ON THE K-THEORY OF C^*-ALGEBRAS ARISING FROM INTEGRAL DYNAMICS

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ABSTRACT. We investigate the K-theory of unital UCT Kirchberg algebras $\mathcal{Q}_S$ arising from families S of relatively prime numbers. It is shown that $K_*(\mathcal{Q}_S)$ is the direct sum of a free abelian group and a torsion group, each of which is realized by another distinct C^*-algebra naturally associated to S. The C^*-algebra representing the torsion part is identified with a natural subalgebra $\mathcal{A}_S$ of $\mathcal{Q}_S$. For the K-theory of $\mathcal{Q}_S$, the cardinality of S determines the free part and is also relevant for the torsion part, for which the greatest common divisor g_S of $\{p-1 : p \in S\}$ plays a central role as well. In the case where $|S| \leq 2$ or $g_S = 1$ we obtain a complete classification for $\mathcal{Q}_S$. Our results support the conjecture that $\mathcal{A}_S$ coincides with $\otimes_{p\in S}\mathcal{O}_p$. This would lead to a complete classification of $\mathcal{Q}_S$, and is related to a conjecture about k-graphs.

1. Introduction

Suppose S is a non-empty family of relatively prime natural numbers and consider the submonoid of $\mathbb{N}^\times$ generated by S. Its action on $\mathbb{Z}$ by multiplication can be represented on $\ell^2(\mathbb{Z})$ by the bilateral shift U and isometries $(S_p)_{p\in S}$ defined by $U\xi_n = \xi_{n+1}$ and $S_p\xi_n = \xi_{pn}$. The associated C^*-algebra $C^*\big(U,(S_p)_{p\in S}\big)$ admits a universal model $\mathcal{Q}_S$ that is generated by a unitary u and isometries $(s_p)_{p\in S}$, subject to

$$s_ps_q = s_qs_p \text{ for } q \in S, \quad s_pu = u^ps_p, \quad \text{and} \quad \sum_{m=0}^{p-1} u^m s_p s_p^* u^{-m} = 1.$$

By results of [21] or [43], $\mathcal{Q}_S$ is isomorphic to $C^*\big(U,(S_p)_{p\in S}\big)$ and belongs to the class of unital UCT Kirchberg algebras. In view of the Kirchberg-Phillips classification theorem [24, 37], the information on S encoded in $\mathcal{Q}_S$ can therefore be read off from its K-theory.

In special cases, $\mathcal{Q}_S$ and its K-theory have been considered before: If S is the set of all primes, then $\mathcal{Q}_S$ coincides with the algebra $\mathcal{Q}_\mathbb{N}$ from [12] and it follows that $K_i(\mathcal{Q}_S) = \mathbb{Z}^\infty$ for $i = 0, 1$ and $[1] = 0$. The other extreme case, where $S = \{p\}$ for some $p \geq 2$, appeared already in [20]: Hirshberg showed that $(K_0(\mathcal{Q}_{\{p\}}), [1], K_1(\mathcal{Q}_{\{p\}})) = (\mathbb{Z} \oplus \mathbb{Z}/(p-1)\mathbb{Z}, (0, 1), \mathbb{Z})$. This result was recovered later in [23] and [14] as a byproduct. Note that $\mathcal{Q}_{\{p\}}$ coincides with Katsura's algebra $\mathcal{O}(E_{p,1})$, see [23, Example A.6]. Moreover, Larsen and Li analyzed the situation for $p = 2$ in great detail, see [28]. The similarities and differences among these known cases raise several questions:

(i) Is $K_1(\mathcal{Q}_S)$ always torsion free?
(ii) Is $2 \in S$ the only obstruction to torsion in $K_0(\mathcal{Q}_S)$?
(iii) What is the K-theory of $\mathcal{Q}_S$ in the general case of $|S| \geq 2$?
(iv) What does $\mathcal{Q}_S \cong \mathcal{Q}_T$ reveal about the relationship between S and T?

Through the present work, we provide a complete description in the case of $|S| = 2$, for which the K-theory of $\mathcal{Q}_S$ satisfies

$$(K_0(\mathcal{Q}_S), [1], K_1(\mathcal{Q}_S)) = (\mathbb{Z}^2 \oplus \mathbb{Z}/g_S\mathbb{Z}, (0, 1), \mathbb{Z}^2 \oplus \mathbb{Z}/g_S\mathbb{Z}),$$

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where $g_S = \gcd(\{p-1 : p \in S\})$, see Theorem 6.1 (c). Thus we see that the first two questions from above have a negative answer (for instance, consider $S = \{3,5\}$ and $S = \{5,6\}$, respectively). More generally, we completely determine $K_*(\mathcal{Q}_S)$ in the case of $|S| \leq 2$ or $g_S = 1$, see Theorem 6.1, and conclude that $\mathcal{Q}_S \cong \mathcal{Q}_T$ if and only if $|S| = |T|$ and $g_S = g_T$ in this case. In addition, Theorem 6.1 substantially reduces the problem in the remaining case of $|S| \geq 3$ and $g_S > 1$. Thereby we also make progress towards a general answer to the remaining questions (iii) and (iv) from above.

In order to prove Theorem 6.1, we first compare the stabilization of $\mathcal{Q}_S$ to the C^*-algebra $C_0(\mathbb{R}) \rtimes N \rtimes H$, where $N = \mathbb{Z}\big[\{\frac{1}{p} : p \in S\}\big]$, H is the subgroup of $\mathbb{Q}_+^\times$ generated by S, and the action comes from the natural $ax+b$-action of $N \rtimes H$ on $\mathbb{R}$, see Section 3. This approach is inspired by methods of Cuntz and Li from [13]. However, the final part of their strategy is to use the Pimsner-Voiculescu sequence iteratively, see [13, Remark 3.16], and depends on having free abelian K-groups, which does not work in our situation. Instead, we show that $K_*(\mathcal{Q}_S)$ decomposes as a direct sum of a free abelian group and a torsion group, both arising in a natural way from two distinguished C^*-algebras related to $\mathcal{Q}_S$, see Theorem 4.4 and Corollary 4.7. The determination of the torsion free part of $K_*(\mathcal{Q}_S)$ uses a homotopy argument, and thereby benefits heavily from the comparison with real dynamics. This allows us to prove that the rank of the torsion free subgroup of $K_i(\mathcal{Q}_S)$ equals $2^{|S|-1}$ for both $i = 0,1$, see Proposition 4.5.

The torsion subgroup of $K_*(\mathcal{Q}_S)$ is realized by the semigroup crossed product $M_{d^\infty} \rtimes_\alpha^e H^+$, where d is the product of all primes dividing some element of S, H^+ is the submonoid of $\mathbb{N}^\times$ generated by S, and the action α is inherited from a semigroup crossed product description of $\mathcal{Q}_S$, see Corollary 4.7. Appealing to the recently introduced machinery for equivariantly sequentially split $*$-homomorphisms from [2], we show that $M_{d^\infty} \rtimes_\alpha^e H^+$ is a unital UCT Kirchberg algebra, just like $\mathcal{Q}_S$, see Corollary 5.2. Quite intriguingly, this paves the way to identify $M_{d^\infty} \rtimes_\alpha^e H^+$ with the subalgebra $\mathcal{A}_S = C^*(\{u^m s_p : p \in S, 0 \leq m \leq p-1\})$ of $\mathcal{Q}_S$, see Corollary 5.4. That is why we decided to name $\mathcal{A}_S$ the *torsion subalgebra*. This C^*-algebra is interesting in its own right as, for instance, it admits a model as the boundary quotient $\mathcal{Q}(U)$ of a particular right LCM submonoid U of $\mathbb{N} \rtimes H^+$, see Proposition 5.5. As explained in Remark 5.6, this gives rise to a remarkable diagram for the semigroup C^*-algebras and boundary quotients related to the inclusion of right LCM semigroups $U \subset \mathbb{N} \rtimes H^+$.

With regards to the K-theory of $\mathcal{A}_S$ and hence $\mathcal{Q}_S$, the k-graph description for finite S obtained in Corollary 5.8 is more illuminating: The canonical k-graph for $\mathcal{A}_S$ has the same skeleton as the standard k-graph for $\bigotimes_{p\in S} \mathcal{O}_p$, but uses different factorization rules, see Remark 5.9. It is apparent from the given presentation that $\mathcal{A}_S$ is isomorphic to $\mathcal{O}_p$ for $S = \{p\}$. If S consists of two relatively prime numbers p and q, then a result from [16] shows that $\mathcal{A}_S$ coincides with $\mathcal{O}_p \otimes \mathcal{O}_q$. For the remaining cases, we extract vital information on $\mathcal{A}_S$ by applying Kasparov's spectral sequence [22] (see also [1]) to the H^+-action α on M_{d^∞}, see Theorem 6.4. More precisely, we obtain that $\mathcal{A}_S$ is isomorphic to $\bigotimes_{p\in S} \mathcal{O}_p$ if $|S| \leq 2$ or $g_S = 1$. In the latter case, it actually coincides with $\mathcal{O}_2$. Additionally, we show that the order of every element in $K_*(\mathcal{A}_S)$ divides $g_S^{2^{|S|-2}}$. As we remark at the end of this work, the same results can be obtained by employing the k-graph representation of $\mathcal{A}_S$ and using Evans' spectral sequence [16] for the K-theory of k-graph C^*-algebras. In view of these results, it is very plausible that $\mathcal{A}_S$ always coincides with $\bigotimes_{p\in S} \mathcal{O}_p$. This would be in accordance with Conjecture 5.11, which addresses independence of K-theory from the factorization rules for k-graphs under certain constraints. If $\mathcal{A}_S \cong \bigotimes_{p\in S} \mathcal{O}_p$ holds for all S, then we get a complete classification for $\mathcal{Q}_S$ with the rule that $\mathcal{Q}_S$ and $\mathcal{Q}_T$ are isomorphic if and only if $|S| = |T|$ and $g_S = g_T$, see Conjecture 6.5.

At a later stage, the authors learned that Li and Norling obtained interesting results for the multiplicative boundary quotient for $\mathbb{N} \rtimes H^+$ by using completely different methods, see [30, Subsection 6.5]. Briefly speaking, the multiplicative boundary quotient related to $\mathcal{Q}_S$ is obtained by replacing the unitary u by an isometry v, see Subsection 2.2 for details. As a consequence, the K-theory of the multiplicative boundary quotient does not feature a non-trivial free part. It seems that $\mathcal{A}_S$ is the key to reveal a deeper connection between the K-theoretical structure of these two C^*-algebras. As this is beyond the scope of the present work, we only note that the inclusion map from $\mathcal{A}_S$ into $\mathcal{Q}_S$ factors through the multiplicative boundary quotient as an embedding of $\mathcal{A}_S$ and the natural quotient map. The results of [30] together with our findings indicate that this embedding might be an isomorphism in K-theory. This idea is explored further in [44, Section 5].

The paper is organized as follows: In Section 2, we set up the relevant notation and list some useful known results in Subsection 2.1. We then link $\mathcal{Q}_S$ to boundary quotients of right LCM semigroups, see Subsection 2.2, and a-adic algebras, see Subsection 2.3. These parts explain the central motivation behind our interest in the K-theory of $\mathcal{Q}_S$. In addition, the connection to a-adic algebras allows us to apply a duality theorem from [21], see Theorem 3.1, making it possible to invoke real dynamics. This leads to a decomposition result for $K_*(\mathcal{Q}_S)$ presented in Section 4, which essentially reduces the problem to determining the K-theory of $\mathcal{A}_S$. The structure of the torsion subalgebra $\mathcal{A}_S$ is discussed in Section 5. Finally, the progress on the classification of $\mathcal{Q}_S$ we obtain via a spectral sequence argument for the K-theory of $\mathcal{A}_S$ is presented in Section 6.

2. Preliminaries

2.1. **Notation and basics.** Throughout this paper, we assume that $S \subset \mathbb{N}^\times \setminus \{1\}$ is a non-empty family of relatively prime numbers. We write $p|q$ if $q \in p\mathbb{N}^\times$ for $p, q \in \mathbb{N}^\times$. Given S, we let $P := \{p \in \mathbb{N}^\times : p \text{ prime and } p|q \text{ for some } q \in S\}$. Also, we define $d := \prod_{p \in P} p$ (which is a supernatural number in case S is infinite, see Remark 2.15) and g_S to be the greatest common divisor of $\{p-1 : p \in S\}$, i.e. $g_S := \gcd(\{p-1 : p \in S\})$.

Recall that $\mathbb{N}^\times$ is an Ore semigroup with enveloping group $\mathbb{Q}_+^\times$, that is, $\mathbb{N}^\times$ embeds into $\mathbb{Q}_+^\times$ (in the natural way) so that each element $q \in \mathbb{Q}_+^\times$ can be displayed as $p^{-1}q$ with $p, q \in \mathbb{N}^\times$. The subgroup of $\mathbb{Q}_+^\times$ generated by S is denoted by H. Note that the submonoid of $\mathbb{N}^\times$ generated by S, which we refer to as H^+, forms a positive cone inside H. As the elements in S are relatively prime, H^+ is isomorphic to the free abelian monoid in $|S|$ generators. Finally, we let H_k be the subgroup of H generated by the k smallest elements of S for $1 \leq k \leq |S|$, and define H_k^+ as the analogous submonoid of H^+.

Though the natural action of H^+ on $\mathbb{Z}$ given by multiplication is irreversible, it has a natural extension to an action of H by automorphisms, namely by acting upon the ring extension $\mathbb{Z}\big[\{\frac{1}{p} : p \in S\}\big] = \mathbb{Z}\big[\{\frac{1}{p} : p \in P\}\big]$, that will be denoted by N. Within this context we will consider the collection of cosets

$$\mathcal{F} := \{m + h\mathbb{Z} : m \in \mathbb{Z}, h \in \mathbb{N}^\times, \tfrac{1}{h} \in N\}.$$

Definition 2.1. $\mathcal{Q}_S$ is defined to be the universal C^*-algebra generated by a unitary u and isometries $(s_p)_{p \in S}$ subject to the relations:

$$\text{(i)} \quad s_p^* s_q = s_q s_p^*, \qquad \text{(ii)} \quad s_p u = u^p s_p, \quad \text{and} \qquad \text{(iii)} \quad \sum_{m=0}^{p-1} e_{m+p\mathbb{Z}} = 1$$

for all $p, q \in S, p \neq q$, where $e_{m+p\mathbb{Z}} = u^m s_p s_p^* u^{-m}$.

Observe that the notation $e_{m+p\mathbb{Z}}$ is unambiguous, i.e. it does not depend on the representative m of the coset $m+p\mathbb{Z}$, as

$$u^{m+pn}s_p s_p^* u^{-m-pn} \overset{(ii)}{=} u^m s_p u^{n-n} s_p^* u^{-m} = u^m s_p s_p^* u^{-m}.$$

Remark 2.2. Let us briefly discuss the defining relations for $\mathcal{Q}_S$:

a) Condition (i) is known as the double commutation relation for the isometries s_p and s_q with $p \neq q$. In particular, they commute as $s_p^* s_q = s_q s_p^*$ implies that $(s_p s_q)^* s_q s_p = 1$, which forces $s_p s_q = s_q s_p$. Thus the family $(s_p)_{p\in S}$ gives rise to a representation of the monoid H^+ by isometries and we write s_h for $s_{p_1}\cdots s_{p_n}$ whenever $h = p_1 \cdots p_n \in H^+$ with $p_i \in S$. In fact, u and $(s_p)_{p\in S}$ yield a representation of $\mathbb{Z} \rtimes H^+$ due to Definition 2.1 (i),(ii).
b) $\mathcal{Q}_S$ can also be defined as the universal C^*-algebra generated by a unitary u and isometries $(s_p)_{p\in H^+}$ subject to (ii), (iii) and

$$\text{(i')}\ s_p s_q = s_{pq} \text{ for all } p, q \in H^+.$$

By a), we only need to show that (i') implies (i) for $p \neq q$. Note that (i') and (ii) imply that (iii) holds for all $p \in H^+$. In addition, (iii) implies the following: If $r \in S$ and $k \in \mathbb{Z}$ satisfy $s_r^* u^k s_r \neq 0$, then $k \in r\mathbb{Z}$. As $pq = qp$ and $p\mathbb{Z} \cap q\mathbb{Z} = pq\mathbb{Z}$, we get

$$s_p^* s_q \overset{(iii)}{=} \sum_{k=0}^{pq-1} s_p^* u^k s_{pq} s_{pq}^* u^{-k} s_q \overset{(i')}{=} \sum_{k=0}^{pq-1} (s_p^* u^k s_p) s_q s_p^* (s_q^* u^{-k} s_q) = s_q s_p^*.$$

Remark 2.3. The C^*-algebra $\mathcal{Q}_S$ has a canonical representation on $\ell^2(\mathbb{Z})$: Let $(\xi_n)_{n\in\mathbb{Z}}$ denote the standard orthonormal basis for $\ell^2(\mathbb{Z})$. If we define $U\xi_n := \xi_{n+1}$ and $S_p\xi_n := \xi_{pn}$, then it is routine to verify that U and $(S_p)_{p\in S}$ satisfy (i)–(iii) from Definition 2.1. $\mathcal{Q}_S$ is known to be simple, see [43, Example 3.29 (a) and Proposition 3.2] for proofs and Proposition 2.10 for the connection to [43]. Therefore, the representation from above is faithful and $\mathcal{Q}_S$ can be regarded as a subalgebra of $B(\ell^2(\mathbb{Z}))$.

