

# **$C^*$ -ALGEBRAS GENERATED BY PROJECTIVE REPRESENTATIONS OF FREE NILPOTENT GROUPS**

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ABSTRACT. We compute the two-cocycles (or multipliers) of the free nilpotent groups of class 2 and rank  $n$  and give conditions for simplicity of the corresponding twisted group  $C^*$ -algebras. These groups are representation groups for  $\mathbb{Z}^n$  and can be considered as a family of generalized Heisenberg groups with higher-dimensional center. Their group  $C^*$ -algebras are in a natural way isomorphic to continuous fields over  $\mathbb{T}^{\frac{1}{2}n(n-1)}$  with the noncommutative  $n$ -tori as fibers. In this way, the twisted group  $C^*$ -algebras associated with the free nilpotent groups of class 2 and rank  $n$  may be thought of as “second order” noncommutative  $n$ -tori.

## INTRODUCTION

The discrete Heisenberg group may be described as the group generated by three elements  $u_1$ ,  $u_2$ , and  $v_{12}$  satisfying the commutation relations

$$[u_1, v_{12}] = [u_2, v_{12}] = 1 \quad \text{and} \quad [u_1, u_2] = v_{12}.$$

The group has received much attention in the literature, partly because it is one of the easiest examples of a nonabelian torsion-free group. Moreover, the continuous Heisenberg group (see below) is a connected nilpotent Lie group that arises in certain quantum mechanical systems.

As a natural consequence of this attention, several classes of generalized Heisenberg groups have been investigated. For example, in [14, 15] Milnes and Walters describe the four- and five-dimensional nilpotent groups, and in [11, 12] Lee and Packer study the finitely generated torsion-free two-step nilpotent groups with one-dimensional center.

In this paper, on the other hand, we will consider a family of generalized Heisenberg groups, denoted by  $G(n)$  for  $n \geq 2$ , with larger center. The groups  $G(n)$  are the so-called free nilpotent groups of class 2 and rank  $n$  and will be defined properly in Section 1. Here we also provide further motivation for our investigation of these groups. Inspired by the work of Packer [21] we compute the second cohomology group  $H^2(G(n), \mathbb{T})$  of  $G(n)$  and study the structure of the twisted group  $C^*$ -algebras  $C^*(G(n), \sigma)$  associated with two-cocycles  $\sigma$  of  $G(n)$ .

Section 2 is devoted to two-cocycle calculations, where we decompose  $G(n)$  into a semidirect product and apply techniques introduced by Mackey [13]. In particular, we will see that

$$H^2(G(n), \mathbb{T}) \cong \mathbb{T}^{\frac{1}{2}(n+1)n(n-1)},$$

and in Theorem 2.6 we give explicit formulas for the two-cocycles of  $G(n)$  up to similarity.

Next, in Section 3 we describe  $C^*(G(n), \sigma)$  as a universal  $C^*$ -algebra of a set of generators and relations. Then we construct the algebra that in a natural way appear as a continuous field over the compact space  $H^2(G(n), \mathbb{T})$  with  $C^*(G(n), \sigma)$  as fibers. We also explain that for  $n = 2$ , this algebra is the group  $C^*$ -algebra of the free nilpotent group of class 3 and rank 2.

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In Section 4 we investigate the center of  $C^*(G(n), \sigma)$  and give conditions for simplicity of these twisted group  $C^*$ -algebras in Theorem 4.4 and Corollary 4.6.

Finally, in Section 5 we study the automorphism group of  $G(n)$  and discuss isomorphism invariants of  $C^*(G(n), \sigma)$  coming from  $\text{Aut } G(n)$ .

### 1. THE FREE NILPOTENT GROUPS $G(n)$ OF CLASS 2 AND RANK $n$

For each natural number  $n \geq 2$ , let  $G(n)$  be the group generated by elements  $\{u_i\}_{1 \leq i \leq n}$  and  $\{v_{jk}\}_{1 \leq j < k \leq n}$  subject to the relations

$$[v_{jk}, v_{lm}] = [u_i, v_{jk}] = 1 \quad \text{and} \quad [u_j, u_k] = v_{jk} \quad (1)$$

for  $1 \leq i \leq n$ ,  $1 \leq j < k \leq n$ , and  $1 \leq l < m \leq n$ . Clearly,  $G(2)$  is the usual discrete Heisenberg group. For some purposes, it can be useful to set  $G(1) = \langle u_1 \rangle \cong \mathbb{Z}$ . Note that  $G(n)$  is generated by  $n$  elements, while its dimension equals  $n + \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1)$ .

The group  $G(n)$  is called the free nilpotent group of class 2 and rank  $n$ . Indeed,  $G(n)$  is a free object on  $n$  generators in the category of nilpotent groups of step at most two. To see this, note first that  $G(n)$  is the group generated by  $\{u_i\}_{i=1}^n$  subject to the relations that all commutators of order greater than two involving the generators are trivial. Let  $G'(n)$  be any other nilpotent group of step at most two and let  $\{u'_i\}_{i=1}^n$  be any set of  $n$  elements in  $G'(n)$ . Then there is a unique homomorphism from  $G(n)$  to  $G'(n)$  that maps  $u_i$  to  $u'_i$  for  $1 \leq i \leq n$ . Of course, every free object on  $n$  generators in this category is isomorphic to  $G(n)$ . For a more extensive treatment of free nilpotent groups, see the article on Terence Tao's website [27] (see also 2. in the list below).

Furthermore, we will need the following concrete realization, say  $\tilde{G}(n)$ , of  $G(n)$ . For  $n \geq 2$ , we denote the elements of  $\tilde{G}(n)$  by

$$r = (r_1, \dots, r_n, r_{12}, r_{13}, \dots, r_{n-1,n})^1,$$

where all entries are integers, and define multiplication by

$$r \cdot s = (r_1 + s_1, \dots, r_n + s_n, r_{12} + s_{12} + r_1 s_2, r_{13} + s_{13} + r_1 s_3, \dots, r_{n-1,n} + s_{n-1,n} + r_{n-1} s_n).$$

By letting  $u_i$  have 1 in the  $i$ 'th spot and 0 else and  $v_{jk}$  have 1 in the  $jk$ 'th spot and 0 else, the relations (1) are satisfied for these elements. Next, we define the map

$$\tilde{G}(n) \longrightarrow G(n), \quad r \longmapsto v_{12}^{r_{12}} \cdots v_{n-1,n}^{r_{n-1,n}} \cdot u_n^{r_n} \cdots u_1^{r_1},$$

and then it is not difficult to see that  $\tilde{G}(n)$  is isomorphic to  $G(n)$ . Henceforth, we will not distinguish between  $G(n)$  and the realization  $\tilde{G}(n)$  just described, but this should cause no confusion.

Denote by  $V(n)$  the subgroup of  $G(n)$  generated by the  $v_{jk}$ 's. Then  $V(n)$  coincides with the center  $Z(G(n))$  of  $G(n)$  and

$$V(n) = Z(G(n)) \cong \mathbb{Z}^{\frac{1}{2}n(n-1)}.$$

Indeed, both this and the next observations follow after noticing that

$$r \cdot s \cdot r^{-1} = (s_1, \dots, s_n, s_{12} + r_1 s_2 - s_1 r_2, \dots, s_{n-1,n} + r_{n-1} s_n - s_{n-1} r_n).$$

Moreover, consider the subgroups  $G(n-1)$  and  $H(n)$  of  $G(n)$  defined by

$$G(n-1) = \langle u_i, v_{jk} : 1 \leq i \leq n-1, 1 \leq j < k \leq n-1 \rangle, \\ H(n) = \langle u_n, v_{jn} : 1 \leq j < n \rangle.$$

<sup>1</sup>To be absolutely precise, the entries with double index are colexicographically ordered, that is,  $(i, j) < (k, l)$  if  $j < l$  or if  $j = l$  and  $i < k$ .

Note that  $G(n-1)$  sits inside  $G(n)$  as a subgroup and that  $H(n) \cong \mathbb{Z}^n$  is a normal subgroup of  $G(n)$ . Clearly, we have  $G(n)/V(n) \cong \mathbb{Z}^n$  and  $G(n)/H(n) \cong G(n-1)$ . Therefore, there are short exact sequences

$$1 \longrightarrow V(n) \longrightarrow G(n) \longrightarrow \mathbb{Z}^n \longrightarrow 1$$

and

$$1 \longrightarrow H(n) \longrightarrow G(n) \longrightarrow G(n-1) \longrightarrow 1,$$

where the second one splits and the first does not. In particular,  $G(n)$  is a central extension of  $\mathbb{Z}^n$  by  $\mathbb{Z}^{\frac{1}{2}n(n-1)}$ , showing that  $G(n)$  is indeed a two-step nilpotent group.

To motivate our investigation of  $G(n)$ , we present a few aspects about these groups and some appearances in the literature.

1. Consider in the first place the *continuous* Heisenberg group. We will represent this group in two different ways,  $G_{\text{matrix}}$  and  $G_{\text{wedge}}$ , both with elements  $(x, x') = (x_1, x_2, x') \in \mathbb{R}^3$ , i.e.  $x = (x_1, x_2) \in \mathbb{R}^2$ , and with multiplication as follows. For  $G_{\text{matrix}}$  we define

$$(x_1, x_2, x')(y_1, y_2, y') = (x_1 + y_1, x_2 + y_2, x' + y' + x_1 y_2),$$

and for  $G_{\text{wedge}}$  we set

$$(x_1, x_2, x')(y_1, y_2, y') = (x_1 + y_1, x_2 + y_2, x' + y' + \frac{1}{2}(x_1 y_2 - x_2 y_1)).$$

One can deduce that  $G_{\text{matrix}} \cong G_{\text{wedge}}$ . To motivate the notation, note that  $G_{\text{matrix}}$  can be represented as matrix multiplication in  $M_3(\mathbb{R})$  if one identifies

$$(x_1, x_2, x') \longleftrightarrow \begin{bmatrix} 1 & x_1 & x' \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{bmatrix},$$

and that the multiplication in  $G_{\text{wedge}}$  may be written as

$$(x, x')(y, y') = (x + y, x' + y' + \frac{1}{2}(x \wedge y)).$$

In general, the wedge product on  $\mathbb{R}^n$  is defined as a certain bilinear map (see e.g. [25, p. 79])

$$\mathbb{R}^n \times \mathbb{R}^n \rightarrow \bigwedge^2(\mathbb{R}^n),$$

where  $\bigwedge^2(\mathbb{R}^n)$  is a  $\frac{1}{2}n(n-1)$ -dimensional real vector space. The elements of  $\bigwedge^2(\mathbb{R}^n)$  are called bivectors and if  $\{e_i\}_{i=1}^n$  is a basis for  $\mathbb{R}^n$ , then  $\{e_i \wedge e_j\}_{i < j}$  is a basis for  $\bigwedge^2(\mathbb{R}^n)$ . For every  $n \geq 2$ , define the group  $\widehat{G}(n, \mathbb{R})$  with elements

$$(x, x') \in \mathbb{R}^n \oplus \bigwedge^2(\mathbb{R}^n), \quad \text{where } x = (x_1, \dots, x_n), \quad x' = (x'_{12}, x'_{13}, \dots, x'_{n-1, n}),$$

and where multiplication is given by

$$(x, x')(y, y') = (x + y, x' + y' + \frac{1}{2}(x \wedge y)).$$

This group is of dimension  $n + \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1)$ . Remark especially that if  $n = 3$ , the wedge product can be identified with the vector cross product on  $\mathbb{R}^3$ . That is, the product in  $\widehat{G}(3, \mathbb{R})$  is given by

$$(x, x')(y, y') = (x + y, x' + y' + \frac{1}{2}(x \times y)).$$

It is not hard to see that  $\widehat{G}(n, \mathbb{R})$  is isomorphic to the group consisting of the same elements, but with multiplication given by

$$(x, x')(y, y') = (x + y, x' + y' + (x_1 y_2, x_1 y_3, \dots, x_{n-1} y_n)). \quad (2)$$

Let  $G(n, \mathbb{R})$  denote the group defined by (2). Then  $G(n)$  is the integer version of  $G(n, \mathbb{R})$ .

