since each of Γ_U and Γ_V is contained in a conjugate of the other, we get that Γ_U is amenable (resp. trivial) if and only if Γ_V is amenable (resp. trivial). $\qquad\square$

Corollary 2.15. *Let $\Gamma \curvearrowright X$ be an extreme boundary action such that $N = \ker(\Gamma \curvearrowright X)$ is C^*-simple. If $\Gamma/N \curvearrowright X$ is either topologically free or not amenably free, then Γ is C^*-simple.*

In particular, if N is trivial, then Γ is C^*-simple if and only if $\Gamma \curvearrowright X$ is either topologically free or not amenably free.

Proof. Note first that $\Gamma/N \curvearrowright X$ is a faithful extreme boundary action. If the action is topologically free, then Γ/N is C^*-simple by [22, Theorem 6.2]. If the action is not amenably free, then Γ is C^*-simple by Proposition 2.9

The first part follows from Proposition 2.9 and Lemma 2.14, and the second by Corollary 2.10 and Lemma 2.14. $\qquad\square$

Note that the second statement above is just [27, Theorem 1.4] without the countability condition. We now get the following characterization of non-C^*-simplicity for an extreme boundary action.

Corollary 2.16. *Let $\Gamma \curvearrowright X$ be a faithful extreme boundary action. The following are equivalent:*
   (i) *Γ is non-C^*-simple;*
   (ii) *Γ_x° is non-trivial and amenable for some $x \in X$;*
   (iii) *Γ_U is non-trivial and amenable for some non-empty open $U \subseteq X$.*

3. Groups acting on trees

Let G be a graph, let V be its set of vertices and E its set of edges, and let $s, r \colon E \to V$ denote the source and range maps. All graphs considered will be unoriented, meaning that we assume the existence of an inversion map $E \to E$, $e \mapsto \overline{e}$, such that $\overline{\overline{e}} = e$ and $s(\overline{e}) = r(e)$ for any edge $e \in E$. The *degree* of a vertex $v \in V$ is the cardinality of $s^{-1}(v)$ or $r^{-1}(v)$ (the numbers coincide), and a vertex is a *leaf* if it has degree 1.

A *tree* is a connected graph without circuits. If V is the set of vertices of a tree T, then given any two vertices $u, v \in V$ there are two paths between them (one starting in u and ending in v, and one in the opposite direction), and the (equal) length of these paths is the combinatorial distance $d(u, v)$, yielding the *path metric* on V. For the remainder of this section, we will always consider a tree T with vertices V, edges E and source, range and inversion maps as above. A tree is called *locally finite* if every vertex has finite degree.

A *geodesic* in T between two vertices $v, w \in V$ is a finite sequence $v = v_0, \dots, v_n = w$ such that $d(v_i, v_j) = |i - j|$ for all $i, j \in \{0, \dots, n\}$. A *ray* in T is an infinite sequence of vertices $r = (r_i)_{i \geq 0}$ such that $d(r_i, r_{i+n}) = n$ for all $i, n \geq 0$. Two rays r, s are said to be *cofinal*, written $x \sim y$, if there exist $k, n \geq 0$ such that $r_{i+k} = s_{i+n}$ for all $i \geq 0$. The set of equivalence classes of cofinal rays in T is denoted ∂T, and called the *boundary* of T.

In the following, we will identify T with the set of vertices V equipped with its path metric. For an edge $e \in E$, define the *shadow* $Z_0(e) = \{v \in T \mid d(v, s(e)) > d(v, r(e))\}$, i.e., the set of all vertices that are closer to the range of e than the source. Moreover, define $Z_\infty(e)$ as the set of all rays $r = (r_i)_{i \geq 0}$ such that $r_j = s(e)$ and $r_{j+1} = r(e)$ for some $j \geq 0$, and then define $Z_B(e) \subset \partial T$ as $Z_\infty(e)/\sim$. Finally, define the *extended shadow* $Z(e) = Z_0(e) \cup Z_B(e)$. The collection $\{Z(e) \mid e \in E\}$ forms a subbase of compact open sets for a totally disconnected compact Hausdorff topology on $X = T \cup \partial T$, sometimes called the "shadow topology" on X. We refer to [28, Section 4.1, especially Proposition 4.4] in this regard (they assume that T is countable, but their proofs hold also without this hypothesis, although then the resultant topology is not metrizable, see Appendix A for details).

By removing an edge from T, we obtain two components, known as *half-trees*. An *extended half-tree* is a half-tree together with all its associated boundary points. In this terminology, as explained in [27, Section 4.3] (which does not require countability), half-trees and extended shadows are the same notion, so that the shadow topology is generated by all the extended half-trees in $T \cup \partial T$.

Next, define $F \subseteq T$ as the set of all vertices v of finite degree. The following can be deduced from the mentioned sections of [27, 28]:

Proposition 3.1. *All points in F are isolated in $T \cup \partial T$. The closure $\overline{\partial T}$ of ∂T in $T \cup \partial T$ is $(T \setminus F) \cup \partial T$, and is compact. Moreover, ∂T is closed in $T \cup \partial T$ if and only if $F = T$, if and only if T is locally finite.*

Lemma 3.2. *The sets $Z_B(e)$ constitute a basis for the shadow topology restricted to ∂T.*

Proof. We need to prove that whenever $x \in Z_B(e_1) \cap Z_B(e_2)$ for $x \in \partial T$ and edges e_1, e_2 in T, then there exists an edge e such that $x \in Z_B(e) \subseteq Z_B(e_1) \cap Z_B(e_2)$. With this set-up, x is the equivalence class of a ray $(r_i)_{i \geq 0}$ such that $r_0 = s(e_1)$ and $r_1 = r(e_1)$, which is cofinal to a ray $(s_i)_{i \geq 0}$ such that $s_0 = s(e_2)$ and $s_1 = r(e_2)$. Now there exist $k, n \geq 0$ such that $Z_0(e_1) \ni r_{i+k} = s_{i+n} \in Z_0(e_2)$ for all $i \geq 0$. Letting e be the edge defined by $s(e) = r_k$ and $r(e) = r_{k+1}$, then x is clearly contained in $Z_B(e)$.

Any ray in $Z_\infty(e)$ is cofinal to a ray $(t_i)_{i \geq 0}$ satisfying $t_0 = r_k = s_n$ and $t_1 = r_{k+1} = s_{n+1}$. For such a ray $(t_i)_{i \geq 0}$, construct a new ray $(t'_i)_{i \geq 0}$ by defining $t'_i = r_i$ for $0 \leq i \leq k - 1$ and $t'_{k+m} = t_m$ for all $m \geq 0$ in $Z_\infty(e_1)$. Then $(t'_i)_{i \geq 0}$ is cofinal to $(t_i)_{i \geq 0}$, so that $Z_B(e) \subseteq Z_B(e_1)$. A similar argument shows that $Z_B(e) \subseteq Z_B(e_2)$. $\square$

Lemma 3.3. *Every non-empty open subset of $\overline{\partial T}$ contains $Z(e) \cap \overline{\partial T}$ for some edge e in T.*

Proof. Notice first that any edge e in T satisfies $\overline{Z_B(e)} = Z(e) \cap \overline{\partial T}$. Indeed, for any open neighbourhood $V \subseteq \overline{\partial T}$ of a point $x \in Z(e)$, then $V \cap Z(e)$ is an open neighbourhood of $x \in \overline{\partial T}$ since $Z(e)$ is clopen in the shadow topology (see Lemma A.3), so that

$$V \cap Z_B(e) = V \cap Z(e) \cap \partial T \neq \emptyset.$$

For edges $e_1, \ldots, e_n$ in T such that $\bigcap_{i=1}^n Z(e_i)$ intersects ∂T, then as $\bigcap_{i=1}^n Z(e_i) \cap \partial T$ is open in ∂T, Lemma 3.2 yields an edge e such that $Z_B(e) \subseteq \bigcap_{i=1}^n Z(e_i)$. Hence

$$Z(e) \cap \overline{\partial T} = \overline{Z_B(e)} \subseteq \bigcap_{i=1}^n Z(e_i) \cap \overline{\partial T}$$

by what we have seen above. Finally, if $U \subseteq \overline{\partial T}$ is a non-empty open set, then $U \cap \partial T \neq \emptyset$. As there exist edges $e_1, \ldots, e_n$ in T such that $\bigcap_{i=1}^n Z(e_i) \cap \overline{\partial T} \subseteq U$ and $\bigcap_{i=1}^n Z(e_i) \cap \partial T \neq \emptyset$, we infer that $Z(e) \cap \overline{\partial T} \subseteq U$ for some edge e in T. $\qquad\square$

A *morphism* γ of two trees $T_1 = (V_1, E_1)$ and $T_2 = (V_2, E_2)$ is a tuple of maps $\gamma_V \colon V_1 \to V_2$ and $\gamma_E \colon E_1 \to E_2$ such that

$$s(\gamma_E(e)) = \gamma_V(s(e)), \ \gamma_E(\bar{e}) = \overline{\gamma_E(e)}, \quad e \in E_1.$$

If γ_V and γ_E are bijections, γ is called an *isomorphism*, and if $T_1 = T_2$ we say that γ is an *automorphism*. The group of automorphisms of the tree T is denoted by $\mathrm{Aut}(T)$. Furthermore, any automorphism of T is a surjective isometry with respect to the path metric on T, so that we may easily extend $\sigma \in \mathrm{Aut}(T)$ to $T \cup \partial T$ by defining $\sigma(x)$ for any equivalence class x of a ray $(x_i)_{i \geq 0}$ to be the equivalence class of the ray $(\sigma(x_i))_{i \geq 0}$.

We say that an automorphism $\sigma \in \mathrm{Aut}(T)$ is *elliptic* if it fixes a vertex of T, and an *inversion* if it fixes no vertices but does fix an edge (i.e., $\sigma(e) = \bar{e}$ for some $e \in E$). The fixed point set T^σ of an elliptic automorphism σ of T is easily seen to be a subtree of T, and if σ is an inversion of T, then $\sigma(e) = \bar{e}$ for a unique edge $e \in E$. An automorphism $\sigma \in \mathrm{Aut}(T)$ is said to be *hyperbolic* if it is neither elliptic nor an inversion. We will not make the common assumption here that a given group acts without inversions on a tree as most of our results concern fixed points, and an inversion of a tree T fixes no points in $T \cup \partial T$.

For any automorphism $\sigma \in \mathrm{Aut}(T)$, the amplitude of σ is $\ell(\sigma) = \min\{d(v, \sigma(v)) \mid v \in V\}$. The characteristic set of σ is the σ-invariant set

$$T^\sigma = \{v \in V \mid d(v, \sigma(v)) = \ell(\sigma)\}.$$

A bi-infinite path in a graph is a subgraph isomorphic to the graph with vertex set $\mathbb{Z}$ and edge set $\{e_n \mid n \in \mathbb{Z}\} \cup \{\overline{e_n} \mid n \in \mathbb{Z}\}$, with $s(e_n) = r(\overline{e_n}) = n$ and $r(e_n) = s(\overline{e_n}) = n + 1$ for all $n \in \mathbb{Z}$. A fundamental result of Tits states that for a hyperbolic automorphism $\sigma \in \mathrm{Aut}(T)$, the characteristic set T^σ is always the vertex set of a bi-infinite path L in T, called the *axis* of σ, and any non-empty subtree of T which is invariant under σ and σ^{-1} always contains L. We refer to [32, Proposition 6.4.24] for details. Moreover, σ admits exactly two fixed points in $T \cup \partial T$, namely the two points in ∂T arising from the σ-invariant axis of σ. Two hyperbolic automorphisms are said to be *transverse* if they have disjoint fixed point sets.

Henceforth, we assume that the action of a discrete group Γ on a tree T is *minimal*, i.e., that T contains no proper Γ-invariant subtrees, and *strongly hyperbolic* (cf. [18]), that is, of *general type* in the sense of [27, Section 4.3], meaning that Γ contains two transverse hyperbolic automorphisms of T.

Remark 3.4. If Γ acts minimally on a tree T and T has at least 3 vertices, then T has no leaves. Indeed, the subgraph T' arising from removing all vertices of degree 1 from T is a Γ-invariant subtree. In fact, if we assume additionally that Γ contains at least one hyperbolic automorphism of T, then T is the union of all axes of hyperbolic automorphisms of T in Γ (combine the proof of [8, Proposition 3.1] with [1, Proposition 8.1]).

Lemma 3.5. *For a tree T, let $\Gamma \curvearrowright T$ be a minimal, strongly hyperbolic action. Then the induced action $\Gamma \curvearrowright \overline{\partial T}$ is an extreme boundary action. In particular, $\overline{\partial T}$ is a Γ-boundary.*

Proof. The action of Γ on $\overline{\partial T}$ is an extreme boundary action by [27, Section 4.3] (they also assume faithfulness, but that is not needed in their proof), and $\overline{\partial T}$ is a Γ-boundary, by Lemma 2.13. □

Observe that the fixator subgroup of a half-tree (or shadow) in T coincides with the fixator subgroup of the associated extended half-tree. Indeed, for any edge e in T, $g \in \Gamma$ fixing all vertices in $Z_0(e)$ and any ray in $Z_\infty(e)$, there is $k \geq 0$ such that $gr_i = r_i$ for all $i \geq k$, which implies that g fixes $Z(e)$ pointwise. In order to determine fixator subgroups of open subsets of the Γ-boundary $\overline{\partial T}$, we need the following lemma.

Lemma 3.6. *Let e be an edge in T and suppose that $g \in \Gamma$ fixes $Z_B(e)$ pointwise. Then g fixes $Z(e)$ pointwise. In particular, the fixator subgroups of $Z_0(e)$, $Z_B(e)$ and $Z(e) \cap \overline{\partial T}$ coincide.*

Proof. Notice that $Z(e) \cap \overline{\partial T} = \overline{Z_B(e)}$ as seen in Lemma 3.3. Define $v_0 = r(e)$ and $v_1 = s(e)$. Now notice that g is not hyperbolic; if it were, then g would have only two fixed points in all of $\overline{\partial T}$, implying that $Z(e) \cap \overline{\partial T}$ would be an open set in $\overline{\partial T}$ that contained at most two points. Hence $\overline{\partial T}$ would have an isolated point, contradicting that $\overline{\partial T}$ is a non-trivial Γ-boundary. We conclude that g is elliptic (since g cannot be an inversion), so it fixes at least one vertex, say, $v \in T$. Let $m = d(v, v_0)$.

Notice first that if $(r_i)_{i \geq 0}$ is a ray in T such that $gr_0 = r_0$ and there exists $n \geq 0$ such that $r_k \in Z_0(e)$ for all $k \geq n$, then g fixes each vertex r_i. Indeed, there exist $k, n \geq 0$ such that $r_{i+k} = gr_{i+n}$ for all $i \geq 0$, then $k = d(r_k, r_0) = d(gr_n, gr_0) = d(r_n, r_0) = n$. By geodesics in T being unique (so that the paths $r_0, \ldots, r_k$ and $gr_0, \ldots, gr_k = r_k$ coincide), we conclude that $gr_i = r_i$ for all $i \geq 0$. We now have two cases:

(1) If $v \notin Z_0(e)$, then as T is leafless by Remark 3.4, any vertex in $Z_0(e)$ belongs to the image of a ray $(r_i)_{i \geq 0}$ such that $r_0 = v$, $r_m = v_0$ and $r_k \in Z_0(e)$ for all $k \geq m$. In particular, g fixes v_0.

(2) If $v \in Z_0(e)$, assume first that v_0 has degree at least 3. Then there exists a ray $(r_i)_{i \geq 0}$ such that $r_0 = v$, $r_m = v_0$ and $r_k \in Z_0(e)$ for all $k \geq m$. From the above argument, g fixes v_0.