Remark 2.4. For the case of $S = \{\text{all primes}\}$, the algebra $\mathcal{Q}_S$ coincides with $\mathcal{Q}_\mathbb{N}$ as introduced by Cuntz in [12]. Moreover, $S = \{2\}$ yields the 2-*adic ring* C^*-*algebra of the integers* that has been studied in detail by Larsen and Li in [28].

Definition 2.5. The commutative subalgebra of $\mathcal{Q}_S$ generated by the projections $e_{m+h\mathbb{Z}} = u^m s_h s_h^* u^{-m}$ with $m \in \mathbb{Z}$ and $h \in H^+$ is denoted by $\mathcal{D}_S$.

Remark 2.6. We record the following observations:

a) In view of Remark 2.3, $e_{m+h\mathbb{Z}}$ can be regarded as the orthogonal projection from $\ell^2(\mathbb{Z})$ onto $\ell^2(m+h\mathbb{Z})$.
b) With regards to a), the projections $e_{m+h\mathbb{Z}}$ correspond to certain cosets from $\mathcal{F}$. However, projections arising as sums of such elementary projections may lead to additional cosets: If $h \in \mathbb{N}^\times$ belongs to the submonoid generated by P, then there is $h' \in H^+$ so that $h' = h\ell$ for some $\ell \in \mathbb{N}^\times$. Therefore, we get

$$e_{m+h\mathbb{Z}} = \sum_{k=0}^{\ell-1} e_{m+hk+h'\mathbb{Z}}$$

and $e_{m+h\mathbb{Z}} \in \mathcal{D}_S$ for all such h. In fact, $\mathcal{F}$ equals the collection of all cosets for which the corresponding projection appears in $\mathcal{D}_S$, that is, the projection is expressible as a finite sum of projections $e_{m_i+h_i\mathbb{Z}}$ with $m_i \in \mathbb{Z}$ and $h_i \in H^+$.

Definition 2.7. The subalgebra of $\mathcal{Q}_S$ generated by $\mathcal{D}_S$ and u is denoted by $\mathcal{B}_S$.

Remark 2.8. The C^*-algebra $\mathcal{B}_S$ is isomorphic to the Bunce-Deddens algebra of type d^∞. If $p \in H^+$ and $(e_{i,j}^{(p)})_{0 \leq i,j \leq p-1}$ denote the standard matrix units in $M_p(\mathbb{C})$, then there is a unital $*$-homomorphism $M_p(\mathbb{C}) \otimes C^*(\mathbb{Z}) \to \mathcal{B}_S$ mapping $e_{m,n}^{(p)} \otimes u^k$ to $e_{m+p\mathbb{Z}}u^{m-n+pk}$. Given another $q \in H^+$, the so constructed $*$-homomorphisms associated with p and pq are compatible with the embedding $\iota_{p,pq} \colon M_p(\mathbb{C}) \otimes C^*(\mathbb{Z}) \to M_{pq}(\mathbb{C}) \otimes C^*(\mathbb{Z})$ given by $e_{i,j}^{(p)} \otimes 1 \mapsto \sum_{k=0}^{q-1} e_{i+pk,j+pk}^{(pq)} \otimes 1$ and $1 \otimes u \mapsto 1 \otimes u^q$. The inductive limit associated with $(M_p(\mathbb{C}) \otimes C^*(\mathbb{Z}), \iota_{p,pq})_{p,q \in H^+}$, where $H^+ \subset \mathbb{N}^\times$ is directed set in the usual way, is isomorphic to the Bunce-Deddens algebra of type d^∞. Moreover, under this identification, the natural UHF subalgebra M_{d^∞} of the Bunce-Deddens algebra corresponds to the C^*-subalgebra of $\mathcal{B}_S$ generated by all elements of the form $e_{m+p\mathbb{Z}}u^{m-n}$ with $p \in H^+$ and $0 \leq m,n \leq p-1$.

There is a natural action α of $\mathbb{Z} \rtimes H^+$ on $\mathcal{B}_S$ given by $\alpha_{(k,p)}(x) = u^k s_p x s_p^* u^{-k}$ for $(k,p) \in \mathbb{Z} \rtimes H^+$. Under the above identification, $M_{d^\infty} \subset \mathcal{B}_S$ is invariant under the restricted H^+-action, as for $p,q \in H^+$ and $0 \leq m,n \leq p-1$,

$$s_q e_{m+p\mathbb{Z}} u^{m-n} s_q^* = s_q u^m s_p s_p^* u^{-n} s_q^* = u^{qm} s_{pq} s_{pq}^* u^{-qn} = e_{qm+pq\mathbb{Z}} u^{qm-qn}.$$

Another way to present the algebra $\mathcal{Q}_S$ is provided by the theory of semigroup crossed products. Recall that, for an action β of a discrete, left cancellative semigroup T on a unital C^*-algebra B by $*$-endomorphisms, a unital, covariant representation of (B,β,T) is given by a unital $*$-homomorphism π from B to some unital C^*-algebra C and a semigroup homomorphism φ from T to the isometries in C such that the covariance condition $\varphi(t)\pi(b)\varphi(t)^* = \pi(\beta_t(b))$ holds for all $b \in B$ and $t \in T$. The semigroup crossed product $B \rtimes_\beta^e T$ is then defined as the C^*-algebra generated by a universal unital, covariant representation (ι_B, ι_T) of (B,β,T). We refer to [27] for further details. Note that if T is a group, then this crossed product agrees with the full group crossed product $B \rtimes_\beta T$. Semigroup crossed products may be pathological or extremely complicated in some cases. But we will only be concerned with crossed products of left Ore semigroups acting by injective endomorphisms so that we maintain a close connection to group crossed products, see [26]. With respect to $\mathcal{Q}_S$, we get isomorphisms

$$\mathcal{Q}_S \cong \mathcal{D}_S \rtimes_\alpha^e \mathbb{Z} \rtimes H^+ \cong \mathcal{B}_S \rtimes_\alpha^e H^+ \quad \text{and} \quad \mathcal{B}_S \cong \mathcal{D}_S \rtimes_\alpha \mathbb{Z}, \tag{1}$$

see [43, Proposition 3.18 and Theorem A.5]. Remark 2.8 reveals that the canonical subalgebra $M_{d^\infty} \subset \mathcal{B}_S$ and the isometries $(s_p)_{p \in H^+}$ give rise to a unital, covariant representation of $(M_{d^\infty}, \alpha, H^+)$ on $\mathcal{Q}_S$. We will later see that this representation is faithful so that we can view $M_{d^\infty} \rtimes_\alpha^e H^+$ as a subalgebra of $\mathcal{Q}_S$, see Corollary 5.4.

2.2. **Boundary quotients.** The set $S \subset \mathbb{N}^\times \setminus \{1\}$ itself can be thought of as a data encoding a dynamical system, namely the action θ of the free abelian monoid $H^+ \subset \mathbb{N}^\times$ on the group $\mathbb{Z}$ given by multiplication. θ_h is injective for $h \in H^+$ and surjective only if $h = 1$. Furthermore, as every two distinct elements p and q in S are relatively prime, we have $\theta_p(\mathbb{Z}) + \theta_q(\mathbb{Z}) = \mathbb{Z}$. Hence $(\mathbb{Z}, H^+, \theta)$ forms an *irreversible algebraic dynamical system* in the sense of [43, Definition 1.5], compare [43, Example 1.8 (a)]. In fact, dynamics of this form were one of the key motivations for [43]. In order to compare the C^*-algebra $\mathcal{O}[\mathbb{Z}, H^+, \theta]$ from [43, Definition 3.1] with $\mathcal{Q}_S$, let us recall the definition:

Definition 2.9. Let (G,P,θ) be an irreversible algebraic dynamical system. Then $\mathcal{O}[G,P,\theta]$ is the universal C^*-algebra generated by a unitary representation u of the group G and a representation s of the semigroup P by isometries subject to the relations:

$$\text{(CNP 1)} \qquad s_p u_g = u_{\theta_p(g)} s_p,$$

$$\text{(CNP 2)} \quad s_p^* u_g s_q = \begin{cases} u_{g_1} s_{\gcd(p,q)^{-1}q} s^*_{\gcd(p,q)^{-1}p} u_{g_2} & \text{if } g = \theta_p(g_1)\theta_q(g_2), \\ 0 & \text{else,} \end{cases}$$

$$\text{(CNP 3)} \qquad 1 = \sum_{[g] \in G/\theta_p(G)} e_{g,p} \quad \text{if } [G : \theta_p(G)] < \infty,$$

where $e_{g,p} = u_g s_p s_p^* u_g^*$.

Clearly, Definition 2.1 (ii) is the same as (CNP 1). As $[\mathbb{Z} : h\mathbb{Z}] = h < \infty$ for every $h \in H^+$, (iii) corresponds to (CNP 3) once we use Remark 2.2 a) and note that it is enough to have the summation relation for a set of generators of H^+. The case of distinct $p, q \in S$ and $g = 0$ in (CNP 2) yields (i). On the other hand, a slight modification of the argument in Remark 2.2 b) with $s_p^* u^m s_q$ in place of $s_p^* s_q$ establishes (CNP 2) based on (i)–(iii). Thus we arrive at:

Proposition 2.10. *The C^*-algebras $\mathcal{Q}_S$ and $\mathcal{O}[\mathbb{Z}, H^+, \theta]$ are canonically isomorphic.*

According to [43, Corollary 3.28 and Example 3.29 (a)] $\mathcal{Q}_S$ is therefore a unital UCT Kirchberg algebra. While classification of $\mathcal{O}[G, P, \theta]$ by K-theory was achieved in [43] for irreversible algebraic dynamical systems (G, P, θ) under mild assumptions, and even generalized to *algebraic dynamical systems* in [8], the range of the classifying invariant remained a mystery beyond the case of a single group endomorphism, where the techniques of [14] apply. It thus seemed natural to go back to examples of dynamical systems involving $P = \mathbb{N}^k$ and try to understand the invariant in this case. In other words, our path lead back to $\mathcal{Q}_S$, and the present work aims at making progress precisely in this direction.

There is also an alternative way of constructing $\mathcal{Q}_S$ directly from either of the semigroups $\mathbb{N} \rtimes H^+$ or $\mathbb{Z} \rtimes H^+$ using the theory of boundary quotients of semigroup C^*-algebras. To begin with, let us note that $(\mathbb{N} \rtimes H^+, N \rtimes H)$ forms a quasi lattice-ordered group. Hence we can form the Toeplitz algebra $\mathcal{T}(\mathbb{N} \rtimes H^+, N \rtimes H)$ using the work of Nica, see [34]. But $\mathbb{Z} \rtimes H^+$ has non-trivial units, so it cannot be part of a quasi lattice-ordered pair. In order to treat both semigroups within the same framework, let us instead employ the theory of semigroup C^*-algebras from [29], which generalizes Nica's approach tremendously.

We note that both $\mathbb{N} \rtimes H^+$ and $\mathbb{Z} \rtimes H^+$ are cancellative, countable, discrete semigroups with unit. Moreover, they are *right LCM* semigroups, meaning that the intersection of two principal right ideals is either empty or another principal right ideal (given by a right least common multiple for the representatives of the two intersected ideals). Thus their semigroup C^*-algebras both enjoy a particularly nice and tractable structure, see [5, 6]. Additionally, both are left Ore semigroups with amenable enveloping group $N \rtimes H \subset \mathbb{Q} \rtimes \mathbb{Q}_+^\times$. However, we would like to point out that $\mathbb{N} \rtimes H^+$ and $\mathbb{Z} \rtimes H^+$ are not left amenable (but right amenable) as they fail to be left reversible, see [29, Lemma 4.6] for details.

Roughly speaking, semigroup C^*-algebras have the flavor of Toeplitz algebras. In particular, they tend to be non-simple except for very special situations. Still, we might hope for $\mathcal{Q}_S$ to be a quotient of $C^*(\mathbb{N} \rtimes H^+)$ or $C^*(\mathbb{Z} \rtimes H^+)$ obtained through some systematic procedure. This was achieved in [27] for $\mathbb{N} \rtimes \mathbb{N}^\times$, i.e. S consisting of all primes, by showing that the boundary quotient of $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times, \mathbb{Q} \rtimes \mathbb{Q}_+^\times) = C^*(\mathbb{N} \rtimes \mathbb{N}^\times)$ in the sense of [10] coincides with $\mathcal{Q}_\mathbb{N}$. Recently, this concept of a boundary quotient for a quasi lattice-ordered group from [10] was transferred to semigroup C^*-algebras in the context of right LCM semigroups, see [7, Definition 5.1]:

Definition 2.11. Let T be a right LCM semigroup. A finite set $F \subset T$ is called a *foundation set* if, for all $t \in T$, there is $f \in F$ satisfying $tT \cap fT \neq \emptyset$. The *boundary quotient* $\mathcal{Q}(T)$ of a right LCM semigroup T is the quotient of $C^*(T)$ by the relation

$$\prod_{f \in F} (1 - e_{fT}) = 0 \quad \text{for all foundation sets } F. \tag{2}$$

To emphasize the relevance of this approach, let us point out that right LCM semigroups are much more general than quasi lattice-ordered groups. For instance, right cancellation may fail, so right LCM semigroups need not embed into groups.

On the one hand, this notion of a quotient of a semigroup C^*-algebra seems suitable as $\mathbb{N} \rtimes H^+$ and $\mathbb{Z} \rtimes H^+$ are right LCM semigroups. On the other hand, the abstract condition (2) prohibits an immediate identification of $\mathcal{Q}_S$ with $\mathcal{Q}(\mathbb{N} \rtimes H^+)$ or $\mathcal{Q}(\mathbb{Z} \rtimes H^+)$. This gap has been bridged successfully through [8]:

Proposition 2.12. *There are canonical isomorphisms $\mathcal{Q}_S \cong \mathcal{Q}(\mathbb{Z} \rtimes H^+) \cong \mathcal{Q}(\mathbb{N} \rtimes H^+)$.*

Proof. For $\mathbb{Z} \rtimes H^+$, [8, Corollary 4.2] shows that $\mathcal{Q}(\mathbb{Z} \rtimes H^+) \cong \mathcal{O}[\mathbb{Z}, H^+, \theta]$, and hence $\mathcal{Q}_S \cong \mathcal{Q}(\mathbb{Z} \rtimes H^+)$ by Proposition 2.10. Noting that H^+ is directed, this can also be seen immediately from [8, Remark 2.2 and Proposition 4.1]. For $\mathbb{N} \rtimes H^+$, we infer from [8, Example 2.8 (b)] that it suffices to consider *accurate foundation sets* F for (2) by [8, Proposition 2.4], that is, F consists of elements with mutually disjoint principal right ideals. Now $F \subset \mathbb{N} \rtimes H^+$ is an accurate foundation set if and only if it is an accurate foundation set for $\mathbb{Z} \rtimes H^+$. Conversely, given an accurate foundation set $F' = \{(m_1, h_1), \ldots, (m_n, h_n)\} \subset \mathbb{Z} \rtimes H^+$, we can replace each m_i by some $m_i' \in m_i + h_i\mathbb{N}$ with $m_i' \in \mathbb{N}$ to get an accurate foundation set $F \subset \mathbb{N} \rtimes H^+$ which uses the same right ideals as F'. This allows us to conclude that $\mathcal{Q}(\mathbb{Z} \rtimes H^+)$ and $\mathcal{Q}(\mathbb{N} \rtimes H^+)$ are isomorphic. ☐

The fact that $\mathcal{Q}(\mathbb{N} \rtimes H^+)$ and $\mathcal{Q}(\mathbb{Z} \rtimes H^+)$ coincide is not at all surprising if we take into account [4] and view $C^*(\mathbb{Z} \rtimes H^+)$ as the *additive boundary quotient* of $\mathbb{N} \rtimes H^+$. Where there is an additive boundary, there is also a multiplicative boundary, see the boundary quotient diagram in [4, Section 4]: The *multiplicative boundary quotient* of $C^*(\mathbb{N} \rtimes H^+)$ is obtained by imposing the analogous relation to (iii) from Definition 2.1, i.e. $\sum_{k=0}^{p-1} e_{k+p\mathbb{N}} = 1$ for each $p \in S$. In comparison with Definition 2.1, the essential difference is that the semigroup element $(1,1)$ is implemented by a proper isometry $v_{(1,1)}$ instead of a unitary u. This multiplicative boundary quotient has been considered in [30, Subsection 6.5]. As it turns out, its K-theory is hard to compute for larger S as it leads to increasingly complicated extension problems of abelian groups. It is quite remarkable that there seems to be a deep common theme underlying the structure of the K-theory for both the multiplicative boundary quotient and $\mathcal{Q}_S$.

