HW #1 Solution

From Section 1.1

1. Propositions must have clearly defined truth values, so a proposition must be a declarative sentence with no free variables.
a) This is a true proposition.
b) This is a false proposition (Tallahassee is the capital).
c) This is a true proposition.
d) This is a false proposition.
e) This is not a proposition (it contains a variable; the truth value depends on the value assigned to x).
f) This is not a proposition, since it does not assert anything.

2. In each case, we simply need to determine the truth value of the hypothesis and the conclusion, and then use the definition of the truth value of the conditional statement. The conditional statement is true in every case except when the hypothesis (the "if" part) is true and the conclusion (the "then" part) is false.
a) Since the hypothesis is true and the conclusion is false, this conditional statement is false.
b) Since the hypothesis is false and the conclusion is true, this conditional statement is true.
c) Since the hypothesis is false and the conclusion is false, this conditional statement is true. Note that the conditional statement is false in both part (b) and part (c); as long as the hypothesis is false, we need look no further to conclude that the conditional statement is true.
d) Since the hypothesis is false, this conditional statement is true.

3.
The techniques are the same as in Exercises 31–36, except that there are now three variables and therefore eight rows. For part **(a)**, we have

p	q	r	$\neg q$	$\neg q \vee r$	$p \to (\neg q \vee r)$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	T	F	F	F	T
F	F	T	T	T	T
F	F	F	T	T	T

For part **(b)**, we have

p	q	r	$\neg p$	$q \to r$	$\neg p \to (q \to r)$
T	T	T	F	T	T
T	T	F	F	F	T
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

Parts **(c)** and **(d)** we can combine into a single table.

p	q	r	$p \to q$	$\neg p$	$\neg p \to r$	$(p \to q) \vee (\neg p \to r)$	$(p \to q) \wedge (\neg p \to r)$
T	T	T	T	F	T	T	T
T	T	F	T	F	T	T	T
T	F	T	F	F	T	T	F
T	F	F	F	F	T	T	F
F	T	T	T	T	T	T	T
F	T	F	T	T	F	T	F
F	F	T	T	T	T	T	T
F	F	F	T	T	F	T	F

For part **(e)** we have

p	q	r	$p \leftrightarrow q$	$\neg q$	$\neg q \leftrightarrow r$	$(p \leftrightarrow q) \vee (\neg q \leftrightarrow r)$
T	T	T	T	F	F	T
T	T	F	T	F	T	T
T	F	T	F	T	T	T
T	F	F	F	T	F	F
F	T	T	F	F	F	F
F	T	F	F	F	T	T
F	F	T	T	T	T	T
F	F	F	T	T	F	T

Finally, for part **(f)** we have

p	q	r	$\neg p$	$\neg q$	$\neg p \leftrightarrow \neg q$	$q \leftrightarrow r$	$(\neg p \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow r)$
T	T	T	F	F	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	T	F	F	T
T	F	F	F	T	F	T	F
F	T	T	T	F	F	T	F
F	T	F	T	F	F	F	T
F	F	T	T	T	T	F	F
F	F	F	T	T	T	T	T

4.

We construct a truth table for each conditional statement and note that the relevant column contains only T's. For parts **(a)** and **(b)** we have the following table (column four for part **(a)**, column six for part **(b)**).

p	q	$p \wedge q$	$(p \wedge q) \to p$	$p \vee q$	$p \to (p \vee q)$
T	T	T	T	T	T
T	F	F	T	T	T
F	T	F	T	T	T
F	F	F	T	F	T

For parts **(c)** and **(d)** we have the following table (columns five and seven, respectively).

p	q	$\neg p$	$p \to q$	$\neg p \to (p \to q)$	$p \wedge q$	$(p \wedge q) \to (p \to q)$
T	T	F	T	T	T	T
T	F	F	F	T	F	T
F	T	T	T	T	F	T
F	F	T	T	T	F	T

For parts **(e)** and **(f)** we have the following table. Column five shows the answer for part **(e)**, and column seven shows the answer for part **(f)**.

p	q	$p \to q$	$\neg(p \to q)$	$\neg(p \to q) \to p$	$\neg q$	$\neg(p \to q) \to \neg q$
T	T	T	F	T	F	T
T	F	F	T	T	T	T
F	T	T	F	T	F	T
F	F	T	F	T	T	T

5. This fact was observed in Section 1.1 when the biconditional was first defined. Each of these is true precisely when p and q have the same truth values.

6.

To show that these are *not* logically equivalent, we need only find one assignment of truth values to p, q, and r for which the truth values of $(p \to q) \to r$ and $p \to (q \to r)$ differ. One such assignment is F for all three. Then $(p \to q) \to r$ is false and $p \to (q \to r)$ is true.

Part 2

1.

We will show that if $p \to q$ and $q \to r$ are both true, then $p \to r$ is true. Thus we want to show that if p is true, then so is r. Given that p and $p \to q$ are both true, we conclude that q is true; from that and $q \to r$ we conclude that r is true, as desired. This can also be done with a truth table.

2.

a) There is a student who spends more than five hours every weekday in class.
b) Every student spends more than five hours every weekday in class.
c) There is a student who does not spend more than five hours every weekday in class.
d) No student spends more than five hours every weekday in class. (Or, equivalently, every student spends less than or equal to five hours every weekday in class.)

3.

a) Since adding 1 to a number makes it larger, this is true.
b) Since $2 \cdot 0 = 3 \cdot 0$, this is true.
c) This statement is true, since $0 = -0$.
d) This is true for the nonnegative integers but not for the negative integers. For example, $3(-2) \not\leq 4(-2)$. Therefore the universally quantified statement is false.

