Math 475 **Text: Brualdi, Introductory Combinatorics 5th Ed.**
Prof: Paul Terwilliger
Selected solutions for Chapter 5

1. For an integer k and a real number n, we show

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

First assume $k \le -1$. Then each side equals 0. Next assume $k = 0$. Then each side equals 1. Next assume $k \ge 1$. Recall

$$P(n,k) = n(n-1)(n-2)\cdots(n-k+1).$$

We have

$$\binom{n}{k} = \frac{P(n,k)}{k!} = \frac{nP(n-1,k-1)}{k!}.$$

$$\binom{n-1}{k-1} = \frac{P(n-1,k-1)}{(k-1)!} = \frac{kP(n-1,k-1)}{k!}.$$

$$\binom{n-1}{k} = \frac{P(n-1,k)}{k!} = \frac{(n-k)P(n-1,k-1)}{k!}.$$

The result follows.

2. Pascal's triangle begins

$$\begin{array}{ccccccccccccccccccccc}
&&&&&&&&&&1&&&&&&&&&&\\
&&&&&&&&&1&&1&&&&&&&&&\\
&&&&&&&&1&&2&&1&&&&&&&&\\
&&&&&&&1&&3&&3&&1&&&&&&&\\
&&&&&&1&&4&&6&&4&&1&&&&&&\\
&&&&&1&&5&&10&&10&&5&&1&&&&&\\
&&&&1&&6&&15&&20&&15&&6&&1&&&&\\
&&&1&&7&&21&&35&&35&&21&&7&&1&&&\\
&&1&&8&&28&&56&&70&&56&&28&&8&&1&&\\
&1&&9&&36&&84&&126&&126&&84&&36&&9&&1&\\
1&&10&&45&&120&&210&&252&&210&&120&&45&&10&&1\\
&&&&&&&&&\cdot&\cdot&\cdot&&&&&&&&&
\end{array}$$

3. Let $\mathbb{Z}$ denote the set of integers. For nonnegative $n \in \mathbb{Z}$ define $F(n) = \sum_{k\in\mathbb{Z}} \binom{n-k}{k}$. The sum is well defined since finitely many summands are nonzero. We have $F(0) = 1$ and $F(1) = 1$. We show $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$. Let n be given. Using Pascal's formula and a change of variables $k = h + 1$,

$$\begin{aligned} F(n) &= \sum_{k\in\mathbb{Z}} \binom{n-k}{k} \\ &= \sum_{k\in\mathbb{Z}} \binom{n-k-1}{k} + \sum_{k\in\mathbb{Z}} \binom{n-k-1}{k-1} \\ &= \sum_{k\in\mathbb{Z}} \binom{n-k-1}{k} + \sum_{h\in\mathbb{Z}} \binom{n-h-2}{h} \\ &= F(n-1) + F(n-2). \end{aligned}$$

Thus $F(n)$ is the nth Fibonacci number.

4. We have

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

and

$$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6.$$

5. We have

$$(2x-y)^7 = \sum_{k=0}^{7} \binom{7}{k} 2^{7-k}(-1)^k x^{7-k}y^k.$$

6. The coefficient of x^5y^{13} is $3^5(-2)^{13}\binom{18}{5}$. The coefficient of x^8y^9 is 0 since $8 + 9 \neq 18$.

7. Using the binomial theorem,

$$3^n = (1+2)^n = \sum_{k=0}^{n} \binom{n}{k} 2^k.$$

Similarly, for any real number r,

$$(1+r)^n = \sum_{k=0}^{n} \binom{n}{k} r^k.$$

8. Using the binomial theorem,

$$2^n = (3-1)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k}.$$

9. We have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}10^k = (-1)^n\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}10^k = (-1)^n(10-1)^n = (-1)^n 9^n.$$

The sum is 9^n for n even and -9^n for n odd.

10. Given integers $1 \le k \le n$ we show

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Let S denote the set of ordered pairs (x, y) such that x is a k-subset of $\{1, 2, \ldots, n\}$ and y is an element of x. We compute $|S|$ in two ways. (i) To obtain an element (x, y) of S there are $\binom{n}{k}$ choices for x, and for each x there are k choices for y. Therefore $|S| = k\binom{n}{k}$. (ii) To obtain an element (x, y) of S there are n choices for y, and for each y there are $\binom{n-1}{k-1}$ choices for x. Therefore $|S| = n\binom{n-1}{k-1}$. The result follows.

11. Given integers $n \ge 3$ and $1 \le k \le n$. We show

$$\binom{n}{k} - \binom{n-3}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1}.$$

Let S denote the set of k-subsets of $\{1, 2, \ldots, n\}$. Let S_1 consist of the elements in S that contain 1. Let S_2 consist of the elements in S that contain 2 but not 1. Let S_3 consist of the elements in S that contain 3 but not 1 or 2. Let S_4 consist of the elements in S that do not contain 1 or 2 or 3. Note that $\{S_i\}_{i=1}^4$ partition S so $|S| = \sum_{i=1}^4 |S_i|$. We have

$$|S| = \binom{n}{k}, \quad |S_1| = \binom{n-1}{k-1}, \quad |S_2| = \binom{n-2}{k-1}, \quad |S_3| = \binom{n-3}{k-1}, \quad |S_4| = \binom{n-3}{k}.$$

The result follows.

12. We evaluate the sum

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2.$$

First assume that $n = 2m+1$ is odd. Then for $0 \le k \le m$ the k-summand and the $(n-k)$-summand are opposite. Therefore the sum equals 0. Next assume that $n = 2m$ is even. To evaluate the sum in this case we compute in two ways the the coefficient of x^n in $(1-x^2)^n$. (i) By the binomial theorem this coefficient is $(-1)^m\binom{2m}{m}$. (ii) Observe $(1-x^2) = (1+x)(1-x)$. We have

$$(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k,$$
$$(1-x)^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k x^k.$$

By these comments the coefficient of x^n in $(1-x^2)^n$ is

$$\sum_{k=0}^{n} \binom{n}{n-k} (-1)^k \binom{n}{k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2.$$

Therefore

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 = (-1)^m \binom{2m}{m}.$$

13. We show that the given sum is equal to

$$\binom{n+3}{k}.$$

The above binomial coefficient is in row $n+3$ of Pascal's triangle. Using Pascal's formula, write the above binomial coefficient as a sum of two binomial coeffients in row $n+2$ of Pascal's triangle. Write each of these as a sum of two binomial coeffients in row $n+1$ of Pascal's triangle. Write each of these as a sum of two binomial coeffients in row n of Pascal's triangle. The resulting sum is

$$\binom{n}{k} + 3\binom{n}{k-1} + 3\binom{n}{k-2} + \binom{n}{k-3}.$$

14. Given a real number r and integer k such that $r \neq k$. We show

$$\binom{r}{k} = \frac{r}{r-k}\binom{r-1}{k}.$$

First assume that $k \leq -1$. Then each side is 0. Next assume that $k = 0$. Then each side is 1. Next assume that $k \geq 1$. Observe

$$\binom{r}{k} = \frac{P(r,k)}{k!} = \frac{rP(r-1,k-1)}{k!},$$

and

$$\binom{r-1}{k} = \frac{P(r-1,k)}{k!} = \frac{(r-k)P(r-1,k-1)}{k!}.$$

The result follows.