2.3. **The a-adic algebras.** Our aim is to compute the K-theory of $\mathcal{Q}_S$, and for this we need to make use of a certain duality result [21, Theorem 4.1] that allows us to translate our problem into real dynamics. This will be explained in the next section, but let us first recall the definition and some facts about a-adic algebras from [21] and [35], see also [19, Sections 10 and 25] for more on a-adic numbers.

Let $a = (a_k)_{k \in \mathbb{Z}}$ be a sequence of numbers in $\mathbb{N}^\times \setminus \{1\}$, and define the *$a$-adic numbers* as the abelian group of sequences

$$\Omega_a = \left\{ x \in \prod_{k=-\infty}^{\infty} \{0, 1, \ldots, a_k - 1\} : \text{there exists } \ell \in \mathbb{Z} \text{ such that } x_k = 0 \text{ for all } k < \ell \right\}$$

under addition with carry (that is, like a doubly infinite odometer). The family of all subgroups $\{x \in \Omega_a : x_k = 0 \text{ for } k < \ell\}$ form a neighborhood basis of the identity. This induces a topology that makes Ω_a a totally disconnected, locally compact Hausdorff group. The *a-adic integers* is the compact open subgroup

$$\Delta_a = \{x \in \Omega_a : x_k = 0 \text{ for } k < 0\} \subset \Omega_a. \tag{3}$$

For $k \in \mathbb{Z}$, define the sequence $(e_k)_\ell = \delta_{k\ell}$. For $k \geq 1$, we may associate the rational number $(a_{-1}a_{-2}\cdots a_{-k})^{-1}$ with e_{-k} to get an injective group homomorphism from the non-cyclic subgroup

$$N_a = \left\{ \tfrac{j}{a_{-1}a_{-2}\cdots a_{-k}} : j \in \mathbb{Z}, k \geq 1 \right\} \subset \mathbb{Q}$$

into Ω_a with dense range. Note that N_a contains $\mathbb{Z} \subset \mathbb{Q}$, and by identifying N_a and $\mathbb{Z} \subset N_a$ with their images under the embedding into Ω_a, it follows that $N_a \cap \Delta_a = \mathbb{Z}$.

The subgroups $N_a \cap \{x \in \Omega_a : x_k = 0 \text{ for } k < \ell\}$ for $\ell \in \mathbb{Z}$ give rise to a subgroup topology of N_a, and Ω_a is the Hausdorff completion (i.e. inverse limit completion) of N_a with respect to this filtration. Therefore, the class of a-adic numbers Ω_a comprises all groups that are Hausdorff completions of non-cyclic subgroups of $\mathbb{Q}$. Loosely speaking, the negative part of the sequence a determines a subgroup N_a of $\mathbb{Q}$, and the positive part determines a topology that gives rise to a completion of N_a. Given a sequence a, let a^* denote the dual sequence defined by $a^*_k = a_{-k}$, and write N^*_a and Ω^*_a for the associated groups.

Let H_a be any non-trivial subgroup of $\mathbb{Q}^\times_+$ acting on N_a by continuous multiplication, meaning that for all $h \in H_a$, the map $N_a \to N_a$, $x \mapsto hx$ is continuous with respect to the topology described above. The largest subgroup with this property is generated by the primes dividing infinitely many terms of both the positive and negative tail of the sequence a, see [21, Corollary 2.2], so we must assume that this subgroup is non-trivial (which holds in the cases we study). Then H_a also acts on Ω_a by multiplication, and therefore $N_a \rtimes H_a$ acts on Ω_a by an $ax + b$-action.

Definition 2.13. For a sequence $a = (a_k)_{k\in\mathbb{Z}}$ in $\mathbb{N}^\times \setminus \{1\}$ and a non-trivial subgroup H_a of $\mathbb{Q}^\times_+$ acting by continuous multiplication on N_a, the crossed product $\overline{\mathcal{Q}}(a, H_a) := C_0(\Omega_a) \rtimes N_a \rtimes H_a$ is called the *a-adic algebra* of (a, H_a).

Clearly, interchanging a and a^* and manipulating the position of a_0 will not affect any structural property on the level of algebras. In fact, for our purposes, it will usually be convenient to assume that $a = a^*$. Therefore, we will often use the positive tail of the sequence a in the description of N_a, and think of N_a as the inductive limit of the system $\{(\mathbb{Z}, \cdot a_k) : k \geq 0\}$ via the isomorphism induced by

$$\begin{tikzcd} \mathbb{Z} \arrow[rr, "\cdot a_k"] \arrow[dr, swap, "\cdot\frac{1}{a_0a_1a_2\cdots a_{k-1}}"] & & \mathbb{Z} \arrow[dl, "\cdot\frac{1}{a_0a_1a_2\cdots a_{k-1}a_k}"] \\ & N_a & \end{tikzcd} \tag{4}$$

Remark 2.14. By [21, Corollary 2.8] the a-adic algebra $\overline{\mathcal{Q}}(a, H)$ is always a non-unital UCT Kirchberg algebra, hence it is stable by Zhang's dichotomy, see [46] or [40, Proposition 4.1.3].

An immediate consequence of (4) is that

$$C_0(\Omega_a) \rtimes N_a \cong \overline{\bigcup_{k=0}^{\infty} C(\tfrac{1}{a_0\cdots a_k}\Delta_a) \rtimes \tfrac{1}{a_0\cdots a_k}\mathbb{Z}}.$$

Moreover, by writing $\frac{1}{a_0\cdots a_k}\Delta_a = \Delta_a + (\frac{a_0\cdots a_k - 1}{a_0\cdots a_k} + \Delta_a) + \cdots + (\frac{1}{a_0\cdots a_k} + \Delta_a)$ and checking how the translation action of $\frac{1}{a_0\cdots a_k}\mathbb{Z}$ interchanges the components of this sum, one sees that

$$C(\tfrac{1}{a_0\cdots a_k}\Delta_a) \rtimes \tfrac{1}{a_0\cdots a_k}\mathbb{Z} \cong M_{a_0\cdots a_k}\left(C(\Delta_a) \rtimes \mathbb{Z}\right).$$

In particular, the natural embeddings of the increasing union above translates into embeddings into the upper left corners. Hence, it follows that $C_0(\Omega_a) \rtimes N_a$ is also stable.

Remark 2.15. A supernatural number is a function $d\colon \{\text{all primes}\} \to \mathbb{N} \cup \{\infty\}$, such that $\sum_{p \text{ prime}} d(p) = \infty$, and often written as a formal product $\prod_{p \text{ prime}} p^{d(p)}$. It is well known that there is a one-to-one correspondence between supernatural numbers and non-cyclic subgroups of $\mathbb{Q}$ containing 1, and that the supernatural numbers form a complete isomorphism invariant both for the UHF algebras and the Bunce-Deddens algebras, see [18] and [3].

Every sequence $a = (a_k)_{k\geq 0}$ defines a function $d_a\colon \{\text{all primes}\} \to \mathbb{N} \cup \{\infty\}$ given by $d_a(p) = \sup\{n \in \mathbb{N} : p^n | a_0 a_1 \cdots a_k \text{ for some } k \geq 0\}$. More intuitively, d_a is thought of as the infinite product $d_a = a_0 a_1 a_2 \cdots$. Moreover (see e.g. [21, Lemma 5.1]), we have

$$\Delta_a \cong \prod_{p \in d_a^{-1}(\infty)} \mathbb{Z}_p \times \prod_{p \in d_a^{-1}(\mathbb{N})} \mathbb{Z}/p^{d_a(p)}\mathbb{Z}, \tag{5}$$

and thus the supernatural numbers are a complete isomorphism invariant for the homeomorphism classes of a-adic integers.

Now, as in Section 2, let S be a set consisting of relatively prime numbers, and let H^+ and H denote the submonoid of $\mathbb{N}^\times$ and the subgroup of $\mathbb{Q}_+^\times$ generated by S, respectively. The sequence a_S is defined as follows: Since H^+ is a subset of $\mathbb{N}^\times$, its elements can be sorted into increasing order $1 < a_{S,0} < a_{S,1} < \cdots$, where $a_{S,0} = \min S$. Finally, we set $a_{S,k} = a_{S,-k}$ for $k < 0$. If S is a finite set, an easier way to form a suitable sequence a_S is to let q denote the product of all elements of S, and set $a_{S,k} = q$ for all $k \in \mathbb{Z}$. In both cases, $a_S^* = a_S$ and $N_{a_S} = N$. Henceforth we fix such a sequence a_S and denote Ω_{a_S} and Δ_{a_S} by Ω and Δ, respectively. The purpose of self-duality of a_S is to have $N^* = N$, making the statement of Theorem 3.1 slightly more convenient by avoiding the explicit use of N^*. The sequences a_S are the ones associated with supernatural numbers d for which $d(p) \in \{0, \infty\}$ for every prime p. In this case (5) implies that

$$\Delta \cong \prod_{p \in P} \mathbb{Z}_p \quad \text{and} \quad \Omega \cong \prod_{p \in P}{}' \, \mathbb{Q}_p = \prod_{p \in P} (\mathbb{Q}_p, \mathbb{Z}_p),$$

where the latter denotes the restricted product with respect to $\{\mathbb{Z}_p : p \in P\}$.

Remark 2.16. The spectrum of the commutative subalgebra $\mathcal{D}_S$ of $\mathcal{Q}_S$ from Definition 2.5 coincides with the a-adic integers Δ described in (3). Indeed, for every $X = m + h\mathbb{Z} \in \mathcal{F}$, the projection e_X in $\mathcal{D}_S$ corresponds to the characteristic function on the compact open subset $m + h\Delta$ of Δ. Moreover, this correspondence extends to an isomorphism between the C^*-algebra $\mathcal{B}_S \cong \mathcal{D}_S \rtimes \mathbb{Z}$ of Definition 2.7 and $C(\Delta) \rtimes \mathbb{Z}$, which is equivariant for the natural H^+-actions on the algebras, both denoted by α.

Let us write e for the projection in $\overline{\mathcal{Q}}(a_S, H)$ representing the characteristic function on Δ in $C_0(\Omega)$. It is explained in [35, Section 11.6] that e is a full projection, and thus, by using Remark 2.16 together with (1), we have

$$e\overline{\mathcal{Q}}(a_S, H)e \cong (C(\Delta) \rtimes \mathbb{Z}) \rtimes_\alpha^e H^+ \cong (\mathcal{D}_S \rtimes \mathbb{Z}) \rtimes_\alpha^e H^+ \cong \mathcal{B}_S \rtimes_\alpha^e H^+ \cong \mathcal{Q}_S. \tag{6}$$

In fact, since N coincides with $(H^+)^{-1}\mathbb{Z}$, the above also follows from [26]. Moreover, the argument in [35] does not require H to be non-trivial, so it can be used together with Remark 2.16 and (1) to get

$$e(C_0(\Omega) \rtimes N)e \cong C(\Delta) \rtimes \mathbb{Z} \cong \mathcal{D}_S \rtimes \mathbb{Z} \cong \mathcal{B}_S. \tag{7}$$

Hence, by applying Remark 2.14 we arrive at the following result:

Proposition 2.17. *The stabilization of $\mathcal{Q}_S$ is isomorphic to $\overline{\mathcal{Q}}(a_S, H)$, and the stabilization of $\mathcal{B}_S$ is isomorphic to $C_0(\Omega) \rtimes N$.*

Therefore Proposition 2.17 gives an alternative way to see that $\mathcal{Q}_S$ is a unital UCT Kirchberg algebra, which is also a consequence of Proposition 2.10.

3. Comparison with real dynamics

Let S and H_k be as specified in Section 2, $a = (a_k)_{k\in\mathbb{Z}}$ in $\mathbb{N}^\times \setminus \{1\}$, and H_a a non-trivial subgroup of $\mathbb{Q}_+^\times$ that acts on N_a by continuous multiplication. For convenience, we will assume $a^* = a$ so that $N_a^* = N_a$. Moreover, N_a acts by translation and H_a acts by multiplication on $\mathbb{R}$, respectively, giving rise to an $ax+b$-action of $N_a \rtimes H_a$ on $\mathbb{R}$. Let $\widehat{N}_a$ denote the Pontryagin dual of N_a. By [21, Theorem 3.3], the diagonal embedding $N_a \to \mathbb{R}\times\Omega_a$ has discrete range, and gives an isomorphism

$$(\mathbb{R}\times\Omega_a)/N_a \cong \widehat{N}_a.$$

By applying Green's symmetric imprimitivity theorem, see e.g. [45, Corollary 4.11], we obtain that

$$C_0(\Omega_a) \rtimes N_a \sim_M C_0(\mathbb{R}) \rtimes N_a,$$

and this Morita equivalence is equivariant for the actions of H_a by multiplication on one side and inverse multiplication on the other. The inverse map on H_a does not have any impact on the crossed products, and thus

$$\overline{\mathcal{Q}}(a, H_a) \sim_M C_0(\mathbb{R}) \rtimes N_a \rtimes H_a.$$

All the above is explained in detail in [21, Proof of Theorem 4.1]. Moreover, recall that UCT Kirchberg algebras are either unital or stable, so by using Proposition 2.17 we get:

Theorem 3.1. *The a-adic algebra $\overline{\mathcal{Q}}(a, H_a)$ is isomorphic to $C_0(\mathbb{R}) \rtimes N_a \rtimes H_a$. In particular, the stabilization of $\mathcal{Q}_S$ is isomorphic to $C_0(\mathbb{R}) \rtimes N \rtimes H$.*

Remark 3.2. It follows from Theorem 3.1, based on [21, Theorem 4.1], that any a-adic algebra $\overline{\mathcal{Q}}(a, H_a)$ is isomorphic to a crossed product $C_0(\mathbb{R}) \rtimes N_a \rtimes H_a$. Recall that N_a can be any non-cyclic subgroup of $\mathbb{Q}$ and H_a can be any non-trivial subgroup of $\mathbb{Q}_+^\times$ that acts on N_a by multiplication. In the present work, we limit our scope to the case where N_a and H_a can be obtained from a family S of relatively prime numbers for the benefit of a more concise exposition. In a forthcoming project, we aim at establishing analogous results to the ones proven here for all a-adic algebras.

Remark 3.3. By employing the description of N_a from (4), we can write $C_0(\mathbb{R}) \rtimes N_a$ as an inductive limit. For $k \geq 0$, define the automorphism γ_k of $C_0(\mathbb{R})$ by

$$\gamma_0(f)(s) = f(s-1) \text{ and } \gamma_{k+1}(f)(s) = f\big(s - \tfrac{1}{a_0a_1\cdots a_k}\big), \quad f \in C_0(\mathbb{R}).$$

Under the identification in (4), these automorphisms give rise to the natural N_a-action on $C_0(\mathbb{R})$, where γ_k corresponds to the generator for the kth copy of $\mathbb{Z}$. For $k \geq 0$, let $u_k \in \mathcal{M}(C_0(\mathbb{R}) \rtimes_{\gamma_k} \mathbb{Z})$ denote the canonical unitary implementing γ_k and consider the $*$-homomorphism $\phi_k\colon C_0(\mathbb{R}) \rtimes_{\gamma_k} \mathbb{Z} \to C_0(\mathbb{R}) \rtimes_{\gamma_{k+1}} \mathbb{Z}$ given by $\phi_k(f) = f$ and $\phi_k(fu_k) = fu_{k+1}^{a_k}$ for every $f \in C_0(\mathbb{R})$. The inductive limit description (4) of N_a now yields an isomorphism $\varphi\colon \varinjlim \{C_0(\mathbb{R}) \rtimes_{\gamma_k} \mathbb{Z}, \phi_k\} \xrightarrow{\cong} C_0(\mathbb{R}) \rtimes N_a$.

Remark 3.4. A modification of [12, Lemma 6.7], using the inductive limit description from Remark 3.3, shows that $C_0(\mathbb{R}) \rtimes N_a$ is stable. Hence, it follows from the above together with Remark 2.14 that $C_0(\Omega_a) \rtimes N_a$ is isomorphic to $C_0(\mathbb{R}) \rtimes N_a$. In particular, Proposition 2.17 shows that the stabilization of $\mathcal{B}_S$ is isomorphic to $C_0(\mathbb{R}) \rtimes N$.

We will make use of this fact below.