We also mention that Nielsen [17] has classified all the six-dimensional connected, simply connected, nilpotent Lie groups. In this setting,  $G(3, \mathbb{R})$  is the group denoted by  $G_{6,15}$ .

2. One may define the free nilpotent group  $G(m, n)$  of class  $m$  and rank  $n$  for every  $m \geq 1$ . Indeed,  $G(m, n)$  is the group generated by  $\{u_i\}_{i=1}^n$  subject to the relations that all commutators of order greater than  $m$  involving the generators are trivial. More precisely, for  $m = 1, 2, 3$  and  $n \geq 2$ , we have that  $G(m, n)$  can be described as the groups with presentations

$$\begin{aligned} G(1, n) &= \langle \{u_i\}_{i=1}^n : [u_i, u_j] = 1 \rangle \cong \mathbb{Z}^n, \\ G(2, n) &= \langle \{u_i\}_{i=1}^n : [[u_i, u_j], u_k] = 1 \rangle = G(n), \\ G(3, n) &= \langle \{u_i\}_{i=1}^n : [[[u_i, u_j], u_k], u_l] = 1 \rangle, \end{aligned} \quad (3)$$

and it should now be clear how to define  $G(m, n)$  for all  $m \geq 1$  and  $n \geq 2$ . Finally, we set  $G(m, 1) = \langle u_1 \rangle \cong \mathbb{Z}$  for each  $m \geq 1$ . One can also formally define  $G(m, n)$  as follows: For  $n \geq 1$ , let  $\mathbb{F}_n$  denote the free group on  $n$  generators, set  $\gamma_1(\mathbb{F}_n) = \mathbb{F}_n$  and for  $m \geq 1$ , set  $\gamma_{m+1}(\mathbb{F}_n) = [\mathbb{F}_n, \gamma_m(\mathbb{F}_n)]$ . Then  $G(m, n) = \mathbb{F}_n / \gamma_{m+1}(\mathbb{F}_n)$ .

Moreover, for all  $m \geq 1$  and  $n \geq 2$ , the group  $G(m, n)$  is the free object on  $n$  generators in the category of nilpotent groups of step at most  $m$ . In particular, notice that  $G(m, n)$  is  $m$ -step nilpotent and that

$$G(m, n) \cong G(m+1, n) / Z(G(m+1, n)). \quad (4)$$

Again, we refer to [27] for additional details.

In [15, Section 4] Milnes and Walters describe the simple quotients of the  $C^*$ -algebra associated with a five-dimensional group denoted by  $H_{5,4}$ . One can check that  $H_{5,4}$  is isomorphic to the group  $G(3, 2)$ . See Remark 3.2 for more about this group.

3. The group  $G(3)$  is briefly discussed by Baggett and Packer [4, Example 4.3]. The purpose of that paper is to describe the primitive ideal space of group  $C^*$ -algebras of some two-step nilpotent groups. However,  $G(3)$  only serves as an example of a group the authors could not handle.
4. Let  $n \geq 2$ . It is well-known that the group  $C^*$ -algebra  $A = C^*(G(n))$  may be described as the universal  $C^*$ -algebra generated by unitaries  $\{U_i\}_{1 \leq i \leq n}$  and  $\{V_{jk}\}_{1 \leq j < k \leq n}$  satisfying the relations

$$[V_{jk}, V_{lm}] = [U_i, V_{jk}] = I \quad \text{and} \quad [U_j, U_k] = V_{jk}$$

for all  $1 \leq i \leq n$ ,  $1 \leq j < k \leq n$ , and  $1 \leq l < m \leq n$ .

For  $\lambda = (\lambda_{12}, \lambda_{13}, \dots, \lambda_{n-1,n}) \in \mathbb{T}^{\frac{1}{2}n(n-1)}$ , let  $\mathcal{A}_\lambda$  be the noncommutative  $n$ -torus. It is the universal  $C^*$ -algebra generated by unitaries  $\{W_i\}_{i=1}^n$  and relations  $[W_i, W_j] = \lambda_{ij}I$  for  $1 \leq i < j \leq n$ . The universal property of  $A$  gives that for each  $\lambda$  in  $\mathbb{T}^{\frac{1}{2}n(n-1)}$  there is a surjective  $*$ -homomorphism

$$\pi_\lambda : A \rightarrow \mathcal{A}_\lambda$$

satisfying  $\pi_\lambda(U_i) = W_i$  for  $1 \leq i \leq n$  and  $\pi_\lambda(V_{jk}) = \lambda_{jk}I$  for  $1 \leq j < k \leq n$ .

Furthermore,  $A$  has center  $Z(A) = C^*(\{V_{jk}\}_{1 \leq j < k \leq n}) \cong C^*(V(n))$ . Indeed, this is the case since  $G(n)$  is amenable and its finite conjugacy classes are precisely the one-point sets of central elements (see Lemma 4.1 below). Therefore, we set

$$T = \text{Prim } Z(A) \cong \widehat{Z(A)} = \mathbb{T}^{\frac{1}{2}n(n-1)}.$$

Let  $\lambda$  be a primitive ideal of  $Z(A)$  identified with an element of  $\mathbb{T}^{\frac{1}{2}n(n-1)}$ . Let  $\mathcal{I}_\lambda$  be the ideal of  $A$  generated by  $\lambda$ , that is, the ideal generated by  $\{V_{jk} - \lambda_{jk}I : 1 \leq j < k \leq n\}$ . It is clear that  $\mathcal{I}_\lambda \subseteq \ker \pi_\lambda$ . By the universal property of  $\mathcal{A}_\lambda$ , there is a  $*$ -homomorphism

$$\rho : \mathcal{A}_\lambda \rightarrow A / \mathcal{I}_\lambda$$

such that  $\rho(W_i) = U_i + \mathcal{I}_\lambda$  for  $1 \leq i \leq n$ . Hence,  $\rho \circ \pi_\lambda$  coincides with the quotient map  $A \rightarrow A/\mathcal{I}_\lambda$  and consequently,  $\ker \pi_\lambda \subseteq \mathcal{I}_\lambda$ . Therefore,  $\mathcal{A}_\lambda \cong A/\mathcal{I}_\lambda$  and  $\pi_\lambda$  may be regarded as the quotient map  $A \rightarrow A/\mathcal{I}_\lambda$ .

For an element  $a$  of  $A$ , let  $\tilde{a}$  be the section  $T \rightarrow \bigsqcup_T \mathcal{A}_\lambda$  given by  $\tilde{a}(\lambda) = \pi_\lambda(a)$  and let  $\tilde{A} = \{\tilde{a} \mid a \in A\}$  be the set of all such sections. Then the following can be deduced from the Dauns-Hofmann Theorem [5].

**Theorem 1.1.** *The triple  $(T, \{\mathcal{A}_\lambda\}, \tilde{A})$  consisting of the base space  $T$ ,  $C^*$ -algebras  $\mathcal{A}_\lambda$  for each  $\lambda$  in  $T$ , and the set of sections  $\tilde{A}$ , is a full continuous field of  $C^*$ -algebras. Moreover, the  $C^*$ -algebra associated with this continuous field is naturally isomorphic to  $A$ .*

This result may be obtained as a corollary to [24, Theorem 1.2] which employs tools of Williams [29] related to Fell bundle theory, by taking  $G = G(n)$  and  $\sigma = 1$  in that theorem. It is also a special case of [3, Corollary 2.3]. Our proof is more direct and partly inspired by [1, Theorem 1.1] which covers the case where  $n = 2$ .

From the above discussion it now follows that  $G(n)$  is a *representation group* for  $\mathbb{Z}^n$  in the sense of Moore [16]. In this case, that means  $G(n)$  is (up to isomorphism) the unique central extension of  $\mathbb{Z}^n$  by  $H^2(\mathbb{Z}^n, \mathbb{T})$  such that the ordinary irreducible representation theory of  $G(n)$  coincides with the projective irreducible representation theory of  $\mathbb{Z}^n$ .

This fact plays an important role in [6], where the noncommutative principal torus bundles over locally compact spaces are classified up to equivariant Morita equivalence. As explained in [6, Section 2], the group  $C^*$ -algebra of  $G(n)$  serves as a “universal” bundle in this classification.

We refer to [7, Section 4] for more information on representation groups, where the groups  $G(n, \mathbb{R})$  and  $G(n)$  are treated particularly in [7, Example 4.7].

## 2. THE TWO-COCYCLES OF THE FREE NILPOTENT GROUPS $G(n)$

Let  $G$  be *any* discrete group with identity  $e$ . A function  $\sigma: G \times G \rightarrow \mathbb{T}$  satisfying

$$\begin{aligned}\sigma(r, s)\sigma(rs, t) &= \sigma(r, st)\sigma(s, t) \\ \sigma(r, e) &= \sigma(e, r) = 1\end{aligned}$$

for all elements  $r, s, t \in G$  is called a *two-cocycle of  $G$  with values in  $\mathbb{T}$*  (or a *multiplier of  $G$* ). Moreover, two two-cocycles  $\sigma$  and  $\tau$  are said to be *similar*, written  $\sigma \sim \tau$ , if

$$\tau(r, s) = \beta(r)\beta(s)\overline{\beta(rs)}\sigma(r, s)$$

for all  $r, s \in G$  and some function  $\beta: G \rightarrow \mathbb{T}$ . The set of similarity classes of two-cocycles of  $G$  is an abelian group under pointwise multiplication. This group is the second cohomology group  $H^2(G, \mathbb{T})$ .

Let  $G$  be a semidirect product of a normal subgroup  $H$  and a subgroup  $K$ . By properties of the semidirect product, the elements of  $G$  can be uniquely written as products  $ab$ , where  $a$  belongs to  $H$  and  $b$  belongs to  $K$ . Define the action  $\alpha$  of  $K$  on  $H$  by  $\alpha_b(a) = bab^{-1}$ . One often writes  $G = H \rtimes_\alpha K$ , but to simplify the notation, we will still denote the elements of  $G$  by  $ab$  instead  $(a, b)$  and write the group product in  $G$  as  $(ab)(a'b') = a\alpha_b(a')bb'$  for  $a, a' \in H$  and  $b, b' \in K$ . Hopefully, the reader is familiar with semidirect products so that this does not cause any confusion.

Next, we apply Mackey’s theorem [13, Theorem 9.4] and obtain the following result.

**Theorem 2.1.** *Every two-cocycle of  $G$  is similar to a two-cocycle  $\sigma$  of  $G$  of the form*

$$\sigma(a'b, ab') = \sigma_H(a', \alpha_b(a))g(a, b)\sigma_K(b, b'), \quad (5)$$

where  $\sigma_H$  and  $\sigma_K$  are two-cocycles of  $H$  and  $K$ , respectively,

$$g: H \times K \rightarrow \mathbb{T}$$

is a function such that  $g(a, e) = g(e, b) = 1$  for all  $a \in H$ ,  $b \in K$ , and  $\sigma_H$  and  $g$  satisfy

$$\begin{aligned} g(aa', b) &= \sigma_H(\alpha_b(a), \alpha_b(a')) \overline{\sigma_H(a, a')} \cdot g(a, b)g(a', b), \\ g(a, bb') &= g(\alpha_{b'}(a), b)g(a, b'). \end{aligned} \tag{6}$$

Moreover, for every choice of  $\sigma_H$ ,  $g$ , and  $\sigma_K$  satisfying the conditions above,  $\sigma$  is a two-cocycle of  $G$ .