4.

Existential quantifiers are like disjunctions, and universal quantifiers are like conjunctions. See Examples 11 and 16.

a) We want to assert that $P(x)$ is true for some x in the universe, so either $P(1)$ is true or $P(2)$ is true or $P(3)$ is true or $P(4)$ is true or $P(5)$ is true. Thus the answer is $P(1) \vee P(2) \vee P(3) \vee P(4) \vee P(5)$.

b) $P(1) \wedge P(2) \wedge P(3) \wedge P(4) \wedge P(5)$

c) This is just the negation of part **(a)**: $\neg(P(1) \vee P(2) \vee P(3) \vee P(4) \vee P(5))$

d) This is just the negation of part **(b)**: $\neg(P(1) \wedge P(2) \wedge P(3) \wedge P(4) \wedge P(5))$

e) The formal translation is as follows: $(((1 \neq 3) \rightarrow P(1)) \wedge ((2 \neq 3) \rightarrow P(2)) \wedge ((3 \neq 3) \rightarrow P(3)) \wedge ((4 \neq 3) \rightarrow P(4)) \wedge ((5 \neq 3) \rightarrow P(5))) \vee (\neg P(1) \vee \neg P(2) \vee \neg P(3) \vee \neg P(4) \vee \neg P(5))$. However, since the hypothesis $x \neq 3$ is false when x is 3 and true when x is anything other than 3, we have more simply $(P(1) \wedge P(2) \wedge P(4) \wedge P(5)) \vee (\neg P(1) \vee \neg P(2) \vee \neg P(3) \vee \neg P(4) \vee \neg P(5))$. Thinking about it a little more, we note that this statement is always true, since if the first part is not true, then the second part must be true.

5.

a) This says that there exists a real number x such that for every real number y, the product xy equals y. That is, there is a multiplicative identity for the real numbers. This is a true statement, since $x = 1$ is the identity.

b) The product of two negative real numbers is always a positive real number.

c) There exist real numbers x and y such that x^2 exceeds y but x is less than y. This is true, since we can take $x = 2$ and $y = 3$, for instance.

d) This says that for every pair of real numbers x and y, there exists a real number z that is their sum. In other words, the real numbers are closed under the operation of addition, another true fact. (Some authors would include the uniqueness of z as part of the meaning of the word *closed*.)

6.

a) $P(1,1) \wedge P(1,2) \wedge P(1,3) \wedge P(2,1) \wedge P(2,2) \wedge P(2,3) \wedge P(3,1) \wedge P(3,2) \wedge P(3,3)$

b) $P(1,1) \vee P(1,2) \vee P(1,3) \vee P(2,1) \vee P(2,2) \vee P(2,3) \vee P(3,1) \vee P(3,2) \vee P(3,3)$

c) $(P(1,1) \wedge P(1,2) \wedge P(1,3)) \vee (P(2,1) \wedge P(2,2) \wedge P(2,3)) \vee (P(3,1) \wedge P(3,2) \wedge P(3,3))$

d) $(P(1,1) \vee P(2,1) \vee P(3,1)) \wedge (P(1,2) \vee P(2,2) \vee P(3,2)) \wedge (P(1,3) \vee P(2,3) \vee P(3,3))$

Note the crucial difference between parts **(c)** and **(d)**.

7.

As we push the negation symbol toward the inside, each quantifier it passes must change its type. For logical connectives we either use De Morgan's laws or recall that $\neg(p \rightarrow q) \equiv p \wedge \neg q$.

a)

$$\begin{aligned} \neg\forall x \exists y \forall z\, T(x,y,z) &\equiv \exists x \neg\exists y \forall z\, T(x,y,z) \\ &\equiv \exists x \forall y \neg\forall z\, T(x,y,z) \\ &\equiv \exists x \forall y \exists z\, \neg T(x,y,z) \end{aligned}$$

b)

$$\begin{aligned} \neg(\forall x \exists y\, P(x,y) \vee \forall x \exists y\, Q(x,y)) &\equiv \neg\forall x \exists y\, P(x,y) \wedge \neg\forall x \exists y\, Q(x,y) \\ &\equiv \exists x \neg\exists y\, P(x,y) \wedge \exists x \neg\exists y\, Q(x,y) \\ &\equiv \exists x \forall y\, \neg P(x,y) \wedge \exists x \forall y\, \neg Q(x,y) \end{aligned}$$

c)

$$
\begin{aligned}
\neg\forall x\exists y(P(x,y)\wedge\exists z\,R(x,y,z)) &\equiv \exists x\neg\exists y(P(x,y)\wedge\exists z\,R(x,y,z))\\
&\equiv \exists x\forall y\,\neg(P(x,y)\wedge\exists z\,R(x,y,z))\\
&\equiv \exists x\forall y(\neg P(x,y)\vee\neg\exists z\,R(x,y,z))\\
&\equiv \exists x\forall y(\neg P(x,y)\vee\forall z\,\neg R(x,y,z))
\end{aligned}
$$

d)

$$
\begin{aligned}
\neg\forall x\exists y(P(x,y)\rightarrow Q(x,y)) &\equiv \exists x\,\neg\exists y(P(x,y)\rightarrow Q(x,y))\\
&\equiv \exists x\forall y\,\neg(P(x,y)\rightarrow Q(x,y))\\
&\equiv \exists x\forall y(P(x,y)\wedge\neg Q(x,y))
\end{aligned}
$$