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Master's Thesis

INFLATION IN THE ERA OF PRECISION COSMOLOGY

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Certificate

This is to certify that the thesis entitled "**Inflation in the era of Precision Cosmology**" is a record of the bonafide work done by **Mr. Ommair Ishaque**, Roll No. MS0821011. Thesis is submitted to the University of Karachi in partial fulfilment of the requirements for the award of the degree of **Master of Science**.

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Abstract

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INFLATION IN THE ERA OF PRECISION COSMOLOGY

by Ommair Ishaque

In this thesis polynomial chaotic and hybrid inflationary model has been investigated using scalar field. Since the scalar field has vital role in particle physics, investigating inflation by inflaton(scalar) field can provide a bridge between cosmological and particle physics phenomena. Role of quantum corrections has been studied during the inflationary era, employed to tree level polynomial chaotic models (ϕ^2 and ϕ^4), which arose due to the Yukawa interactions of ϕ. Radiatively corrected ϕ^2 model looks quite compatible compared with PLANCK+WP+highL bounds with limits, $0.939 \lesssim n_s \lesssim 0.966$ and $0.030 \lesssim r \lesssim 0.131$. While ϕ^4 model has limits, $0.944 \lesssim n_s \lesssim 0.956$ and $0.065 \lesssim r \lesssim 0.120$. The corrections terms for hybrid model originate from Yukawa interaction between ϕ and ν_R. In contrast to TLHI model, red tilted spectral index n_s is obtained for RCHI model, consistent with PLANCK data, for sub-Planckian inflaton field. The predicted tensor-to-scalar ratio by RCHI model for sub-Planckian inflaton field lies well below the precision of PLANCK mission to observe primordial gravitational waves, while for radiatively corrected polynomial chaotic models, r is appreciable enough to detect gravity waves.

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Contents

To my mother and father

Chapter 1

Introduction

Cosmology is one of the oldest branch of science which investigates the properties of the universe i.e. origin, evolution and how the universe will going to end or live forever. It provides us methods to develop theories for the structure of the universe from past to future. Theories validation depends on the experimental observations whether it is correct or not or need some improvement. Our current understanding is based on Standard Hot Big Bang Model (SBB) (Big Bang was not a explosion in real sense), or Friedmann-Robertson-Walker (FRW) cosmological model. This model is very simple in its mathematical structure and explains the evolution of the universe by just two differential equations which are derived from Einstein field equations. Qualitatively, in SBB, universe at its early stages assumed to be very hot and dense, dominated by radiation or matter undergoing rapid expansion. The rate of expansion slows down with time due to the gravitational attraction. Energy density of matter and radiation decreases and the temperature drops down with the expansion of the universe. The average properties of the universe, energy density and temperature are assumed to have same value everywhere. SBB model is based on the assumption of the homogeneity and isotropy of the universe on large scales, combination of which is known as Cosmological Principle, a key assumption of the Standard Big Bang Model [1].

This framework is based on three experimental observations: (a) Receding velocities of the galaxies is proportional to the distant from the observer ($V = Hd$), where H is the Hubble constant; (b) Weak electromagnetic (Radio) signals has been observed from all parts of the sky at same temperature of $2.7K$, known as Cosmic Microwave Background Radiation (CMBR), which is assumed to be the remnant of the big bang; (c) Abundance of light nuclei like hydrogen ($\sim 75\%$), helium($\sim 25\%$) and traces of some other elements, and all heavy elements and their isotopes are constituents of protons and neutrons, known as big bang nucleosynthesis; (d) and structure formation of the galaxies.

Beside the successes of the SBB model, still requires various modifications because its mathematical structure is derived from Einstein's theory of general relativity, which gives singularities when it is applied to the space-time point of subatomic levels, for that quantum mechanics is required and still today we don't have any such model which unifies general relativity and quantum mechanics, and also don't have experimental verification.

Apart from the above mention problem, SBB model has some serious shortcomings: why the energy density of the universe is so close to the critical value of the density?, it means universe was flat from its early stages, which requires fine tuning (Flatness Problem). Why CMBR is so uniform (same temperature) coming from regions of the universe which are casually disconnected (Horizon Problem)? In Grand Unified theories whenever symmetry breaks (like Higgs mechanism) it is connected to phase transition of the universe at early times. And during such phase transition different kinds of topological defects, magnetic monopoles and super-symmetric particles arise which is associated with the huge production of super-heavy stable particles which interacts very weakly. SBB model fails to explain the existence of such types of super-heavy particles. The model also fails to give any firm explanation of the origin of small density fluctuations required for the structure formation like galaxies, in the universe and the cause of the observed temperature fluctuations in the CMBR [15].

To solve the fundamental problems in the SBB model, the idea of accelerated expansion at the early times of the universe, i.e. *inflation*, was put forward by Alan Guth and Katsuhiko Sato independently in early 1980's, today which is known as *old inflation*. Guth's model was based on the process of bubble nucleation, universe has gone through the inflationary era which took place in the false vacuum state and leaves the universe big and flat. Inflation continues until false vacuum decays through quantum mechanical tunnelling to true vacuum state, bubbles collide and leaving the hot universe at the end. The main problem was in the concept of false vacuum state. It was believed that quantum mechanical tunnelling could end inflation in an elegant fashion, but the formation of the bubbles was highly random process in the model i.e. they could be formed near (not enough inflation to solve the cosmological problems) or far (too much inflation would leave the universe empty) from each other, both the options were unsatisfactory. This is known as "graceful exit" problem.

Linde, and Albrecht and Steinhardt in 1982 gave the successful solution of graceful exit problem, which is known as *new inflation*. In this model, inflation commence in a false vacuum state, or in a plateau shaped rickety state (where the value of the field is zero) of the potential. Instead of tunnelling mechanism, concept of slow roll mechanism of the field was used for the end of inflation, field continue to slow roll until it reaches the steeper part of the potential. Idea of new inflation was also abandoned later because of fine tuning problem. In 1983 Linde came up with more improved version of the slow roll inflation which is known as *chaotic inflation*. This model gave logical explanations, homogeneous and isotropic universe can be achieved if enough amount of inflation take place independent of shape of the potential. But in the old and new inflation, the state of thermal equilibrium of the universe was assumed from the beginning [2].

Plenty of inflationary models have been formulated since idea of inflation has taken birth. Each model has its implication limits and motivation on which they are build. Currently Planck satellite is exploring the secrets of the universe by measuring the anisotropies of CMB and also verifying the validity of the SBB model, with great accuracy than (Wilkinson Microwave Anisotropy Probe) WMAP and (Cosmic Background Explore) COBE which were observing the universe for almost last two decades. Planck latest data has put strong bounds on the existing models and some of the models are in danger i.e. touching the exclusion limits and supports the idea of inflation.

Throughout this thesis *Planck units*$(\hbar = c = 1)$ has been used. The structure of this thesis is as follows. In **chapter 2**, pedagogical review of Friedmann-Lemaitre-Robertson-Walker cosmology has been presented and discussed the standard cosmological problems, flatness, horizon and monopole problems.
Idea of inflation is introduced in **chapter 3** as a possible solid solution to the cosmological puzzles. Then introduced the concepts of inflation driven by scalar fields and slow roll inflation.
In **chapter 4** we studied the cosmological perturbations generated during inflationary phase, which explains why universe is inhomogeneous at small scales. Then discussed power spectrum of scalar and tensor perturbations. And the most important inflationary parameters, spectral index, tensor-to-scalar ratio and running of spectral index has been discussed.
The main essence of this thesis lie in **chapter 5** and **chapter 6**, in which tree level and radiatively corrected analysis of chaotic polynomial and hybrid model has been done, campare with recent PLANCK data.

Chapter 2

Standard Model of Cosmology And Its Shortcomings

2.1 Friedmann-Robertson-Walker-Lemaitre Universe

Standard Big Bang cosmology which explains the evolution of the universe since it was born, lies on the assumption of isotropy and homogeneity (Cosmological Principle) which is well supported by the observations. General relativity provides the mathematical formalism, which gives the quantitative illustration of the cosmology and the key equation in it is the Einstein field equation as,

$$G^{\mu}_{\nu} \equiv R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}R = 8\pi G_N T^{\mu}_{\nu} \tag{2.1}$$

The general form of the metric can be written as

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} = -dt + a(t)^2 g_{ij}dx^i dx^j, \tag{2.2}$$

where g_{ij} is the metric tensor and specify the spatial part of the above metric, and $a(t)$ is the scale factor, a function of time, which tells us by what factor universe is expanding or contracting. Coherent with the cosmological principle, what plausible forms are valid for g_{ij}. One of the most obvious possibility is Euclidean like metric tensor i.e. $g_{ij} = \delta_{ij}$. Second possibility can be evaluated by visualizing the three-spatial dimensions embedded in a 4D Euclidean space. Assuming that those three dimensions are the part of the surface of the 3D sphere of unit radius such that,

$$x^2 + y^2 + z^2 + w^2 = 1, \tag{2.3}$$

where w is a fabricated coordinate of the embedding space. Now line element corresponding to such a space is

$$dl^2 = dx^2 + dy^2 + dz^2 + dw^2 \tag{2.4}$$

Now eliminating dw from equation (2.4) by taking the differential of equation (2.3)

$$dw^2 = \frac{(xdx + ydy + zdz)^2}{1 - (x^2 + y^2 + z^2)}. \tag{2.5}$$

Using the transformation equations so that x, y, z in terms of r, θ, ϕ as

$$x = rsin\theta cos\phi, \qquad y = rsin\theta sin\phi \qquad z = rcos\theta, \tag{2.6}$$

then the line element dl^2 will become

$$dl^2 = \frac{dr^2}{1 - r^2} + r^2(d\theta^2 + r^2 sin^2\theta d\phi^2) \tag{2.7}$$

The third possibility of the metric tensor g_{ij} can be derived similarly by putting ir as radius in equation (2.3) then dl^2 can be written as,

$$dl^2 = \frac{dr^2}{1 + r^2} + r^2(d\theta^2 + r^2 sin^2\theta d\phi^2) \tag{2.8}$$

On combining all three possibilities of spatial metric dl^2 we have,

$$dl^2 = \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + r^2 sin^2\theta d\phi^2), \tag{2.9}$$

where k can take values $0, \pm 1$. Now substituting the equation (2.9) in equation (2.2) we get,

$$ds^2 = -dt^2 + a(t)^2[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + r^2 sin^2\theta d\phi^2)] \tag{2.10}$$

The coordinates (r, θ, ϕ) in FRW metric are called co-moving coordinates, i.e. they remains fixed with the evolution of the universe. The physics distance between any two regions or points can be calculated as, $s = a(t)x$, where s is the physical distance and x is co-moving distance. And t is the physical time of the co-moving observer whose 4-velocity in the co-moving coordinates can be written as, $U^\alpha = \frac{dx^\alpha}{ds}$ [3].

Equation (2.10) can also be written in the form for convenience, we have
Let us redefine g_{rr} component as

$$d\chi = \frac{dr}{\sqrt{1 - kr^2}} \tag{2.11}$$

Such that,

$$ds^2 = -dt^2 + a(t)^2[d\chi^2 + \Phi_k^2(\chi)(d\theta^2 + sin^2\theta d\phi^2)] \tag{2.12}$$

Where,

$$\Phi_k(\chi) = \begin{cases} sinh\chi & k = -1 \\ \chi & k = 0 \\ sinh\chi & k = +1 \end{cases} \tag{2.13}$$

The matrix form of equation (2.10)

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a(t)^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & a(t)^2r^2 & 0 \\ 0 & 0 & 0 & a(t)^2r^2sin^2\theta \end{pmatrix} \tag{2.14}$$

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1-kr^2}{a(t)^2} & 0 & 0 \\ 0 & 0 & a(t)^{-2}r^{-2} & 0 \\ 0 & 0 & 0 & a(t)^{-2}r^{-2}sin^{-2}\theta \end{pmatrix} \tag{2.15}$$

To solve the equation(2.1) we are required to calculate Ricci tensor and Ricci scalar, respectively

$$R^{\lambda}_{\mu\lambda\nu} = R_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\lambda}_{\mu\lambda,\nu} + \Gamma^{\lambda}_{\rho\lambda}\Gamma^{\rho}_{\mu\nu} - \Gamma^{\lambda}_{\rho\mu}\Gamma^{\rho}_{\lambda\nu} \quad , \quad R = g^{\mu\nu}R_{\mu\nu} \tag{2.16}$$

Where $R^{\lambda}_{\mu\nu\alpha}$ is Reimann Curvarture tensor. The next step is to evaluate components of the Christofell symbols $\Gamma^{\lambda}_{\mu\nu}$:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\alpha}(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}) \quad , \quad g_{\mu\nu,\alpha} = \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \tag{2.17}$$

From equation(2.17) we can calculate the non-zero components of the Christofell symbol. The time- and curvature-independent components of the Christofell connection are:

$$\Gamma^{\theta}_{r\theta} = \Gamma^{\phi}_{r\phi} = \frac{1}{r} \quad , \quad \Gamma^{\theta}_{\phi\phi} = -sin\theta cos\theta \quad , \quad \Gamma^{\phi}_{\phi\theta} = cot\theta. \tag{2.18}$$

The time-independent but curvature-dependent components are:

$$\Gamma^{r}_{rr} = \frac{kr}{(1-kr^2)} \quad , \quad \Gamma^{r}_{\theta\theta} = -r(1-kr^2) \quad , \quad \Gamma^{r}_{\phi\phi} = -rsin^2\theta(1-kr^2), \tag{2.19}$$

and the time-dependent ones are:

$$\Gamma^t_{rr} = \frac{a\dot{a}}{(1-kr^2)} \quad , \quad \Gamma^t_{\theta\theta} = a\dot{a}r^2 \quad , \quad \Gamma^t_{\phi\phi} = a\dot{a}r^2 sin^2\theta, \tag{2.20}$$

and

$$\Gamma^r_{tr} = \Gamma^\theta_{t\theta} = \Gamma^\phi_{t\phi} = \frac{\dot{a}}{a}. \tag{2.21}$$

Now using equation(2.16) the components of the Ricci tensor are:

$$\begin{aligned} R_{tt} &= -\partial_t(\Gamma^r_{rt} + \Gamma^\theta_{\theta t} + \Gamma^\phi_{\phi t}) - (\Gamma^r_{rt}\Gamma^r_{rt} + \Gamma^\theta_{\theta t}\Gamma^\theta_{\theta t} + \Gamma^\phi_{\phi t}\Gamma^\phi_{\phi t}) \\ &= -[(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}) + (\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}) + (\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2})] - [(\frac{\dot{a}}{a})^2 + (\frac{\dot{a}}{a})^2 + (\frac{\dot{a}}{a})^2] \\ &= -3\frac{\ddot{a}}{a} \end{aligned} \tag{2.22}$$

$$\begin{aligned} R_{rr} &= (\partial_t\Gamma^t_{rr} + \partial_r\Gamma^r_{rr}) - (\partial_r\Gamma^r_{rr} + \partial_r\Gamma^\theta_{\theta r} + \partial_r\Gamma^\phi_{\phi r}) \\ &+ [\Gamma^t_{rr}(\Gamma^r_{rt} + \Gamma^\theta_{\theta t} + \Gamma^\phi_{\phi t}) + \Gamma^r_{rr}(\Gamma^r_{rr} + \Gamma^\theta_{\theta r} + \Gamma^\phi_{\phi r}) \\ &- (\Gamma^r_{rr}\Gamma^r_{rr} + \Gamma^\theta_{\theta r}\Gamma^\theta_{\theta r} + \Gamma^\phi_{\phi r}\Gamma^\phi_{\phi r} + 2\Gamma^r_{tr}\Gamma^t_{rr})] \\ &= (\frac{a\ddot{a} + \dot{a}^2}{1-kr^2}) + (\frac{1}{r^2} + \frac{1}{r^2}) + [\frac{a\dot{a}}{1-kr^2}(-\frac{\dot{a}}{a} + \frac{\dot{a}}{a} + \frac{\dot{a}}{a}) \\ &+ \frac{kr}{1-kr^2}(\frac{1}{r} + \frac{1}{r}) - (\frac{1}{r^2} + \frac{1}{r^2})] \\ &= \frac{a^2}{1-kr^2}(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2}) \end{aligned} \tag{2.23}$$

$$\begin{aligned} R_{\theta\theta} &= (\partial_t\Gamma^t_{\theta\theta} + \partial_r\Gamma^r_{\theta\theta} - \partial_\theta\Gamma^\phi_{\phi\theta}) + [\Gamma^t_{\theta\theta}(\Gamma^r_{rt} + \Gamma^\theta_{\theta t} + \Gamma^\phi_{\phi t}) \\ &+ \Gamma^r_{\theta\theta}(\Gamma^r_{rr} + \Gamma^\theta_{\theta r} + \Gamma^\phi_{\phi r}) - (\Gamma^\phi_{\phi\theta}\Gamma^\phi_{\phi\theta} + 2\Gamma^t_{\theta\theta}\Gamma^\theta_{t\theta} + 2\Gamma^r_{\theta\theta}\Gamma^\theta_{r\theta})] \\ &= r^2(a\ddot{a} + \dot{a}^2) - (1 - 3kr^2) + (1 + \frac{cos^2\theta}{sin^2\theta}) \\ &+ [r^2 a\dot{a}(\frac{\dot{a}}{a} - \frac{\dot{a}}{a} + \frac{\dot{a}}{a}) - r(1-kr^2)(\frac{kr}{1-kr^2} - \frac{1}{r} + \frac{1}{r}) - \frac{cos^2\theta}{sin^2\theta}] \\ &= a^2r^2(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2}) \end{aligned} \tag{2.24}$$

$$\begin{aligned} R_{\phi\phi} &= (\partial_t\Gamma^t_{\phi\phi} + \partial_r\Gamma^r_{\phi\phi} + \partial_\theta\Gamma^\theta_{\phi\phi}) + [\Gamma^t_{\phi\phi}(\Gamma^r_{rt} + \Gamma^\theta_{\theta t} + \Gamma^\phi_{\phi t}) + (\Gamma^\theta_{\phi\phi}\Gamma^\phi_{\phi\theta}) \\ &+ \Gamma^r_{\phi\phi}(\Gamma^r_{rr} + \Gamma^\theta_{\theta r} + \Gamma^\phi_{\phi r}) - (2\Gamma^t_{\phi\phi}\Gamma^\phi_{t\phi} + 2\Gamma^r_{\phi\phi}\Gamma^\phi_{r\phi} + 2\Gamma^\theta_{\phi\phi}\Gamma^\phi_{\theta\phi})] \end{aligned}$$

$$
\begin{aligned}
&= ((a\ddot{a}+\dot{a}^2)r^2sin^2\theta-(1-3kr^2)sin2\theta+(sin^2\theta-cos^2\theta)) \\
&+ [a\dot{a}r^2sin^2\theta(\frac{\dot{a}}{a}+\frac{\dot{a}}{a}-\frac{\dot{a}}{a})-(-cos^2\theta) \\
&-(r-kr^3)sin^2\theta(\frac{kr}{1-kr^2}+\frac{1}{r}-\frac{1}{r})] \\
&= a^2r^2sin^2\theta(\frac{\ddot{a}}{a}+2\frac{\dot{a}^2}{a^2}+2\frac{k}{a^2}) \qquad (2.25)
\end{aligned}
$$

Multiplying the convariant components of the Ricci tensor by corresponding components of the contravariant metric tensor, we then have components of the mixed Ricci tensor as

$$R^{\mu}_{\nu}=g^{\mu\tau}R_{\tau\nu} \qquad (2.26)$$

or we can have

$$R^t_t=g^{tt}R_{tt}=3\frac{\ddot{a}}{a} \qquad (2.27)$$

$$R^r_r=g^{rr}R_{rr}=(\frac{\ddot{a}}{a}+2\frac{\dot{a}^2}{a^2}+2\frac{k}{a^2}) \qquad (2.28)$$

$$R^\theta_\theta=g^{\theta\theta}R_{\theta\theta}=(\frac{\ddot{a}}{a}+2\frac{\dot{a}^2}{a^2}+2\frac{k}{a^2}) \qquad (2.29)$$

$$R^\phi_\phi=g^{\phi\phi}R_{\phi\phi}=(\frac{\ddot{a}}{a}+2\frac{\dot{a}^2}{a^2}+2\frac{k}{a^2}) \qquad (2.30)$$

From equation(2.16) the Ricci scalar is

$$
\begin{aligned}
R &= g^{tt}R_{tt}+g^{rr}R_{rr}+g^{\theta\theta}R_{\theta\theta}+g^{\phi\phi}R_{\phi\phi} \\
&= 6(\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}) \qquad (2.31)
\end{aligned}
$$

Components of the Einstein's tensors is

$$G^t_t\equiv R^t_t-\frac{1}{2}\delta^t_tR=-3(\frac{\dot{a}}{a}+\frac{k}{a^2}) \qquad (2.32)$$

and

$$G^i_j\equiv R^i_j-\frac{1}{2}\delta^i_jR=-(2\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}) \quad , \quad i,j=r,\theta,\phi \qquad (2.33)$$

To solve the equation(2.1) finally we need the energy momentum tensor of the cosmic fluid. Based on the assumption of homogeneity and isotropy the energy-momentum tensor of an ideal cosmic fluid is given by

$$T^\mu_\nu=g^{\mu\alpha}T_{\mu\alpha}=(\rho+p)u^\mu u_\nu+p\delta^\mu_\nu, \qquad (2.34)$$

where ρ is the matter energy density, p is the isotropic pressure and $u^\beta\equiv\frac{dx^\beta}{d\tau}=(1,0,0,0)$ is a four velocity of the cosmic fluid in a co-moving frame. Here τ is representing the proper time of an observer, satisfying $g_{\mu\nu}u^\mu u^\nu=-1$.

Using these results T^{μ}_{ν} of an ideal cosmic fluid can be written in the matrix form:

$$T^{\mu}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{2.35}$$

Substituting the results from equations(2.32),(2.33) and (2.35) in equation(2.1) in components, we have time-time component:

$$\begin{aligned} G^{t}_{t} &= -3(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}) = -8\pi G_N \rho \\ H^2 &= \frac{8\pi G_N}{3}\rho - \frac{k}{a^2} \end{aligned} \tag{2.36}$$

Where $H(t) \equiv \frac{da(t)/dt}{a}$ is define as Hubble parameter. And space-space component:

$$G^{r}_{r} = G^{\theta}_{\theta} = G^{\phi}_{\phi} = -(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}) = 8\pi G_N p$$

Using Eq(2.36)

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}(\rho + 3p) \tag{2.37}$$

or

$$\dot{H} + H^2 = -\frac{4\pi G_N}{3}(\rho + 3p) \tag{2.38}$$

Equation(2.36) is called the *Friedmann Equation* and equation(2.37) or (2.38) is called *Raychaudhuri Equation*. The solution of these equations explains the evolution of the Universe in the Standard Big Bang Model.

To get continuity equation differentiate equation(2.36) w.r.t time and divide it by $a\dot{a}$,we have

$$\frac{\ddot{a}}{a} = \frac{4\pi G_N}{3}\dot{\rho}\frac{a}{\dot{a}} + \frac{8\pi G_N}{3}\rho \tag{2.39}$$

Now inserting equation(2.39) in equation(2.37), we get

$$\dot{\rho} + 3(\rho + p)H = 0 \tag{2.40}$$

Which is energy-momentum conservation equation in the cosmology and implies that it is not independent of equations (2.36) and (2.37) because it is formulated from them. This result can be derived from the zeroth component of $\nabla_{\mu}T^{\mu}_{\nu} = 0$, which shows that Einstein tensor is covarianlty conserved, i.e. $\nabla_{\mu}G^{\mu}_{\nu} = 0$.