Lemma 3.5. *Let $\widetilde{\alpha}$ and β denote the actions of H_a on $C_0(\Omega_a) \rtimes N_a$ and $C_0(\mathbb{R}) \rtimes N_a$, respectively. Then β^{-1} is exterior equivalent to an action $\widetilde{\beta}$ for which there is an $\widetilde{\alpha}$ - $\widetilde{\beta}$-equivariant isomorphism $C_0(\Omega_a) \rtimes N_a \xrightarrow{\cong} C_0(\mathbb{R}) \rtimes N_a$.*

Proof. The respective actions $\widetilde{\alpha}$ and β^{-1} of H_a are Morita equivalent by [21, Proof of Theorem 4.1]. Moreover, both C^*-algebras are separable and stable, see Remark 3.4. Therefore, [9, Proposition on p. 16] implies that the actions are also outer conjugate, and the statement follows. □

In the following, we denote by $\iota_{N_a} \colon C_0(\mathbb{R}) \hookrightarrow C_0(\mathbb{R}) \rtimes N_a$ the canonical embedding, which is equivariant for the respective H_a-actions β (and also β^{-1}). We conclude this section by proving that ι_{N_a} induces an isomorphism between the corresponding K_1-groups.

Proposition 3.6. *The canonical embedding $\iota_{N_a} \colon C_0(\mathbb{R}) \hookrightarrow C_0(\mathbb{R}) \rtimes N_a$ induces an isomorphism between the corresponding K_1-groups.*

Proof. Recall the isomorphism $\varphi \colon \varinjlim \{C_0(\mathbb{R}) \rtimes_{\gamma_k} \mathbb{Z}, \phi_k\} \xrightarrow{\cong} C_0(\mathbb{R}) \rtimes N_a$ from Remark 3.3. For $k \geq 0$, let $\iota_k \colon C_0(\mathbb{R}) \to C_0(\mathbb{R}) \rtimes_{\gamma_k} \mathbb{Z}$ be the canonical embedding. As $\iota_{k+1} = \phi_k \circ \iota_k$, we obtain the following commutative diagram

$$\begin{tikzcd} C_0(\mathbb{R}) \arrow[r, "\iota_{N_a}"] \arrow[d, "\phi_{k,\infty} \circ \iota_k"'] & C_0(\mathbb{R}) \rtimes N_a \\ \varinjlim \{C_0(\mathbb{R}) \rtimes_{\gamma_m} \mathbb{Z}, \phi_m\} \arrow[ur, "\varphi"'] & \end{tikzcd} \tag{8}$$

Here, $\phi_{k,\infty} \colon C_0(\mathbb{R}) \rtimes_{\gamma_k} \mathbb{Z} \to \varinjlim \{C_0(\mathbb{R}) \rtimes_{\gamma_m} \mathbb{Z}, \phi_m\}$ denotes the canonical $*$-homomorphism given by the universal property of the inductive limit.

As $K_0(C_0(\mathbb{R})) = 0$, the Pimsner-Voiculescu sequence [38] for $\gamma_k \in \mathrm{Aut}(C_0(\mathbb{R}))$ reduces to an exact sequence

$$K_0(C_0(\mathbb{R}) \rtimes_{\gamma_k} \mathbb{Z}) \hookrightarrow K_1(C_0(\mathbb{R})) \xrightarrow{\mathrm{id} - K_1(\gamma_k)} K_1(C_0(\mathbb{R})) \overset{K_1(\iota_k)}{\twoheadrightarrow} K_1(C_0(\mathbb{R}) \rtimes_{\gamma_k} \mathbb{Z}).$$

For each $k \geq 0$, the automorphism γ_k is homotopic to the identity on $\mathbb{R}$, so that $K_1(\gamma_k) = \mathrm{id}$. It thus follows that $K_1(\iota_k)$ is an isomorphism. As $\iota_{k+1} = \phi_k \circ \iota_k$, we therefore get that $K_1(\phi_k)$ is an isomorphism as well. Hence, by continuity of K-theory, $K_1(\phi_{k,\infty})$ is an isomorphism. It now follows from (8) that $K_1(\iota_{N_a})$ is an isomorphism, which completes the proof. □

4. A decomposition of the K-theory of $\mathcal{Q}_S$

In this section, we show that $K_*(\mathcal{Q}_S)$ decomposes as a direct sum of a free abelian group and a torsion group, see Theorem 4.4 and Corollary 4.7. We would like to highlight that this is not just an abstract decomposition of $K_*(\mathcal{Q}_S)$, but a result that facilitates a description of the two parts by distinguished C^*-algebras associated to S, namely $M_{d^\infty} \rtimes_\alpha^e H^+$ for the torsion part, and $C_0(\mathbb{R}) \rtimes_\beta H$ for the free part. The free abelian part is then shown to have rank $2^{|S|-1}$, see Proposition 4.5, so that $\mathcal{Q}_S$ and $\mathcal{Q}_T$ can only be isomorphic if S and T have the same cardinality.

The following is the key tool for the proof of this section's main result, and we think it is of interest in its own right.

Proposition 4.1. *Let $k \in \mathbb{N} \cup \{\infty\}$, A, B, C C^*-algebras, and $\alpha \colon \mathbb{Z}^k \curvearrowright A$, $\beta \colon \mathbb{Z}^k \curvearrowright B$, and $\gamma \colon \mathbb{Z}^k \curvearrowright C$ actions. Let $v \colon \mathbb{Z}^k \to \mathcal{U}(\mathcal{M}(C))$ be a γ-cocycle and denote by $\tilde{\gamma} \colon \mathbb{Z}^k \curvearrowright C$ the induced action given by $\tilde{\gamma}_h = \mathrm{Ad}(v_h) \circ \gamma_h$ for $h \in \mathbb{Z}^k$. Let $\kappa \colon C \rtimes_{\tilde{\gamma}} \mathbb{Z}^k \xrightarrow{\cong} C \rtimes_\gamma \mathbb{Z}^k$ be the $*$-isomorphism induced by the γ-cocycle v. Assume that $\varphi \colon A \to C$ is a non-degenerate*

α - γ-equivariant $*$-homomorphism and $\psi\colon B \to C$ a non-degenerate β - $\tilde{\gamma}$-equivariant $*$-homomorphism such that $K_0(\varphi)$ and $K_1(\psi)$ are isomorphisms and $K_1(\varphi)$ and $K_0(\psi)$ are trivial. Then

$$K_*(\varphi \rtimes \mathbb{Z}^k) \oplus K_*(\kappa \circ (\psi \rtimes \mathbb{Z}^k))\colon K_*(A \rtimes_\alpha \mathbb{Z}^k) \oplus K_*(B \rtimes_\beta \mathbb{Z}^k) \to K_*(C \rtimes_\gamma \mathbb{Z}^k)$$

is an isomorphism.

Proof. Consider the amplified action $\gamma^{(2)}\colon \mathbb{Z}^k \curvearrowright M_2(C)$ given by entrywise application of γ. Let $w\colon \mathbb{Z}^k \to \mathcal{U}(\mathcal{M}(M_2(C)))$ be the $\gamma^{(2)}$-cocycle given by $w_h = \operatorname{diag}(1, v_h)$ for $h \in \mathbb{Z}^k$. The induced $\mathbb{Z}^k$-action $\delta = \operatorname{Ad}(w) \circ \gamma^{(2)}$ satisfies $\delta_h(\operatorname{diag}(c, c')) = \operatorname{diag}(\gamma_h(c), \tilde{\gamma}_h(c'))$ for all $h \in \mathbb{Z}^k$ and $c, c' \in C$. Thus, $\eta = \varphi \oplus \psi\colon A \oplus B \to M_2(C)$ is a non-degenerate $\alpha \oplus \beta$ - δ-equivariant $*$-homomorphism.

By additivity of K-theory, $K_*(\eta) = K_*(\varphi) + K_*(\psi)$. Hence, $K_*(\eta)$ is an isomorphism, as $K_0(\eta) = K_0(\varphi)$ and $K_1(\eta) = K_1(\psi)$. If $k \in \mathbb{N}$, an iterative use of the naturality of the Pimsner-Voiculescu sequence and the Five Lemma yields that $K_*(\eta \rtimes \mathbb{Z}^k)$ is an isomorphism. If $k = \infty$, it follows from continuity of K-theory that $K_*(\eta \rtimes \mathbb{Z}^\infty)$ is an isomorphism, since $K_*(\eta \rtimes \mathbb{Z}^k)$ is an isomorphism for every $k \in \mathbb{N}$.

Let $u\colon \mathbb{Z}^k \to \mathcal{U}(\mathcal{M}((A \oplus B) \rtimes_{\alpha\oplus\beta} \mathbb{Z}^k))$ and $\tilde{u}\colon \mathbb{Z}^k \to \mathcal{U}(\mathcal{M}(M_2(C) \rtimes_\delta \mathbb{Z}^k))$ denote the canonical representations, respectively. The covariant pair given by the natural inclusion $A \hookrightarrow 1_{\mathcal{M}(A)}((A \oplus B) \rtimes_{\alpha\oplus\beta} \mathbb{Z}^k)1_{\mathcal{M}(A)}$ and the unitary representation $1_{\mathcal{M}(A)}u_h$, $h \in \mathbb{Z}^k$, gives rise to a $*$-homomorphism $\Phi_A\colon A \rtimes_\alpha \mathbb{Z}^k \to (A \oplus B) \rtimes_{\alpha\oplus\beta} \mathbb{Z}^k$. Similarly, we define $\Phi_B\colon B \rtimes_\beta \mathbb{Z}^k \to (A \oplus B) \rtimes_{\alpha\oplus\beta} \mathbb{Z}^k$. It is easy to check that Φ_A and Φ_B are orthogonal and

$$\Phi_A \oplus \Phi_B\colon A \rtimes_\alpha \mathbb{Z}^k \oplus B \rtimes_\beta \mathbb{Z}^k \to (A \oplus B) \rtimes_{\alpha\oplus\beta} \mathbb{Z}^k$$

is an isomorphism. Moreover, let $\tilde{\varphi}\colon A \rtimes_\alpha \mathbb{Z}^k \to M_2(C) \rtimes_\delta \mathbb{Z}^k$ be the $*$-homomorphism induced by the covariant pair in $\mathcal{M}(e_{11}(M_2(C) \rtimes_\delta \mathbb{Z}^k)e_{11})$ given by the composition of the embedding $C \hookrightarrow M_2(C)$ into the upper left corner with φ and the unitary representation $e_{11}\tilde{u}_h$, $h \in \mathbb{Z}^k$. Define $\tilde{\psi}\colon B \rtimes_\beta \mathbb{Z}^k \to M_2(C) \rtimes_\delta \mathbb{Z}^k$ analogously by considering the embedding $C \hookrightarrow M_2(C)$ into the lower right corner. By construction, the following diagram commutes

$$\begin{tikzcd}
A \rtimes_\alpha \mathbb{Z}^k \oplus B \rtimes_\beta \mathbb{Z}^k \arrow[r, "\Phi_A\oplus\Phi_B", "\cong"'] \arrow[dr, "\tilde{\varphi}\oplus\tilde{\psi}"'] & (A \oplus B) \rtimes_{\alpha\oplus\beta} \mathbb{Z}^k \arrow[d, "\eta\rtimes\mathbb{Z}^k"] \\
 & M_2(C) \rtimes_\delta \mathbb{Z}^k
\end{tikzcd}$$

which shows that

$$K_*(\tilde{\varphi}) \oplus K_*(\tilde{\psi})\colon K_*(A \rtimes_\alpha \mathbb{Z}^k) \oplus K_*(B \rtimes_\beta \mathbb{Z}^k) \to K_*(M_2(C) \rtimes_\delta \mathbb{Z}^k)$$

is an isomorphism.

Let $\kappa'\colon M_2(C) \rtimes_\delta \mathbb{Z}^k \xrightarrow{\cong} M_2(C) \rtimes_{\gamma^{(2)}} \mathbb{Z}^k$ denote the isomorphism induced by the $\gamma^{(2)}$-cocycle w. Then the following diagram commutes and the proof is complete:

$$\begin{tikzcd}
A \rtimes_\alpha \mathbb{Z}^k \arrow[r, "\varphi\rtimes\mathbb{Z}^k"] \arrow[d, "\tilde{\varphi}"'] & C \rtimes_\gamma \mathbb{Z}^k \arrow[dr, "\mathrm{id}_{C\rtimes_\gamma\mathbb{Z}^k}\oplus 0"] & \\
M_2(C) \rtimes_\delta \mathbb{Z}^k \arrow[r, "\kappa'", "\cong"'] & M_2(C) \rtimes_{\gamma^{(2)}} \mathbb{Z}^k \arrow[r, "\cong"] & M_2(C \rtimes_\gamma \mathbb{Z}^k) \\
B \rtimes_\beta \mathbb{Z}^k \arrow[r, "\psi\rtimes\mathbb{Z}^k"] \arrow[u, "\tilde{\psi}"] & C \rtimes_{\tilde{\gamma}} \mathbb{Z}^k \arrow[r, "\kappa", "\cong"'] & C \rtimes_\gamma \mathbb{Z}^k \arrow[u, "0\oplus\mathrm{id}_{C\rtimes_\gamma\mathbb{Z}^k}"']
\end{tikzcd}$$

□

Remark 4.2. Proposition 4.1 is true in a more general setting. In fact, $\mathbb{Z}^k$ could be replaced by any locally compact group G with the following property: If $\varphi\colon A \to B$ is an α - β-equivariant $*$-homomorphism such that $K_*(\varphi)$ is an isomorphism, then $K_*(\varphi \rtimes G)$ is an isomorphism as well.

Remark 4.3. Note that $K_1(M_{d^\infty}) = 0$ and the natural embedding $j\colon M_{d^\infty} \hookrightarrow \mathcal{B}_S$ induces an isomorphism between the corresponding K_0-groups. The invariance of $M_{d^\infty} \subset \mathcal{B}_S$ under the H^+-action α, see Remark 2.8, yields a non-degenerate $*$-homomorphism $j_\infty\colon M_{d^\infty,\infty} \to \mathcal{B}_{S,\infty}$ between the minimal automorphic dilations for α, which is equivariant for the induced H-actions α_∞. From the concrete model of the minimal automorphic dilation as an inductive limit, see [26, Proof of Theorem 2.1], we conclude that $K_1(M_{d^\infty,\infty}) = 0$ and $K_0(j_\infty)$ is an isomorphism. Moreover, there is an isomorphism between $\mathcal{B}_S$ and $C(\Delta) \rtimes \mathbb{Z}$ that intertwines the actions of H^+, see Remark 2.16. It then follows from (7) and [26, Theorem 2.1] that $\alpha\colon H^+ \curvearrowright C(\Delta) \rtimes \mathbb{Z}$ dilates to $\widetilde{\alpha}\colon H \curvearrowright C_0(\Omega) \rtimes N$, where $\widetilde{\alpha}$ coincides with the H-action from Lemma 3.5. Consequently, there is an α_∞ - $\widetilde{\alpha}$-equivariant isomorphism $\mathcal{B}_{S,\infty} \xrightarrow{\cong} C_0(\Omega) \rtimes N$.

As in Section 3, let $\iota_N\colon C_0(\mathbb{R}) \hookrightarrow C_0(\mathbb{R}) \rtimes N$ denote the canonical embedding. Note that ι_N is non-degenerate and equivariant with respect to the H-actions β (and also β^{-1}).

Theorem 4.4. *The map*

$$K_*(j \rtimes^e H^+) \oplus K_*(\iota_N \rtimes H)\colon K_*(M_{d^\infty} \rtimes^e_\alpha H^+) \oplus K_*(C_0(\mathbb{R}) \rtimes_\beta H) \to K_*(\mathcal{Q}_S),$$

induced by the identifications $\mathcal{B}_S \rtimes^e_\alpha H^+ \cong \mathcal{Q}_S$ from (1) *and $(C_0(\mathbb{R}) \rtimes N) \rtimes_\beta H \cong \mathcal{Q}_S \otimes \mathcal{K}$ from Theorem 3.1, is an isomorphism.*

Proof. By combining Remark 4.3 with Lemma 3.5, there exist an H-action $\widetilde{\beta}$ on $C_0(\mathbb{R}) \rtimes N$ that is exterior equivalent to β^{-1}, and a non-degenerate α_∞ - $\widetilde{\beta}$-equivariant $*$-homomorphism $\psi\colon M_{d^\infty,\infty} \to C_0(\mathbb{R}) \rtimes N$, namely the one coming from the composition

$$M_{d^\infty,\infty} \xrightarrow{j_\infty} \mathcal{B}_{S,\infty} \xrightarrow{\cong} C_0(\Omega) \rtimes N \xrightarrow{\cong} C_0(\mathbb{R}) \rtimes N.$$

Since $K_1(j_\infty) = 0$ and $K_0(j_\infty)$ is an isomorphism by Remark 4.3, the same also holds for $K_1(\psi)$ and $K_0(\psi)$, respectively. Now Proposition 3.6 gives that $K_0(\iota_N)$ is trivial and $K_1(\iota_N)$ is an isomorphism. As ψ and ι_N are non-degenerate,

$$\begin{aligned} &K_*(\kappa \circ (\psi \rtimes H)) \oplus K_*(\iota_N \rtimes H)\colon \\ &\qquad K_*(M_{d^\infty,\infty} \rtimes_{\alpha_\infty} H) \oplus K_*(C_0(\mathbb{R}) \rtimes_{\beta^{-1}} H) \to K_*((C_0(\mathbb{R}) \rtimes N) \rtimes_{\beta^{-1}} H) \end{aligned}$$

is an isomorphism by Proposition 4.1, where $\kappa\colon (C_0(\mathbb{R}) \rtimes N) \rtimes_{\widetilde{\beta}} H \xrightarrow{\cong} (C_0(\mathbb{R}) \rtimes N) \rtimes_{\beta^{-1}} H$ denotes the isomorphism induced by a fixed β^{-1}-cocycle defining $\widetilde{\beta}$. Since $K_*(j \rtimes^e H^+)$ corresponds to $K_*(j_\infty \rtimes H)$ under the isomorphisms induced by the minimal automorphic dilations, we also get that $K_*(j \rtimes^e H^+)$ corresponds to $K_*(\kappa \circ (\psi \rtimes H))$ under the isomorphism $K_*(\mathcal{B}_S \rtimes^e_\alpha H^+) \cong K_*((C_0(\mathbb{R}) \rtimes N) \rtimes_{\beta^{-1}} H)$. As $\mathcal{B}_S \rtimes^e_\alpha H^+ \cong \mathcal{Q}_S$ by (1) and $(C_0(\mathbb{R}) \rtimes N) \rtimes_{\beta^{-1}} H \cong (C_0(\mathbb{R}) \rtimes N) \rtimes_\beta H \cong \mathcal{Q}_S \otimes \mathcal{K}$ by Theorem 3.1, the conclusion follows. □

We will now show that the two summands appearing in Theorem 4.4 correspond to the torsion and the free part of $K_*(\mathcal{Q}_S)$, respectively.

