BINARY LOGIC

Boolean Algebra

Today the computing (information) technology is based on Binary logic

The Binary logic is based on the Aristotelian Logic

Binary and Aristotelian logic:

- [1] = True
- [0] = False

True and False [logic]

- The Greek philosopher Aristotle (384-322 BC) founded a system of logic based on two types of propositions: **True** and **False**.

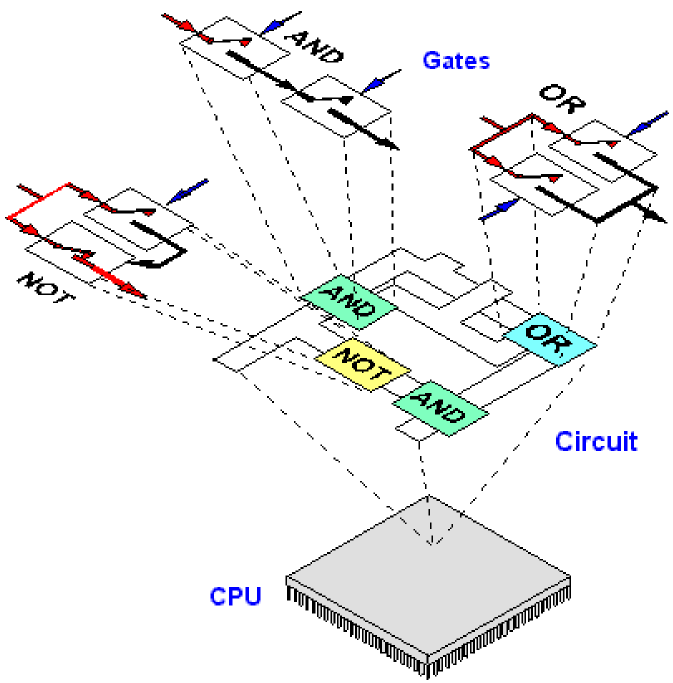
True-False (Binary logic)

- **True** and **False**. Lead to the four foundational laws of logic:
  - **Law of Identity**: (“A” is “A”) or (“A” = “A”);
  - **Law of Non-contradiction**: (“A” is not “non-A”);
  - **Law of the Excluded Middle**: (Something is either “A” or “non-A”);
  - **Law of Rational Inference**…
    - All Letters are Characters
      - A is a Letter
      - A is a Character

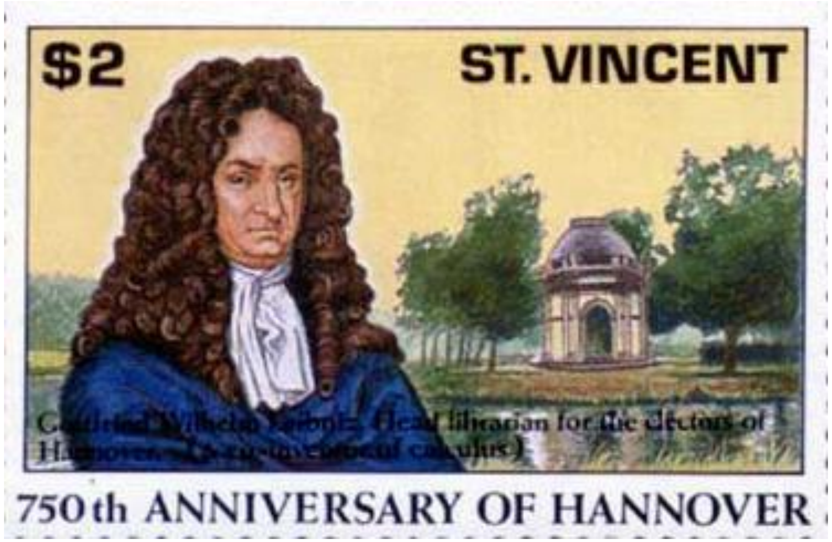
Aristotle (right) gestures to the earth, representing his belief in knowledge through empirical observation and experience, while holding a copy of his Nicomachean Ethics in his hand, while **Plato** (left) gestures to the heavens, representing his belief in The Forms.

Centuries later…

Mathematicians (Leibniz, Boole, …) and Engineers (Shannon, Shestakov, …) extended the Aristotelian Logic to Symbolic Logic … to Algebra of Logic to … Logic Circuits …



Gottfried Wilhelm von LEIBNIZ (1646-1716)



Mathematician born in Saxony (now Germany)

De la numération binaire

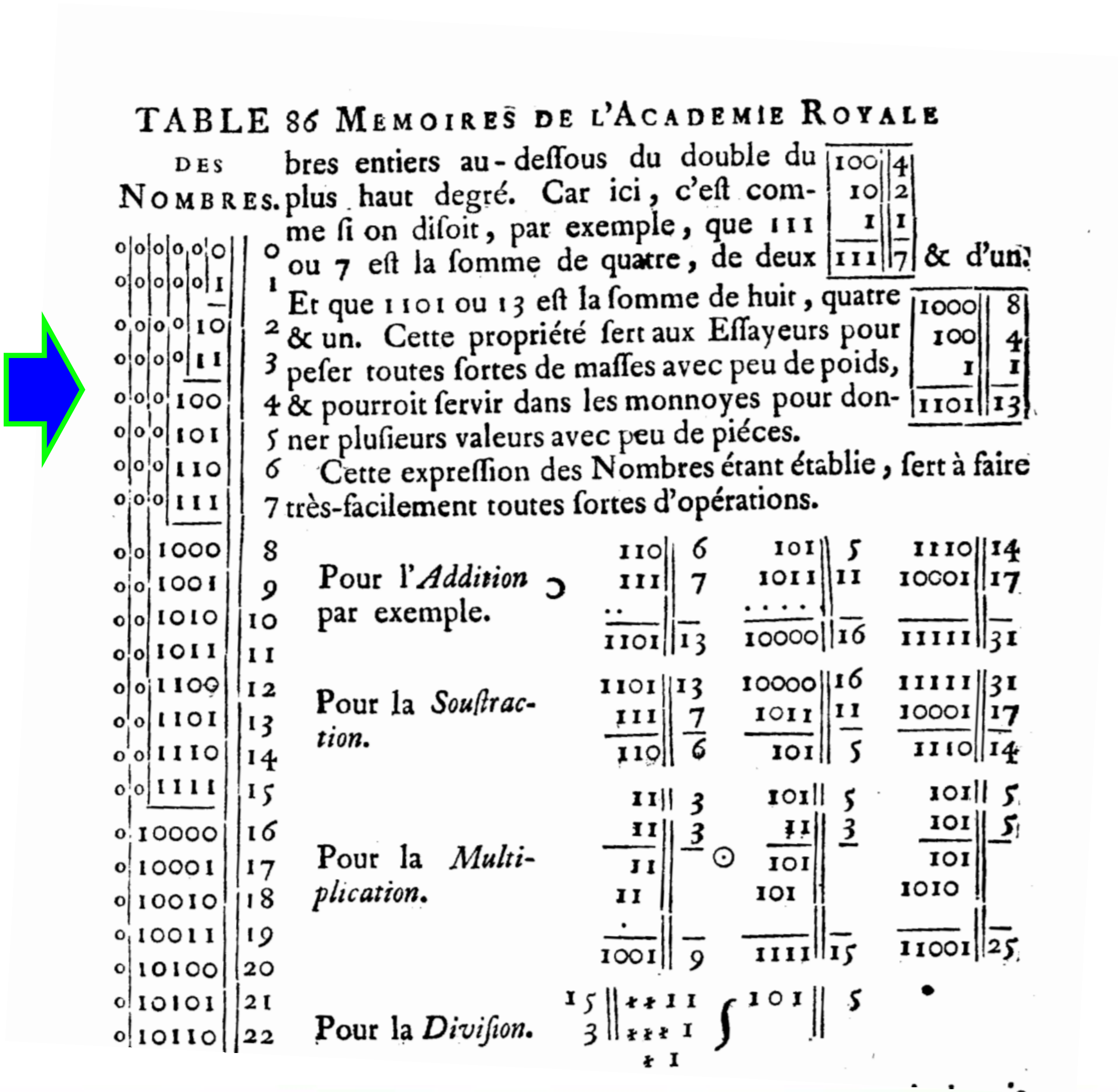


TABLE DES NOMBRES. 86 MEMOIRES DE L'ACADEMIE ROYALE

bres entiers au-deſſous du double du plus haut degré. Car ici, c'eſt comme ſi on diſoit, par exemple, que 111 ou 7 eſt la ſomme de quatre, de deux & d'un.

100	4
10	2
1	1
111	7

Et que 1101 ou 13 eſt la ſomme de huit, quatre & un. Cette propriété ſert aux Eſſayeurs pour peſer toutes ſortes de maſſes avec peu de poids, & pourroit ſervir dans les monnoyes pour donner pluſieurs valeurs avec peu de piéces.

1000	8
100	4
1	1
1101	13

Cette expreſſion des Nombres étant établie, ſert à faire très-facilement toutes ſortes d'opérations.

0 0 0 0 0 0	0
0 0 0 0 0 1	1
0 0 0 0 10	2
0 0 0 0 11	3
0 0 0 100	4
0 0 0 101	5
0 0 0 110	6
0 0 0 111	7
0 0 1000	8
0 0 1001	9
0 0 1010	10
0 0 1011	11
0 0 1100	12
0 0 1101	13
0 0 1110	14
0 0 1111	15
0 10000	16
0 10001	17
0 10010	18
0 10011	19
0 10100	20
0 10101	21
0 10110	22

Pour l'*Addition* par exemple.