The Friedmann equation (2.36) contains k whose sign is important because evolution of scale factor a depends on it. The *critical density* is define as the density of the Universe for which $k = 0$ i.e. geometrically flat Universe,

$$\rho_c \equiv \frac{3H^2}{8\pi G_N} \tag{2.41}$$

Different universes can be characterized as, for $\rho > \rho_c$ universe is positively curved and closed, with finite volume, for $\rho < \rho_c$ the universe is negatively curved and open, with infinite volume and for $\rho = \rho_c$ the universe is flat and open [4].
Now Friedmann equation can be rewritten in terms of density parameter Ω as

$$\Omega(t) - 1 = \frac{k}{a^2 H^2} \tag{2.42}$$

where Ω is define as the ratio of actual energy density ρ to the critical density ρ_c

$$\Omega \equiv \frac{\rho}{\rho_c} \tag{2.43}$$

2.2 Solution of the Friedmann Equation

To study the dynamics of the FRW universe, i.e. how does the energy-momentum tensor explain the expansion of the universe, solution of the continuity equation is required. Here in this thesis only three basic types of cosmological energy-momentum are studied: matter, radiation and vacuum.
To evaluate the solutions of equation(2.40), we first rewrite it in terms of equation of state parameter as:

$$dln\rho = -3(1+\omega)dlna, \quad with \quad \omega \equiv p/\rho \tag{2.44}$$

On integrating the equation(2.44), we have

$$\begin{gathered}\int_{\rho}^{\rho_o} dln\rho = -3(1+\omega)\int_{a}^{a_o} dlna \\ \frac{\rho(a)}{\rho_o} = a_o^{3(1+\omega)} a^{-3(1+\omega)}\end{gathered} \tag{2.45}$$

here a_o and ρ_o represents the present values of the scale factor and energy density. $a_o = 1$ is taken so that scale factor can be normalized. Experimentally it is observed that the present energy density is nearly equally to the critical value of the energy density i.e. $\rho_o = \rho_c$.
Then we can write,

$$\rho(a) = \rho_o a^{-3(1+\omega)} \tag{2.46}$$

Inserting equation(2.46) in (2.36) with $k = 0$ and integrate, we have

$$\int_0^a a^{\frac{3}{2}(1+\omega)-1} da = \sqrt{\frac{8\pi G_N \rho_o}{3}} \int_0^t dt$$

$$a(t) = [\frac{3}{2}(1+\omega)H_o]^{\frac{2}{3(1+\omega)}} t^{\frac{2}{3(1+\omega)}} \tag{2.47}$$

equation (2.46) is for $\omega \neq -1$.
For the universe dominated by the pressureless non-relativistic matter and for the relativistic matter (radiation), equations (2.46) and (2.47) yields,

$$Matter \quad dominant \quad : \quad \rho \propto a^{-3}, \quad a \propto t^{\frac{2}{3}} \quad for \quad \omega = 0 \quad (p = 0) \tag{2.48}$$

$$Radiation \quad dominant \quad : \quad \rho \propto a^{-4}, \quad a \propto t^{\frac{1}{2}} \quad for \quad \omega = 1/3 \quad (p = \tfrac{1}{3}\rho) \tag{2.49}$$

Equations (2.48) and (2.49) shows that energy density of the universe decreases in both the radiation and matter dominant eras as the scale factor grows with time i.e. $V \propto a^3$, where V is the volume of the sphere or box which contains these matters. In addition to the mentioned fact relativistic matter also loses energy because of Doppler redshift (cosmological redshift), $\nu \propto a^{-1}$, that's why energy density in the radiation dominant era is proportional to inverse fourth of the scale factor.
Now if we evaluate the equation (2.46) for $\omega = -1$ (Vacuum dominant era or cosmological constant) and insert the result in equation (2.36) and integrating for $k = 0$ we have,

$$\rho = \rho_o, \quad a(t) \propto \exp Ht \tag{2.50}$$

Equation (2.50) shows that energy density of the universe remains constant in vacuum energy state which implies Hubble parameter remains independent of time, as the scale factor grows exponentially.

2.3 Hubble Radius and Particle Horizon

The Concept of Hubble radius and particle horizon are very important to understand the evaluation of the universe according to SBB model and shortcomings it contains. One can calculate particle horizon using FRW metric. Light or photons which are coming from the past of the universe is the only method to calculate the distances and time it has taken to reach us at present, photons follows the light-like space-time, i.e. $ds^2 = 0$. Let us redefine FRW metric in terms of conformal time τ for convenience such that

$$ds^2 = 0 = a(\tau)[-d\tau^2 + (d\chi^2 + \Phi_k^2(\chi)d\Omega)] \tag{2.51}$$

where $d\Omega = d\theta^2 + sin^2\theta d\phi^2$ and,

$$\tau \equiv \int \frac{dt}{a(t)} \tag{2.52}$$

Because of isotropy of the space, we can take $d\theta = d\phi = 0$ and consider only radial propagation of photons, equation (2.43) becomes

$$0 = a(\tau)[-d\tau^2 + (d\chi^2] \tag{2.53}$$

integrating we have,

$$\chi_p(t) = \tau - \tau_i = \int_{t_i}^{t} \frac{dt}{a(t)} \tag{2.54}$$

Equation (2.54) tells us, the maximum co-moving length or distance photon could have covered since initial singularity (usually taken $t_i \equiv 0$ such that $a(t_i \equiv 0) \equiv 0$) to some time t later, known as co-moving particle horizon. Or simply whether the past occurred events in the universe are observable or not depends on this maximum co-moving distance, if they lie outside co-moving particle horizon then those events could never been observed.

As mentioned before physical length of the co-moving particle horizon can calculated by just multiplying it by the scale factor $a(t)$ at some time t,

$$d_p = a(t) \int_{t_i}^{t} \frac{dt'}{a(t')} \tag{2.55}$$

Let us represent co-moving particle horizon in a way which has more physical insight as,

$$\chi_p(t) \equiv \triangle\tau = \int_{a_i}^{a} \frac{1}{aH}\frac{da}{a} = \int_{a_i}^{a} \left(\frac{1}{aH}\right) dlna \tag{2.56}$$

The quantity $(aH)^{-1}$ in the above equation is called the co-moving Hubble radius or co-moving Hubble horizon. The physical Hubble radius or Hubble horizon is H^{-1}. Every observer in the universe has it own co-moving or physical Hubble radius which evolves with time, it is a distance beyond which we can't observe our universe at time t. According SBB model co-moving particle horizon and co-moving Hubble radius are of the same order (approximately equal) and sometimes they are used interchangeable with a name *Horizon*. It is crucial to understand the conceptual distinction between $\chi_p(\tau)$ and $(aH)^{-1}$ because the heart of the big bang puzzles lie in them:

- If the regions of the universe are apart by lengths or distances greater, compare to χ_p, they never could have shared any information or communicated with one another.

- If they are apart by lengths or distances greater than $(aH)^{-1}$, they can't exchange any signal between them or communicate now.

But it could be possible that regions which are not communicating today, could have communicated early on only if $\chi_p > (aH)^{-1}$ at present.

Using the definition of conformal time we can express equations (2.47) and (2.50) as,

$$a(t) \propto \begin{cases} \tau^{2/(1+3\omega)} & \omega \neq -1 \\ (-\tau)^{-1} & \omega = -1 \end{cases} \tag{2.57}$$

We can see easily equation (2.57) gives τ, τ^2 and $(-\tau)^{-1}$ for $\omega = 0, 1/3, -1$ respectively. Now, if we equate equation (2.47) in the definition of co-moving Hubble radius, simplifying we have

$$(aH)^{-1} = H_o a^{(1+3\omega)/2}, \tag{2.58}$$

where H_o represents the present value of the Hubble parameter.

2.4 Standard Cosmological Puzzles

Standard Big Bang model is pretty successful and simple in it approach, explains the mechanism of the universe and observationally well supported. But the problem arises when the question has been asked, why the universe looks like the way it is? Such a question can be analysed by extrapolating the model back in time, which tells model has some fundamental theoretical flaws i.e. it requires some particular initial conditions which results in the universe what we see today. There are many unresolved questions in the SBB model, few of which are discussed here.

2.4.1 Flatness Problem

Why universe's present energy density is approximately equal to critical density? To analyse this problem let us write Friedmann equation in terms of density parameter Ω and co-moving Hubble $(aH)^{-1}$ radius as given in the equation (2.42). As mentioned before in Standard big bang model, universe has only gone through radiation and matter dominated era and velocities of the regions (galaxies etc.) in the universe has decreased with time i.e. deceleration ($\ddot{a} < 0$), which can be seen from equation (2.37) for $p = 0$ and $p = \frac{\rho}{3}$. It is important here to calculate co-moving Hubble radius using equation (2.58) so that we can get more clear picture of the universe's evolution from Friedmann equation (2.42) as,

$$(aH)^{-1} \propto \begin{cases} a^{1/2} & \omega = 0 \\ a & \omega = 1/3 \end{cases} \tag{2.59}$$

It shows that as the decelerating universe gets older and older, co-moving Hubble Radius increases with time, the observer which draws a co-moving Hubble radius around him will include more and more co-moving observers inside his co-moving Hubble radius. In other words, in decelerating universe $\dot{a} = aH$ decreases and $\frac{k}{(aH)^2}$ increases with time means ρ should go away from ρ_c but still observed universe today has $\rho \sim \rho_c$ i.e. $\frac{k}{(aH)^2}$ has less than 1% contribution in comparison $\frac{\rho}{\rho_c}$,why it is so? But if we go back in time, $\frac{k}{(aH)^2}$ becomes more smaller than it is today. Let us see this quantitatively, using equation (2.59) in equation (2.42) we get,

$$\Omega(t) - 1 \propto \begin{cases} a & MD \\ a^2 & RD \end{cases} \tag{2.60}$$

From above equation it is clear, for both radiation and matter dominant phases $(\Omega(t)-1)$ decreases if we go back in time. Now taking a ratio of equation (2.42) evaluating at some initial time t_i and at present time t_o, we get

$$\frac{|\Omega(t_i) - 1|}{|\Omega(t_o) - 1|} = \left(\frac{(a(t_i)H(t_i))^{-1}}{(a(t_o)H(t_o))^{-1}} \right)^2 \tag{2.61}$$

Again using equation (2.59) in equation (2.61), evaluating at Big Bang nucleosynthesis(BBN) time $t_i = t_{BBN}$ and using $(aT = constant)$ we get,

$$\frac{|\Omega(t_{BBN}) - 1|}{|\Omega(t_o) - 1|} \approx \left(\frac{a(t_{BBN})}{a(t_o)} \right)^2 \approx \left(\frac{T_o}{T_{BBN}} \right)^2 \approx \mathcal{O}(10^{-16}) \tag{2.62}$$

Here, $(T_{BBN} \sim 10^{-3} GeV)$ representing temperature at BBN epoch and $(T_o \sim 10^{-13} GeV)$ is the temperature of the CMB radiation today. Since $(|\Omega(t_o) - 1|) \approx 10^{-2}$, we have roughly taken it ~ 1.

Now if we go back further in time, $t_i = t_{GUT} \sim 10^{16} GeV$ then,

$$\frac{|\Omega(t_{GUT}) - 1|}{|\Omega(t_o) - 1|} \approx \left(\frac{a(t_{GUT})}{a(t_o)} \right)^2 \approx \left(\frac{T_o}{T_{GUT}} \right)^2 \approx \mathcal{O}(10^{-58}) \tag{2.63}$$

And at $t_i = t_{Pl} \sim 10^{19} GeV$ we have,

$$\frac{|\Omega(t_{Pl}) - 1|}{|\Omega(t_o) - 1|} \approx \left(\frac{a(t_{Pl})}{a(t_o)} \right)^2 \approx \left(\frac{T_o}{T_{Pl}} \right)^2 \approx \mathcal{O}(10^{-64}) \tag{2.64}$$

From equations (2.62), (2.63) and (2.64), ρ was extremely close to the ρ_c in the early universe. What is the reason behind this?

2.4.2 Horizon Problem

In Standard big model, initially universe was hot and dense, production of particle-antiparticles and photons had equal rates. But as the universe cooled down to the energy scale of $0.1eV$ due to the expansion, photons didn't had sufficient energy to produce more pairs particles and then electrons could combine with proton to form a neutral atom. Photons which left were permanent and could travel freely on average and today we are observing them as cosmic microwave background radiation having temperature of about $0.3eV$ because of cosmological red shift. Observations showed that CMB radiation has very small temperature variation $\mathcal{O}(10^{-5})$, means CMB radiation has almost uniform temperature across the visible universe. Why the temperature fluctuations in the CMB radiation are so small coming from regions which entered our horizon in the recent past observed by COBE, WMAP and currently PLANCK mission?
From equation (2.58), co-moving Hubble radius $(aH)^{-1}$ depends on the exponent of the scale factor $(1+3\omega)$. For $\omega = 0$ and $\omega = 1/3$ strong energy condition(SEC) is satisfied , $1+3\omega > 0$, means co-moving Hubble radius increases with the expansion of the universe. Now using equation (2.58) in equation (2.56) and solving the integral we have,

$$\triangle\,\tau = \frac{2H_o}{1+3\omega}\left[a^{(1+3\omega)/2} - a_i^{(1+3\omega)/2}\right] \tag{2.65}$$

For ordinary matter and radiation, all contributions in the integral is coming from the upper limit and becomes zero at lower limit for $a_i = 0$ (singularity),

$$\tau_i \propto a_i^{(1+3\omega)/2} = 0, \qquad for \quad \omega > -\frac{1}{3} \tag{2.66}$$

Above result shows that, scale factor becomes zero $(a \to 0)$ at finite value of conformal time $(\tau_i \to 0)$. Then we can write equation (2.65) as,

$$\chi_p(t) = \frac{2H_o}{1+3\omega}a^{(1+3\omega)/2} \qquad for \quad \omega > -\frac{1}{3} \tag{2.67}$$

Using equation (2.58) in equation (2.67) we get,

$$\chi_p(t) = \frac{2}{1+3\omega}(aH)^{-1} \qquad for \quad \omega > -\frac{1}{3} \tag{2.68}$$

Equation (2.68) tells us that in standard big bang cosmology $\chi_p(\tau) \sim (aH)^{-1}$. From equation (2.67), co-moving particle horizon increases with time as the universe expands, implies that the regions which are entering our horizon in recent times were long way outside the horizon when for the first time photons decoupled from matter. But the extreme uniformity in the temperature of CMB photons illustrates that universe was in thermal equilibrium at the time of decoupling, but in SBB model extremely large number of regions have past light-cone which do not intersect and were causally independent [5].

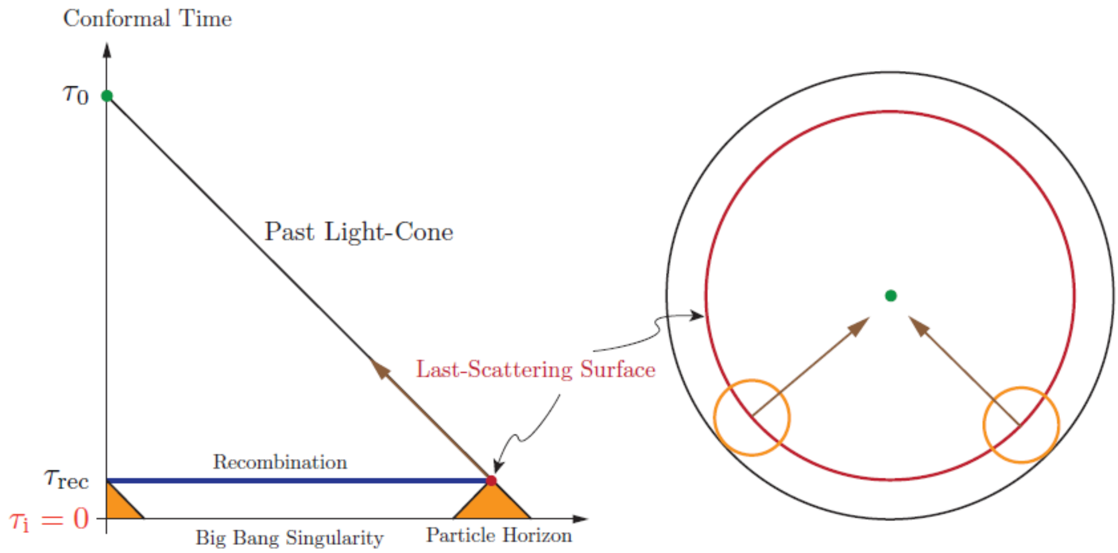

FIGURE 2.1: Horizon Problem in the Standard Cosmology.

2.4.3 Monopole Problem

Grand unified theories predicts the existence of magnetic monopoles, topological defects, domain wall etc, due to the phase transitions taken place at the early universe, which corresponds to the stable massive particles, having very low interaction cross-sections. Because of their stability, they decouple and freeze out in the early universe, becomes the dominant part of the universe and should be observe today ($\rho \propto a^{-3}$ for non-relativistic particles). But experimentally no magnetic monopoles has been detected so far which is inconsistent with the GUT's [6].

Chapter 3

Inflationary Paradigm And Scalar Field Dynamics

3.1 Idea of Inflation

The problems of the standard cosmological model (discussed in the chapter 2) indicates that modification should be made in the initial phases of evolution of the universe which SBB model predicts, to overcome the fine tuned initial conditions which give rise to the required smoothness and the flatness of the experimentally observed universe, give the mechanism behind the structure formation and explain why the probability of detection of magnetic monopoles, super-symmetric particles and domain walls is negligible at present [7].

All the puzzles can be resolved if an era of superluminal (accelerated) expansion take place in the very early stages of the universe, known as *Inflationary era.* From Friedman Equation (2.37),

$$\ddot{a} = \frac{d\dot{a}}{dt} > 0 \Longleftrightarrow \rho + 3p < 0 \tag{3.1}$$

gives negative pressure

$$p < -\frac{\rho}{3} \quad or \quad \omega < -\frac{1}{3} \tag{3.2}$$

which is the required condition for the inflationary phase. Here energy density is taken positive. Another important consequence of accelerated expansion is shrinking of co-moving Hubble radius w.r.t time,

$$\begin{aligned} \frac{d(aH)^{-1}}{dt} &= -(aH)^{-2}[\dot{a}H + \dot{H}a] \\ &= -(aH)^{-2}a[H^2 + \dot{H}] \\ &= -(aH)^{-2}\ddot{a} \end{aligned} \tag{3.3}$$

above result shows that, only if

$$\frac{d(aH)^{-1}}{dt} < 0 \Longrightarrow \ddot{a} > 0 \tag{3.4}$$

The relation between ρ and p plays an important role in explaining the different phases of the universe's evolution. The two cases which satisfies equation (3.3) are pure and quasi *de Sitter stage*. In pure de Sitter stage p and ρ are equal having opposite sign ($p = -\rho$). Such a case is already discussed in the chapter two, in which universe is dominated by constant vacuum energy density i.e. ($\dot{\rho} = 0$), gives rise to the exponential growth in the scale factor while H remains constant. And *quasi de Sitter stage* is the one in which $p \approx -\rho$ and Hubble parameter varies slowly. It is more convenient do calculations in the pure de Sitter stage [3].

3.2 Resolution of Standard Big Bang Model Puzzles

3.2.1 Solution to Flatness Problem

In the accelerating universe $a \propto e^{H_i t}$ which implies, $\dot{a} = aH_i$ increases then $\frac{k}{(aH)^2}$ will decrease. Friedman equation (2.42) will become,

$$\Omega - 1 = \frac{k}{(aH_i)^2} \propto \frac{1}{(\dot{a})^2} \propto e^{-2H_i(t_e - t_i)} \longrightarrow 0 \tag{3.5}$$

It shows that before the RD era may be $\rho \neq \rho_c$, but then there was long period of inflation, which drove ρ towards ρ_c, drove it so far towards ρ_c that by the beginning of the RD era even though ρ is drifting away from ρ_c ever since between the beginning of RD era and today, it hasn't drifted away enough to observable yet. Here H_i represents the Hubble parameter which remains constant in time during inflation and, t_i and t_e are times when inflation was initiated and end respectively.

3.2.2 Solution to Horizon Problem

The fundamental reason of horizon problem lie in the state of co-moving Hubble radius, it increases with time in the standard big bang model. An elegant solution to this problem is to consider a phase at an early times of the universe in which comoving Hubble radius shrink, i.e. inflation ($\ddot{a} > 0$). If the certain amount of inflation took place horizon problem can be solved. As discussed earlier, strong energy condition (SEC) has to be violated i.e. $1 + 3\omega < 0$, for decreasing co-moving Hubble radius with time. Now the lower limit of the integral (2.65) doesn't become zero at singularity ($a = 0$),

$$\tau_i \propto a_i^{(1+3\omega)/2} = -\infty, \qquad for \quad \omega < -\frac{1}{3} \tag{3.6}$$

Equation(3.6) shows that conformal time become negative infinity for $\omega = -1$,instead of zero at $a = 0$, i.e. inflationary phase has provided enough conformal time for the regions which were thought to be causally disconnected. As shown in Fig 3.1 the past light cones of the regions which were disconnected at the time, when for the first time photons(CMB radiation) left the matter, had enough conformal time to overlap somewhere between $\tau = -\infty$ and $\tau = 0$. In SBB model $\tau = 0$ represents the initial singularity but in the inflationary cosmology it is just the reheating time. This explains why CMB radiation have uniform temperature.

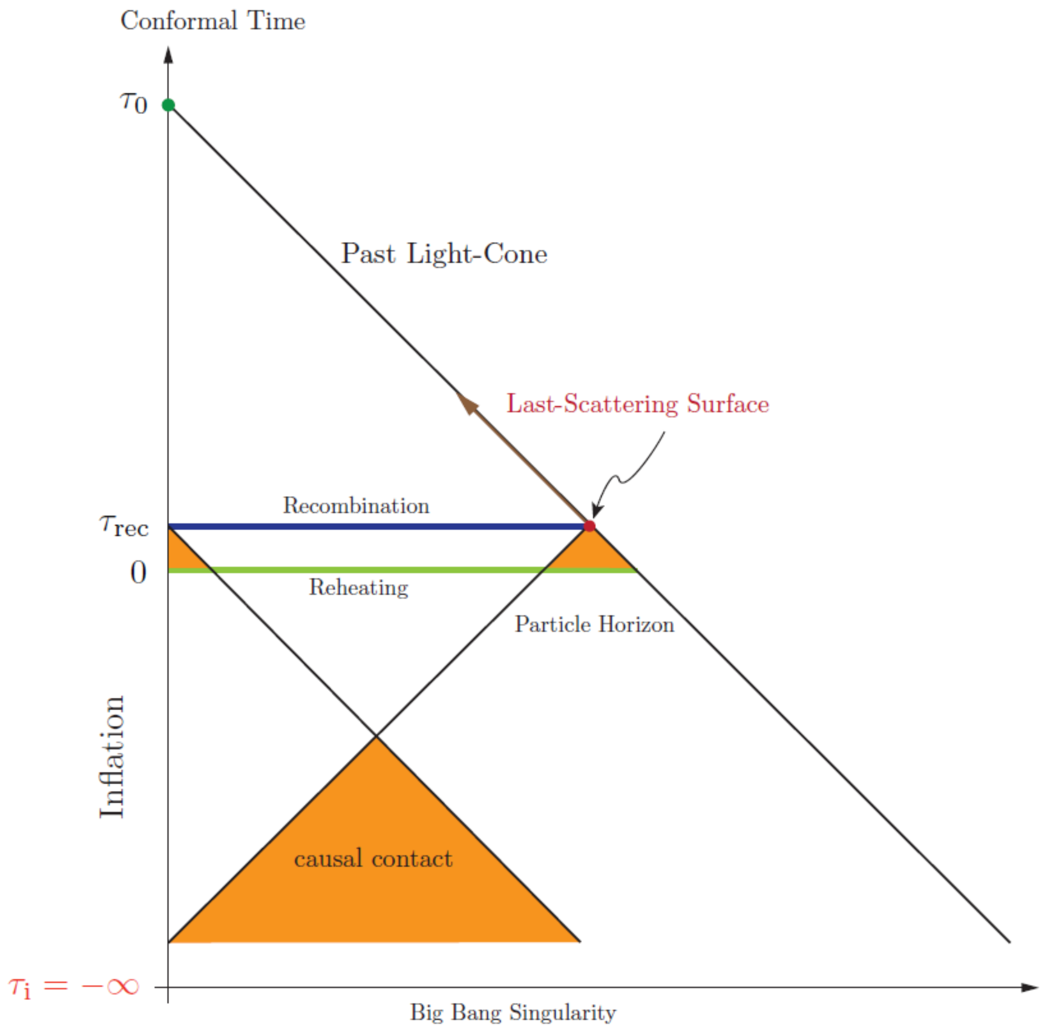

FIGURE 3.1: Solution to the Horizon Problem.

3.2.2.1 More About Shrinking of Co-moving Hubble Radius

Suppose there are two regions or particles which are separated by the co-moving length λ. Before the inflation has started, regions were causally connected, satisfying the condition $\lambda < (a_i H_i)^{-1}$ (*Sub-Horizon*). As the co-moving Hubble radius shrinks during inflation, communication between regions and the information they contain ceases because then $\lambda > (aH)^{-1}$ (*Super-Horizon*). It means before inflation regions were with in the co-moving Hubble radius and had enough time to establish equilibrium but as the long period of inflation took place regions went out of the co-moving Hubble radius which is termed as *horizon exit*. And then after inflationary phase, standard cosmological processes has taken place in which co-moving Hubble radius increases with time, regions which went out of the co-moving Hubble radius are entering again today showing almost same temperature. Figure(3.2) is expressing the above discussion in a more effective way [4].

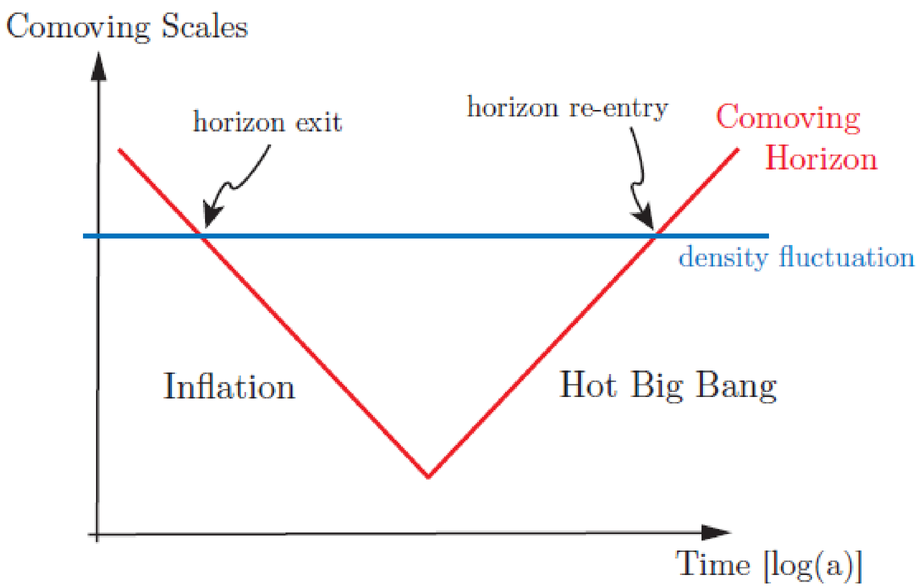

FIGURE 3.2: Horizon Exit And Horizon Re-entry.