**Proposition 2.2.** *Let  $(\sigma_H, g, \sigma_K)$  and  $(\sigma'_H, g', \sigma'_K)$  be triples satisfying the conditions of Theorem 2.1 and let  $\sigma$  and  $\sigma'$  be the corresponding two-cocycles of  $G$ . Then  $\sigma \sim \sigma'$  if and only if the following conditions hold:*

- (i)  $\sigma_K \sim \sigma'_K$ ,
- (ii) there exists a function  $\beta: H \rightarrow \mathbb{T}$  such that

$$\begin{aligned} \sigma'_H(a, a') &= \overline{\beta(a)\beta(a')}\beta(aa')\sigma_H(a, a'), \\ g'(a, b) &= \beta(\alpha_b(a))\overline{\beta(a)}g(a, b). \end{aligned}$$

**Remark 2.3.** If (ii) holds, then  $\sigma_H \sim \sigma'_H$ . If  $\sigma_H \sim \sigma'_H$  and  $\beta$  and  $\beta'$  are two functions implementing the similarity, then  $\beta' = f \cdot \beta$  for some homomorphism  $f: H \rightarrow \mathbb{T}$ .

*Proof of Proposition 2.2.* Suppose  $\sigma \sim \sigma'$ . Then there exists some  $\gamma: G \rightarrow \mathbb{T}$  such that

$$\sigma(a'b, ab') = \gamma(a'b)\gamma(ab')\overline{\gamma(a'bab')} \sigma'(a'b, ab') \tag{7}$$

for all  $a, a' \in H$  and  $b, b' \in K$ . In particular, if  $a = a' = e$ , then

$$\sigma_K(b, b') = \gamma(b)\gamma(b')\overline{\gamma(bb')}\sigma'_K(b, b')$$

for all  $b, b' \in K$ , so  $\sigma_K \sim \sigma'_K$ . Moreover, the formula (5) from Theorem 2.1 with  $a = e$  and  $b = e$  gives that

$$\sigma(a', b') = 1 = \sigma'(a', b')$$

for all  $a' \in H$  and  $b' \in K$ . Applying this fact to (7) shows that  $\gamma(a'b') = \gamma(a')\gamma(b')$  for all  $a' \in H$  and  $b' \in K$ . Define  $\beta$  on  $H$  by  $\beta(a) = \gamma(a)$ . Then, by letting  $b = b' = e$  in (5) and (7), we get

$$\sigma'_H(a', a) = \overline{\beta(a')\beta(a)}\beta(a'a)\sigma_H(a', a)$$

for all  $a', a \in H$ . Furthermore, by letting  $a' = e$  and  $b' = e$  in (5) and (7), we compute

$$\begin{aligned} g(a, b) &= \gamma(b)\gamma(a)\overline{\gamma(ba)}g'(a, b) \\ &= \gamma(b)\gamma(a)\overline{\gamma(\alpha_b(a)b)}g'(a, b) \\ &= \gamma(b)\gamma(a)\overline{\gamma(\alpha_b(a))}\gamma(b)g'(a, b) \\ &= \gamma(a)\overline{\gamma(\alpha_b(a))}g'(a, b) \\ &= \beta(a)\overline{\beta(\alpha_b(a))}g'(a, b) \end{aligned}$$

for all  $a \in H$  and  $b \in K$ .

Assume next that  $\beta$  is such that (ii) holds, and that (i) holds through  $\delta$ , that is,

$$\sigma_K(b, b') = \delta(b)\delta(b')\overline{\delta(bb')}\sigma'_K(b, b').$$

Define  $\gamma$  on  $G$  by  $\gamma(ab) = \beta(a)\delta(b)$ . Then

$$\begin{aligned}\sigma(a'b, ab') &= \sigma_H(a', \alpha_b(a))g(a, b)\sigma_K(b, b') \\ &= \beta(a')\beta(\alpha_b(a))\overline{\beta(a'\alpha_b(a))}\sigma'_H(a', \alpha_b(a)) \\ &\quad \cdot \beta(a)\overline{\beta(\alpha_b(a))}g'(a, b) \cdot \delta(b)\delta(b')\overline{\delta(bb')}\sigma'_K(b, b') \\ &= \beta(a')\delta(b) \cdot \beta(a)\delta(b') \cdot \overline{\beta(a'\alpha_b(a))\delta(bb')}\sigma'(a'b, ab') \\ &= \gamma(a'b)\gamma(ab')\overline{\gamma(a'bab')}\sigma'(a'b, ab').\end{aligned}$$

□

Fix  $n \geq 2$ . To compute the two-cocycles of  $G(n)$  up to similarity, we will proceed in the following way. Consider  $G(n)$  as the split extension of  $G(n-1)$  by  $H(n)$  as described in Section 1. We will identify the elements

$$\begin{aligned}a &= (0, \dots, 0, a_n, 0, \dots, 0, a_{1n}, \dots, a_{n-1, n}), \\ b &= (b_1, \dots, b_{n-1}, 0, b_{12}, \dots, b_{n-2, n-1}, 0, \dots, 0),\end{aligned}$$

of  $H(n)$  and  $G(n-1)$ , respectively, with ones of the form

$$\begin{aligned}a &\longleftrightarrow (a_n, a_{1n}, \dots, a_{n-1, n}), \\ b &\longleftrightarrow (b_1, \dots, b_{n-1}, b_{12}, \dots, b_{n-2, n-1}).\end{aligned}$$

The elements of  $G(n)$  will be written as products  $ab$ , where  $a$  belongs to  $H(n)$  and  $b$  belongs to  $G(n-1)$ , and the action  $\alpha$  of  $G(n-1)$  on  $H(n)$  is then given by

$$\alpha_b(a) = bab^{-1} = (a_n, a_{1n} + b_1a_n, \dots, a_{n-1, n} + b_{n-1}a_n).$$

**Remark 2.4.** In the published version, Theorem 2.1 and Proposition 2.2 were only shown to hold for  $G(n)$ , not for any semidirect product.

Proposition 2.2 can be deduced from [24, Appendix 2], but in any case it may be useful to give a proof by a direct computation.

Let  $\tau_n$  be a two-cocycle of  $G(n)$  coming from a pair  $(\sigma_{H(n)}, g_n)$ , that is,

$$\tau_n(a'b, ab') = \sigma_{H(n)}(a', \alpha_b(a))g_n(a, b), \quad (8)$$

where  $(\sigma_{H(n)}, g_n)$  satisfies (6). By Theorem 2.1 and Proposition 2.2, every two-cocycle of  $G(n)$  that is trivial on  $G(n-1)$  is similar to one of this form. Denote the abelian group of similarity classes of two-cocycles of this type by  $\tilde{H}^2(G(n), \mathbb{T})$ .

**Corollary 2.5.** *The second cohomology group of  $G(n)$  may be decomposed as*

$$H^2(G(n), \mathbb{T}) = \tilde{H}^2(G(n), \mathbb{T}) \oplus H^2(G(n-1), \mathbb{T}) = \bigoplus_{k=2}^n \tilde{H}^2(G(k), \mathbb{T}).$$

*Proof.* It follows from Theorem 2.1 and Proposition 2.2 (see our comment above) that

$$H^2(G(n), \mathbb{T}) = \tilde{H}^2(G(n), \mathbb{T}) \oplus H^2(G(n-1), \mathbb{T}).$$

Thus, the second inequality is proven by induction after noticing that

$$\{1\} = H^2(\mathbb{Z}, \mathbb{T}) = H^2(G(1), \mathbb{T}) = \tilde{H}^2(G(1), \mathbb{T}). \quad \square$$

**Theorem 2.6.** *We have*

$$H^2(G(n), \mathbb{T}) \cong \mathbb{T}^{\frac{1}{3}(n+1)n(n-1)},$$

*and for each set of  $\frac{1}{3}(n+1)n(n-1)$  parameters*

$$\{\lambda_{i,jk} : 1 \leq i \leq k, 1 \leq j < k \leq n\} \subseteq \mathbb{T},$$

the associated  $[\sigma]$  in  $H^2(G(n), \mathbb{T})$  may be represented by

$$\begin{aligned} \sigma(r, s) = & \prod_{i < j < k} \lambda_{i,jk}^{s_{jk}r_i + s_k r_{ij}} \lambda_{j,ik}^{s_{ik}r_j + s_k(r_i r_j - r_{ij})} \\ & \cdot \prod_{j < k} \lambda_{j,jk}^{s_{jk}r_j + \frac{1}{2}s_k r_j(r_j - 1)} \lambda_{k,jk}^{r_k(s_{jk} + r_j s_k) + \frac{1}{2}r_j s_k(s_k - 1)}. \end{aligned} \quad (9)$$

The proof of this theorem will be given in Section 2.1.

See the paragraph following Theorem 3.1 for an explanation of why  $\lambda_{i,jk}$  for  $i > k$  is not involved in the above.

**Example 2.7.** For  $G(1) \cong \mathbb{Z}$  there are no nontrivial two-cocycles. The two-cocycles of the usual Heisenberg group  $G(2)$  are, up to similarity, given by two parameters (as computed in [21, Proposition 1.1]):

$$\sigma(r, s) = \lambda_{1,12}^{s_{12}r_1 + \frac{1}{2}s_2 r_1(r_1 - 1)} \lambda_{2,12}^{r_2(s_{12} + r_1 s_2) + \frac{1}{2}r_1 s_2(s_2 - 1)} \quad (10)$$

The two-cocycles of  $G(3)$  are, up to similarity, given by eight parameters:

$$\begin{aligned} \sigma(r, s) = & \lambda_{1,23}^{s_{23}r_1 + s_3 r_{12}} \lambda_{2,13}^{s_{13}r_2 + s_3(r_1 r_2 - r_{12})} \\ & \cdot \lambda_{1,12}^{s_{12}r_1 + \frac{1}{2}s_2 r_1(r_1 - 1)} \lambda_{2,12}^{r_2(s_{12} + r_1 s_2) + \frac{1}{2}r_1 s_2(s_2 - 1)} \\ & \cdot \lambda_{1,13}^{s_{13}r_1 + \frac{1}{2}s_3 r_1(r_1 - 1)} \lambda_{3,13}^{r_3(s_{13} + r_1 s_3) + \frac{1}{2}r_1 s_3(s_3 - 1)} \\ & \cdot \lambda_{2,23}^{s_{23}r_2 + \frac{1}{2}s_3 r_2(r_2 - 1)} \lambda_{3,23}^{r_3(s_{23} + r_2 s_3) + \frac{1}{2}r_2 s_3(s_3 - 1)} \end{aligned}$$

**Remark 2.8.** One may associate a Lyndon-Hochschild-Serre spectral sequence with the extension (see e.g. [28, 6.8.2]):

$$1 \longrightarrow V(n) \longrightarrow G(n) \longrightarrow \mathbb{Z}^n \longrightarrow 1$$

By applying [10, Theorem 4] to this sequence, one can compute the second homology group of  $G(n)$  (which is recently also done more generally for  $G(m, n)$  in [26, Proposition 2.1]), and deduce that

$$H_2(G(n), \mathbb{Z}) \cong \mathbb{Z}^{\frac{1}{3}(n+1)n(n-1)},$$

which gives that  $H^2(G(n), \mathbb{T}) \cong \mathbb{T}^{\frac{1}{3}(n+1)n(n-1)}$  after dualizing, using the universal coefficient theorem for cohomology. However, this does not give an explicit description of  $H^2(G(n), \mathbb{T})$ .