110	6	101	5	1110	14
111	7	1011	11	10001	17
1101	13	10000	16	11111	31

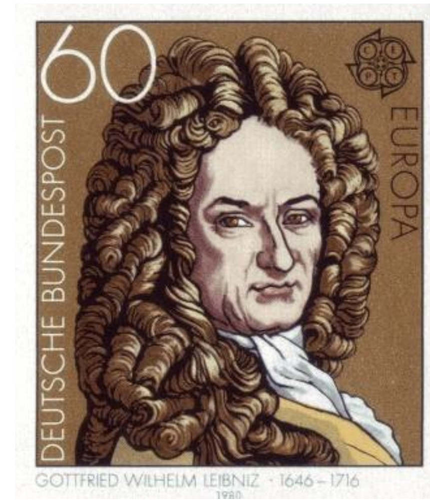
Pour la *Souſtraction.*

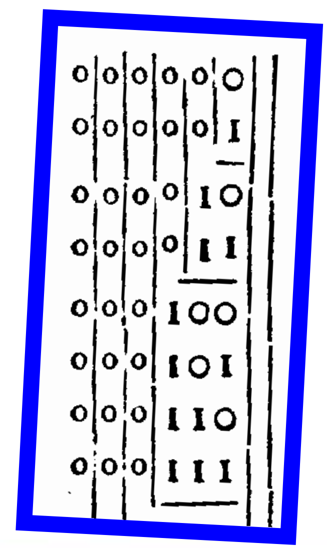
1101	13	10000	16	11111	31
111	7	1011	11	10001	17
110	6	101	5	1110	14

Pour la *Multiplication.*

11	3	101	5	101	5
11	3	11	3	101	5
11		101		101	
11		101		1010	
1001	9	1111	15	11001	25

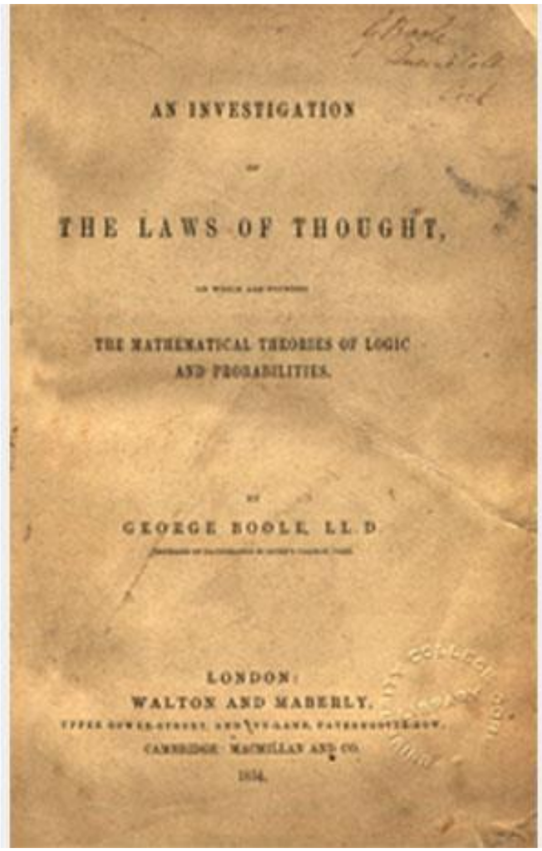
Pour la *Diviſion.* 15 | 3 | 1111 | 11 | 101 | 5





George Boole, Mathematician (1815-1864)

«The Mathematical Analysis of Logic» (1847)


AN INVESTIGATION

OF

THE LAWS OF THOUGHT,

THE MATHEMATICAL THEORIES OF LOGIC
AND PROBABILITIES.

BY

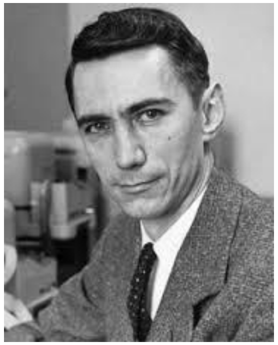
GEORGE BOOLE, LL.D.

LONDON:
WALTON AND MABERLY,
CAMBRIDGE: MACMILLAN AND CO.

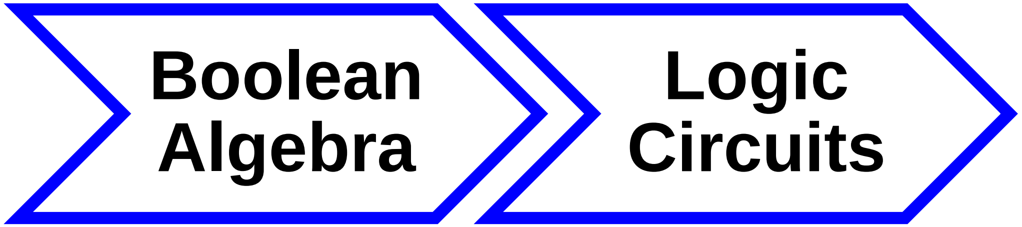
Symbolic logic → Algebra of logic → Boolean algebra

Claude Shannon, Victor Ivanovich Shestakov

- **Claude Shannon** (1916-2001): «*A symbolic analysis of relay and switching circuits*», Thesis (M.S.E.E)-Massachusetts Institute of Technology, 1940.

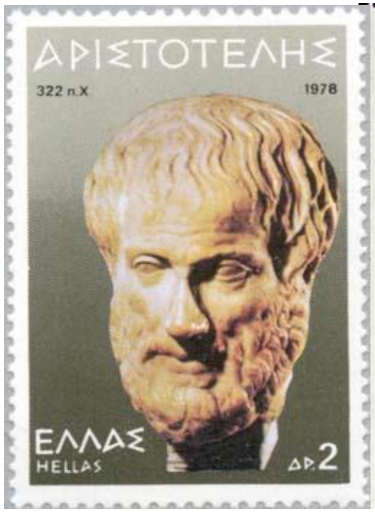


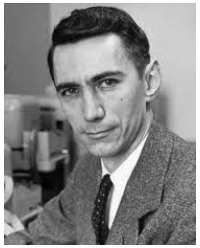
- **Victor Ivanovich Shestakov** (1907-1987): «*Mathematical logic and foundations*», Ph.D. Dissertation-Lomonosov Moscow State University, 1939.



https://www.genealogy.math.ndsu.nodak.edu/id.php?id=113382

Logic … logic circuits



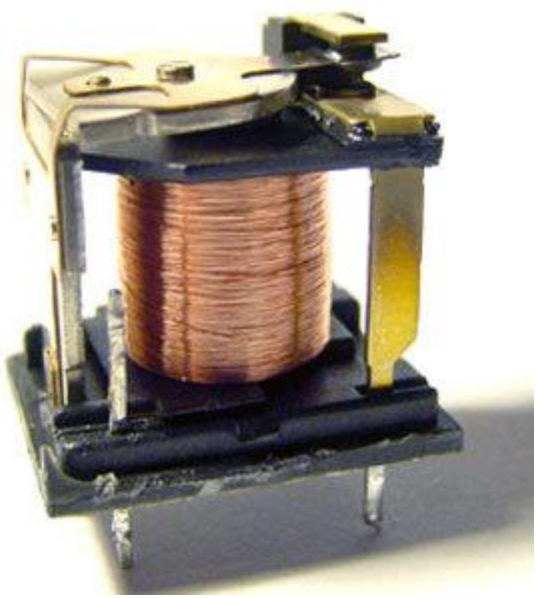

- Aristotle (400 B.C) : Logic (True and False)
- Muslim mathematicians (middle ages) → survived Aristotelian and other manuscripts
- Leibniz (1679-1701): Aristotelian logic → Mathematical Logic
- Boole (1854): Gave a meaning to Mathematical Logic → Algebra of Logic
- Claude Shannon (1937) and Victor Ivanovich Shestakov (1935): Applied the Algebra of logic → Logic Circuits

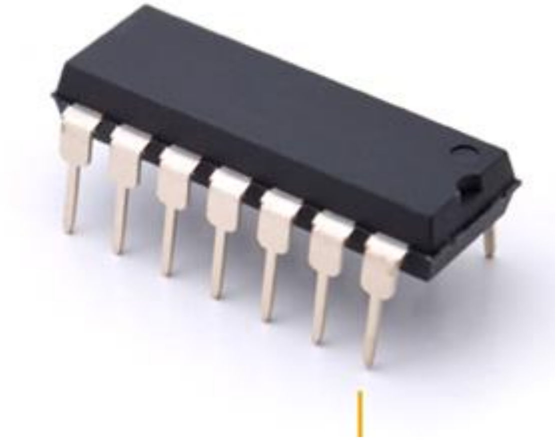
Electronic Computers

Binary logic

The binary logic is implemented with switches

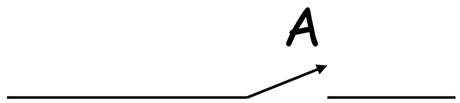
- Relays → ElectroMechanical
- Vacuum Tubes → Electrical
- Semiconductors → Electronic



Basic Principle: Switch

Basic Principle: Switch



Switch: OFF (open)

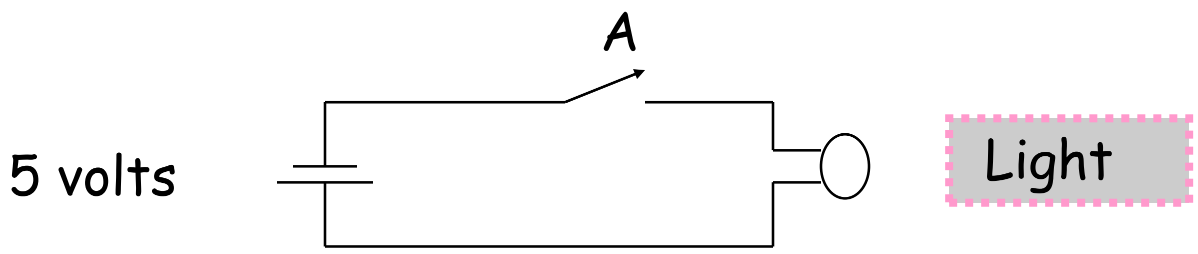
Basic Principle: Switch

Switch: ON ()

Basic Principle: “ON-OFF” Switch

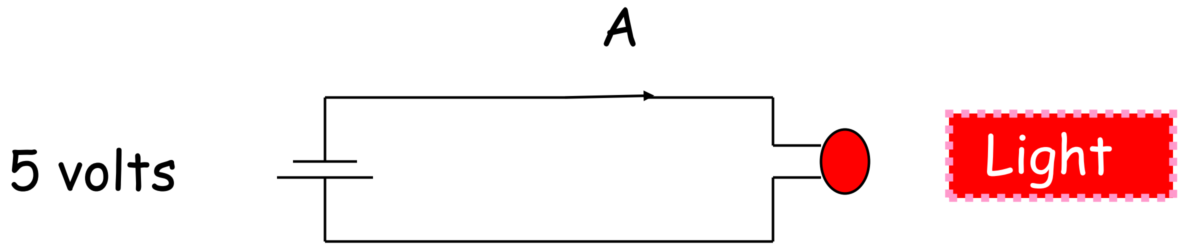
Basic Principle: Digital and Analog Switch

