HW #3 Solution

1.

a) The expression $1/x$ is meaningless for $x = 0$, which is one of the elements in the domain; thus the "rule" is no rule at all. In other words, $f(0)$ is not defined.
b) Things like $\sqrt{-3}$ are undefined (or, at best, are complex numbers).
c) The "rule" for f is ambiguous. We must have $f(x)$ defined uniquely, but here there are two values associated with every x, the positive square root and the negative square root of $x^2 + 1$.

2.

The floor function rounds down and the ceiling function rounds up.
a) 1 **b)** 0 **c)** 0 **d)** -1 **e)** 3 **f)** -1 **g)** $\lfloor \frac{1}{2} + \lceil \frac{3}{2} \rceil \rfloor = \lfloor \frac{1}{2} + 2 \rfloor = \lfloor 2\frac{1}{2} \rfloor = 2$
h) $\lfloor \frac{1}{2} \lfloor \frac{5}{2} \rfloor \rfloor = \lfloor \frac{1}{2} \cdot 2 \rfloor = \lfloor 1 \rfloor = 1$ if you finish (h) as "3/2" in the homework pdf, then the answer is 0

3.

Obviously there are an infinite number of correct answers to each part. The problem asked for a "formula." Parts **(a)** and **(c)** seem harder here, since we somehow have to fold the negative integers into the positive ones without overlap. Therefore we probably want to treat the negative integers differently from the positive integers. One way to do this with a formula is to make it a two-part formula. If one objects that this is not "a formula," we can counter as follows. Consider the function $g(x) = \lfloor 2^x \rfloor / 2^x$. Clearly if $x \geq 0$, then 2^x is a positive integer, so $g(x) = 2^x/2^x = 1$. If $x < 0$, then 2^x is a number between 0 and 1, so $g(x) = 0/2^x = 0$. If we want to define a function that has the value $f_1(x)$ when $x \geq 0$ and $f_2(x)$ when $x < 0$, then we can use the formula $g(x) \cdot f_1(x) + (1 - g(x)) \cdot f_2(x)$.
a) We could map the positive integers (and 0) into the positive multiples of 3, say, and the negative integers into numbers that are 1 greater than a multiple of 3, in a one-to-one manner. This will give us a function that leaves some elements out of the range. So let us define our function as follows:

$$f(x) = \begin{cases} 3x + 3 & \text{if } x \geq 0 \\ 3|x| + 1 & \text{if } x < 0 \end{cases}$$

The values of f on the inputs 0 through 4 are then $3, 6, 9, 12, 15$; and the values on the inputs -1 to -4 are $4, 7, 10, 13$. Clearly this function is one-to-one, but it is not onto since, for example, 2 is not in the range.
b) This is easier. We can just take $f(x) = |x| + 1$. It is clearly onto, but $f(n)$ and $f(-n)$ have the same value for every positive integer n, so f is not one-to-one.
c) This is similar to part **(a)**, except that we have to be careful to hit all values. Mapping the nonnegative integers to the odds and the negative integers to the evens will do the trick:

$$f(x) = \begin{cases} 2x + 1 & \text{if } x \geq 0 \\ 2|x| & \text{if } x < 0 \end{cases}$$

d) Here we can use a trivial example like $f(x) = 17$ or a simple nontrivial one like $f(x) = x^2 + 1$. Clearly these are neither one-to-one nor onto.

4.

a) One way to determine whether a function is a bijection is to try to construct its inverse. This function is a bijection, since its inverse (obtained by solving $y = 2x + 1$ for x) is the function $g(y) = (y-1)/2$. Alternatively, we can argue directly. To show that the function is one-to-one, note that if $2x+1 = 2x'+1$, then $x = x'$. To show that the function is onto, note that $2((y-1)/2)+1 = y$, so every number is in the range.

b) This function is not a bijection, since its range is the set of real numbers greater than or equal to 1 (which is sometimes written $[1, \infty)$), not all of $\mathbf{R}$. (It is not injective either.)

c) This function is a bijection, since it has an inverse function, namely the function $f(y) = y^{1/3}$ (obtained by solving $y = x^3$ for x).

d) This function is not a bijection. It is easy to see that it is not injective, since x and $-x$ have the same image, for all real numbers x. A little work shows that the range is only $\{y \mid 0.5 \le y < 1\} = [0.5, 1)$.

5.

To establish the setting here, let us suppose that $g : A \to B$ and $f : B \to C$. Then $f \circ g : A \to C$. We are told that f and $f \circ g$ are onto. Thus all of C gets "hit" by the images of elements of B; in fact, each element in C gets hit by an element from A under the composition $f \circ g$. But this does not seem to tell us anything about the elements of B getting hit by the images of elements of A. Indeed, there is no reason that they must. For a simple counterexample, suppose that $A = \{a\}$, $B = \{b_1, b_2\}$, and $C = \{c\}$. Let $g(a) = b_1$, and let $f(b_1) = c$ and $f(b_2) = c$. Then clearly f and $f \circ g$ are onto, but g is not, since b_2 is not in its range.

6.

In each case we simply evaluate the given function at $n = 0, 1, 2, 3$.

a) $a_0 = 2^0 + 1 = 2$, $a_1 = 2^1 + 1 = 3$, $a_2 = 2^2 + 1 = 5$, $a_3 = 2^3 + 1 = 9$

b) $a_0 = 1^1 = 1$, $a_1 = 2^2 = 4$, $a_2 = 3^3 = 27$, $a_3 = 4^4 = 256$

c) $a_0 = \lfloor 0/2 \rfloor = 0$, $a_1 = \lfloor 1/2 \rfloor = 0$, $a_2 = \lfloor 2/2 \rfloor = 1$, $a_3 = \lfloor 3/2 \rfloor = 1$

d) $a_0 = \lfloor 0/2 \rfloor + \lceil 0/2 \rceil = 0 + 0 = 0$, $a_1 = \lfloor 1/2 \rfloor + \lceil 1/2 \rceil = 0 + 1 = 1$, $a_2 = \lfloor 2/2 \rfloor + \lceil 2/2 \rceil = 1 + 1 = 2$, $a_3 = \lfloor 3/2 \rfloor + \lceil 3/2 \rceil = 1 + 2 = 3$. Note that $\lfloor n/2 \rfloor + \lceil n/2 \rceil$ always equals n.

7.

a) $2+3+4+5+6 = 20$ **b)** $1-2+4-8+16 = 11$ **c)** $3+3+\cdots+3 = 10 \cdot 3 = 30$

d) This series "telescopes": each term cancels part of the term before it (see also Exercise 35). The sum is $(2-1)+(4-2)+(8-4)+\cdots+(512-256) = -1+512 = 511$.

8.

a) The bit strings not containing 0 are just the bit strings consisting of all 1's, so this set is $\{\lambda, 1, 11, 111, 1111, \ldots\}$, where λ denotes the empty string (the string of length 0). Thus this set is countably infinite, where the correspondence matches the positive integer n with the string of $n-1$ 1's.

b) This is a subset of the set of rational numbers, so it is countable (see Exercise 16). To find a correspondence, we can just follow the path in Example 4, but omit fractions in the top three rows (as well as continuing to omit those fractions that are duplicates of rational numbers already encountered).

c) This set is uncountable, as can be shown by applying the diagonal argument of Example 5.

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