   If v_0 has degree 2, then let $x_0 = v, x_1, \ldots, x_m = v_0$ be the geodesic from v to v_0. Let $0 \leq k \leq m$ be smallest such that x_j has degree 2 for all $k \leq j \leq m$. If $k > 0$, then let $v' \in T$ be a vertex adjacent to x_{k-1}, but $v' \neq x_k$. As T is leafless, there is a ray $(r_i)_{i \geq 0}$ with image in $Z_0(e)$ such that $r_0 = x_0 = v$, $r_{k-1} = x_{k-1}$ and $r_k = v'$. The same argument as above implies $gv' = v'$ and $gx_{k-1} = x_{k-1}$. Since g is an automorphism fixing x_{k-1} and all vertices adjacent to x_{k-1} bar x_k, it must fix x_k as well. By x_k having degree 2, x_{k+1} is also fixed by g (since g fixes x_{k-1} and x_k), and we continue this way until we reach $x_m = v_0$, concluding that g fixes v_0. If $k = 0$, then let v' be the vertex adjacent to v that is not x_1. By taking a ray in $Z_0(e)$ emanating in v and passing through v', we see that g fixes v', so it fixes x_1. By the same method as above, we conclude that g fixes v_0.

Now, for any $w \in Z_0(e)$, then there exists a ray $(r_i)_{i \geq 0}$ such that $r_0 = v_0$, $r_i \in Z_0(e)$ for all $i \geq 0$, and $r_k = w$ for some $k \geq 0$. As above, we see that g fixes w, so that g fixes $Z_0(e)$ and hence $Z(e)$ pointwise. □

Note that by Remark 2.1,
$$\ker(\Gamma \curvearrowright T) = \ker(\Gamma \curvearrowright \partial T) = \ker(\Gamma \curvearrowright \overline{\partial T}),$$
using Lemma 3.6 for the first identity, and
$$\mathrm{int}(\Gamma \curvearrowright \partial T) = \mathrm{int}(\Gamma \curvearrowright \overline{\partial T}).$$

The following is now an immediate consequence of Lemmas 2.14, 3.5, and 3.6.

Corollary 3.7. *Let T_1 and T_2 be any two half-trees of T, and denote by K_1 and K_2 the fixator subgroups of T_1 and T_2, respectively. Then the normal closures of K_1 and K_2 coincide, and equal $\mathrm{int}(\Gamma \curvearrowright \partial T)$. Moreover, K_1 is amenable if and only if K_2 is amenable.*

We say that a continuous action of a group G on a topological space X is *strongly faithful* [16, Lemma 4] if for all finite subsets $F \subseteq G \setminus \{1\}$ there exists $x \in X$ such that $fx \neq x$ for all $f \in F$.

Proposition 3.8. *The following are equivalent:*

   (i) $\mathrm{int}(\Gamma \curvearrowright \partial T) = \{1\}$;
   (ii) $\Gamma \curvearrowright T$ *is strongly faithful;*
   (iii) $\Gamma \curvearrowright \partial T$ *is strongly faithful;*
   (iv) $\Gamma \curvearrowright \partial T$ *is topologically free, i.e., $\Gamma \curvearrowright T$ is slender in the sense of* [18];
   (v) *the fixator subgroups of all half-trees are trivial;*
   (vi) *the fixator subgroup of some half-tree is trivial.*

If these conditions hold, then Γ is a Powers group, and in particular C^*-simple.

Proof. The equivalence between (i) and (iv) follows from the definition, and the equivalence between (i), (v) and (vi) follows from Corollary 3.7.

(ii) $\implies$ (v): Suppose that T admits a half-tree T_1 for which the fixator subgroup K_1 is non-trivial. Let T_2 be the component obtained by removing the edge defining T_1 from T (and so that T_1 and T_2 are disjoint), and let K_2 be the fixator subgroup of T_2. Note that $K_1 \cap K_2 = \ker(\Gamma \curvearrowright T)$. If $K_1 = \ker(\Gamma \curvearrowright T)$, pick $a \in K_1 \setminus \{1\} = \ker(\Gamma \curvearrowright T) \setminus \{1\}$, and set $F = \{a\}$. Then $a \cdot x = x$ for all $x \in T \cup \partial T$. Otherwise, let $a_1 \in K_1 \setminus \ker(\Gamma \curvearrowright T)$. Notice that K_1 is contained in a conjugate of K_2, as seen in the proof of Lemma 2.14. Hence $\ker(\Gamma \curvearrowright T) \subsetneq K_1$ implies $\ker(\Gamma \curvearrowright T) \subsetneq K_2$ since $\ker(\Gamma \curvearrowright T)$ is normal. Let $a_2 \in K_2 \setminus \ker(\Gamma \curvearrowright T)$, and set $F = \{a_1, a_2\}$. For every $x \in T \cup \partial T$, then we will either have $a_1 \cdot x = x$ or $a_2 \cdot x = x$. In particular, (ii) does not hold.

(iv) $\implies$ (iii): For any finite subset $F \subseteq \Gamma \setminus \{1\}$, $\bigcap_{f \in F} \partial T \setminus (\partial T)^f$ is open and dense in ∂T. Hence there exists $x \in \partial T$ such that $fx \neq x$ for all $f \in F$, so the action of G on ∂T is strongly faithful.

(iii) $\implies$ (ii): Let F be a finite subset of $\Gamma \setminus \{1\}$ and take $x \in \partial T$ such that $fx \neq x$ for all $f \in F$. By Lemma 3.2, there is an edge e in T such that $fZ_B(e) \cap Z_B(e) = \emptyset$ for all $f \in F$. For each $f \in F$, the set of f-fixed points in $Z_0(e)$ is now a bounded subtree of the unbounded subtree $Z_0(e)$ in T. Hence $Z_0(e)$ contains a vertex not fixed by any $f \in F$.

[18, Corollary 15] ensures the Powers property. $\qquad \square$

Theorem 3.9. *Let K be the a fixator subgroup of a half-tree of T and set $N = \ker(\Gamma \curvearrowright T)$. The following hold:*

   (i) *If Γ is C^*-simple, then K is either trivial or non-amenable;*
   (ii) *If $K = N$ and N is C^*-simple, then Γ is C^*-simple;*
   (iii) *If K/N is non-amenable and N is C^*-simple, then Γ is C^*-simple.*

In particular, if N is trivial, then Γ is C^*-simple if and only if K is trivial or non-amenable. Also, if K is amenable, then Γ is C^*-simple if and only if K is trivial.

Proof. By Lemma 3.6, K is the fixator subgroup Γ_U of a non-empty and non-dense open subset U of $\overline{\partial T}$. Then part (i) follows from Proposition 2.7, while part (ii) and (iii) follow from Corollary 2.15, using that $K/N = \Gamma_U/N = (\Gamma/N)_U$. $\qquad \square$

Example 3.10. Let T be a regular tree of degree $d \geq 3$, i.e., all vertices in T have degree d, and let Γ be the subgroup of $\mathrm{Aut}(T)$ consisting of all automorphisms that are either elliptic or hyperbolic with an even translation length. The group Γ is also considered in [18, Remarks (iv),

p. 468], and it is uncountable and has finite index in $\mathrm{Aut}(T)$. We will now show that $\Gamma \curvearrowright \overline{\partial T}$ is an extreme boundary action. As T is regular, $\overline{\partial T} = \partial T$ if $d < \infty$ and $\overline{\partial T} = T \cup \partial T$ if $d = \infty$.

We start by explaining that for every two edges e and f an element $g \in \Gamma$ can be found, such that $gZ_0(f) \subseteq Z_0(e)$. The canonical binary partitioning of T induces an orientation of the edges. There are two cases: e and f have the same orientation, or e and f have opposite orientation. In the latter case, take an edge $e_1 \in Z_0(e)$ with $r(e) = s(e_1)$. Then it is easy to see that e_1 has the same orientation as f and, of course, $Z_0(e) \supseteq Z_0(e_1)$. So it is enough to show that there is a group element $g \in \Gamma$ for which $e = gf$ (in the first case), or $e_1 = gf$ (in the second case). But if p and q are edges of the same orientation, then either the geodesic connecting them has even number of edges, and $|d(s(p), s(q)) - d(r(p), r(q))| = 2$ (in this case there is an elliptic element g with $p = gq$), or the geodesic connecting them has odd number of edges, and $|d(s(p), s(q)) - d(r(p), r(q))| = 0$ (in this case there is a hyperbolic element g with $p = gq$). But Γ contains all elements that preserve the orientation, so $gZ_0(f) = Z_0(gf) \subseteq Z_0(e)$.

Let $K \subsetneq \overline{\partial T}$ be a closed set and $\emptyset \neq U \subseteq \overline{\partial T}$ be an open set. Since $\overline{\partial T} \setminus K$ is open and non-empty, then by Lemma 3.3 there exist edges e and f such that $Z(e) \subseteq U$ and $Z(\overline{f}) \subseteq \overline{\partial T} \setminus K$, that is, $K \subseteq Z(f)$. The element g constructed above, for which $gZ_0(f) \subseteq Z_0(e)$, satisfies $gZ(f) \subseteq Z(e)$ by continuity. Therefore $gK \subseteq gZ(f) \subseteq Z(e) \subseteq U$. It follows that the actions of Γ and $\mathrm{Aut}(T)$ on $\overline{\partial T}$ are extreme boundary actions.

Since Γ acts faithfully on T, Γ has the unique trace property by Proposition 2.2. Moreover, the action is not topologically free (i.e., not slender), so $\mathrm{int}(\Gamma \curvearrowright \partial T)$ is non-trivial, and thus the simplicity of Γ implies that $\mathrm{int}(\Gamma \curvearrowright \partial T) = \Gamma$. Note that any half-tree of T is a regular rooted tree, so the fixator subgroup in $\mathrm{Aut}(T)$ of a half-tree is isomorphic to the full automorphism group of a regular rooted tree of branching degree d. This group contains the full automorphism group of a regular rooted tree of branching degree 3 (i.e., of a rooted binary tree), which in turn contains a free subgroup on two generators according to [33]. Note that all automorphisms of a binary rooted tree preserve the orientation, therefore this group is contained in Γ. We can conclude that the fixator subgroup in both Γ and $\mathrm{Aut}(T)$ of any half-tree is non-amenable. Therefore, both Γ and $\mathrm{Aut}(T)$ are C^*-simple by Theorem 3.9.

Remark 3.11. In [21, Lemma 5.8] by the second and third author it was incorrectly stated that ∂T is always a Γ-boundary, which was later used the proof of [21, Theorem 5.9]. The statement of [21, Theorem 5.9] is still correct; in fact, a much more general result holds, as shown in Theorem 3.9.

4. Graphs of groups and HNN extensions

In this section we give some of the preliminaries for the branch of geometric group theory now known as *Bass-Serre theory*. Much of our exposition is based on the original source [32, §I.5] for this topic, but is also inspired by the approach taken in [2] and [18].

4.1. **Graphs of groups.** A *graph of groups* (G, Y) consists of a non-empty connected graph $Y = (V, E, s, r)$, families of groups $(G_v)_{v \in V}$ and $(G_e)_{e \in E}$ such that $G_{\overline{e}} = G_e$ for all $e \in E$, and a family of injective group homomorphisms $\varphi_e \colon G_e \to G_{s(e)}$, $e \in E$. We pick an *orientation* $E_+ \subseteq E$ of Y, meaning that $E_+ \cap \{e, \overline{e}\}$ contains only one edge for all $e \in E$, and define $E_- = E \setminus E_+$.

Now, let $M = (V(M), E(M))$ be a maximal subtree of Y. We define the *fundamental group* $\Gamma = \pi_1(G, Y, M)$ of (G, Y) by

$$\Gamma = \left\langle \{G_v\}_{v \in V}, \ \{\tau_e\}_{e \in E} \ \middle| \ \begin{array}{ll} \tau_{\overline{e}} = \tau_e^{-1} & \text{for all } e \in E, \\ \tau_e \varphi_{\overline{e}}(g) \tau_e^{-1} = \varphi_e(g) & \text{for all } e \in E, \ g \in G_e, \\ \tau_e = 1 & \text{for all } e \in E(M) \end{array} \right\rangle.$$

We have natural group homomorphisms $G_y \to \Gamma$ for all $y \in V$, and they are all injective. Moreover, $\tau_e \in \Gamma$ has infinite order for all $e \in E \setminus E(M)$.

We next define a graph $T = (V(T), E(T), s, r)$ as follows. For any edge $e \in E$, let $|e|$ be the unique edge in $\{e, \overline{e}\} \cap E^+$ and define $\Gamma_e = \varphi_{|e|}(G_e)$. We define $V(T)$ and $E(T)$ by means of left coset spaces in the following way:

$$V(T) = \bigsqcup_{v \in V} \Gamma/G_v, \quad E(T) = \bigsqcup_{e \in E} \Gamma/\Gamma_e.$$

The source, range and inversion maps are given by

$$s(g\Gamma_e) = \left\{ \begin{array}{ll} gG_{s(e)} & \text{for } e \in E^+ \\ g\tau_e^{-1}G_{s(e)} & \text{for } e \in E^-, \end{array} \right. \quad r(g\Gamma_e) = \left\{ \begin{array}{ll} g\tau_e G_{r(e)} & \text{for } e \in E^+ \\ gG_{r(e)} & \text{for } e \in E^-, \end{array} \right.$$

$$\overline{g\Gamma_e} = g\Gamma_{\overline{e}}$$

for all $g \in \Gamma$ and $e \in E$.

Remark 4.1. Note that in [32] the set-up is slightly different: there φ_e is a map $G_e \to G_{r(e)}$, and the relation is $\tau_e\varphi_e(g)\tau_e^{-1} = \varphi_{\overline{e}}(g)$. However, above we follow the convention used in [2], which also coincides with the description in [5, Example E.13].

Theorem 4.2 ([32, §I.5]). *For (G, Y), M and E^+ as above, the graph T constructed above is a tree. Up to isomorphisms, Γ and T are independent of the choice of maximal subtree M and orientation E^+.*

The tree T is the so-called *Bass-Serre tree* of the graph of groups (G, Y); we will also say that T is the Bass-Serre tree of the fundamental group Γ of (G, Y).

The definition of T incites us to define an action of Γ on T by left translation of cosets, and Γ acts on T by automorphisms without inversions, i.e., $ge \neq \overline{e}$ for all $g \in \Gamma$ and $e \in E(T)$.

Theorem 4.3 (Fundamental theorem of Bass-Serre theory). *Suppose that Γ is a group acting without inversions on a tree T. Then there exists a graph of groups with fundamental group Γ' isomorphic to Γ such that the action of Γ' on its Bass-Serre tree T' is isomorphic to the action of Γ on T. Moreover, the stabilizer subgroups of vertices (resp. edges) of T are isomorphic to the vertex (resp. edge) groups of the graph of groups under this isomorphism.*

Remark 4.4. Let $\mathcal{G} = (G, Y)$ be a graph of groups, and let Γ and T be the corresponding fundamental group and Basse-Serre tree, as described above. In [6], inspired by the well-known construction of graph C^*-algebras, and under the assumption that T is locally finite and without leaves, the authors define a C^*-algebra $C^*(\mathcal{G})$ by generators and relations that are encoded by $\mathcal{G}$ in a natural way. Then they prove a C^*-algbraic version of Theorem 4.3, showing in [6, Theorem 4.1] that $C^*(\mathcal{G})$ is isomorphic to the stabilization of the full crossed product $C(\partial T) \rtimes \Gamma$. One of their goals is to find conditions ensuring that $C^*(\mathcal{G})$ is a UCT Kirchberg algebra, which in our case, i.e., assuming only that $\Gamma \curvearrowright T$ is minimal and strongly hyperbolic, can be characterized completely for the associated crossed products; namely, we have that the following are equivalent:

   (i) $C(\overline{\partial T}) \rtimes \Gamma$ is a UCT Kirchberg algebra,
   (ii) $C(\overline{\partial T}) \rtimes_r \Gamma$ is a UCT Kirchberg algebra,
   (iii) Γ and T are countable and $\Gamma \curvearrowright \overline{\partial T}$ is amenable and topologically free.

