Fallacies

MP $((p \to q) \land p) \to q$

Several common fallacies arise in incorrect arguments. These fallacies resemble rules of inference, but are based on contingencies rather than tautologies.

The proposition $((p \to q) \land q) \to p$ is not a tautology, because it is false when p is false and q is true. However, there are many incorrect arguments that treat this as a tautology. In other words, they treat the argument with premises $p \to q$ and q and conclusion p as a valid argument form, which it is not. This type of incorrect reasoning is called the fallacy of affirming the conclusion.

Introduction to Proofs

Section 1.7

Section Summary

Introduction

Some Terminology

Methods of Proofs

- Direct Proofs
- Proof by Contraposition
- Vacuous and Trivial Proofs
- Proofs by Contradiction
- Proofs of Equivalence

Mistakes in Proofs

Introduction

In this section, we will learn the notion of a proof and methods for constructing proofs.

Formal Proofs: the arguments we learned in Section 1.6 involving propositions and quantified statements were formal proof. In formal proof, all steps were supplied and the rules for each step in argument were given. However, it could be very long and hard to follow.

Informal Proofs: more than one rule of inference may be used in each step. Some steps may be skipped, some axioms may be assumed, the rule of inference used are not explicitly stated, easier to understand and to explain to people.

Introduction

Methods of proof are important and have many practical applications:

- Verifying that computer programs are correct
- Establishing that operating systems are secure
- Making inferences in artificial intelligence
- Showing that system specifications are consistent

Some Terminology

Axiom: a statement or proposition which is regarded as being established, accepted, or self-evidently true.

Lemma: a lemma is a generally minor, proven proposition which is used as a stepping stone to a larger result.

Theorem (facts/results): is a statement which is non self evident but can be proven to be true.

Proof: is a valid argument that establishes the truth of a theorem. The statements used in proof can include axioms or postulates

Corollary: a theorem of less importance which can be readily deduced from a previous, more notable statement

Conjecture: a conclusion or a proposition which is suspected to be true due to preliminary supporting evidence, but for which no proof or disproof has yet been found.

When a proof of a conjecture is found, then conjecture becomes a theorem.

Methods of Proofs

- Direct Proofs
- Proofs by Contraposition
- Vacuous and Trivial Proofs
- Proofs by Contradiction
- Proofs of Equivalence

Direct Proofs and Examples

Direct Proofs

A **Direct Proof** shows that a conditional statement $p \rightarrow q$ is true by showing that if p is true, then q must also be true.

- ✓ In a direct proof, the first step is to assume that p is true, and then use axioms, definitions, previously proven theorems, and rules of inference to finally show that q must also be true.
- ✓ Begin with the premises, continue with a sequence of deductions, and end with the conclusion.
- ✓ Straightforward: Starting with the hypothesis and leading to the conclusion, the way forward is essentially dictated by the premises available at that step.

Direct Proofs-Examples

Definition 1: even and odd integers

The integer n is even if there exists an integer k such that n = 2k, and n is odd if there exists an integer k such that n = 2k + 1. (Note that every integer is either even or odd, and no integer is both even and odd.) Two integers have the same parity when both are even or both are odd; they have opposite parity when one is even and the other is odd.

Direct Proofs-Examples

Example 1:

Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

Solution:

Step 1: Assume that n is odd.

Step 2: Then $n = 2k + 1$ for an integer k. Squaring both sides of the equation, we get: $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2r + 1$, where $r = 2k^2 + 2k$, an integer.

Step 3: Based on step 1 and 2, we have proved that if n is an odd integer, then n^2 is an odd integer.

Direct Proofs-Examples

Definition 2: rational number

The real number r is rational if there exist integers p and q with $q \neq 0$ such that $r = p/q$. A real number that is not rational is called irrational.

Direct Proofs-Examples

Example 2

Prove that the sum of two rational numbers is rational.

Solution:

Step 1: Assume r and s are two rational numbers.

Step 2: Then there must be integers p, q and also t, u such that

$$r = p / q, \quad s = t / u, \quad u \neq 0, \quad q \neq 0$$

$$r + s = \frac{p}{q} + \frac{t}{u} = \frac{pu + qt}{qu} = \frac{v}{w}$$

where $v = pu + qt$

$w = qu \neq 0$

Step 3: Based on definition of rational number, we have proved sum of two rational numbers (r+s) is rational

Proof by Contraposition and Examples

Proof by Contraposition

Sometimes, attempts at direct proofs often reach dead ends. Then we can change our way to use <u>indirect proofs</u>.