15. For a variable x consider

$$(1-x)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k.$$

Take the derivative with respect to x and obtain

$$-n(1-x)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k k x^{k-1}.$$

Now set $x = 1$ to get

$$0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k k.$$

The result follows.

16. For a variable x consider

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Integrate with respect to x and obtain

$$\frac{(1+x)^{n+1}}{n+1} = \sum_{k=0}^{n} \binom{n}{k} \frac{x^{k+1}}{k+1} + C$$

for a constant C. Set $x = 0$ to find $C = 1/(n+1)$. Thus

$$\frac{(1+x)^{n+1} - 1}{n+1} = \sum_{k=0}^{n} \binom{n}{k} \frac{x^{k+1}}{k+1}.$$

Now set $x = 1$ to get

$$\frac{2^{n+1} - 1}{n+1} = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1}.$$

17. Routine.

18. For a variable x consider

$$(x-1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} x^k.$$

Integrate with respect to x and obtain

$$\frac{(x-1)^{n+1}}{n+1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{x^{k+1}}{k+1} + C$$

for a constant C. Set $x = 0$ to find $C = (-1)^{n+1}/(n+1)$. Thus

$$\frac{(x-1)^{n+1} - (-1)^{n+1}}{n+1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{x^{k+1}}{k+1}.$$

Now set $x = 1$ to get

$$\frac{(-1)^n}{n+1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{1}{k+1}.$$

Therefore

$$\frac{1}{n+1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k} \frac{1}{k+1}.$$

19. One readily checks

$$2\binom{m}{2} + \binom{m}{1} = m(m-1) + m = m^2.$$

Therefore

$$\begin{aligned} \sum_{k=1}^{n} k^2 &= \sum_{k=0}^{n} k^2 \\ &= 2\sum_{k=0}^{n} \binom{k}{2} + \sum_{k=0}^{n} \binom{k}{1} \\ &= 2\binom{n+1}{3} + \binom{n+1}{2} \\ &= \frac{(n+1)n(2n+1)}{6}. \end{aligned}$$

20. One readily checks

$$m^3 = 6\binom{m}{3} + 6\binom{m}{2} + \binom{m}{1}.$$

Therefore

$$\begin{aligned} \sum_{k=1}^{n} k^3 &= \sum_{k=0}^{n} k^3 \\ &= 6\sum_{k=0}^{n} \binom{k}{3} + 6\sum_{k=0}^{n} \binom{k}{2} + \sum_{k=0}^{n} \binom{k}{1} \\ &= 6\binom{n+1}{4} + 6\binom{n+1}{3} + \binom{n+1}{2} \\ &= \frac{(n+1)^2 n^2}{4} \\ &= \binom{n+1}{2}^2. \end{aligned}$$

21. Given a real number r and an integer k. We show

$$\binom{-r}{k} = (-1)^k \binom{r+k-1}{k}.$$

First assume that $k < 0$. Then each side is zero. Next assume that $k \geq 0$. Observe

$$\begin{aligned} \binom{-r}{k} &= \frac{(-r)(-r-1)\cdots(-r-k+1)}{k!} \\ &= (-1)^k \frac{r(r+1)\cdots(r+k-1)}{k!} \\ &= (-1)^k \binom{r+k-1}{k}. \end{aligned}$$

22. Given a real number r and integers k, m. We show

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}.$$

First assume that $m < k$ or $k < 0$. Then each side is zero. Next assume that $0 \leq k \leq m$. Observe

$$\begin{aligned} \binom{r}{m}\binom{m}{k} &= \frac{r(r-1)\cdots(r-m+1)}{m!}\,\frac{m!}{k!(m-k)!} \\ &= \frac{r(r-1)\cdots(r-m+1)}{k!(m-k)!} \\ &= \frac{r(r-1)\cdots(r-k+1)}{k!}\,\frac{(r-k)(r-k-1)\cdots(r-m+1)}{(m-k)!} \\ &= \binom{r}{k}\binom{r-k}{m-k}. \end{aligned}$$

23. (a) $\binom{24}{10}$.

(b) $\binom{9}{4}\binom{15}{6}$.

(c) $\binom{9}{4}\binom{9}{3}\binom{6}{3}$.

(d) $\binom{9}{4}\binom{15}{6} - \binom{9}{4}\binom{9}{3}\binom{6}{3}$.

24. The number of walks of length 45 is equal to the number of words of length 45 involving 10 x's, 15 y's, and 20 z's. This number is

$$\frac{45!}{10! \times 15! \times 20!}.$$

25. Given integers $m_1, m_2, n \geq 0$. Show

$$\sum_{k=0}^{n} \binom{m_1}{k}\binom{m_2}{n-k} = \binom{m_1+m_2}{n}.$$

Let A denote a set with cardinality m_1+m_2. Partition A into subsets A_1, A_2 with cardinalities m_1 and m_2 respectively. Let S denote the set of n-subsets of A. We compute $|S|$ in two ways. (i) By construction

$$|S| = \binom{m_1+m_2}{n}.$$

(ii) For $0 \leq k \leq n$ let the set S_k consist of the elements in S whose intersection with A_1 has cardinality k. The sets $\{S_k\}_{k=0}^{n}$ partition S, so $|S| = \sum_{k=0}^{n} |S_k|$. For $0 \leq k \leq n$ we now compute $|S_k|$. To do this we construct an element $x \in S_k$ via the following 2-stage procedure:

stage	to do	# choices
1	pick $x \cap A_1$	$\binom{m_1}{k}$
2	pick $x \cap A_2$	$\binom{m_2}{n-k}$

The number $|S_k|$ is the product of the entries in the right-most column above, which comes to $\binom{m_1}{k}\binom{m_2}{n-k}$. By these comments

$$|S| = \sum_{k=0}^{n} \binom{m_1}{k}\binom{m_2}{n-k}.$$

The result follows.