(i) $\implies$ (ii): Nuclearity implies that the action is amenable, so the full and reduced crossed products are isomorphic.

(ii) $\implies$ (iii): Separability implies that Γ and T are countable, and nuclearity implies that $\Gamma \curvearrowright \overline{\partial T}$ is amenable, and thus amenably free by Remark 2.6. Since $C(\overline{\partial T}) \rtimes_r \Gamma$ is simple, then Γ is C^*-simple by [22, Theorem 6.2], so by Proposition 2.7, the action must be topologically free.

(iii) $\implies$ (i): Amenability implies that the full and reduced crossed products are isomorphic, and they are nuclear, while countability implies that they are separable. Moreover, if a countable group acts amenably on a commutative C^*-algebra, then the crossed product staisfies the UCT

property (see [34]). Finally, topological freeness also implies that $C(\overline{\partial T}) \rtimes_r \Gamma$ is simple and purely infinite by [24, Theorem 5].

Remark 4.5. Let Γ be a countable discrete group acting on a countable tree T. Then, following the argument given in [6, Theorem 5.29], applying [5, Theorem 5.2.1 and Lemma 5.2.6], and using Remark 2.6, the following are equivalent:

   (i) Γ_x is amenable for all $x \in T$,
  (ii) $\Gamma \curvearrowright T \cup \partial T$ is amenable,
 (iii) Γ_x is amenable for all $x \in T \cup \partial T$.

If any of these equivalent conditions hold, then $\Gamma \curvearrowright \overline{\partial T}$ is amenable.

We complete this subsection by discussing a few well-known normal subgroups. For a group G, let $FC(G)$ denote the FC-center of G, i.e., normal subgroup of elements in G of finite conjugacy class, and let $NF(G)$ denote the largest normal subgroup of G that does not contain any non-abelian free subgroup. Now let Γ be a fundamental group of graphs of groups acting on its Bass-Serre tree T and just write $\ker \Gamma$ for $\ker(\Gamma \curvearrowright T)$. As usual, we assume that the action is minimal and strongly hyperbolic. Then we have the following sequence of subgroups:

$$(4.1) \qquad FC(\Gamma) \subseteq FC(\ker \Gamma) \subseteq R(\ker \Gamma) = R(\Gamma) \subseteq NF(\Gamma) = NF(\ker \Gamma) \subseteq \ker \Gamma.$$

The inclusion $R(\Gamma) \subseteq NF(\Gamma)$ is well-known to hold for any group [9], and the two equalities follow from [21, Examples 6.4, 6.6, and Lemma 6.7]. The last inclusion follows from [7, Proposition 19]. Note that this implies that Γ always contains a non-abelian free subgroup. The first containment is clear since $FC(\Gamma)$ is a subgroup of $\ker \Gamma$ and the second holds since $FC(\ker \Gamma)$ is an amenable normal subgroup of $\ker \Gamma$. In general, $FC(\ker \Gamma)$ is always normal in Γ (since an FC-center is a characteristic subgroup), but it may be bigger than $FC(\Gamma)$.

Moreover, if $\ker \Gamma$ is finite, then $FC(\Gamma) = \ker \Gamma$, and the sequence (4.1) collapses. To see why, note that $h \in \ker \Gamma = g^{-1}(\ker \Gamma)g$ for all $g \in \Gamma$, so $ghg^{-1} \in \ker \Gamma$ for all $g \in \Gamma$. Therefore the conjugacy class of h is contained in $\ker \Gamma$, which is finite, so $h \in FC(\Gamma)$.

4.2. **HNN extensions.** We now zoom in on the case of HNN extensions, first from an algebraic point of view, for which we provide some structure results and define subgroups which we shall interpret geometrically in the next subsection. Our work should be compared with similar results by the second and third author for free products with amalgamation [21]: whereas HNN extensions are fundamental groups of loops, free amalgamated products are fundamental groups of a segment of length 1.

Suppose that (F, Y) is a graph of groups, where Y is a loop with one vertex v and one pair of edges $\{e, \bar{e}\}$. We let $\{e\}$ be the orientation of Y and $G = F_v$. As the homomorphisms $\varphi_e \colon F_e \to G$ and $\varphi_{\bar{e}} \colon F_e \to G$ of F_e are both injective, we may define $H = \varphi_e(F_e)$ and an injective group homomorphism $\theta = \varphi_{\bar{e}}\varphi_e^{-1} \colon H \to G$. Defining the *stable letter* $\tau = \tau_e$, the fundamental group Γ is the well-known *HNN extension* $\mathrm{HNN}(G, H, \theta)$:

$$\Gamma = \mathrm{HNN}(G, H, \theta) = \langle G, \tau \mid \tau^{-1} h \tau = \theta(h) \text{ for all } h \in H \rangle.$$

These groups are named after Higman, Neumann, and Neumann, who first studied them in [19].

Let $\Gamma = \mathrm{HNN}(G, H, \theta)$ be an HNN extension and let T be the Bass-Serre tree of Γ. The vertices in T are left cosets of Γ/G and the edge set of T consists of two disjoint copies of Γ/H, say, $\Gamma/H \sqcup \overline{\Gamma/H}$, where the inversion map sends gH to $\overline{gH}$ and vice versa. The source and range maps are given by

$$s(gH) = gG = r(\overline{gH}), \quad r(gH) = g\tau G = s(\overline{gH}), \quad g \in \Gamma.$$

The action of Γ on T by left translation is transitive. Moreover, T is regular, i.e., all vertices have the same degree $[G : H] + [G : \theta(H)]$, so that in particular, T is leafless, and T is countable if and only if G/H and $G/\theta(H)$ are of at most countably infinite cardinality.

The picture below illustrates part of the Bass-Serre tree of the HNN extension of a group G, and subgroups $H \cong \theta(H)$ such that $[G : H] = 2$ and $[G : \theta(H)] = 3$. We let $S_{-1} = \{1, s\}$ and $S_1 = \{1, t_1, t_2\}$ be sets of left coset representatives for H and $\theta(H)$, respectively. Observe that for $g \in S_{-1}$, if we want to add $g\tau$ to the right in one of the vertex cosets mG (e.g., going from mG to $mg\tau G$), we traverse an edge *emanating* from mG, and when adding $g\tau^{-1}$ to the right for $g \in S_1$, one traverses an edge *ending* in mG.
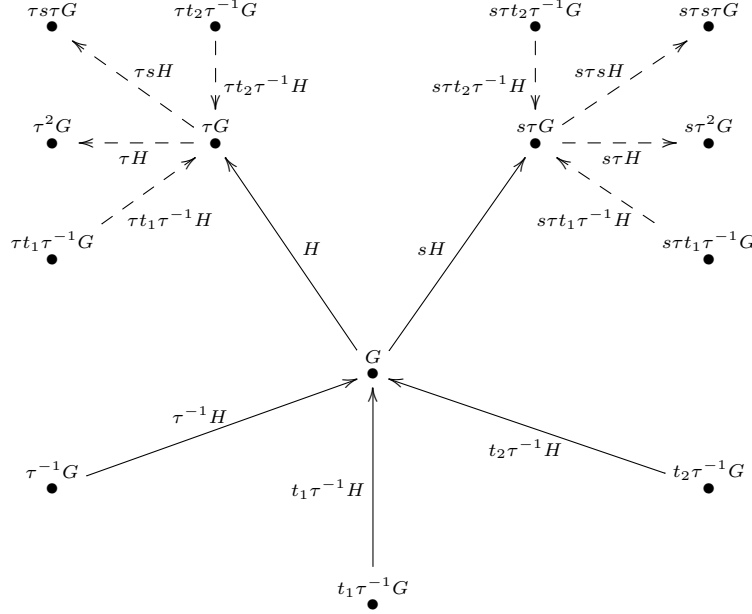
Let $g \in \Gamma$ be an element with normal form $g = g_1\tau^{\varepsilon_1}g_2\tau^{\varepsilon_2}\cdots g_n\tau^{\varepsilon_n}$. Defining $t_0 = 1$ and

$$(4.2) \qquad t_k = \prod_{i=1}^{k} g_i\tau^{\varepsilon_i}, \quad e_k = \begin{cases} t_k\tau^{-1}H = t_{k-1}g_kH & \text{if } \varepsilon_k = 1, \\ \overline{t_kH} & \text{if } \varepsilon_k = -1 \end{cases}$$

for $1 \leq k \leq n$, the unique path from G to gG is given by

$$G \xrightarrow{\quad e_1 \quad} t_1G \xrightarrow{\quad e_2 \quad} \cdots \xrightarrow{\quad e_n \quad} gG$$

Indeed, if $\varepsilon_k = 1$, then $s(t_{k-1}g_kH) = t_{k-1}G$ and $r(t_{k-1}g_kH) = t_{k-1}g_k\tau G = t_kG$, and if $\varepsilon_k = -1$ we notice that $s(\overline{t_kH}) = t_k\tau G = t_{k-1}G$ and $r(\overline{t_kH}) = t_kG$.

If we let S_{-1} and S_1 be systems of representatives for the *left* cosets of H and $\theta(H)$ in G, respectively, such that $1 \in S_{-1} \cap S_1$, the *unique normal form* of an element $g \in \Gamma$ (see, e.g., [3, Theorem 2.14.3]) can be written

$$g = g_1\tau^{\varepsilon_1}g_2\tau^{\varepsilon_2}\cdots g_n\tau^{\varepsilon_n}g_{n+1},$$

where $n \geq 0$, and the following conditions are satisfied:

(i) $g_{n+1} \in G$ and $\varepsilon_i \in \{\pm 1\}$ for $1 \leq i \leq n$;
(ii) $g_i \in S_{-\varepsilon_i}$ for all $1 \leq i \leq n$;
(iii) $g_i = e$ implies $\varepsilon_{i-1} = \varepsilon_i$ for $2 \leq i \leq n$.

This also entails that the natural map $G \to \Gamma$ is actually an injection. With g as above, we say that $n = |g|$ is the *length* of g, and if $n \geq 1$, we say that ε_1 and ε_n is the *type* and *direction* of g, respectively. The *initial letter* of g is $g_1 \in S_{-\varepsilon_1}$ and the *end letter* of g is $g_{n+1} \in G$. We define the length of any $g \in G$ to be 0, and the initial letter and end letter of g as an element in Γ are given by 1 and g, respectively.

For $n \geq 1$, $g_1, \ldots, g_{n+1} \in G$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\}$, the word

$$g_1 \tau^{\varepsilon_1} \cdots g_n \tau^{\varepsilon_n} g_{n+1}$$

is said to be *reduced* (in Γ) if for all $1 \leq i \leq n-1$ we have

  (a) $g_{i+1} \notin H$ whenever $\varepsilon_i = -1$ and $\varepsilon_{i+1} = 1$, and
  (b) $g_{i+1} \notin \theta(H)$ whenever $\varepsilon_i = 1$ and $\varepsilon_{i+1} = -1$.

If we define

$$H_{-1} = H, \quad H_1 = \theta(H),$$

the conditions (a) and (b) can be rephrased as follows: for all $1 \leq i \leq n-1$, $g_{i+1} \notin H_{\varepsilon_i}$ whenever $\varepsilon_{i+1} = -\varepsilon_i$. Notice that

(4.3) $$\tau^{-\varepsilon} H_{-\varepsilon} \tau^{\varepsilon} = H_{\varepsilon}.$$

We say that $g_1 \in G$ is *reduced* if $g_1 \neq 1$. A fundamental result for HNN extensions, also known as *Britton's lemma*, is that reduced words always define non-identity elements. The result itself can be derived from the uniqueness of the normal form. Indeed, if for $n \geq 1$ the word $g = g_1 \tau^{\varepsilon_1} \cdots g_n \tau^{\varepsilon_n} g_{n+1} \in \Gamma$ is reduced, let $s_1 \in S_{-\varepsilon_1}$ and $h_1 \in H_{-\varepsilon_1}$ such that $g_1 = s_1 h_1$ and rewrite

$$g = s_1 \tau^{\varepsilon_1} (\tau^{-\varepsilon_1} h_1 \tau^{\varepsilon_1}) g_2 \tau^{\varepsilon_2} \cdots g_n \tau^{\varepsilon_n} g_{n+1}.$$

The remainder of the proof divides into two possible situations, depending on whether consecutive powers of τ in the word coincide. Indeed, define $g_2' = (\tau^{-\varepsilon_1} h_1 \tau^{\varepsilon_1}) g_2$. If $\varepsilon_2 = \varepsilon_1$, then write $g_2' = s_2 h_2$ for $s_2 \in S_{-\varepsilon_2}$ and $h_2 \in H_{-\varepsilon_2}$ as above, and write

$$g = s_1 \tau^{\varepsilon_1} g_2' \tau^{\varepsilon_2} \cdots g_n \tau^{\varepsilon_n} g_{n+1} = s_1 \tau^{\varepsilon_1} s_2 \tau^{\varepsilon_2} (\tau^{-\varepsilon_2} h_2 \tau^{\varepsilon_2}) g_3 \cdots g_n \tau^{\varepsilon_n} g_{n+1}.$$

If $\varepsilon_2 = -\varepsilon_1$, then $g_2 \notin H_{-\varepsilon_2} = H_{\varepsilon_1}$ by assumption, so that due to (4.3), $g_2' = (\tau^{-\varepsilon_1} h_1 \tau^{\varepsilon_1}) g_2 \in H_{\varepsilon_1} g_2$ and $g_2' \notin H_{\varepsilon_1} = H_{-\varepsilon_2}$. We then proceed as for g_1, noting that the coset representative of g_2' with respect to $H_{-\varepsilon_2}$ is not 1. Iterating the process yields the normal form of g, which contains at least $2n-1$ terms. If $n \geq 2$ then $g \neq 1$ due to uniqueness of the normal form, and if $n = 1$, then $g = g_1 \tau^{\varepsilon_1} g_2 \neq 1$ would imply $\tau \in G$, a contradiction.

The above proof of Britton's lemma also proves the following lemma.

Lemma 4.6. *Let $n \geq 1$ and let $g = g_1 \tau^{\varepsilon_1} \cdots g_n \tau^{\varepsilon_n} g_{n+1} \in \Gamma$ be a reduced word. Then*

  (i) *$g \notin G$ and $|g| = n$;*
  (ii) *the type of g is ε_1;*
  (iii) *the direction of g is ε_n;*
  (iv) *the initial letter of g is 1 if and only if $g_1 \in H_{-\varepsilon_1}$;*
  (v) *if $g_{n+1} \in H_{\varepsilon_n}$, the end letter of the normal form is contained in H_{ε_n} as well.*

In some cases when working with large subsets of an HNN extension, it will prove helpful to be able to uncover properties of elements in these subsets without having to reduce. We introduce a simple lemma to remedy this situation, the proof modeled after the proof of the preceding lemma.

Lemma 4.7. *For an HNN extension $\Gamma = \mathrm{HNN}(G, H, \theta)$, $n \geq 1$, $g_1, \ldots, g_{n+1} \in G$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\}$, define*

$$g = g_1 \tau^{\varepsilon_1} \cdots g_n \tau^{\varepsilon_n} g_{n+1} \in \Gamma.$$

If n is odd, then $g \notin G$, and if $\varepsilon_1 = \ldots = \varepsilon_k$ for some $k > \frac{n}{2}$, the type of g is ε_1.