3.2.3 Solution to the Magnetic Monopole Problem

Inflation gives the very efficient way out to resolve the magnetic monopole problem. Since magnetic monopoles were produced when the temperature of the universe was about 10^{16}GeV and they might have last until inflationary phase began, because during this phase number density of the magnetic monopoles would have dropped exponentially, diluting it to the undetectable levels.

3.3 Amount of Inflation required to Solve the Cosmological Problems

Amount of inflation tells us how much *number of e-folds* we require to solve the SBB problems i.e. by what factor of exponential expansion is required (e^N). Here is N is the number of the e-folds. It is required that the co-moving Hubble radius today (t_o) should be smaller than the co-moving Hubble radius at the beginning of the inflationary phase (t_i), so that the universe which is observable today, atleast fits in the co-moving Hubble radius at t_i [5],

$$\frac{1}{a_o H_o} < \frac{1}{a_i H_i} \tag{3.7}$$

or we can write,

$$\frac{a_i H_i}{a_e H_e} \frac{a_e H_e}{a_o H_o} < 1 \tag{3.8}$$

Where $(a_e H_e)^{-1}$ is the co-moving Hubble radius at the end of the inflation, $a_i H_i / a_e H_e$ is the amount by which $(aH)^{-1}$ shrank during inflation, and $a_e H_e / a_o H_o$ is the amount by which $(aH)^{-1}$ grew between the end of the inflation and decoupling time, and till present.

Assume that universe was radiation dominated after the end of inflationary phase. Then we express Hubble parameter as, $H \propto a^{-2}$ in radiation dominant era. Second term of equation(3.8) will become, using $aT = const$,

$$\frac{a_e H_e}{a_o H_o} = \frac{a_e}{a_o} \frac{a_o^2}{a_e^2} \sim \frac{T_e}{T_o} \sim 10^{28} \tag{3.9}$$

Where it is assumed, $T_e \sim T_R \sim 10^{15}$GeV (energy at the time of reheating). Current temperature in terms of energy is $T_o \sim 10^{-13}$GeV.

First term of equation(3.8) can be written as, using equation(2.58) for $\omega = -1$,

$$\frac{a_i H_i}{a_e H_e} = e^{-H_i(t_e - t_i)} = e^{-N} \tag{3.10}$$

where $N \equiv H(t_e - t_i)$. Now using equations (3.9) and (3.10) in (3.8) we have,

$$N > ln(10^{28}) \Rightarrow N > 64 \tag{3.11}$$

It shows that atleast 60 or greater number of e-folds is required for the solution of the cosmological problems.

3.4 Dynamics of Single Field Inflation

Cosmological constant is a good candidate for inflation to occur, this gives an elegant solution of the cosmological problems. But assuming a cosmological constant on *ad hoc* basis and claiming that it will decay out after solving the puzzles doesn't looks reasonable. A true inflationary model should have a reasonable theoretical framework for the genesis of the cosmological constant and, a logical way how inflation will end. It is believed, that Inflation should have taken placed at the time when the universe was about 10^{-34}s old and very hot, fundamental interactions of particle physics were the most dominant one. As we know in particle physics, whenever a phase transition take place in the universe, symmetry breaks down (like electroweak symmetry breaking) and it is govern by special type of matter known as scalar field (Higgs field). For inflation, there should be a scalar field known as *inflaton*, whose true nature is still unknown yet. Character of the scalar field depends on the nature of the phase transition. There are scalar fields which satisfies inflationary condition $(1+3\omega) < 0$, and act as a cosmological constant. Potential energy of such a scenario has special feature that it decreases very slowly with the expansion of the universe. Immediately after the completion of the phase transition, scalar field decays out, with inflationary phase comes to an end, leaving the universe in the state from where standard big bang model processes can take place [8][9].

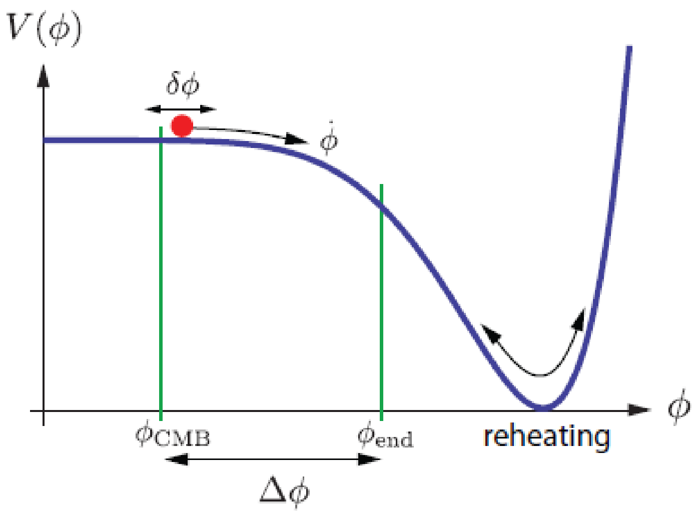

FIGURE 3.3: potential.

Action S in terms of all fundamental fields is given by [4],

$$S = \int d^4x\sqrt{-g}[\frac{M_{pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)] \tag{3.12}$$

Where,

$$\int d^4x\sqrt{-g}[\frac{M_{pl}^2}{2}R] \quad = \quad S_{EH} \tag{3.13}$$

S_{EH} is the *Einstein-Hilbert* action, in which the fundamental field is metric $g_{\mu\nu}(x,t)$. Variation of S_{EH} with respect to $g_{\mu\nu}$ yields Einstein field equation in vacuum, $G_{\mu\nu}=0$. Here R is the 4-dimensional Ricci scalar and $d^4x\sqrt{-g}$ is the invariant volume element.

$$\delta S_{EH} = \int d^4x\sqrt{-g}\left[R_{\mu} - \frac{1}{2}g_{\mu\nu}R\right]\delta g^{\mu\nu} = 0 \tag{3.14}$$

Thus,

$$T_{\mu\nu} \equiv \frac{1}{\sqrt{-g}}\frac{\delta S_{EH}}{\delta g^{\mu\nu}} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \tag{3.15}$$

It is the stress-energy tensor of the vacuum. And,

$$\int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] \quad = \quad S_\phi \tag{3.16}$$

S_ϕ represents the action for inflaton field ϕ. Variation of S_ϕ with respect to ϕ gives

$$\delta S_\phi = -\int d^4x\sqrt{-g}\left[g^{\mu\nu}\partial_\nu\phi\delta\partial_\mu\phi + \frac{\partial V}{\partial\phi}\delta\phi\right] \tag{3.17}$$

using product rule of differentiation we have,

$$\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\partial_\mu\delta\phi = \partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\delta\phi\right) - \partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right)\delta\phi \tag{3.18}$$

equating equation (3.18) in (3.17) and putting an integral equal to zero we get,

$$\begin{aligned}\delta S_\phi = \int d^4x\left[\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) + \sqrt{-g}\frac{\partial V}{\partial\phi}\right]\delta\phi = 0 \\ \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\phi - \frac{\partial V(\phi)}{\partial\phi} = 0\end{aligned} \tag{3.19}$$

Using the FRW metric $(-1, a^{-2}, a^{-2}, a^{-2})$ and $\sqrt{-g} = a^{-3}$ we get,

$$\ddot{\phi} + 3H\dot{\phi} - \bigtriangledown^2\phi + \frac{\partial V(\phi)}{\partial\phi} = 0 \tag{3.20}$$

Since in FRW cosmology it is assumed the space is isotropic and homogeneous i.e., $\bigtriangledown\phi = 0$, equation (3.20) will become,

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V(\phi)}{\partial\phi} = 0 \tag{3.21}$$

Equation (3.21) is a equation of motion for the inflaton field and it is like *Klein-Gordon equation.* Here the term $3H\dot{\phi}$ is important, acting as damping or friction term and it is proportional to the velocity $\dot{\phi}$ and the Hubble parameter H. It tells us, as the inflaton rolls down to the valley of the potential energy curve it will suffer friction. If this term becomes zero then equation (3.21) will reduce to equation of a harmonic oscillator with frequency m, if we use $V(\phi) = \frac{1}{2}m^2\phi^2$.

3.4.1 Energy Momentum Tensor of a Inflaton Field

Energy-momentum tensor is define as [5],

$$\mathrm{T}_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta S_\phi}{\delta g^{\mu\nu}} \tag{3.22}$$

Rewriting S_ϕ as,

$$S_\phi = \int d^4x \sqrt{-g}\mathcal{L} \tag{3.23}$$

Variation of action in equation(3.16) with respect to the metric $g^{\mu\nu}$

$$\begin{aligned} \delta S_\phi &= \int d^4x \left[\sqrt{-g} \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} \delta g^{\mu\nu} + \mathcal{L} \frac{\partial \sqrt{-g}}{\partial g} \delta g \right] \\ &= \int d^4x \left[\sqrt{-g} \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} \delta g^{\mu\nu} - \frac{1}{2} \mathcal{L} \frac{1}{\sqrt{-g}} \delta g \right] \end{aligned} \tag{3.24}$$

We can write δg as follows,

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu} = -g g_{\mu\nu} \delta g^{\mu\nu} \tag{3.25}$$

Using equation(3.24) in equation(3.23),

$$\delta S_\phi = \int d^4x \left[\sqrt{-g} \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} \delta g^{\mu\nu} - \frac{1}{2} \mathcal{L} g_{\mu\nu} \sqrt{-g} \right] \delta g^{\mu\nu} \tag{3.26}$$

Then we can write as

$$\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \sqrt{-g} \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} \delta g^{\mu\nu} - \frac{1}{2} \mathcal{L} g_{\mu\nu} \sqrt{-g} \tag{3.27}$$

After rearranging we get,

$$\mathrm{T}_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta S_\phi}{\delta g^{\mu\nu}} = \mathcal{L} g_{\mu\nu} - 2 \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} \tag{3.28}$$

From equation(3.16) one can identify $\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi)$. Taking derivative w.r.t metric $g^{\mu\nu}$,

$$\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} = -\frac{1}{2} \partial_\mu \phi \partial_\nu \phi \tag{3.29}$$

Plugging it in equation (3.27) we get,

$$\mathrm{T}_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu}[\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + V(\phi)] \tag{3.30}$$

Multiplying equation(3.29) with $g^{\alpha\mu}$ to raise the index

$$\mathrm{T}^\mu_\nu = g^{\alpha\mu} \partial_\alpha \phi \partial_\nu \phi - \delta^\mu_\nu [\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + V(\phi)] \tag{3.31}$$

As mentioned in chapter 1, equation(2.35) represents the energy momentum tensor for an ideal fluid, we get

$$\mathrm{T}^0_0 = -\rho, \quad \mathrm{T}^i_0 = \mathrm{T}^0_i = 0, \quad \mathrm{T}^i_j = \delta^i_j p \tag{3.32}$$

Inserting equation(3.31) in (3.30) and solving for components,

$$\begin{aligned} \mathrm{T}^0_0 &= -\frac{1}{2}\dot{\phi}^2 - \frac{1}{2} a^{-2} (\bigtriangledown \phi)^2 - V(\phi) \\ \mathrm{T}^i_0 &= -\dot{\phi} \partial^i \phi, \quad \mathrm{T}^0_i = -\dot{\phi} \partial_i \phi \\ \mathrm{T}^i_j &= \delta^i_j [\frac{1}{2}\dot{\phi}^2 - \frac{1}{2} a^{-2} (\bigtriangledown \phi)^2 - V(\phi)] \end{aligned} \tag{3.33}$$

Again using the homogeneity and isotropy principle,

$$\partial_i \phi = \partial^i \phi = 0 \tag{3.34}$$

Finally we will get energy density and pressure of the ideal cosmic fluid in the form,

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{3.35}$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{3.36}$$

Now equation of the state can be written as

$$\omega = \frac{p}{\rho} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \tag{3.37}$$

equation(3.37) shows that, inflaton field can give rise to a negative pressure i.e. inflation $\omega < -\frac{1}{3}$, only if potential energy $V(\phi)$ leads the kinetic energy $\frac{1}{2}\dot{\phi}^2$,

$$\dot{\phi}^2 \quad << \quad V(\phi) \tag{3.38}$$

Equations (3.35) and (3.36) are useful, using them in Friedmann equation (2.36) and Raychaudhuri equation (2.37) can be rewritten in terms of scalar field,

$$H^2 = \frac{1}{3M_{pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) - \frac{k}{a^2} \tag{3.39}$$

And

$$\frac{\ddot{a}}{a} = -\frac{1}{3M_{pl}^2}\left(\dot{\phi}^2 - V(\phi)\right) \tag{3.40}$$

Where we have used $M_{pl} = \frac{1}{\sqrt{8\pi G_N}}$. Here M_{pl} is called reduced Planck mass.
We can also rewrite continuity equation (2.40) in terms of scalar field ϕ as,

$$\begin{aligned} \dot{\phi}\ddot{\phi} + \frac{\partial V}{\partial \phi}\frac{d\phi}{dt} + 3H\dot{\phi}^2 &= 0 \\ \ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} &= 0 \end{aligned} \tag{3.41}$$

Which is exactly same as equation(3.21) i.e., Klein Gordon equation.

3.4.2 Hubble Parameters for Slow Roll Inflation

If the universe is dominated by scalar field ϕ then, the acceleration equations (2.37) and (2.38) can be written as,

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_{pl}^2}(\rho + 3p) = H^2\left(1 + \frac{\dot{H}}{H^2}\right) \tag{3.42}$$

Where,

$$\varepsilon_H \equiv -\frac{\dot{H}}{H^2} \tag{3.43}$$

ε_H is called Hubble slow roll parameter and for inflationary phase it must be $\varepsilon_H < 1$. Now rewriting equation(3.42) in terms of ε_H as,

$$\frac{\ddot{a}}{a} = -\frac{\rho}{6M_{pl}^2}(1 + 3\omega) = H^2(1 - \varepsilon_H) \tag{3.44}$$

Using Friedmann equation $\rho = 3M_{pl}^2H^2$ for $k = 0$ in equation(3.44) and solving for ε_H, we get

$$\varepsilon_H = \frac{3}{2}(1 + \omega) \tag{3.45}$$

Equation(3.45) can also be represented in Hubble parameter H as,

$$\varepsilon_H = -\frac{\dot{H}}{H^2} = -\frac{dlnH}{Hdt} = -\frac{dlnH}{dN} \tag{3.46}$$

Where $dN = Hdt = dlna$, which measures the number of e-folds N during inflationary era. Equation(3.46) implies that, the Hubble parameter changes very slowly per e-fold. Another useful expression of the ε_H can be obtain by differentiating equation(3.39) w.r.t time for $k = 0$ and using $\ddot{\phi} = -3H\dot{\phi} - \frac{\partial V(\phi)}{\partial \phi}$, we have

$$\dot{H} = -\frac{1}{2}\frac{\dot{\phi}^2}{M_{pl}^2} \tag{3.47}$$

Substituting equation(3.43) in the definition of ε_H, we get

$$\varepsilon_H = \frac{1}{2}\frac{\dot{\phi}^2}{M_{pl}^2 H^2} \tag{3.48}$$

For the solution of cosmological puzzles, long period of accelerated expansion is required i.e. atleast 60 e-folds. To accomplish this, we need $\varepsilon_H < 1$ and variation in ε_H should remain small w.r.t e-folds. This is define by second Hubble slow roll parameter [4].

$$\eta_H = \frac{\dot{\varepsilon}_H}{H\varepsilon_H} = \frac{dln\varepsilon_H}{dN} \tag{3.49}$$

accelerated expansion persists as long as $\mid \eta_H \mid < 1$.
As mentioned before for an inflation to occur $\dot{\phi}^2 << V(\phi)$, implies that $\ddot{\phi}$ should also be small. To analyse this, lets define a dimensionless acceleration per Hubble time [4].

$$\delta = -\frac{\ddot{\phi}}{H\dot{\phi}} \tag{3.50}$$

Differentiating equation(3.48) with respect to time and using equation(3.49) i.e. $\dot{\varepsilon}_H = \eta_H H \varepsilon_H$, we get

$$\eta_H H \varepsilon_H = \frac{\dot{\phi}^2}{M_{pl}^2 H^2}\left[\frac{\ddot{\phi}}{\dot{\phi}} - \frac{\dot{H}}{H}\right]$$

$$\eta_H = 2\left[\varepsilon_H - \delta\right] \tag{3.51}$$

As far as $\{\varepsilon_H, \mid \delta \mid \ll 1\}$,the fractional variation in both H and ε_H would be smaller.

3.4.2.1 Slow-Roll Approximation

The standard technique for the analysis of the inflationary models is *slow roll approximation*. It is assumed that for inflation, inflaton field vary slowly enough so that potential energy supersede kinetic energy. And the second derivative of the inflaton field w.r.t time should remain small for sufficient time so that slow roll condition persist.

Using equation(3.38) in Friedmann equation(3.39), we get

$$H^2 \approx \frac{V(\phi)}{3M_{pl}^2} \tag{3.52}$$

and condition $\mid \delta \mid \ll 1$ reduce the equation(3.41) to,

$$3H\dot{\phi} \approx -V(\phi) \tag{3.53}$$

Using equation(3.52) and (3.53) in equation(3.48), we get

$$\boxed{\varepsilon_V \equiv \frac{M_{pl}^2}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2} \tag{3.54}$$

where ε_V is slow roll parameter.
Taking time derivative of equation(3.53), we get

$$\ddot{\phi} = \frac{-V''(\phi)\dot{\phi} - 3\dot{H}\dot{\phi}}{3H} \tag{3.55}$$

Now putting equation(3.55) in (3.50) and adding equation(3.43), we have

$$\delta + \varepsilon_H = \frac{V''(\phi)}{3H^2} + \frac{\dot{H}}{H^2} - \frac{\dot{H}}{H^2} \tag{3.56}$$

Using $3H^2 = \frac{V(\phi)}{M_{pl}^2}$, we get

$$\boxed{\eta_V \equiv \; M_{pl}^2\frac{V''(\phi)}{V(\phi)}} \tag{3.57}$$

η_V is the second slow roll parameter. For Inflation potential slow roll should follow the condition,

$$\varepsilon_V \ll 1, \qquad \mid \eta_V \mid \ll 1 \tag{3.58}$$

ε_V and η_V measures the curvature of the potential energy function and relative slope. The rate of expansion and the causal length of the horizon is set by the potential energy, while the slope of the potential function controls the kinetic energy of the inflaton field and the duration of inflationary period in which slow roll approximation holds.
The relation between Hubble and potential slow roll parameters are,

$$\varepsilon_V \approx \varepsilon_H \qquad \eta_V \approx 2\varepsilon_H - \frac{1}{2}\eta_H \tag{3.59}$$

In the slow roll regime, inflation ends as the conditions given in the equation(3.58) are violated depending on the model i.e.,

$$\varepsilon_V(\phi_e) \sim 1 \quad or \quad \mid \eta_V(\phi_e) \mid \sim 1 \tag{3.60}$$

ϕ_e represents the value of the field at the end of the inflation.

3.5 Amount of Inflation

As mentioned previously the sufficient amount of inflation N is required for the solution of cosmological puzzles. It is defined as,

$$\begin{aligned} N = ln\left(\frac{a_e}{a_i}\right) &= \int_{a_i}^{a_e} dlna = \int_{t_i}^{t_e} H(t)dt = \int_{\phi_i}^{\phi_e} \frac{H}{\dot{\phi}} d\phi \\ &\simeq -\int_{\phi_i}^{\phi_e} \frac{3H^2}{V'(\phi)} d\phi \end{aligned} \tag{3.61}$$

Using $3H^2 = \frac{V}{M_{pl}^2}$, one can rewrite equation(3.61) in terms of potential,

$$\boxed{N(\phi) \simeq \int_{\phi_e}^{\phi_i} \frac{1}{M_{pl}^2} \frac{V(\phi)}{V'(\phi)}} \tag{3.62}$$

Chapter 4

Quantum Fluctuation During Inflation

4.1 ADM Formalism

To study the power spectrum of scalar and tensor perturbations, we need to evaluate the second order action, containing inflaton field weakly coupled with gravity,

$$S = \frac{1}{2} \int dx^4 \sqrt{-g} \left[M_{pl}^2 R^{(4)} - (\bigtriangledown\phi)^2 - 2V(\phi) \right] \tag{4.1}$$

Here, the first term in the square bracket represents the total curvature and the other belong to the matter part.
There are various methods to study primordial fluctuations, simpler approach is to use ADM formmulation if we are interested in the canonical quantization. Metric takes the form in the ADM formalism as [10],

$$ds^2 = -N^2 dt^2 + h_{ij} \left(dx^i + N^i dt \right) \left(dx^j + N^j dt \right) \tag{4.2}$$

$$\begin{aligned} ds^2 &= -N^2 dt^2 + h_{ij} dx^i dx^j + h_{ij} N^j dx^i dt + h_{ij} N^i dt dx^j + h_{ij} N^i N^j dt^2 \\ &= \left(-N^2 + h_{ij} N^i N^j \right) dt^2 + \left(h_{ij} N^i \right) dt dx^j + \left(h_{ij} N^j \right) dx^i dt + h_{ij} dx^i dx^j \end{aligned} \tag{4.3}$$

Where, N and N^i are representing lapse function and shift vector in the spatial direction. And these quantities are not dynamical degree of freedom because they are arbitrary choice of slicing. While h^{ij} is dynamical and we want to solve for h^{ij} because it is a function of time. we know that,

$$ds^2 = g_{00}dt^2 + g_{0j}dtdx^j + g_{i0}dx^i dt + g_{ij}dx^i dx^j \tag{4.4}$$

comparing equations(4.3) and (4.4), we get

$$\begin{aligned} g_{00} &= -N^2 + h_{ij}N^iN^j \\ g_{0j} &= h_{ij}N^i \\ g_{i0} &= h_{ij}N^j \\ g_{ij} &= h_{ij} \end{aligned}$$

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + h_{ij}N^iN^j & h_{ij}N^i \\ h_{ij}N^j & h_{ij} \end{pmatrix} \tag{4.5}$$

One can find the inverse of the equation(4.5), using

$$\begin{aligned} (\bigtriangledown\phi)^2 &= g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \\ &= g^{00}\partial_0\phi\partial_0\phi + g^{0i}\partial_0\phi\partial_i\phi + g^{i0}\partial_i\phi\partial_0\phi + g^{ij}\partial_i\phi\partial_j\phi \end{aligned} \tag{4.6}$$

$$g_{\alpha\beta} = \begin{pmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{pmatrix} \tag{4.7}$$

and,

$$g_{\mu\nu}g^{\nu\alpha} = \delta^\alpha_\mu \tag{4.8}$$

$$g_{0\nu}g^{\nu 0} = \delta^0_0 = 1 \tag{4.9}$$

$$g_{0\nu}g^{\nu i} = \delta^i_0 = 0 \tag{4.10}$$

$$g_{i\nu}g^{\nu 0} = \delta^0_i = 0 \tag{4.11}$$

$$g_{i\nu}g^{\nu j} = \delta^j_i \tag{4.12}$$

Now solving for all the components. Using equation(4.9)

$$-N^2g^{00} + h_{ij}g^{00}N^iN^j + h_{ij}N^jg^{i0} = 1 \tag{4.13}$$

Using equation(4.10)

$$-N^2g^{0l} + h_{ij}g^{0l}N^iN^j + h_{ij}N^ig^{ml} = 0 \tag{4.14}$$

Using equation(4.11)

$$h_{ij}N^i g^{00} + h_{ij}g^{i0} = 0 \tag{4.15}$$

Using equation(4.12)

$$g_{ij}N^i g^{0k} + h_{ij}g^{ik} = \delta_j^k \tag{4.16}$$

from equation(4.15)

$$\begin{aligned}
&h^{kj}h_{ij}N^i g^{00} + h^{kj}h_{ij}g^{i0} = 0\\
&\delta_i^k N^i g^{00} + \delta_i^k g^{i0} = 0\\
&g^{i0} = -N^i g^{00}
\end{aligned} \tag{4.17}$$

putting the value of g^{i0} from equation(4.17) in (4.13)

$$\begin{aligned}
&- N^2 g^{00} + h_{ij}g^{00}N^iN^j - h_{ij}N^iN^jg^{00} = 1\\
&g^{00} = -N^{-2}
\end{aligned} \tag{4.18}$$

Then equation(4.17)⇒

$$g^{i0} = N^i N^{-2} \tag{4.19}$$

Now using equation(4.16)

$$\begin{aligned}
&g_{i0}g^{0j} + g_{ik}g^{kj} = \delta_i^j\\
&h_{ik}N^kN^jN^{-2} + h_{ik}g^{kj} = \delta_i^j\\
&h^{il}h_{ik}g^{kj} = h^{il}\delta_i^j - h^{il}h_{ik}N^{-2}N^kN^j\\
&\delta_k^l g^{kl} = h^{il}\delta_i^j - \delta_k^l N^{-2}N^kN^j\\
&g^{jl} = h^{jl} - N^{-2}N^jN^l
\end{aligned} \tag{4.20}$$