Basic Principle: Switch with Light



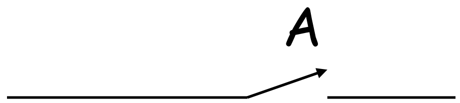
A	Light
Open	0

Basic Principle: Switch



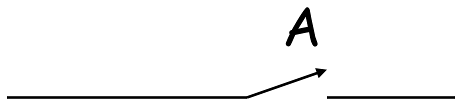
A	Light
Open	0
Closed	1

Truth Table



A Switch in computing can also be implemented with a…

… a transistor

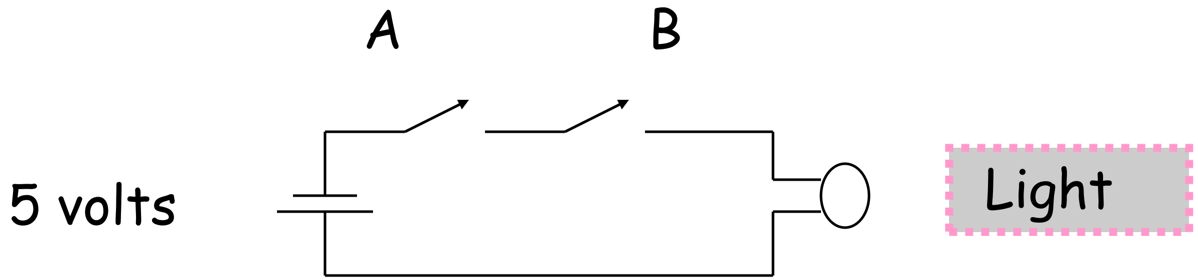


Transistor

We will talk about it later …

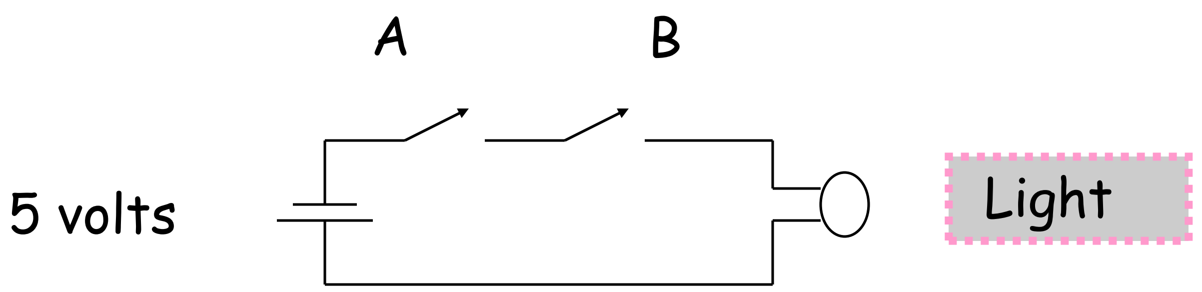
Let’s put two switches in series …

AND operation



Truth Table ?

AND

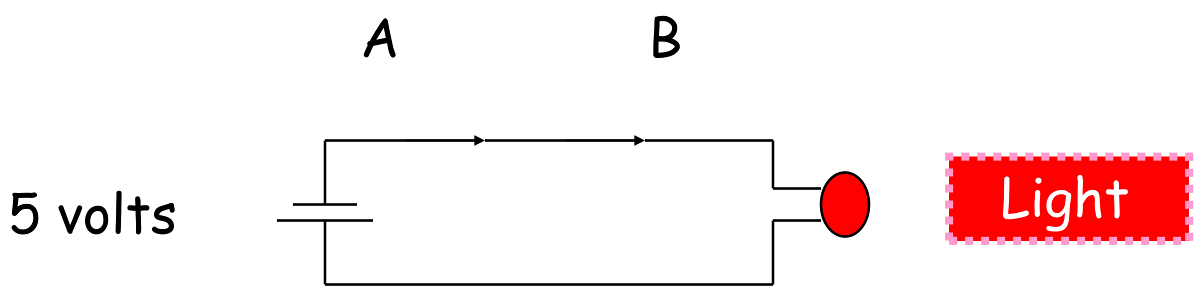


A	B	Light
0	0	
0	1	
1	0	
1	1	

Truth Table

AND

A (and) B = A • B = AB

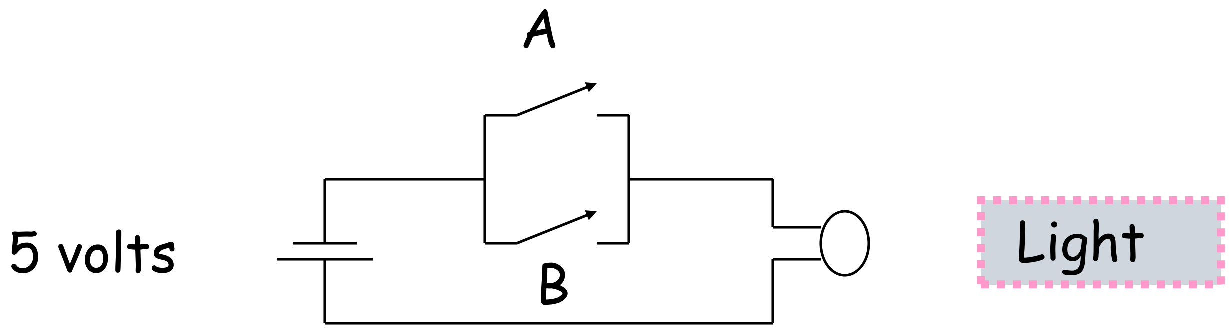


A	B	Light
0	0	0
0	1	0
1	0	0
1	1	1

Truth Table

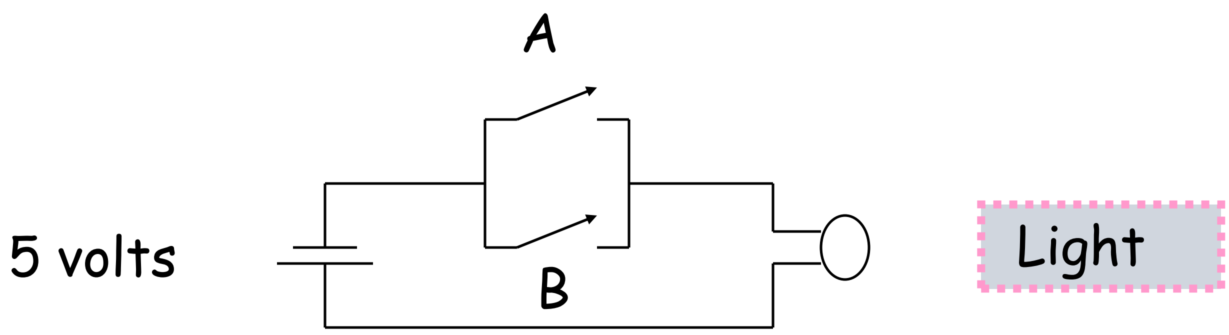
Let’s put two switches in parallel …

OR operation



Truth Table ?

OR

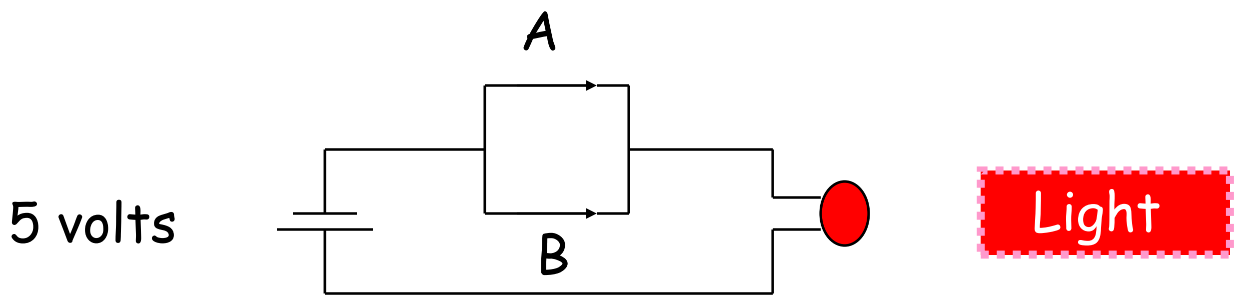


A	B	Light
0	0	
0	1	
1	0	
1	1	

Truth Table

OR

$A \text{ (or) } B = A + B$

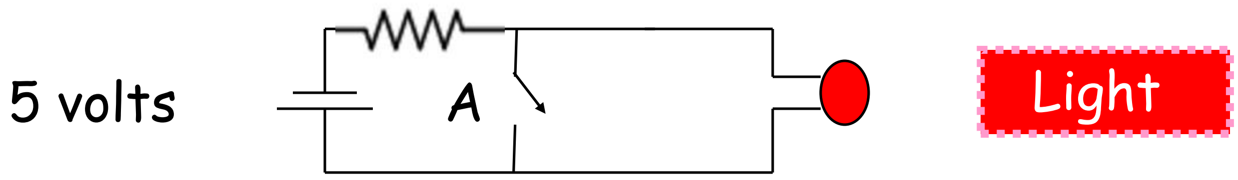


A	B	Light
0	0	0
0	1	1
1	0	1
1	1	1

Truth Table

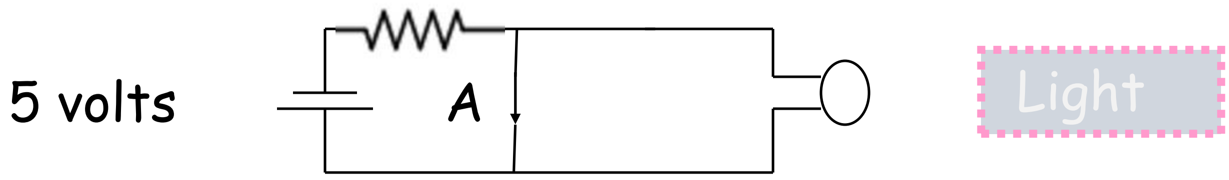
The last basic operation …

NOT (A = 0 = Open)



A	Light
Open (0)	1

NOT (A = 1 = Closed)



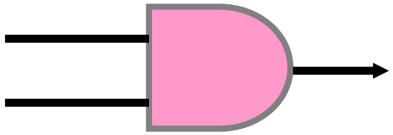
A	Light
Open (0)	1
Closed(1)	0

NOT (truth table)

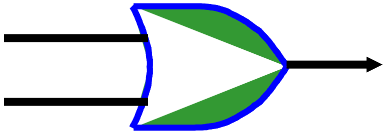
A_1	$\overline{A_1}$
0	1
1	0

The 3 basic operations and their symbols (gates)