Proof. Notice that the above expression of g is reduced if and only if $\tau^{\varepsilon_j} g_{j+1} \tau^{\varepsilon_{j+1}} \notin G$ for all $1 \leq j \leq n-1$. We may therefore assume that there is $1 \leq j \leq n-1$ such that $\tau^{\varepsilon_j} g_{j+1} \tau^{\varepsilon_{j+1}} \in G$. Let $1 \leq i \leq \min\{j, n-j\}$ be largest such that

$$h = g_{j-(i-1)} \tau^{\varepsilon_{j-(i-1)}} g_{j-(i-1)+1} \cdots g_{j+i} \tau^{\varepsilon_{j+i}} g_{j+i+1} \in G.$$

We may now write $g = g_1\tau^{\varepsilon_1}\cdots g_{j-i}\tau^{\varepsilon_{j-i}}h\tau^{\varepsilon_{j+i+1}}\cdots g_n\tau^{\varepsilon_n}g_{n+1}$. If this word is not reduced, we can continue this process for this new expression of g. After a finite number of iterations, the word must be reduced, so because this process always removes an even number of powers of τ from the preceding expression of g, n being odd implies that $g \notin G$ by Lemma 4.6 (i). If $\varepsilon_1 = \ldots = \varepsilon_k$ for some $k > \frac{n}{2}$, then reduction removes at most $n - k < k$ of the identical first k powers of τ in the original expression of g. Therefore the type of g is ε_1 by Lemma 4.6 (ii). $\quad\square$

The *kernel* of the HNN extension Γ is the normal subgroup

$$\ker\Gamma = \bigcap_{r \in \Gamma} rHr^{-1}.$$

For $\varepsilon \in \{\pm 1\}$, let T_ε be the collection of elements $g \in \Gamma$ of length $n \geq 1$ and type ε. Let $T_\varepsilon^\dagger$ be the subset of $g \in T_\varepsilon$ of length $n \geq 1$ with initial letter 1. We then define the *quasi-kernel*

$$K_\varepsilon = \bigcap_{r \in \Gamma \setminus T_\varepsilon^\dagger} rHr^{-1}.$$

Notice that $(\Gamma \setminus T_{-1}^\dagger) \cup (\Gamma \setminus T_1^\dagger) = \Gamma$, so that $\ker\Gamma = K_{-1} \cap K_1$.

We will consider criteria for HNN extensions to be C^*-simple and to have the unique trace property in the following. An HNN extension $\Gamma = \mathrm{HNN}(G, H, \theta)$ is *ascending* if either $H = G$ or $\theta(H) = G$. In order to make the most of Britton's lemma, we will mostly consider *non-ascending* HNN extensions, which is not too restrictive of a property, meaning that both S_1 and S_{-1} are non-trivial.

Lemma 4.8. *If Γ is a non-ascending HNN extension, the normal closures of the quasi-kernels in Γ coincide.*

Proof. For any $s \in S_{-1} \setminus \{1\}$, $s\tau(\Gamma \setminus T_{-1}^\dagger) \subseteq \Gamma \setminus T_1^\dagger$, so that

$$K_1 = \bigcap_{r \in \Gamma \setminus T_1^\dagger} rHr^{-1} \subseteq s\tau\left(\bigcap_{r \in \Gamma \setminus T_{-1}^\dagger} rHr^{-1}\right)\tau^{-1}s^{-1} = s\tau K_{-1}\tau^{-1}s^{-1}.$$

Hence the normal closure of K_{-1} contains that of K_1. The reverse inclusion is seen in a similar manner. $\quad\square$

As the normal closures of K_1 and K_{-1} in Γ coincide by the above lemma, it follows that K_1 is trivial if and only if K_{-1} is trivial. Moreover, if one of K_1 and K_{-1} is equal to the normal subgroup $\ker\Gamma$, then the other one is also equal to $\ker\Gamma$.

For a non-ascending HNN extension Γ, let $\mathrm{int}\,\Gamma$ be the normal closure of either of the quasi-kernels. We call $\mathrm{int}\,\Gamma$ the *interior* of Γ.

Remark 4.9. Let $\Gamma = \mathrm{HNN}(G, H, \theta)$ be a non-ascending HNN extension. Then $\mathrm{int}\,\Gamma = \ker\Gamma$ if and only if the quasi-kernels of Γ coincide with the kernel of Γ, if and only if there exists $g_\varepsilon \in G \setminus H_{-\varepsilon}$ such that $g_\varepsilon K_\varepsilon g_\varepsilon^{-1} = K_\varepsilon$ for $\varepsilon \in \{\pm 1\}$. To see this, note first that $\varepsilon \in \{\pm 1\}$ and all $g_\varepsilon \in G \setminus H_{-\varepsilon}$ we have

$$\Gamma \setminus T_{-\varepsilon}^\dagger \subseteq (\Gamma \setminus T_\varepsilon^\dagger) \cup g_\varepsilon(\Gamma \setminus T_\varepsilon^\dagger).$$

Indeed, if $x \in T_\varepsilon^\dagger \subseteq \Gamma \setminus T_{-\varepsilon}^\dagger$, then $g_\varepsilon^{-1}x$ has type ε and initial letter different from 1. Therefore $\Gamma = (\Gamma \setminus T_\varepsilon^\dagger) \cup g_\varepsilon(\Gamma \setminus T_\varepsilon^\dagger)$, so that

$$\ker\Gamma = \bigcap_{r \in \Gamma \setminus T_\varepsilon^\dagger} rHr^{-1} \cap \bigcap_{r \in \Gamma \setminus T_\varepsilon^\dagger} g_\varepsilon rHr^{-1}g_\varepsilon^{-1} = K_\varepsilon \cap g_\varepsilon K_\varepsilon g_\varepsilon^{-1}.$$

The conclusions now follow from normality.

The following theorem (and the main result of this section) is motivated by condition (SF')
in [16, Proposition 11] and [21, Theorem 3.2].

Theorem 4.10. *Let* $\Gamma = \mathrm{HNN}(G, H, \theta)$ *be a non-ascending HNN extension. The following are
equivalent:*

   (i) $\mathrm{int}\,\Gamma = \{1\}$*;*
  (ii) *the quasi-kernel* K_ε *is trivial for some or both* ε*;*
 (iii) *for every finite subset* $F \subseteq \Gamma \setminus \{1\}$*, there exists* $g \in \Gamma$ *such that* $gFg^{-1} \cap G = \emptyset$*;*
 (iv) *for every finite subset* $F \subseteq \Gamma \setminus \{1\}$*, there exists* $g \in \Gamma$ *such that* $gFg^{-1} \cap H = \emptyset$*;*
  (v) *for every finite subset* $F \subseteq G \setminus \{1\}$*, there exists* $g \in \Gamma$ *such that* $gFg^{-1} \cap H = \emptyset$*;*
 (vi) *for every finite subset* $F \subseteq H \setminus \{1\}$*, there exists* $g \in \Gamma$ *such that* $gFg^{-1} \cap H = \emptyset$*.*

Moreover, if these conditions hold, then Γ *is a Powers group, and in particular* C^**-simple.*

Proof. (iii) $\implies$ (iv) $\implies$ (v) $\implies$ (vi) and (i) $\iff$ (ii) are obvious.

(vi) $\implies$ (ii): If K_ε is trivial for some $\varepsilon \in \{\pm 1\}$, then $K_{-\varepsilon}$ is trivial by Lemma 4.8. Therefore
suppose that K_{-1} and K_1 are both non-trivial. Pick $f_\varepsilon \in K_\varepsilon \setminus \{1\}$ for $\varepsilon \in \{\pm 1\}$ and set
$F = \{f_{-1}, f_1\}$. For an arbitrary $g \in \Gamma$, we have $g^{-1} \in \Gamma \setminus T_\varepsilon^\dagger$ for some ε. Then $gf_\varepsilon g^{-1} \in H$,
i.e., $gf_\varepsilon g^{-1} \in gFg^{-1} \cap H$.

(ii) $\implies$ (iii): Choose a finite $F \subseteq \Gamma \setminus \{1\}$. Assume first there is an element $f_1 \in F \cap G$
(otherwise, there is nothing to show). Since $f_1 \neq 1$, we may pick $g_1 \in \Gamma \setminus T_1^\dagger$ such that
$g_1^{-1} f_1 g_1 \notin H$. We may now assume that $g_1^{-1} f_1 g_1 \notin G$; if $g_1^{-1} f_1 g_1 \in G$, we can freely replace g_1
by $g_1 \tau$. In particular, $g_1 \notin G$, and so we may let ε_1 be the direction of g_1. We then see that
the type and direction of $g_1^{-1} f_1 g_1$ are $-\varepsilon_1$ and ε_1, respectively, since we can write $g_1^{-1} f_1 g_1$ as
a reduced word by means of the normal form of g_1 and then apply Lemma 4.6. In this way,
we also see that replacing g_1 by $g_1 h^{-1}$ where h is the end letter of g_1 does not change this
conclusion, so we may assume that $g_1^{-1} f_1 g_1$ has initial letter 1 and end letter contained in H_{ε_1}.

We now assume that there is an element $f_2 \in F$ such that $g_1^{-1} f_2 g_1 \in G$ (otherwise, we are
done). Pick $g_2 \in \Gamma \setminus T_{-\varepsilon_1}^\dagger$ such that $g_2^{-1} g_1^{-1} f_2 g_1 g_2 \notin H$. In the same manner as above, we may
assume that $g_2^{-1} g_1^{-1} f_2 g_1 g_2 \notin G$, $g_2 \notin G$ and g_2 has end letter 1, and if ε_2 is the direction of
g_2, then $g_2^{-1} g_1^{-1} f_2 g_1 g_2$ has type $-\varepsilon_2$ and direction ε_2. We now claim that $g_2^{-1} g_1^{-1} f_1 g_1 g_2 \notin G$
as well. Indeed, since $g_1^{-1} f_1 g_1 \notin G$ has type $-\varepsilon_1$ and direction ε_1 with initial letter 1 and end
letter in H_{ε_1}, then $g_2^{-1} g_1^{-1} f_1 g_1 g_2$ has type $-\varepsilon_2$ and direction ε_2, with initial letter 1 and end
letter contained in H_{ε_2}. To realize this, we consider the normal forms of $g_1^{-1} f_1 g_1$ and of g_2, say,
$g_2 = h_1 \tau^\varepsilon \cdots h_m \tau^{\varepsilon_2}$. Then

$$g_2^{-1}(g_1^{-1} f_1 g_1)g_2 = \tau^{-\varepsilon_2} h_m^{-1} \cdots \tau^{-\varepsilon} h_1^{-1}(g_1^{-1} f_1 g_1)h_1 \tau^\varepsilon \cdots h_m \tau^{\varepsilon_2}.$$

Hence reduction is only possible if $\varepsilon = -\varepsilon_1$, but $h_1 \notin H_{-\varepsilon} = H_{\varepsilon_1}$ by assumption since
$g_2 \in \Gamma \setminus T_{-\varepsilon_1}^\dagger$. Therefore the above word is always reduced, so Lemma 4.6 applies.

It should be clear how this process continues, choosing g_i from the set $\Gamma \setminus T_\varepsilon^\dagger$, depending on
the direction of g_{i-1}, and since F is finite, we take g to be the product of the g_i's, and then
$g^{-1} f g \notin G$ for every $f \in F$.

We refer to [16, Proposition 11] for a proof that any of the six conditions implies that Γ is a
Powers group, if Γ is non-ascending. $\qquad\square$

Remark 4.11. In the 2011 article [18, Theorem 3 (ii)], a sufficient criterion to ensure C^*-
simplicity of a non-ascending, countable HNN extension was given by de la Harpe and Préaux,
formulated as follows. For $\Gamma = \mathrm{HNN}(G, H, \theta)$ non-ascending and G countable, define $H_0 = H$,
and recursively define a descending sequence of subgroups $(H_k)_{k \geq 1}$ of $H_0 = H$ by $H_k' = H_k \cap \tau^{-1} H_k \tau$ and

$$H_k = \left(\bigcap_{g \in G} g H_k' g^{-1} \right) \cap \tau \left(\bigcap_{g \in G} g H_k' g^{-1} \right) \tau^{-1}.$$

The criterion ensuring C^*-simplicity of Γ was that $H_k = \{1\}$ for some $k \geq 0$ (in fact, Γ is a Powers group).

We claim that Theorem 4.10 is a stronger result. Indeed, for $k \geq 1$, let C_k be the set of elements in Γ of length $\leq k+1$. Then $H_k = \{1\}$ implies $\bigcap_{r \in C_k} rHr^{-1} = \{1\}$, since each H_k is obtained by taking intersections of sets of the form rHr^{-1}, r running through a subset of C_k. For $\varepsilon \in \{\pm 1\}$ and $s \in S_{-\varepsilon} \setminus \{1\}$, then $s\tau^{(k+2)\varepsilon}C_k \subseteq \Gamma \setminus T_\varepsilon^\dagger$ due to Lemma 4.7. Therefore

$$K_\varepsilon = \bigcap_{r \in \Gamma \setminus T_\varepsilon^\dagger} rHr^{-1} \subseteq s\tau^{(k+2)\varepsilon} \left(\bigcap_{r \in C_k} rHr^{-1} \right) \tau^{-(k+2)\varepsilon}s^{-1} = \{1\}.$$

The following result, similar to [21, Theorem 3.7], now holds.

Proposition 4.12. *Let $\Gamma = \mathrm{HNN}(G, H, \theta)$ be a non-ascending HNN extension such that $H \cap \theta(H)$ is finite. Then the following are equivalent:*

   (i) Γ *is icc;*
  (ii) $\ker \Gamma = \{1\}$*;*
 (iii) $\mathrm{int}\,\Gamma = \{1\}$*;*
 (iv) Γ *is a Powers group;*
  (v) Γ *is C^*-simple;*
 (vi) Γ *has the unique trace property.*

Proof. It is clear that (iii) $\implies$ (iv) $\implies$ (v) $\implies$ (vi) $\implies$ (i).

Note that $\ker \Gamma \subseteq H \cap \theta(H)$ so the kernel is finite, and therefore (i) $\implies$ (ii) by (4.1) and the preceding comment.

Thus only (ii) $\implies$ (iii) remains. Suppose that $\ker \Gamma = \{1\}$. Note that $\ker \Gamma$ coincides with the intersection of the decreasing sequence

$$H_0 \supseteq H_1 \supseteq \cdots H_k \supseteq H_{k+1} \supseteq \cdots$$

of Remark 4.11. For $k \geq 1$, H_k is a subgroup of $H \cap \theta(H)$ and therefore finite, meaning that H_k must be trivial for some $k \geq 1$. As in Remark 4.11, we conclude that K_ε is trivial for $\varepsilon \in \{\pm 1\}$. $\qquad\square$

The above result indicates that in order to search for examples of non-ascending HNN extensions $\mathrm{HNN}(G, H, \theta)$ that are not C^*-simple but do satisfy the unique trace property, we have to ensure that the image of θ inside G is not too far away from H. We give such an example in Section 5.

Next, combining the notion of residual normality from [23, Section 5] with Theorem 4.10, we obtain the following result.

Lemma 4.13. *Let $\Gamma = \mathrm{HNN}(G, H, \theta)$ be a non-ascending HNN extension. The following are equivalent:*

   (i) K_{-1} *and K_1 are non-trivial;*
  (ii) H *is residually normal in Γ;*
 (iii) G *is residually normal in Γ.*

Corollary 4.14. *Let $\Gamma = \mathrm{HNN}(G, H, \theta)$ be a non-ascending HNN extension. If H is not residually normal, then Γ is C^*-simple. Consequently, if H is amenable, then Γ is C^*-simple if and only if H is* not *residually normal in Γ.*

Proof. If H is not residually normal, then K_{-1} and K_1 are trivial by Lemma 4.13, and therefore Γ is C^*-simple by Theorem 4.10. The last part now follows from [23, Theorem 5.3]. $\qquad\square$

Remark 4.15. There is also another way of seeing that $\Gamma = \mathrm{HNN}(G, H, \theta)$ is not C^*-simple when H is amenable and the quasi-kernels K_{-1} and K_1 are non-trivial.

Note first that if $K_1 = \ker \Gamma$, then $K_{-1} = \ker \Gamma$. Thus, by assumption, $\ker \Gamma$ is a non-trivial normal amenable subgroup of Γ, and hence Γ cannot be C^*-simple. Hence, we may assume that both K_1 and K_{-1} are different from $\ker \Gamma$.