Equation(4.7) will become,

$$g_{\alpha\beta} = \begin{pmatrix} -N^{-2} & N^iN^{-2} \\ N^iN^{-2} & h^{ij} - N^iN^jN^{-2} \end{pmatrix} \tag{4.21}$$

Inserting all information in equation(4.6), we have

$$\begin{aligned}
(\bigtriangledown\phi)^2 &= -N^{-2}\dot{\phi}^2 + 2N^iN^{-2}\dot{\phi}\partial_i\phi + \left(h^{ij} - N^iN^jN^{-2}\right)\partial_i\phi\partial_j\phi\\
&= -N^{-2}\left(\dot{\phi}^2 - 2N^i\dot{\phi}\partial_i\phi + N^iN^j\partial_i\phi\partial_j\phi\right) + h^{ij}\partial_i\phi\partial_j\phi\\
&= -N^{-2}\left(\dot{\phi} - N^i\partial_i\phi\right)^2 + h^{ij}\partial_i\phi\partial_j\phi
\end{aligned} \tag{4.22}$$

4.1.1 Scalar Perturbation

Perturbations in the scalar part can be written as [4],

$$ds^2 = a(\tau)^2\left[-(1+2\Phi)d\tau^2 + 2B_{,i}d\tau dx^i + \left[(1-2\Psi)\delta_{ij} + D_{ij}E\right]dx^i dx^j\right] \tag{4.23}$$

Plugging all the values in the equation(4.1), we get

$$S = \frac{1}{2}\int dx^4\sqrt{-g}\left[M_{pl}^2 R^{(4)} + N^{-2}\left(\dot{\phi} - N^i\partial_i\phi\right)^2 - h^{ij}\partial_i\phi\partial_j\phi - 2V(\phi)\right] \tag{4.24}$$

To simplify the action we need to evaluate $\sqrt{-g}$ and $R^{(4)}$.
Since,

$$\sqrt{-g} = N\sqrt{h} \tag{4.25}$$

and 4D Ricci scalar can be expressed in terms of 3D Ricci scalar plus some terms proportional to the extrinsic curvature as follows:

$$R^{(4)} = R^{(3)} + N^{-2}\left(E_{ij}E^{ij} - E^2\right) \tag{4.26}$$

and

$$E_{ij} = \frac{1}{2}\left[\partial_0 h_{ij} - \bigtriangledown_i N_j - \bigtriangledown_j N_i\right] = NK_{ij}, \qquad E = E_i^i = h^{ij}E_{ij} \tag{4.27}$$

Equation(4.24) will become,

$$S = \frac{1}{2}\int dx^4\sqrt{h}[M_{pl}^2 NR^{(3)} + N^{-1}\left(E_{ij}E^{ij} - E^2\right) + N^{-1}\left(\dot{\phi} - N^i\partial_i\phi\right)^2 - Nh^{ij}\partial_i\phi\partial_j\phi - 2NV(\phi)] \tag{4.28}$$

To fix the time and spatial representations, we have chosen the following gauge for the dynamical fields h_{ij} and ϕ.
Comoving Gauge: (moving along the field)

$$\delta\phi = 0, \quad h_{ij} = a^2\left[(1+2\zeta)\delta_{ij} + \gamma_{ij}\right] + \mathcal{O}(2) + ..., \quad \partial_i\gamma_{ij} = 0, \quad \gamma_{ii} = 0 \tag{4.29}$$

Similarly, expanding the action,

$$S = S_0 + S_1 + S_2 + S_3 + ... \tag{4.30}$$

Where, S_0 = Background term (no perturbations),
$S_1 = 0$, because of equation of motion,
S_2 = quadratic in fluctuations, gives equation of motion for ζ, $\delta\phi$ and yields power spectrum,

$S_3 =$ gives rise to non-gaussianity. The perturbation in the inflaton field can be written as,

$$\phi(x,t) = \phi(t) + \delta\phi(x,t) \tag{4.31}$$

But because of our choice of co-moving gauge our computation will become convenient and equation of motion for N and N^i can be obtained by using Lagrangian inside action S in equation(4.28)

$$\mathcal{L} = M_{pl}^2 N R^{(3)} + N^{-1}\left(E_{ij}E^{ij} - E^2\right) + N^{-1}\left(\dot{\phi} - N^i\partial_i\phi\right)^2 - Nh^{ij}\partial_i\phi\partial_j\phi - 2NV(\phi) \tag{4.32}$$

<u>EOM for N</u>:

$$\frac{\partial\mathcal{L}}{\partial N} = M_{pl}^2 R^{(3)} - N^{-2}\left[\left(E_{ij}E^{ij} - E^2\right) + \left(\dot{\phi} - N^i\partial_i\phi\right)^2\right] - h^{ij}\partial_i\phi\partial_j\phi - 2V(\phi) \tag{4.33}$$

and

$$\frac{\partial\mathcal{L}}{\partial\dot{N}} = 0 \tag{4.34}$$

Then using Euler-Lagrange equation, we get

$$M_{pl}^2 R^{(3)} - N^{-2}\left[\left(E_{ij}E^{ij} - E^2\right) + \left(\dot{\phi} - N^i\partial_i\phi\right)^2\right] - h^{ij}\partial_i\phi\partial_j\phi - 2V(\phi) = 0 \tag{4.35}$$

Since in our choice of the gauge $\delta\phi = 0$ i.e. ϕ only depends on time, then the terms $N^i\partial_i\phi$ and $h^{ij}\partial_i\phi\partial_j\phi$ will vanish because it contains space derivatives of ϕ. Equation(4.35) will become,

$$\boxed{M_{pl}^2 R^{(3)} - N^{-2}\left(E_{ij}E^{ij} - E^2 + \dot{\phi}^2\right) - 2V(\phi) = 0} \tag{4.36}$$

<u>EOM for N^i</u>: Again using Lagrangian from equation(4.32), only term that will contribute $N^{-1}\left(E_{ij}E^{ij} - E^2\right)$ and other will vanish. The Euler-Lagrange equation will become

$$\partial_l\left(\frac{\partial\mathcal{L}}{\partial(\partial_l N^m)}\right) - \frac{\partial\mathcal{L}}{\partial N^m} = 0 \tag{4.37}$$

Solving first term of equation(4.37), we have using definition of covariant derivative $\nabla_i N_j = \partial_i N_j - \Gamma_{ij}^k N_k$, we have

$$\partial_l\left(\frac{\partial\mathcal{L}}{\partial(\partial_l N^m)}\right) = \partial_l\left[N^{-1}\left(E_m^l - \delta_m^l E\right)\right] \tag{4.38}$$

And,

$$\frac{\partial\mathcal{L}}{\partial N^m} = 0 \tag{4.39}$$

Equation of motion for N^i will become,

$$\boxed{\partial_l \left[N^{-1}\left(E^l_m - \delta^l_m E\right)\right] = 0} \tag{4.40}$$

The strategy is now to solve the constraint equations(4.36) and (4.40) order by order in terms of the actual dynamical variables (ϕ, h_{ij}) and then plug the solution back into action.
Now expanding Lapse function N and shift vector N_i,

$$N_i \equiv B_{,i} + \widetilde{N_i}, \qquad \partial_i \widetilde{N_i} = 0 \tag{4.41}$$

and

$$N \equiv 1 + \alpha \tag{4.42}$$

The quantities α, B and $\widetilde{N_i}$ can be expressed in terms of ζ as,

$$\begin{aligned}
\alpha &= \alpha_1 + \alpha_2 + \alpha_3 + ..., \\
B &= B_1 + B_2 + B_3 + ..., \\
\widetilde{N_i} &= \widetilde{N_i}^{(1)} + \widetilde{N_i}^{(2)} + \widetilde{N_i}^{(3)} + ...,
\end{aligned} \tag{4.43}$$

At $\mathcal{O}(0)$ order, $N^{(0)} = 1$ and $N_i^{(0)} = 0$, using this information equation(4.40) will vanish and equation(4.36) becomes, using $R^{(3)} = 6\frac{\dot{a}^2}{a^2}$ from equation(2.31)

$$H^2 = \frac{1}{3M_{pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{4.44}$$

Now for $\mathcal{O}(1)$ order, γ_{ij} is transverse and traceless which implies that it doesn't contain information about scalar modes, so we can neglect it. Then the metric for scalar degree of freedom will become [10],

$$h_{ij} = a^2 e^{2\zeta}\delta_{ij}, \qquad h^{ij} = a^{-2}e^{-2\zeta}\delta^{ij} \tag{4.45}$$

Taking time derivative of equation(4.45), we get

$$\dot{h_{ij}} = 2a^2\left(H + \dot{\zeta}\right)e^{2\zeta}\delta_{ij}, \quad \dot{h^{ij}} = -2a^{-2}\left(H + \dot{\zeta}\right)e^{-2\zeta}\delta^{ij} \tag{4.46}$$

Since E_{ij} contains covariant derivative and for that we need to solve connection,

$$\begin{aligned}
\Gamma^k_{ij} &= \frac{1}{2}h^{kl}\left(\partial_j h_{il} + \partial_i h_{jl} - \partial_l h_{ij}\right) \\
&= \delta^{kl}\left(\partial_j\zeta\delta_{il} + \partial_i\zeta\delta_{jl} - \partial_l\zeta\delta_{ij}\right)
\end{aligned} \tag{4.47}$$

Using equation(4.46) and (4.47) in equation(4.27) to get the expression for the extrinsic curvature E_{ij}, we have

$$\begin{aligned} E_{ij} &= \frac{M_{pl}}{2}\left[2a^2\left(H+\dot{\zeta}\right)e^{2\zeta}\delta_{ij} - 2\partial\left(iN_j\right) + 2\Gamma^k_{ij}N_k\right] \\ &= M_{pl}\left[2a^2\left(H+\dot{\zeta}\right)e^{2\zeta}\delta_{ij} - 2\partial_{(i}N_{j)} + 2N_{(i}\partial_{j)}\zeta - N_k\partial_k\zeta\delta_{ij}\right] \end{aligned} \tag{4.48}$$

where,

$$2\partial_{(i}N_{j)} = \partial_i N_j + \partial_j N_i, \quad 2N_{(i}\partial_{j)}\zeta = N_i\partial_j\zeta + N_j\partial_i\zeta \tag{4.49}$$

And

$$E^{ij} = h^{ik}h^{jl}E_{kl} \tag{4.50}$$

After inserting all information, kinetic term becomes [16],

$$\begin{aligned} E_{ij}E^{ij} - E^2 = &\left[h^{ik}h^{jl} - h^{ij}h^{kl}\right]E_{ij}E_{kl} \\ &= -6H^2 - 12H\dot{\zeta}^2 - 6\dot{\zeta}^2 + 4a^{-2}H\partial_i\partial_i B + 4a^{-2}\dot{\zeta}\partial_i\partial_i B \\ &+ 4a^{-2}H\partial_i B\partial_i\zeta + a^{-4}\left(\partial_i\partial_j B\right)^2 - a^{-4}\partial_i\partial_i B\partial_j\partial_j B \\ &- 8a^{-2}\zeta H\partial_i\partial_i B \end{aligned} \tag{4.51}$$

At first order equation(4.51) reduces to,

$$E_{ij}E^{ij} - E^2 = -6H^2 - 12H\dot{\zeta} + 4a^{-2}H\partial_i\partial_i B \tag{4.52}$$

Now $R^{(3)}$ can be expressed in terms of ζ by using the definition of Ricci scalar in 3-D,

$$R^{(3)} = h^{ik}\partial_l\Gamma^l_{ik} - h^{ik}\partial_k\Gamma^l_{il} + h^{ik}\Gamma^l_{ik}\Gamma^m_{lm} - h^{ik}\Gamma^m_{il}\Gamma^l_{km} \tag{4.53}$$

Solving equation(4.53) by inserting equation(4.47), we get

$$R^{(3)} = -2a^{-2}e^{-2\zeta}\left[2\partial_l\partial_l\zeta + \partial_l\zeta\partial_l\zeta\right] \tag{4.54}$$

Second term of equation(4.54) is $\mathcal{O}(2)$ so we can neglect it.
Now inserting equation(4.42), (4.52), (4.54) and $V = 3H^2 - \frac{1}{2}\dot{\phi}^2$ in equation(4.36) and taking $M_{pl} = 1$ for simplification, we have

$$\begin{aligned} \Rightarrow &-4a^{-2}e^{-2\zeta}\partial_l\partial_l\zeta - (1+\alpha_1)\left(-6H^2 - 12H\dot{\zeta} + 4a^{-2}H\partial_i\partial_i B + 12H^2\alpha_1\right) \\ &-6H^2 + (1-1+2\alpha_1) = 0 \end{aligned}$$

$$\boxed{\Rightarrow 2a^{-2}\partial_l\partial^l\left(HB+\zeta\right) - 6H\left(H\alpha_1 - \dot{\zeta}\right) + \alpha_1\dot{\phi}^2 = 0} \tag{4.55}$$

Now solving equation(4.40) for $\mathcal{O}(1)$, we require

$$\begin{aligned} E_i^j = h^{jk}E_{ik} &= \delta^{jk}a^{-2}e^{-2\zeta}\left[a^2e^{2\zeta}\delta_{ik}\left(H+\dot{\zeta}\right)-\partial_{(i}N_{k)}\right] \\ &= \delta^{jk}a^{-2}\left(1-2\zeta\right)\left[a^2\left(1+2\zeta\right)\delta_{ik}\left(H+\dot{\zeta}\right)-\frac{\partial_i N_k}{2}-\frac{\partial_k N_i}{2}\right] \\ &= \left(H+\dot{\zeta}\right)\delta_i^j - a^{-2}\partial_i\partial_k B\delta^{jk} \end{aligned} \tag{4.56}$$

And

$$E = 3\left(H+\dot{\zeta}\right) - a^{-2}\partial_k\partial_k B \tag{4.57}$$

After inserting equation(4.56) and (4.57) in (4.40), we have

$$\partial_j\left[(1+\alpha)^{-1}\left[\left(H+\dot{\zeta}\right)\delta_i^j - a^{-2}\partial_i\partial_k B\delta^{jk} - 3\left(H+\dot{\zeta}\right)+a^{-2}\partial_k\partial_k B\right]\right]=0$$
$$2\partial_j\left[-\left(H+\dot{\zeta}\right)\delta_j^i + H\alpha_1\delta_i^j\right]=0$$

$$\boxed{\Phi = \alpha_1 = \frac{\dot{\zeta}}{H}} \tag{4.58}$$

equating equation(4.58) in (4.55), we have

$$\begin{aligned} 2a^{-2}\partial_l\partial^l HB + 2a^{-2}\partial_l\partial^l\zeta &= -\frac{\dot{\zeta}}{H}\dot{\phi}^2 \\ \partial_l\partial_l B = a^2\frac{\dot{\zeta}}{2H^2}\dot{\phi}^2 &- \frac{\partial_l\partial_l\zeta}{H} \end{aligned} \tag{4.59}$$

$$\boxed{B = -\frac{\zeta}{H} + a^2\frac{\dot{\phi}^2}{2H^2}\partial^{-2}\dot{\zeta}} \tag{4.60}$$

Insert Φ and B in the action (4.28) and expand it to second order, after doing some integration by parts, we get [10]

$$\begin{aligned} S = \int ae^{\zeta}\left(1+\frac{\dot{\zeta}}{H}\right)\left(-4\partial^2\zeta - 2(\partial\zeta)^2 - 2Va^2e^{2\zeta}\right) \\ + e^{3\zeta}a^3\left(1-\frac{\dot{\zeta}}{H}+\frac{\dot{\zeta}^2}{H^2}\right)\left[-6\left(H+\dot{\zeta}\right)^2+\dot{\phi}^2+4a^{-2}e^{-2\zeta}\left(H+\dot{\zeta}\right)\left(\partial_i B\partial_i\zeta+\partial^2 B\right)\right] \end{aligned} \tag{4.61}$$

Further again doing some integration by parts and using $3H^2 = V + \frac{1}{2}\dot{\phi}^2$, we get [4]

$$S_2 = \int dtd^3x\frac{\dot{\phi}^2}{H^2}\left[\frac{a^3}{2}\dot{\zeta}^2 - \frac{a}{2}(\partial\zeta)^2\right] \tag{4.62}$$

Let us defined auxiliary field as,

$$\chi \equiv z\zeta, \qquad z^2 \equiv a^2 \frac{\dot{\phi}^2}{H^2} = 2a^2\varepsilon \tag{4.63}$$

Using equation(4.63) in (4.62) and converting it to conformal time τ, we get

$$S_2 = \frac{1}{2} \int d\tau d^3x \left[\left(\chi'\right)^2 - \left(\partial_i \chi\right)^2 + \frac{z''}{z}\chi^2 \right] \tag{4.64}$$

Action in equation(4.64) resembles the action of an oscillator in the Minkowski spacetime if the third term in the our action act as time-dependent effective mass in the de Sitter spacetime. The interaction of the gravitational field with the scalar field ζ lies in the time dependent mass.

$$m^2_{eff} \equiv -\frac{z''}{z} = -\frac{a''}{a} = \frac{2}{\tau^2} \tag{4.65}$$

Field χ can be expanded in the Fourier modes as,

$$\chi(x,\tau) = \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \chi_{\mathbf{k}}(\tau) e^{i\mathbf{k}.x} \tag{4.66}$$

Using equation(4.66) in equation(4.64), we get

$$S_2 = \frac{1}{2} \int \frac{d\tau d^3x d^3\mathbf{k} d^3\mathbf{k}'}{(2\pi)^3} [\chi'_{\mathbf{k}}\chi'_{\mathbf{k}'} e^{-i[-(\mathbf{k}+\mathbf{k}')]x} - \chi_{\mathbf{k}}\chi_{\mathbf{k}'}(-\mathbf{k}\mathbf{k}') e^{-i[-(\mathbf{k}+\mathbf{k}')]x}$$
$$+ \chi_{\mathbf{k}}\chi_{\mathbf{k}'} e^{-i[-(\mathbf{k}+\mathbf{k}')]x} \frac{z''}{z}]$$

using $\int d^3x e^{-i[-(\mathbf{k}+\mathbf{k}')]x} = (2\pi)\delta^3(-\mathbf{k}-\mathbf{k}')$ and now we will drop $\mathbf{k}$ to k because action only depends on the magnitude of the k , we get

$$S_2 = \frac{1}{2} \int d\tau d^3k \left[\left(\chi'_k\right)^2 - k^2\chi_k^2 + \frac{z''}{z}\chi_k^2 \right] \tag{4.67}$$

Now using the Euler-Lagrange equation,

$$\chi''_k + k^2\chi_k - \frac{z''}{z}\chi_k = 0 \tag{4.68}$$

Equation(4.68) is called Mukhanov-Sasaki Mode equation. And the solution of this equation gives qualitative behaviours at sub-horizon and super-horizon scales.
The general solution $\chi(\tau)$ can be written as a linear combination of v_k and v_k^* as [11]

$$\chi_{\mathbf{k}}(\tau) = \left[a^-_{\mathbf{k}} v_k^*(\tau) + a^+_{-\mathbf{k}} v_k(\tau) \right] \tag{4.69}$$

Where, $v_k(\tau)$ and $v_k^*(\tau)$ are two linearly independent solution of equation(4.68) and $a_{\mathbf{k}}^{\pm}$ are complex constant of integration. Insert the equation(4.69) in (4.66) and $a_{\mathbf{k}}^{+} = (a_{\mathbf{k}}^{-})^*$.

$$\begin{aligned}\chi(\mathbf{x},\tau) &= \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \left[a_{\mathbf{k}}^{-} v_k^*(\tau) + a_{-\mathbf{k}}^{+} v_k(\tau)\right] e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \left[a_{\mathbf{k}}^{-} v_k^*(\tau) e^{i\mathbf{k}\cdot\mathbf{x}} + a_{\mathbf{k}}^{+} v_k(\tau) e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \end{aligned} \tag{4.70}$$

Equation(4.70) can be written in the form of quantized field, just replace χ by $\hat{\chi}$ and $a_{\mathbf{k}}^{\pm}$ by operators $\hat{a}_{\mathbf{k}}^{\pm}$ satisfying commutation relations.

$$\left[\hat{a}_{\mathbf{k}}^{-}, \hat{a}_{\mathbf{k}'}^{+}\right] = \delta(\mathbf{k}-\mathbf{k}'), \quad \left[\hat{a}_{\mathbf{k}}^{-}, \hat{a}_{\mathbf{k}'}^{-}\right] = \left[\hat{a}_{\mathbf{k}}^{+}, \hat{a}_{\mathbf{k}'}^{+}\right] = 0 \tag{4.71}$$

And mode function v_k can be normalized as

$$v_k' v_k^* - v_k (v_k^*)' = -i \tag{4.72}$$

Sub-horizon scales: If the co-moving wavelength of the perturbation is much smaller in comparison with Hubble radius i.e. $k \gg aH$ implies $k^2 \gg | \frac{z''}{z} |$, then mode equation becomes an equation of harmonic oscillator,

$$v_k'' + k^2 v_k = 0 \tag{4.73}$$

Locally space-time is Minkowski i.e. if the scales are smaller than the radius of curvature of de Sitter space-time. The unique solution in Minkowski space (gravity free case) exist for only positive frequency part, implies this is the minimum energy eigenstate known as Bunch-Davies vacuum,

$$v_k = \frac{1}{\sqrt{2k}} k e^{-ik\tau}, \quad (k \gg aH) \tag{4.74}$$

Super-horizon scales: In contrast for the scales much larger than the co-moving Hubble Radius i.e. $k \ll aH$ implies $k^2 \ll | \frac{z''}{z} |$, then mode equation reduces to

$$v_k'' - \frac{z''}{z} v_k = 0 \tag{4.75}$$

Assume a solution as, $v_k \propto \tau^n$, where n is an integer. Inserting it in equation(4.75), we have

$$\begin{aligned} 0 &= n(n-1)\tau^{n-2} - 2\tau^{n-2} \\ &= (n-2)(n+1) \end{aligned} \tag{4.76}$$

For $n = 2$ and $n = -1$ solutions are $v_k \propto z \propto \tau^2$(decaying mode solution) and $v_k \propto \tau^{-1}$(growing mode solution). Second solution shows that on super-horizon scales ζ freezes i.e. $\zeta_{\mathbf{k}} = z^{-1}\chi_{\mathbf{k}} \propto const.$
In de Sitter space Mukhanov-Sasaki equation can be written as, using equation(4.65)

$$v_k'' + \left(k^2 - \frac{2}{\tau^2}\right) v_k = 0 \tag{4.77}$$

General solution of the Bessel equation(4.77) is

$$v_k(\tau) = \left[A(k)H^{(1)}_{\frac{3}{2}}(-k\tau) + B(k)H^{(2)}_{\frac{3}{2}}(-k\tau)\right]\sqrt{-\tau} \tag{4.78}$$

Where $H^{(1,2)}_{\frac{3}{2}}$ represents first and second kind Hankel functions.

$$H^{(2)}_{\frac{3}{2}}(x) = \left[H^{(1)}_{\frac{3}{2}}(x)\right]^* = -\sqrt{\frac{2}{\pi x}}e^{-ix}\left(1 + \frac{1}{ix}\right) \tag{4.79}$$

inserting equation(4.79) in (4.78), we get

$$\begin{aligned} v_k(\tau) &= A(k)e^{-ik\tau}\left(1 + \frac{1}{ik\tau}\right) + B(k)e^{ik\tau}\left(1 - \frac{1}{ik\tau}\right) \\ &= A(k)e^{-ik\tau}\left(1 - \frac{i}{k\tau}\right) + B(k)e^{ik\tau}\left(1 + \frac{i}{k\tau}\right) \end{aligned} \tag{4.80}$$

Imposing initial conditions to determine constants $A(k)$ and $B(k)$, using equation(4.74), we get

$$A(k) = \frac{1}{\sqrt{2k}}, \qquad B(k) = 0 \tag{4.81}$$

Then equation(4.80) will become,

$$v_k(\tau) = \frac{1}{\sqrt{2k}}e^{-ik\tau}\left(1 - \frac{i}{k\tau}\right) \tag{4.82}$$

Equation(4.82) is valid for all wavelengths i.e. for $\mid k\tau \mid \gg 1$ it reduces to equation(4.74) and $\mid k\tau \mid \ll 1$ mode function will become,

$$\begin{aligned} v_k(\tau) &= \frac{1}{\sqrt{2k}}(1 - ik\tau)\left(1 - \frac{i}{k\tau}\right) \\ &= \frac{1}{i\sqrt{2}k^{\frac{3}{2}}}\frac{1}{\tau} \\ &= const \quad a \end{aligned} \tag{4.83}$$

which is identical to growing mode solution.

4.1.2 Gravitational Waves

Tensor perturbation or gravitational waves(primordial) is equally important like scalar perturbations. Its detection will help to probe the early universe physics and about most important parameter of inflationary cosmology, energy scale of inflation. Its is assumed that the Primordial gravitational waves are due to the vacuum quantum fluctuations. These waves can be detected via clean signature in the polarization states(transverse and longitudinal) of the CMBR. [12]

Now in a similar fashion like we did for scalar perturbations, one can solve mode function for tensor perturbations. Considering only tensor part of the perturbed metric,

$$ds^2 = a^2(\tau)\left[-d\tau^2 + (1 + H_{ij})\, dx^i dx^j\right] \tag{4.84}$$

where H_{ij} is the tensor perturbation.