26. For an integer $n \geq 1$ show

$$\sum_{k=1}^{n} \binom{n}{k}\binom{n}{k-1} = \frac{1}{2}\binom{2n+2}{n+1} - \binom{2n}{n}.$$

Using Problem 25,

$$\begin{aligned}
\sum_{k=1}^{n} \binom{n}{k}\binom{n}{k-1} &= \sum_{k=0}^{n} \binom{n}{k}\binom{n}{k-1} \\
&= \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n+1-k} \\
&= \binom{2n}{n+1} \\
&= \frac{1}{2}\binom{2n}{n-1} + \frac{1}{2}\binom{2n}{n+1}.
\end{aligned}$$

It remains to show

$$\frac{1}{2}\binom{2n}{n-1} + \frac{1}{2}\binom{2n}{n+1} = \frac{1}{2}\binom{2n+2}{n+1} - \binom{2n}{n}.$$

This holds since

$$\begin{aligned}\binom{2n}{n-1} + 2\binom{2n}{n} + \binom{2n}{n+1} &= \binom{2n+1}{n} + \binom{2n+1}{n+1} \\ &= \binom{2n+2}{n+1}.\end{aligned}$$

27. Given an integer $n \geq 1$. We show

$$n(n+1)2^{n-2} = \sum_{k=1}^{n} k^2 \binom{n}{k}.$$

Let S denote the set of 3-tuples (s, x, y) such that s is a nonempty subset of $\{1, 2, \ldots, n\}$ and x, y are elements (not necessarily distinct) in s. We compute $|S|$ in two ways. (i) Call an element (s, x, y) of S *degenerate* whenever $x = y$. Partition S into subsets S^+, S^- with S^+ (resp. S^-) consisting of the degenerate (resp. nondegenerate) elements of S. So $|S| = |S^+| + |S^-|$. We compute $|S^+|$. To obtain an element (s, x, x) of S^+ there are n choices for x, and given x there are 2^{n-1} choices for s. Therefore $|S^+| = n2^{n-1}$. We compute $|S^-|$. To obtain an element (s, x, y) of S^- there are n choices for x, and given x there are $n - 1$ choices for y, and given x, y there are 2^{n-2} choices for s. Therefore $|S^-| = n(n-1)2^{n-2}$. By these comments

$$|S| = n2^{n-1} + n(n-1)2^{n-2} = n(n+1)2^{n-2}.$$

(ii) For $1 \leq k \leq n$ let S_k denote the set of elements (s, x, y) in S such that $|s| = k$. The sets $\{S_k\}_{k=1}^n$ give a partition of S, so $|S| = \sum_{k=1}^n |S_k|$. For $1 \leq k \leq n$ we compute $|S_k|$. To obtain an element (s, x, y) of S_k there are $\binom{n}{k}$ choices for s, and given s there are k^2 ways to choose the pair x, y. Therefore $|S_k| = k^2\binom{n}{k}$. By these comments

$$|S| = \sum_{k=1}^{n} k^2 \binom{n}{k}.$$

The result follows.

28. Given an integer $n \geq 1$. We show

$$\sum_{k=1}^{n} k \binom{n}{k}^2 = n\binom{2n-1}{n-1}.$$

Let S denote the set of ordered pairs (s, x) such that s is a subset of $\{\pm 1, \pm 2, \ldots, \pm n\}$ and x is a positive element of s. We compute $|S|$ in two ways. (i) To obtain an element (s, x) of S There are n choices for x, and given x there are $\binom{2n-1}{n-1}$ choices for s. Therefore

$$|S| = n\binom{2n-1}{n-1}.$$

(ii) For $1 \leq k \leq n$ let S_k denote the set of elements (s, x) in S such that s contains exactly k positive elements. The sets $\{S_k\}_{k=1}^n$ partition S, so $|S| = \sum_{k=1}^n |S_k|$. For $1 \leq k \leq n$ we compute $|S_k|$. To obtain an element (s, x) of S_k there are $\binom{n}{k}$ ways to pick the positive elements of s and $\binom{n}{n-k} = \binom{n}{k}$ ways to pick the negative elements of s. Given s there are k ways to pick x. Therefore $|S_k| = k\binom{n}{k}^2$. By these comments

$$|S| = \sum_{k=1}^{n} k \binom{n}{k}^2.$$

The result follows.

29. The given sum is equal to

$$\binom{m_2 + m_2 + m_3}{n}.$$

To see this, compute the coefficient of x^n in each side of

$$(1+x)^{m_1}(1+x)^{m_2}(1+x)^{m_3} = (1+x)^{m_1+m_2+m_3}.$$

In this computation use the binomial theorem.

30, 31, 32. We refer to the proof of Theorem 5.3.3 in the text. Let $\mathcal{A}$ denote an antichain such that

$$|\mathcal{A}| = \binom{n}{\lfloor n/2 \rfloor}.$$

For $0 \leq k \leq n$ let α_k denote the number of elements in $\mathcal{A}$ that have size k. So

$$\sum_{k=0}^{n} \alpha_k = |\mathcal{A}| = \binom{n}{\lfloor n/2 \rfloor}.$$

As shown in the proof of Theorem 5.3.3,

$$\sum_{k=0}^{n} \frac{\alpha_k}{\binom{n}{k}} \leq 1,$$

with equality if and only if each maximal chain contains an element of $\mathcal{A}$. By the above comments

$$\sum_{k=0}^{n} \alpha_k \frac{\binom{n}{\lfloor n/2 \rfloor} - \binom{n}{k}}{\binom{n}{k}} \leq 0,$$

with equality if and only if each maximal chain contains an element of $\mathcal{A}$. The above sum is nonpositive but each summand is nonnegative. Therefore each summand is zero and the sum is zero. Consequently (a) each maximal chain contains an element of $\mathcal{A}$; (b) for $0 \leq k \leq n$ either α_k is zero or its coefficient is zero. We now consider two cases.

Case: n is even. We show that for $0 \le k \le n$, $\alpha_k = 0$ if $k \ne n/2$. Observe that for $0 \le k \le n$, if $k \ne n/2$ then the coefficient of α_k is nonzero, so $\alpha_k = 0$.
Case: n is odd. We show that for $0 \le k \le n$, either $\alpha_k = 0$ if $k \ne (n-1)/2$ or $\alpha_k = 0$ if $k \ne (n+1)/2$. Observe that for $0 \le k \le n$, if $k \ne (n \pm 1)/2$ then the coefficient of α_k is nonzero, so $\alpha_k = 0$. We now show that $\alpha_k = 0$ for $k = (n-1)/2$ or $k = (n+1)/2$. To do this, we assume that $\alpha_k \ne 0$ for both $k = (n \pm 1)/2$ and get a contradiction. By assumption $\mathcal{A}$ contains an element x of size $(n+1)/2$ and an element y of size $(n-1)/2$. Define $s = |x \cap y|$. Choose x, y such that s is maximal. By construction $0 \le s \le (n-1)/2$. Suppose $s = (n-1)/2$. Then $y = x \cap y \subseteq x$, contradicting the fact that x, y are incomparable. So $s \le (n-3)/2$. Let y' denote a subset of x that contains $x \cap y$ and has size $(n-1)/2$. Let x' denote a subset of $y' \cup y$ that contains y' and has size $(n+1)/2$. By construction $|x' \cap y| = s+1$. Observe y' is not in $\mathcal{A}$ since x, y' are comparable. Also x' is not in $\mathcal{A}$ by the maximality of s. By construction x' covers y' so they are together contained in a maximal chain. This chain does not contain an element of $\mathcal{A}$, for a contradiction.