Choose $a \in K_1 \setminus \ker \Gamma$ and $b \in K_{-1} \setminus \ker \Gamma$. Then

$$\{gH \mid gH \neq agH\} \subseteq \{gH \mid g \in T_{-1}\} \quad \text{and}$$
$$\{gH \mid gH \neq bgH\} \subseteq \{gH \mid g \in T_1\},$$

which are clearly disjoint. By using the technique from Proposition 4.8 in [14], also explained in [29, p. 12], the action of Γ on Γ/H gives rise to a unitary representation $\pi \colon G \to \ell^2(\Gamma/H)$, that extends to a continuous representation of $C_r^*(\Gamma)$. It follows that $(1 - \lambda(a))(1 - \lambda(b))$ generates a proper two-sided closed ideal of $C_r^*(\Gamma)$. Hence, Γ is not C^*-simple.

4.3. **Boundary actions of non-ascending HNN extensions.** Now let T denote the Bass-Serre tree of an HNN extension $\Gamma = \mathrm{HNN}(G, H, \theta)$. It was remarked in the beginning of the previous subsection that the action of Γ on T is transitive, so it is also minimal. Moreover, we either have $\overline{\partial T} = \partial T$ or $\overline{\partial T} = T \cup \partial T$ in $T \cup \partial T$ in the shadow topology, since T is regular. If $|\partial T| \leq 2$, every vertex in T has degree 2, so that $H = \theta(H) = G$ and $\Gamma = G \rtimes_\theta \mathbb{Z}$.

Proposition 4.16. *Let $\Gamma = \mathrm{HNN}(G, H, \theta)$ be an HNN extension, and let T be the Bass-Serre tree of Γ. If $|\partial T| \geq 3$, then the following are equivalent:*

 (i) *The action of Γ on T is strongly hyperbolic.*
 (ii) *Γ is non-ascending.*

If any of these two conditions is satisfied, $\overline{\partial T}$ is a Γ-boundary in the shadow topology.

Proof. The equivalence follows from [18, Proposition 20], and since the action of Γ on T is minimal, it follows from Lemma 3.5 that $\overline{\partial T}$ is indeed a Γ-boundary. $\qquad\square$

The shadows in $T \cup \partial T$ can be described as follows. If g has normal form

$$g = g_1 \tau^{\varepsilon_1} g_2 \tau^{\varepsilon_2} \cdots g_n \tau^{\varepsilon_n} g_{n+1}$$

for $n \geq 1$, let $U(g)$ be the subset of all elements hG where the normal form of h begins with $g g_{n+1}^{-1} = g_1 \tau^{\varepsilon_1} g_2 \tau^{\varepsilon_2} \cdots g_n \tau^{\varepsilon_n}$, as well as all equivalence classes of rays identifiable with a ray beginning with $g_1 \tau^{\varepsilon_1} g_2 \tau^{\varepsilon_2} \cdots g_n \tau^{\varepsilon_n}$. Then $U(g)$ is an extended half-tree, and by letting $V(g)$ be the complement extended half-tree resulting from removing the edge connecting $g g_{n+1}^{-1} \tau^{-\varepsilon_n} G$ and gG, the collection of all sets of the form $U(g)$ and $V(g)$, $g \in \Gamma$, generates the shadow topology.

For any $g \in \Gamma$, let $K(g)$ (resp. $L(g)$) be the fixator subgroup of the extended half-tree $U(g)$ (resp. $V(g)$). Then $K(g)$ is the fixator of the half-tree $U(g) \cap T$, and $L(g)$ is the fixator of $V(g) \cap T$.

Lemma 4.17. *Let $\Gamma = \mathrm{HNN}(G, H, \theta)$ be a non-ascending HNN extension with Bass-Serre tree T. For any $g \in \Gamma \setminus G$ with direction ε,*

$$K(g) = g K_{-\varepsilon} g^{-1}, \quad L(g) = g t^{-1} \tau^{-\varepsilon} K_\varepsilon \tau^\varepsilon t g^{-1},$$

where $t \in G$ is the end letter of g. Moreover, $g \in \Gamma$ fixes all $x \in T$ if and only if $g \in \ker \Gamma$.

Proof. Suppose that $g' \in \Gamma$ is a fixator of all elements in $U(g)$ where

$$g = g_1 \tau^{\varepsilon_1} g_2 \tau^{\varepsilon_2} \cdots g_n \tau^{\varepsilon_n} g_{n+1} \in \Gamma$$

in the normal form, and let ε be the direction of g. Then g' fixes all edges in the subtree spanned by $U(g)$, i.e., $g'(mH) = mH$ for all $m \in \Gamma$ with normal form beginning with $g_1 \tau^{\varepsilon_1} g_2 \tau^{\varepsilon_2} \cdots g_n \tau^{\varepsilon}$. Therefore $g' \in K(g)$ if and only if $g'(grH) = grH$ for all $r \in \Gamma$ with the normal form $r = r_1 \tau^{f_1} r_2 \tau^{f_2} \cdots r_m \tau^{f_m} r_{m+1}$ where either

 (1) $m = 0$, or

(2) $m \geq 1$ and either (2a) $f_1 = \varepsilon$, or (2b) $f_1 = -\varepsilon$ and $r_1 \notin H_\varepsilon$.

In case (2b), the fact $r_1 \in S_{-f_1} = S_\varepsilon$ implies that $r_1 \neq 1$. Therefore the $r \in \Gamma$ of the above form constitute the set $\Gamma \setminus T_{-\varepsilon}^\dagger$, so

$$g' \in g \left(\bigcap_{r \in \Gamma \setminus T_{-\varepsilon}^\dagger} rHr^{-1} \right) g^{-1} = gK_{-\varepsilon}g^{-1}.$$

Hence $K(g) = gK_{-\varepsilon}g^{-1}$.

Now let $s = g_1\tau^{\varepsilon_1}g_2\tau^{\varepsilon_2} \cdots g_n\tau^{\varepsilon_{n-1}}g_n$ and notice that $sU(g_n\tau^{\varepsilon_n}) \cap T = U(g) \cap T$. Indeed, $m \in \Gamma$ has normal form beginning with $g_n\tau^{\varepsilon_n}$ if and only sm has normal form beginning with $g_1\tau^{\varepsilon_1}g_2\tau^{\varepsilon_2} \cdots g_n\tau^{\varepsilon_n}$. In particular, $sV(g_n\tau^{\varepsilon_n}) \cap T = V(g) \cap T$, so that $L(g) = sL(g_n\tau^{\varepsilon_n})s^{-1}$.

We next observe that $L(g_n\tau^{\varepsilon_n})$ is the subgroup of elements in Γ fixing all vertices of half-trees of the form $U(r\tau^{\varepsilon_n})$ for $r \in S_{-\varepsilon_n} \setminus \{g_n\}$ and $U(t\tau^{-\varepsilon_n})$ for $t \in S_{\varepsilon_n}$, as well as the vertex G. In particular, any $s \in L(g_n\tau^{\varepsilon_n})$ must fix all the edges emanating from and ending in the vertex G, i.e., $\{rH\}_{r \in S_{-1}}$ and $\{t\tau^{-1}H\}_{t \in S_1}$, so that $s \in \bigcap_{g \in G} g(H \cap \theta(H))g^{-1}$. Thus

$$L(g_n\tau^{\varepsilon_n}) = \bigcap_{r \in S_{-\varepsilon_n} \setminus \{g_n\}} r\tau^{\varepsilon_n}K_{-\varepsilon_n}\tau^{-\varepsilon_n}r^{-1} \cap \bigcap_{t \in S_{\varepsilon_n}} t\tau^{-\varepsilon_n}K_{\varepsilon_n}\tau^{\varepsilon_n}t^{-1} \cap \bigcap_{g \in G} g(H \cap \theta(H))g^{-1}$$

$$= \bigcap_{\substack{r \in \Gamma \setminus g_n T_{\varepsilon_n}^\dagger \\ |r| \geq 1}} rHr^{-1} \cap \bigcap_{g \in G} g(H \cap \theta(H))g^{-1}$$

$$= \bigcap_{\substack{r \in \Gamma \setminus T_{\varepsilon_n}^\dagger \\ |r| \geq 1}} g_nrHr^{-1}g_n^{-1} \cap \bigcap_{g \in G} g(H \cap \theta(H))g^{-1} \subseteq g_nK_{\varepsilon_n}g_n^{-1}.$$

Conversely, assume that $(g_nr)^{-1}s(g_nr) \in H$ for all $r \in \Gamma \setminus T_{\varepsilon_n}^\dagger$. Then for any $g' \in \Gamma$ such that $g'G$ belongs to one of the half-trees $U(r\tau^{\varepsilon_n})$ for $r \in S_{-\varepsilon_n} \setminus \{g_n\}$ and $U(t\tau^{-\varepsilon_n})$ for $t \in S_{\varepsilon_n}$, we have $g_n^{-1}g' \notin \Gamma \setminus T_{\varepsilon_n}^\dagger$, so that $sg'G = g'G$, and clearly $s \in G$. Hence $s \in L(g_n\tau^{\varepsilon_n})$. We conclude that

$$L(g) = sL(g_n\tau^{\varepsilon_n})s^{-1} = sg_nK_{\varepsilon_n}g_n^{-1}s^{-1} = gg_{n+1}^{-1}\tau^{-\varepsilon_n}K_{\varepsilon_n}\tau^{\varepsilon_n}g_{n+1}g^{-1},$$

which completes the proof. $\qquad\square$

In the Bass-Serre tree of a non-ascending HNN extension $\Gamma = \mathrm{HNN}(G, H, \theta)$, remove the edge H connecting G and $\tau^{-1}G$. By the above lemma, the fixator subgroups of the resulting half-trees are $K(\tau^{-1}) = \tau^{-1}K_1\tau = \theta(K_1)$ and $L(\tau^{-1}) = K_{-1}$. Similarly, if we remove the edge τH connecting G and τG, then the elements that fix the two resulting half-trees are $\theta^{-1}(K_{-1})$ and K_1. By Lemma 4.8, the normal closures in Γ of K_{-1}, $\theta(K_1)$, $\theta^{-1}(K_{-1})$, and K_1 all coincide.

Proposition 4.18. *Let Γ be a non-ascending HNN extension with Bass-Serre tree T, and consider $T \cup \partial T$ equipped with the shadow topology. The action of Γ on T satisfies*

$$\ker \Gamma = \ker(\Gamma \curvearrowright T) = \ker(\Gamma \curvearrowright \partial T) = \ker(\Gamma \curvearrowright \overline{\partial T})$$

and

$$\mathrm{int}\,\Gamma = \mathrm{int}(\Gamma \curvearrowright \partial T) = \mathrm{int}(\Gamma \curvearrowright \overline{\partial T}).$$

Proof. By Lemma 4.17, we have $\ker \Gamma = \ker(\Gamma \curvearrowright T)$. The above paragraph says that both quasi-kernels K_{-1} and K_1 are fixator subgroups of half-trees in T, so by Corollary 3.7, their normal closures, or the interior $\mathrm{int}\,\Gamma$, coincides with the interior of the action of Γ on ∂T. The remaining identities are explained after the proof of Lemma 3.6. $\qquad\square$

Theorem 4.19. *Let Γ be a non-ascending HNN extension with Bass-Serre tree T. Then Γ has the unique trace property if and only if $\ker \Gamma$ has the unique trace property, and Γ is C^*-simple if and only if $\mathrm{int}\,\Gamma$ is C^*-simple.*

Proof. This follows from Proposition 4.18 and Proposition 2.2. □

Theorem 4.20. *Let Γ be a non-ascending HNN extension, with quasi-kernels K_{-1} and K_1.*
If $\ker \Gamma$ is trivial, then Γ is C^*-simple if and only if K_{-1} and K_1 are trivial or non-amenable.
If K_{-1} and K_1 are amenable, then Γ is C^*-simple if and only if K_{-1} and K_1 are trivial.

Proof. This follows from Proposition 4.18 and Theorem 3.9, since both K_{-1} and K_1 are fixator subgroups for half-trees in the Bass-Serre tree T. □

Example 4.21. Let $G = \mathbb{Z}$ and let $g \in G$ be a generator of G. For $m, n \in \mathbb{Z} \setminus \{0\}$, define $H = \langle g^m \rangle$ (thus corresponding to $m\mathbb{Z}$) and an injective homomorphism $\theta\colon H \to G$ by $\theta(g^{km}) = g^{kn}$ for $k \in \mathbb{Z}$. Then the HNN extension $\mathrm{HNN}(G, H, \theta) = \mathrm{HNN}(\mathbb{Z}, m\mathbb{Z}, km \mapsto kn)$ is the *Baumslag-Solitar group*

$$\Gamma = \mathrm{BS}(m, n) = \langle g, \tau \mid \tau^{-1} g^m \tau = g^n \rangle.$$

Clearly, Γ is non-ascending if and only if $\min\{|m|, |n|\} \geq 2$. A result of the second author [20, Theorem 4.9] states that $\mathrm{BS}(m, n)$ is C^*-simple if and only if $\min\{|m|, |n|\} \geq 2$ and $|m| \neq |n|$.

We give a new proof using the C^*-simplicity criterion for HNN extensions given above. Notice first that if $|m| = |n|$, then $H = \langle g^m \rangle$ is a normal abelian subgroup of Γ. Furthermore, $\mathrm{BS}(\pm 1, n)$ and $\mathrm{BS}(m, \pm 1)$ are solvable. Indeed, in the case $m = 1$, $N = \langle \{\tau^k g \tau^{-k} \mid k \in \mathbb{Z}\} \rangle$ is a normal abelian subgroup of Γ, with Γ/N infinite and cyclic.

Next assume that $\min\{|m|, |n|\} \geq 2$ and $|m| \neq |n|$ and let d be the greatest common divisor of m and n, so that we may write $m = dm'$ and $n = dn'$ for $m', n' \in \mathbb{Z}$. We may assume that $|n| > |m|$ so that $|n'| > 1$. Clearly $\tau^{-1} H \tau = \langle g^n \rangle = \langle g^{dn'} \rangle$. For $k \in \mathbb{Z}$ and $i \geq 1$, write $kd(n')^i = qm + r$ for $q \in \mathbb{Z}$ and $0 \leq r < m$. If

$$G \ni \tau^{-1} g^{kd(n')^i} \tau = g^{qn} \tau^{-1} g^r \tau,$$

then $r = 0$, so $m \mid kd(n')^i$ and $m \mid kd$. Hence $m(n')^i \mid kd(n')^i = qm$, meaning that $(n')^i$ divides q and

$$\tau^{-1} g^{kn(n')} \tau \in \langle g^{n(n')} \rangle = \langle g^{d(n')^2} \rangle, \tau^{-i} g^{kn} \tau^i \in \langle g^{n(n')^i} \rangle = \langle g^{d(n')^{i+1}} \rangle.$$

This shows that $\tau^{-i} H \tau^i \cap G \subseteq \langle g^{d(n')^{i+1}} \rangle$ for $i \geq 1$, meaning that $K_1 = \{1\}$. Therefore Γ is C^*-simple by Theorem 4.10 or Theorem 4.20.