Proposition 4.5. *For $i = 0, 1$, $K_i(C_0(\mathbb{R}) \rtimes_\beta H)$ is the free abelian group in $2^{|S|-1}$ generators.*

Proof. The result holds for any non-trivial subgroup H of $\mathbb{Q}^\times_+$, and we prove it in generality, not necessarily requiring H to be generated by S. Suppose that k is the (possibly infinite) rank of H. Let $\{h_i : 0 \leq i \leq k\}$ be a minimal generating set for H. For $t \in [0, 1]$ and

$1 \leq i \leq k$, define $\tilde{\beta}_{h_i,t} \in \mathrm{Aut}(C_0(\mathbb{R}))$ by $\tilde{\beta}_{h_i,t}(f)(s) = f((th_i^{-1} + 1 - t)s)$. Note that $\tilde{\beta}_{h_i,t}$ is indeed an automorphism as $h_i > 0$. Since multiplication on $\mathbb{R}$ is commutative, we see that for each $t \in [0,1]$, $\{\tilde{\beta}_{h_i,t}\}_{1 \leq i \leq k}$ defines an H-action. Let $\gamma\colon H \curvearrowright C_0([0,1], C_0(\mathbb{R}))$ be the action given by $\gamma_{h_i}(f)(t) = \tilde{\beta}_{h_i,t}(f(t))$. We have the following short exact sequence of C^*-algebras:

$$C_0((0,1], C_0(\mathbb{R})) \rtimes_\gamma H \hookrightarrow C_0([0,1], C_0(\mathbb{R})) \rtimes_\gamma H \overset{\mathrm{ev}_0 \rtimes H}{\twoheadrightarrow} C_0(\mathbb{R}) \rtimes_{\mathrm{id}} H$$

The Pimsner-Voiculescu sequence shows that $K_*(C_0((0,1], C_0(\mathbb{R})) \rtimes_\gamma H) = 0$, where we also use continuity of K-theory if $k = \infty$. The six-term exact sequence corresponding to the above extension now yields that $K_*(\mathrm{ev}_0 \rtimes H)$ is an isomorphism. A similar argument shows that $K_*(\mathrm{ev}_1 \rtimes H)$ is an isomorphism. We therefore conclude that for $i = 0, 1$,

$$K_i(C_0(\mathbb{R}) \rtimes_\beta H) \cong K_i(C_0(\mathbb{R}) \rtimes_{\mathrm{id}} H) \cong K_i(C_0(\mathbb{R}) \otimes C^*(H)).$$

This completes the proof as $K_i(C_0(\mathbb{R}) \otimes C^*(H))$ is the free abelian group in 2^{k-1} generators. □

Proposition 4.6. *$K_*(M_{d^\infty} \rtimes^e_\alpha H^+)$ is a torsion group, which is finite if S is finite.*

Proof. As in Remark 2.8, we think of M_{d^∞} as the inductive limit $(M_p(\mathbb{C}), \iota_{p,pq})_{p,q \in H^+}$ with $\iota_{p,pq}\colon M_p(\mathbb{C}) \to M_{pq}(\mathbb{C})$ given by $e^{(p)}_{i,j} \otimes 1 \mapsto \sum_{k=0}^{q-1} e^{(pq)}_{i+pk,j+pk} \otimes 1$. With this perspective, α satisfies $\alpha_q(e^{(p)}_{m,m}) = e^{(pq)}_{qm,qm}$ for all $p, q \in H^+$ and $0 \leq m \leq p-1$. From this, one concludes that for $q \in H^+$, $K_0(\alpha_q)$ is given by multiplication with $1/q$ on $K_0(M_{d^\infty}) \cong N$. Hence, for $p \in S$, there exists a Pimsner-Voiculescu type exact sequence, see [36, Theorem 4.1] and also [11, Proof of Proposition 3.1],

$$0 \longrightarrow K_1(M_{d^\infty} \rtimes^e_{\alpha_p} \mathbb{N}) \longrightarrow N \xrightarrow{\frac{p-1}{p}} N \longrightarrow K_0(M_{d^\infty} \rtimes^e_{\alpha_p} \mathbb{N}) \longrightarrow 0$$

This shows that $K_1(M_{d^\infty} \rtimes^e_{\alpha_p} \mathbb{N}) = 0$ and $K_0(M_{d^\infty} \rtimes^e_{\alpha_p} \mathbb{N}) \cong N/(p-1)N$. In particular, $K_*(M_{d^\infty} \rtimes^e_{\alpha_p} \mathbb{N})$ is a torsion group. If S is finite, we can write $M_{d^\infty} \rtimes^e_\alpha H^+$ as an $|S|$-fold iterative crossed product by $\mathbb{N}$ and apply the Pimsner-Voiculescu type sequence repeatedly to get that $K_*(M_{d^\infty} \rtimes^e_\alpha H^+)$ is a torsion group. If S is infinite, we may use continuity of K-theory to conclude the claim from the case of finite S.

Finiteness of S implies finiteness of $K_*(M_{d^\infty} \rtimes^e_\alpha H^+)$ because $N/(p-1)N$ is finite for all $p \in S$, which follows from the forthcoming Lemma 6.11. □

Using Proposition 4.5 and 4.6, we record the following immediate consequence of the decomposition of $K_*(\mathcal{Q}_S)$ given in Theorem 4.4.

Corollary 4.7. *$K_*(\mathcal{Q}_S)$ decomposes as a direct sum of a free abelian group and a torsion group. More precisely, $K_*(j \rtimes^e H^+)$ is a split-injection onto the torsion subgroup and $K_*(\iota_N \rtimes H)$ is a split-injection onto the torsion free part of $K_*(\mathcal{Q}_S)$, respectively.*

5. The torsion subalgebra

Within this section we analyze the structure of $M_{d^\infty} \rtimes^e_\alpha H^+$ and its role relative to $\mathcal{Q}_S$ more closely. First, we show that the inclusion $M_{d^\infty} \hookrightarrow \mathcal{B}_S$ is equivariantly sequentially split with respect to the H^+-actions α in the sense of [2, Remark 3.17], see Proposition 5.1. According to [2], we thus get that $M_{d^\infty} \rtimes^e_\alpha H^+$ shares many structural properties with $\mathcal{B}_S \rtimes^e_\alpha H^+ \cong \mathcal{Q}_S$. Most importantly, $M_{d^\infty} \rtimes^e_\alpha H^+$ is a unital UCT Kirchberg algebra, see Corollary 5.2. By simplicity of $M_{d^\infty} \rtimes^e_\alpha H^+$, we conclude that this C^*-algebra is in fact isomorphic to the natural subalgebra $\mathcal{A}_S$ of $\mathcal{Q}_S$ that is generated by all the isometries $u^m s_p$ with $p \in S$ and $0 \leq m \leq p-1$, see Corollary 5.4. By Corollary 4.7, it thus follows that the canonical inclusion $\mathcal{A}_S \hookrightarrow \mathcal{Q}_S$ induces a split-injection onto the torsion subgroup of $K_*(\mathcal{Q}_S)$. Due to this remarkable feature, we call $\mathcal{A}_S$ the *torsion subalgebra* of $\mathcal{Q}_S$.

We then present two additional interesting perspectives on the torsion subalgebra $\mathcal{A}_S$. Firstly, $\mathcal{A}_S$ can be described as the boundary quotient of the right LCM subsemigroup $U = \{(m,p) : p \in H^+, 0 \leq m \leq p-1\}$ of $\mathbb{N} \rtimes H^+$ in the sense of [7], see Proposition 5.5. This yields a commutative diagram which might be of independent interest, see Remark 5.6.

Secondly, the boundary quotient perspective allows us to identify $\mathcal{A}_S$ for $k := |S| < \infty$ with the C^*-algebra of the k-graph $\Lambda_{S,\theta}$ consisting of a single vertex with p loops of color p for every $p \in S$, see Corollary 5.8. Quite intriguingly, $\Lambda_{S,\theta}$ differs from the canonical k-graph model $\Lambda_{S,\sigma}$ for $\bigotimes_{p\in S} \mathcal{O}_p$ only with respect to its factorization rules, see Remark 5.9. In fact, the corresponding C^*-algebras coincide for $|S| \leq 2$, see Proposition 5.10. After obtaining these intermediate results, we were glad to learn from Aidan Sims that, in view of Conjecture 5.11, it is reasonable to expect that the results for $|S| \leq 2$ already display the general form, i.e. that $\mathcal{A}_S$ is always isomorphic to $\bigotimes_{p\in S} \mathcal{O}_p$.

Proposition 5.1. *The embedding $M_{d^\infty} \hookrightarrow \mathcal{B}_S$ is α-equivariantly sequentially split.*

Proof. Let $\iota \colon M_{d^\infty} \hookrightarrow \prod_{p\in H^+} M_{d^\infty} \big/ \bigoplus_{p\in H^+} M_{d^\infty}$ denote the canonical inclusion as constant sequences and $\bar{\alpha}$ the induced action of H^+ on $\prod_{p\in H^+} M_{d^\infty} \big/ \bigoplus_{p\in H^+} M_{d^\infty}$ given by componentwise application of α_h for $h \in H$. Clearly, $\prod_{p\in H^+} M_{d^\infty} \big/ \bigoplus_{p\in H^+} M_{d^\infty}$ is canonically isomorphic to the sequence algebra of M_{d^∞}, $\prod_{n\in\mathbb{N}} M_{d^\infty} \big/ \bigoplus_{n\in\mathbb{N}} M_{d^\infty}$. In particular, this isomorphism intertwines $\bar{\alpha}$ and the natural H^+-action on the sequence algebra induced by α. We therefore need to construct a α - $\bar{\alpha}$-equivariant $*$-homomorphism $\chi \colon \mathcal{B}_S \to \prod_{p\in H^+} M_{d^\infty} \big/ \bigoplus_{p\in H^+} M_{d^\infty}$ making the following diagram commute:

$$\begin{tikzcd} M_{d^\infty} \arrow[rr, "\iota"] \arrow[dr, hook] & & \prod\limits_{p\in H^+} M_{d^\infty} \Big/ \bigoplus\limits_{p\in H^+} M_{d^\infty} \\ & \mathcal{B}_S \arrow[ur, dashed, "\chi"'] & \end{tikzcd} \tag{9}$$

Recall the inductive system $(M_p(\mathbb{C}) \otimes C^*(\mathbb{Z}), \iota_{p,pq})_{p,q\in H^+}$ from Remark 2.8 whose inductive limit is isomorphic to $\mathcal{B}_S$. The canonical subalgebra $M_p(\mathbb{C}) \subset M_p(\mathbb{C}) \otimes C^*(\mathbb{Z})$ can in this way be considered as a subalgebra of $M_{d^\infty} \subset \mathcal{B}_S$ in a natural way. For each $p \in H^+$, the map $\chi_p \colon M_p(\mathbb{C}) \otimes C^*(\mathbb{Z}) \to M_p(\mathbb{C})$ given by $\sum_{k=1}^n a_k \otimes u \mapsto \sum_{k=1}^n a_k$ is a $*$-homomorphism. Thus, the family $(\chi_p)_{p\in H^+}$ gives rise to a $*$-homomorphism

$$\chi' \colon \prod_{p\in H^+} M_p(\mathbb{C}) \otimes C^*(\mathbb{Z}) \to \prod_{p\in H^+} M_{d^\infty}.$$

Clearly, $\chi'\big(\bigoplus_{p\in H^+} M_p(\mathbb{C}) \otimes C^*(\mathbb{Z})\big) \subset \bigoplus_{p\in H^+} M_{d^\infty}$, so χ' induces a map

$$\chi \colon \prod_{p\in H^+} M_p(\mathbb{C}) \otimes C^*(\mathbb{Z}) \big/ \big(\bigoplus_{p\in H^+} M_p(\mathbb{C}) \otimes C^*(\mathbb{Z}) \big) \to \prod_{p\in H^+} M_{d^\infty} \big/ \bigoplus_{p\in H^+} M_{d^\infty}.$$

Using the inductive limit description of $\mathcal{B}_S$ from Remark 2.8, we can think of $\mathcal{B}_S$ as a subalgebra of $\prod_{p\in H^+} M_p(\mathbb{C}) \otimes C^*(\mathbb{Z}) \big/ \big(\bigoplus_{p\in H^+} M_p(\mathbb{C}) \otimes C^*(\mathbb{Z})\big)$. Moreover, because of the concrete realization of M_{d^∞} as the inductive limit associated with $(M_p(\mathbb{C}), \iota_{p,pq})_{p,q\in H^+}$, we have that χ restricts to the canonical embedding ι on M_{d^∞}. Hence, (9) is commutative, when we ignore the question of equivariance, or, in other words, ι is sequentially split as an ordinary $*$-homomorphism. However, we claim that we also have a commutative diagram

$$\begin{tikzcd} \prod\limits_{p\in H^+} M_{d^\infty}/\bigoplus\limits_{p\in H^+} M_{d^\infty} \arrow[r, "\bar{\alpha}_p"] & \prod\limits_{p\in H^+} M_{d^\infty}/\bigoplus\limits_{p\in H^+} M_{d^\infty} \\ \mathcal{B}_S \arrow[u, "\chi"] \arrow[r, "\alpha_p"'] & \mathcal{B}_S \arrow[u, "\chi"'] \end{tikzcd} \tag{10}$$

for each $p \in H^+$. Let us expand this diagram for fixed p and arbitrary $q \in H^+$ to:

$$\begin{tikzcd} \prod\limits_{p\in H^+} M_{d^\infty}/\bigoplus\limits_{p\in H^+} M_{d^\infty} \arrow[rrr, "\bar{\alpha}_p"] & & & \prod\limits_{p\in H^+} M_{d^\infty}/\bigoplus\limits_{p\in H^+} M_{d^\infty} \\ & M_q(\mathbb{C}) \arrow[ul, hook] \arrow[r, "\alpha_p"] & M_{pq}(\mathbb{C}) \arrow[ur, hook] & \\ & M_q(\mathbb{C})\otimes C^*(\mathbb{Z}) \arrow[u, "\chi_q"] \arrow[r, "\alpha_p"'] \arrow[dl, hook] & M_{pq}\otimes C^*(\mathbb{Z}) \arrow[u, "\chi_{pq}"'] \arrow[dr, hook] & \\ \mathcal{B}_S \arrow[uuu, "\chi"] \arrow[rrr, "\alpha_p"'] & & & \mathcal{B}_S \arrow[uuu, "\chi"'] \end{tikzcd} \tag{11}$$

It is clear that the four outer chambers are commutative, so we only need to check the centre. For every $0 \leq i, j \leq q-1$, we get

$$\chi_{pq} \circ \alpha_p(e_{i,j}^{(q)} \otimes u) = \chi_{pq}(e_{pi,pj}^{(pq)} \otimes u^p) = e_{pi,pj}^{(pq)} = \alpha_p \circ \chi_q(e_{i,j}^{(q)} \otimes u)$$

and therefore $\chi_{pq} \circ \alpha_p = \alpha_p \circ \chi_q$ on $M_q(\mathbb{C}) \otimes C^*(\mathbb{Z})$. This establishes the claim as we have $M_{d^\infty} = \varinjlim(M_q(\mathbb{C}), q \in H^+)$ and $\mathcal{B}_S = \varinjlim(M_q(\mathbb{C}) \otimes C^*(\mathbb{Z}), q \in H^+)$. ☐

Corollary 5.2. *The inclusion $M_{d^\infty} \rtimes_\alpha^e H^+ \to \mathcal{B}_S \rtimes_\alpha^e H^+$ is sequentially split. In particular, $M_{d^\infty} \rtimes_\alpha^e H^+$ is a UCT Kirchberg algebra.*

Proof. By Proposition 5.1, we know that $M_{d^\infty} \hookrightarrow \mathcal{B}_S$ is α-equivariantly sequentially split. As this inclusion preserves the units, we can use the universal property of the semigroup crossed products $M_{d^\infty} \rtimes_\alpha^e H^+$ and $\mathcal{B}_S \rtimes_\alpha^e H^+$ to obtain a commutative diagram of $*$-homomorphisms

$$\begin{tikzcd} M_{d^\infty} \rtimes_\alpha^e H^+ \arrow[rr, "\iota \rtimes^e H^+"] \arrow[dr] & & \left(\prod\limits_{p\in H^+} M_{d^\infty}/\bigoplus\limits_{p\in H^+} M_{d^\infty}\right) \rtimes_{\bar{\alpha}}^e H^+ \\ & \mathcal{B}_S \rtimes_\alpha^e H^+ \arrow[ur] & \end{tikzcd}$$

Again by the universal property of semigroup crossed products, there is a natural $*$-homomorphism

$$\psi\colon \left(\prod_{p\in H^+} M_{d^\infty}/\bigoplus_{p\in H^+} M_{d^\infty}\right) \rtimes_{\bar{\alpha}}^e H^+ \to \prod_{p\in H^+} M_{d^\infty} \rtimes_\alpha^e H^+/\bigoplus_{p\in H^+} M_{d^\infty} \rtimes_\alpha^e H^+$$

such that $\psi \circ (\iota \rtimes^e H^+)$ coincides with the standard embedding. This shows that the inclusion $M_{d^\infty} \rtimes_\alpha^e H^+ \to \mathcal{B}_S \rtimes_\alpha^e H^+$ is sequentially split. It now follows from [2, Theorem 2.9 (1)+(8)] that $M_{d^\infty} \rtimes_\alpha^e H^+$ is a Kirchberg algebra. Moreover, $M_{d^\infty} \rtimes_\alpha^e H^+$ satisfies the UCT by [2, Theorem 2.10]. We note that this part also follows from standard techniques combined with the central result of [26]. ☐

We will now see that simplicity enables us to identify $M_{d^\infty} \rtimes_\alpha^e H^+$ with the following natural subalgebra of $\mathcal{Q}_S$, whose name is justified by the next result.