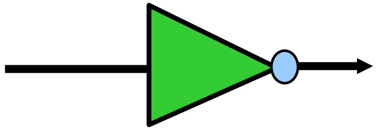
AND



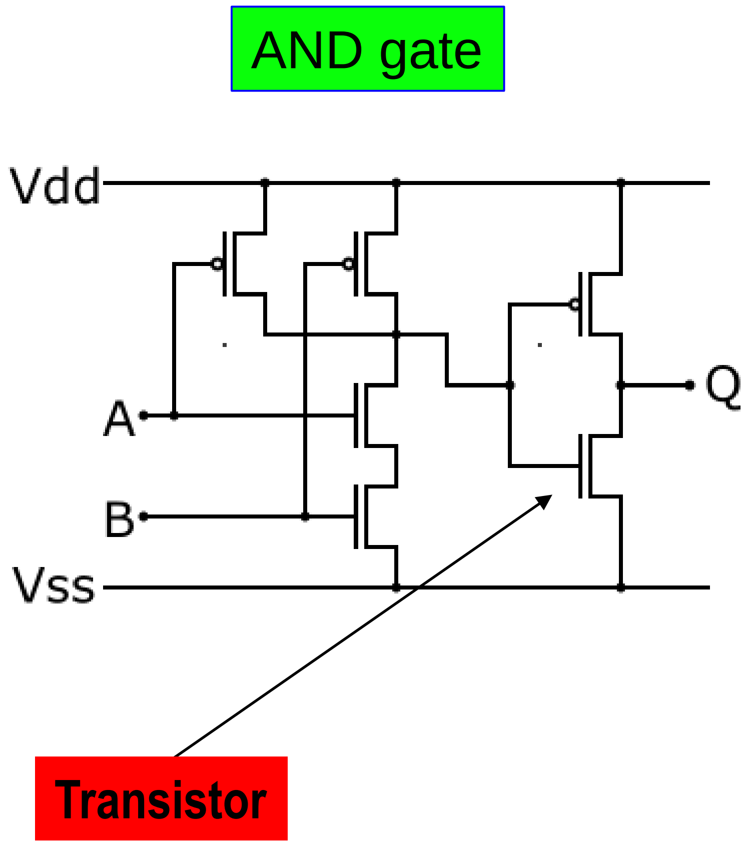
OR

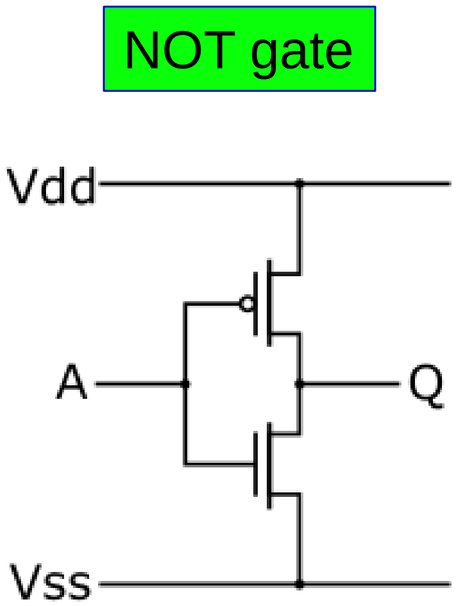


NOT



The reality – Transistors (CMOS)



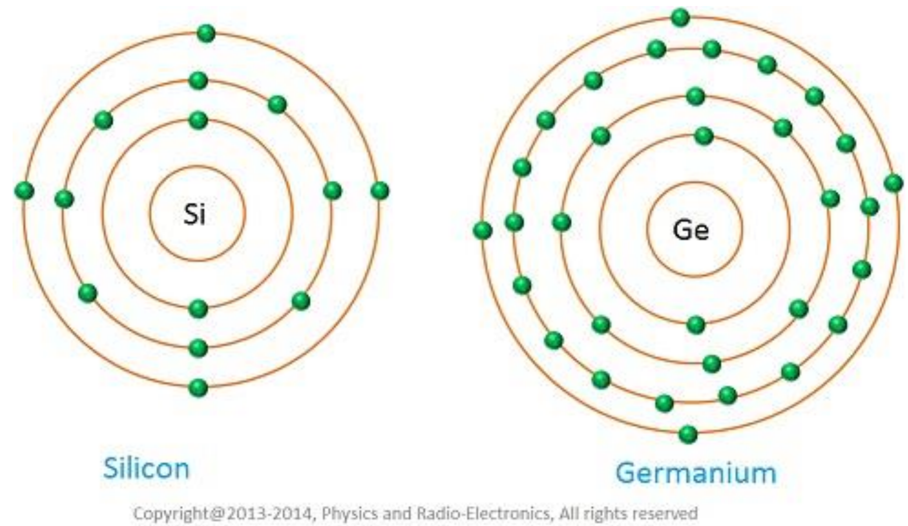


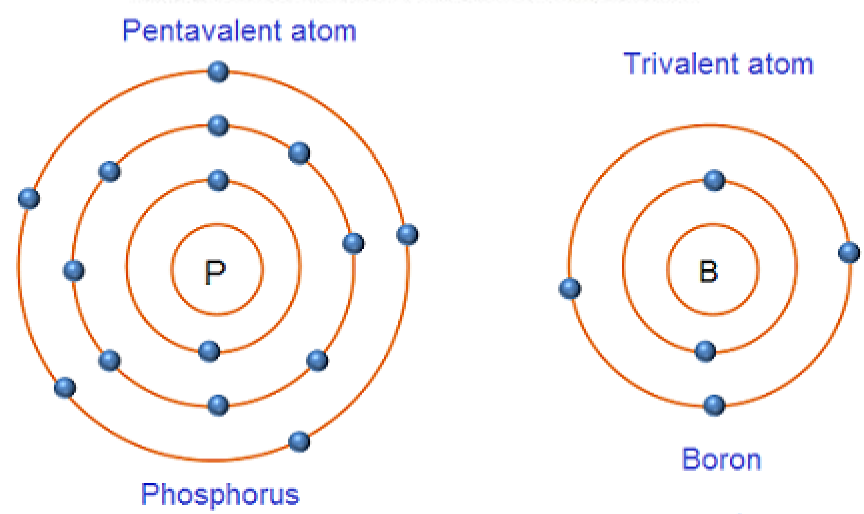
CMOS (**C**omplementary **M**etal–**O**xide–**S**emiconductor)

Chemistry basics ….