**2.1. Proof of Theorem 2.6.** We will in this proof first compute  $\tilde{H}^2(G(n), \mathbb{T})$  through several lemmas and then use Corollary 2.5 to conclude the argument.

**Lemma 2.1.1.** *Every element of  $\tilde{H}^2(G(n), \mathbb{T})$  may be represented by a pair  $(\sigma_{H(n)}, g_n)$ , where  $\sigma_{H(n)}$  is a two-cocycle of  $H(n)$  given by*

$$\sigma_{H(n)}(a', a) = \prod_{i=1}^{n-1} \lambda_i^{a'_i a_{in}} \quad (11)$$

for some  $\lambda_1, \dots, \lambda_{n-1} \in \mathbb{T}$ , and  $g_n$  satisfies

$$g_n(a + a', b) = \left( \prod_{i=1}^{n-1} \lambda_i^{b_i a_n a'_i} \right) g_n(a, b) g_n(a', b) \quad (12)$$

for all  $a, a' \in H(n)$  and  $b \in G(n-1)$ .



*Proof.* Every element of  $\tilde{H}^2(G(n), \mathbb{T})$  may be represented by a two-cocycle of the form (8), that is, by a pair  $(\sigma_{H(n)}, g_n)$  satisfying (6).

Moreover, it is well-known (see e.g. [2]) that every two-cocycle of  $H(n) \cong \mathbb{Z}^n$  is similar to one of the form

$$\sigma_{H(n)}(a', a) = \prod_{1 \leq i \leq n-1} \lambda_i^{a'_n a_{in}} \cdot \prod_{1 \leq j < k \leq n-1} \mu_{jk}^{a'_{jn} a_{kn}}$$

for some sets of scalars  $\{\lambda_i\}_{1 \leq i \leq n-1}, \{\mu_{jk}\}_{1 \leq j < k \leq n-1} \subseteq \mathbb{T}$ . Since  $H(n)$  is abelian, (6) gives that

$$\begin{aligned} \sigma_{H(n)}(\alpha_b(a), \alpha_b(a')) \overline{\sigma_{H(n)}(a, a')} &= g_n(a + a', b) \overline{g_n(a, b) g_n(a', b)} \\ &= g_n(a' + a, b) \overline{g_n(a', b) g_n(a, b)} \\ &= \sigma_{H(n)}(\alpha_b(a'), \alpha_b(a)) \overline{\sigma_{H(n)}(a', a)} \end{aligned}$$

for all  $a, a' \in H(n)$  and  $b \in G(n-1)$ . Furthermore, we have

$$\begin{aligned} \sigma_H(\alpha_b(a), \alpha_b(a')) \overline{\sigma_H(a, a')} &= \prod_{1 \leq i \leq n-1} \lambda_i^{a_n(a'_{in} + b_i a'_n) - a_n a'_{in}} \cdot \prod_{1 \leq j < k \leq n-1} \mu_{jk}^{(a_{jn} + b_j a_n)(a'_{kn} + b_k a'_n) - a_{jn} a'_{kn}} \\ &= \prod_{1 \leq i \leq n-1} \lambda_i^{b_i a_n a'_n} \cdot \prod_{1 \leq j < k \leq n-1} \mu_{jk}^{b_j a'_{kn} a_n + b_k a_{jn} a'_n + b_j b_k a_n a'_n}. \end{aligned}$$

This is equal to  $\sigma_H(\alpha_b(a'), \alpha_b(a)) \overline{\sigma_H(a', a)}$  for all  $a, a' \in H(n)$  and  $b \in G(n-1)$  if and only if the expression remains unchanged under the substitution  $a \longleftrightarrow a'$ , that is, if and only if all the  $\mu_{jk}$ 's are 1.  $\square$

**Lemma 2.1.2.** *For every element of  $\tilde{H}^2(G(n), \mathbb{T})$  there is a unique associated pair  $(\sigma_{H(n)}, g_n)$  satisfying the conditions of Lemma 2.1.1 such that*

$$g_n(u_n, u_i) = 1 \quad \text{for all } 1 \leq i \leq n-1. \quad (13)$$

*Proof.* Suppose that  $(\sigma_{H(n)}, g_n)$  satisfies (11) and (12). Let  $f: H(n) \rightarrow \mathbb{T}$  be the homomorphism determined by  $f(u_n) = 1$  and  $f(v_{in}) = \overline{g_n(u_n, u_i)}$  for all  $1 \leq i \leq n-1$  and define  $g'_n$  by  $g'_n(a, b) = f(\alpha_b(a)) \overline{f(a)} g_n(a, b)$ . Then,  $g'_n(u_n, u_i) = 1$  for all  $1 \leq i \leq n-1$  and by Proposition 2.2,  $(\sigma_{H(n)}, g'_n)$  determines a two-cocycle of  $H(n)$  in the same similarity class as the one coming from  $(\sigma_{H(n)}, g_n)$ .

Suppose now that there are two pairs  $(\sigma_{H(n)}, g_n)$  and  $(\sigma'_{H(n)}, g'_n)$  both satisfying the conditions of Lemma 2.1.1. Then  $\sigma'_{H(n)} = \sigma_{H(n)}$ , so by Proposition 2.2 and the succeeding remark, there is a homomorphism  $f: H(n) \rightarrow \mathbb{T}$  such that

$$g'_n(a, b) = f(\alpha_b(a)) \overline{f(a)} g_n(a, b) = \left( \prod_{i=1}^{n-1} f(v_{in})^{a_n b_i} \right) g_n(a, b)$$

for all  $a \in H(n)$  and  $b \in G(n-1)$ . In particular,

$$g'_n(u_n, u_i) = f(v_{in}) g_n(u_n, u_i) \quad \text{for all } 1 \leq i \leq n-1,$$

so that  $g'_n = g_n$  if  $g'_n(u_n, u_i) = g_n(u_n, u_i)$  for all  $1 \leq i \leq n-1$ .  $\square$

In the forthcoming lemmas we fix an element of  $\tilde{H}^2(G(n), \mathbb{T})$ , and let  $(\sigma_{H(n)}, g)$  be the unique associated pair satisfying (11), (12), and (13) for some set of scalars  $\{\lambda_i\}_{i=1}^{n-1} \subseteq \mathbb{T}$ .

For computational reasons, we introduce the following notation. For  $a = (a_n, a_{1n}, \dots, a_{n-1,n})$  in  $H(n)$ , we write  $a = w(a) + z(a)$ , where  $w(a) = (a_n, 0, \dots, 0)$ , and  $z(a)$  is the “central part”, i.e.  $z(a) = (0, a_{1n}, \dots, a_{n-1,n})$ . Similarly, for  $b = (b_1, \dots, b_{n-1}, b_{12}, \dots, b_{n-2,n-1})$

in  $G(n-1)$ , we write  $b = w(b)z(b)$ , where  $w(b) = (b_1, \dots, b_{n-1}, 0, \dots, 0)$  and  $z(b) = (0, \dots, 0, b_{12}, \dots, b_{n-2, n-1})$ . Note that  $\alpha_b(a) = a$  if either  $w(a)$  or  $w(b)$  is trivial, i.e. if either  $a$  or  $b$  is central.

**Lemma 2.1.3.** *For all  $a \in H(n)$  and  $b \in G(n-1)$  we have*

$$g(a, b) = g(w(a), w(b))g(w(a), z(b))g(z(a), w(b)).$$

*Proof.* It follows immediately from Lemma 2.1.1 that if  $a, a' \in H(n)$  and  $w(a)$  or  $w(a')$  is 0, then

$$g(a + a', b) = g(a, b)g(a', b), \quad (14)$$

hence,

$$g(a, b) = g(w(a) + z(a), b) = g(w(a), b)g(z(a), b)$$

for all  $a \in H(n)$  and  $b \in G(n-1)$ . If  $b' \in G(n-1)$  and  $w(b') = e$ , then  $b'$  is central and  $\alpha_{b'}(a) = a$  for all  $a \in H(n)$ . Therefore,

$$g(a, b)g(a, b') = g(a, bb') = g(a, b'b) = g(\alpha_b(a), b')g(a, b) \quad (15)$$

for all  $a \in H(n)$ ,  $b \in G(n-1)$ . By (14), we then get

$$\begin{aligned} 1 &= g(\alpha_b(a), b')\overline{g(a, b')} \\ &= g(a + (0, b_1 a_n, \dots, b_{n-1} a_n), b')\overline{g(a, b')} \\ &= g(a, b')g((0, b_1 a_n, \dots, b_{n-1} a_n), b')\overline{g(a, b')} \\ &= g((0, b_1 a_n, \dots, b_{n-1} a_n), b') \end{aligned}$$

for all  $a \in H(n)$  and  $b \in G(n-1)$ . Consequently, since this holds for all  $a \in H(n)$  and  $b \in G(n-1)$ , and central  $b' \in G(n-1)$ , we get that if  $\tilde{a}$  and  $\tilde{b}$  are *any* elements in  $H(n)$  and  $G(n-1)$ , respectively, then  $g(z(\tilde{a}), z(\tilde{b})) = 1$ . Moreover, (15) also imply that if  $b, b' \in G(n-1)$  and *either*  $w(b)$  or  $w(b')$  is equal to  $e$ , that is, either  $b$  or  $b'$  is central, then

$$g(a, bb') = g(a, b)g(a, b'). \quad (16)$$

Hence, by (16) and (14),

$$\begin{aligned} g(a, b) &= g(a, w(b)z(b)) = g(a, w(b))g(a, z(b)) \\ &= g(w(a), w(b))g(z(a), w(b))g(w(a), z(b)) \cdot 1 \end{aligned}$$

for all  $a \in H(n)$  and  $b \in G(n-1)$ . □

**Lemma 2.1.4.** *For all  $a \in H(n)$  and  $b, b' \in G(n-1)$  we have*

$$\begin{aligned} g(z(a), w(b)) &= \prod_{i,j=1}^{n-1} g(v_{in}, u_j)^{a_{in}b_j}, \\ g(w(a), z(b)) &= \prod_{1 \leq i < j \leq n} g(u_n, v_{ij})^{a_n b_{ij}} = \prod_{1 \leq i < j \leq n} \left( \overline{g(v_{in}, u_j)} g(v_{jn}, u_i) \right)^{a_n b_{ij}}, \end{aligned}$$

and

$$g(a, bb') = \left( \prod_{i,j=1}^{n-1} g(v_{in}, u_j)^{b'_i b_j a_n} \right) g(a, b)g(a, b'). \quad (17)$$

*Proof.* Let  $z(H(n)) = \{z(a) \mid a \in H(n)\}$  and  $z(G(n-1)) = \{z(b) \mid b \in G(n-1)\}$ . Then  $g$  is a bihomomorphism when restricted to  $z(H(n)) \times G(n-1)$  or  $H(n) \times z(G(n-1))$ . Therefore, the first two identities hold. Indeed, this follows directly from (6) after noticing that since  $z(a)$  and  $z(b)$  are central,

$$\alpha_{w(b)}(z(a)) = z(a) \quad \text{and} \quad \alpha_{z(b)}(w(a)) = w(a).$$