For equation of motion for the primordial gravitational waves, we will equation(3.14) and evaluate it up to second order in terms of H_{ij},

$$\delta S_{EH} = \int d^4x\sqrt{-g}\frac{M_{pl}^2}{2}G_{\mu\nu}\delta g^{\mu\nu} = 0 \tag{4.85}$$

where,

$$g_{\mu\nu} = a^2(\tau)\left(g_{\mu\nu}^{RW} + H_{ij}\delta_\mu^i\delta_\nu^j\right), \quad g^{\mu\nu} = a^{-2}(\tau)\left(g_{RW}^{\mu\nu} - H^{ij}\delta_i^\mu\delta_j^\nu\right), \quad H_i^i = H_{ij;m}^m = 0 \tag{4.86}$$

After inserting all components of Einstein tensor and using in equation(4.85) up to second order, we get

$$S_2 = \frac{M_{pl}^2}{8}\int d^3x d\tau a^2\left[\left(H_{ij}'\right)^2 - \left(\partial_l H_{ij}\right)^2\right] \tag{4.87}$$

Defining the Fourier representation of $H_{ij}(\tau, x)$,

$$H_{ij}(\tau, x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\sum_{\gamma=+,\times}\epsilon_{ij}^\gamma(k)H_{\mathbf{k},\gamma}(\tau)e^{i\mathbf{k}.x} \tag{4.88}$$

where $\epsilon_{ii}^\gamma = k^i\epsilon_{ij}^\gamma = 0$ and $\epsilon_{ij}^\gamma\epsilon_{ij}^{\gamma'} = 2\delta_{\gamma\gamma'}$, and $\gamma = +, \times$ represents the two modes of polarization of tensor perturbations. Inserting equation(4.88) in (4.87), we get

$$S_2 = \frac{M_{pl}^2}{4}\int d^3\mathbf{k} d\tau a^2\left[\left(H_{\mathbf{k},\gamma}'\right)^2 - k^2\left(H_{\mathbf{k},\gamma}\right)^2\right] \tag{4.89}$$

Now redefining $H_{\mathbf{k},\gamma}$ in terms of an auxiliary field $v_{\mathbf{k},\gamma}$,

$$v_{\mathbf{k},\gamma} \equiv \frac{a}{2} M_{pl} H_{\mathbf{k},\gamma} \tag{4.90}$$

Using equation(4.90) in (4.89), we get

$$S_2 = \sum_{\gamma=+,\times} \frac{1}{2} \int d\tau d^3\mathbf{k} \left[\left(v'_{\mathbf{k},\gamma}\right)^2 - \left(k^2 - \frac{a''}{a}\right) \left(v_{\mathbf{k},\gamma}\right)^2 \right] \tag{4.91}$$

Using Euler–Lagrange equation, we get

$$v''_{\mathbf{k},\gamma} + \left(k^2 - \frac{a''}{a}\right) \left(v_{\mathbf{k},\gamma}\right) = 0 \tag{4.92}$$

Since in de Sitter space $\frac{a''}{a} = \frac{2}{\tau^2}$. Equation(4.92) is similar to (4.77), so we can write solution of equation(4.92) in a similar manner as,

$$v_{\mathbf{k},\gamma}(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right) \tag{4.93}$$

On super-horizon scales i.e. $\mid k\tau \mid \ll 1$,

$$v_{\mathbf{k},\gamma}(\tau) = \frac{1}{i\sqrt{2}k^{\frac{3}{2}}} \frac{1}{\tau} \tag{4.94}$$

4.2 Power Spectrum

A mathematical function which measures the characteristics of perturbations or distribution of any quantity in a space in known as power spectrum. In Quasi de Sitter space, power spectrum of the scalar perturbations can be calculated as, using $\hat{\zeta}_{\mathbf{k}} = z^{-1}\hat{\chi}_{\mathbf{k}}$

$$\begin{aligned} < 0 \mid \hat{\zeta}_{\mathbf{k}} \hat{\zeta}_{\mathbf{k'}} \mid 0 > &= \frac{1}{z^2} < 0 \mid \hat{\chi}_{\mathbf{k}} \hat{\chi}_{\mathbf{k'}} \mid 0 > \\ &=< 0 \mid \left(a^-_{\mathbf{k}} v_k + a^+_{-\mathbf{k}} v^*_k\right) \left(a^-_{\mathbf{k'}} v_{k'} + a^+_{-\mathbf{k'}} v^*_{k'}\right) \mid 0 > \\ &= v_k v^*_{k'} < 0 \mid \left[a^-_{\mathbf{k}}, a^+_{-\mathbf{k'}}\right] + a^+_{-\mathbf{k'}} a^-_{\mathbf{k}} \mid 0 > \\ &=\mid v_k \mid^2 \delta^3 \left(k + k'\right) \end{aligned} \tag{4.95}$$

Where, P_ζ is the power spectrum of the curvature perturbation and $P_v =\mid v_k \mid^2$ so we can write,

$$P_\zeta = \frac{1}{z^2} P_v \tag{4.96}$$

Equating equation(4.82) in (4.96) and, using $z^2 = 2a^2\varepsilon$ and $a = -(aH)^{-1}$, we get

$$< \hat{\zeta}_{\mathbf{k}} \hat{\zeta}_{\mathbf{k'}} >= \frac{H^2}{4\varepsilon_H k^3} \left(1 + k^2\tau^2\right) \delta^3(k + k') \tag{4.97}$$

On super-horizon scales equation(4.96), using $\varepsilon = \frac{\dot{\phi}^2}{2M_{pl}^2 H^2}$ yields,

$$< \hat{\zeta}_{\mathbf{k}} \hat{\zeta}_{\mathbf{k'}} >= \frac{H^4}{2k^3 \dot{\phi}^2} \delta^3(k + k') \tag{4.98}$$

At horizon crossing ζ freezes and power spectrum can be written as

$$< \hat{\zeta}_{\mathbf{k}} \hat{\zeta}_{\mathbf{k'}} >= \left. \frac{H^2}{4k^3 M_{pl}^2 \varepsilon} \right|_{k=aH} \tag{4.99}$$

The dimensionless power spectrum is define as

$$\triangle_s^2 (k) \equiv \frac{k^3}{2\pi^2} P_\zeta(k) = \left. \frac{H^2}{8\pi^2 M_{pl}^2 \varepsilon} \right|_{k=aH} \tag{4.100}$$

Equation(4.100) shows that $\triangle_s^2(k)$ depends on H and ε which are the function of time, which implies power spectrum will vary slightly in the quasi de Sitter space, from the scale invariant spectrum. To quantify the slowly varying spectrum, let us define an important quantity known as scalar spectral index n_s as [4],

$$n_s - 1 \equiv \frac{dln \triangle_s^2 (k)}{dlnk} \tag{4.101}$$

Using chain rule we can write R.H.S of equation(4.101) as,

$$\frac{dln\triangle_s^2}{dlnk} = \frac{dln\triangle_s^2}{dN} \times \frac{dN}{dlnk} \tag{4.102}$$

Inserting equation(4.100) in first time of equation(4.102), we get

$$\frac{dln\triangle_s^2}{dlnk} = \left[2\frac{dlnH}{dN} - \frac{dln\varepsilon}{dN} \right] \times \frac{dN}{dlnk} \tag{4.103}$$

Using the results from the chapter two, $\varepsilon = -\frac{dlnH}{dN}$ and $\eta = \frac{dln\varepsilon}{dN}$, then equation(4.103) becomes

$$\frac{dln\triangle_s^2}{dlnk} = [-2\varepsilon - \eta] \times \frac{dN}{dlnk} \tag{4.104}$$

Now evaluating last factor of equation(4.103) at horizon crossing $k = aH$

$$lnk = lna + lnH \Rightarrow N + lnH \tag{4.105}$$

Hence,

$$\frac{dN}{dlnk} = \left[\frac{dlnk}{dN} \right]^{-1} = \left[1 + \frac{dlnH}{dN} \right]^{-1} \approx 1 + \varepsilon \tag{4.106}$$

Finally equation(4.101) will become,

$$\begin{aligned} n_s(k) - 1 &= (-2\varepsilon - \eta)(1+\varepsilon) \\ &= -2\varepsilon - 2\varepsilon^2 - \eta - \eta\varepsilon \end{aligned} \tag{4.107}$$

considering first order approximation

$$n_s(k) - 1 \approx -2\varepsilon - \eta \tag{4.108}$$

From equation(3.51) we write equation(4.108) as,

$$\boxed{n_s(k) - 1 \approx 2\delta - 4\varepsilon} \tag{4.109}$$

Similarly power spectrum of the tensor perturbations can be evaluated as,

$$\begin{aligned} < 0 \mid \hat{H}_{\mathbf{k}',\gamma}\hat{H}_{\mathbf{k},\gamma} \mid 0 > &= \frac{4}{a^2 M_{pl}^2} < 0 \mid \hat{v}_{\mathbf{k}',\gamma}\hat{v}_{\mathbf{k},\gamma} \mid 0 > \\ &= \frac{4}{a^2 M_{pl}^2}\frac{(aH)^2}{2k^3}\delta^3(k+k') \end{aligned} \tag{4.110}$$

$$P_h =< \hat{H}_{\mathbf{k}',\gamma}\hat{H}_{\mathbf{k},\gamma} > = \frac{4}{a^2 M_{pl}^2} P_v \tag{4.111}$$

Here $P_v = \frac{(aH)^2}{2k^3}$ and $P_H = \frac{4}{a^2 M_{pl}^2}\frac{(aH)^2}{2k^3}$. Adding the power spectra of the polarization modes of H_{ij} gives the power spectrum of the tensor perturbations.

$$P_t = 2P_H = \frac{8}{a^2 M_{pl}^2} P_v = \frac{4}{k^3}\frac{H^2}{M_{pl}^2} \tag{4.112}$$

dimensionless form of the power spectrum is,

$$\triangle_t^2(k) \equiv \frac{k^3}{2\pi^2} P_t = \frac{2}{\pi^2}\frac{H^2}{M_{pl}^2}\bigg|_{k=aH} \tag{4.113}$$

Equation(4.112) depends on H which varies slightly implies power spectrum of tensor perturbations will be nearly scale invariant on super-horizon scales, defining the tensor spectral index n_t as,

$$n_t \equiv \frac{dln\triangle_t^2}{dlnk} \tag{4.114}$$

Using chain rule and $\frac{dN}{dlnk} \approx 1 + \varepsilon$ in equation(4.112), we have

$$\begin{aligned} n_t(k) &= \frac{dln\triangle_t^2}{dN}\frac{dN}{dlnk} \\ &= -2\varepsilon - 2\varepsilon^2 \end{aligned} \tag{4.115}$$

First order approximation will give,

$$\boxed{n_t \approx -2\varepsilon} \tag{4.116}$$

For comparison between scalar and tensor power spectrum, a tensor to scalar ratio plays an important role,

$$r \equiv \frac{\triangle_t^2}{\triangle_s^2} = \frac{\frac{2}{\pi^2}\frac{H^2}{M_{pl}^2}}{\frac{1}{8\pi^2}\frac{H^2}{M_{pl}^2\varepsilon}} = 16\varepsilon \tag{4.117}$$

4.2.1 Slow Roll Results

The relation between the potential and Hubble slow roll parameters in the approximation of slow roll are as,

$$\varepsilon \approx \epsilon_V, \quad \delta \approx \eta_V - \epsilon_V \tag{4.118}$$

And the spectra of scalar and tensor can be written in terms of potential using equation(3.48) and (3.52) as,

$$\triangle_s^2(k) \approx \frac{1}{24\pi^2 M_{pl}^2}\frac{V}{\epsilon_V}\bigg|_{k=aH}, \quad \triangle_t^2(k) \approx \frac{2}{3\pi^2}\frac{V}{M_{pl}^2}\bigg|_{k=aH} \tag{4.119}$$

Using equation(4.118) in equations(4.109) and (4.116), scalar and tensor spectral indices will become,

$$\boxed{n_s - 1 = 2\eta_V - 6\epsilon_V} \tag{4.120}$$

$$\boxed{n_t = -2\epsilon_V} \tag{4.121}$$

And the tensor to scalar ratio becomes,

$$\boxed{r = 16\epsilon_V \approx -8n_t} \tag{4.122}$$

Running of spectral index can be calculated as,

$$\begin{aligned}\alpha_s = \frac{dn_s}{dlnk} &= 16\left[\frac{M_{pl}^4}{2}\frac{V'^2V''}{V^3}\right] - 24\left[\frac{M_{pl}^4}{4}\left(\frac{V'}{V}\right)^4\right] - 2\left[M_{pl}^4\frac{V'V''}{V^2}\right] \\ &= 16\epsilon_V\eta_V - 24\epsilon_V^2 - 2\xi_V^2 \end{aligned}\tag{4.123}$$

where,

$$\xi_V^2 = M_{pl}^4\frac{V'V'''}{V^2} \tag{4.124}$$

It represents second order slow roll parameter.

4.3 Energy Scale Of The Inflation and The Lyth Bound

The observational constraint on the amplitude of scalar power spectrum measured by The Cosmic Background Explorer (COBE) is [12],

$$\frac{2}{5}P_{\zeta}^{\frac{1}{2}}(k) = 1.91 \times 10^{-5} \tag{4.125}$$

Its is assumed that the contribution of the gravity waves are suppressed in comparison with scalar perturbations, which is why tensor perturbations are still undetected, shown from the CMB data. Since $\triangle_t^2 \propto H^2 \approx V$, the energy scale of the inflation can be estimated by r and $\triangle_s^2$ is fixed. Rewriting equation(4.119) as,

$$\frac{4}{25}P_{\zeta}(k) \equiv \triangle_{\zeta}^2(k) = \frac{1}{150\pi^2 M_{pl}^4}\frac{V}{\epsilon_V} \tag{4.126}$$

Comparing equation(4.125) and (4.126), we get

$$\left(\frac{V}{\epsilon_V}\right)^{\frac{1}{4}} = 0.027M_{pl} = 6.7 \times 10^{16}GeV \tag{4.127}$$

using equation(4.122),

$$V^{\frac{1}{4}} \simeq \left(\frac{r}{0.01}\right)^{\frac{1}{4}} \times 10^{16}GeV \tag{4.128}$$

It shows that inflationary energy scale is couple of order less than the Planck's scale i.e. inflation would have occurred at GUT scale. Only the detection of the gravitational waves will help us to estimate the Hubble scale of the inflation which is related to the size of tensor to scalar ratio. So it is important to understand the bounds of r which determines whether the primordial gravity waves are observable in near future or not. PLANCK satellite has placed upper bound $r < 0.12$ at 95% C.L. In the near future Gravity waves has very little chances to be detected unless $r \gtrsim 0.01$. Large field models like quadratic model predicts large enough value of r which has put it in danger of exclusion and quartic model is already excluded after PLANCK data and most of the inflationary models predicts much smaller r [13].

The Lyth gave the relation of variation in the scalar field ϕ with respect to the number of the e-folds N in terms of tensor-to-scalar ratio r, known as the Lyth bound, using equation(4.117)

$$r = \frac{\frac{8}{M_{pl}^2}\left(\frac{H}{2\pi}\right)^2}{\left(\frac{H}{\dot{\phi}}\right)^2\left(\frac{H}{2\pi}\right)^2} = \frac{8}{M_{pl}^2}\left(\frac{d\phi}{dN}\right)^2 \tag{4.129}$$

or,

$$\frac{d\phi}{dN} = M_{pl}\sqrt{\frac{r}{8}} \tag{4.130}$$

In the slow roll regime, tensor to scalar ratio vary very slightly for 60 e-folds,

$$\frac{\triangle\phi}{M_{pl}} \simeq 2\left(\frac{r}{0.01}\right)^{\frac{1}{2}} \tag{4.131}$$

Lyth bound allows us to distinguish between large field models $\triangle\phi \gtrsim M_{pl}$ and small field models $\triangle\phi < M_{pl}$.

4.4 Density and Curvature Perturbations During Inflation

During inflation major source of energy density is scalar field(inflaton), and perturbations in it causes perturbations in the stress-energy tensor and produces metric perturbations. These small quantum fluctuations are due to intrinsic uncertainties in all microscopic processes. Since the inflation has taken place in the very beginning of the universe so the major part would have came from quantum fluctuations. As the inflaton rolls down on the potential energy curve where kinetic energy of the inflaton is negligible compared to $V(\phi)$ to the valley, inflaton can be found be near end of the plateau or may be near to the hill because of the uncertainty in position and energy, but over all movement of the inflaton is towards valley. As a result inflation would have ended at different times in different patches of the universe, implies different patches has went through different amount of inflation leaving the universe inhomogeneous(matter density varies in the different regions). These perturbations grew with the expansion of the universe and their amplitude freezes after becoming bigger than Hubble scale, then renter the horizon and appears as macroscopic objects. It is expected that it is true for the perturbations of the curvature and energy density of the universe. Photons carries information about the distribution of matter and radiation since decoupling. Denser regions emits photons of energy little less than average because photon loses it energy in performing work against strong gravitational pull and opposite is the case for less denser regions giving rise to temperature fluctuations. Depending on the density of the regions attract more matter from the surrounding which we observe as super-clusters, cluster and galaxies etc. Thus inflation provides the mechanism for the formation of large scale structures [14]. First order perturbation of the FRW metric as,

$$g(\tau, x) = \bar{g}_{\mu\nu}(\tau) + \delta g_{\mu\nu}(\tau, x) \tag{4.132}$$

Where,

$$\delta g_{\mu\nu} = \begin{pmatrix} -2\Phi & B_{,i} - S_i \\ B_{,j} - S_j & -2\Psi\delta_{ij} + E_{ij} \end{pmatrix} \tag{4.133}$$

which contains four scalars (Φ, B, Ψ, E) has four degree of freedom, two vectors (F, S) has two degree of freedom for each and one tensor (H_{ij}) has two degree of freedom,

$$\begin{aligned}
&B_i = \partial_i B - S_i \quad \text{where} \quad \partial^i S_i = 0 \\
&E_{ij} \equiv 2D_{ij}E + 2\partial_{(i}F_{j)} + H_{ij} \quad \text{where} \quad D_{ij} = \partial_{ij} - \frac{1}{3}\delta_{ij}\bigtriangledown^2, \\
&\partial^i F_i = 0, \text{and} \quad H^i_i = \partial^i H_{ij} = 0
\end{aligned} \tag{4.134}$$

At $\mathcal{O}(1)$, the scalars, vectors and tensor perturbations can be treated separately because they acts as independent entities. Only scalar and tensor perturbations are important here because they appears as density perturbations and the primordial gravitational waves in the later stages of the universe, and vector perturbations decays out with the expansion of the universe.
Scalar perturbations do change under coordinate transformations while tensor perturbations acts as invariant under gauge transformation.
Consider a gauge transformation as,

$$\begin{aligned}
&x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu(x,t) \quad \text{or} \quad x^\mu = \tilde{x}^\mu - \xi^\mu \\
&\xi^i = \xi^i_\perp + \partial^i \xi_s, \quad \text{where} \quad \partial_i(\xi^i_\perp) = 0
\end{aligned} \tag{4.135}$$

We know that metric tensor transforms from one coordinate system to another as follows,

$$\tilde{g}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(x) \tag{4.136}$$

using $x^\mu = \tilde{x}^\mu - \xi^\mu$, in equation(4.136), we have

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \left[\delta^\alpha_\mu - \partial_\mu \xi^\alpha\right]\left[\delta^\beta_\nu - \partial_\nu \xi^\beta\right] g_{\alpha\beta}(x) \tag{4.137}$$

Neglecting second order terms, equation(4.137) becomes

$$\tilde{g}_{\mu\nu}(\tilde{x}) = g_{\mu\nu} - g_{\mu\beta}\partial_\nu \xi^\beta - g_{\alpha\nu}\partial_\mu \xi^\alpha \tag{4.138}$$

Since $\tilde{g}_{\mu\nu}(\tilde{x})$ and $g_{\mu\nu}(x)$ are the combination of background plus perturbed metric,

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \bar{\tilde{g}}_{\mu\nu}(\tilde{x}) + \delta\tilde{g}_{\mu\nu}(\tilde{x}) \quad \text{and} \quad g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x) \tag{4.139}$$

Using equation(4.139) in equation(4.138), we have

$$\bar{\tilde{g}}_{\mu\nu}(\tilde{x}) + \delta\tilde{g}_{\mu\nu}(\tilde{x}) = \bar{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x) - \bar{g}_{\mu\beta}\partial_\nu \xi^\beta - \bar{g}_{\alpha\nu}\partial_\mu \xi^\alpha \tag{4.140}$$

Now applying coordinate transformation $x^\mu = \tilde{x}^\mu - \xi^\mu$ in the first term of the L.H.S and expanding it using Taylor expansion, we have

$$\bar{\tilde{g}}_{\mu\nu}(\tilde{x}) = \bar{\tilde{g}}_{\mu\nu}(x+\xi) = \bar{g}_{\mu\nu}(x) + \xi^\lambda \partial_\lambda \bar{g}_{\mu\nu}(x) \tag{4.141}$$

We need background terms should remain equal in all coordinate systems,

$$\bar{\tilde{g}}_{\mu\nu}(\tilde{x}) = \bar{g}_{\mu\nu}(x) \tag{4.142}$$

Then equation(4.140) will become,

$$\bar{g}_{\mu\nu}(x) + \xi^\lambda \partial_\lambda \bar{g}_{\mu\nu}(x) + \delta\tilde{g}_{\mu\nu}(\tilde{x}) = \bar{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x) - \bar{g}_{\mu\beta}\partial_\nu \xi^\beta - \bar{g}_{\alpha\nu}\partial_\mu \xi^\alpha \tag{4.143}$$

After cancelling first terms both sides, we get

$$\delta\tilde{g}_{\mu\nu}(\tilde{x}) = \delta g_{\mu\nu}(x) - \xi^\lambda \partial_\lambda \bar{g}_{\mu\nu}(x) - \bar{g}_{\mu\beta}\partial_\nu \xi^\beta - \bar{g}_{\alpha\nu}\partial_\mu \xi^\alpha \tag{4.144}$$

Now components of the equation(4.144), we have

$$\begin{aligned}
\delta\tilde{g}_{00} &= \delta g_{00} - \xi^\lambda \partial_\lambda \bar{g}_{00}(x) - \bar{g}_{0\beta}\partial_0 \xi^\beta - \bar{g}_{\alpha 0}\partial_0 \xi^\alpha \\
\tilde{\Phi} &= \Phi - \frac{a'}{a}\xi^0 - (\xi^0)'
\end{aligned} \tag{4.145}$$

$$\begin{aligned}
\delta\tilde{g}_{0i} &= \delta g_{0i} - \xi^\lambda \partial_\lambda \bar{g}_{0i}(x) - \bar{g}_{0\beta}\partial_i \xi^\beta - \bar{g}_{\alpha i}\partial_0 \xi^\alpha \\
\tilde{B}_{,i} &= B_{,i} + \xi^0_{,i} - (\xi^i)' \\
\tilde{B} &= B + \xi^0 + \xi_s'
\end{aligned} \tag{4.146}$$

And

$$\begin{aligned}
\delta\tilde{g}_{ij} &= \delta g_{ij} - \xi^\lambda \partial_\lambda \bar{g}_{ij}(x) - \bar{g}_{i\beta}\partial_j \xi^\beta - \bar{g}_{\alpha j}\partial_i \xi^\alpha \\
-\tilde{\Psi}\delta_{ij} + \tilde{E}_{ij} &= -\Psi\delta_{ij} + E_{ij} - \frac{a'}{a}\delta_{ij}\xi^0 - \frac{1}{2}\left(\xi^i_{,j} + \xi^j_{,i}\right)
\end{aligned} \tag{4.147}$$

The trace of $\frac{1}{2}\left(\xi^i_{,j} + \xi^j_{,i}\right)$ is $\xi^k_{,k}$ so we can split it into traceless and a diagonal part,

$$\frac{1}{2}\left(\xi^i_{,j} + \xi^j_{,i}\right) = \frac{1}{3}\delta_{ij}\xi^k_{,k} + \frac{1}{2}\left(\xi^i_{,j} + \xi^j_{,i}\right) - \frac{1}{3}\delta_{ij}\xi^k_{,k} \tag{4.148}$$

then separating terms, trace of the equation(4.146) gives transformation as,

$$
\begin{aligned}
-\tilde{\Psi}\delta_{ij} &= -\Psi\delta_{ij} - \frac{1}{3}\delta_{ij}\xi^{k}_{,k} - \frac{a'}{a}\delta_{ij}\xi^{0} \\
\tilde{\Psi} &= \Psi + \frac{1}{3}\xi^{k}_{,k} + \frac{a'}{a}\xi^{0}
\end{aligned}
$$

And

$$\tilde{E} = E + \xi_s \tag{4.149}$$

And tensor part H_{ij}

$$\tilde{H}_{ij} = H_{ij} \tag{4.150}$$

Similarly energy-momentum tensor can be written in the combination of background and perturbed part using,

$$T^{\mu}_{\nu} = (\rho + p)\, u^{\mu} u_{\nu} + p\delta^{\mu}_{\nu} + \Pi^{\mu}_{\nu} \tag{4.151}$$

where Π^{μ}_{ν} is an anisotropic stress and the only non-zero components of Π^{μ}_{ν} are Π^{i}_{j}. For the components of the T^{μ}_{ν} we need contravariant and covariant components of the four-velocity, written as

$$u^{\mu} = a^{-1}\left[1 - \Phi, \quad v^{i}\right] \quad \text{and} \quad u_{\mu} = a\left[-\left(1 + \Phi\right), \quad B_i + v_i\right] \tag{4.152}$$

Using equation(4.151) in (4.152), we get

$$T^{0}_{0} = -\left(\bar{\rho} + \delta\rho\right) \tag{4.153}$$

$$T^{0}_{i} = \left(\bar{\rho} + \bar{p}\right)\left(v^{i} + B^{i}\right) \tag{4.154}$$

$$T^{0}_{i} = -\left(\bar{\rho} + \bar{p}\right) v_i \tag{4.155}$$

$$T^{0}_{i} = \left(\bar{p} + \delta p\right) + \Pi^{i}_{j} \tag{4.156}$$

Following the similar procedure as for the perturbed metric, coordinate transformation of the perturbed energy-momentum can be written as,

$$\delta\tilde{T}^{\mu}_{\nu} = \delta T^{\mu}_{\nu} - \xi^{\gamma}\partial_{\gamma}\bar{T}^{\mu}_{\nu} + \partial_{\gamma}\xi^{\mu}\bar{T}^{\gamma}_{\nu} - \partial_{\nu}\xi^{\gamma}\bar{T}^{\mu}_{\gamma} \tag{4.157}$$

Now solving for components, we get

$$\delta\tilde{\rho} = \delta\rho - \xi^0\bar{\rho}' \tag{4.158}$$

$$\delta\tilde{p} = \delta p - \xi^0\bar{p}' \tag{4.159}$$

$$\tilde{q}^i = q^i + \xi^{i'}\left(\bar{\rho} + \bar{p}\right) \tag{4.160}$$

$$\tilde{\Pi}^i_j = \Pi^i_j \tag{4.161}$$

Above transformations shows that metric and energy momentum tensor perturbations are gauge dependent and it is convenient to work with quantities whci dont depend on the particular choice of coordinates.