33. Define a poset $(X, \le)$ as follows. The set X consists of the subsets of $\{1, 2, \ldots, n\}$. For $x, y \in X$ define $x \le y$ whenever $x \subseteq y$. For $n = 3, 4, 5$ we display a symmetric chain decomposition of this poset. We use the inductive procedure from the text.
For $n = 3$,

$$\begin{aligned}
&\emptyset, 1, 12, 123 \\
&2, 23 \\
&3, 13.
\end{aligned}$$

For $n = 4$,

$$\begin{aligned}
&\emptyset, 1, 12, 123, 1234 \\
&4, 14, 124 \\
&2, 23, 234 \\
&24, \\
&3, 13, 134 \\
&34.
\end{aligned}$$

For $n = 5$,

$$\begin{aligned}
&\emptyset, 1, 12, 123, 1234, 12345 \\
&5, 15, 125, 1235 \\
&4, 14, 124, 1245 \\
&45, 145 \\
&2, 23, 234, 2345 \\
&25, 235 \\
&24, 245 \\
&3, 13, 134, 1345 \\
&35, 135 \\
&34, 345.
\end{aligned}$$

34. For $0 \leq k \leq \lfloor n/2 \rfloor$ there are exactly $\binom{n}{k} - \binom{n}{k-1}$ symmetric chains of length $n - 2k + 1$.

35. Let S denote the set of 10 jokes. Each night the talk show host picks a subset of S for his repertoire. It is required that these subsets form an antichain. By Corollary 5.3.2 each antichain has size at most $\binom{10}{5}$, which is equal to 252. Therefore the talk show host can continue for 252 nights.

36. Compute the coefficient of x^n in either side of

$$(1+x)^{m_1}(1+x)^{m_2} = (1+x)^{m_1+m_2},$$

In this computation use the binomial theorem.

37. In the multinomial theorem (Theorem 5.4.1) set $x_i = 1$ for $1 \leq i \leq t$.

38. $(x_1 + x_2 + x_3)^4$ is equal to

$$\begin{aligned} &x_1^4 + x_2^4 + x_3^4 + 4(x_1^3 x_2 + x_1^3 x_3 + x_1 x_2^3 + x_2^3 x_3 + x_1 x_3^3 + x_2 x_3^3) \\ &+ 6(x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2) + 12(x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2). \end{aligned}$$

39. The coefficient is

$$\frac{10!}{3! \times 1! \times 4! \times 0! \times 2!}$$

which comes to 12600.

40. The coefficient is

$$\frac{9!}{3! \times 3! \times 1! \times 2!} \times 1^3 \times (-1)^3 \times 2 \times (-2)^2$$

which comes to -40320.

41. One routinely obtains the multinomial theorem (Theorem 5.4.1) with $t = 3$.

42. Given an integer $t \geq 2$ and positive integers $n_1, n_2, \ldots, n_t$. Define $n = \sum_{i=1}^{t} n_i$. We show

$$\binom{n}{n_1\ n_2\ \cdots\ n_t} = \sum_{k=1}^{t} \binom{n-1}{n_1\ \cdots\ n_{k-1}\ n_k - 1\ n_{k+1}\ \cdots\ n_t}.$$

Consider the multiset

$$\{n_1 \cdot x_1, n_2 \cdot x_2, \ldots, n_t \cdot x_t\}.$$

Let P denote the set of permutations of this multiset. We compute $|P|$ in two ways.
(i) We saw earlier that

$$|P| = \frac{n!}{n_1! \times n_2! \times \cdots \times n_t!} = \binom{n}{n_1\ n_2\ \cdots\ n_t}.$$

(ii) For $1 \leq k \leq t$ let P_k denote the set of elements in P that have first coordinate x_k. The sets $\{P_k\}_{k=1}^t$ partition P, so $|P| = \sum_{k=1}^t |P_k|$. For $1 \leq k \leq t$ we compute $|P_k|$. Observe that $|P_k|$ is the number of permutations of the multiset

$$\{n_1 \cdot x_1, \ldots, n_{k-1} \cdot x_{k-1}, (n_k - 1) \cdot x_k, n_{k+1} \cdot x_{k+1}, \ldots, n_t \cdot x_t\}.$$

Therefore

$$|P_k| = \binom{n-1}{n_1 \ \cdots \ n_{k-1} \ n_k - 1 \ n_{k+1} \ \cdots \ n_t}.$$

By these comments

$$|P| = \sum_{k=1}^{t} \binom{n-1}{n_1 \ \cdots \ n_{k-1} \ n_k - 1 \ n_{k+1} \ \cdots \ n_t}.$$

The result follows.

43. Given an integer $n \geq 1$. Show by induction on n that

$$\frac{1}{(1-z)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} z^k, \qquad |z| < 1.$$

The base case $n = 1$ is assumed to hold. We show that the above identity holds with n replaced by $n+1$, provided that it holds for n. Thus we show

$$\frac{1}{(1-z)^{n+1}} = \sum_{\ell=0}^{\infty} \binom{n+\ell}{\ell} z^\ell, \qquad |z| < 1.$$

Observe

$$\begin{aligned}
\frac{1}{(1-z)^{n+1}} &= \frac{1}{(1-z)^n} \frac{1}{1-z} \\
&= \left(\sum_{k=0}^{\infty} \binom{n+k-1}{k} z^k\right)\left(\sum_{h=0}^{\infty} z^h\right) \\
&= \sum_{\ell=0}^{\infty} c_\ell z^\ell
\end{aligned}$$

where

$$\begin{aligned}
c_\ell &= \binom{n-1}{0} + \binom{n}{1} + \binom{n+1}{2} + \cdots + \binom{n+\ell-1}{\ell} \\
&= \binom{n+\ell}{\ell}.
\end{aligned}$$

The result follows.