5. A NON-C^*-SIMPLE HNN EXTENSION WITH THE UNIQUE TRACE PROPERTY

In the following, let $A = \bigcup_{n \in \mathbb{N}} \{0, 1\}^{2n}$ and define $X \subseteq A$ by

$$X = \{(i_1, j_1, \ldots, i_n, j_n) \mid n \in \mathbb{N}, \ (j_k, i_{k+1}, j_{k+1}) \notin \{(0, 0, 1), (1, 0, 0)\} \text{ for all } 1 \leq k \leq n - 1\}.$$

We say that an element $x = (i_1, j_1, \ldots, i_n, j_n) \in X$ has *length* $\ell(x) = n$. For $1 \leq k \leq n$, we define $\pi_k\colon X \to \{0, 1\}^2$ by

$$\pi_k(i_1, j_1, \ldots, i_n, j_n) = (i_k, j_k).$$

We let $\underline{H}$ be the group with generators $\{h(x) \mid x \in X\}$ satisfying the following relations:

(R1) $h(x)^2 = 1$ for all $x \in X$;
(R2) $h(x)h(y) = h(y)h(x)$ whenever $\ell(x) = \ell(y)$;
(R3) for all $2 \leq k + 1 \leq n$,

$$h(i_1', j_1', \ldots, i_k', j_k')h(i_1, j_1, \ldots, i_n, j_n)h(i_1', j_1', \ldots, i_k', j_k') =$$

$$\begin{cases} h(i_1, j_1, \ldots, i_k, j_k, i_{k+1} \oplus 1, j_{k+1}, \ldots, i_n, j_n), & \text{if } i_1' = i_1, j_1' = j_1, \ldots, i_k' = i_k, j_k' = j_k = j_{k+1}, \\ h(i_1, j_1, \ldots, i_k, j_k, i_{k+1}, j_{k+1}, \ldots, i_n, j_n), & \text{otherwise,} \end{cases}$$

where $\oplus$ denotes addition modulo 2. We now let G be the group containing $\underline{H}$ as a subgroup as well as elements g_0, g_1 such that $\langle \underline{H}, g_0, g_1 \rangle = G$,

$$g_0^2 = 1 = g_1^2, \quad g_0 g_1 = g_1 g_0,$$

and such that for all $n \in \mathbb{N}_0$,

- $g_j h(i, j \oplus 1, i_1, j_1, \ldots, i_n, j_n) = h(i, j \oplus 1, i_1, j_1, \ldots, i_n, j_n) g_j$ and
- $g_j h(i, j, i_1, j_1, \ldots, i_n, j_n) = h(i \oplus 1, j, i_1, j_1, \ldots, i_n, j_n) g_j$.

For all $n \in \mathbb{N}$ and $x = (i_1, j_1, \ldots, i_n, j_n) \in X$, let us write

$$H(i_1, j_1, \ldots, i_n, j_n) = \{h(x') \mid x' \in X, \ \ell(x') \geq \ell(x), \ \pi_k(x') = \pi_k(x) \text{ for all } k = 1, \ldots, n\}.$$

For $i \in \{0,1\}$, we define $H_i = \langle \underline{H}, g_i \rangle$ and a map $\theta_i \colon \{h(x) \mid x \in X\} \cup \{g_i\} \to G$ by

$$\theta_i(g_i) = h(0,i), \quad \theta_i(h(0, i \oplus 1)) = g_{i \oplus 1},$$

and for all $n \in \mathbb{N}_0$,

- $\theta_i(h(i_1, j_1, \ldots, i_n, j_n)) = h(0, i, i_1, j_1, \ldots, i_n, j_n)$ whenever $(i_1, j_1) \neq (0, i \oplus 1)$;
- $\theta_i(h(0, i \oplus 1, i_1, j_1, \ldots, i_n, j_n)) = h(i_1, j_1, \ldots, i_n, j_n)$ whenever $n \geq 1$.

Lemma 5.1. *The map θ_i extends to a group homomorphism $H_i \to G$ with image $H_{i \oplus 1}$.*

Proof. Let $x, y \in X$. If $\pi_1(x) \neq \pi_1(y)$, then (R2) and (R3) together imply that $h(x)$ and $h(y)$ commute. If $\pi_1(x), \pi_1(y) \notin \{(0, i \oplus 1)\}$, then (R2) and (R3) together imply that $\theta_i(h(x))$ and $\theta_i(h(y))$ commute. We may therefore assume $\pi_1(y) = (0, i \oplus 1)$. If $\ell(y) \geq 2$, then $\pi_2(y) \neq (0, i)$ so that $\theta_i(h(x))$ and $\theta_i(h(y))$ commute by (R2) and (R3) again, and if $\ell(y) = 1$, then $\theta_i(h(x)) \in H(0, i)$ and $\theta(h(y)) = g_{i \oplus 1}$ commute.

We may therefore assume from now on that $\pi_1(x) = \pi_1(y)$. If $\ell(x) = \ell(y)$, then by (R2) we have $h(x)h(y) = h(y)h(x)$ and $\theta_i(h(x))\theta_i(h(y)) = \theta_i(h(y))\theta_i(h(x))$, so we may assume that $\ell(x) > \ell(y)$ without loss of generality. Write $x = (i_1, j_1, \ldots, i_n, j_n)$, so that $\ell(x) = n > k = \ell(y)$. Let $m \leq k$ be largest such that $\pi_r(x) = \pi_r(y)$ for all $1 \leq r \leq m$. If $m < k$, then $h(x)$ and $h(y)$ commute, and so do $\theta_i(h(x))$ and $\theta_i(h(y))$. Therefore we may assume $m = \ell(y)$. If $j_{k+1} \neq j_k$, then $h(x)$ and $h(y)$ commute, as do $\theta_i(h(x))$ and $\theta_i(h(y))$ once more. We may therefore also assume $j_{k+1} = j_k$ as well, as well as $\pi_1(y) = \pi_1(x) = (0, i \oplus 1)$ (the case $\pi_1(y) \neq (0, i \oplus 1)$ is completely analogous). Then $j_2 = i \oplus 1$. There are now two cases:

- If $k = 1$, then

$$\begin{aligned} \theta_i(h(y))\theta_i(h(x))\theta_i(h(y)) &= g_{i \oplus 1} h(i_2, j_2, \ldots, i_n, j_n) g_{i \oplus 1} \\ &= h(i_2 \oplus 1, j_2, \ldots, i_n, j_n) \\ &= \theta_i(h(x)h(y)h(x)). \end{aligned}$$

- If $k \geq 2$, then

$$\begin{aligned} \theta_i(h(y))\theta_i(h(x))\theta_i(h(y)) &= h(i_2, j_2, \ldots, i_k, j_k) h(i_2, j_2, \ldots, i_n, j_n) h(i_2, j_2, \ldots, i_k, j_k) \\ &= h(i_2 \oplus 1, j_2, \ldots, i_n, j_n) = \theta_i(h(x)h(y)h(x)). \end{aligned}$$

Finally, if $j_1 = i \oplus 1$, then g_i commutes with $h(x)$, and $\theta_i(g_i) = h(0, i)$ commutes with $\theta_i(h(x))$. If $j_1 = i$, then

$$\begin{aligned} \theta_i(g_i)\theta_i(h(x))\theta_i(g_i) &= h(0, i) h(0, i, i_1, j_1, \ldots, i_n, j_n) h(0, i) \\ &= h(0, i, i_1 \oplus 1, j_1, \ldots, i_n, j_n) \\ &= \theta_i(h(i_1 \oplus 1, j_1, \ldots, i_n, j_n)) \\ &= \theta_i(g_i h(i_1, j_1, \ldots, i_n, j_n) g_i). \end{aligned}$$

It follows that there is a homomorphism $\theta_i \colon H_i \to G$ with the desired requirements, and $\theta_i(H_i) = \langle \underline{H}, g_{i \oplus 1} \rangle$. $\qquad\square$

Letting $H = H_0$ and $\theta = \theta_0 \colon H \to G$ be the group homomorphism of the above lemma (for $i = 0$), then $\theta_1 \circ \theta = \mathrm{id}_H$. In particular, θ is injective.

The following are some easy properties of the group G.

Lemma 5.2. (i) $[G : H] = [G : \theta(H)] = 2$ and therefore $H \lhd G$ and $\theta(H) \lhd G$;
(ii) $\underline{H} = \langle H(0,0) \rangle \cdot \langle H(1,0) \rangle \cdot \langle H(0,1) \rangle \cdot \langle H(1,1) \rangle$;
(iii) For each $x \in \{0,1\}^2$ there exists a homomorphism $\underline{H} \to \langle H(x) \rangle$ which is the identity map on $\langle H(x) \rangle$ and maps each element of $H(y)$ to 1 for all $y \in \{0,1\}^2 \setminus \{x\}$. In particular, $\langle H(x) \rangle \cap \langle H(y) \rangle = \{1\}$ for distinct $x, y \in \{0,1\}^2$.

Proof. (i) is evident and (ii) follows from the commutation relations. To see that (iii) holds, notice for $x \in \{0,1\}^2$ that the universal property of $\underline{H}$ yields a surjective homomorphism φ_x onto the group $H'(x)$ generated by $\{h(y) \mid y \in X \ \pi_1(y) = x\}$ with the same relations as $\underline{H}$; moreover, there is a natural homomorphism $\iota_x \colon H'(x) \to \underline{H}$ such that $\iota_x(h(y)) = h(y)$ for all $y \in X$ with $\pi_1(y) = x$. As $\varphi_x \circ \iota_x$ is the identity map on $H'(x)$, (iii) follows. $\square$

We now consider the HNN extension $\Gamma = \mathrm{HNN}(G, H, \theta) = \langle G, \tau \mid \tau^{-1} h \tau = \theta(h) \rangle$. We choose the coset representatives $\{1, g_1\}$ for H and $\{1, g_0\}$ for $\theta(H)$. Therefore, any $g \in \Gamma$ is of the form

$$g = r_1 \tau^{\varepsilon_1} \cdots r_n \tau^{\varepsilon_n} r_{n+1},$$

where $r_i \in \{1, g_1\}$ if $\varepsilon_i = 1$ and $r_i \in \{1, g_0\}$ if $\varepsilon_i = -1$, for all $1 \leq i \leq n$.

Let K_{-1} and K_1 be the quasi-kernels of the HNN extension Γ.

Lemma 5.3. The quasi-kernels of Γ satisfy $\langle H(0,0) \rangle \subseteq K_{-1}$ and $\langle H(0,1) \rangle \subseteq K_1$.

Proof. We first define $M_{-1} = H(0,1)$ and $M_1 = H(0,0)$. Let $i \in \{0,1\}$ and $s \in H(0,i)$. Let $\varepsilon = -1$ if $i = 0$ and $\varepsilon = 1$ if $i = 1$. We claim first that $g^{-1} s g \in M_{\varepsilon'}$ for all $g \in \Gamma \setminus T_\varepsilon^\dagger$ of length ≥ 1, end letter 1 and direction ε'.

Assume that $|g| = 1$. We separate the cases $i = 0$ and $i = 1$:

- $i = 0$. For $\varepsilon_1 = -1$ we have $r_1 = g_0$, in which case $r_1 s r_1 \in g_0 H(0,0) g_0 = H(1,0)$ and $g^{-1} s g \in \theta^{-1}(H(1,0)) \subseteq H(0,1)$. If $\varepsilon_1 = 1$, then $s r_1 = r_1 s$ and

$$g^{-1} s g = \theta(s) \in H(0,0).$$

- $i = 1$. For $\varepsilon_1 = -1$ we have $s r_1 = r_1 s$ and $g^{-1} s g = \theta^{-1}(s) \in H(0,1)$, and for $\varepsilon_1 = 1$ we have $r_1 = g_1$ so that $r_1 s r_1 \in g_1 H(0,1) g_1 = H(1,1)$ and

$$g^{-1} s g \in \theta(H(1,1)) \subseteq H(0,0).$$

Next assume that we have proved the claim for all $g \in \Gamma \setminus T_{-1}^\dagger$ with $|g| = n - 1$, $n \geq 2$. Let $g \in \Gamma \setminus T_{-1}^\dagger$ with $|g| = n$ and write

$$g^{-1} s g = (\tau^{-\varepsilon_n} r_n \cdots \tau^{-\varepsilon_1} r_1) s (r_1 \tau^{\varepsilon_1} \cdots r_n \tau^{\varepsilon_n}).$$

Letting ε' be the direction of $g' = r_1 \tau^{\varepsilon_1} \cdots r_{n-1} \tau^{\varepsilon_{n-1}}$, then $s' = (g')^{-1} s g' \in M_{\varepsilon'}$.

If $\varepsilon_n = -1$, then for $r_n = 1$ we have $\varepsilon_n = \varepsilon'$, so that $s' \in M_{-1} = H(0,1)$. Hence $g^{-1} s g = \tau s' \tau^{-1} = \theta^{-1}(s') \in H(0,1)$. If $r_n = g_0$, then for $\varepsilon = 1$ we have $r_n s' r_n \in g_0 H(0,0) g_0 = H(1,0)$ and $g^{-1} s g = \theta^{-1}(H(1,0)) \in H(0,1)$, and for $\varepsilon' = -1$ we have $r_n s' r_n \in g_0 H(0,1) g_0 \subseteq H(0,1)$ and $g^{-1} s g \in H(0,1)$.

If $\varepsilon_n = 1$, the procedure is exactly the same: if $r_n = 1$, then $s' \in M_1 = H(0,0)$. Thus $g^{-1} s g = \tau^{-1} s' \tau = \theta(s') \in H(0,0)$. If $r_n = g_1$, then $\varepsilon' = 1$ implies $r_n s' r_n \in g_1 H(0,0) g_1 = H(0,0)$ and $g^{-1} s g \in \theta(H(0,0)) = H(0,0)$, and $\varepsilon' = -1$ implies $r_n s' r_n \in g_1 H(0,1) g_1 = H(1,1)$ and $g^{-1} s g \in \theta(H(1,1)) \subseteq H(0,0)$. This finishes the proof of the claim.

For any $g \in \Gamma \setminus T_\varepsilon^\dagger$ of length ≥ 1, then there exists $t \in G$ such that $gt \in \Gamma$ has end letter 1. Thus $g^{-1} s g \in t(H(0,0) \cup H(0,1)) t^{-1} \subseteq t H t^{-1} = H$, since H is normal in G. Therefore $H(0,0) \subseteq K_{-1}$ and $H(0,1) \subseteq K_1$. $\square$

To prove the reverse inclusions, we need two preparatory lemmas that describe how we may "reduce" a given element in $\{h(x) \mid x \in X\}$ by means of conjugation by elements of Γ.

Lemma 5.4. *Let $x \in X$ and $n = \ell(x)$. Define*

$$r_j(x) = \begin{cases} \tau^{-1} & \text{when } \pi_j(x) = (0,0), \\ g_0\tau^{-1} & \text{when } \pi_j(x) = (1,0), \\ \tau & \text{when } \pi_j(x) = (0,1), \\ g_1\tau & \text{when } \pi_j(x) = (1,1), \end{cases}$$

and let $r(x) = r_1(x) \cdots r_n(x)$. Furthermore, let $i \colon X \to \{0,1\}$ be given by

$$i(x) = \begin{cases} 0 & \text{if } \pi_n(x) \in \{(0,0),(1,0)\}, \\ 1 & \text{if } \pi_n(x) \in \{(0,1),(1,1)\}. \end{cases}$$

Then $r(x) \in \Gamma$ is the unique element in Γ with end letter 1, such that $|r(x)| = n$ and $r(x)^{-1}h(x)r(x) \in \{g_0, g_1\}$, in which case $r(x)^{-1}h(x)r(x) = g_{i(x)}$. The resulting map $r \colon X \to \Gamma$ is injective.

Proof. It is easy to check that $r(x)$ and $i(x)$ satisfy the desired requirements. Notice also that for all $x \in X$, the inner automorphisms $\mathrm{Ad}(\tau), \mathrm{Ad}(g_1\tau), \mathrm{Ad}(\tau^{-1}), \mathrm{Ad}(g_0\tau^{-1})$ map an element of the form $h(y)$, $\ell(y) \geq 2$, to an element of the form $h(y')$ for which $\ell(y') \in \{\ell(y) \pm 1\}$.