- Conductors
- Insulators
- Semiconductors

Semiconductor basics





Now Read This

https://www.halbleiter.org/en/fundamentals/conductors-insulators-semiconductors/

Watch the video

https://youtu.be/60Qz051rD_w

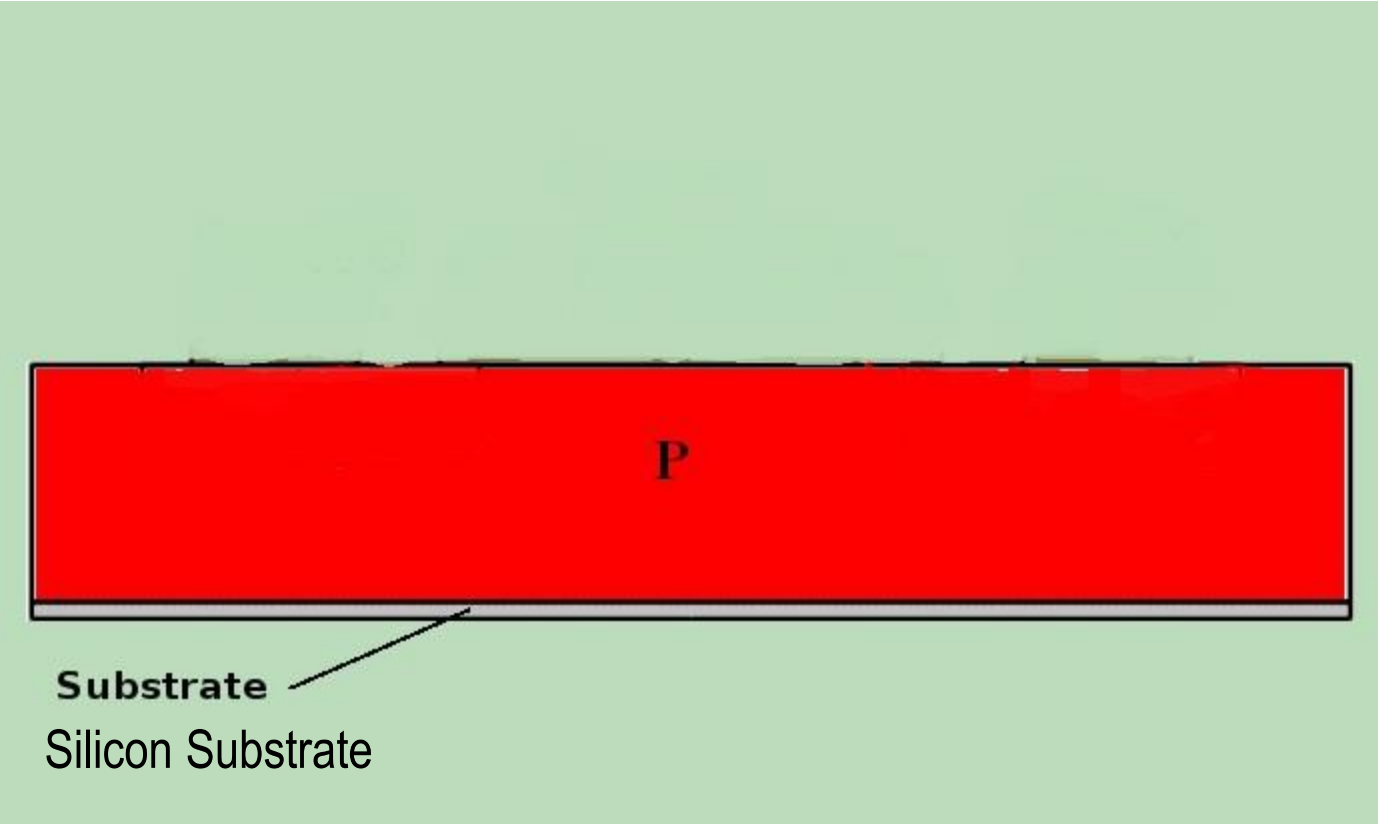
https://youtu.be/k12GMjtN8aA

https://youtu.be/ethnHSgVbHs

Transistor

A semiconductor switch

P-type semiconductor Silicon material

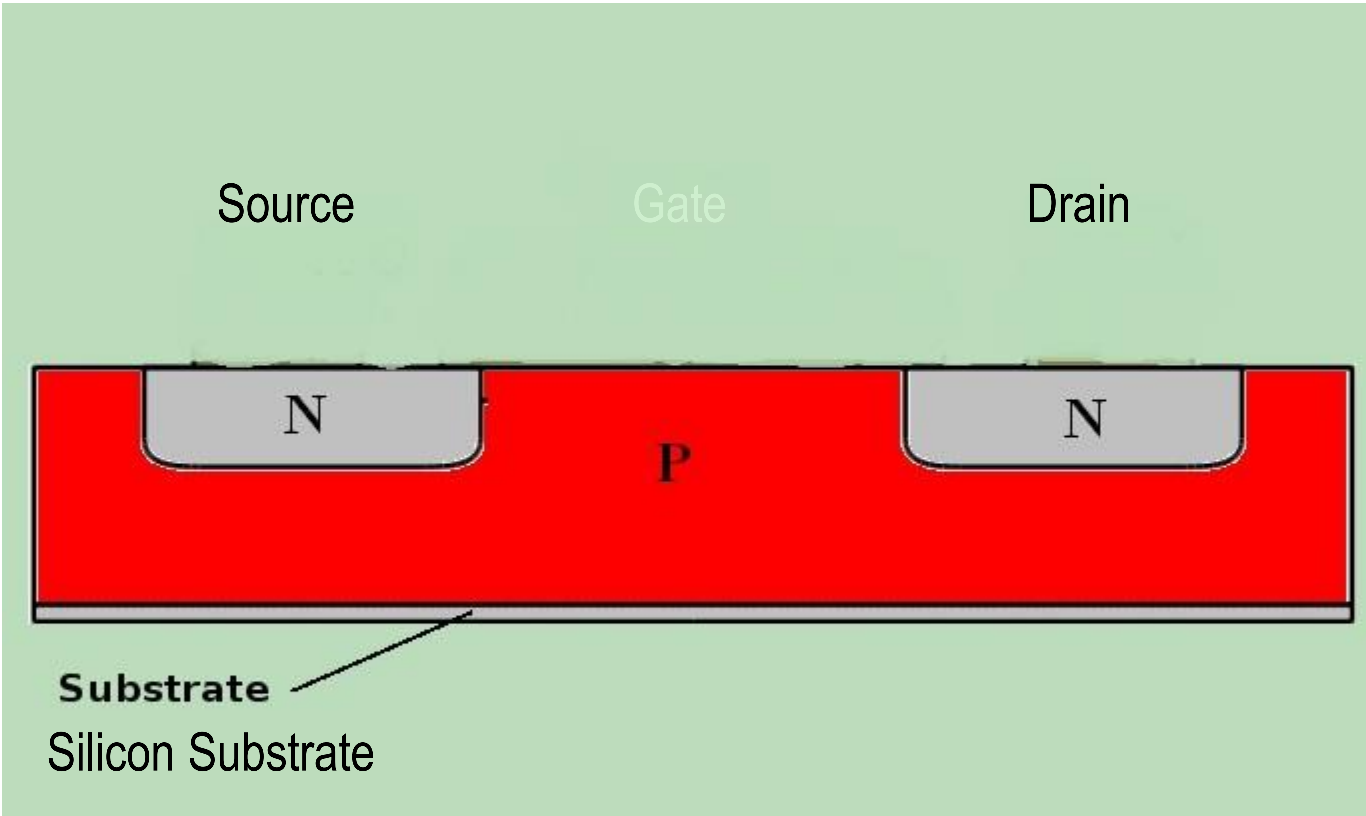


P = Positive

Silicon is a chemical element with symbol Si and atomic number 14

Add N-type semiconductor material

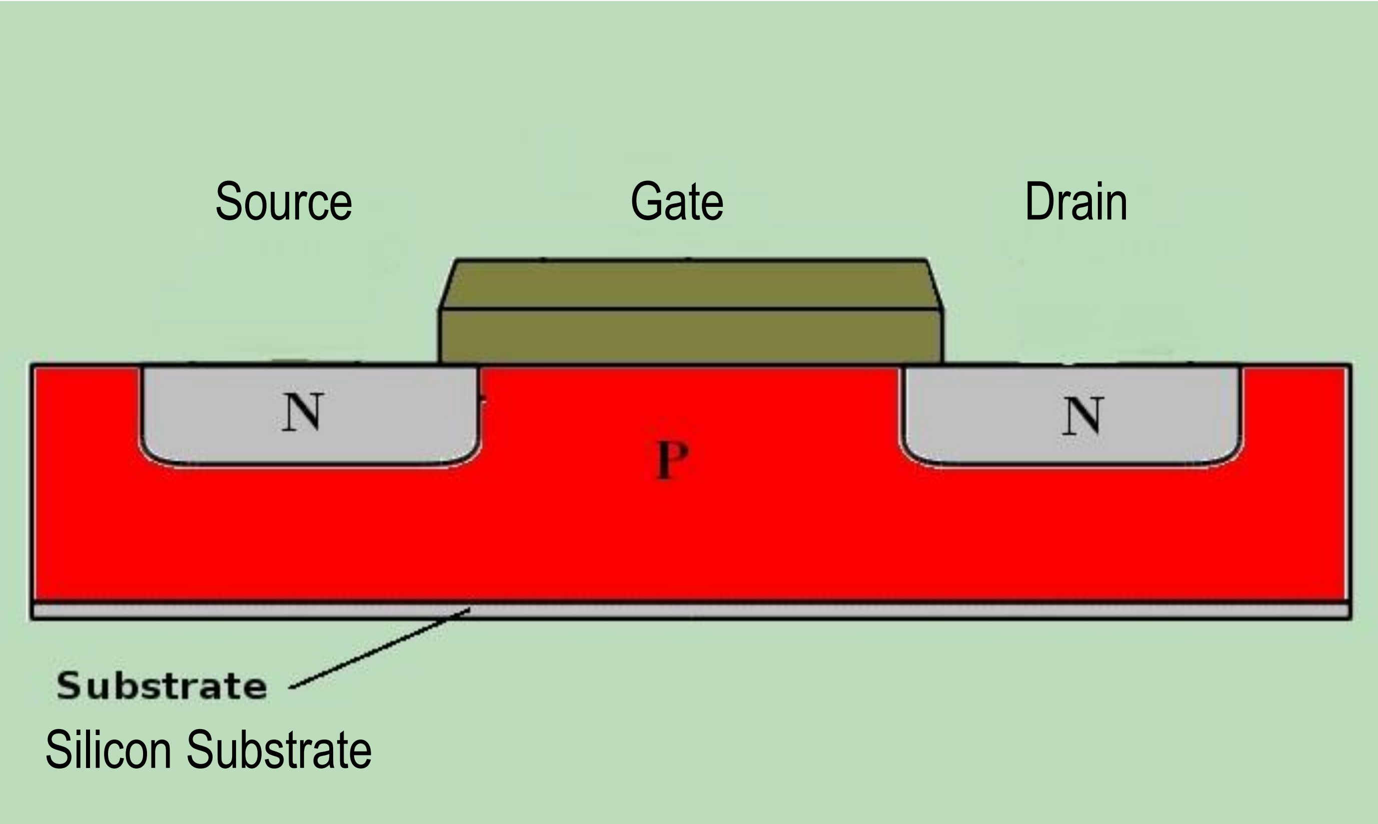
NPN=Negative-Positive-Negative type of configuration



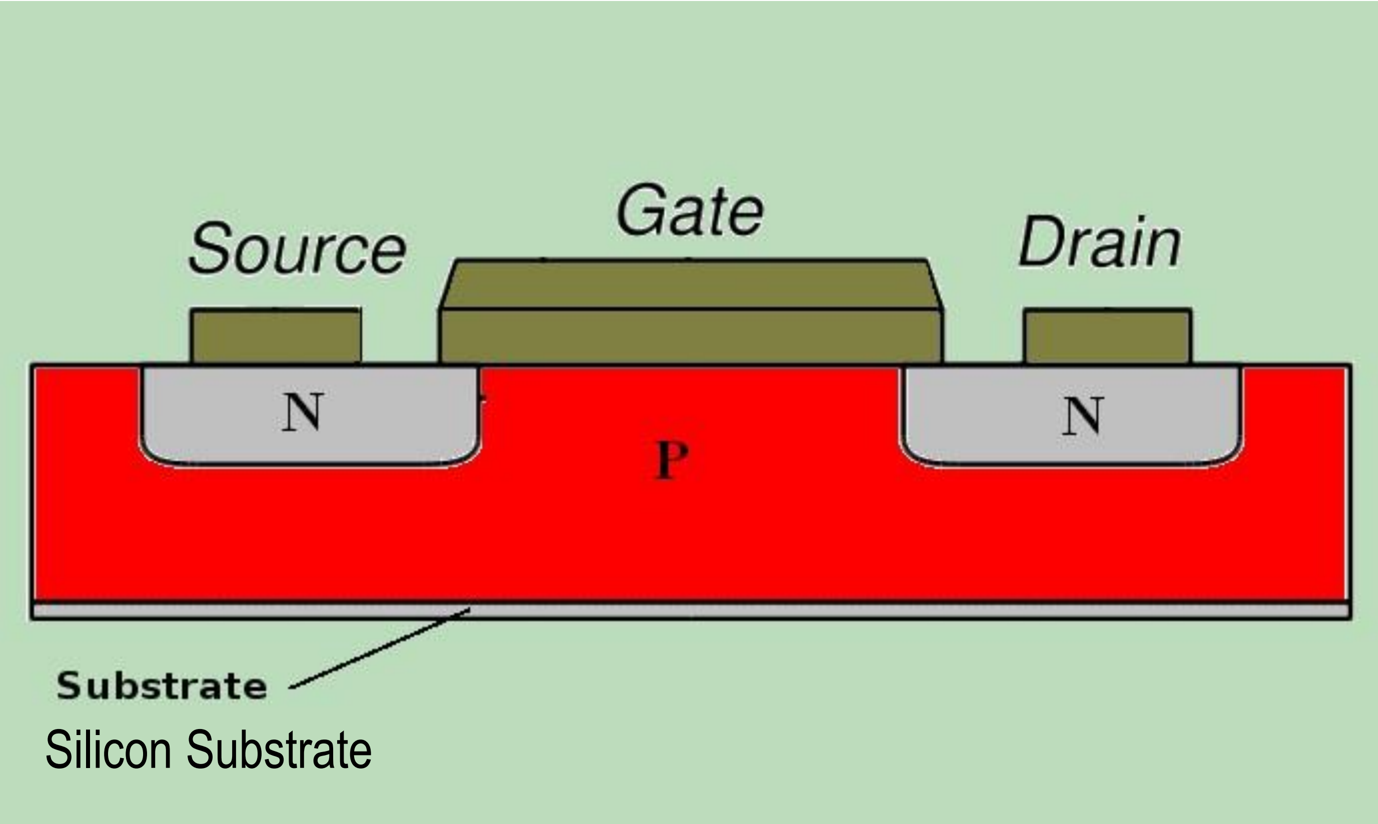
N = Negative

The two N-type semiconductor should communicate ….

