1.

a) We must prove the contrapositive: If n is odd, then n^3+5 is even. Assume that n is odd. Then we can write $n=2k+1$ for some integer k. Then $n^3+5=(2k+1)^3+5=8k^3+12k^2+6k+6=2(4k^3+6k^2+3k+3)$. Thus n^3+5 is two times some integer, so it is even.

b) Suppose that n^3+5 is odd and that n is odd. Since n is odd, and the product of odd numbers is odd, in two steps we see that n^3 is odd. But then subtracting we conclude that 5, being the difference of the two odd numbers n^3+5 and n^3, is even. This is not true. Therefore our supposition was wrong, and the proof by contradiction is complete.

2.

The difference of two squares can be factored: $a^2-b^2=(a+b)(a-b)$. If we can arrange for our given odd integer to equal $a+b$ and for $a-b$ to equal 1, then we will be done. But we can do this by letting a and b be the integers that straddle $n/2$. For example, if $n=11$, then we take $a=6$ and $b=5$. Specifically, if $n=2k+1$, then we let $a=k+1$ and $b=k$. Here, then, is our proof. Since n is odd, we can write $n=2k+1$ for some integer k. Then $(k+1)^2-k^2=k^2+2k+1-k^2=2k+1=n$. This expresses n as the difference of two squares.

3.

We give an proof by contraposition. The contrapositive of this statement is "If $1/x$ is rational, then x is rational" so we give a direct proof of this contrapositive. Note that since $1/x$ exists, we know that $x\neq 0$. If $1/x$ is rational, then by definition $1/x=p/q$ for some integers p and q with $q\neq 0$. Since $1/x$ cannot be 0 (if it were, then we'd have the contradiction $1=x\cdot 0$ by multiplying both sides by x), we know that $p\neq 0$. Now $x=1/(1/x)=1/(p/q)=q/p$ by the usual rules of algebra and arithmetic. Hence x can be written as the quotient of two integers with the denominator nonzero. Thus by definition, x is rational.

4.

a) Plugging in $n=1$ we have that $P(1)$ is the statement $1^2=1\cdot 2\cdot 3/6$.

b) Both sides of $P(1)$ shown in part **(a)** equal 1.

c) The inductive hypothesis is the statement that

$$1^2+2^2+\cdots+k^2=\frac{k(k+1)(2k+1)}{6}.$$

d) For the inductive step, we want to show for each $k\geq 1$ that $P(k)$ implies $P(k+1)$. In other words, we want to show that assuming the inductive hypothesis (see part **(c)**) we can show

$$1^2+2^2+\cdots+k^2+(k+1)^2=\frac{(k+1)(k+2)(2k+3)}{6}.$$

e) The left-hand side of the equation in part **(d)** equals, by the inductive hypothesis, $k(k+1)(2k+1)/6+(k+1)^2$. We need only do a bit of algebraic manipulation to get this expression into the desired form: factor out $(k+1)/6$ and then factor the rest. In detail,

$$\begin{aligned}(1^2+2^2+\cdots+k^2)+(k+1)^2&=\frac{k(k+1)(2k+1)}{6}+(k+1)^2 \quad \text{(by the inductive hypothesis)}\\ &=\frac{k+1}{6}(k(2k+1)+6(k+1))=\frac{k+1}{6}(2k^2+7k+6)\\ &=\frac{k+1}{6}(k+2)(2k+3)=\frac{(k+1)(k+2)(2k+3)}{6}.\end{aligned}$$

f) We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer n.

5.

We proceed by induction. The basis step, $n = 0$, is true, since $1^2 = 1 \cdot 1 \cdot 3/3$. For the inductive step assume the inductive hypothesis that

$$1^2 + 3^2 + 5^2 + \cdots + (2k+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}.$$

We want to show that

$$1^2 + 3^2 + 5^2 + \cdots + (2k+1)^2 + (2k+3)^2 = \frac{(k+2)(2k+3)(2k+5)}{3}$$

(the right-hand side is the same formula with $k+1$ plugged in for n). Now the left-hand side equals, by the inductive hypothesis, $(k+1)(2k+1)(2k+3)/3 + (2k+3)^2$. We need only do a bit of algebraic manipulation to get this expression into the desired form: factor out $(2k+3)/3$ and then factor the rest. In detail,

$$\begin{aligned}
&\left(1^2 + 3^2 + 5^2 + \cdots + (2k+1)^2\right) + (2k+3)^2 \\
&\qquad = \frac{(k+1)(2k+1)(2k+3)}{3} + (2k+3)^2 \quad \text{(by the inductive hypothesis)} \\
&\qquad = \frac{2k+3}{3}\left((k+1)(2k+1) + 3(2k+3)\right) = \frac{2k+3}{3}(2k^2 + 9k + 10) \\
&\qquad = \frac{2k+3}{3}\left((k+2)(2k+5)\right) = \frac{(k+2)(2k+3)(2k+5)}{3}.
\end{aligned}$$

6.

a) $P(2)$ is the statement that $1 + \frac{1}{4} < 2 - \frac{1}{2}$.

b) This is true because $5/4$ is less than $6/4$.

c) The inductive hypothesis is the statement that

$$1 + \frac{1}{4} + \cdots + \frac{1}{k^2} < 2 - \frac{1}{k}.$$

d) For the inductive step, we want to show for each $k \geq 2$ that $P(k)$ implies $P(k+1)$. In other words, we want to show that assuming the inductive hypothesis (see part **(c)**) we can show

$$1 + \frac{1}{4} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}.$$

e) Assume the inductive hypothesis. Then we have

$$\begin{aligned}
1 + \frac{1}{4} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} &< 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \\
&= 2 - \left(\frac{1}{k} - \frac{1}{(k+1)^2}\right) \\
&= 2 - \left(\frac{k^2 + 2k + 1 - k}{k(k+1)^2}\right) \\
&= 2 - \frac{k^2 + k}{k(k+1)^2} - \frac{1}{k(k+1)^2} \\
&= 2 - \frac{1}{k+1} - \frac{1}{k(k+1)^2} < 2 - \frac{1}{k+1}.
\end{aligned}$$

f) We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer n greater than 1.

7.

a) $P(8)$ is true, because we can form 8 cents of postage with one 3-cent stamp and one 5-cent stamp. $P(9)$ is true, because we can form 9 cents of postage with three 3-cent stamps. $P(10)$ is true, because we can form 10 cents of postage with two 5-cent stamps.

b) The inductive hypothesis is the statement that using just 3-cent and 5-cent stamps we can form j cents postage for all j with $8 \leq j \leq k$, where we assume that $k \geq 10$.

c) In the inductive step we must show, assuming the inductive hypothesis, that we can form $k+1$ cents postage using just 3-cent and 5-cent stamps.

d) We want to form $k+1$ cents of postage. Since $k \geq 10$, we know that $P(k-2)$ is true, that is, that we can form $k-2$ cents of postage. Put one more 3-cent stamp on the envelope, and we have formed $k+1$ cents of postage, as desired.