4.4.1 Gauge Invariant Curvature Perturbation

The quantity Φ is known as curvature perturbation which is not gauge invariant, and it is important to determine the intrinsic spatial curvature on hyper-surfaces of fixed conformal time τ, for $k = 0$,

$$R^{(3)} = \frac{4}{a^2}\bigtriangledown^2\Psi \tag{4.162}$$

Gauge invariant curvature perturbation can be evaluated on the slices of uniform density i.e. $\delta\rho_{unif} = 0$. Then using $\xi^0 = \delta\tau$,

$$\begin{aligned}\delta\rho_{unif} &= \delta\rho - \bar{\rho}'\delta\tau = 0\\ \delta\tau &= \frac{\delta\rho}{\bar{\rho}'}\end{aligned} \tag{4.163}$$

equating equation(4.163) in the transformation equation of curvature perturbation Ψ, we get

$$\boxed{-\zeta \equiv \Psi + H\frac{\delta\rho}{\dot{\bar{\rho}}}} \tag{4.164}$$

Similarly, another gauge invariant quantity co-moving curvature perturbation can be written by considering the hyper-surfaces of the constant-ϕ, i.e. $\delta\phi = 0$

$$\boxed{\mathcal{R} \equiv \Psi + H\frac{\delta\phi}{\dot{\bar{\phi}}}} \tag{4.165}$$

Both $-\zeta$ and $\mathcal{R}$ becomes equal on super-horizon scales. It can be shown by using $\dot{\rho} = -3H\left(\rho + p\right)$, equation(4.164) becomes $-\zeta = \Psi - \frac{\delta\rho}{3(\rho+p)}$. During slow roll inflation $\rho + p = \dot{\phi}^2$ and $\dot{\phi} \approx -\frac{V'(\phi)}{3H}$, which implies $\delta\rho = \dot{\phi}\delta\phi + V'(\phi)\delta\phi \approx V'(\phi)\delta\phi = -3H\dot{\phi}\delta\phi$. On super-horizon scales inflaton fluctuation freezes, then leads to $-\zeta \approx \mathcal{R}$.

Why ζ is so important? Because this quantity remains constant i.e. $\dot{\zeta} = 0$ since wavelength of the perturbations becomes larger than the Hubble horizon (horizon exit), to when Hubble horizon again becomes large than perturbation wavelength (horizon re-entry). To study the evaluation of the universe after the end of inflation to say big bang nucleosynthesis, during which scientist have no information. It is required something to be remain fixed during this era, so that we can relate the universe outside the horizon to inside the horizon.

To understand the generation of the density perturbation during inflationary era, different regions of the universe went through slow roll inflation by similar way but, inflation ends at different times because of quantum effects i.e. $\delta t = \frac{\delta\phi}{\dot{\phi}}$. As a result different patches will expand slightly differently in amount and give rise to spectrum of energy density which is proportional to the perturbations in the scale factor at the end of the inflation.

$$\frac{\delta\rho}{\rho} \sim \frac{\delta a}{a} = H\delta t = H\frac{\delta\phi}{\dot{\phi}} = \frac{H^2}{2\pi\dot{\phi}} \tag{4.166}$$

Where we have taken $\delta\phi = \frac{H}{2\pi}$ representing the average amplitude of perturbations during inflationary era in the time interval of H^{-1}.

4.5 Reheating

Inflation ends as the strong energy condition are violated which leads to increase of kinetic energy of the inflaton and becomes the dominant energy. The inflaton field moves towards the minimum of the potential and performs damped oscillations about the minimum of the potential. During inflationary phase, it is assumed that interactions of the scalar field with other fields was negligible, but as the inflation ends transfer of inflaton energy takes place to ordinary matter particles and standard cosmology processes becomes dominant(radiation era) leading to matter dominant phase. Standard model of the particle physics requires the interactions of the inflaton with other fields such as $\lambda^2\sigma^2\phi^2$ (hybrid model) or Yukawa interactions with fermions $\lambda\phi\bar{\psi}\psi$ which makes the inflaton unstable and allow it decay into other degree of freedom. During inflaton oscillations about minimum, the equation motion for the scalar field becomes [4][7],

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma_\phi\dot{\phi} + V'(\phi) = 0 \tag{4.167}$$

where third term represents the friction term proportional to the decay width Γ_ϕ. During slow rolling of the inflaton $\Gamma_\phi \ll H$.

Chapter 5

Polynomial Chaotic Inflation

Chaotic inflationary model was proposed by Linde which is independent of any constraint on the universe to be in thermal equilibrium before inflation and based on slow roll of the inflaton rather than high temperature phase transition, in contrast to old and new inflationary model. There are many models which lies in the category of chaotic model but the most simplest models among them are quadratic and quartic models. Here '*Chaotic*' means, inflaton can take any value until $\phi \geq M_{pl}$, and sufficient amount of accelerated expansion is required so that processes of conventional big bang can be started, giving rise to the present state of the universe [17].
Generally chaotic potentials has the form $V(\phi) = \lambda_p \phi^p$, where p is a number having integral values, ϕ is a super-Planckian scalar field and λ_p represents small self-coupling constant of inflaton i.e $\lambda_p \ll 1$, so that the amplitude of density fluctuation remain small. Due to this potential energy automatically becomes sub-Planckian, $V \ll M_{pl}^4$.
In this chapter we have presented the predictions of the polynomial chaotic models at tree level and the role of quantum corrections in these models. After the end of an inflation, process of reheating starts and coupling of the inflaton field with matter(fermions) fields give rise to the modification in the potential energy V, i.e radiative correction. To illustrate the role of these correction terms in the polynomial chaotic models i.e. ϕ^2 and ϕ^4, Colemann-Weinberg method has been used.
For the quantum corrections consider the Lagrangian density [18],

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi_B \partial_B \phi_B + \frac{i}{2}\bar{N}\gamma^\mu \partial_\mu N - \frac{1}{2}m_B^2 \phi_B^2 - \frac{\lambda_B}{4!}\phi_B^4 - \frac{1}{2}h\phi_B \bar{N}N - \frac{1}{2}m_N \bar{N}N \tag{5.1}$$

where $'B'$ in the subscript represents the bare quantities, and N is the field denoting Standard Model singlet fermion (like right handed neutrino). One loop corrected inflationary potential is given by

$$V = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + V_{loop}, \tag{5.2}$$

Using Colemann-Weinberg method V_{loop} can be written as [18],

$$V_{loop} = \frac{1}{64\pi^2}\left[\left(m^2 + \frac{\lambda}{2}\phi^2\right)^2 ln\left(\frac{m^2 + (\lambda/2)\,\phi^2}{\mu^2}\right) - 2\,(h\phi + m_N)^4\, ln\left(\frac{(h\phi + m_N)^2}{\mu^2}\right)\right] \tag{5.3}$$

Considering $h\phi \gg m_N$ and $h\phi \gg m$ and $h^2 \gg \lambda$ during inflationary era, then the one loop correction to equation(5.2) will become,

$$V = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 - \kappa\phi^4 ln\left(\frac{\phi}{M_{pl}}\right) \tag{5.4}$$

where we have used renormalization scale $\mu = hM_{pl}$. Applying slow roll formalism to the potential given in equation(5.4) in two different limits, i.e. for $\lambda \ll \left(m^2/\phi^2\right)$ which will produce effectively quadratic potential plus radiative corrections, and for $\lambda \gg \left(m^2/\phi^2\right)$ yields quartic potential plus one loop corrections.

5.1 Case I: $\lambda \ll \left(m^2/\phi^2\right)$

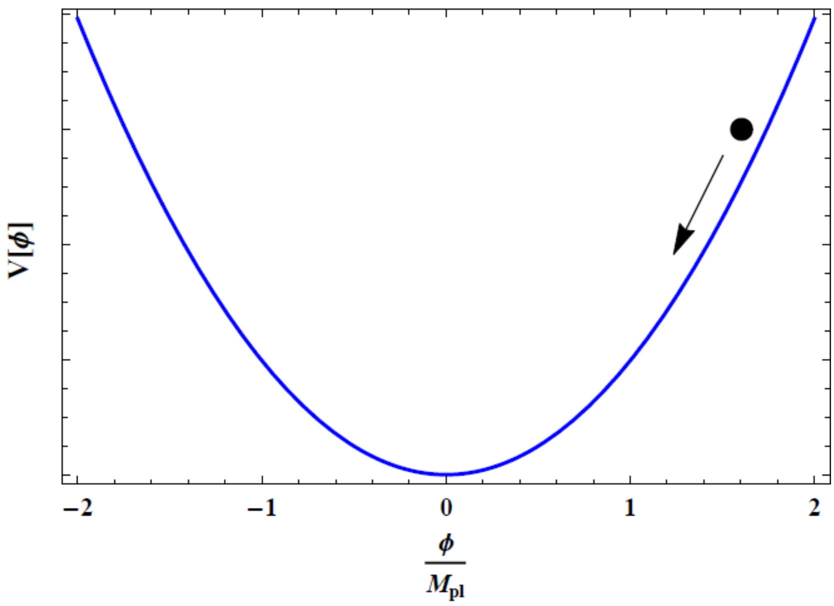

FIGURE 5.1: Inflaton field in the Quadratic Potential

At the tree level Quadratic potential is,

$$V(\phi) = \frac{1}{2}m^2\phi^2, \tag{5.5}$$

equations(3.54),(3.57), and (4.124) gives,

$$\epsilon_V(\phi) = \frac{2M_{pl}^2}{\phi^2} \tag{5.6}$$

$$\eta_V(\phi) = \frac{2M_{pl}^2}{\phi^2} \tag{5.7}$$

$$\xi_V^2 = 0 \tag{5.8}$$

Since $\eta_V = \epsilon_V$, inflation ends with $\epsilon_V(\phi_e) = 1$, i.e.

$$\phi_e = \sqrt{2}M_{pl} \tag{5.9}$$

Then equation(3.62) gives the number of e-folds,

$$N = \frac{1}{4M_{pl}^2}\left(\phi_0^2 - \phi_e^2\right) \tag{5.10}$$

using equation(5.9),

$$\phi_0 = 2M_{pl}\sqrt{N + \frac{1}{2}} \tag{5.11}$$

Now dropping subscript $'0'$ from ϕ and, expressing spectral index n_s and tensor-to-scalar ratio r in terms of no of e-folds N.

$$r = \frac{8}{\left(N + \frac{1}{2}\right)} \tag{5.12}$$

and

$$n_s = 1 - \frac{2}{\left(N + \frac{1}{2}\right)} \tag{5.13}$$

$$\alpha = -\frac{32}{\left(N + \frac{1}{2}\right)^2} \tag{5.14}$$

Amplitude of curvature perturbation becomes,

$$\triangle_\zeta^2(k) = \frac{1}{6\pi^2}\frac{m^2}{M_{pl}^2}\left(N + \frac{1}{2}\right)^2 \tag{5.15}$$

If we solve equations (5.12),(5.12), (5.13) and (5.15) for $N = 60$ and using experimental value $\triangle_\zeta^2(k) = 2.43 \times 10^{-9}$ yields, $n_s \simeq 0.9669$, $r \simeq 0.1322$, and $m \simeq 1.5271 \times 10^{13}$GeV.

Quadratic potential with quantum correction has the form,

$$V(\phi) = \frac{1}{2}m^2\phi^2 - \kappa\phi^4 ln\left(\frac{\phi}{M_{pl}}\right) \tag{5.16}$$

To find the predictions for the potential in equation(5.16) numerical computation has been used with the help of MATHEMATICA because it is not easy to solve analytically. Solutions has been found by using slow roll results at $N = 60$ by fixing m and scan over κ. This potential has two solutions for each value of κ. One of the solution exist in the limit of $m^2 \gg 4\kappa\phi_0^2 ln\left(\frac{\phi_0}{M_{pl}}\right)$, which is the quadratic solution and other at $m^2 \approx 4\kappa\phi_0^2 ln\left(\frac{\phi_0}{M_{pl}}\right)$, the hilltop solution. Two solutions coincides where κ reached it maximum value i.e. 7.16143×10^{-15}. Numerical results of quadratic potential with quantum correction are given below.

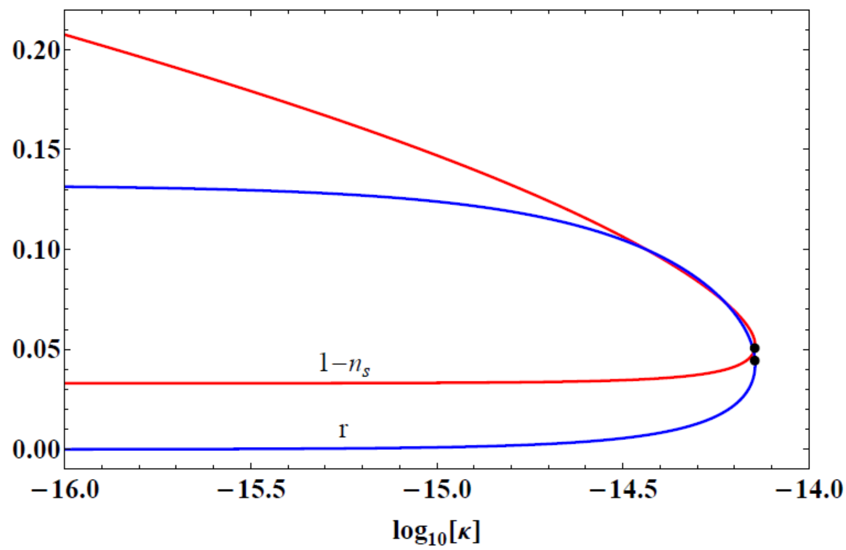

FIGURE 5.2: $1 - n_s$ and r vs $Log_{10}[\kappa]$. Black dots where two solutions meet.

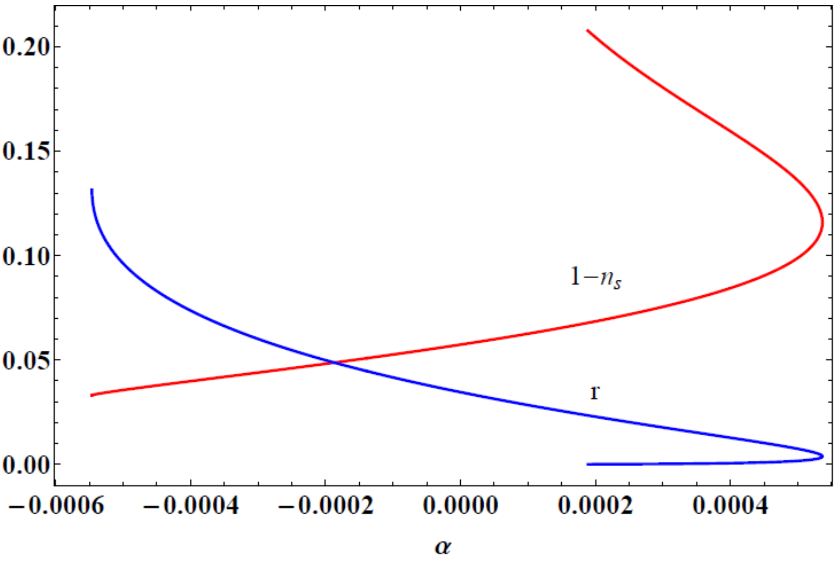

FIGURE 5.3: $1 - n_s$ and r vs α

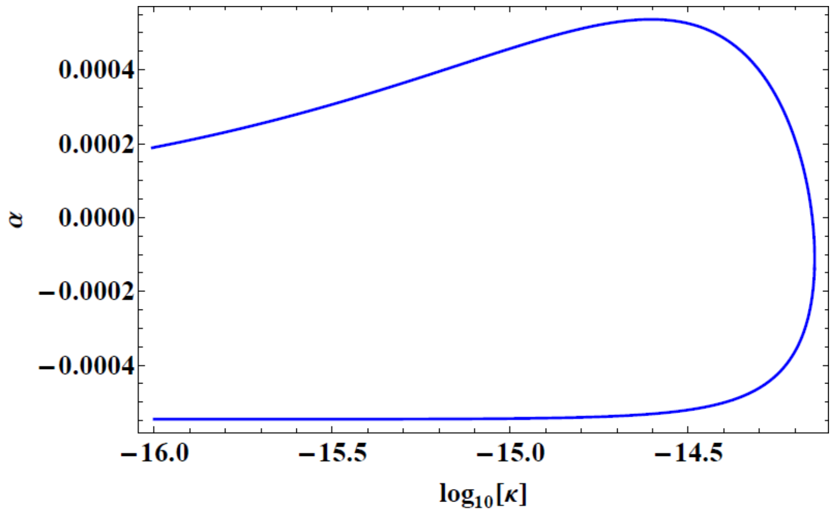

FIGURE 5.4: κ vs α

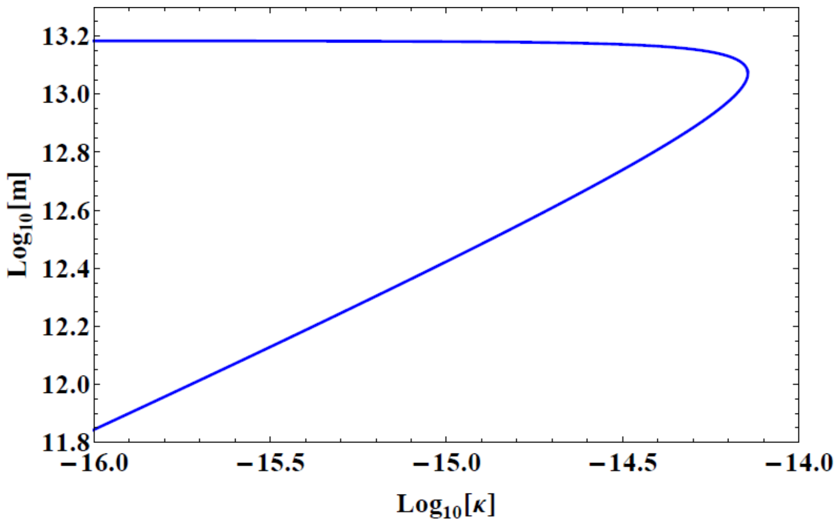

FIGURE 5.5: κ vs m

5.2 Case II: $\lambda \gg \left(m^2/\phi^2\right)$

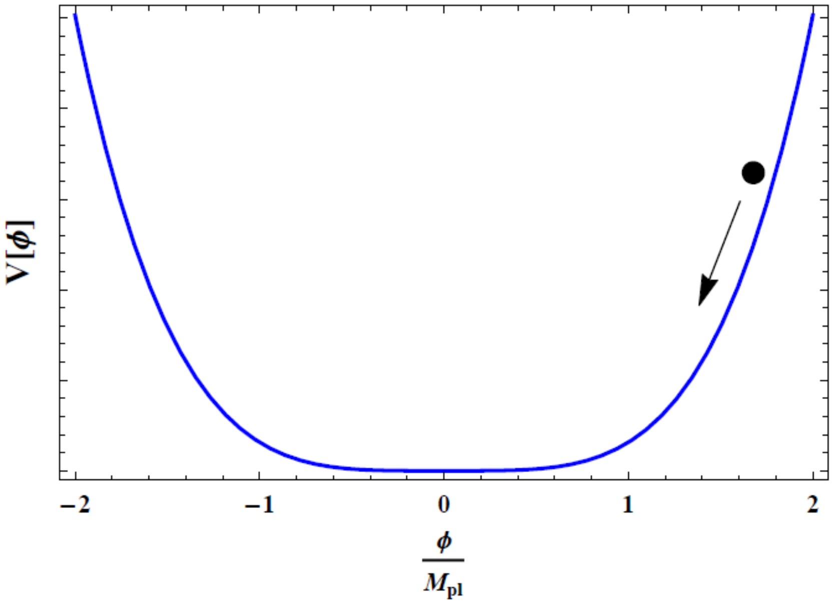

FIGURE 5.6: Inflaton field in the Quartic Potential

Tree level quartic potential is

$$V(\phi) = \frac{\lambda}{4!}\phi^4 \tag{5.17}$$

similarly using this potential in slow roll parameters, we get

$$\epsilon_V(\phi) = \frac{8M_{pl}^2}{\phi^2} \tag{5.18}$$

$$\eta_V(\phi) = \frac{12M_{pl}^2}{\phi^2} \tag{5.19}$$

$$\xi_V^2 = \frac{96M_{pl}^4}{\phi^4} \tag{5.20}$$

Since, inflation ends with $\eta_V(\phi_e) = 1$, i.e.

$$\phi_e = \sqrt{12}M_{pl} \tag{5.21}$$

And the number of e-folds,

$$N = \frac{1}{8M_{pl}^2}\left(\phi_0^2 - \phi_e^2\right) \tag{5.22}$$

using equation(5.21),

$$\phi_0 = M_{pl}\sqrt{8\left(N + \frac{1}{2}\right)} \tag{5.23}$$

Then we can write,

$$r = \frac{16}{(N+\frac{3}{2})} \tag{5.24}$$

and

$$n_s = 1 - \frac{3}{(N+\frac{3}{2})} \tag{5.25}$$

$$\alpha = -\frac{192}{(N+\frac{3}{2})^2} \tag{5.26}$$

Amplitude of curvature perturbation becomes,

$$\triangle_\zeta^2(k) = \frac{\lambda}{192\pi^2}\left[8\left(N+\frac{3}{2}\right)\right]^3 \tag{5.27}$$

Solving equations from (5.24) to (5.27) for $N = 60$, we get, $n_s \simeq 0.9512$, $r \simeq 0.2601$, and $\lambda \simeq 9.192 \times 10^{-13}$. And one loop quantum corrected form of the quartic potential is,

$$V(\phi) = \frac{\lambda\phi^4}{4} - \phi^4\kappa ln\left(\frac{\phi}{M_{pl}}\right) \tag{5.28}$$

For the predictions again numerical computation has been used, which yields two solutions for each value of κ like previously seen for quadratic potential. Predictions of quartic potential dominates when $\lambda \gg 24\kappa ln\left(\frac{\phi_0}{M_{pl}}\right)$ and for $\lambda \approx 24\kappa ln\left(\frac{\phi_0}{M_{pl}}\right)$ hilltop solution becomes dominant. And two solutions overlap when κ reached it peak value i.e. 6.123×10^{-14}. Numerical results are as follows,

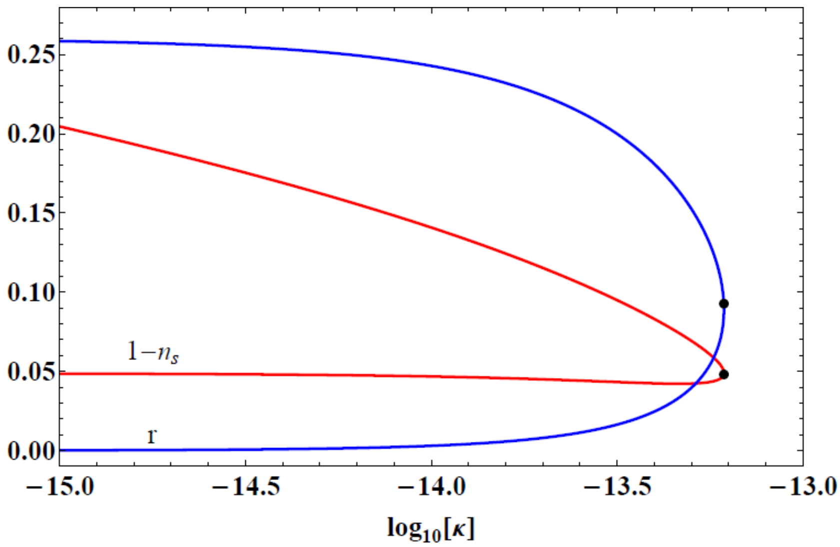

FIGURE 5.7: $1 - n_s$ and r vs $Log_{10}[\kappa]$ for ϕ^4. Black dots where two solutions meet.