Add a metal bar in between, named Gate

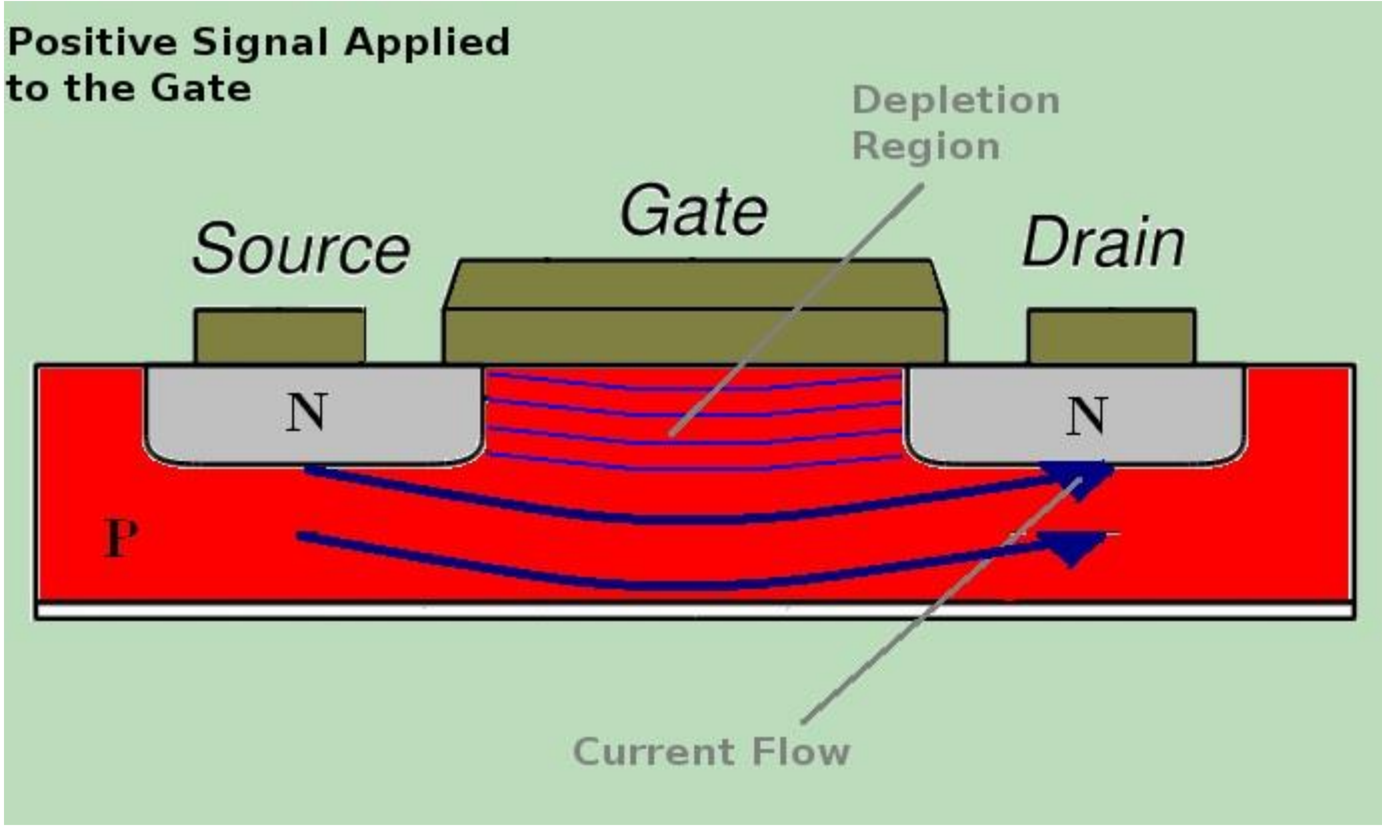


Add metal nodes … source, drain, gate



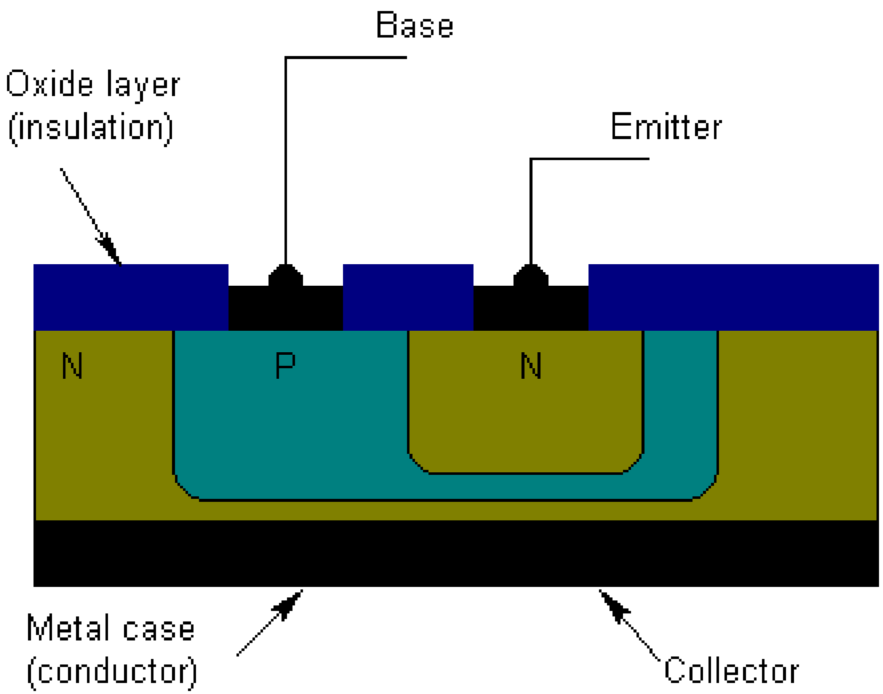
Source is the input … Drain is the Output …
We need to go from Source to Drain via the P-type