Moreover, for  $i < j$  we have  $u_i u_j = v_{ij} u_j u_i$ . By (6) and the previous lemma, one calculates

$$\begin{aligned} g(u_n, u_i u_j) &= g(\alpha_{u_j}(u_n), u_i) g(u_n, u_j) \\ &= g(u_n v_{jn}, u_i) g(u_n, u_j) \\ &= g(u_n, u_i) g(v_{jn}, u_i) g(u_n, u_j) \end{aligned}$$

and

$$\begin{aligned} g(u_n, v_{ij} u_j u_i) &= g(u_n, v_{ij}) g(u_n, u_j u_i) \\ &= g(u_n, v_{ij}) g(\alpha_{u_i}(u_n), u_j) g(u_n, u_i) \\ &= g(u_n, v_{ij}) g(u_n v_{in}, u_j) g(u_n, u_i) \\ &= g(u_n, v_{ij}) g(u_n, u_j) g(v_{in}, u_j) g(u_n, u_i), \end{aligned}$$

so that

$$g(v_{jn}, u_i) = g(u_n, v_{ij}) g(v_{in}, u_j), \quad (18)$$

which gives the last identity in the second line of the statement. Finally, we compute

$$\begin{aligned} g(a, bb') &= g(\alpha_{b'}(a), b) g(a, b') = g(a + (0, b'_1 a_n, \dots, b'_{n-1} a_n), b) g(a, b') \\ &= g((0, b'_1 a_n, \dots, b'_{n-1} a_n), w(b)) g(a, b) g(a, b') \\ &= \left( \prod_{i=1}^{n-1} g(v_{in}, w(b))^{b'_i a_n} \right) g(a, b) g(a, b') \\ &= \left( \prod_{i=1}^{n-1} \left( \prod_{j=1}^{n-1} g(v_{in}, u_j)^{b_j} \right)^{b'_i a_n} \right) g(a, b) g(a, b'). \end{aligned}$$

□

**Lemma 2.1.5.** *For all  $a \in H(n)$  and  $b \in G(n-1)$  we have*

$$\begin{aligned} g(w(a), w(b)) &= \left( \prod_{i=1}^{n-1} \lambda_i^{\frac{1}{2} b_i a_n (a_n - 1)} g(v_{in}, u_i)^{\frac{1}{2} a_n b_i (b_i - 1)} \right) \\ &\quad \cdot \prod_{1 \leq i < j \leq n-1} g(v_{in}, u_j)^{b_i b_j a_n}. \end{aligned}$$

*Proof.* First we see from (17) that if  $b_j \geq 1$ , then

$$\begin{aligned} g(u_n, u_j^{b_j}) &= g(u_n, u_j^{b_j-1} u_j) \\ &= g(v_{jn}, u_j)^{b_j-1} g(u_n, u_j^{b_j-1}) g(u_n, u_j) \\ &= \dots = g(v_{jn}, u_j)^{\frac{1}{2} b_j (b_j-1)} g(u_n, u_j)^{b_j} \end{aligned}$$

and then it is not hard to see that

$$g(u_n, u_j^{b_j}) = g(v_{jn}, u_j)^{\frac{1}{2} b_j (b_j-1)} g(u_n, u_j)^{b_j}$$

for negative  $b_j$  as well, for example by applying (17) again.

Moreover, note that  $w(b) = u_{n-1}^{b_{n-1}} \cdots u_1^{b_1}$ , so that by (17),

$$\begin{aligned} g(u_n, w(b)) &= g(u_n, u_{n-1}^{b_{n-1}} \cdots u_1^{b_1}) \\ &= \left( \prod_{j=2}^{n-1} g(v_{1n}, u_j)^{b_1 b_j} \right) g(u_n, u_{n-1}^{b_{n-1}} \cdots u_2^{b_2}) g(u_n, u_1^{b_1}) \\ &= \cdots = \left( \prod_{1 \leq i < j \leq n-1} g(v_{in}, u_j)^{b_i b_j} \right) \left( \prod_{j=1}^{n-1} g(u_n, u_j^{b_j}) \right). \end{aligned}$$

Then by (12) for  $a_n \geq 1$ ,

$$\begin{aligned} g(w(a), w(b)) &= g(a_n u_n, u_{n-1}^{b_{n-1}} \cdots u_1^{b_1}) \\ &= \left( \prod_{i=1}^{n-1} \lambda_i^{b_i(a_n-1)} \right) \cdot g((a_n-1)u_n, u_{n-1}^{b_{n-1}} \cdots u_1^{b_1}) g(u_n, u_{n-1}^{b_{n-1}} \cdots u_1^{b_1}) \\ &= \cdots = \left( \prod_{i=1}^{n-1} \lambda_i^{b_i \cdot \frac{1}{2} a_n(a_n-1)} \right) \cdot g(u_n, u_{n-1}^{b_{n-1}} \cdots u_1^{b_1})^{a_n} \\ &= \left( \prod_{i=1}^{n-1} \lambda_i^{\frac{1}{2} b_i a_n(a_n-1)} \right) \cdot \left( \prod_{1 \leq i < j \leq n-1} g(v_{in}, u_j)^{b_i b_j a_n} \right) \\ &\quad \cdot \left( \prod_{j=1}^{n-1} g(v_{jn}, u_j)^{\frac{1}{2} a_n b_j(b_j-1)} g(u_n, u_j)^{a_n b_j} \right). \end{aligned}$$

Again, it is not hard to see that a similar argument also works for negative  $a_n$ . Finally, recall that we have chosen  $g$  so that  $g(u_n, u_j) = 1$  by (13).  $\square$

**Lemma 2.1.6.** *We have*

$$\tilde{H}^2(G(n), \mathbb{T}) \cong \mathbb{T}^{n(n-1)},$$

and for each set of  $n(n-1)$  parameters

$$\{\lambda_{i,jn} : 1 \leq i \leq n, 1 \leq j \leq n-1\} \subseteq \mathbb{T},$$

the associated  $[\tau]$  in  $\tilde{H}^2(G(n), \mathbb{T})$  may be represented by

$$\begin{aligned} \tau(a'b, ab') &= \prod_{1 \leq i < j \leq n-1} \lambda_{i,jn}^{a_{jn} b_i + a_n b_{ij}} \lambda_{j,in}^{a_{in} b_j + a_n(b_i b_j - b_{ij})} \prod_{j=1}^{n-1} \lambda_{j,jn}^{a_{jn} b_j + \frac{1}{2} a_n b_j(b_j-1)} \\ &\quad \cdot \prod_{j=1}^{n-1} \lambda_{n,jn}^{a'_n(a_{jn} + b_j a_n) + \frac{1}{2} b_j a_n(a_n-1)}. \end{aligned}$$

*Proof.* If one puts  $\lambda_{i,jn} = g(v_{jn}, u_i)$  for  $i, j < n$  and  $\lambda_{n,jn} = \lambda_j$  for  $j < n$ , then this is a consequence of the preceding lemmas. Indeed, by (8) we can represent  $\tau$  as a pair  $(\sigma_{H(n)}, g)$ . Here  $\sigma_{H(n)}$  is of the form (11) and  $g$  can be decomposed as in Lemma 2.1.3 with factors computed in Lemma 2.1.4 and Lemma 2.1.5.  $\square$

To complete the proof of Theorem 2.6, we set  $r = a'b$  and  $s = ab'$  and recall that by Corollary 2.5 we can compute  $\sigma_n$  inductively as  $[\sigma_n] = \prod_{k=2}^n [\tau_k]$ .

Finally, we can also check that  $\sum_{k=2}^n k(k-1) = \frac{1}{3}(n+1)n(n-1)$ .

### 3. THE TWISTED GROUP C\*-ALGEBRAS $C^*(G(n), \sigma)$ OF $G(n)$

Again, let  $G$  be *any* discrete group,  $\sigma$  a two-cocycle of  $G$  and  $\mathcal{H}$  a nontrivial Hilbert space. A map  $U$  from  $G$  into the unitary group of  $\mathcal{H}$  satisfying

$$U(r)U(s) = \sigma(r, s)U(rs)$$

for all  $r, s \in G$  is called a  $\sigma$ -projective unitary representation of  $G$  on  $\mathcal{H}$ .

We recall the following facts about twisted group C\*-algebras and refer to Zeller-Meier [30] for further details of the construction.

To each pair  $(G, \sigma)$ , we may associate the full twisted group C\*-algebra  $C^*(G, \sigma)$ . Denote the canonical injection of  $G$  into  $C^*(G, \sigma)$  by  $i_\sigma$ . Then  $C^*(G, \sigma)$  satisfies the following universal property. Every  $\sigma$ -projective unitary representation of  $G$  on some Hilbert space  $\mathcal{H}$  (or in some unital C\*-algebra  $A$ ) factors uniquely through  $i_\sigma$ .

The reduced twisted group C\*-algebra  $C_r^*(G, \sigma)$  is generated by the left regular  $\sigma$ -projective unitary representation  $\lambda_\sigma$  of  $G$  on  $B(\ell^2(G))$ . Consequently,  $\lambda_\sigma$  extends to a \*-homomorphism of  $C^*(G, \sigma)$  onto  $C_r^*(G, \sigma)$ . If  $G$  is amenable, then  $\lambda_\sigma$  is faithful. Note especially that every nilpotent group is amenable, so that  $C^*(G(n), \sigma) \cong C_r^*(G(n), \sigma)$  through  $\lambda_\sigma$  for every  $n \geq 1$  and all two-cocycles  $\sigma$  of  $G(n)$ .

Finally, we remark that if  $\tau \sim \sigma$  through some function  $\beta: G \rightarrow \mathbb{T}$ , then the assignment  $i_\tau(r) \mapsto \beta(r)i_\sigma(r)$  induces an isomorphism  $C^*(G, \tau) \rightarrow C^*(G, \sigma)$ .

**Theorem 3.1** (Remark 3.1 in the published version). *Fix  $n \geq 2$ , let  $\sigma$  be a two-cocycle of  $G(n)$  of the form (9), that is, determined by the  $\frac{1}{3}(n+1)n(n-1)$  parameters*

$$\{\lambda_{i,jk} : 1 \leq i \leq k, 1 \leq j < k \leq n\} \subseteq \mathbb{T},$$

and set

$$\lambda_{k,ij} = \overline{\lambda_{i,jk}} \lambda_{j,ik} \quad (19)$$

when  $1 \leq i < j < k \leq n$ .

*Then the twisted group C\*-algebra  $C^*(G(n), \sigma)$  is the universal C\*-algebra generated by unitaries  $\{U_i\}_{1 \leq i \leq n}$  and  $\{V_{jk}\}_{1 \leq j < k \leq n}$  satisfying the relations*

$$[V_{jk}, V_{lm}] = I, \quad [U_i, V_{jk}] = \lambda_{i,jk} I, \quad [U_j, U_k] = V_{jk} \quad (20)$$

for  $1 \leq i \leq n$ ,  $1 \leq j < k \leq n$ , and  $1 \leq l < m \leq n$ .

*Proof.* Set  $U_i = i_\sigma(u_i)$  and  $V_{jk} = i_\sigma(v_{jk})$  and note that (9) gives that  $\sigma(u_i, v_{jk}) = \lambda_{i,jk}$  and  $\sigma(v_{jk}, u_i) = 1$  for all  $1 \leq i \leq n$  and  $1 \leq j < k \leq n$ . Thus,

$$[U_i, V_{jk}] = \sigma(u_i, v_{jk}) \overline{\sigma(v_{jk}, u_i)} I = \lambda_{i,jk} I \text{ for all } 1 \leq i \leq n, 1 \leq j < k \leq n.$$

Moreover, note that  $\sigma(u_i, u_j) = 1$  for all  $1 \leq i, j \leq n$  and  $\sigma(v_{jk}, v_{lm}) = 1$  for all  $1 \leq j < k \leq n$  and  $1 \leq l < m \leq n$ . Hence, it is clear that  $C^*(G(n), \sigma)$  is generated as a C\*-algebra by unitaries satisfying (20).