44. (Problem statement contains typo) The given sum is equal to $(-3)^n$. Observe

$$\begin{aligned} (-3)^n &= (-1-1-1)^n \\ &= \sum_{n_1+n_2+n_3=n} \binom{n}{n_1\ n_2\ n_3} (-1)^{n_1+n_2+n_3} \\ &= \sum_{n_1+n_2+n_3=n} \binom{n}{n_1\ n_2\ n_3} (-1)^{n_1-n_2+n_3}. \end{aligned}$$

Also

$$\begin{aligned} 1 &= (1-1+1)^n \\ &= \sum_{n_1+n_2+n_3=n} \binom{n}{n_1\ n_2\ n_3} (-1)^{n_2}. \end{aligned}$$

45. (Problem statement contains typo) The given sum is equal to $(-4)^n$. Observe

$$\begin{aligned} (-4)^n &= (-1-1-1-1)^n \\ &= \sum_{n_1+n_2+n_3+n_4=n} \binom{n}{n_1\ n_2\ n_3\ n_4} (-1)^{n_1+n_2+n_3+n_4} \\ &= \sum_{n_1+n_2+n_3+n_4=n} \binom{n}{n_1\ n_2\ n_3\ n_4} (-1)^{n_1-n_2+n_3-n_4}. \end{aligned}$$

Also

$$\begin{aligned} 0 &= (1-1+1-1)^n \\ &= \sum_{n_1+n_2+n_3+n_4=n} \binom{n}{n_1\ n_2\ n_3\ n_4} (-1)^{n_2+n_4}. \end{aligned}$$

46. Observe

$$\begin{aligned} \sqrt{30} &= 5\sqrt{\frac{30}{25}} \\ &= 5(1+z)^{1/2} \qquad\qquad z = 1/5, \\ &= 5\sum_{k=0}^{\infty} \binom{1/2}{k} z^k. \end{aligned}$$

For $n = 0, 1, 2, \ldots$ the nth approximation to $\sqrt{30}$ is

$$a_n = 5\sum_{k=0}^{n} \binom{1/2}{k} 5^{-k}.$$

We have

n	a_n
0	5
1	5.5
2	5.475
3	5.4775
4	5.4771875
5	5.47723125
6	5.477224688
7	5.477225719
8	5.477225551
9	5.477225579

47. Observe

$$\begin{aligned} 10^{1/3} &= 2\left(\frac{10}{8}\right)^{1/3} \\ &= 2(1+z)^{1/3} \qquad\qquad z = 1/4, \\ &= 2\sum_{k=0}^{\infty}\binom{1/3}{k} z^k. \end{aligned}$$

For $n = 0, 1, 2, \ldots$ the nth approximation to $10^{1/3}$ is

$$a_n = 2\sum_{k=0}^{n}\binom{1/3}{k} 4^{-k}.$$

We have

n	a_n
0	2
1	2.166666667
2	2.152777778
3	2.154706790
4	2.154385288
5	2.154444230
6	2.154432769
7	2.154435089
8	2.154434605
9	2.154434708

48. We show that a poset with $mn+1$ elements has a chain of size $m+1$ or an antichain of size $n+1$. Our strategy is to assume the result is false, and get a contradiction. By assumption each chain has size at most m and each antichain has size at most n. Let r denote the size of the longest chain. So $r \leq m$. By Theorem 5.6.1 the elements of the poset can be partitioned into r antichains $\{A_i\}_{i=1}^r$. We have $|A_i| \leq n$ for $1 \leq i \leq r$. Therefore

$$mn+1 = \sum_{i=1}^{r} |A_i| \leq rn \leq mn,$$

for a contradiction. Therefore, the poset has a chain of size $m + 1$ or an antichain of size $n + 1$.

49. We are given a sequence of $mn + 1$ real numbers, denoted $\{a_i\}_{i=0}^{mn}$. Let X denote the set of ordered pairs $\{(i, a_i)|0 \leq i \leq mn\}$. Observe $|X| = mn + 1$. Define a partial order $\leq$ on X as follows: for distinct $x = (i, a_i)$ and $y = (j, a_j)$ in X, declare $x < y$ whenever $i < j$ and $a_i \leq a_j$. For the poset $(X, \leq)$ the chains correspond to the (weakly) increasing subsequences of $\{a_i\}_{i=0}^{mn}$, and the antichains correspond to the (strictly) decreasing subsequences of $\{a_i\}_{i=0}^{mn}$. By Problem 48, there exists a chain of size $m+1$ or an antichain of size $n+1$. Therefore the sequence $\{a_i\}_{i=0}^{mn}$ has a (weakly) increasing subsequence of size $m+1$ or a (strictly) decreasing subsequence of size $n + 1$.

50. (i) Here is a chain of size four: $1, 2, 4, 8$. Here is a partition of X into four antichains:

$$8, 12$$
$$4, 6, 9, 10$$
$$2, 3, 5, 7, 11$$
$$1$$

Therefore four is both the largest size of a chain, and the smallest number of antichains that partition X.

(ii) Here is an antichain of size six: $7, 8, 9, 10, 11, 12$. Here is a partition of X into six chains:

$$1, 2, 4, 8$$
$$3, 6, 12$$
$$9$$
$$5, 10$$
$$7$$
$$11$$

Therefore six is both the largest size of an antichain, and the smallest number of chains that partition X.

51. There exists a chain $x_1 < x_2 < \cdots < x_t$ of size $t \geq 2$ in the poset S such that $x_1 \not< x_t$ in the poset R. Indeed we could take $t = 2$ and let x_1, x_2 be elements of X such that $x_1 < x_2$ in S but $x_1 \not< x_2$ in R. Pick a chain as above with t maximal. Define $p = x_1$ and $q = x_t$. Then the pair (p, q) meets the requirements of the problem.

Math 475 **Text: Brualdi, Introductory Combinatorics 5th Ed.**
Prof: Paul Terwilliger
Selected solutions for Chapter 3

1. For $1 \leq k \leq 22$ we show that there exists a succession of consecutive days during which the grandmaster plays exactly k games. For $1 \leq i \leq 77$ let b_i denote the number of games played on day i. Consider the numbers $\{b_1 + b_2 + \cdots + b_i + k\}_{i=0}^{76} \cup \{b_1 + b_2 + \cdots + b_j\}_{j=1}^{77}$. There are 154 numbers in the list, all among $1, 2, \ldots, 153$. Therefore the numbers $\{b_1 + b_2 + \cdots + b_i + k\}_{i=0}^{76} \cup \{b_1 + b_2 + \cdots + b_j\}_{j=1}^{77}$. are not distinct. Therefore there exist integers i, j $(0 \leq i < j \leq 77)$ such that $b_{i+1} + \cdots + b_j = k$. During the days $i+1, \ldots, j$ the grandmaster plays exactly k games.