For any $x \in X$, $r(x) = r_1(x) \cdots r_n(x)$ is a reduced word in Γ. Indeed, for all $1 \leq k \leq n-1$ then $\pi_k(x) = (0,1)$ or $\pi_k = (1,1)$ implies $\pi_{k+1}(x) \neq (0,0)$, meaning $r_k(x) \in \{\tau, g_1\tau\}$ implies $r_{k+1}(x) \in \{g_0\tau^{-1}, \tau, g_1\tau\}$. Similarly, one sees that $r_k(x) \in \{\tau^{-1}, g_0\tau^{-1}\}$ implies $r_{k+1}(x) \in \{\tau^{-1}, g_0\tau^{-1}, g_1\tau\}$. Hence injectivity of r will follow from Britton's lemma, once we show that $r(x)$ is uniquely determined for all $x \in X$.

For uniqueness, let us first assume that $n = 1$. For $(i,j) \neq (0,0)$ assume that $t = t_1\tau^{\varepsilon_1}$ satisfies $\tau^{-\varepsilon_1}t_1^{-1}h(i,j)t_1\tau^{\varepsilon_1} = g_i$. As $t_1^{-1}h(i,j)t_1 = h(x')$ for some $x' \in X$ of length 1, then $\tau^{\varepsilon_1}g_i\tau^{-\varepsilon_1} = h(x')$. Hence $i = 0$ implies $\varepsilon_1 = -1$, in which case the equation $h(i,j) = t_1h(0,0)t_1^{-1}$ determines t_1; if $i = 1$, then $\varepsilon_1 = 1$ and $h(i,j) = t_1h(0,1)t_1^{-1}$ also determines t_1. Hence $t = r_1(x)$.

For $n \geq 2$, write $x = (i_1, j_1, \ldots, i_n, j_n)$ and assume that uniqueness holds for $x' = (i_2, j_2, \ldots, i_n, j_n)$. If $t = t_1\tau^{\varepsilon_1} \cdots t_n\tau^{\varepsilon_n}$ satisfies $t^{-1}h(x)t = g_i$ for some i, then

$$(t_1\tau^{\varepsilon_1})^{-1}h(x)(t_1\tau^{\varepsilon_1}) = h(x').$$

Now $g_i = t^{-1}h(x)t = (t_2\tau^{\varepsilon_2} \cdots t_n\tau^{\varepsilon_n})^{-1}h(x')(t_2\tau^{\varepsilon_2} \cdots t_n\tau^{\varepsilon_n})$, so that uniqueness yields $r(x') = t_2\tau^{\varepsilon_2} \cdots t_n\tau^{\varepsilon_n}$ and $t = t_1\tau^{\varepsilon_1}r(x')$. Now $(t_1\tau^{\varepsilon_1})^{-1}h(x)(t_1\tau^{\varepsilon_1}) = h(i_2, j_2, \ldots, i_n, j_n)$ determines t_1 and ε_1, completing the proof. $\square$

Lemma 5.5. *Let $x, y \in X$ such that either of the following conditions hold:*

    (i) $\ell(x) < \ell(y)$;
    (ii) $\ell(y) \leq \ell(x)$ and $\pi_i(x) = \pi_i(y)$ for all $1 \leq i \leq k$ implies $k < \ell(y)$.

Then for the element $r(x) \in \Gamma$ defined in Lemma 5.4, we have $r(y)^{-1}h(x)r(y) = h(x')$ for some $x' \in X$. In fact, in either of the above cases, the sequence x' satisfies

$$\ell(x') = \ell(x) + \ell(y) - 2\max\{k \mid \pi_i(x) = \pi_i(y) \text{ for all } i \leq k\}.$$

Proof. Assume first that we have proved that the equation holds for $x, y \in X$ such that $\pi_1(y) \neq \pi_1(x)$. If $\pi_1(y) = \pi_1(x)$, let $1 \leq k \leq \min\{\ell(x), \ell(y)\}$ be largest such that $\pi_i(y) = \pi_i(x)$ for all $i \leq k$. Remember that we deliberately choose $r_i(x)$ for $1 \leq i \leq \ell(x)$ such that applying the inner automorphisms $\mathrm{Ad}(r_1(x)^{-1}), \mathrm{Ad}(r_2(x)^{-1}), \ldots, \mathrm{Ad}(r_{\ell(x)}(x)^{-1})$ in that order removes the first tuple, then the second tuple, until we have removed all tuples from x. Hence the inner automorphism $\mathrm{Ad}(r_1(x) \cdots r_k(x))^{-1}$ will remove the first k tuples from both $h(x)$ and $h(y)$.

If $\ell(y) \leq \ell(x)$, then $k < \ell(y)$. Let x_0 and y_0 be x and y respectively, but without their first k tuples, and observe that $r(x_0)^{-1}h(y_0)r(x_0) = h(y')$ for some $y' \in X$ such that $\ell(y') = \ell(y_0) + \ell(x_0) = \ell(y) + \ell(x) - 2k$. We then have

$$
\begin{aligned}
r(x_0)^{-1}h(y_0)r(x_0) &= r(x_0)^{-1}\varphi(h(y))r(x_0) \\
&= r(x_0)^{-1}(r_1(x) \cdot r_k(x))^{-1}h(y)r_1(x) \cdot r_k(x)r(x_0) \\
&= r(x)^{-1}h(y)r(x).
\end{aligned}
$$

If $\ell(y) > \ell(x)$ and $k < \ell(x)$, then the same argument applies; finally, for $k = \ell(x)$, then the first k tuples of y constitute x, so that the $y' \in X$ such that $r(x)^{-1}h(y)r(x)^{-1}$ has length $\ell(y) - \ell(x) = \ell(x) + \ell(y) - 2k$.

Now assume that $\pi_1(y) \neq \pi_1(x)$. Write $x = (i_1, j_1, \ldots, i_n, j_n)$ and define $r_0(x) = 1$. Define $\varphi_i = \mathrm{Ad}(r_i(x)^{-1})$ for all $0 \leq i \leq n$ and $\Phi_i = \varphi_i \circ \cdots \circ \varphi_0$, so that $\Phi_n(h(x)) = r(x)^{-1}h(x)r(x)$. We claim for all $0 \leq i \leq n$ that the first tuple of $\Phi_i(h(y))$ never agrees with the first tuple of $\Phi_i(h(x))$. This will prove that $h(y)$ gets one tuple longer for each i, while $h(x)$ will eventually collapse to either g_0 or g_1.

Let $1 \leq i \leq n$. The first line in one of the 12 cells in the table below is one of the possibilities of what the first tuple of $\Phi_{i-1}(h(x))$ could be, the arrow pointing to what the first tuple of $\Phi_i(h(x))$ may be. The next line considers $\Phi_{i-1}(h(y))$ with distinct first tuple from $\Phi_{i-1}(h(x))$, the arrow pointing to what the first tuple of $\Phi_i(h(y))$ will then be (which depends on the first tuple of $\Phi_{i-1}(h(x))$). Notice that the pair of two digits after the arrow on the lower line inside each cell never coincides with any pair above it.

10	$\rightarrow$	00,10,11	01	$\rightarrow$	01,10,11	11	$\rightarrow$	01,10,11
00	$\rightarrow$	01	00	$\rightarrow$	00	00	$\rightarrow$	00
00	$\rightarrow$	00,10,11	10	$\rightarrow$	00,10,11	11	$\rightarrow$	01,10,11
01	$\rightarrow$	01	01	$\rightarrow$	01	01	$\rightarrow$	00
00	$\rightarrow$	00,10,11	01	$\rightarrow$	01,10,11	11	$\rightarrow$	01,10,11
10	$\rightarrow$	01	10	$\rightarrow$	00	10	$\rightarrow$	00
00	$\rightarrow$	00,10,11	01	$\rightarrow$	01,10,11	10	$\rightarrow$	00,10,11
11	$\rightarrow$	01	11	$\rightarrow$	00	11	$\rightarrow$	01

This completes the proof. $\square$

Lemma 5.6. *The quasi-kernels of Γ are given by $K_{-1} = \langle H(0,0) \rangle$ and $K_1 = \langle H(0,1) \rangle$.*

Proof. We established the inclusions "$\supseteq$" in Lemma 5.3. Showing that $K_{-1} \subseteq \langle H(0,0) \rangle$ is equivalent to showing that for all $g \notin \langle H(0,0) \rangle$, there exists $r \in \Gamma \setminus T_{-1}^{\dagger}$ such that $r^{-1}gr \notin H$.

Of course, if $g \notin H$, just take $r = e$. If $g = hg_0$ for $h \in \underline{H}$, then take $r = g_0\tau^{-1}$, so that $\tau g_0 g g_0 \tau^{-1} = \tau g_0 \tau^{-1}\theta^{-1}(h) \notin H$. Since $H = \underline{H} \cup g_0\underline{H}$, we may assume from this point on that $g \in \underline{H}$. Due to Lemma 5.2, we may write

$$
g = \prod_{x \in \{0,1\}^2} g_x,
$$

where g_x is a product of the form $\prod_{k=1}^{n_x} h(\omega_{x,k})$ where $\pi_1(\omega_{x,k}) = x$. We can assume that n_x is the smallest positive integer such that g_x is a product of n_x elements from $H(x)$.

Let $x \in \{0,1\}^2$. By means of the commutation relations in $\underline{H}$, we may assume that $\ell(\omega_{x,k}) \leq \ell(\omega_{x,k+1})$ for all $1 \leq k < n_x$. If $n_x \geq 2$, assume that $\omega_{x,k} = \omega_{x,m}$ for some $1 \leq k < m \leq n_x$. Then $\ell(\omega_{x,k}) = \ell(\omega_{x,k'})$ for all $k \leq k' \leq m$, so that

$$
h(\omega_{x,k})h(\omega_{x,k+1})\cdots h(\omega_{x,m}) = h(\omega_{x,k})^2 h(\omega_{x,k+1})\cdots h(\omega_{x,m-1}),
$$

contradicting the assumption that n_x is smallest possible. Hence $\omega_{x,k} \neq \omega_{x,m}$, so that $h(\omega_{x,k}) \neq h(\omega_{x,m})$ for all distinct $k, m \in \{1, \ldots, n_x\}$.

Since $h(\omega_{x,k})$ and $h(\omega_{y,k'})$ commute for distinct $x, y \in \{0,1\}^2$ and k, k', we can reorder the factors and write

$$g = h(x_1)h(x_2)\cdots h(x_n),$$

where n is smallest possible, $\ell(x_i) \leq \ell(x_{i+1})$ for all $1 \leq i \leq n-1$ and $h(x_i) \neq h(x_j)$ for all $i \neq j$ by what we saw above. Since $g \notin \langle H(0,0) \rangle$, there is a smallest $1 \leq i \leq n$ such that $\pi_1(x_i) \neq (0,0)$. Then the element $r(x_i) \in \Gamma$ of Lemma 5.4 satisfies $r(x_i)^{-1}h(x_i)r(x_i) = g_m$ for some $m \in \{0,1\}$, and $r(x_i) \in \Gamma \setminus T_{-1}^\dagger$. Moreover, $r(x_i)^{-1}h(x_j)r(x_i) \in \underline{H}$ for all $j \neq i$ by Lemma 5.5, since the existence of $1 \leq j < i$ such that $\pi_k(x_j) = \pi_k(x_i)$ for all $1 \leq k \leq \ell(\omega_j)$ would contradict the minimality of i. Hence

$$r(x_i)^{-1}gr(x_i) \in \underline{H}g_m\underline{H} = g_m\underline{H}.$$

If $m = 0$, let $r = r(x_i)g_0\tau^{-1}$, and if $m = 1$, let $r = r(x_i)g_1\tau$.

The proof that $K_1 \subseteq \langle H(0,1) \rangle$ is completely analoguous. $\qquad\square$

Lemma 5.7. *The group G is locally finite and hence amenable.*

Proof. For any non-negative integer $n \geq 0$, consider the subgroup $\underline{H}_n$ of $\underline{H}$ generated by the union of the sets $H_{x,n} = \{H(y) \mid y \in X, \ \pi_1(y) = x, \ \ell(y) \leq n\}$, $x \in \{0,1\}^2$. As in the proof of Lemma 5.6, any element in the latter subgroup can be decomposed into factors g_x for $x \in \{0,1\}^2$. By assuming the number of factors in each g_x to be the smallest possible, we can write $g_x = \prod_{s \in H_{x,n}} s^{\varepsilon_s}$ for numbers $\varepsilon_s \in \{0,1\}$, $s \in H_{x,n}$. Therefore there are only finitely many elements in $\langle H_{x,n} \rangle$, and since the sets $H_{x,n}$ and $H_{y,n}$ commute in $\underline{H}$ for $x \neq y$, it follows that $\underline{H}_n$ is finite.

Every finitely generated subgroup of G is contained in the subgroup G_n generated by g_0, g_1 and $\underline{H}_n$ for some $n \geq 0$. As $\underline{H}_n$ is invariant under conjugation by g_0 and g_1, any element in G_n has a unique decomposition $g = g_0^{\varepsilon_0}g_1^{\varepsilon_1}h$ for $\varepsilon_0, \varepsilon_1 \in \{0,1\}$ and $h \in \underline{H}_n$, and so G_n is also finite. $\qquad\square$

Theorem 5.8. *The HNN extension Γ has the unique trace property, but is not C^*-simple.*

Proof. Since $K_{-1} = \langle H(0,0) \rangle$ and $K_1 = \langle H(0,1) \rangle$ by Lemma 5.6, we see that

$$\ker \Gamma = \langle H(0,0) \rangle \cap \langle H(0,1) \rangle = \{1\}$$

by Lemma 5.2 (iii), meaning that Γ has the unique trace property by Theorem 4.19. However, K_{-1} and K_1 are both amenable by Lemma 5.7 and non-trivial, so Γ is not C^*-simple by Theorem 4.20. $\qquad\square$

Remark 5.9. The interior of Γ coincides with the normal closure of G in Γ. Indeed, $H(1,0) = \tau H(0,0,1,0)\tau^{-1}$, $H(1,1) = \tau H(0,0,1,1)\tau^{-1}$, $g_0 = \tau h(0,0)\tau^{-1}$, and $g_1 = \tau^{-1}h(0,1)\tau$ all belong to the normal closure of $\langle H(0,0), H(0,1) \rangle$. Thus $G \subseteq \text{int}\,\Gamma$, so the normal closure of G belongs to $\text{int}\,\Gamma$. Conversely, $H(0,0), H(0,1) \subseteq G$, and the containment clearly still holds when passing to normal closures. Moreover, consider the map $\Gamma \to \mathbb{Z}$ defined by $g \mapsto 0$ for $g \in G$ and $\tau \mapsto 1$. The normal closure of G coincides with the kernel of this map, and it follows that Γ is isomorphic to a semidirect product $(\text{int}\,\Gamma) \rtimes \mathbb{Z}$.

Remark 5.10. The tree of Γ is the regular tree of branching degree 4. Let T_0 denote the subtree consisting of all vertices gG, where $g \in \Gamma \setminus G$ starts with either $g_0\tau^{-1}$ or τ^{-1} on reduced form (that is, of type -1), and let T_1 denote the subtree consisting of all vertices gG, where $g \in \Gamma \setminus G$ starts with either $g_1\tau$ or τ on reduced form (that is, of type 1). Then we can find a coloring of the edges of T in such a way that the local permutation of g_0 is (12) on T_0 and id on T_1, the local permutation of g_1 is id on T_0 and (34) on T_1, and the local permutation of τ is (23) everywhere (cf. [26]). Therefore Γ is not one of the examples of [26, Theorem C].