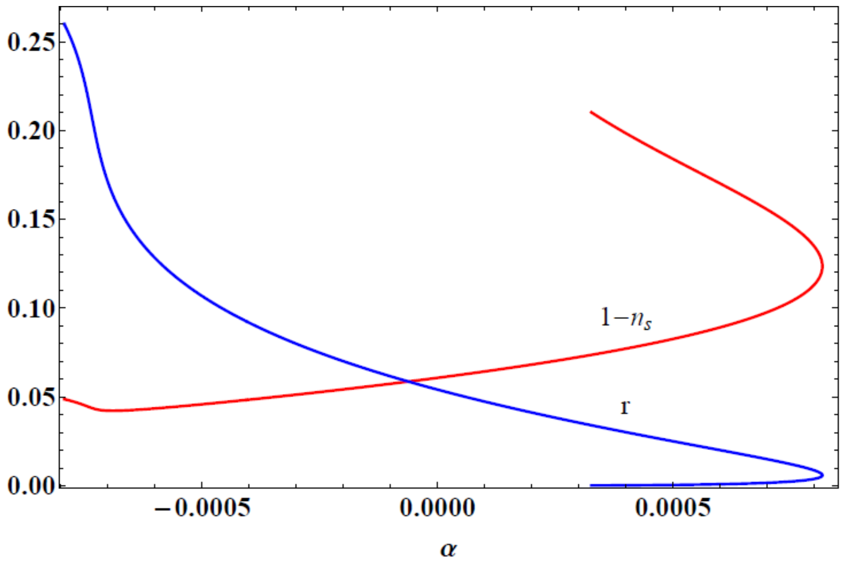

FIGURE 5.8: $1 - n_s$ and r vs α for ϕ^4

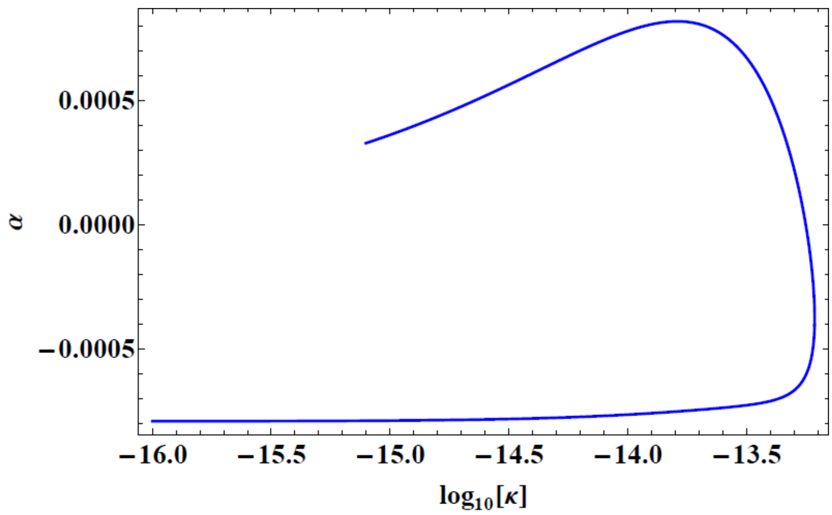

FIGURE 5.9: κ vs α for ϕ^4

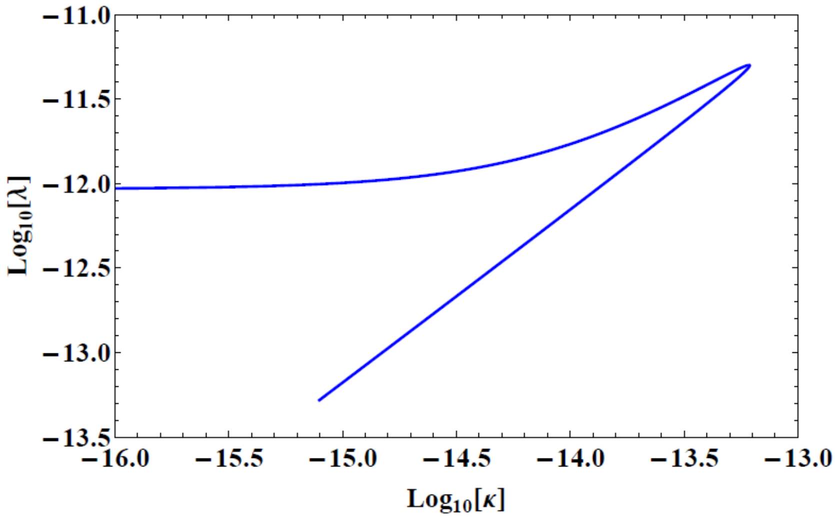

FIGURE 5.10: κ vs λ for ϕ^4

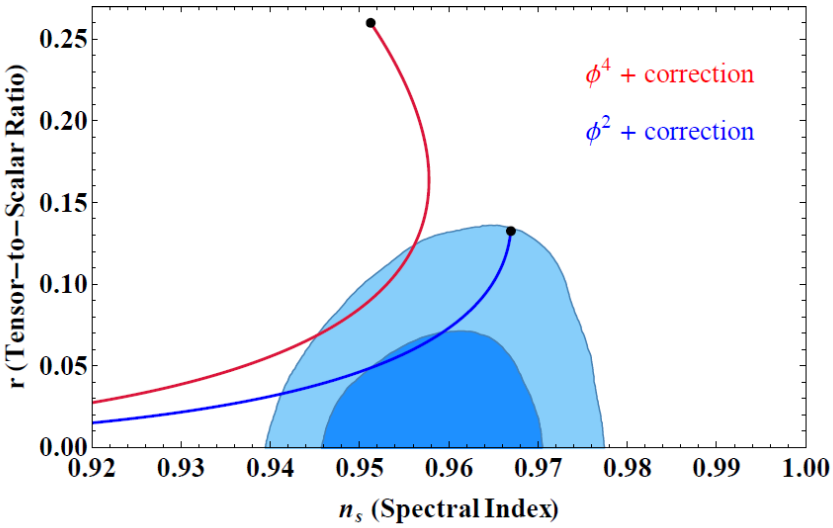

FIGURE 5.11: r vs n_s

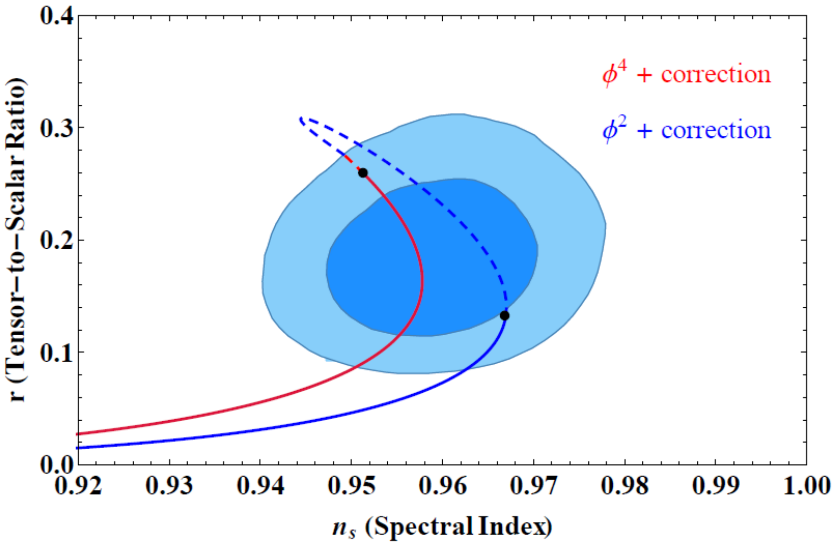

FIGURE 5.12: r vs n_s

5.3 Discussion

Results shows that the tree level prediction of tensor to scalar ratio (black spots) for quadratic potential lies on the boundary of 95% CL contour of PLANCK+WP+highL, i.e. this model is in danger of exclusion. And tree level quartic potential was already excluded by WMAP and also lies well outside the the bounds of PLANCK's dataset. But the inclusion of one loop quantum correction to the ϕ^2 model has made it look quiet compatible to current data and predicts appreciable enough tensor-to-scalar ratio which favours the detections of gravity waves. But the prediction for ϕ^4 still lies on the edge of 95% CL contour which puts it in danger. While on BICEP2 data predictions for both bosonic and fermionic interactions are quite consistent for both mentioned models.

Chapter 6

Non-SUSY Hybrid inflation

Hybrid inflationary model has more than one field and during inflationary phase inflaton slowly rolls down the potential, while the other field remains massive and frozen, and effectively make it a single field model. End of inflation take place because of the other field, i.e. water fall mechanism leading to the phase transition. Hybrid inflation is not a small nor the large field model instead it is the intermediate of the latter.

6.1 Tree Level Hybrid Inflation

The quadratic hybrid inflation at tree level and one loop corrected form has been studied in the light of the PLANCK data, and the latter case shows good agreement with the experimental data. Tree level quadratic hybrid inflationary potential can be written as [21],

$$V\left(\psi,\phi\right) = \kappa^2\left(M^2 - \frac{\psi^2}{4}\right)^2 + \frac{\lambda^2\psi^2\phi^2}{4} + \frac{m^2\phi^2}{2}, \tag{6.1}$$

where ϕ is the inflaton field, ψ represents the waterfall field and λ is a coupling constant, and m and M are mass parameters. The effective mass squared of the field ψ is $m_\psi^2 = -\kappa^2M^2 + \lambda^2\phi^2/2$ at $\psi = 0$. Inflation persists until $\phi > \phi_c = \frac{\sqrt{2}\kappa M}{\lambda}$ where ψ field has its stable value i.e. $\psi = 0$. And when $\phi < \phi_c$, phase transition occurs with the symmetry breaking [19][20]. The equation(6.1) will take the form at $\psi = 0$ as,

$$V(\phi) = V_0 + \frac{m^2\phi^2}{2} = V_o[1 + \tilde{\phi}^2], \tag{6.2}$$

where $\tilde{\phi} = \frac{\phi_o m}{\sqrt{2V_o}}$, and $V_0 = \kappa^2M^4$ is the constant energy of the vacuum and the second term provides a slope in the flat potential due to which inflaton can rolls down to the valley at $\psi = 0$.

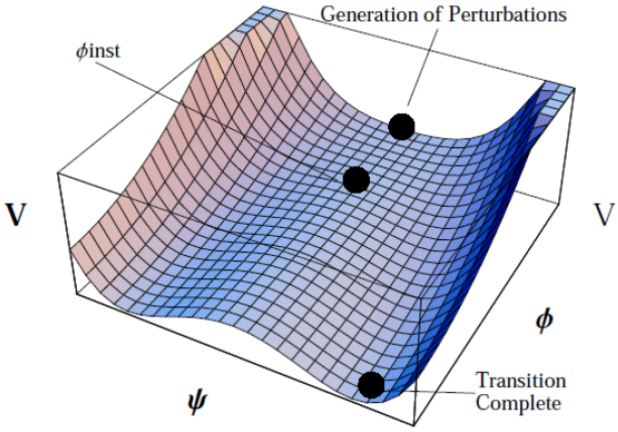

FIGURE 6.1: Inflaton field in the Hybrid inflationary model.

For the tree level predictions of the hybrid potential, slow roll parameter are given as,

$$\epsilon = \frac{M_p^2}{2}\left(\frac{\partial_\phi V}{V}\right)^2 = \frac{\eta_o\tilde{\phi}^2}{(\tilde{\phi}^2+1)^2} \tag{6.3}$$

$$\eta = M_p^2\left(\frac{\partial_\phi^2 V}{V}\right) = \frac{\eta_o}{\tilde{\phi}^2+1} \tag{6.4}$$

where $\eta_o(\phi = 0) = m^2 M_{pl}^2/V_o$. The number of e-folds between horizon crossing and at the end of inflation is given by

$$N = \frac{1}{M_{pl}^2}\int_{\phi_c}^{\phi_o}\left(\frac{V}{\partial_\phi V}\right) = \frac{1}{2\eta_o}\left[ln\left(\frac{\tilde{\phi_0}^2}{\tilde{\phi}_c^2}\right) + \left(\tilde{\phi}_0^2 - \tilde{\phi}_c^2\right)\right] \tag{6.5}$$

where $\phi_c = \sqrt{\frac{2\kappa}{\lambda^2}}V_0^{1/4}\left(\text{or} \quad \tilde{\phi_c} = \sqrt{\frac{\eta_o}{2}}(\phi_c/M_{pl})\right)$ with $\kappa \sim \lambda \sim 10^{-3}$ at the end of the inflation. We can eliminate η_o by using expression of amplitude of curvature perturbation

$$\triangle_\zeta = \frac{1}{\sqrt{6\eta_o}\pi M_{pl}^2}\frac{V^{3/2}}{\partial_{\tilde{\phi}}V}\bigg|_{\tilde{\phi}=\tilde{\phi_o}} \Rightarrow \eta_o = \frac{(1+\tilde{\phi}_o^2)^3}{24\pi^2\,\triangle_\zeta^2\,\tilde{\phi}_o^2}\frac{V_o}{M_p^4} \tag{6.6}$$

where $\triangle_\zeta(k_0) = 4.91\times10^{-5}$ observed by PLANCK satellite. The leading order expression for scalar spectral index n_s and tenor-scalar ratio r can be written as

$$n_s \simeq 1 - 6\epsilon + 2\eta = 1 - 4\eta_o\frac{(\tilde{\phi}_o^2 - 1/2)}{(1+\tilde{\phi}_o^2)^2} = 1 - 4\frac{(\tilde{\phi}_o^2-1/2)(1+\tilde{\phi}_o^2)}{24\pi^2\,\triangle_\zeta^2\,\tilde{\phi}_o^2}\frac{V_o}{M_p^4} \tag{6.7}$$

and

$$r \simeq 16\epsilon = \frac{16\eta_o\tilde{\phi}_o^2}{(1+\tilde{\phi}_o^2)} = 16(1+\tilde{\phi}_o^2)\frac{V_o}{24\pi^2\,\triangle_\zeta^2\,M_p^4} \tag{6.8}$$

Using above equations numerical computing has been performed for 60 e-folds on MATHEMATICA, we get

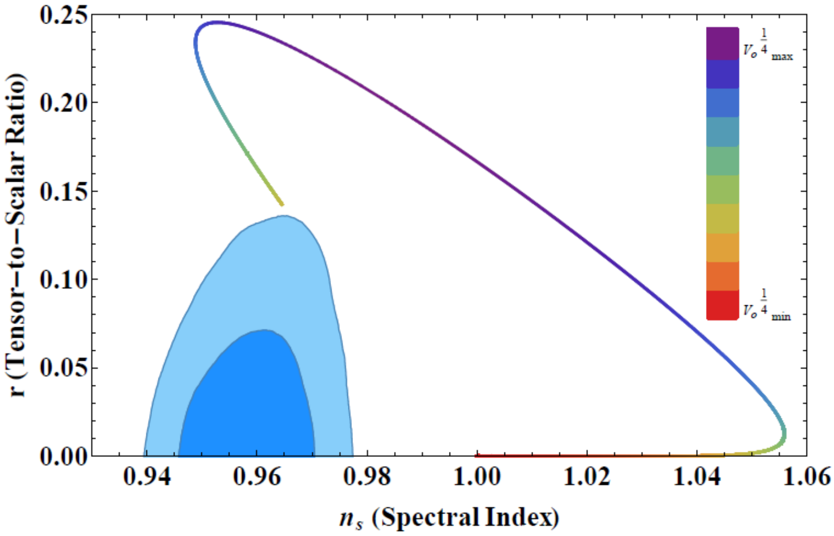

FIGURE 6.2: Tree level prediction for Hybrid inflation, r vs n_s

It shows that TLHI potential predictions lies outside the PLANCK+WP+highL contours of 65% and 95% CL. Predicting very high tensor to scalar ratio and $n_s > 1$ i.e. blue titled spectral index for trans-Planckian field and red tilted for sub-Plankian field.

6.1.1 Tree Level Approximations

Tree level prediction can be divided in different cases as follows:

- $\tilde{\phi}_o \gg 1$(Quadratic limit corresponds to red tilted spectral index, $\eta \gg \epsilon$)

$$n_s = 1 - 4\tilde{\phi}_o^2 \frac{V_o}{24\pi^2 \triangle_{\mathcal{R}}^2 M_p^4} = 1 - \frac{2\phi_o^2 m^2}{24\pi^2 \triangle_{\mathcal{R}}^2 M_p^4} \tag{6.9}$$

$$r = 16\tilde{\phi}_o^2 \frac{V_o}{24\pi^2 \triangle_{\mathcal{R}}^2 M_p^4} = \frac{8\phi_o^2 m^2}{24\pi^2 \triangle_{\mathcal{R}}^2 M_p^4} \tag{6.10}$$

- $\tilde{\phi}_o \sim 1$

$$n_s = 1 - 4\frac{V_o}{24\pi^2 \triangle_{\mathcal{R}}^2 M_p^4} = 1 - 4\frac{\kappa^2 M^4}{24\pi^2 \triangle_{\mathcal{R}}^2 M_p^4} \tag{6.11}$$

$$r = 32\frac{V_o}{24\pi^2 \triangle_{\mathcal{R}}^2 M_p^4} = 32\frac{\kappa^2 M^4}{24\pi^2 \triangle_{\mathcal{R}}^2 M_p^4} \tag{6.12}$$

- $\tilde{\phi}_o \ll 1$(Flat potential corresponds to blue tilted spectral index, $\eta \backsim \epsilon$)

$$n_s \quad = 1 + \frac{2}{\tilde{\phi}_o^2} \frac{V_o}{24\pi^2 \triangle_{\mathcal{R}}^2 M_p^4} \quad = 1 + \frac{4\kappa^4 M^8}{\phi_o^2 m^2} \frac{1}{24\pi^2 \triangle_{\mathcal{R}}^2 M_p^4} \tag{6.13}$$

$$r \quad = 16 \frac{V_o}{24\pi^2 \triangle_{\mathcal{R}}^2 M_p^4} \quad = \frac{16\kappa^2 M^4}{24\pi^2 \triangle_{\mathcal{R}}^2 M_p^4} \tag{6.14}$$

6.2 Radiatively-Corrected Hybrid Inflation

Tree level results can be improved by adding one loop radiative corrections in equation(6.1). Consider the Lagrangian density [21]

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}\partial^\mu \phi_B \partial_\mu \phi_B + \frac{1}{2}\partial^\mu \psi_B \partial_\mu \psi_B + \frac{i}{2}\bar{N}\gamma^\mu \partial_\mu N - \kappa^2 \left(M^2 - \frac{\psi_B^2}{4}\right)^2 - \frac{m_B^2 \phi_B^2}{2} \\ & - \frac{\lambda_B^2 \psi_B^2 \phi_B^2}{4} - \frac{1}{2} y_B \phi_B \bar{N} N - \frac{1}{2} Y_B \psi_B \bar{N} N - \frac{1}{2} m_N \bar{N} N \end{aligned} \tag{6.15}$$

where $'B'$ in the subscript represents the bare quantities and a single Yukawa coupling exists, represented by N,and each of ϕ and ψ. And N is the standard model field for singlet fermion. In the inflationary era $\psi = 0$ i.e. there is no effect of interaction between ψ and N. Contribution of N becomes significant after the inflation and oscillations of ψ gives rise to process of reheating.
Adding one loop corrections to equation(6.1) yields,

$$V(\psi, \phi) = \kappa^2 \left(M^2 - \frac{\psi^2}{4}\right)^2 + \frac{\lambda^2 \psi^2 \phi^2}{4} + \frac{m^2 \phi^2}{2} + V_{loop}, \tag{6.16}$$

where V_{loop} represents the one loop corrected term added to the tree level hybrid potential. During the inflationary era i.e. at $\psi = 0$,

$$\begin{aligned}V_{loop} =& \frac{1}{64\pi^2}\left[m^4 ln\left(\frac{m^2}{\mu^2}\right) + \frac{\lambda^4}{4}\left(\phi^2 - \phi_c^2\right)^2 ln\left(\frac{\frac{\lambda^2}{2}\left(\phi^2 - \phi_c^2\right)}{\mu^2}\right)\right. \\ & \left. - 2\left(m_N + y\phi\right)^4 ln\left(\frac{m_N + y\phi}{\mu}\right)^2\right]\end{aligned} \tag{6.17}$$

During inflation, $\phi > \phi_c$, therefore for $y\phi_c \gg (m_N, m)$ and $y \gtrsim \frac{\lambda}{\sqrt{2}}$, then the potential in the equation(6.8) reduces to

$$V_{loop} = -A\,\phi^4\, ln\left(\frac{y\phi}{\mu}\right), \quad \text{with} \quad A = \frac{y^4}{16\pi^2} \tag{6.18}$$

Setting convenient renormalization scale as $\mu = y\phi_c$ so that log factor remain positive during inflationary epoch. Final form of equation(6.7) at $\psi = 0$ becomes,

$$V(\phi) = V_0 + \frac{m^2\phi^2}{2} - \phi^4 A\, ln\left(\frac{\phi}{\phi_c}\right) \tag{6.19}$$

Radiatively corrected hybrid inflationary model has two solutions namely, hilltop and non-hilltop types. If the potential have concave down curvature at the beginning of the inflation, then is known as hilltop solutions while non-hilltop solutions has concave up curvature of potential at the start of the inflation. Initially numerical computation has been performed to calculate the solutions, while keeping the A fixed for each curve and scan over V_0 for $N = 60$. The predictions of potential given in the equation(6.10) are follows:

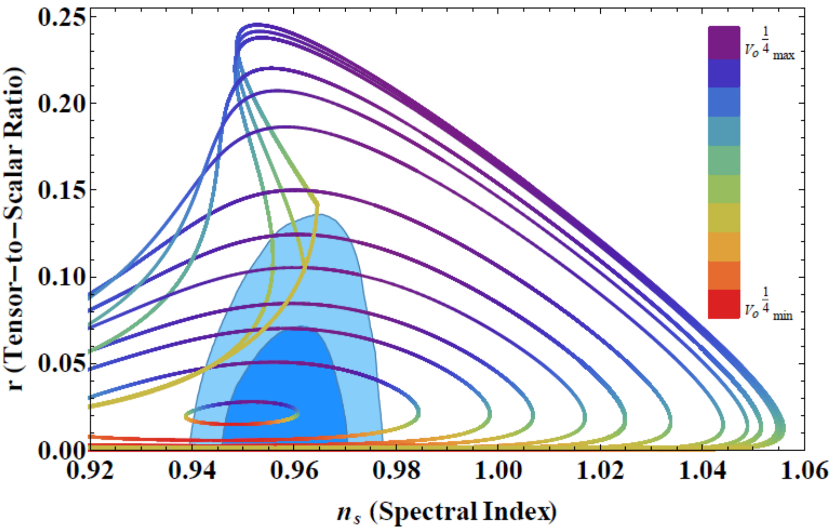

FIGURE 6.3: Prediction for radiatively corrected hybrid inflation, scan over $V_0^{1/4}$ for fixed A on each curve. r vs n_s

In Fig(6.3) each contour has fix value of A and $V_0^{1/4}$ varies from maximum(violet color) to minimum(red color). Here the outermost curve corresponds to smallest value of the $A = 10^{-16}$, which resembles the tree level predictions of $r - n_s$. As the A increases tensor-scalar ratio and spectral index both becomes smaller, comes inside the PLANCK+WP+highL contours of 65% and 95% CL.