Definition 5.3. The *torsion subalgebra* $\mathcal{A}_S$ of $\mathcal{Q}_S$ is the C^*-subalgebra of $\mathcal{Q}_S$ generated by $\{u^m s_p : p \in S, 0 \leq m \leq p-1\}$.

Note that for $S = \{p\}$, the subalgebra $\mathcal{A}_S$ is canonically isomorphic to $\mathcal{O}_p$.

Corollary 5.4. *The isomorphism $\mathcal{B}_S \rtimes_\alpha^e H^+ \xrightarrow{\cong} \mathcal{Q}_S$ from* (1) *restricts to an isomorphism $M_{d^\infty} \rtimes_\alpha^e H^+ \xrightarrow{\cong} \mathcal{A}_S$. In particular, the canonical inclusion $\mathcal{A}_S \hookrightarrow \mathcal{Q}_S$ induces a split-injection onto the torsion subgroup of $K_*(\mathcal{Q}_S)$.*

Proof. $\mathcal{A}_S$ contains the copy of $M_{d^\infty} \subset \mathcal{B}_S$ described in Remark 2.8. Together with $s_p, p \in S$, which are also contained in $\mathcal{A}_S$, this defines a covariant representation of (M_{d^∞}, α) inside $\mathcal{A}_S$. The resulting $*$-homomorphism $M_{d^\infty} \rtimes_\alpha^e H^+ \to \mathcal{A}_S$ is surjective. By Corollary 5.2, $M_{d^\infty} \rtimes_\alpha^e H^+$ is simple, so this map is an isomorphism. The second claim is due to Corollary 4.7. □

Let us continue with the representation of $\mathcal{A}_S$ as a boundary quotient. When $|S| = k < \infty$, this will lead us to a k-graph model for $\mathcal{A}_S$ that is closely related to the canonical k-graph representation for $\bigotimes_{p \in S} \mathcal{O}_p$, see Remark 5.9. Consider the subsemigroup $U := \{(m,h) \in \mathbb{N} \rtimes H^+ : 0 \leq m \leq h-1\}$ of $\mathbb{N} \rtimes H^+$. Observe that U is a right LCM semigroup because

$$(m,h)U \cap (m',h')U = \big((m,h)(\mathbb{N} \rtimes H^+) \cap (m',h')(\mathbb{N} \rtimes H^+)\big) \cap U \tag{12}$$

for all $(m,h), (m',h') \in U$, and $\mathbb{N} \rtimes H^+$ is right LCM. We note that U can be used to describe $\mathbb{N} \rtimes H^+$ as a Zappa-Szép product $U \bowtie \mathbb{N}$, where action and restriction are given in terms of the generator $1 \in \mathbb{N}$ and $(m,h) \in U$ by

$$1.(m,h) = \begin{cases} (m+1,h) & \text{if } m < h-1, \\ (0,h) & \text{if } m = h-1, \end{cases} \qquad 1|_{(m,h)} = \begin{cases} 0 & \text{if } m < h-1, \text{ and} \\ 1 & \text{if } m = h-1. \end{cases}$$

In the case of $H^+ = \mathbb{N}^\times$ this has been discussed in detail in [7, Subsection 3.2] and the very same arguments apply for the cases we consider here.

Proposition 5.5. *$\mathcal{A}_S$ is canonically isomorphic to the boundary quotient $\mathcal{Q}(U)$.*

Proof. Recall that $\mathcal{Q}(U)$ is the quotient of the full semigroup C^*-algebra $C^*(U)$ by relation (2). In particular, it is generated as a C^*-algebra by a representation v of U by isometries whose range projections are denoted $v_{(m,h)}v_{(m,h)}^* = e_{(m,h)U}$ for $(m,h) \in U$.

For every $h \in H^+$, we get a family of matrix units $(v_{(m,h)}v_{(n,h)}^*)_{0 \leq m,n \leq h-1}$ because

$$v_{(n,h)}^* v_{(m,h)} = v_{(n,h)}^* e_{(n,h)U \cap (m,h)U} v_{(m,h)} = \delta_{m,n} \quad \text{and} \quad \sum_{m=0}^{h-1} e_{(m,h)U} = 1$$

as $\{(m,h) : 0 \leq m \leq h-1\}$ is an accurate foundation set for U. That is to say that, for each $u \in U$, there is $0 \leq m \leq h-1$ such that $uU \cap (m,h)U \neq \emptyset$, and $(m,h)U \cap (n,h)U = \emptyset$ unless $m = n$, see [8] for further details. Since

$$\begin{aligned} v_{(m,h)}v_{(n,h)}^* &= v_{(m,h)}\big(\sum_{k=0}^{h'-1} e_{(k,h')U}\big)v_{(n,h)}^* \\ &= \sum_{k=0}^{h'-1} v_{(m+hk,hh')}v_{(n+hk,hh')}^* \end{aligned}$$

for each $h' \in H^+$, we see that $C^*(\{v_{(m,h)}v_{(n,h)}^* : h \in H^+, 0 \leq m,n \leq h-1\}) \subset \mathcal{Q}(U)$ is isomorphic to M_{d^∞}. In fact, we get a covariant representation for $(M_{d^\infty}, H^+, \alpha)$ as

$$v_{(0,p)}v_{(m,h)}v_{(n,h)}^*v_{(0,p)}^* = v_{(pm,ph)}v_{(pn,ph)}^*.$$

Thus we get a $*$-homomorphism $\varphi\colon \mathcal{A}_S \cong M_{d^\infty} \rtimes_\alpha^e H^+ \to \mathcal{Q}(U)$ given by $u^m s_h \mapsto v_{(m,h)}$, see Corollary 5.4. The map is surjective, and due to Corollary 5.2, the domain is simple so that φ is an isomorphism. $\square$

Remark 5.6. Conceptually, it seems that there is more to Proposition 5.5 than the proof entails: There is a commutative diagram

$$\begin{tikzcd} C^*(U) \arrow[r, hook, "\iota"] \arrow[d, two heads, "\pi_U"'] & C^*(\mathbb{N} \rtimes H^+) \arrow[d, two heads, "\pi_{\mathbb{N}\rtimes H^+}"] \\ \mathcal{Q}(U) \arrow[r, hook, "\varphi^{-1}"'] & \mathcal{Q}(\mathbb{N} \rtimes H^+) \end{tikzcd} \tag{13}$$

with ι induced by $U \subset \mathbb{N} \rtimes H^+$ and φ as in the proof of Proposition 5.5. The fact that ι is an injective $*$-homomorphism follows from [6, Proposition 3.6]: $N \rtimes H$ is amenable and hence $C^*(\mathbb{N} \rtimes H^+) \cong C^*_r(\mathbb{N} \rtimes H^+)$, see [5, Example 6.3], and similarly $C^*(U) \cong C^*_r(U)$. Note that the bottom row of (13) is given by $\mathcal{A}_S \hookrightarrow \mathcal{Q}_S$, see Proposition 2.12 and Proposition 5.5.

By Corollary 5.4 and Proposition 5.5, the torsion part of the K-theory of the boundary quotient of $\mathbb{N} \rtimes H^+$ arises from the boundary quotient of the distinguished submonoid U, which in fact sits inside $\mathcal{Q}(\mathbb{N} \rtimes H^+)$ in the natural way.

For the remainder of this section, we will assume that S is finite with cardinality k. This restriction is necessary in order to derive a k-graph model for $\mathcal{A}_S$, which we obtain via the boundary quotient representation of $\mathcal{A}_S$. Note that for $p, q \in \mathbb{N}^\times$ and $(m, n) \in \{0, \ldots, p-1\} \times \{0, \ldots, q-1\}$, there is a unique pair $(n', m') \in \{0, \ldots, q-1\} \times \{0, \ldots, p-1\}$ such that $m + pn = n' + qm'$. In other words, the map

$$\theta_{p,q}\colon \{0, \ldots, p-1\} \times \{0, \ldots, q-1\} \to \{0, \ldots, q-1\} \times \{0, \ldots, p-1\}$$

with $(m, n) \mapsto (n', m')$ determined by $n' + qm' = m + pn$ is bijective.

Remark 5.7. For each $p \in S$, we can consider the 1-graph given by a single vertex with p loops $(m, p), 0 \leq m \leq p-1$. If we think of the collection of these 1-graphs as the skeleton of a k-graph, i.e. the set of all edges of length at most 1, where the vertices for different p are identified, then the maps $\theta_{p,q}$ satisfy condition (2.8) in [17, Remark 2.3], and hence define a row-finite k-graph $\Lambda_{S,\theta}$. Indeed, this is obvious for $k = 2$. For $k \geq 3$, let $p, q, r \in S$ be pairwise distinct elements and fix $0 \leq m_t \leq t-1$ for $t = p, q, r$. We compute

$$\begin{aligned} m_p + p(m_q + qm_r) &= m_p + p(m_r^{(1)} + rm_q^{(1)}) &&= m_r^{(2)} + r(m_p^{(1)} + pm_q^{(1)}) \\ &= m_r^{(2)} + r(m_q^{(2)} + qm_p^{(2)}) &&= m_q^{(3)} + q(m_r^{(3)} + rm_p^{(2)}) \\ &= m_q^{(3)} + q(m_p^{(3)} + pm_r^{(4)}) &&= m_p^{(4)} + p(m_q^{(4)} + qm_r^{(4)}), \end{aligned}$$

where $0 \leq m_t^{(i)} \leq t-1$ for $t = p, q, r$ and $i = 1, \ldots, 4$ are uniquely determined by the $\theta_{s,t}$ for the respective values of s and t. The bijection from (2.8) in [17, Remark 2.3] now maps $((m_p, p), (m_q, q), (m_r, r))$ to $((m_p^{(4)}, p), (m_q^{(4)}, q), (m_r^{(4)}, r))$. It is easy to check that $m_t^{(4)} = m_t$ for $t = p, q, r$, which shows that condition (2.8) in [17, Remark 2.3] is valid. Applying [25, Definition 1.5] to the case of $\Lambda_{S,\theta}$, we see that $C^*(\Lambda_{S,\theta})$ is the universal C^*-algebra generated by isometries $(t_{(m,p)})_{p \in S, 0 \leq m \leq p-1}$ subject to the relations:

$$\text{(i) } t_{(m,p)} t_{(n,q)} = t_{(n',q)} t_{(m',p)} \text{ if } m + pn = n' + qm' \quad \text{and} \quad \text{(ii) } \sum_{m=0}^{p-1} t_{(m,p)} t^*_{(m,p)} = 1$$

for all $p, q \in S$.

Corollary 5.8. *$\mathcal{A}_S$ is isomorphic to $C^*(\Lambda_{S,\theta})$.*

Proof. We will work with $\mathcal{Q}(U)$ in place of $\mathcal{A}_S$ and invoke Proposition 5.5. Condition (i) guarantees that $(m,p) \mapsto t_{(m,p)}$ yields a representation of U by isometries as U is generated by (m,p) with $p \in S, 0 \leq m \leq p-1$, and $(m,p)(n,q) = (m+pn, pq) = (n',q)(m',p)$. (ii) holds for arbitrary $p \in H^+$ if we write $t_{(m,p)}$ for the product $t_{(m_1,p_1)} \cdots t_{(m_k,p_k)}$ where $(m_1,p_1)\cdots(m_k,p_k) = (m,p) \in U$ with $p_i \in S$. It is then straightforward to verify that we get a $*$-homomorphism $C^*(U) \to C^*(\Lambda_{S,\theta})$. Now let $F \subset U$ be a foundation set and set $h := \operatorname{lcm}(\{h' : (m',h') \in F \text{ for some } 0 \leq m' \leq h'-1\})$. Then $F_a := \{(m,h) : 0 \leq m \leq h-1\}$ is a foundation set that refines F. Therefore, it suffices to establish (2) for F_a in place of F. But as F_a is accurate, (2) takes the form $\sum_{m=0}^{h-1} t_{(m,h)} t_{(m,h)}^* = 1$, which follows from (ii) as explained in the proof of Proposition 5.5. Thus $v_{(m,p)} \mapsto t_{(m,p)}$ defines a surjective $*$-homomorphism $\mathcal{Q}(U) \to C^*(\Lambda_{S,\theta})$. By simplicity, see Corollary 5.2 and Proposition 5.5, this map is also injective. $\square$

Remark 5.9. Similar to $\Lambda_{S,\theta}$, we can also consider the row-finite k-graph $\Lambda_{S,\sigma}$ with $\sigma_{p,q}$ being the flip, i.e. $\sigma_{p,q}(m,n) := (n,m)$. That is to say, we keep the skeleton of $\Lambda_{S,\theta}$, but replace θ by σ. In this case, it is easy to see that $C^*(\Lambda_{S,\sigma}) \cong \bigotimes_{p \in S} \mathcal{O}_p$.

With regards to the K-theory of $\mathcal{Q}_S$, it is interesting to ask whether $C^*(\Lambda_{S,\theta})$ and $C^*(\Lambda_{S,\sigma})$ are isomorphic or not. At least for $|S| \leq 2$, the answer is known to be positive.

Proposition 5.10. *Let $p,q \geq 2$ be two relatively prime numbers and $S = \{p,q\}$. Then $\mathcal{A}_S \cong C^*(\Lambda_{S,\theta}) \cong C^*(\Lambda_{S,\sigma}) \cong \mathcal{O}_p \otimes \mathcal{O}_q$.*

Proof. We have seen in Corollary 5.8 and Remark 5.9 that the UCT Kirchberg algebras $\mathcal{A}_S$ and $\mathcal{O}_p \otimes \mathcal{O}_q$ are both expressible as C^*-algebras associated with row-finite 2-graphs $\Lambda_{S,\theta}$ and $\Lambda_{S,\sigma}$ sharing the same skeleton. The claim therefore follows from [16, Corollary 5.3]. $\square$

Concerning a generalization of Proposition 5.10 to the case of $|S| \geq 3$, we learned from Aidan Sims that the following conjecture for k-graphs might be true:

Conjecture 5.11. Suppose Λ and Λ' are row-finite k-graphs without sources such that $C^*(\Lambda)$ and $C^*(\Lambda')$ are unital, purely infinite and simple. If Λ and Λ' have the same skeleton, then the associated C^*-algebras are isomorphic.

Note that $C^*(\Lambda)$ and $C^*(\Lambda')$ are indeed unital UCT Kirchberg algebras, as separability, nuclearity and the UCT are automatically satisfied, see [25, Theorem 5.5]. We will come back to Conjecture 5.11 at the end of the next section.

