1. In each case we merely have to compute the expression on the right mod 13. This means dividing it by 13 and taking the (nonnegative) remainder.
   a. $9 \cdot 4 \bmod 13 = 36 \bmod 13 = 10$
   b. $11 \cdot 9 \bmod 13 = 99 \bmod 13 = 8$
   c. $4 + 9 \bmod 13 = 13 \bmod 13 = 0$
   d. $2 \cdot 4 + 3 \cdot 9 \bmod 13 = 35 \bmod 13 = 9$
   e. 4^2 + 9^2 mod 13 = 97 mod 13 = 6
   f. 4^3 - 9^3 mod 13 = -665 mod 13 = 11 (because $-665 = -52 \cdot 13 + 11$)

2. a) Working modulo 23, we have $-133 + 261 = 128 = 13$, so the answer is 13.

   b) Working modulo 23, we have $457 \cdot 182 = 20 \cdot 21 = 420 = 6$.

3. In each case we can use trial division, starting with the smallest prime and increasing to the next prime once we find that a given prime no longer is a divisor of what is left. A calculator comes in handy. Alternatively, one could use a factor tree.

a) We note that 2 is a factor of 88, and the quotient upon division by 2 is 44. We divide by 2 again, and then again, leaving a quotient of 11. Since 11 is prime, we are done, and we have found the prime factorization: $88 = 2^3 \cdot 11$.

b) $126 = 2 \cdot 63 = 2 \cdot 3 \cdot 21 = 2 \cdot 3 \cdot 3 \cdot 7 = 2 \cdot 3^2 \cdot 7$

c) $729 = 3 \cdot 243 = 3 \cdot 3 \cdot 81 = 3 \cdot 3 \cdot 3 \cdot 27 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 9 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^6$

d) $1001 = 7 \cdot 143 = 7 \cdot 11 \cdot 13$

e) $1111 = 11 \cdot 101$ (we know that 101 is prime because we have already tried all prime factors less than $\sqrt{101}$)

f) $909090 = 2 \cdot 454545 = 2 \cdot 3 \cdot 151515 = 2 \cdot 3 \cdot 3 \cdot 50505 = 2 \cdot 3 \cdot 3 \cdot 3 \cdot 16835 = 2 \cdot 3 \cdot 3 \cdot 3 \cdot 5 \cdot 3367 = 2 \cdot 3 \cdot 3 \cdot 3 \cdot 5 \cdot 7 \cdot 481 = 2 \cdot 3 \cdot 3 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 37 = 2 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13 \cdot 37$

4.

To find the greatest common divisor of two numbers whose prime factorizations are given, we just need to take the smaller exponent for each prime.

a) The first number has no prime factors of 2, so the gcd has no 2's. Since the first number has seven factors of 3, but the second number has only five, the gcd has five factors of 3. Similarly the gcd has a factor of 5^3. So the gcd is $3^5 \cdot 5^3$.

b) These numbers have no common prime factors, so the gcd is 1. **c)** 23^{17} **d)** $41 \cdot 43 \cdot 53$

e) These numbers have no common prime factors, so the gcd is 1.

f) The gcd of any positive integer and 0 is that integer, so the answer is 1111.

5.

We follow the hint. Adding 6 to both sides gives the equivalent congruence $15x^2 + 19x + 6 \equiv 0 \pmod{11}$, because $5 + 6 = 11 \equiv 0 \pmod{11}$. This factors as $(5x + 3)(3x + 2) \equiv 0 \pmod{11}$. Because there are no non-zero divisors of 0 working modulo 11, we conclude that the solutions are precisely the solutions of $5x + 3 \equiv 0 \pmod{11}$ and $3x + 2 \equiv 0 \pmod{11}$. We solve these by the method of Example 3. By inspection (trial-and-error) or working it out through the Euclidean algorithm and back-substituting, we find that an inverse of 5 modulo 11 is 9, and multiplying both sides of $5x+3 \equiv 0 \pmod{11}$ by 9 yields $x+27 \equiv 0 \pmod{11}$, so $x \equiv -27 \equiv 6 \pmod{11}$. Similarly, an inverse of 3 modulo 11 is 4, and we get $x \equiv -8 \equiv 3 \pmod{11}$. So the solution set is $\{3, 6\}$ (and anything congruent to these modulo 11). Plugging these values into the original equation to check, we have $15 \cdot 3^2 + 19 \cdot 3 + 6 = 198 \equiv 0 \pmod{11}$ and $15 \cdot 6^2 + 19 \cdot 6 + 6 = 660 \equiv 0 \pmod{11}$.

6.

a) We need to compute k **mod** 31 in each case. A good way to do this on a calculator is as follows. Enter k and divide by 31. The result will be a number with an integer part and a decimal fractional part. Subtract off the integer part, leaving a decimal fraction between 0 and 1. This is the remainder expressed as a decimal. To find out what whole number remainder that really represents, multiply by 31. The answer will be a whole number (or nearly so—it may require rounding, say from 4.9999 or 5.0001 to 5), and that number is k **mod** 31.

(i) 317 **mod** $31 = 7$ (ii) 918 **mod** $31 = 19$ (iii) 007 **mod** $31 = 7$

(iv) 100 **mod** $31 = 7$ (v) 111 **mod** $31 = 18$ (vi) 310 **mod** $31 = 0$

b) Take the next available space, where the next space is computed by adding 1 to the space number and pretending that $30 + 1 = 0$.