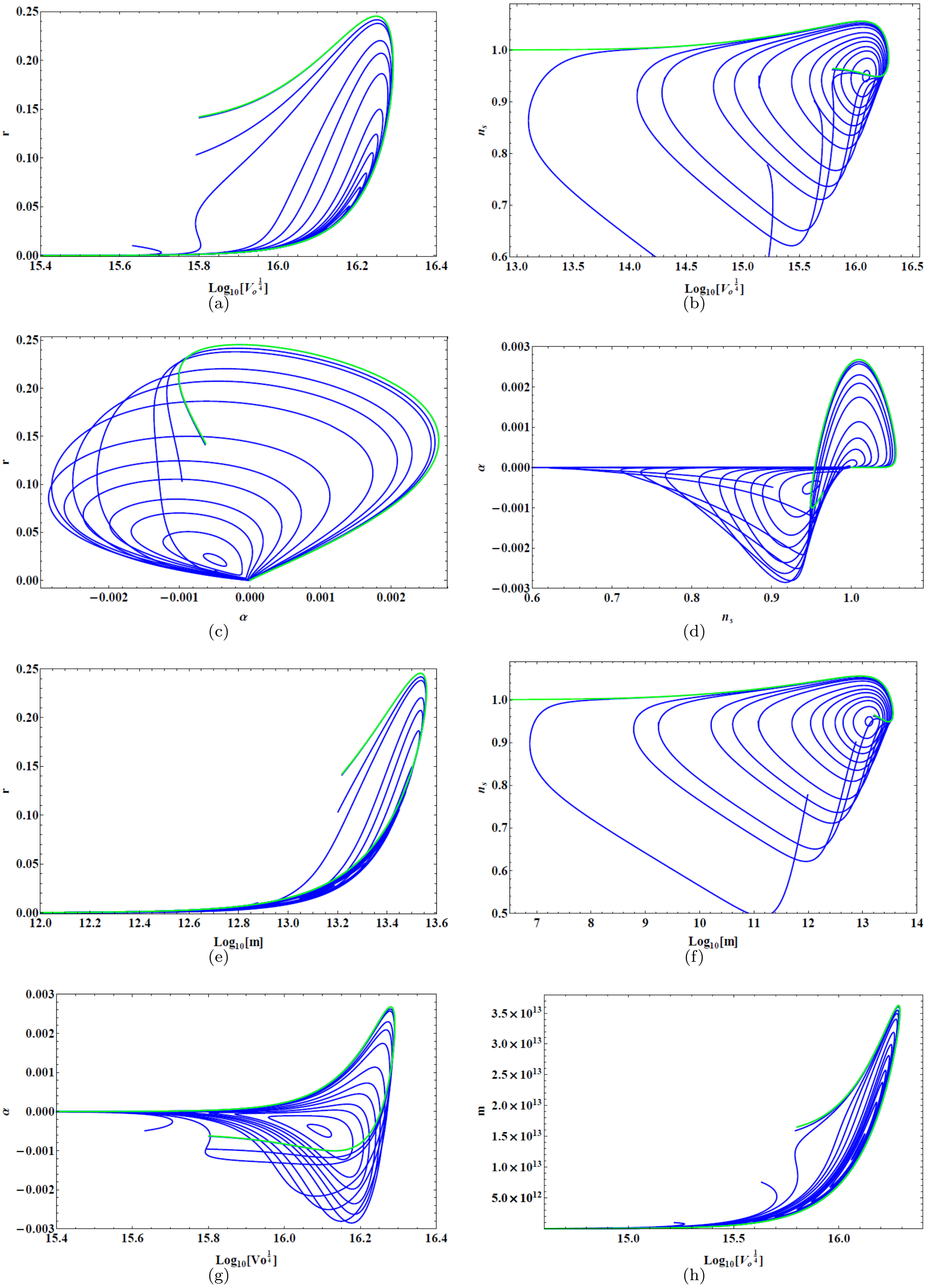

FIGURE 6.4: From fig(a)-(h) represents the predictions of the RCHI model keeping the A fixed for each curve.

Now another type of solutions can be obtained by fixing the V_0 for each curve and scanning over A for $N = 60$, we get

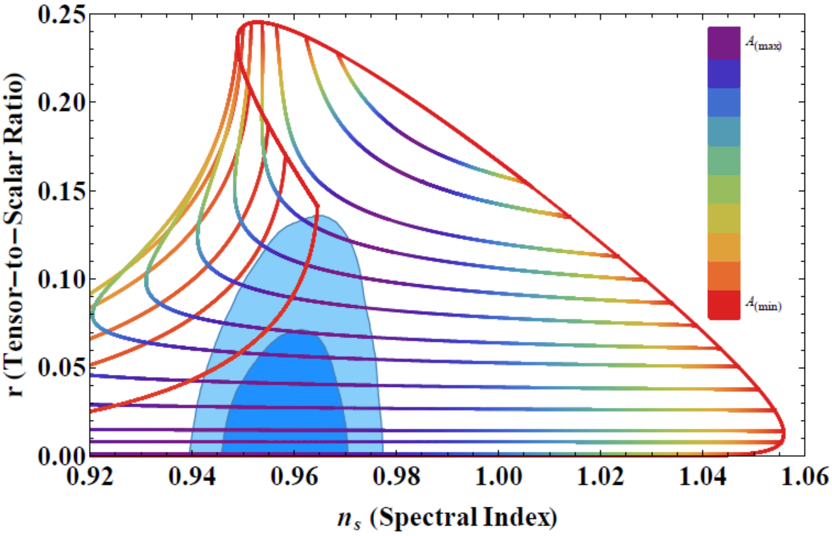

FIGURE 6.5: Prediction for radiatively corrected hybrid inflation, scan over A for fixed $V_0^{1/4}$ on each curve. r vs n_s

In Fig(6.5) each curve has fixed $V_0^{1/4}$. At the maximum value of $V_0^{1/4} = 10^{16}$ prediction matches the tree level but as the $V_0^{1/4}$ decreases, $r - n_s$ predictions passes through the PLANCK+WP+highL contours of 65% and 95% CL.

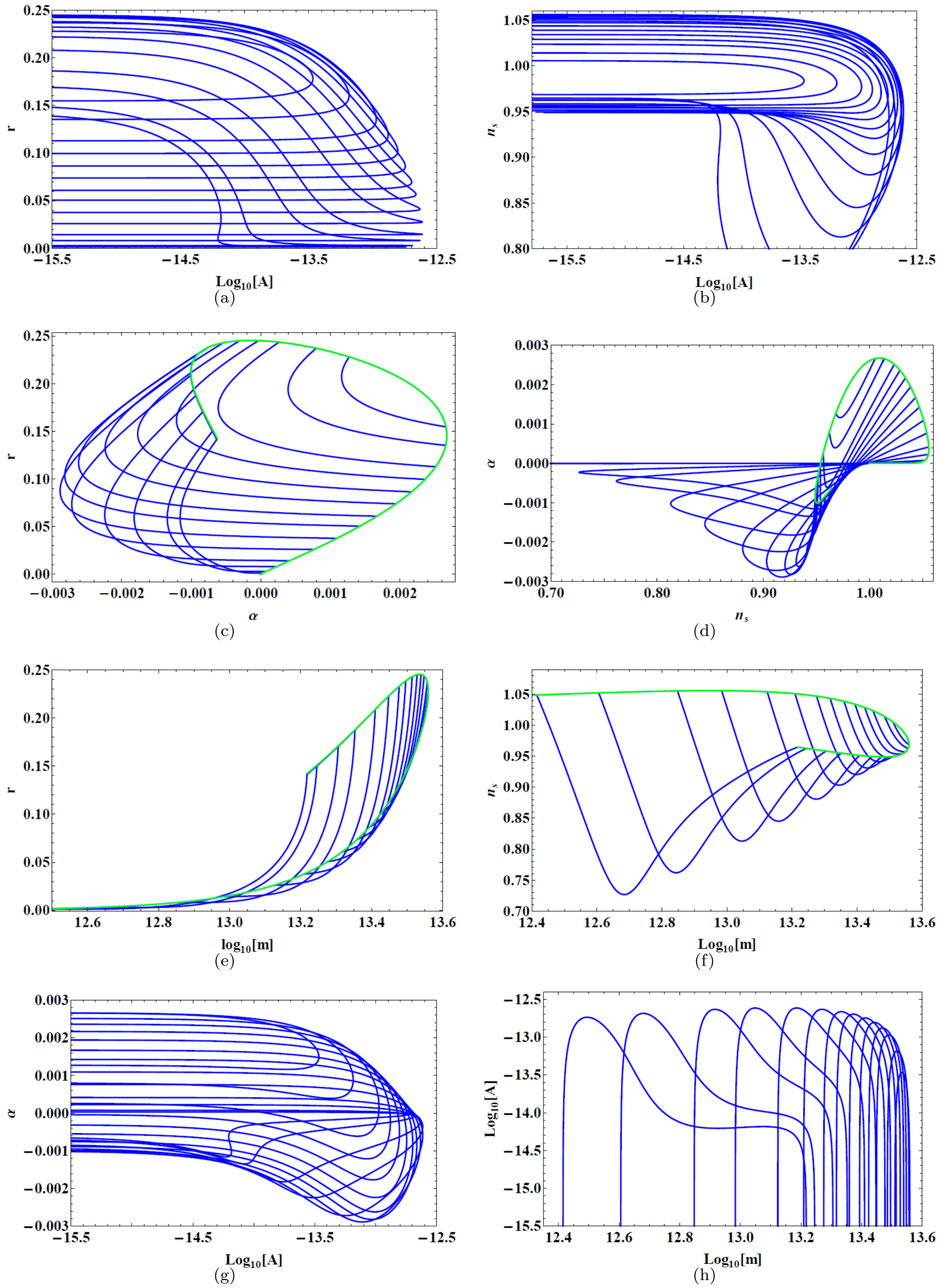

FIGURE 6.6: From fig(a)-(h) represents the predictions of the RCHI model keeping the $V_0^{1/4}$ fixed for each curve.

Variation of scalar spectral index and tensor-to-scalar ratio with respect to inflaton field is important to know whether the sub-Planckian or trans-Planckian predictions has significant role, as shown below

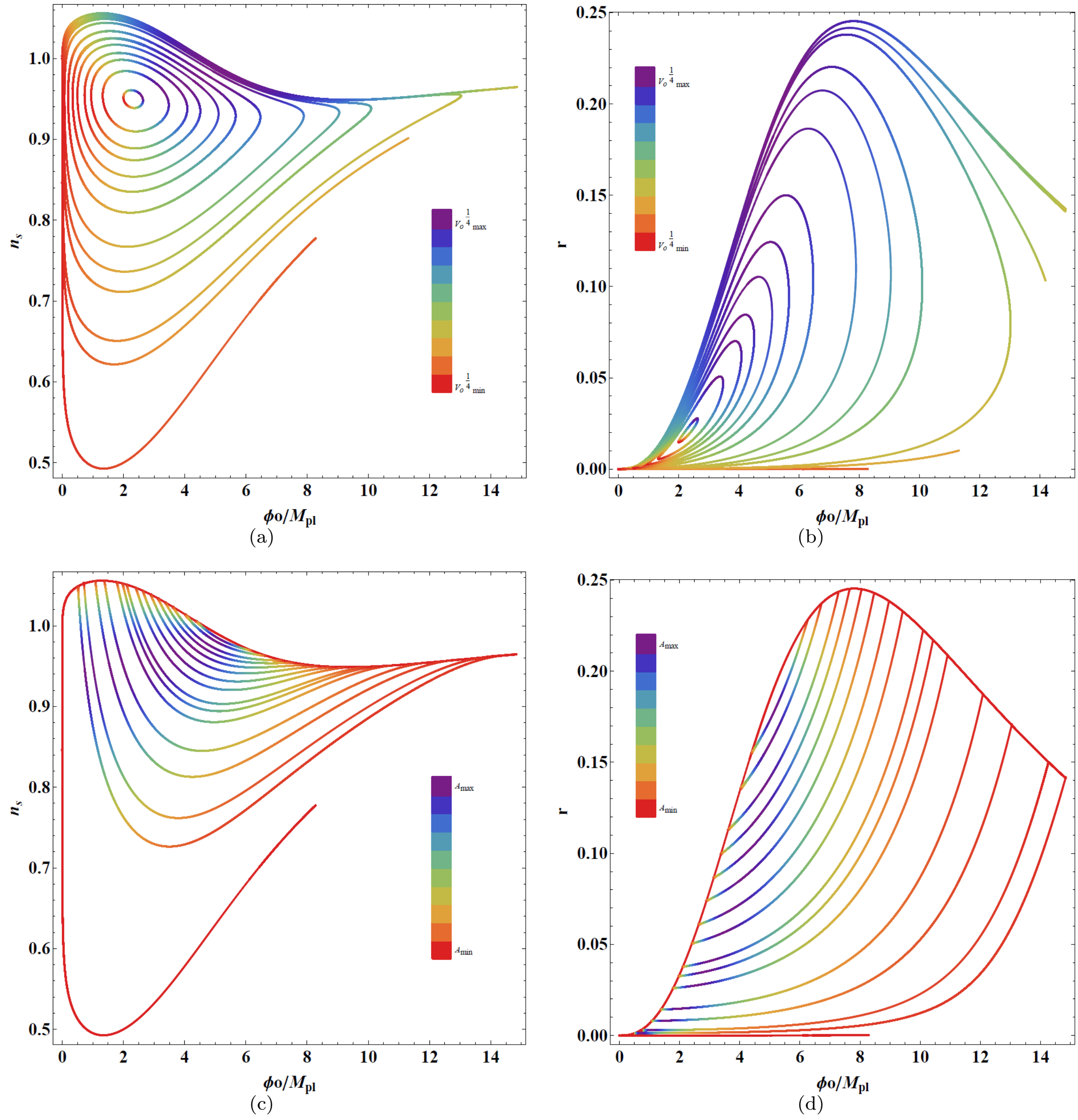

FIGURE 6.7: (a)-(d) represents prdictions for r and n_s vs ϕ_0/M_{pl}. Top: Scan over $V_0^{1/4}$, fixing A for each curve. Bottom: Oppsite of upper case.

It is more interesting to compare the predictions of r vs n_s at sub-Planckian and trans-Planckian inflation. Here we have taken three values as the pivoting point of inflation: $\phi_P = 0.25$,$\phi_P = 1$ and $\phi_P = 2.5$. For each case ϕ_0 is kept fixed and scan over V_0 and A for $N = 60$. Here $\phi_P = \frac{\phi_0}{M_{pl}}$.

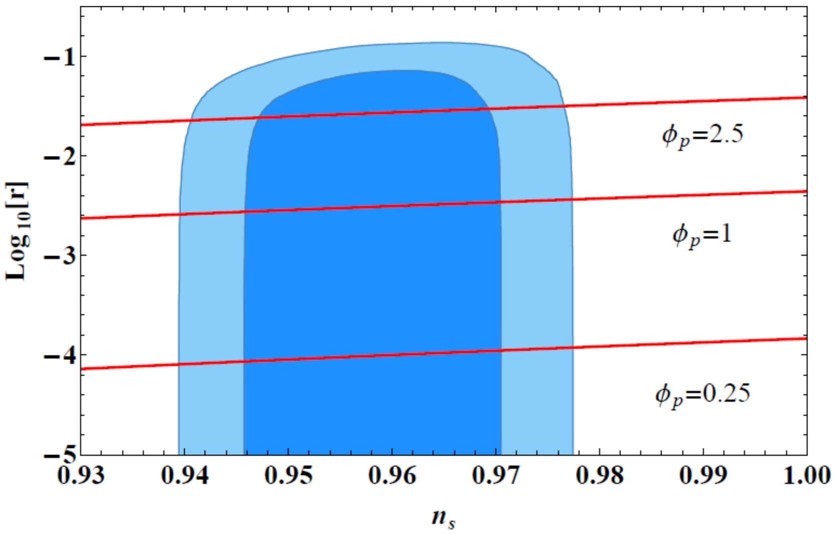

FIGURE 6.8: $Log_{10}r$ vs n_s for the cases ϕ_P =0.25,1 and 2.5 on PLACNK+WP+highL contours.

Numerical results for the RCHI potential are compared with 1σ and 2σ bounds of PLANCK+WP+highL. On each curve as the A increases, V_0 decreases and n_s starts to decrease from unity. As A increases more, V_0 decreases further and each curve becomes consistent with PLANCK data. Predictions for r varies for each case. And for the trans-Planckian case $\phi_P = 2.5$, r has appreciable value, while for other two cases r is very small to detect even by PLANCK mission, also shown in fig(6.5). But it is hope that PLANCK mission will able to detect tran-Planckian values of scalar field with greater precision.

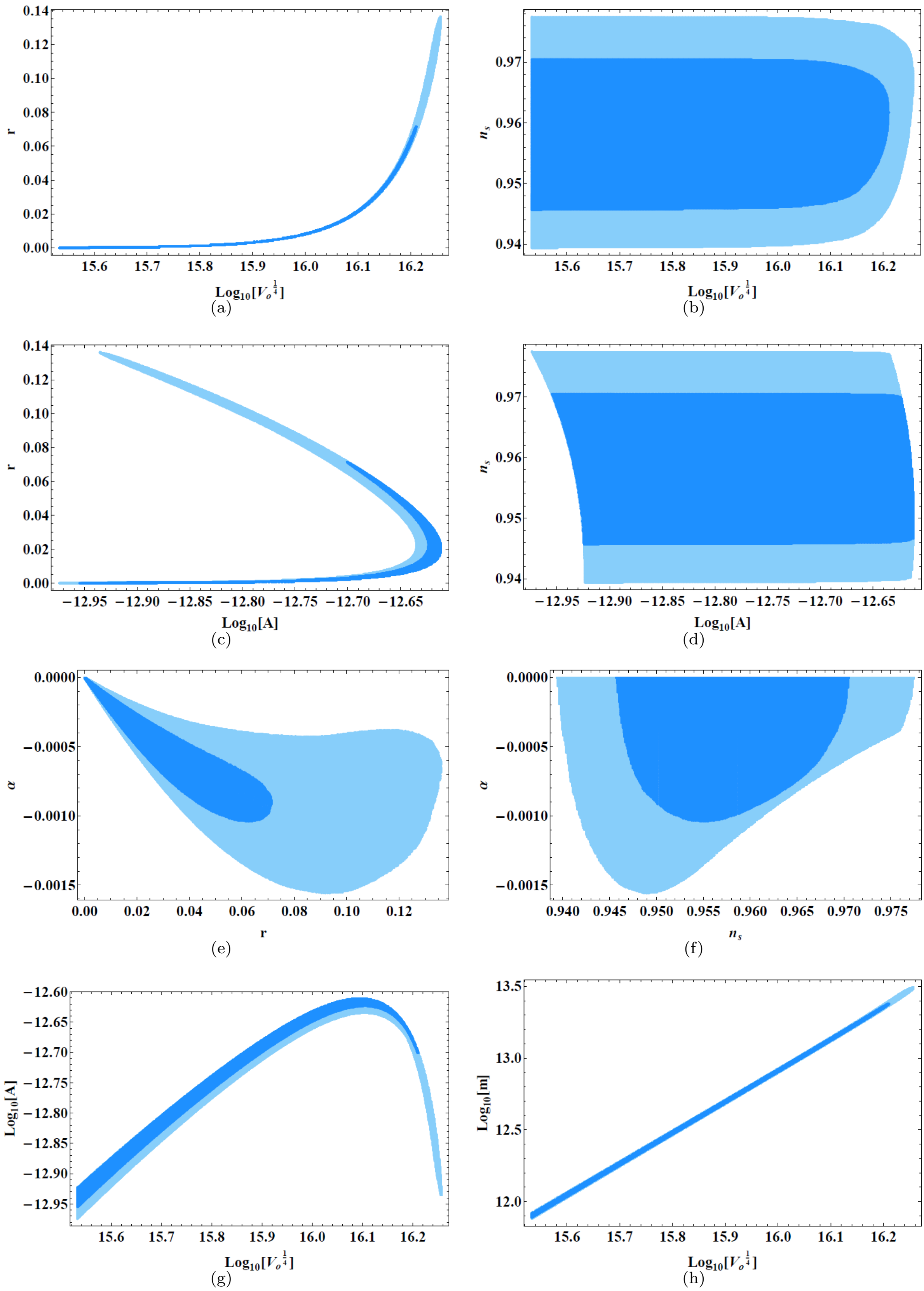

FIGURE 6.9: Range of inflationary parameters lies with in 1σ and 2σ bounds of PLANCK+WP +highL.

6.3 Comparison with BICEP2 results

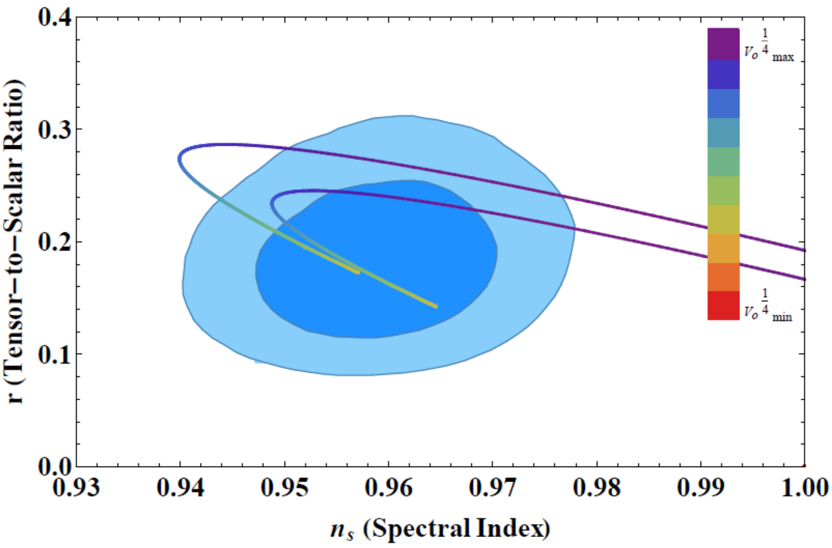

FIGURE 6.10: Tree level prediction for Hybrid inflation at 50(upper) and 60(lower) e-folds, r vs n_s

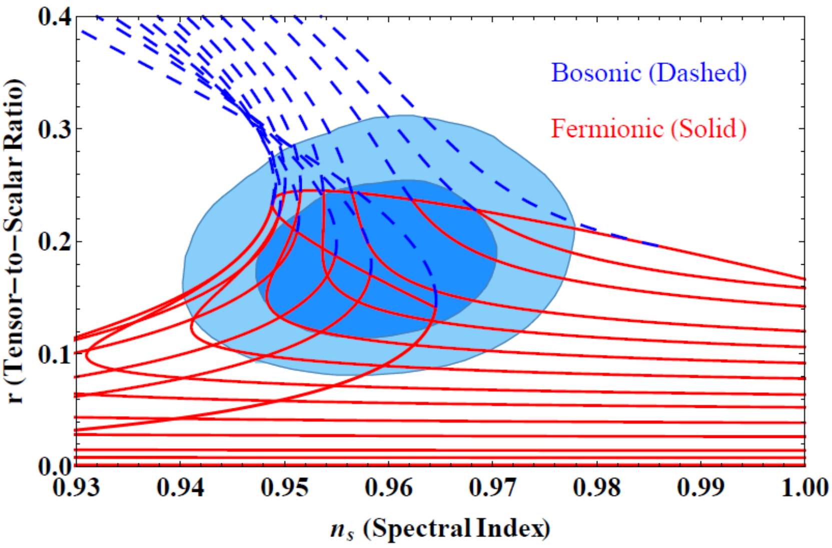

FIGURE 6.11: Radiatively Corrected prediction for Hybrid inflation for Bosonic and Fermionic contributions, r vs n_s

6.4 Discussion

Results of tree level and one loop corrected hybrid inflation shows that, the predictions of tree level hybrid inflation are inconsistent with recent PLANCK data. And in contrast one loop quantum correction in non-susy hybrid model has brought its predictions with in PLANCK+WP+highL bounds of 65% and 95% confidence level. Quantum corrections were originated by Yukawa coupling between right handed neutrinos and inflaton and made it possible to access both hilltop and non-hilltop solutions in RCHI. It is observed, spectral index showed a red tilt for sub-Planckian infalton field, which was blue tilted at tree level. But tensor to scalar ratio r has very small value for sub-Planckian field, implies it is highly unfavourable to detect primordial gravity waves in near future. But BICEP2 team has claimed about the detection of tensor-to-scalar r at about 0.2 and will get confirmed by Planck's polarization data in later part of the year 2014.

Chapter 7

Conclusion

Inflation provides an efficient mechanism which explains the early Universe in an elegant fashion and serves as an initial seeds for the quantum fluctuations and generate nearly scale invariant spectrum, which have been studied with great precision, consistent with CMB data observed by the PLANCK and previous mission like WMAP. The most appealing prediction of this framework is, the current structure of the universe is due to the amplification of tensor fluctuations generated at the very early times and that can be confirmed if in the near future PLANCK detects gravitational waves.

There are large number of inflationary models based on varieties of motivations, but the best model would be the one which can make link between particle physics and cosmological effects. In this thesis, predictions of two types of inflationary models have been analyzed, namely large field (chaotic polynomial models) and hybrid model together with one loop quantum corrections. The future of the inflationary paradigm depends greatly on the detection of primordial gravity waves. PLANCK mission is expected to detect gravity waves only if tensor-to-scalar ratio $r \geq 0.01$ and some next generation CMB experiments (gravitational waves interferometers) might able to detect B-modes, i.e. beyond PLANCK, if $r \geq 10^{-3} - 10^{-4}$.

If primordial gravitational waves is failed to be detected then in particle physics view, the only way to imply inflation will be via small field models and can be explained by standard quantum field theory, with small radiative corrections. Then the energy scale of the inflation could never be measured and implementation of inflation in high energy particle physics models will remain an vague conjectures.

Tensor-to-Scalar Ratio has better chance to detected in large field models and tensor tilt n_t can be measured with some precision. By this a comparison between r and n_t can be made using $r = -8n_t$, which will be a direct proof of inflation. But this case has problem of large radiative corrections and non-renormalizable terms, and doesn't support supersymmetric theories, which requires new version of particle physics theories.

If r lies between 10^{-4} and 10^{-2}, primordial gravitational waves still can be detected, but tensor tilt will not easy to be measured. Inflationary Models which predicts such a range of r like RCHI model at sub-Planckian inflaton field can serve as a good particle physics motivated model.

Beside the success of inflationary paradigm, still there are some unresolved problems in it: What is the origin of scalar field(inflaton)? Who ordered the inflation? What was before inflationary era? Whether inflaton can ever be detected like Higgs field or can it be part of known particles of high energy physics?

Recent results of PLANCK+WMAP has disfavoured many highly motivated multi-field models and discovery of the Higgs with $m_H \simeq 125 GeV$ at LHC has raised some serious issues against inflationary idea and standard cosmology. These issues has opened the doors for alternatives to inflation, like cyclic model, is a famous one. Future experiments for new particles at LHC, B-modes searches and forthcoming PLANCK results will be decisive.

Recent claim of detection of primordial gravitational waves by BICEP2 experiments has given new life to the large field models like quadratic and quartic models which were in danger of exclusion, and has opened door for new physics at GUT scale. It will get confirmed after the release of polarization data of PLANCK satellite later this year.

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