6. Towards a classification of $\mathcal{Q}_S$

This final section provides a survey of the progress on the classification of $\mathcal{Q}_S$ that we achieve through the preceding sections and a spectral sequence argument for $K_*(\mathcal{A}_S)$, see Theorem 6.1 and Theorem 6.4. Recall that $N = \mathbb{Z}\big[\{\frac{1}{p} : p \in S\}\big]$ and g_S denotes the greatest common divisor of $\{p-1 : p \in S\}$. We begin by stating our main result.

Theorem 6.1. *Let $S \subset \mathbb{N}^\times \setminus \{1\}$ be a non-empty family of relatively prime numbers. Then the K-theory of $\mathcal{Q}_S$ satisfies*

$$K_i(\mathcal{Q}_S) \cong \mathbb{Z}^{2^{|S|-1}} \oplus K_i(\mathcal{A}_S), \quad i = 0,1,$$

where $K_i(\mathcal{A}_S)$ is a torsion group. Moreover, the following statements hold:

(a) If $g_S = 1$, then $K_i(\mathcal{Q}_S)$ is free abelian in $2^{|S|-1}$ generators for $i = 0,1$, and $[1] = 0$.
(b) If $|S| = 1$, then $(K_0(\mathcal{Q}_S), [1], K_1(\mathcal{Q}_S)) \cong (\mathbb{Z} \oplus \mathbb{Z}/g_S\mathbb{Z}, (0,1), \mathbb{Z})$.
(c) If $|S| = 2$, then $(K_0(\mathcal{Q}_S), [1], K_1(\mathcal{Q}_S)) \cong (\mathbb{Z}^2 \oplus \mathbb{Z}/g_S\mathbb{Z}, (0,1), \mathbb{Z}^2 \oplus \mathbb{Z}/g_S\mathbb{Z})$.

Remark 6.2. Note that for $S = \{p\}$, the torsion subalgebra $\mathcal{A}_S$ is canonically isomorphic to the Cuntz algebra $\mathcal{O}_p$. Therefore, Theorem 6.1 (b) recovers known results by Hirshberg [20, Example 1, p. 106] and Katsura [23, Example A.6]. Indeed, it is already clear from the presentation for $\mathcal{O}(E_{p,1})$ described in [23, Example A.6] that it coincides with $\mathcal{Q}_S$. Theorem 6.1 (c) shows an unexpected result for the K-groups of $\mathcal{Q}_S$ in the case of $S = \{p, q\}$ for two relatively prime numbers p and q with $g_S > 1$: $K_1(\mathcal{Q}_S)$ has torsion and is therefore, for instance, not a graph C^*-algebra, see [39, Theorem 3.2]. By virtue of (a), Theorem 6.1 also explains why $\mathcal{Q}_\mathbb{N}$ and $\mathcal{Q}_2$ have torsion free K-groups. More importantly, it shows that the presence of 2 in the family S is not the only way to achieve this. Indeed, S can contain at most one even number. If $g_S = 1$, then S must contain an even number, and there are many examples, e.g. S with $2^m + 1, 2n \in S$ for some $m, n \geq 1$.

In view of the Kirchberg-Phillips classification theorem [24, 37], we get the following immediate consequence of Theorem 6.1.

Corollary 6.3. *Let $S, T \subset \mathbb{N}^\times \setminus \{1\}$ be non-empty families of relatively prime numbers. Then $\mathcal{Q}_S \cong \mathcal{Q}_T$ implies $|S| = |T|$. Moreover, the following statements hold:*

- *(a) If $g_S = 1 = g_T$, then $\mathcal{Q}_S$ is isomorphic to $\mathcal{Q}_T$ if and only if $|S| = |T|$.*
- *(b) If $|S| \leq 2$, then $\mathcal{Q}_S$ is isomorphic to $\mathcal{Q}_T$ if and only if $|S| = |T|$ and $g_S = g_T$.*

Observe that the decomposition of $K_*(\mathcal{Q}_S)$ claimed in Theorem 6.1 follows from Corollary 4.7, Proposition 4.5 and Corollary 5.4. To prove our main result, it is therefore enough to establish the following theorem reflecting our present knowledge on the torsion subalgebra $\mathcal{A}_S$, an object which is certainly of interest in its own right.

Theorem 6.4. *Let $S \subset \mathbb{N}^\times \setminus \{1\}$ be a non-empty family of relatively prime numbers. Then the following statements hold:*

- *(a) If $g_S = 1$, then $\mathcal{A}_S \cong \mathcal{O}_2 \cong \bigotimes_{p \in S} \mathcal{O}_p$.*
- *(b) If $S = \{p\}$, then $\mathcal{A}_S \cong \mathcal{O}_p$.*
- *(c) If $S = \{p, q\}$ with $p \neq q$, then $\mathcal{A}_S \cong \mathcal{O}_p \otimes \mathcal{O}_q$.*
- *(d) For $|S| \geq 3$ and $g_S > 1$, $K_i(\mathcal{A}_S)$ is a torsion group in which the order of any element divides $g_S^{2^{|S|-2}}$. Moreover, $K_i(\mathcal{A}_S)$ is finite whenever S is finite.*

Note that in the case of infinite S with $g_S > 1$, part (d) still makes sense within the realm of supernatural numbers. Based on Theorem 6.1 and Theorem 6.4, we suspect that the general situation is in accordance with Conjecture 5.11:

Conjecture 6.5. For a family $S \subset \mathbb{N}^\times \setminus \{1\}$ of relatively prime numbers with $|S| \geq 2$, $\mathcal{A}_S$ is isomorphic to $\bigotimes_{p \in S} \mathcal{O}_p$. Equivalently, $\mathcal{Q}_S$ is the unital UCT Kirchberg algebra with

$$(K_0(\mathcal{Q}_S), [1], K_1(\mathcal{Q}_S)) = (\mathbb{Z}^{2^{|S|-1}} \oplus (\mathbb{Z}/g_S\mathbb{Z})^{2^{|S|-2}}, (0, e_1), \mathbb{Z}^{2^{|S|-1}} \oplus (\mathbb{Z}/g_S\mathbb{Z})^{2^{|S|-2}}),$$

where $e_1 = (\delta_{1,j})_j \in (\mathbb{Z}/g_S\mathbb{Z})^{2^{|S|-2}}$. In particular, if $S, T \subset \mathbb{N}^\times \setminus \{1\}$ are non-empty sets of relatively prime numbers, then $\mathcal{Q}_S$ is isomorphic to $\mathcal{Q}_T$ if and only if $|S| = |T|$ and $g_S = g_T$.

Remark 6.6. It follows from Theorem 6.1 and Theorem 6.4 (d) that the K-theory of $\mathcal{Q}_S$ is finitely generated if and only if S is finite. Consequently, when S is finite the defining relations of $\mathcal{Q}_S$ from Defintion 2.1 are *stable*, see [15, Corollary 4.6] and [31, Chapter 14].

For the proof of Theorem 6.4, we will employ the isomorphism $\mathcal{A}_S \cong M_{d^\infty} \rtimes_\alpha^e H$ and make use of a spectral sequence by Kasparov constructed in [22, 6.10]. Let us briefly review the relevant ideas and refer to [1] for a detailed exposition. Given a C^*-dynamical system $(B, \beta, \mathbb{Z}^k)$, we can consider its *mapping torus*

$$\mathcal{M}_\beta(B) := \left\{ f \in C(\mathbb{R}^k, B) \ : \ \beta_z(f(x)) = f(x+z) \text{ for all } x \in \mathbb{R}^k,\ z \in \mathbb{Z}^k \right\}.$$

It is well-known that $K_*(\mathcal{M}_\beta(B))$ is isomorphic to $K_{*+k}(B \rtimes_\beta \mathbb{Z}^k)$, see e.g. [1, Section 1]. The mapping torus admits a finite cofiltration

$$\mathcal{M}_\beta(B) = F_k \xrightarrow{\pi_k} F_{k-1} \xrightarrow{\pi_{k-1}} \cdots \xrightarrow{\pi_1} F_0 = A \xrightarrow{\pi_0} F_{-1} = 0 \tag{14}$$

arising from the filtration of $\mathbb{R}^k$ by its skeletons

$$\emptyset = X_{-1} \subset \mathbb{Z}^k = X_0 \subset X_1 \subset \cdots \subset X_k = \mathbb{R}^k,$$

where $X_\ell := \{(x_1, \ldots, x_k) \in \mathbb{R}^k : |\{1 \leq i \leq k : x_i \in \mathbb{R} \setminus \mathbb{Z}\}| \leq \ell\}$.

As for filtrations of C^*-algebras by closed ideals [42], there is a standard way relying on Massey's technique of exact couples [32, 33] of associating a spectral sequence to a given finite cofiltration of a C^*-algebra. In this way, the cofiltration (14) yields a spectral sequence $(E_\ell, d_\ell)_{\ell \geq 1}$ that converges to $K_*(\mathcal{M}_\beta(B)) \cong K_{*+k}(B \rtimes_\beta \mathbb{Z}^k)$. Using Savinien-Bellissard's [41] description of the E_1-term, we can summarize as follows.

Theorem 6.7 (cf. [22, 6.10], [41, Theorem 2] and [1, Corollary 2.5])**.**
Let $(B, \beta, \mathbb{Z}^k)$ be a C^*-dynamical system. There exists a cohomological spectral sequence $(E_\ell, d_\ell)_{\ell \geq 1}$ converging to $K_*(\mathcal{M}_\beta(B)) \cong K_{*+k}(B \rtimes_\beta \mathbb{Z}^k)$. The E_1-term is given by

$$\begin{aligned} &E_1^{p,q} := K_q(B) \otimes_{\mathbb{Z}} \Lambda^p(\mathbb{Z}^k), \ \textit{with} \\ &d_1^{p,q} \colon E_1^{p,q} \to E_1^{p+1,q}, \quad x \otimes e \mapsto \sum_{j=1}^{k} (K_q(\beta_j) - \mathrm{id})(x) \otimes (e_j \wedge e). \end{aligned}$$

Furthermore, the spectral sequence collapses at the $(k+1)$th page, so that $E_\infty = E_{k+1}$.

By Bott periodicity, we have that $(E_\ell^{p,q+2}, d_\ell^{p,q+2}) = (E_\ell^{p,q}, d_\ell^{p,q})$ for all $p, q \in \mathbb{Z}$. In particular, the E_∞-term reduces to $E_\infty^{p,q}$ with $p \in \mathbb{Z}$ and $q = 0, 1$.

Remark 6.8. Let us recall the meaning of convergence of the spectral sequence $(E_\ell, d_\ell)_{\ell \geq 1}$. For $q = 0, 1$, consider the diagram

$$K_q(\mathcal{M}_\gamma(B)) = K_q(F_k) \longrightarrow K_q(F_{k-1}) \longrightarrow \cdots \longrightarrow K_q(F_0) \longrightarrow K_q(F_{-1}) = 0.$$

Define $\mathcal{F}_p K_q(\mathcal{M}_\beta(B)) := \ker(K_q(\mathcal{M}_\beta(B)) \to K_q(F_p))$ for $p = -1, \ldots, k$, and observe that this gives rise to a filtration of abelian groups

$$0 \hookrightarrow \mathcal{F}_{k-1} K_q(\mathcal{M}_\beta(B)) \hookrightarrow \cdots \hookrightarrow \mathcal{F}_{-1} K_q(\mathcal{M}_\beta(B)) = K_q(\mathcal{M}_\beta(B)).$$

One can now show the existence of exact sequences

$$0 \longrightarrow \mathcal{F}_p K_{p+q}(\mathcal{M}_\beta(B)) \longrightarrow \mathcal{F}_{p-1} K_{p+q}(\mathcal{M}_\beta(B)) \longrightarrow E_\infty^{p,q} \longrightarrow 0, \tag{15}$$

or in other words, there are isomorphisms

$$E_\infty^{p,q} \cong \mathcal{F}_{p-1} K_{p+q}(\mathcal{M}_\beta(B)) / \mathcal{F}_p K_{p+q}(\mathcal{M}_\beta(B)).$$

Hence, the E_∞-term determines the K-theory of $\mathcal{M}_\beta(B)$, and thus of $B \rtimes_\beta \mathbb{Z}^k$, up to group extension problems.

Let us now turn to the K-theory of $M_{d^\infty} \rtimes_\alpha^e H^+$. By Laca's dilation theorem [26], see also Remark 4.3, we may and will determine the K-theory of the dilated crossed product $M_{d^\infty,\infty} \rtimes_{\alpha_\infty} H$ instead. Fix a natural number $1 \leq k \leq |S|$ and observe that $H_k \cong \mathbb{Z}^k$. Let $\alpha_\infty(k)$ be the H_k-action on $M_{d^\infty,\infty}$ induced by the k smallest elements $p_1 < p_2 < \cdots < p_k$ of S. It follows from the proof of Proposition 4.6 that $K_0(\alpha_{\infty,p_\ell})$ is given by multiplication with $1/p_\ell$ on $K_0(M_{d^\infty}) \cong N$. It turns out to be more convenient to work with the action $\alpha_\infty^{-1}(k)$ given by the inverses of the α_ℓ, whose crossed product is canonically isomorphic to $M_{d^\infty,\infty} \rtimes_{\alpha_\infty(k)} H_k$.

Let $(E_\ell, d_\ell)_{\ell\geq 1}$ denote the spectral sequence associated with $\alpha_\infty^{-1}(k)$. As $K_1(M_{d^\infty}) = 0$, it follows directly from Theorem 6.7 that $E_1^{p,1} = 0$ for all $p \in \mathbb{Z}$. Moreover, according to Theorem 6.7, $d_1^{p,0}\colon N \otimes_{\mathbb{Z}} \Lambda^p(\mathbb{Z}^k) \to N \otimes_{\mathbb{Z}} \Lambda^p(\mathbb{Z}^k)$, $p \in \mathbb{Z}$, is given by

$$d_1^{p,0}(x \otimes e) = \textstyle\sum_{\ell=1}^k (p_\ell - 1)x \otimes e_\ell \wedge e = \sum_{\ell=1}^k x \otimes (p_\ell - 1)e_\ell \wedge e.$$

In other words, $d_1^{p,0} = \mathrm{id}_N \otimes h^p$ with

$$h^p\colon \Lambda^p(\mathbb{Z}^k) \to \Lambda^{p+1}(\mathbb{Z}^k), \quad h^p(e) = \textstyle\sum_{\ell=1}^k (p_\ell - 1)e_\ell \wedge e. \tag{16}$$

To obtain $E_2^{p,0}$, we therefore compute the cohomology of the complex $(\Lambda^p(\mathbb{Z}^k), h^p)_{p\in\mathbb{Z}}$. To do so, we consider h^p as a matrix $A_p \in M_{\binom{k}{p+1}\times\binom{k}{p}}(\mathbb{Z})$, where the identification is taken with respect to the canonical bases of $\Lambda^p(\mathbb{Z}^k)$ and $\Lambda^{p+1}(\mathbb{Z}^k)$ in lexicographical ordering. The computation then mainly reduces to determining the Smith normal form of A_p.

Theorem 6.9 (Smith normal form)**.** *Let A be a non-zero $m \times n$-matrix over a principal ideal domain R. There is an invertible $m \times m$-matrix S and an invertible $n \times n$-matrix T over R, so that*

$$D := SAT = \mathrm{diag}(\delta_1, \dots, \delta_r, 0, \dots, 0)$$

for some $r \leq \min(m, n)$ and non-zero $\delta_i \in R$ satisfying $\delta_i | \delta_{i+1}$ for $1 \leq i \leq r - 1$. The elements δ_i are unique up to multiplication with some unit and are called elementary divisors *of A. The diagonal matrix D is called a* Smith normal form *of A. The δ_i can be computed as*

$$\delta_1 = d_1(A), \quad \delta_i = \tfrac{d_i(A)}{d_{i-1}(A)}, \tag{17}$$

where $d_i(A)$, called the i-th determinant divisor*, is the greatest common divisor of all $i \times i$-minors of A.*

Of course, D can only be a diagonal matrix if $m = n$. The notation in Theorem 6.9 is supposed to mean that D is the $m \times n$ matrix over R with the $\min(m, n) \times \min(m, n)$ left upper block matrix being $\mathrm{diag}(\delta_1, \dots, \delta_r, 0, \dots, 0)$ and all other entries being zero.

For each $1 \leq k \leq |S|$, set $g_k := \gcd(\{p_\ell - 1 : \ell = 1, \cdots, k\})$.

Lemma 6.10. *The group $\ker(h^p)/\operatorname{im}(h^{p-1})$ is isomorphic to $(\mathbb{Z}/g_k\mathbb{Z})^{\binom{k-1}{p-1}}$ for $1 \leq p \leq k$ and vanishes otherwise.*

Proof. For $p \in \mathbb{Z}$, let $D_p = S_p A_p T_p$ denote the Smith normal form of A_p with elementary divisors $\delta_1^{(p)}, \dots, \delta_{r_p}^{(p)}$. As $\Lambda^p(\mathbb{Z}^k) = 0$ unless $0 \leq p \leq k$, $\ker(h^p)/\operatorname{im}(h^{p-1})$ vanishes if $p < 0$ or $p \geq k + 1$.

