On the Nonexistence of a Bijection Between $\mathbb{N}$ and $\mathbb{Q}$

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Abstract

This paper presents an argument for the nonexistence of a bijection from the natural numbers $\mathbb{N}$ to the rational numbers $\mathbb{Q}$, using a set-theoretic construction inspired by Cantor's Theorem for the nonexistance of a bijection between the naturl numbers and the power set of the natural numbers. We define a special "deficiency set" and deduce from basic properties of surjective maps that such a bijection cannot exist.

1 Introduction

The claim to the countability of the rational numbers $\mathbb{Q}$ is well-known; Here, we present a proof against that claim in the style of Cantor's Theorem demonstrating that for a class of functions from $\mathbb{N}$ to $\mathbb{Q}$, one cannot construct a bijection, by analyzing the image of such functions with respect to a deficiency set.

2 Definitions

Let $f \colon \mathbb{N} \to \mathbb{Q}$ be a function.

Definition 2.1. Define the *deficiency set* [D $= n \in \mathbb{N} \mid \forall m \in \mathbb{N},\ f(m) \neq n$,]

that is, D consists of precisely those natural numbers which are not in the image of f.

3 Main Result

We analyze the implications of the structure of D for the surjectivity of f.

Theorem 3.1. *Let $f \colon \mathbb{N} \to \mathbb{Q}$ be a function. Then f cannot be a bijection.*

Proof. We proceed by cases based on the nature of the set D:

Case 1: $D = \emptyset$.

If D is empty, then for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $f(m) = n$; that is, f is surjective from $\mathbb{N}$ onto $\mathbb{N} \subseteq \mathbb{Q}$. Since a bijective f must map from $\mathbb{N}$ onto all of $\mathbb{Q}$, it must also map to elements of $\mathbb{Q}$ not in $\mathbb{N}$, such as -1.

However, if $f(m) = -1$ for some $m \in \mathbb{N}$, this is a contradiction to the surjection from $\mathbb{N}$ onto $\mathbb{N} \subseteq \mathbb{Q}$, because there exists no $m \in \mathbb{N}$ such that $f(m) = -1$ if f is surjective onto $\mathbb{N}$ (since $-1 \notin \mathbb{N}$, but $-1 \in \mathbb{Q}$). Thus, it is impossible for f to be surjective onto all of $\mathbb{Q}$.

Case 2: $D \neq \emptyset$.

If D is a nonempty subset of $\mathbb{N}$, then by the well-ordering principle, D has a least element, say n^*. By definition of D, n^* is not in the image of f; that is, there does *not* exist any m with $f(m) = n^*$. Thus, f is not surjective, and therefore there cannot be a bijection.

Conclusion: In either case, f is not a bijection. □

4 Discussion

This theorem demonstrates that for any function from $\mathbb{N}$ to $\mathbb{Q}$, if one attempts to construct a bijection, one will always find an element of $\mathbb{Q}$ that is not in the image of the function, therefore proving that the function is not bijective. By taking the same approach as Cantor when proving the nonexistence of a bijection between the natural numbers and the power set of the natural numbers via a deficiency set and the resulting contradiction to a bijection, this theorem exposes an inconsistent application of proof in the foundations of mathematics. This theorem contradicts the current distinctions between countable and uncountable sets and showcases the inevitable loss of mathematical semantics regarding infinite sets, as identified by Galileo Galilei in 1638. This theorem provides further justification that a philosophical foundation in constructive mathematics is required to avoid contradiction and paradox.

References

- Galilei, G. (1638). *Discourses and Mathematical Demonstrations Relating to Two New Sciences.*
- Cantor, G. (1891). *On a Fundamental Property of All Real Algebraic Numbers.* Journal für die reine und angewandte Mathematik.
- Halmos, P. (1974). *Naive Set Theory.* Springer.

Note: Please find the associated lean4 proof in the gethub repository below:

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https://github.com/omniarelativum/chapter-logic