2. Let S denote a set of 100 integers chosen from $1, 2, \ldots, 200$ such that i does not divide j for all distinct $i, j \in S$. We show that $i \notin S$ for $1 \leq i \leq 15$. Certainly $1 \notin S$ since 1 divides every integer. By construction the odd parts of the elements in S are mutually distinct and at most 199. There are 100 numbers in the list $1, 3, 5, \ldots, 199$. Therefore each of $1, 3, 5, \ldots, 199$ is the odd part of an element of S. We have $3 \times 5 \times 13 = 195 \in S$. Therefore none of $3, 5, 13, 15$ are in S. We have $3^3 \times 7 = 189 \in S$. Therefore neither of $7, 9$ is in S. We have $11 \times 17 = 187 \in S$. Therefore $11 \notin S$. We have shown that none of $1, 3, 5, 7, 9, 11, 13, 15$ is in S. We show neither of $6, 14$ is in S. Recall $3^3 \times 7 = 189 \in S$. Therefore $3^2 \times 7 = 63 \notin S$. Therefore $2 \times 3^2 \times 7 = 126 \in S$. Therefore $2 \times 3 = 6 \notin S$ and $2 \times 7 = 14 \notin S$. We show $10 \notin S$. Recall $3 \times 5 \times 13 = 195 \in S$. Therefore $5 \times 13 = 65 \notin S$. Therefore $2 \times 5 \times 13 = 130 \in S$. Therefore $2 \times 5 = 10 \notin S$. We now show that none of $2, 4, 8, 12$ are in S. Below we list the integers of the form $2^r 3^s$ that are at most 200:

$$
\begin{array}{llllllll}
1, & 2, & 4, & 8, & 16, & 32, & 64, & 128, \\
3, & 6, & 12, & 24, & 48, & 96, & 192, & \\
9, & 18, & 36, & 72, & 144, & & & \\
27, & 54, & 108, & & & & & \\
81, & 162. & & & & & &
\end{array}
$$

In the above array each element divides everything that lies to the southeast. Also, each row contains exactly one element of S. For $1 \leq i \leq 5$ let r_i denote the element of row i that is contained in S, and let c_i denote the number of the column that contains r_i. We must have $c_i < c_{i-1}$ for $2 \leq i \leq 5$. Therefore $c_i \geq 6 - i$ for $1 \leq i \leq 5$. In particular $c_1 \geq 5$ so $r_1 \geq 16$, and $c_2 \geq 4$ so $r_2 \geq 24$. We have shown that none of $2, 4, 8, 12$ is in S. By the above comments $i \notin S$ for $1 \leq i \leq 15$.

3. See the course notes.

4, 5, 6. Given integers $n \geq 1$ and $k \geq 2$ suppose that $n + 1$ distinct elements are chosen from $\{1, 2, \ldots, kn\}$. We show that there exist two that differ by less than k. Partition $\{1, 2, \ldots, nk\} = \cup_{i=1}^{n} S_i$ where $S_i = \{ki, ki - 1, ki - 2, \ldots, ki - k + 1\}$. Among our $n + 1$ chosen elements, there exist two in the same S_i. These two differ by less than k.

7. Partition the set $\{0, 1, \ldots, 99\} = \cup_{i=0}^{50} S_i$ where $S_0 = \{0\}$, $S_i = \{i, 100 - i\}$ for $1 \leq i \leq 49$, $S_{50} = \{50\}$. For each of the given 52 integers, divide by 100 and consider the remainder. The remainder is contained in S_i for a unique i. By the pigeonhole principle, there exist two of the 52 integers for which these remainders lie in the same S_i. For these two integers the sum or difference is divisible by 100.

8. For positive integers m, n we consider the rational number m/n. For $0 \leq i \leq n$ divide the integer $10^i m$ by n, and call the remainder r_i. By construction $0 \leq r_i \leq n - 1$. By the pigeonhole principle there exist integers i, j $(0 \leq i < j \leq n)$ such that $r_i = r_j$. The integer n divides $10^j m - 10^i m$. For notational convenience define $\ell = j - i$. Then there exists a positive integer q such that $nq = 10^i(10^\ell - 1)m$. Divide q by $10^\ell - 1$ and call the remainder r. So $0 \leq r \leq 10^\ell - 2$. By construction there exists an integer $b \geq 0$ such that $q = (10^\ell - 1)b + r$. Writing $\theta = m/n$ we have

$$
\begin{aligned}
10^i \theta &= b + \frac{r}{10^\ell - 1} \\
&= b + \frac{r}{10^\ell} + \frac{r}{10^{2\ell}} + \frac{r}{10^{3\ell}} + \cdots
\end{aligned}
$$

Since the integer r is in the range $0 \leq r \leq 10^\ell - 2$ this yields a repeating decimal expansion for θ.

9. Consider the set of 10 people. The number of subsets is $2^{10} = 1024$. For each subset consider the sum of the ages of its members. This sum is among $0, 1, \ldots, 600$. By the pigeonhole principle the 1024 sums are not distinct. The result follows. The following argument shows that the result still holds for 9 people. Without loss we many assume that all the ages are distinct; otherwise we are done. Now in the above argument, the sum of the ages is at most 504. Then the number of subsets is $2^9 = 512$. By the pigeonhole principle the 512 sums are not distinct.

10. For $1 \leq i \leq 49$ let b_i denote the number of hours the child watches TV on day i. Consider the numbers $\{b_1 + b_2 + \cdots + b_i + 20\}_{i=0}^{48} \cup \{b_1 + b_2 + \cdots + b_j\}_{j=1}^{49}$. There are 98 numbers in the list, all among $1, 2, \ldots, 96$. By the pigeonhole principle the numbers $\{b_1 + b_2 + \cdots + b_i + 20\}_{i=0}^{48} \cup \{b_1 + b_2 + \cdots + b_j\}_{j=1}^{49}$. are not distinct. Therefore there exist integers i, j $(0 \leq i < j \leq 49)$ such that $b_{i+1} + \cdots + b_j = 20$. During the days $i + 1, \ldots, j$ the child watches TV for exactly 20 hours.

11. For $1 \leq i \leq 37$ let b_i denote the number of hours the student studies on day i. Consider the numbers $\{b_1 + b_2 + \cdots + b_i + 13\}_{i=0}^{36} \cup \{b_1 + b_2 + \cdots + b_j\}_{j=1}^{37}$. There are 74 numbers in the list, all among $1, 2, \ldots, 72$. By the pigeonhole principle the numbers $\{b_1 + b_2 + \cdots + b_i + 13\}_{i=0}^{36} \cup \{b_1 + b_2 + \cdots + b_j\}_{j=1}^{37}$ are not distinct. Therefore there exist integers i, j $(0 \leq i < j \leq 37)$ such that $b_{i+1} + \cdots + b_j = 13$. During the days $i + 1, \ldots, j$ the student will have studied exactly 13 hours.

12. Take $m = 4$ and $n = 6$. Pick a among $0, 1, 2, 3$ and b among $0, 1, 2, 3, 4, 5$ such that $a + b$ is odd. Suppose that there exists a positive integer x that yields a remainder of a (resp. b) when divided by 4 (resp. by 6). Then there exist integers r, s such that $x = 4r + a$ and

$x = 6s + b$. Combining these equations we obtain $2x - 4r - 6s = a + b$. In this equation the left-hand side is even and the right-hand side is odd, for a contradiction. Therefore x does not exist.