Proposition 5.11. *The group Γ is amenablish.*

Proof. Let Γ^+ be the subgroup of Γ generated by the stabilizers of the edges of T, and let T_0 and T_1 be as in the previous remark. It is easy to see that the fixator subgroup of T_0 is $\theta(H)$ and the fixator subgroup of T_1 is H. For example the edge H with ends G and τG is stabilized by the subgroup $[H \cap \theta(H)] \times \tau[H \cap \theta(H)]\tau^{-1}$ which is the product of the fixators of the corresponding half-trees. This statement holds for every edge. It is clear that if $g \in \Gamma$, the stabilizer subgroup of the edge grH, where $r \in \{1, g_1, \tau^{-1}, g_0\tau^{-1}\}$ is $\gamma[H \cap \theta(H)]\gamma^{-1} \times gr[H \cap \theta(H)]r^{-1}g^{-1}$ and these groups generate $\operatorname{int}\Gamma$. We can use [25, Corollary 4.6], which is modeled after the Tits criterion, to conclude that a normal subgroup $N < \operatorname{int}\Gamma$ contains the commutator groups $[\Gamma_e, \Gamma_e]$ for every $e \in E(T)$. It easily follows that

$$\underline{\operatorname{int}\Gamma} := \langle g[H \cap \theta(H), H \cap \theta(H)]g^{-1} \mid g \in \Gamma \rangle$$

consists of all elements of $\operatorname{int}\Gamma$ with even number of g_0 and even number of g_1 in their products. Therefore

$$\underline{\operatorname{int}\Gamma} = \ker(\Gamma \to \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2),$$

where the homomorphism is defined on the generators of Γ as follows

$$\tau \mapsto (1, 0, 0), \quad g_0 \mapsto (0, 1, 0), \quad g_1 \mapsto (0, 0, 1).$$

We conclude that $\underline{\operatorname{int}\Gamma}$ is a simple amenablish group. Since the class of amenablish groups is closed under extensions, it follows that Γ is amenablish. $\square$

APPENDIX A. COMPACTNESS OF $T \cup \partial T$

We include a proof of the following result of Monod and Shalom from [28, Section 4.1].

Theorem A.1. *Let T be a tree with boundary ∂T. Then $T \cup \partial T$ is compact and totally disconnected in the shadow topology. Moreover, $T \cup \partial T$ is metrizable if and only if T is countable.*

In [28, Section 4.1] countability of T is assumed, but it is not needed to conclude compactness of the space $T \cup \partial T$. Throughout this section, let T denote a tree and for any edge e in T, we let $Z_0(e) \subseteq T$, $Z_B(e) \subseteq \partial T$ and $Z(e)$ be as defined in Section 3.

Lemma A.2. *Let e, e' be edges in T.*
   (i) *If $Z_B(e) \cap Z_B(e') \neq \emptyset$, then $Z_0(e) \cap Z_0(e')$ is infinite. In particular, $Z_0(e) \cap Z_0(e') = \emptyset$ if and only if $Z(e) \cap Z(e') = \emptyset$.*
   (ii) *$T \cup \partial T$ is the disjoint union of the clopen sets $Z(e)$ and $Z(\bar{e})$.*

Proof. (i): Any $x \in Z_B(e) \cap Z_B(e')$ is the equivalence class of a ray $(r_i)_{i \geq 0}$ such that $r_0 = s(e)$ and $r_1 = r(e)$, which is cofinal to a ray $(s_i)_{i \geq 0}$ such that $s_0 = s(e')$ and $s_1 = r(e')$. Now there exist $k, n \geq 0$ such that $Z_0(e) \ni r_{i+k} = s_{i+n} \in Z_0(e')$ for all $i \geq 0$.

(ii): Clearly $Z_0(e)$ and $Z_0(\bar{e})$ are disjoint (so that $Z(e) \cap Z(\bar{e}) = \emptyset$ by (i)), and their union is T. Therefore let $x \in \partial T$ and assume that $x \notin Z_B(e)$. Let $(r_i)_{i \geq 0}$ be the ray in the equivalence class of x such that $r_0 = s(e)$. Then $r_1 \neq r(e)$, so that we may define a ray $(t_i)_{i \geq 0}$ by $t_i = r_{i-1}$ for $i \geq 1$ and $t_0 = r(e)$. Now (t_i) and (r_i) are cofinal, and $(t_i) \in Z_\infty(\bar{e})$, meaning that $x \in Z_B(\bar{e})$. $\square$

Lemma A.3. *Suppose that two edges e and e' in T satisfy $d(r(e), r(e')) > d(s(e), s(e'))$. Then $Z(e) \cap Z(e') = \emptyset$.*

Proof. We clearly have $e \neq e'$ and $e \neq \overline{e'}$. If we had $\{s(e), r(e)\} \subseteq Z_0(e')$, then the geodesic between $r(e')$ and $s(e)$ would not contain $r(e)$ (otherwise $d(s(e), s(e')) = 1 + d(s(e), r(e')) = d(r(e), r(e')) + 2$), meaning that $d(r(e), r(e')) = d(s(e), r(e')) + 1 = d(s(e), s(e'))$. Hence $\{s(e), r(e)\} \subseteq Z_0(\overline{e'})$.

Now $r(e)$ is not contained in the geodesic between $s(e)$ and $r(e')$ – otherwise

$$d(s(e), s(e')) = d(r(e), s(e')) + 1 = d(r(e), r(e')),$$

a contradiction. Therefore the geodesic between $r(e)$ and $r(e')$ contains $s(e)$. As $r(e') \notin Z_0(e)$ (otherwise $d(s(e'), s(e)) < d(r(e'), r(e)) < d(r(e'), s(e))$, so that $s(e) \in Z_0(e')$), then for all $v \in Z_0(e)$,

$$d(v, r(e')) = d(v, s(e)) + d(s(e), r(e')) = d(v, s(e)) + d(s(e), s(e')) + 1 > d(v, s(e')),$$

i.e., $v \notin Z_0(e')$. Hence $Z_0(e)$ and $Z_0(e')$ are disjoint, so the claim follows from Lemma A.2 (i). $\square$

The following lemma is proved in [32, Lemma 6.4.9], so we omit the proof.

Lemma A.4 ("The bridge lemma"). *If T_1 and T_2 are subtrees of a tree T, and $T_1 \cap T_2$ contains at most one vertex, there are unique vertices $v_1 \in T_1$, $v_2 \in T_2$ with $d(v_1, v_2) = d(T_1, T_2)$. Moreover, we have $d(w_1, w_2) = d(w_1, v_1) + d(v_1, v_2) + d(v_2, w_2)$ for all $w_1 \in T_1$ and $w_2 \in T_2$.*

Proof of Theorem A.1. Let $X = T \cup \partial T$, and let $\mathcal{S} \subseteq 2^X$ be the Boolean algebra generated by the extended shadows $\{Z(e)\}_{e \in E}$. We consider the space $\mathcal{M}$ of Boolean homomorphisms $\mathcal{S} \to \{0, 1\}$ of the compact Hausdorff space $2^\mathcal{S}$, i.e., maps $\mu \colon \mathcal{S} \to \{0, 1\}$ such that $\mu(\emptyset) = 0$, $\mu(X) = 1$, and

$$\mu(A \cup B) = \max\{\mu(A), \mu(B)\}, \ \mu(A \cap B) = \min\{\mu(A), \mu(B)\}, \quad A, B \in \mathcal{S}.$$

We claim that $\mathcal{M}$ is compact. For any $\mu \in 2^\mathcal{S}$, define maps $\varphi_1(\mu) \colon \mathcal{S} \times \mathcal{S} \to \{0, 1\}$ by $\varphi_1(\mu)(A, B) = \mu(A \cup B)$ and $\varphi_2(\mu) \colon \mathcal{S} \times \mathcal{S} \to \{0, 1\}$ by $\varphi_2(\mu)(A, B) = \max\{\mu(A), \mu(B)\}$. These maps are continuous, when viewed as maps $\varphi_1 \colon 2^\mathcal{S} \to 2^{\mathcal{S} \times \mathcal{S}}$ and $\varphi_2 \colon 2^\mathcal{S} \to 2^{\mathcal{S} \times \mathcal{S}}$. Since $2^{\mathcal{S} \times \mathcal{S}}$ is Hausdorff, $\{\mu \in 2^\mathcal{S} \mid \varphi_1(\mu) = \varphi_2(\mu)\}$ is closed. Considering intersections the same way, it follows that $\mathcal{M}$ is closed in $2^\mathcal{S}$.

Now let $\mu \in \mathcal{M}$, and let $\mathcal{O}_\mu = \{e \in E \mid \mu(Z(e)) = 1\}$. Then $\mathcal{O}_\mu$ is an orientation of T by Lemma A.2 (ii). We claim that there exists $x \in X$ such that

$$\{x\} = \bigcap_{e \in \mathcal{O}_\mu} Z(e).$$

Observe that for any $v \in T$ there is at most one edge e such that $s(e) = v$ and $e \in \mathcal{O}_\mu$. (If $s(e) = s(e') = v$ for $e \neq e'$, then $Z_0(e) \cap Z_0(e') = \emptyset$, but then either $\mu(Z(e)) = 0$ or $\mu(Z(e')) = 0$.) There are now two cases:

Case 1: Any $v \in T$ admits one edge $e_v \in \mathcal{O}_\mu$ such that $s(e_v) = v$. For fixed $r_0 \in T$, let $e_0 \in \mathcal{O}_\mu$ such that $s(e_0) = r_0$, let $r_1 = r(e_0)$. We then let $e_1 \in \mathcal{O}_\mu$ such that $s(e_1) = r_1$, and continue taking edges $(e_i)_{i \geq 0}$ and vertices $(r_i)_{i \geq 0}$ in T, until we obtain a sequence $(r_i)_{i \geq 0}$ of vertices in T. Then $(r_i)_{i \geq 0}$ is a ray in T: if $r_{i-1} = s(e_{i-1}) = r(e_i) = r_{i+1}$ and $r(e_{i-1}) = s(e_i) = r_i$, then $\overline{e_i} = e_{i-1} \in \mathcal{O}_\mu$, a contradiction. Let $x \in \partial T$ be the equivalence class of $(r_i)_{i \geq 0}$.

Now $x \in Z(e)$ for all $e \in \mathcal{O}_\mu$. If $e = e_i$ for some $i \geq 0$, the claim is evident, assume that $e \notin \{e_i \mid i \geq 0\}$. Now let $k \geq 0$ be such that r_k is the closest of the vertices in the ray $(r_i)_{i \geq 0}$ to the subtree of endpoints $s(e), r(e)$ of e by the bridge lemma (Lemma A.4). If $s(e)$ belongs to the geodesic between $r(e)$ and r_k, then $d(r(e), r(e_k)) = 1 + d(r(e), s(e_k)) = 2 + d(s(e), s(e_k))$, meaning that $Z(e) \cap Z(e_k) = \emptyset$, contradicting Lemma A.3. Hence we may let $(t_i)_{i \geq 0}$ be a ray emanating from $s(e)$, passing through $r(e)$ and the vertices $(r_i)_{i \geq k}$, meaning that $x \in Z(e)$.

We next note that $\bigcap_{n=0}^\infty Z(e_n) = \{x\}$. Indeed, for $v \in T$, let $n \geq 0$ such that $d(r_n, v) \leq d(r_j, v)$ for all $j \geq 0$ by the bridge lemma. Then $d(v, s(e_n)) = d(v, r_n) < d(v, r_{n+1}) = d(v, r(e_n))$ and $v \notin Z(e_n)$. If $y \in \partial T$ satisfies $y \in Z(e_n)$ for all $n \geq 1$, let $(t_i)_{i \geq 0}$ be the unique ray with equivalence class y and $t_0 = r_0$. For $n \geq 1$, $(t_i)_{i \geq 0}$ is cofinal to a ray (u_j) such that $u_{j+1} = r_{n+1}$ and $u_j = r_n$ for some $j \geq 0$. In particular, there exists $m \geq 0$ for which $t_m \in Z_0(e_n)$. Since $t_0 \notin Z_0(e_n)$, the geodesic from $t_0 = r_0$ to t_m passes through r_n and r_{n+1}, meaning that $t_n = r_n$ and $t_{n+1} = r_{n+1}$. Since n was arbitrary, $y = x$. Therefore $\{x\} = \bigcap_{e \in \mathcal{O}_\mu} Z(e)$.

Case 2. There exists $v \in T$ such that $s(e) \neq v$ for all $e \in \mathcal{O}_\mu$. For some $e \in \mathcal{O}_\mu$, assume that $v \in Z(\overline{e})$, so that the geodesic from v to $r(e)$ passes through $s(e)$. Let $e_0, \ldots, e_n = e$

be the edges constituting this geodesic, so that $r(e_i) = s(e_{i+1})$ for all $0 \le i \le n-1$. Since $s(e_{n-1}) \notin Z(e_n)$, the geodesic from $w \in Z(e_n)$ to $s(e_{n-1})$ must contain $r(e_{n-1}) = s(e_n)$, meaning that $Z(e_n) \subseteq Z(e_{n-1})$. Hence $e_{n-1} \in \mathcal{O}_\mu$. We continue this way until we obtain $e_0 \in \mathcal{O}_\mu$, a contradiction (since $s(e_0) = v$). Hence $v \in Z(e)$ for all $e \in \mathcal{O}_\mu$.

Assume that $w \in T \setminus \{v\}$. Let $e_0, \dots, e_n$ be a path such that $r(e_i) = s(e_{i+1})$ for all $1 \le i \le n-1$ and $s(e_0) = v$, $r(e_n) = w$. Then $e_0 \notin \mathcal{O}_\mu$, so that $\overline{e_0} \in \mathcal{O}_\mu$ but $w \notin Z(\overline{e_0})$. For $x \in \partial T$, let $(r_i)_{i \ge 0}$ be the unique ray representing x such that $r_0 = v$, and let $(e_i)_{i \ge 0}$ be the edges such that $s(e_i) = r_i$, $r(e_i) = r_{i+1}$. Now $x \in Z(e_0)$, so that $x \notin Z(\overline{e_0})$ by Lemma A.2 (ii), but $\overline{e_0} \in \mathcal{O}_\mu$ as above. Hence $\{v\} = \bigcap_{e \in \mathcal{O}_\mu} Z(e)$.

Finally, let $\mathcal{S}_\mu = \{S \in \mathcal{S} \mid \mu(S) = \delta_x(S)\}$. Then S_μ is stable under complements, finite unions and intersections and $Z(e) \in S_\mu$ for all $e \in E$, so $\mathcal{S}_\mu = \mathcal{S}$ by minimality. Hence $\mu = \delta_x$.

We conclude that $\mathcal{M} = \{\delta_x \mid x \in X\}$. For edges $e_1, \dots, e_n$ in T such that $Z = \bigcap_{i=1}^n Z(e_i)$ is non-empty (this is a basis element of X), then $U = \{f \in 2^\mathcal{S} \mid f(Z) = 1\}$ is open in $2^\mathcal{S}$, so that $\mathcal{M} \cap U$ is open in $\mathcal{M}$. As $\delta_x \in \mathcal{M} \cap U$ if and only if $x \in Z$, it follows that $\mathcal{M} \cap U$ is the pre-image of the clopen basis element Z under the bijection $\mathcal{M} \to X$ given by $\delta_x \mapsto x$, so that this map is continuous. In particular X is compact, and since X is Hausdorff, the bijection $\mathcal{M} \to X$ is a homeomorphism.

Notice that $\mathcal{M}$ is totally disconnected by $2^\mathcal{S}$ being totally disconnected. If T is countable, then $\mathcal{S}$ is countable, so that $2^\mathcal{S}$ and $\mathcal{M}$ are metrizable. If $\mathcal{M}$ is metrizable, it has a countable basis $\mathcal{B}$ of clopen subsets. By [15, Corollary, p. 75] the countable algebra generated by $\mathcal{B}$ contains all clopen sets of $\mathcal{M}$, i.e., the edge set of T is countable. Hence T is countable. $\qquad\square$

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Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen, Denmark
  Email address: `rbryder@gmail.com`

Faculty of Mathematics and Informatics, University of Sofia, 5 James Bourchier Blvd., BG-1164 Sofia, Bulgaria
  Email address: `nikolay.antonov.ivanov@gmail.com`

Department of Mathematics, University of Oslo, NO-0316 Oslo, Norway and Department of Computer Science, Oslo Metropolitan University, NO-0130 Oslo, Norway
  Email address: `trono@math.uio.no`