1. The product rule applies here, since a flight is determined by choosing an airline for the flight from New York to Denver (which can be done in 6 ways) and then choosing an airline for the flight from Denver to San Francisco (which can be done in 7 ways). Therefore there are $6 \cdot 7 = 42$ different possibilities for the entire flight.

2. 4 by pigeonhole principle

3. By symmetry we need prove only the first statement. Let A be one of the people. Either A has at least four friends, or A has at least six enemies among the other nine people (because $3 + 5 < 9$). Suppose, in the first case, that B, C, D, and E are all A's friends. If any two of these are friends with each other, then we have found three mutual friends. Otherwise $\{B, C, D, E\}$ is a set of four mutual enemies. In the second case, let $\{B, C, D, E, F, G\}$ be a set of enemies of A. By Example 11, among B, C, D, E, F, and G there are either three mutual friends or three mutual enemies, who form, with A, a set of four mutual enemies.

4.

a) To specify a bit string of length 10 that contains exactly four 1's, we simply need to choose the four positions that contain the 1's. There are $C(10,4) = 210$ ways to do that.

b) To contain at most four 1's means to contain four 1's, three 1's, two 1's, one 1, or no 1's. Reasoning as in part **(a)**, we see that there are $C(10,4)+C(10,3)+C(10,2)+C(10,1)+C(10,0) = 210+120+45+10+1 = 386$ such strings.

c) To contain at least four 1's means to contain four 1's, five 1's, six 1's, seven 1's, eight 1's, nine 1's, or ten 1's. Reasoning as in part **(b)**, we see that there are $C(10,4) + C(10,5) + C(10,6) + C(10,7) + C(10,8) + C(10,9) + C(10,10) = 210 + 252 + 210 + 120 + 45 + 10 + 1 = 848$ such strings. A simpler approach would be to figure out the number of ways not to have at least four 1's (i.e., to have three 1's, two 1's, one 1, or no 1's) and then subtract that from 2^{10}, the total number of bit strings of length 10. This way we get $1024 - (120 + 45 + 10 + 1) = 848$, fortunately the same answer as before. Solving a combinatorial problem in more than one way is a useful check on the correctness of the answer.

d) To have an equal number of 0's and 1's in this case means to have five 1's. Therefore the answer is $C(10,5) = 252$. Incidentally, this gives us another way to do part **(b)**. If we don't have an equal number of 0's and 1's, then we have either at most four 1's or at least six 1's. By symmetry, having at most four 1's occurs in half of these cases. Therefore the answer to part **(b)** is $(2^{10} - C(10,5))/2 = 386$, as above.

5.

a) Each flip can be either heads or tails, so there are $2^{10} = 1024$ possible outcomes.

b) To specify an outcome that has exactly two heads, we simply need to choose the two flips that came up heads. There are $C(10,2) = 45$ such outcomes.

c) To contain at most three tails means to contain three tails, two tails, one tail, or no tails. Reasoning as in part **(b)**, we see that there are $C(10,3) + C(10,2) + C(10,1) + C(10,0) = 120 + 45 + 10 + 1 = 176$ such outcomes.

d) To have an equal number of heads and tails in this case means to have five heads. Therefore the answer is $C(10,5) = 252$.

6.

We need to be careful here, because strings can have repeated letters.

a) We need to choose the position for the vowel, and this can be done in 6 ways. Next we need to choose the vowel to use, and this can be done in 5 ways. Each of the other five positions in the string can contain any of the 21 consonants, so there are 21^5 ways to fill the rest of the string. Therefore the answer is $6 \cdot 5 \cdot 21^5 = 122{,}523{,}030$.

b) We need to choose the position for the vowels, and this can be done in $C(6,2) = 15$ ways (we need to choose two positions out of six). We need to choose the two vowels (5^2 ways). Each of the other four positions in the string can contain any of the 21 consonants, so there are 21^4 ways to fill the rest of the string. Therefore the answer is $15 \cdot 5^2 \cdot 21^4 = 72{,}930{,}375$.

c) The best way to do this is to count the number of strings with no vowels and subtract this from the total number of strings. We obtain $26^6 - 21^6 = 223{,}149{,}655$.

d) As in part **(c)**, we will do this by subtracting from the total number of strings, the number of strings with no vowels and the number of strings with one vowel (this latter quantity having been computed in part **(a)**). We obtain $26^6 - 21^6 - 6 \cdot 5 \cdot 21^5 = 223149655 - 122523030 = 100{,}626{,}625$.