If $p = 0$, then $h^0\colon \mathbb{Z} \to \mathbb{Z}^k$ is given by $A_0 = (p_1 - 1, \dots, p_k - 1)$. Thus we have $r_0 = 1$ and $\delta_1^{(0)} = g_k$. Moreover, h^0 is injective, so $\ker(h^0)/\operatorname{im}(h^{-1}) = 0$.

Likewise, $p = k$ is simple as $h^{k-1}\colon \mathbb{Z}^k \to \mathbb{Z}$ is given by $A_{k-1} = (p_1 - 1, \dots, p_k - 1)^t$ and h^k is zero because $\Lambda^{k+1}(\mathbb{Z}^k) = 0$. Therefore, $r_{k-1} = 1$ and $\delta_1^{(k-1)} = g_k$, and hence

$$\ker(h^k)/\operatorname{im}(h^{k-1}) = \mathbb{Z}/g_k\mathbb{Z} = (\mathbb{Z}/g_k\mathbb{Z})^{\binom{k-1}{k-1}}.$$

As this completes the proof for $p \leq 0$ and $p \geq k$, we will assume $1 \leq p \leq k - 1$ from now on.

We start by showing that for $\ell = 1, \cdots, k$, the matrix A_p contains a $\binom{k-1}{p} \times \binom{k-1}{p}$ diagonal matrix with entries $\pm(p_\ell - 1)$ (obtained by deleting suitable rows and columns). This will allow us to conclude that A_p has a $j \times j$-minor equal to $\pm(p_\ell - 1)^j$ for each $j = 1, \dots, \binom{k-1}{p}$. Thus we obtain that $d_j(A_p)$ divides g_k^j for $j = 1, \dots, \binom{k-1}{p}$:

First, keep only those columns of A_p which correspond to basis elements $e_{i_1} \wedge \ldots \wedge e_{i_p} \in \Lambda^p(\mathbb{Z}^k)$ satisfying $\ell \neq i_j$ for all $j = 1, \dots, p$. As this amounts to choosing p elements out of $k - 1$ without order and repetition, we are left with $\binom{k-1}{p}$ columns (out of $\binom{k}{p}$). Next, we restrict to those rows which correspond to basis elements $e_{i_1} \wedge \ldots \wedge e_{i_{p+1}} \in \Lambda^{p+1}(\mathbb{Z}^k)$ satisfying $\ell = i_j$ for some (necessarily unique) $j = 1, \ldots, p-1$. Here again $\binom{k-1}{p}$ rows (out of $\binom{k}{p+1}$) remain. The resulting matrix describes the linear map

$$\Lambda^p(\mathbb{Z}^k) \supset \mathbb{Z}^{\binom{k-1}{p}} \to \mathbb{Z}^{\binom{k-1}{p}} \subset \Lambda^{p+1}(\mathbb{Z}^k),\ e_{i_1} \wedge \ldots \wedge e_{i_p} \mapsto (p_\ell - 1) \cdot e_\ell \wedge e_{i_1} \wedge \ldots \wedge e_{i_p},$$

which is nothing but a diagonal matrix of size $\binom{k-1}{p}$ with entries $\pm(p_\ell - 1)$. As explained above, we thus obtain that $d_j(A_p) | g_k^j$ for $j = 1, \ldots, \binom{k-1}{p}$.

We will now show that the converse holds as well, i.e. $g_k^j | d_j(A_p)$ for $j = 1, \ldots, \binom{k-1}{p}$. Note that every 1×1-minor is either zero or $p_\ell - 1$ for some $\ell = 1, \ldots, k$. This shows that $d_1(A_p) = g_k$ for $1 \leq p \leq k-1$. Let $1 \leq j \leq \binom{k-1}{p} - 1$ and assume that $d_j(A_p) = g_k^j$. Let L be any $(j+1) \times (j+1)$-matrix arising from A_p by deleting sums and rows. By the Laplace expansion theorem, the determinant of L is given as a linear combination of some of its $j \times j$-minors. The coefficients in the linear combination all are entries of L. The occurring minors are all $j \times j$-minors of A_p. Hence, $g_k^j | \det(L)$ by assumption. In fact, we have $g_k^{j+1} | \det(L)$ because all entries in A_p are divisible by g_k. Altogether, $d_j(A_p) = g_k^j$ for $j = 1, \ldots, \binom{k-1}{p}$ and we have shown that for $p = 1, \ldots, k-1$, $r_p \geq \binom{k-1}{p}$ and $\delta_j^{(p)} = g_k$ for $j = 1, \ldots, \binom{k-1}{p}$.

Since A_p and D_p have isomorphic kernel and image, our considerations show that

$$\binom{k-1}{p} \leq \operatorname{rank}(\operatorname{im}(h^p)) \qquad \text{and} \qquad \operatorname{rank}(\ker(h^p)) \leq \binom{k}{p} - \binom{k-1}{p} = \binom{k-1}{p-1}.$$

By $h^{p+1} \circ h^p = 0$, we conclude that $\operatorname{rank}(\ker(h^{p+1})) = \operatorname{rank}(\operatorname{im}(h^p)) = \binom{k-1}{p}$ which implies $r_p = \binom{k-1}{p}$. Moreover, $h^p \circ h^{p-1} = 0$ forces $T_p^{-1} S_{p-1}^{-1}(\operatorname{im}(D_{p-1})) \subset \ker(D_p)$ or, equivalently, $\operatorname{im}(D_{p-1}) \subset \ker(D_p T_p^{-1} S_{p-1}^{-1})$. Since

$$\operatorname{im}(D_{p-1}) = g_k \mathbb{Z}^{\binom{k-1}{p-1}} \oplus \{0\}^{\binom{k}{p} - \binom{k-1}{p-1}}$$

has the same rank as $\ker(D_p T_p^{-1} S_{p-1}^{-1})$, it means that

$$\ker(D_p T_p^{-1} S_{p-1}^{-1}) = \mathbb{Z}^{\binom{k-1}{p-1}} \oplus \{0\}^{\binom{k}{p} - \binom{k-1}{p-1}}.$$

Moreover, S_{p-1} is an automorphism of $\mathbb{Z}^{\binom{k}{p}}$ that restricts both to an isomorphism $\ker(A_p) \xrightarrow{\cong} \ker(D_p T_p^{-1} S_{p-1}^{-1})$ and to an isomorphism $\operatorname{im}(A_{p-1}) \xrightarrow{\cong} \operatorname{im}(D_{p-1})$. Hence,

$$\begin{aligned} \ker(h^p)/\operatorname{im}(h^{p-1}) \quad &= \quad \ker(A_p)/\operatorname{im}(A_{p-1}) \quad \cong \quad \ker(D_p T_p^{-1} S_{p-1}^{-1})/\operatorname{im}(D_{p-1}) \\ &\cong \quad (\mathbb{Z}/g_k\mathbb{Z})^{\binom{k-1}{p-1}}. \end{aligned}$$

□

Lemma 6.10 now allows us to compute the E_2-term of the spectral sequence associated to $\alpha_\infty^{-1}(k)\colon H_k \curvearrowright M_{d^\infty,\infty}$ by appealing to the following simple, but useful observation.

Lemma 6.11. *The group $N/g_S N$ is isomorphic to $\mathbb{Z}/g_S\mathbb{Z}$. Moreover, for every $1 \leq k \leq |S|$, the group $N/g_k N$ is isomorphic to a subgroup of $\mathbb{Z}/g_k\mathbb{Z}$.*

Proof. Recall that S consists of relatively prime numbers, $N = \mathbb{Z}\big[\{\frac{1}{p} : p \in S\}\big]$ and let us simply write g for $g_S = \gcd(\{p-1 : p \in S\})$. The map

$$N/gN \to \mathbb{Z}/g\mathbb{Z}, \quad \tfrac{1}{r} + gN \mapsto s + g\mathbb{Z},$$

where r is a natural number and s is the unique solution in $\{0, 1, \ldots, g-1\}$ of $rs = 1 \pmod{g}$, defines a group homomorphism. To see this, note first that for every $p \in P$ there is a $q \in S$ such that $p|q$, i.e. $\gcd(q-1, p) = 1$. Therefore, $\gcd(g, p) = 1$ for all $p \in P$. If $\frac{1}{r} \in N$, then all the prime factors of r come from P, and it follows that $\gcd(g,r)=1$. Thus, the above map is well-defined and extends by addition to the whole domain. Moreover, every s appearing as a solution is relatively prime with g, meaning that the kernel is gN, i.e. the map is injective. Finally, the inverse map is given by $1+g\mathbb{Z}\mapsto 1+gN$.

For the second part, set $g_k' = g_k/\max(\gcd(g_k,r))$, where the maximum is taken over all natural numbers r such that $\frac{1}{r}\in N$, i.e. g_k' is the largest number dividing g_k so that $\gcd(g_k',r)=1$ for all such r. Then $g_k'N = g_kN$ and a similar proof as above shows that $N/g_kN = N/g_k'N \cong \mathbb{Z}/g_k'\mathbb{Z}$. $\square$

Proposition 6.12. *For every $1\le k\le |S|$, the respective group $E_2^{p,0}$ is isomorphic to a subgroup of $(\mathbb{Z}/g_k\mathbb{Z})^{\binom{k-1}{p-1}}$ for $1\le p\le k$, and vanishes otherwise. $E_2^{p,1}$ vanishes for $p\in\mathbb{Z}$.*

Proof. Note that N is torsion free and hence a flat module over $\mathbb{Z}$. Thus, an application of Lemma 6.10 yields

$$\begin{aligned} E_2^{p,0} &= \ker(\mathrm{id}_N\otimes h^p)/\operatorname{im}(\mathrm{id}_N\otimes h^{p-1}) \cong N\otimes_{\mathbb{Z}}\ker(h^p)/\operatorname{im}(h^{p-1})\\ &\cong N\otimes_{\mathbb{Z}}(\mathbb{Z}/g_k\mathbb{Z})^{\binom{k-1}{p-1}} \cong (N\otimes_{\mathbb{Z}}\mathbb{Z}/g_k\mathbb{Z})^{\binom{k-1}{p-1}} \cong (N/g_kN)^{\binom{k-1}{p-1}}\end{aligned}$$

and Lemma 6.11 shows that N/g_kN is isomorphic to a subgroup of $\mathbb{Z}/g_k\mathbb{Z}$. The second claim follows from the input data. $\square$

Remark 6.13. Assume that $g_k=1$ for some $1\le k\le |S|$, $k<\infty$, and let $k\le \ell\le |S|$ be a natural number. If $(E_i^{p,q})_{i\ge 1}$ denotes the spectral sequence associated with $\alpha_\infty^{-1}(\ell)$, then Proposition 6.12 yields $E_2^{p,0}=0$ for all $p\in\mathbb{Z}$.

Proof of Theorem 6.4. Let $k\ge 1$ be finite with $k\le |S|$. The main idea is to use the E_∞-term of the spectral sequence associated with $\alpha_\infty^{-1}(k)$ to compute $K_*(M_{d^\infty}\rtimes^e_{\alpha(k)} H_k^+)$, up to certain group by employing convergence of this spectral sequence, see Theorem 6.7. Recall the general form (15) of the extension problems involved. Since $\mathcal{F}_kK_q(\mathcal{M}_{\alpha_\infty}(M_{d^\infty,\infty}))=0$ and hence $\mathcal{F}_{k-1}K_q(\mathcal{M}_{\alpha_\infty}(M_{d^\infty,\infty}))\cong E_\infty^{k,q-k}$, we face k iterative extensions of the form:

$$\begin{array}{ccccc} E_\infty^{k,q-k} & \hookrightarrow & \mathcal{F}_{k-2}K_q(\mathcal{M}_{\alpha_\infty}(M_{d^\infty,\infty})) & \twoheadrightarrow & E_\infty^{k-1,q-k+1}\\ \mathcal{F}_{k-2}K_q(\mathcal{M}_{\alpha_\infty}(M_{d^\infty,\infty})) & \hookrightarrow & \mathcal{F}_{k-3}K_q(\mathcal{M}_{\alpha_\infty}(M_{d^\infty,\infty})) & \twoheadrightarrow & E_\infty^{k-2,q-k+2}\\ \vdots & & \vdots & & \vdots\\ \mathcal{F}_0K_q(\mathcal{M}_{\alpha_\infty}(M_{d^\infty,\infty})) & \hookrightarrow & \mathcal{F}_{-1}K_q(\mathcal{M}_{\alpha_\infty}(M_{d^\infty,\infty})) & \twoheadrightarrow & E_\infty^{0,q}. \end{array} \tag{18}$$

Using

$$\begin{aligned}\mathcal{F}_{-1}K_q(\mathcal{M}_{\alpha_\infty}(M_{d^\infty,\infty})) &= K_q(\mathcal{M}_{\alpha_\infty}(M_{d^\infty,\infty}))\\ &\cong K_{k+q}(M_{d^\infty,\infty}\rtimes_{\alpha_\infty(k)}H_k)\cong K_{k+q}(M_{d^\infty}\rtimes^e_{\alpha(k)}H_k^+),\end{aligned}$$

we will thus arrive at $K_{k+q}(M_{d^\infty}\rtimes^e_{\alpha(k)}H_k^+)$, see Theorem 6.7. Recall that by Bott periodicity, $E_\infty^{p,q+2}\cong E_\infty^{p,q}$ for all $q\in\mathbb{Z}$. In addition, we know from Proposition 6.12 that for $p\in\mathbb{Z}$, the group $E_\infty^{p,1}$ is trivial, and $E_\infty^{p,0}$ vanishes unless $1\le p\le k$, in which case it is a subquotient of $E_2^{p,0}$, and thus a subquotient of $(\mathbb{Z}/g_k\mathbb{Z})^{\binom{k-1}{p-1}}$.

Assume now that $g=1$. Clearly, this holds exactly if $g_k=1$ for some $k\ge 1$. For such $k\ge 1$, the corresponding E_∞-term is trivial, yielding $K_*(M_{d^\infty}\rtimes^e_{\alpha(k)}H_k^+)=0$. Using continuity of K-theory if necessary, we obtain that $\mathcal{A}_S\cong M_{d^\infty}\rtimes^e_\alpha H^+$ has trivial K-theory. It follows from Kirchberg-Phillips classification that $\mathcal{A}_S\cong\mathcal{O}_2$. This proves (a).

If $S=\{p\}$, $\mathcal{A}_S\cong\mathcal{O}_p$ by the definition of $\mathcal{A}_S$, and (b) follows.

Claim (c) is nothing but Proposition 5.10.

Lastly, let us prove claim (d). Let $k \geq 2$, and denote by $(E_\ell, d_\ell)_{\ell \geq 1}$ the spectral sequence associated with $\alpha_\infty^{-1}(k)$. Recall that only those $E_\infty^{\ell,q-\ell}$ with $1 \leq \ell \leq k$ and $q - \ell \in 2\mathbb{Z}$ may be non-trivial subgroups of $(\mathbb{Z}/g_k\mathbb{Z})^{\binom{k-1}{\ell-1}}$. Keeping track of the indices, we get

$$\sum_{\substack{1 \leq \ell \leq k: \\ \ell \text{ even}}} \binom{k-1}{\ell-1} = 2^{k-2} = \sum_{\substack{1 \leq \ell \leq k: \\ \ell \text{ odd}}} \binom{k-1}{\ell-1}.$$

This allows us to conclude that every element in $K_i(M_{d^\infty} \rtimes^e_{\alpha(k)} H_k^+)$ is a divisor of $g_k^{2^{k-2}}$. This concludes the proof, as $g_S = g_k$ for $k \leq |S|$ sufficiently large. □

Remark 6.14. It is possible to say something about the case $|S| = 3$, though the answer is incomplete: Noting that $E_\infty^{2,-2}$ is a subgroup of $(\mathbb{Z}/g\mathbb{Z})^2$, $E_\infty^{3,-2}$ and $E_\infty^{1,0}$ are subgroups of $\mathbb{Z}/g\mathbb{Z}$, and the remaining terms vanish, we know that $K_1(M_{d^\infty} \rtimes^e_\alpha H^+) \cong E_\infty^{2,-2}$ and $K_0(M_{d^\infty} \rtimes^e_\alpha H^+)$ fits into an exact sequence

$$E_\infty^{3,-2} \hookrightarrow K_0(M_{d^\infty} \rtimes^e_\alpha H^+) \twoheadrightarrow E_\infty^{1,0}.$$

But we cannot say more without additional information here.

Remark 6.15. By considering $\mathcal{A}_S$ as the k-graph C^*-algebra $C^*(\Lambda_{S,\theta})$ for finite S, see Corollary 5.8, one could probably also apply Evans' spectral sequence [16, Theorem 3.15] to obtain Theorem 6.4 by performing basically the same proof. In fact, Evans' spectral sequence is the homological counterpart of the spectral sequence used here.

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