13. Since $r(3,3) = 6$ there exists a K_3 subgraph of K_6 that is red or blue. We assume that this K_3 subgraph is unique, and get a contradiction. Without loss we may assume that the above K_3 subgraph is red. Let x denote one of the vertices of this K_3 subgraph, and let $\{x_i\}_{i=1}^5$ denote the remaining five vertices of K_6. Consider the K_5 subgraph with vertices $\{x_i\}_{i=1}^5$. By assumption this subgraph has no K_3 subgraph that is red or blue. The only edge coloring of K_5 with this feature is shown in figure 3.2 of the text. Therefore we may assume that the vertices $\{x_i\}_{i=1}^5$ are labelled such that for distinct i, j $(1 \leq i, j \leq 5)$ the edge connecting x_i, x_j is red (resp. blue) if $i - j = \pm 1$ modulo 5 (resp. $i - j = \pm 2$ modulo 5). By construction and without loss of generality, we may assume that each of x_1, x_2 is connected to x by a red edge. Thus the vertices x, x_1, x_2 give a red K_3 subgraph. Now the edge connecting x and x_3 is blue; otherwise the vertices x, x_2, x_3 give a second red K_3 subgraph. Similarly the edge connecting x and x_5 is blue; otherwise the vertices x, x_1, x_5 give a second red K_3 subgraph. Now the vertices x, x_3, x_5 give a blue K_3 subgraph.

14. After n minutes we have removed n pieces of fruit from the bag. Suppose that among the removed fruit there are at most 11 pieces for each of the four kinds. Then our total n must be at most $4 \times 11 = 44$. After $n = 45$ minutes we will have picked at least a dozen pieces of fruit of the same kind.

15. For $1 \leq i \leq n+1$ divide a_i by n and call the reminder r_i. By construction $0 \leq r_i \leq n-1$. By the pigeonhole principle there exist distinct integers i, j among $1, 2, \ldots, n+1$ such that $r_i = r_j$. Now n divides $a_i - a_j$.

16. Label the people $1, 2, \ldots, n$. For $1 \leq i \leq n$ let a_i denote the number of people aquainted with person i. By construction $0 \leq a_i \leq n - 1$. Suppose the numbers $\{a_i\}_{i=1}^n$ are mutually distinct. Then for $0 \leq j \leq n-1$ there exists a unique integer i $(1 \leq i \leq n)$ such that $a_i = j$. Taking $j = 0$ and $j = n - 1$, we see that there exists a person aquainted with nobody else, and a person aquainted with everybody else. These people are distinct since $n \geq 2$. These two people know each other and do not know each other, for a contradiction. Therefore the numbers $\{a_i\}_{i=1}^n$ are not mutually distinct.

17. We assume that the conclusion is false and get a contradiction. Label the people $1, 2, \ldots, 100$. For $1 \leq i \leq 100$ let a_i denote the number of people aquainted with person i. By construction $0 \leq a_i \leq 99$. By assumption a_i is even. Therefore a_i is among $0, 2, 4, \ldots, 98$. In this list there are 50 numbers. Now by our initial assumption, for each even integer j $(0 \leq j \leq 98)$ there exists a unique pair of integers (r, s) $(1 \leq r < s \leq 100)$ such that $a_r = j$ and $a_s = j$. Taking $j = 0$ and $j = 98$, we see that there exist two people who know nobody else, and two people who know everybody else except one. This is a contradiction.

18. Divide the 2×2 square into four 1×1 squares. By the pigeonhole principle there exists a 1×1 square that contains at least two of the five points. For these two points the distance apart is at most $\sqrt{2}$.

19. Divide the equilateral triangle into a grid, with each piece an equilateral triangle of side length $1/n$. In this grid there are $1 + 3 + 5 + \cdots + 2n - 1 = n^2$ pieces. Suppose we place $m_n = n^2 + 1$ points within the equilateral triangle. Then by the pigeonhole principle there exists a piece that contains two or more points. For these two points the distance apart is at most $1/n$.

20. Color the edges of K_{17} red or blue or green. We show that there exists a K_3 subgraph of K_{17} that is red or blue or green. Pick a vertex x of K_{17}. In K_{17} there are 16 edges that contain x. By the pigeonhole principle, at least 6 of these are the same color (let us say red). Pick distinct vertices $\{x_i\}_{i=1}^{6}$ of K_{17} that are connected to x via a red edge. Consider the K_6 subgraph with vertices $\{x_i\}_{i=1}^{6}$. If this K_6 subgraph contains a red edge, then the two vertices involved together with x form the vertex set of a red K_3 subgraph. On the other hand, if the K_6 subgraph does not contain a red edge, then since $r(3,3) = 6$, it contains a K_3 subgraph that is blue or green. We have shown that K_{17} has a K_3 subgraph that is red or blue or green.

21. Let X denote the set of sequences $(a_1, a_2, a_3, a_4, a_5)$ such that $a_i \in \{1, -1\}$ for $1 \le i \le 5$ and $a_1a_2a_3a_4a_5 = 1$. Note that $|X| = 16$. Consider the complete graph K_{16} with vertex set X. We display an edge coloring of K_{16} with colors red, blue, green such that no K_3 subgraph is red or blue or green. For distinct x, y in X consider the edge connecting x and y. Color this edge red (resp. blue) (resp. green) whenever the sequences x, y differ in exactly 4 coordinates (resp. differ in exactly 2 coordinates i, j with $i - j = \pm 1$ modulo 5) (resp. differ in exactly 2 coordinates i, j with $i - j = \pm 2$ modulo 5). Each edge of K_{16} is now colored red or blue or green. For this edge coloring of K_{16} there is no K_3 subgraph that is red or blue or green.

22. For an integer $k \ge 2$ abbreviate $r_k = r(3, 3, \ldots, 3)$ (k 3's). We show that $r_{k+1} \le (k+1)(r_k - 1) + 2$. Define $n = r_k$ and $m = (k+1)(n-1) + 2$. Color the edges of K_m with $k+1$ colors $C_1, C_2, \ldots, C_{k+1}$. We show that there exists a K_3 subgraph with all edges the same color. Pick a vertex x of K_m. In K_m there are $m-1$ edges that contain x. By the pigeonhole principle, at least n of these are the same color (which we may assume is C_1). Pick distinct vertices $\{x_i\}_{i=1}^{n}$ of K_m that are connected to x by an edge colored C_1. Consider the K_n subgraph with vertices $\{x_i\}_{i=1}^{n}$. If this K_n subgraph contains an edge colored C_1, then the two vertices involved together with x give a K_3 subgraph that is colored C_1. On the other hand, if the K_n subgraph does not contain an edge colored C_1, then since $r_k = n$, it contains a K_3 subgraph that is colored C_i for some i ($2 \le i \le k+1$). In all cases K_m has a K_3 subgraph that is colored C_i for some i ($1 \le i \le k+1$). Therefore $r_k \le m$.