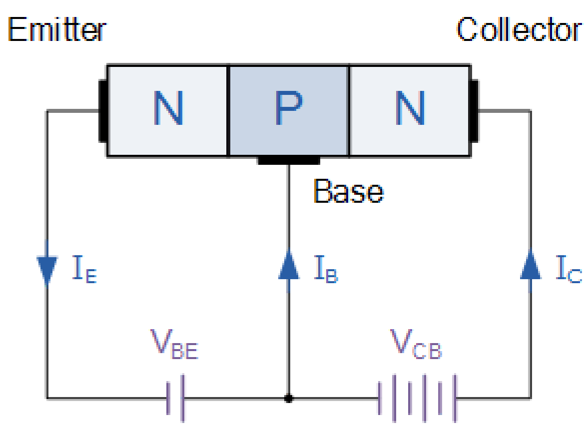
Apply Positive voltage to Gate …

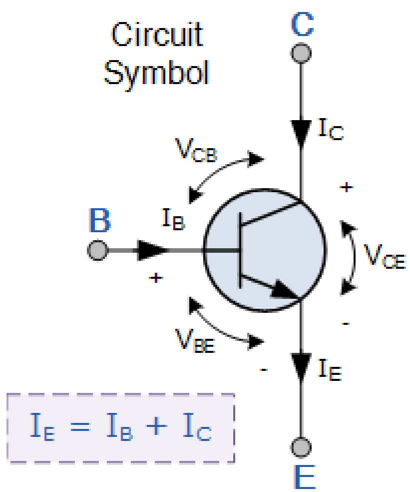


The Gate acts as a Switch … by applying voltage or not

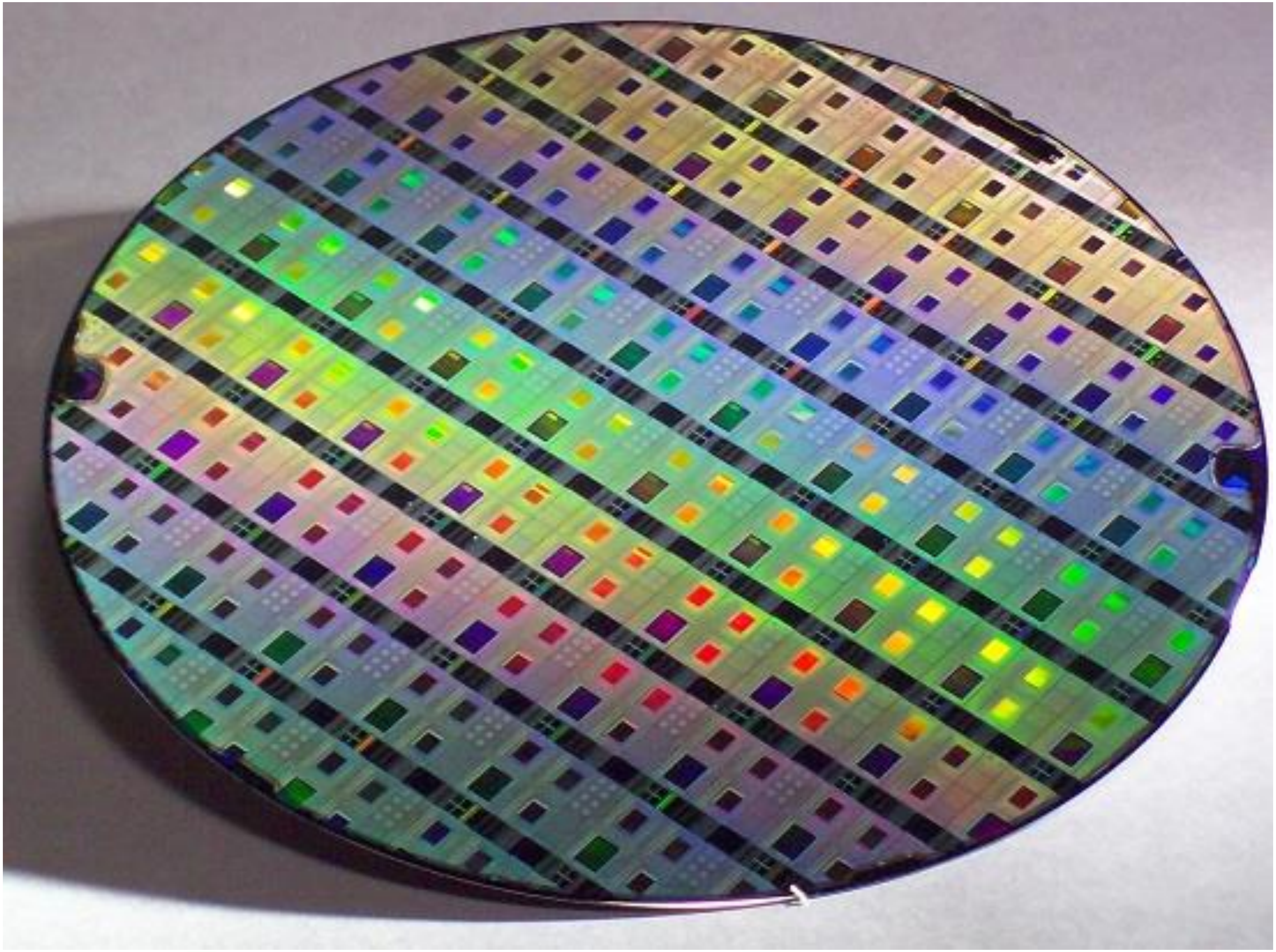
A single real NPN Transistor



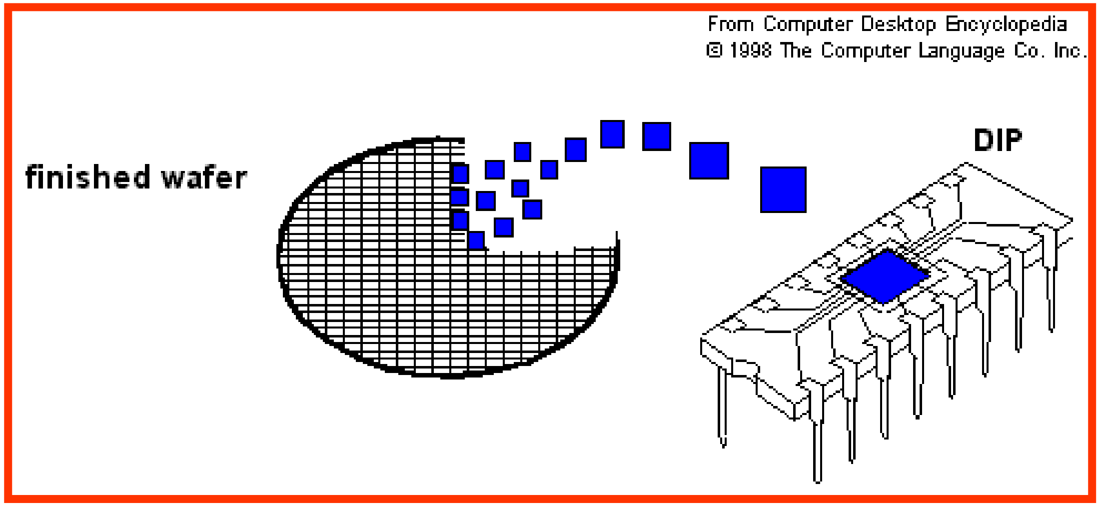


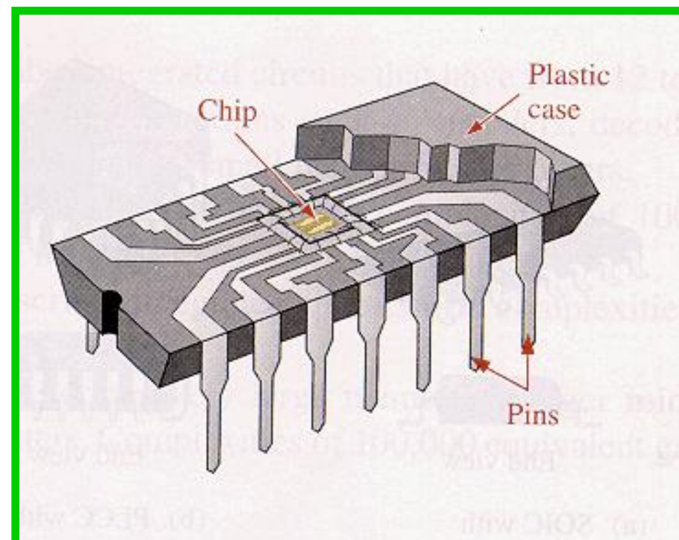


From silicon to chip…

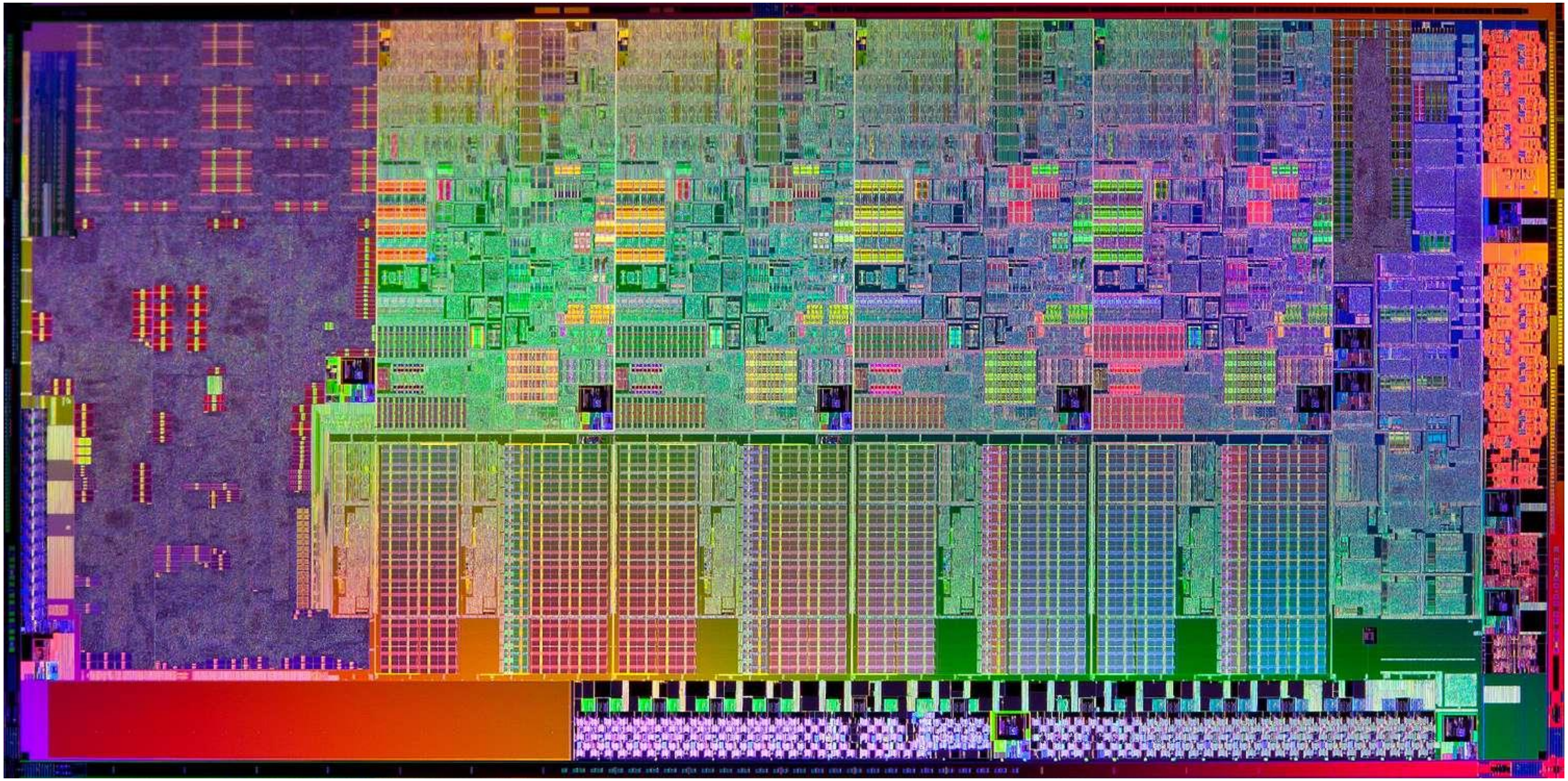


Chip + Housing (simplified view)





Today’s computer technology is based on Boolean algebra …



Intel

Basic Boolean Theorems (Rules)

Boolean Algebra

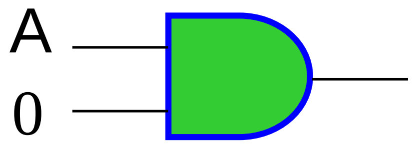
Boolean Theorems

- Single Variable: $f(A)$
- Multiple variable: $f(A,B,C,\ldots)$.

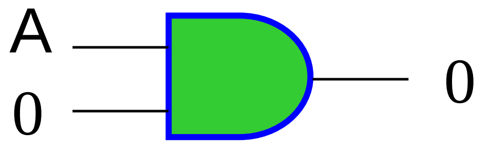
SingleVariable Boolean Theorems

$$f(A) = A \bullet 0$$

Operation with zero (1); A•0 = ?

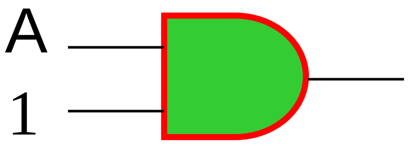


Operation with zero (1); A•0 = 0

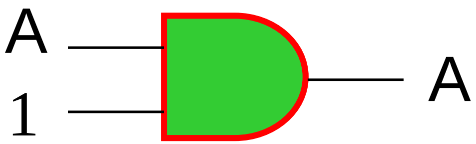


A	0	Output
0	0	0
1	0	0

Operation with one (2); A•1 = ?

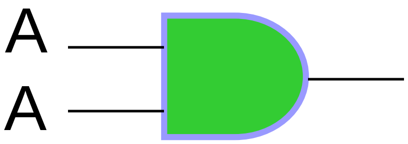


Operation with one (2); A•1 = A

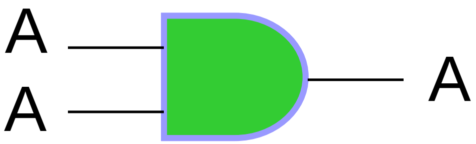


A	1	Output
0	1	0
1	1	1

Idempotent theorem (3) ; A•A = ?

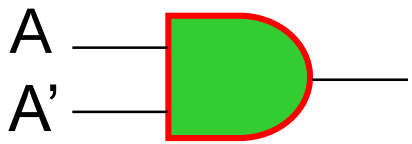


Idempotent theorem (3) ; A•A = A

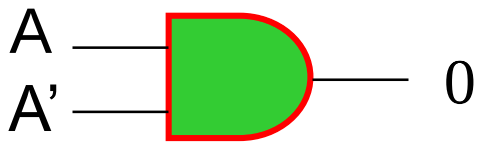


A	A	Output
0	0	0
1	1	1

Complementary (4) ; A•A’ = ?

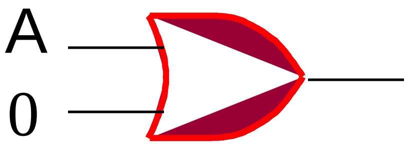


Complementary (4) ; A•A' = 0

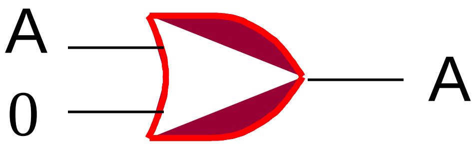


A	A'	Output
0	1	0
1	0	0

Operation with zero (5) ; A + 0 = ?

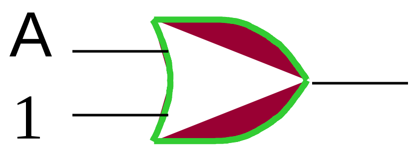


Operation with zero (5) ; A + 0 = A

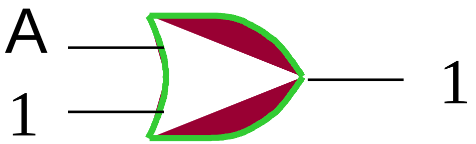


A	0	Output
0	0	0
1	0	1

Operation with one (6) ; A + 1 = ?