Next, suppose that  $A$  is any C\*-algebra generated by a set of unitaries satisfying the relations (20). For each  $r$  in  $G(n)$  we define the unitary  $W_r$  in  $A$  by

$$W_r = V_{12}^{r_{12}} \cdots V_{n-1,n}^{r_{n-1,n}} \cdot U_n^{r_n} \cdots U_1^{r_1}.$$

Then a computation using (20) repeatedly gives that<sup>2</sup>  $W_r W_s = \tau(r, s) W_{rs}$ , where  $\tau(r, s)$  is a scalar in  $\mathbb{T}$  for all  $r, s \in G(n)$ . Now, the associativity of  $A$  immediately implies that  $\tau$  is a two-cocycle of  $G(n)$ , so that  $W$  is a  $\tau$ -projective unitary representation of  $G(n)$  in  $A$ . Furthermore, note that  $\tau$  satisfies

$$\tau(u_i, v_{jk}) \overline{\tau(v_{jk}, u_i)} = \lambda_{i,jk} \text{ for } 1 \leq i \leq n, 1 \leq j < k \leq n.$$

<sup>2</sup>In general, it will require much work to compute the formula for  $\tau$  and it is not needed for this argument. However, for  $n = 2$ , the expression for  $\tau$  is precisely of the form (10).

By the universal property of the full twisted group  $C^*$ -algebra, there exists a unique  $*$ -homomorphism  $\varphi$  of  $C^*(G(n), \tau)$  onto  $A$  such that  $\varphi(i_\tau(r)) = W(r)$  for all  $r \in G(n)$ .

Therefore, it is sufficient to show that  $\tau \sim \sigma$ , because then  $C^*(G(n), \tau)$  is canonically isomorphic to  $C^*(G(n), \sigma)$ . By Theorem 2.6, there is some  $\beta: G(n) \rightarrow \mathbb{T}$  such that  $\sigma'$ , given by

$$\sigma'(r, s) = \beta(r)\beta(s)\overline{\beta(rs)}\tau(r, s),$$

is of the form (9). We calculate that

$$\begin{aligned} & \sigma'(u_i, v_{jk})\overline{\sigma'(v_{jk}, u_i)} \\ &= \beta(u_i)\beta(v_{jk})\overline{\beta(u_i v_{jk})}\tau(u_i, v_{jk})\overline{\beta(v_{jk})\beta(u_i)\beta(v_{jk}u_i)\tau(v_{jk}, u_i)} \\ &= \tau(u_i, v_{jk})\overline{\tau(v_{jk}, u_i)} = \lambda_{i,jk} \end{aligned}$$

for all  $1 \leq i \leq n$  and  $1 \leq j < k \leq n$ . Hence,  $\sigma' = \sigma$ , so  $\tau \sim \sigma$ .  $\square$

The above relation (19) is a consequence of (18) in the proof of Theorem 2.6 and is the reason why  $\lambda_{i,jk}$  for  $i > k$  is not involved in the expression (9). To illustrate this further, consider the case of rank 3. Let  $U_1, U_2, U_3$  and  $V_{12}, V_{13}, V_{23}$  be unitaries in a  $C^*$ -algebra  $B$  satisfying

$$[V_{jk}, V_{lm}] = I, \quad [U_i, V_{jk}] = \mu_{i,jk}I, \quad [U_j, U_k] = V_{jk}$$

for  $1 \leq i \leq 3$ ,  $1 \leq j < k \leq 3$ , and  $1 \leq l < m \leq 3$  where  $\{\mu_{i,jk}\}$  is any set of nine scalars in  $\mathbb{T}$ . Then we can compute that

$$\begin{aligned} U_1 U_2 U_3 &= V_{12} U_2 U_1 U_3 = \cdots = \mu_{2,13} V_{12} V_{13} V_{23} U_3 U_2 U_1, \\ U_1 U_2 U_3 &= U_1 V_{23} U_3 U_2 = \cdots = \mu_{1,23} \mu_{3,12} V_{12} V_{13} V_{23} U_3 U_2 U_1, \end{aligned}$$

that is, we must have  $\mu_{2,13} = \mu_{1,23} \mu_{3,12}$ .

For rank  $n > 3$ , any choice of a triple of unitaries from the family  $\{U\}_{i=1}^n$  gives a similar dependence. In the  $n \cdot \frac{1}{2}n(n-1)$  commutation relations, these  $\binom{n}{3}$  dependencies are the only possible ones since

$$n \cdot \frac{1}{2}n(n-1) - \binom{n}{3} = \frac{1}{2}n(n-1)(n - \frac{1}{3}(n-2)) = \frac{1}{3}(n+1)n(n-1).$$

**Remark 3.2.** For  $n \geq 2$ , let  $\omega$  be the dual two-cocycle of  $G(n)$ , that is,

$$\omega: G(n) \times G(n) \rightarrow H^2(\widehat{G(n)}, \mathbb{T}) \cong \mathbb{Z}^{\frac{1}{3}(n+1)n(n-1)}$$

is determined by  $\omega(r, s)(\sigma) = \sigma(r, s)$  for a two-cocycle  $\sigma$  of  $G(n)$ . Let the group  $R(G(n))$  be defined as the set  $\mathbb{Z}^{\frac{1}{3}(n+1)n(n-1)} \times G(n)$  with product

$$(j, r)(k, s) = (j + k + \omega(r, s), rs).$$

It is not entirely obvious that  $\omega$  and  $R(G(n))$  are well-defined and we refer to [24, p. 689–690] and [7, Section 4] for details on this and the fact that  $R(G(n))$  is a representation group for  $G(n)$ . Moreover, according to [24, Corollary 1.3] we may construct a continuous field  $A$  over  $H^2(G(n), \mathbb{T})$  with fibers  $A_\lambda \cong C^*(G(n), \sigma_\lambda)$  for each  $\lambda \in H^2(G(n), \mathbb{T})$ . Then the  $C^*$ -algebra associated with this continuous field will be naturally isomorphic to the group  $C^*$ -algebra of the group  $R(G(n))$ .

Next, we briefly consider the group  $G(3, 2)$  generated by  $u_1, u_2, v_{12}, w_1, w_2$  satisfying

$$[u_1, u_2] = v_{12}, \quad [u_1, v_{12}] = w_1, \quad [u_2, v_{12}] = w_2, \quad w_1, w_2 \text{ central}.$$

Then we have  $Z(G(3, 2)) \cong \mathbb{Z}^2$  and  $Z(C^*(G(3, 2))) \cong C(\mathbb{T}^2)$ .

The following statement can also be deduced from [24, Theorem 1.2 and Examples 1.4 (3)], but we include the analysis that follows, because it is similar to that used in Theorem 1.1.

Let  $i$  denote the canonical injection of  $G(3, 2)$  into  $C^*(G(3, 2))$ . For each  $\lambda = (\lambda_1, \lambda_2) \in \mathbb{T}^2$ , let  $C^*(G(2), \sigma_\lambda)$  be generated by unitaries satisfying (20). By a similar argument as in Theorem 1.1, there is a surjective  $*$ -homomorphism

$$\pi_\lambda: C^*(G(3, 2)) \rightarrow C^*(G(2), \sigma_\lambda)$$

such that  $i(u_i) = U_i$ ,  $i(v_{12}) = V_{12}$ , and  $i(w_i) = \lambda_i I$  for  $i = 1, 2$ . Moreover, the kernel of  $\pi_\lambda$  coincides with the ideal of  $C^*(G(3, 2))$  generated by

$$\lambda \in \text{Prim } Z(C^*(G(3, 2))) \cong Z(C^*(\widehat{G(3, 2)})) = \mathbb{T}^2 \cong H^2(G(2), \mathbb{T}).$$

Again, similarly as in Theorem 1.1, we define a set of sections and apply the Dauns-Hofmann Theorem. In this way, the triple

$$\left( H^2(G(2), \mathbb{T}), \{C^*(G(2), \sigma_\lambda)\}_\lambda, C^*(\widehat{G(3, 2)}) \right)$$

is a full continuous field of  $C^*$ -algebras, and the  $C^*$ -algebra associated with this continuous field is naturally isomorphic to  $C^*(G(3, 2))$ .

It is not difficult to see that  $R(G(2))$  is isomorphic to  $G(3, 2)$ . We conjecture that  $R(G(n)) \cong G(3, n)$  also for  $n \geq 3$ , where  $G(3, n)$  is the free nilpotent group of class 3 and rank  $n$  as described in (3), so that  $A$  is isomorphic to  $C^*(G(3, n))$ . For  $n \geq 3$ , the complicated part is to construct a homomorphism  $R(G(n)) \rightarrow G(3, n)$ , find an isomorphism  $\mathbb{Z}^{\frac{1}{3}(n+1)n(n-1)} \cong Z(G(3, n))$ , and then use (4) to produce a commuting diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}^{\frac{1}{3}(n+1)n(n-1)} & \longrightarrow & R(G(n)) & \longrightarrow & G(n) & \longrightarrow & 1 \\ & & \downarrow \cong & & \downarrow & & \downarrow = & & \\ 1 & \longrightarrow & Z(G(3, n)) & \longrightarrow & G(3, n) & \longrightarrow & G(n) & \longrightarrow & 1 \end{array}$$

In fact, [26, Proposition 2.2 and Remark 2.3] indicate that the representation group  $R(G(m, n))$  for  $G(m, n)$  defined similarly as above is isomorphic to  $G(m+1, n)$  for all  $m, n \geq 1$ .

#### 4. SIMPLICITY OF THE TWISTED GROUP $C^*$ -ALGEBRAS $C^*(G(n), \sigma)$

Let  $\sigma$  be a two-cocycle of any group  $G$ . An element  $r$  of  $G$  is called  $\sigma$ -regular if  $\sigma(r, s) = \sigma(s, r)$  whenever  $s$  in  $G$  commutes with  $r$ . If  $r$  is  $\sigma$ -regular, then every conjugate of  $r$  is also  $\sigma$ -regular. Therefore, we say that a conjugacy class of  $G$  is  $\sigma$ -regular if it contains a  $\sigma$ -regular element.

Let  $n \geq 2$ . The conjugacy class  $C_r$  of  $r \in G(n)$  is infinite if  $r \notin V(n) = Z(G(n))$ . Indeed, for any  $s \in G(n)$  we have

$$(srs^{-1})_i = r_i \text{ and } (srs^{-1})_{jk} = r_{jk} + s_j r_k - r_j s_k.$$

Hence,  $|C_r| = \infty$  if  $r_i \neq 0$  for some  $i$ . Of course,  $C_r = \{r\}$  if  $r \in V(n)$ .

More generally, it is explained in [9, Section 2.1.1] that if  $G$  is any torsion-free nilpotent group and  $r \in G \setminus Z(G)$ , then  $|C_r| = \infty$ .

In the published version, the following was only shown to hold for  $G(n)$ .

**Lemma 4.1.** *Let  $\sigma$  be a two-cocycle of a group  $G$ . Suppose that  $|C_r| \in \{1, \infty\}$  whenever  $r \in G$  is  $\sigma$ -regular. Define  $S(G)$  to be the set of  $\sigma$ -regular central elements of  $G$ , that is,*

$$S(G) = \{r \in Z(G) \mid \sigma(r, s) = \sigma(s, r) \text{ for all } s \in G\}.$$

*Then  $S(G)$  is a subgroup of  $G$  and  $Z(C_r^*(G), \sigma) \cong C(\widehat{S(G)})$ .*

*Proof.* It is not hard to check that  $S(G)$  is a subgroup of  $Z(G)$ .

Let  $\delta_e$  in  $\ell^2(G)$  be the characteristic function on  $\{e\}$  and for an operator  $T$  in  $B(\ell^2(G))$ , set  $f_T = T\delta_e \in \ell^2(G)$ . If  $T$  belongs to the center of  $C_r^*(G, \sigma)$ , then  $f_T$  can be nonzero only on the finite  $\sigma$ -regular conjugacy classes of  $G$ , that is, on  $S(G)$  (see e.g. [19, Lemmas 2.3 and 2.4]).