23. We proved earlier that

$$r(m,n) \le \binom{m+n-2}{n-1}.$$

Applying this result with $m = 3$ and $n = 4$ we obtain $r(3,4) \le 10$.

24. We show that $r_t(t, t, q_3) = q_3$. By construction $r_t(t, t, q_3) \ge q_3$. To show the reverse inequality, consider the complete graph with q_3 vertices. Let X denote the vertex set of this

graph. Color the t-element subsets of X red or blue or green. Then either (i) there exists a t-element subset of X that is red, or (ii) there exists a t-element subset of X that is blue, or (iii) every t-element subset of X is green. Therefore $r_t(t, t, q_3) \le q_3$ so $r_t(t, t, q_3) = q_3$.

25. Abbreviate $N = r_t(m, m, \ldots, m)$ (k m's). We show $r_t(q_1, q_2, \ldots, q_k) \le N$. Consider the complete graph K_N with vertex set X. Color each t-element subset of X with k colors $C_1, C_2, \ldots, C_k$. By definition there exists a K_m subgraph all of whose t-element subsets are colored C_i for some i $(1 \le i \le k)$. Since $q_i \le m$ there exists a subgraph of that K_m with q_i vertices. For this subgraph every t-element subset is colored C_i.

26. In the $m \times n$ array assume the rows (resp. columns) are indexed in increasing order from front to back (resp. left to right). Consider two adjacent columns $j-1$ and j. A person in column $j-1$ and a person in column j are called *matched* if they occupy the same row of the original formation. Thus a person in column j is taller than their match in column $j-1$. Now consider the adjusted formation. Let L and R denote adjacent people in some row i, with L in column $j-1$ and R in column j. We show that R is taller than L. We assume that L is at least as tall as R, and get a contradiction. In column $j-1$, the people in rows $i, i+1, \ldots, m$ are at least as tall as L. In column j, the people in rows $1, 2, \ldots, i$ are at most as tall as R. Therefore everyone in rows $i, i+1, \ldots, m$ of column $j-1$ is at least as tall as anyone in rows $1, 2, \ldots, i$ of column j. Now for the people in rows $1, 2, \ldots, i$ of column j their match stands among rows $1, 2, \ldots, i-1$ of column $j-1$. This contradicts the pigeonhole principle, so L is shorter than R.

27. Let $s_1, s_2, \ldots, s_k$ denote the subsets in the collection. By assumption these subsets are mutually distinct. Consider their complements $\overline{s_1}, \overline{s_2}, \ldots, \overline{s_k}$. These complements are mutually distinct. Also, none of these complements are in the collection. Therefore $s_1, s_2, \ldots, s_k$, $\overline{s_1}, \overline{s_2}, \ldots, \overline{s_k}$ are mutually distinct. Therefore $2k \le 2^n$ so $k \le 2^{n-1}$. There are at most 2^{n-1} subsets in the collection.

28. The answer is 1620. Note that $1620 = 81 \times 20$. First assume that $\sum_{i=1}^{100} a_i < 1620$. We show that no matter how the dance lists are selected, there exists a group of 20 men that cannot be paired with the 20 women. Let the dance lists be given. Label the women $1, 2, \ldots, 20$. For $1 \le j \le 20$ let b_j denote the number of men among the 100 that listed woman j. Note that $\sum_{j=1}^{20} b_j = \sum_{i=1}^{100} a_i$ so $(\sum_{j=1}^{20} b_j)/20 < 81$. By the pigeonhole principle there exists an integer j $(1 \le j \le 20)$ such that $b_j \le 80$. We have $100 - b_j \ge 20$. Therefore there exist at least 20 men that did not list woman j. This group of 20 men cannot be paired with the 20 women.

Consider the following selection of dance lists. For $1 \le i \le 20$ man i lists woman i and no one else. For $21 \le i \le 100$ man i lists all 20 women. Thus $a_i = 1$ for $1 \le i \le 20$ and $a_i = 20$ for $21 \le i \le 100$. Note that $\sum_{i=1}^{100} a_i = 20 + 80 \times 20 = 1620$. Note also that every group of 20 men can be paired with the 20 women.

29. Without loss we may assume $|B_1| \le |B_2| \le \cdots \le |B_n|$ and $|B_1^*| \le |B_2^*| \le \cdots \le |B_{n+1}^*|$. By assumption $|B_1^*|$ is positive. Let N denote the total number of objects. Thus $N = \sum_{i=1}^{n} |B_i|$ and $N = \sum_{i=1}^{n+1} |B_i^*|$. For $0 \le i \le n$ define

$$\Delta_i = |B_1^*| + |B_2^*| + \cdots + |B_{i+1}^*| - |B_1| - |B_2| - \cdots - |B_i|.$$

We have $\Delta_0 = |B_1^*| > 0$ and $\Delta_n = N - N = 0$. Therefore there exists an integer r $(1 \leq r \leq n)$ such that $\Delta_{r-1} > 0$ and $\Delta_r \leq 0$. Now

$$0 < \Delta_{r-1} - \Delta_r = |B_r| - |B_{r+1}^*|$$

so $|B_{r+1}^*| < |B_r|$. So far we have

$$|B_1^*| \leq |B_2^*| \leq \cdots \leq |B_{r+1}^*| < |B_r| \leq |B_{r+1}| \leq \cdots \leq |B_n|.$$

Thus $|B_i^*| < |B_j|$ for $1 \leq i \leq r+1$ and $r \leq j \leq n$. Define

$$\theta = |(B_1^* \cup B_2^* \cup \cdots \cup B_{r+1}^*) \cap (B_r \cup B_{r+1} \cup \cdots \cup B_n)|.$$

We show $\theta \geq 2$. Using $\Delta_{r-1} > 0$ we have

$$\begin{aligned}
|B_1^*| + |B_2^*| + \cdots + |B_r^*| &> |B_1| + |B_2| + \cdots + |B_{r-1}| \\
&= |B_1 \cup B_2 \cup \cdots \cup B_{r-1}| \\
&\geq |(B_1 \cup B_2 \cup \cdots \cup B_{r-1}) \cap (B_1^* \cup B_2^* \cup \cdots \cup B_{r+1}^*)| \\
&= |B_1^* \cup B_2^* \cup \cdots \cup B_{r+1}^*| - \theta \\
&= |B_1^*| + |B_2^*| + \cdots + |B_{r+1}^*| - \theta \\
&\geq |B_1^*| + |B_2^*| + \cdots + |B_r^*| + 1 - \theta.
\end{aligned}$$

Therefore $\theta > 1$ so $\theta \geq 2$.