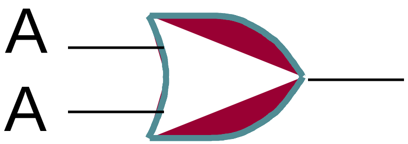


Operation with one (6) ; A + 1 = 1

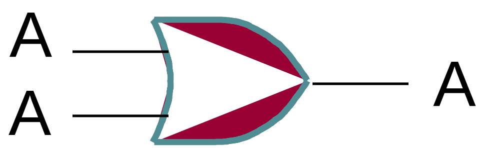


A	1	Output
0	1	1
1	1	1

Idempotent (7) ; A + A = ?

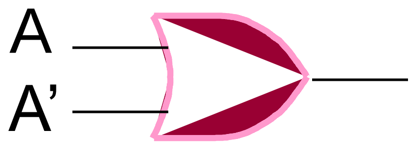


Idempotent (7) ; A + A = A

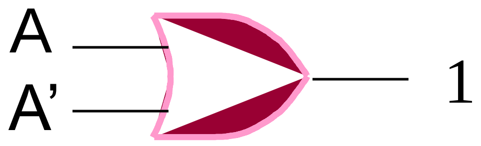


A	A	Output
0	0	0
1	1	1

Complementary (8) ; A + A’ = ?

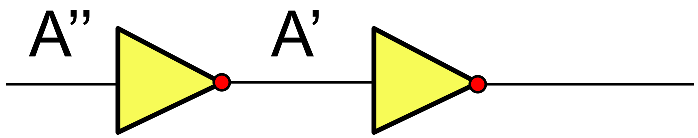


Complementary (8) ; A + A’ = 1

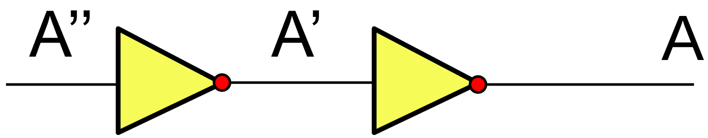


A	A’	Output
0	1	1
1	0	1

Involution theorem (9); A’’ = ?



Involution theorem (9); A'' = A



A''	A'	Output
0	1	0
1	0	1

The 9 basic Boolean theorems

- $A \cdot 0 = 0$
- $A \cdot 1 = A$
- $A \cdot A = A$
- $A \cdot A' = 0$
- $(A')' = A$
- $A + 0 = A$
- $A + 1 = 1$
- $A + A = A$
- $A + A' = 1$

$A' = \overline{A}$

MultiVariable Boolean theorems

$f(A,B) = A + B$

Multivariable theorems(1)

- Commutative Laws:
  - ❖ $A+B = B+A$
  - ❖ $A \bullet B = B \bullet A$

Multivariable theorems(2)

- Associative Laws:
  - ❖ $A+(B+C) = (A+B)+C = A+B+C$
  - ❖ $A \bullet (B \bullet C) = (A \bullet B) \bullet C = A \bullet B \bullet C$

Multivariable theorems(3)

- Distributed Law over Multiplication
  - $(D+A)\bullet(B+C) = D\bullet B+D\bullet C+A\bullet B+A\bullet C$
  - $A\bullet(B+C) = A\bullet B+A\bullet C$

Multivariable theorems(3)

- Distributed Law over Addition
  - ❖ $A+(B\bullet C) = (A+B)\bullet(A+C)$

  - **Is the above equality valid?**

It is not obvious…

- $A+(B\bullet C) \overset{?}{=} (A+B)\bullet(A+C)$

- Prove it … (5 minutes)

Proof … using the Boolean Theorems

A+(B●C) = (A+B)●(A+C)

Distribute

$A+(B \bullet C) = (A+B) \bullet (A+C)$

$= A \bullet A + A \bullet C + A \bullet B + B \bullet C$

$A \bullet A = A$

Factor-out A

$A+(B \bullet C) = (A+B) \bullet (A+C)$

$= A \bullet A + A \bullet C + A \bullet B + B \bullet C$

$= \quad A \quad + A \bullet C + A \bullet B + B \bullet C$

1+C=1

$$A+(B\bullet C) = (A+B)\bullet(A+C)$$

$$= A\bullet A+A\bullet C+A\bullet B+B\bullet C$$

$$= \quad A \quad +A\bullet C+A\bullet B+B\bullet C$$

$$= \quad A\bullet(1+C)+A\bullet B+B\bullet C$$

1+C=1

$A \bullet 1 = 1$

$$A+(B \bullet C) = (A+B) \bullet (A+C)$$

$$= A \bullet A + A \bullet C + A \bullet B + B \bullet C$$

$$= A + A \bullet C + A \bullet B + B \bullet C$$

$$= A \bullet (1+C) + A \bullet B + B \bullet C$$

$$= A \bullet 1 + A \bullet B + B \bullet C$$

$A \bullet 1 = 1$

Factor-out A

$A+(B\bullet C) = (A+B)\bullet(A+C)$

$= A\bullet A+A\bullet C+A\bullet B+B\bullet C$

$= A \quad +A\bullet C+A\bullet B+B\bullet C$

$= A\bullet(1+C)+A\bullet B+B\bullet C$

$= A\bullet 1 \qquad +A\bullet B+B\bullet C$

$= {\color{blue}A} \qquad {\color{blue}+A\bullet B}+B\bullet C$

1+B=1

$$A+(B\bullet C) = (A+B)\bullet(A+C)$$

$$= A\bullet A+A\bullet C+A\bullet B+B\bullet C$$

$$= A \quad +A\bullet C+A\bullet B+B\bullet C$$

$$= A\bullet(1+C)+A\bullet B+B\bullet C$$

$$= A\bullet 1 \quad +A\bullet B+B\bullet C$$

$$= A \quad +A\bullet B+B\bullet C$$

$$= A(1+B) \quad +B\bullet C$$

1+B=1

$A \bullet 1 = 1$

$$A+(B \bullet C) = (A+B) \bullet (A+C)$$

$$= A \bullet A + A \bullet C + A \bullet B + B \bullet C$$

$$= A + A \bullet C + A \bullet B + B \bullet C$$

$$= A \bullet (1+C) + A \bullet B + B \bullet C$$

$$= A \bullet 1 + A \bullet B + B \bullet C$$

$$= A + A \bullet B + B \bullet C$$

$$= A(1+B) + B \bullet C$$

$$= A \bullet 1 + B \bullet C$$

$A \bullet 1 = A$

Done …

$$A+(B\bullet C) = (A+B)\bullet(A+C)$$

$$= A\bullet A+A\bullet C+A\bullet B+B\bullet C$$

$$= A + A\bullet C+A\bullet B+B\bullet C$$

$$= A\bullet(1+C)+A\bullet B+B\bullet C$$

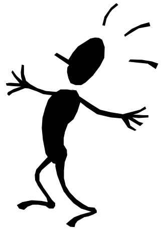
$$= A\bullet 1 + A\bullet B+B\bullet C$$

$$= A + A\bullet B+B\bullet C$$

$$= A(1+B) + B\bullet C$$

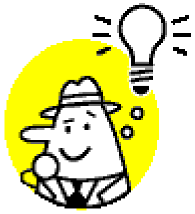
$$= A\bullet 1 + B\bullet C$$

$$= A + B\bullet C$$



$A+(B\bullet C) = (A+B)\bullet(A+C)$

Another way to prove the equation?



$A+(B\bullet C) = (A+B)\bullet(A+C)$

Set-up the truth table for the above expression

A	B	C	A+B	A+C	(A+B)(A+C)	BC	A+(BC)
0	0	0					
0	0	1					
0	1	0					
0	1	1					
1	0	0					
1	0	1					
1	1	0					
1	1	1					

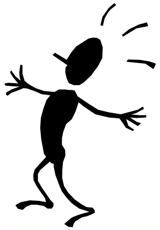
$A+(B \bullet C) = (A+B) \bullet (A+C)$

A	B	C	A+B	A+C	(A+B)(A+C)	BC	A+(BC)
0	0	0	0				
0	0	1	0				
0	1	0	1				
0	1	1	1				
1	0	0	1				
1	0	1	1				
1	1	0	1				
1	1	1	1				

$A+(B \bullet C) = (A+B) \bullet (A+C)$

A	B	C	A+B	A+C	(A+B)(A+C)	BC	A+(BC)
0	0	0	0	0	0	0	0
0	0	1	0	1	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	1	1	1	1
1	0	0	1	1	1	0	1
1	0	1	1	1	1	0	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

Perfect induction



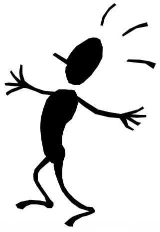
New formula (F-1)

○ $A+A \bullet B = A$ or $A+AB = A$

- Proof …

New formula (F-1)

- $A+A \bullet B = A$
- $A \bullet (1+B)$
- $A \bullet 1$
- A



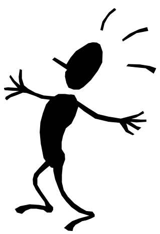
New formula (F-2)

- $A' = A' + A' \bullet B$

- Proof

New formula (F-2)

- $A' = A' + A' \bullet B$
- $\quad = A' \bullet (1+B)$
- $\quad = A'$



More formulas

- $A+A'\bullet B = A+B$ (F-3)
- $A'+A\bullet B = A'+B$ (F-4)
- $A\bullet(A+B) = A$ (F-5)

Let us proof the above 3 formulas

$A + A'\bullet B = A + B$;   (F-3 Proof)

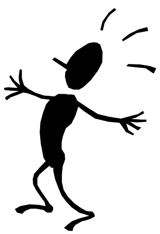
$A + A'B = \ldots$

$A + A'B = A + AB + A'B$ $\quad (A = A + AB)$

$= A + B(A + A')$ $\quad (A + A' = 1)$

$= A + B$

$\boxed{AB = A\bullet B}$



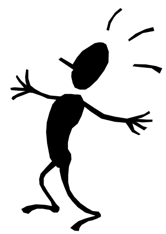
$A' + A \bullet B = A' + B$;   (F-4 proof)

$A' + AB = \ldots$

$A' + AB = A' + A'B + AB$ $(A' = A' + A'B)$

$= A' + B(A' + A)$ $(A + A' = 1)$

$= A' + B$



$A \bullet (A+B) = A$;   (F-5 proof)

$A(A+B) = AA + AB$ (distribute)

$= A + AB$ ($AA = A$)

$= A(1+B)$ (factor-out A)

$= A\,1$ ($1+B = 1$)

$= A$