Next, let  $C^*(S(G), \sigma)$  be identified with  $\{\lambda_\sigma(s) \mid s \in S(G)\} \subseteq B(\ell^2(G))$ . This means that  $Z(C_r^*(G, \sigma)) \subseteq C^*(S(G), \sigma)$ . As the reverse inclusion obviously holds, we have  $Z(C_r^*(G, \sigma)) = C^*(S(G), \sigma)$ .

Now, it is not difficult to see that  $C^*(S(G)) \cong C^*(S(G), \sigma)$ . Indeed, as  $s \mapsto \lambda_\sigma(s)$  is a unitary representation of  $S(G)$  into  $C^*(S(G), \sigma)$  and the canonical tracial state  $\tau$  on  $C^*(S(G), \sigma)$  is faithful and satisfies  $\tau(\lambda_\sigma(s)) = 0$  for each nonzero  $s \in S(G)$ , this is just a consequence of [30, Théorème 4.22]. Altogether, we get

$$Z(C_r^*(G), \sigma) = C^*(S(G), \sigma) \cong C^*(S(G)) \cong C(\widehat{S(G)}). \quad \square$$

Now, we fix a two-cocycle  $\sigma$  of  $G(n)$  of the form (9).

**Remark 4.2.** If  $S(G(n))$  is nontrivial, we can describe  $C^*(G(n), \sigma)$  as a continuous field of  $C^*$ -algebras over the base space  $\widehat{S(G(n))}$ . The fibers will be isomorphic to  $C^*(G(n)/S(G(n)), \omega)$  for some two-cocycle  $\omega$  of  $G(n)/S(G(n))$  (see [11, Theorem 1.1] and [24, Theorem 1.2] for further details).

**Example 4.3** ([11, Lemma 3.8 and Theorem 3.9]). Fix a two-cocycle  $\sigma$  of  $G(2)$  of the form (10) such that both  $\lambda_{1,12}$  and  $\lambda_{2,12}$  are torsion elements. Let  $p$  and  $q$  be the smallest natural numbers such that  $\lambda_{1,12}^p = \lambda_{2,12}^q = 1$  and set  $k = \text{lcm}(p, q)$ . Clearly,  $V(2) = \mathbb{Z}$  and  $S(G(2)) = k\mathbb{Z}$ . Moreover,  $G(2)/S(G(2))$  can be identified with the group with product

$$(r_1, r_2, r_{12})(s_1, s_2, s_{12}) = (r_1 + s_1, r_2 + s_2, r_{12} + s_{12} + r_1 s_2 \bmod k\mathbb{Z})$$

for  $r_1, r_2, s_1, s_2 \in \mathbb{Z}$  and  $r_{12}, s_{12} \in \{0, 1, \dots, k-1\}$ .

Then  $C^*(G(2), \sigma)$  is a continuous field of  $C^*$ -algebras over the base space  $\widehat{S(G(2))} \cong \mathbb{T}$ . The fibers will be isomorphic to  $C^*(G(n)/S(G(n)), \omega_\lambda)$ , where  $\lambda \in \mathbb{T}$  and

$$\omega_\lambda(r, s) = \sigma(r, s)\mu^{r_1 s_2}$$

for some  $\mu \in \mathbb{T}$  with  $\mu^k = \lambda$ .

Characterizations for simplicity of twisted group  $C^*$ -algebras of two-step nilpotent groups have been given in [11, Corollary 1.4] and [24, Corollary 1.6]. For the groups  $G(n)$ , the necessary and sufficient conditions for simplicity are somewhat easier to provide.

**Theorem 4.4.** *The following are equivalent:*

- (i)  $C^*(G(n), \sigma)$  is simple.
- (ii)  $C^*(G(n), \sigma)$  has trivial center.
- (iii) There are no nontrivial central  $\sigma$ -regular elements in  $G(n)$ .

*Proof.* By [22, Theorem 1.7]  $C^*(G(n), \sigma)$  is simple if and only if every nontrivial  $\sigma$ -regular conjugacy class of  $G(n)$  is infinite. Since every finite conjugacy class of  $G(n)$  is a one-point set of a central element, then (i) is equivalent with (iii).

Moreover, (iii) is the same as saying that  $S(G(n))$  is trivial, so therefore, (ii) is equivalent with (iii) by Lemma 4.1. This also follows from [19, Theorem 2.7].  $\square$

**Lemma 4.5.** *A central element  $s = (0, \dots, 0, s_{12}, s_{13}, \dots, s_{n-1,n})$  of  $G(n)$  is  $\sigma$ -regular if and only if*

$$\prod_{1 \leq j < k \leq n} \lambda_{i;jk}^{s_{jk}} = 1$$



for all  $1 \leq i \leq n$ .

*Proof.* Clearly, an element  $s = (0, \dots, 0, s_{12}, s_{13}, \dots, s_{n-1,n}) \in V(n)$  is  $\sigma$ -regular if and only if  $\sigma(s, r) = \sigma(r, s)$  for all  $r \in G(n)$ . By a direct calculation from the cocycle formula (9), we get that

$$\begin{aligned} \sigma(r, s) \overline{\sigma(s, r)} &= \left( \prod_{i < j < k} \lambda_{i,jk}^{s_{jk} r_i} \lambda_{j,ik}^{s_{ik} r_j} \right) \left( \prod_{j < k} \lambda_{j,jk}^{s_{jk} r_j} \lambda_{k,jk}^{r_k s_{jk}} \right) \left( \prod_{i < j < k} \lambda_{i,jk}^{-r_k s_{ij}} \lambda_{j,ik}^{r_k s_{ij}} \right) \\ &= \prod_{i=1}^n \left( \prod_{1 \leq j < k \leq n} \lambda_{i,jk}^{s_{jk}} \right)^{r_i} \end{aligned}$$

is equal to 1 for all  $r \in G(n)$  if and only if the inner parenthesis is 1 for each  $1 \leq i \leq n$ .  $\square$

**Corollary 4.6.** *We have that  $C^*(G(n), \sigma)$  is simple if and only if for each nontrivial central element  $s = (0, \dots, 0, s_{12}, s_{13}, \dots, s_{n-1,n})$  there is some  $1 \leq i \leq n$  such that*

$$\prod_{1 \leq j < k \leq n} \lambda_{i,jk}^{s_{jk}} \neq 1.$$

**Example 4.7.** In particular,  $C^*(G(3), \sigma)$  is simple if and only if for each nontrivial central element  $s = (0, 0, 0, s_{12}, s_{13}, s_{23})$  at least one of the following hold:

$$\begin{aligned} \lambda_{1,12}^{s_{12}} \lambda_{1,13}^{s_{13}} \lambda_{1,23}^{s_{23}} &\neq 1, \\ \lambda_{2,12}^{s_{12}} \lambda_{2,13}^{s_{13}} \lambda_{2,23}^{s_{23}} &\neq 1, \\ \lambda_{3,12}^{s_{12}} \lambda_{3,13}^{s_{13}} \lambda_{3,23}^{s_{23}} &\neq 1. \end{aligned}$$

Next, set  $\lambda_{i,jk} = e^{2\pi i t_{i,jk}}$  for  $t_{i,jk} \in [0, 1)$  and consider the  $n \times \frac{1}{2}n(n-1)$ -matrix  $T$  with entries  $t_{i,jk}$  in the corresponding spots. Then  $T$  induces a linear map

$$\mathbb{R}^{\frac{1}{2}n(n-1)} \rightarrow \mathbb{R}^n.$$

**Corollary 4.8.** *Let  $T$  be the matrix described above. Then following are equivalent:*

- (i)  $C^*(G(n), \sigma)$  is simple
- (ii)  $T^{-1}(\mathbb{Z}^n) \cap \mathbb{Z}^{\frac{1}{2}n(n-1)} = \{0\}$
- (iii)  $T(\mathbb{Z}^{\frac{1}{2}n(n-1)} \setminus \{0\}) \cap \mathbb{Z}^n = \emptyset$

**Remark 4.9.** Clearly, condition (ii) above is equivalent to that  $T$  restricts to an injective map

$$\mathbb{Z}^{\frac{1}{2}n(n-1)} \rightarrow \mathbb{R}^n / \mathbb{Z}^n \cong \mathbb{T}^n.$$

Furthermore, for  $1 \leq j < k \leq n$ , define

$$\Lambda_{jk} = \{t_{i,jk} \in [0, 1), 1 \leq i \leq n \mid e^{2\pi i t_{i,jk}} = \lambda_{i,jk}\}$$

and for  $1 \leq i \leq n$ , define

$$\Lambda_i = \{t_{i,jk} \in [0, 1), 1 \leq j < k \leq n \mid e^{2\pi i t_{i,jk}} = \lambda_{i,jk}\}.$$

**Proposition 4.10.** *If there exists  $i$  such that all the elements of  $\Lambda_i$  are irrational and linearly independent over  $\mathbb{Q}$ , then  $C^*(G(n), \sigma)$  is simple.*

*Proof.* It follows immediately from Lemma 4.5, that “equation  $i$ ” cannot be satisfied unless  $s = 0$ . Hence, no nontrivial  $\sigma$ -regular central elements exists.  $\square$

**Proposition 4.11.** *If there exists  $j < k$  such that  $\Lambda_{jk}$  consists of only rational elements, then  $C^*(G(n), \sigma)$  is not simple.*

*Proof.* Let  $q$  be the least common two-cocycle of the denominators of the elements of  $\Lambda_{jk}$ . Then  $qv_{jk}$  is central and  $\sigma$ -regular. Indeed,

$$\sigma(r, qv_{jk})\overline{\sigma(qv_{jk}, r)} = \prod_{i=1}^{n-1} \lambda_{i,jk}^{qr_i} = 1$$

for all  $r \in G(n)$ . □

**Remark 4.12.** In the case where  $C^*(G(n), \sigma)$  is not simple, some more information about the primitive ideal space can be deduced from [11, Proposition 1.3].

## 5. ON ISOMORPHISMS INVARIANTS OF $C^*(G(n), \sigma)$

Fix  $n \geq 2$  and let  $\sigma$  be a two-cocycle of  $G(n)$ . If  $\varphi$  is an automorphism of  $G(n)$ , define the two-cocycle  $\sigma_\varphi$  of  $G(n)$  by

$$\sigma_\varphi(r, s) = \sigma(\varphi(r), \varphi(s)). \quad (21)$$

Then it is well-known that the associated twisted group  $C^*$ -algebras  $C^*(G(n), \sigma)$  and  $C^*(G(n), \sigma_\varphi)$  are isomorphic. Indeed, the map

$$i_{(G, \sigma)}(r) \mapsto i_{(G, \sigma_\varphi)}(\varphi^{-1}(r))$$

extends to an isomorphism  $C^*(G(n), \sigma) \rightarrow C^*(G(n), \sigma_\varphi)$ . Moreover, for any automorphism  $\varphi$  of  $G(n)$ , it is easily seen that two two-cocycles  $\sigma$  and  $\tau$  of  $G(n)$  are similar if and only if  $\sigma_\varphi$  and  $\tau_\varphi$  are similar. Hence, there is a well-defined group action of the automorphism group  $\text{Aut } G(n)$  on  $H^2(G(n), \mathbb{T})$  defined by  $\varphi \cdot [\sigma] = [\sigma_\varphi]$ .

Therefore, we will now briefly investigate  $\text{Aut } G(n)$ . Let  $V(n)^n$  be the subgroup of  $\text{Aut } G(n)$  consisting of the automorphisms  $G(n) \rightarrow G(n)$  of the form  $u_i \mapsto z_i u_i$  for  $1 \leq i \leq n$  and elements  $z_i \in V(n) = Z(G(n))$ . Clearly,  $V(n)^n$  contains  $\text{Inn } G(n)$ . In fact, in the case  $n = 2$ , we have  $V(2)^2 = \text{Inn } G(2)$ .

**Proposition 5.1.** *There is a split short exact sequence:*

$$1 \longrightarrow V(n)^n \longrightarrow \text{Aut } G(n) \longrightarrow \text{GL}(n, \mathbb{Z}) \longrightarrow 1$$

*Proof.* Assume that  $\varphi$  is any endomorphism  $G(n) \rightarrow G(n)$ . The image of a central element under  $\varphi$  must be central, so  $\varphi$  restricts to an endomorphism  $\varphi_1: V(n) \rightarrow V(n)$ . Therefore,  $\varphi$  also induces an endomorphism  $\varphi_2: G(n)/V(n) \rightarrow G(n)/V(n)$  determined by  $\varphi_2(q(r)) = q(\varphi(r))$ . Consider now the following commutative diagram:

$$\begin{array}{ccccccc} 1 & \longrightarrow & V(n) & \xrightarrow{i} & G(n) & \xrightarrow{q} & \mathbb{Z}^n \longrightarrow 1 \\ & & \varphi_1 \downarrow & & \varphi \downarrow & & \downarrow \varphi_2 \\ 1 & \longrightarrow & V(n) & \xrightarrow{i} & G(n) & \xrightarrow{q} & \mathbb{Z}^n \longrightarrow 1 \end{array}$$

Assume that  $\varphi_2$  is an automorphism. Since  $\varphi_2$  is surjective, then for all  $u_i$  there is some  $s_i \in G(n)$  such that  $\varphi(s_i) = z_i u_i$  for some  $z_i \in V(n)$ . Hence, for all  $j < k$ , we have  $\varphi_1(s_j s_k s_j^{-1} s_k^{-1}) = v_{jk}$  and therefore,  $\varphi_1$  is surjective. Every surjective endomorphism of  $\mathbb{Z}^n$  is also injective, so  $\varphi_1$  is an automorphism as well. Thus, by the “short five lemma”,  $\varphi$  is an automorphism.

The converse obviously holds and hence,  $\varphi$  is an automorphism if and only if  $\varphi_2$  is an automorphism.

Furthermore, the construction of  $G(n)$  in terms of generators and relations means that every endomorphism  $G(n) \rightarrow G(n)$  is uniquely determined by its values at  $\{u_i\}_{i=1}^n$ . In particular, we let  $\varphi: G(n) \rightarrow G(n)$  be determined by the pair of matrices given by its entries

$$(\varphi(u_i)_j), (\varphi(u_i)_{jk}) \in M_n(\mathbb{Z}) \times M_{n, \frac{1}{2}n(n-1)}(\mathbb{Z})$$

so that the induced endomorphism  $\varphi_2$  is coming from a matrix in  $M_n(\mathbb{Z})$ .

By the above argument, the map between endomorphism groups defined by

$$\text{End } G(n) \rightarrow \text{End } \mathbb{Z}^n, \quad (\varphi(u_i)_j), (\varphi(u_i)_{jk}) \mapsto (\varphi(u_i)_j) \quad (22)$$

restricts to a surjective map  $\text{Aut } G(n) \rightarrow \text{Aut } \mathbb{Z}^n = \text{GL}(n, \mathbb{Z})$ .

Before concluding the argument, we need the following.

**Lemma 5.2** (nested inside the proof). *If  $\varphi$  and  $\varphi'$  are two endomorphisms of  $G(n)$ , then*

$$(\varphi \circ \varphi')(u_i)_j = \sum_{k=1}^n \varphi'(u_i)_k \varphi(u_k)_j.$$

*If  $\varphi$  and  $\varphi'$  are two endomorphisms of  $G(n)$  that both induce the trivial map on  $G(n)/V(n)$ , then*

$$(\varphi \circ \varphi')(u_i)_{jk} = \varphi'(u_i)_{jk} + \varphi(u_i)_{jk}.$$

*Proof.* For the moment, set  $\varphi(u_i)_j = r_{ij}$  and  $\varphi'(u_i)_j = s_{ij}$ . Then

$$(\varphi \circ \varphi')(u_i) = \varphi(u_n^{s_{in}} \cdots u_1^{s_{i1}} z) = (u_n^{r_{nn}} \cdots u_1^{r_{n1}})^{s_{in}} \cdots (u_n^{r_{1n}} \cdots u_1^{r_{11}})^{s_{i1}} z'$$

for some elements  $z, z' \in V(n)$ . Moreover, we can change the order of the  $u_i$ 's in the expression just by replacing  $z'$  by another central element  $z''$  and thus,

$$(\varphi \circ \varphi')(u_i)_j = r_{nj}s_{in} + r_{n-1,j}s_{i,n-1} + \cdots + r_{1j}s_{i1} = \sum_{k=1}^n s_{ik}r_{kj}.$$

If both  $\varphi_2$  and  $\varphi'_2$  are trivial, then  $\varphi(u_i) = z_i u_i$  and  $\varphi'(u_i) = z'_i u_i$  for all  $1 \leq i \leq n$  and some elements  $z_i, z'_i \in V(n)$ . Hence,  $\varphi(v_{jk}) = \varphi'(v_{jk}) = v_{jk}$  for all  $j < k$  and thus,

$$(\varphi \circ \varphi')(u_i) = \varphi(z'_i u_i) = z'_i z_i u_i. \quad \square$$

Therefore, (22) restricts to a surjective *homomorphism*  $\text{Aut } G(n) \rightarrow \text{GL}(n, \mathbb{Z})$  with kernel isomorphic to the group  $M_{n, \frac{1}{2}n(n-1)}(\mathbb{Z})$  under addition, that is, to  $V(n)^n$ .

Moreover, if  $A$  is a matrix in  $\text{GL}(n, \mathbb{Z})$  with entries  $a_{ij}$ , then one can define an automorphism  $\varphi_A$  of  $G(n)$  by  $\varphi_A(u_i)_j = a_{ij}$ . Thus, it should be clear that  $\text{GL}(n, \mathbb{Z})$  sits inside  $\text{Aut } G(n)$  as a subgroup so that the sequence splits.  $\square$

**Proposition 5.3.** *If  $\varphi$  belongs to  $V(n)^n$ , then  $\sigma_\varphi$  is similar to  $\sigma$ . Thus, the action of  $V(n)^n$  on  $H^2(G(n), \mathbb{T})$  given by (21) is trivial.*

*Proof.* It is not hard to see that

$$\sigma(u_i, v_{jk}) \overline{\sigma(v_{jk}, u_i)} = \sigma_\varphi(u_i, v_{jk}) \overline{\sigma_\varphi(v_{jk}, u_i)},$$

that is,

$$[i_{(G, \sigma)}(u_i), i_{(G, \sigma)}(v_{jk})] = [i_{(G, \sigma_\varphi)}(u_i), i_{(G, \sigma_\varphi)}(v_{jk})]$$

for all  $1 \leq i \leq n$  and  $1 \leq j < k \leq n$ . It then follows from the universal property of  $C^*(G(n), \sigma)$  described in Theorem 3.1 that  $\sigma_\varphi \sim \sigma$ .  $\square$

To describe the  $\mathrm{GL}(n, \mathbb{Z})$ -action on  $H^2(G(n), \mathbb{T})$  requires more work. To any  $A$  in  $\mathrm{GL}(n, \mathbb{Z})$  we may associate a square matrix  $\tilde{A}$  of dimension  $\frac{1}{2}n(n-1)$ , with entries coming from the determinant of all  $2 \times 2$ -matrices inside  $A$ . More precisely, if  $A = (a_{ij})$ ,  $\tilde{A}$  is given by entries  $\tilde{a}_{ij,kl}$  for  $i < j, k < l$  such that  $\tilde{a}_{ij,kl} = a_{ik}a_{jl} - a_{il}a_{jk}$ . Then  $A$  acts on the matrix  $T$  defined prior to Corollary 4.8 by  $A \cdot T = AT\tilde{A}$ . Tedious computations of commutation relations and use of the universal property of  $C^*(G(n), \sigma)$  from Theorem 3.1 now lead to the following result.

**Proposition 5.4.** *Let  $\sigma$  and  $\sigma'$  be two-cocycles of  $G(n)$  of the form (9) and let  $T$  and  $T'$  be the associated matrices of Corollary 4.8. If there exists a matrix  $A$  in  $\mathrm{GL}(n, \mathbb{Z})$  such that  $A \cdot T = T'$ , then  $C^*(G(n), \sigma)$  and  $C^*(G(n), \sigma')$  are isomorphic.*

For  $n = 2$ , it is shown by Packer [21, Theorem 2.9] that  $C^*(G(2), \sigma)$  and  $C^*(G(2), \sigma')$ , where  $\sigma$  and  $\sigma'$  are of the form (9), are isomorphic if and only if there is a  $\mathrm{GL}(2, \mathbb{Z})$ -matrix  $A$  taking  $\sigma$  to  $\sigma'$ . Note in this case that  $\det \tilde{A} = \det A = \pm 1$ .

For  $n \geq 3$ , it is at the moment not clear whether the  $\mathrm{GL}(n, \mathbb{Z})$ -action on  $H^2(G(n), \mathbb{T})$  described above is such that the orbits represent different isomorphism classes of twisted group  $C^*$ -algebras. Therefore, the problem of determining the isomorphism classes of  $C^*(G(n), \sigma)$  remains open for future investigation.

**Remark 5.5.** Every two-cocycle  $\sigma_\lambda$  of  $G(n)$  is of the form  $e^{2\pi i \tilde{\sigma}_\lambda}$  for some two-cocycle  $\tilde{\sigma}_\lambda$  on  $G(n, \mathbb{R})$ . Moreover, any pair of two-cocycles  $\sigma_\lambda$  and  $\sigma_\mu$  of the form described above are homotopic in the sense of Packer and Raeburn [23, Section 4]. Hence, one may use [23, Theorem 4.2 and Corollary 4.5] to deduce that

$$K_i(C^*(G(n)), \sigma_\lambda) \cong K_i(C^*(G(n)), \sigma_\mu) \cong K_{\mathrm{top}}^{i+\frac{1}{2}n(n-1)}(G(n, \mathbb{R})/G(n)).$$

**Remark 5.6** (new). The groups  $G(n)$ , or more generally all the free nilpotent groups  $G(m, n)$  of class  $m$  and rank  $n$  are finitely generated (torsion-free) nilpotent groups. Let  $\sigma$  be a two-cocycle of  $G(m, n)$  and suppose that there are no nontrivial central  $\sigma$ -regular elements in  $G(m, n)$ , that is,  $C^*(G(m, n), \sigma)$  is simple with a unique trace by [22] (see Theorem 4.4 and the subsequent results above for the case of  $G(n)$ ). Then there exists an irreducible representation  $\pi_\sigma$  of  $G(m+1, n)$  such that  $C^*(G(m, n), \sigma) \cong C^*(\pi_\sigma(G(m+1, n)))$  by using Theorem 1.1, Remark 3.2, and (4). Thus it follows from [9] and [8] that  $C^*(G(m, n), \sigma)$  is classified by its ordered  $K$ -theory within the class of simple, unital, nuclear  $C^*$ -algebras with finite nuclear dimension that satisfy the universal coefficient theorem.

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This document includes several computations and details that were left out in the published version of the article from 2015, in addition to a reformulation of Proposition 2.2 in terms of arbitrary semidirect products, a more general statement in Lemma 4.1, and a new Remark 5.6 that makes a connection to recent work. The latest arxiv version of the paper [20] includes all these updates, except Lemma 4.1 and a few other minor details.

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