<u>Proof by contraposition:</u> the conditional statement $p \rightarrow q$ is equivalent to its contrapositive, $\neg q \rightarrow \neg p$. This means that the conditional statement $p \rightarrow q$ can be proved by showing that its contrapositive, $\neg q \rightarrow \neg p$, is true.

- ✓ In proof by contraposition, the first step is to assume that premise $\neg p$ is true, and then use axioms, definitions, previously proven theorems, and rules of inference to finally show that $\neg q$ must also be true.

Proof by Contraposition-Examples

Can we use direct proof to conclude that n is odd?

Example 3:

Prove that if $3n + 2$ is odd, then n is odd. n is an integer

Solution:

Step 1: Assume n is even.

Step 2: Then $n = 2k$ for some integer k. Thus $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1) = 2j$ for $j = 3k + 1$. Therefore $3n + 2$ is even.

Step 3: Since we have shown $\neg q \rightarrow \neg p$, $p \rightarrow q$ must hold as well. If n is an integer and $3n + 2$ is odd (not even), then n is odd (not even).

A fact: if $0 < s < t$ and $0 < u < v, then$ $su < tv$

Proof by Contraposition-Examples

Example 4:

Prove that if n = ab, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Solution:

Step 1: Assume "$a \leq \sqrt{n}$ or $b \leq \sqrt{n}$" is false.

Step 2: Then $a > \sqrt{n}$ and $b > \sqrt{n}$, we can multiply this inequalities together to get $ab > \sqrt{n} \cdot \sqrt{n} = \mathrm{n}$. So wen have n$\neq$ ab which contradicts the statement n=ab.

Step 3: Based on step 1 and 2, we used proof by contraposition to show that the original statement is true.

Vacuous and Trivial Proofs

Vacuous and Trivial Proofs

Vacuous Proof: If we know p is false then $p \to q$ is true as well.

"If 2 + 2 = 5 then I am playing computer games ."

Trivial Proof: If we know q is true, then $p \to q$ is true as well.

"If it is raining then 1=1."

Proofs by Contradiction and Examples

Proofs by Contradiction

Proof by Contradiction can be used to

- ✓ To prove a statement p to be true,

2) To prove conditional statement ($p \rightarrow q$) to be true.

To prove a statement p to be true: we can show that $\neg p \rightarrow (r \wedge \neg r)$ is true for some proposition r.

1. Assume $\neg p$ is true.
2. To see if $\neg p$ is leads to a contradiction ($r \wedge \neg r = F$)

Since we have shown that $\neg p \rightarrow F$ is true , it follows that the contrapositive $T \rightarrow p$ also true. Then we can have p to be true.

Proofs by Contradiction[1]

$\neg p \rightarrow (r \wedge \neg r) \equiv (\neg p \rightarrow r) \wedge (\neg p \rightarrow \neg r)$
Section 1.3 table 7

Example 5

Prove that "at least four of any 22 days must fall on the same day of the week".

Solution:

Step 1: Assume that $\neg p$: "at most 3 of the 22 days fall on the same day of the week" is true

Step 2: Because there are 7 days of the week, this implies that at most 21 days could have been chosen. This contradicts the premise that we have 22 days under consideration.

Step 3: If r is "22 days are chosen", then we have shown that $\neg p \rightarrow (r \wedge \neg r)$. Consequently, we know that p is true. We proved that ...

Proof by Contradiction[1]

Example: Use a proof by contradiction to give a proof that √2 is irrational.

Solution: Suppose √2 is rational. Then there exists integers *a* and *b* with √2 = *a/b*, where *b*≠ *0* and *a* and *b* have no common factors. Then

$$2 = \frac{a^2}{b^2} \qquad 2b^2 = a^2$$

Therefore a^2 must be even. If a^2 is even then *a* must be even (an exercise). Since *a* is even, *a* = 2*c* for some integer *c*. Thus,

$$2b^2 = 4c^2 \qquad b^2 = 2c^2$$

Therefore b^2 is even. Again then *b* must be even as well.

But then 2 must divide both *a* and *b*. This contradicts our assumption that *a* and *b* have no common factors. We have proved by contradiction that our initial assumption must be false and therefore √2 is irrational.

Proofs by Contradiction[2]

p	q	p→q	(p∧¬q)→F
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Proof by Contradiction can be used to

1) To prove a statement p to be true,

✓ To prove conditional statement (p →q) to be true.

To prove conditional statement (p →q) to be true: In such proofs, we first assume the negation of the conclusion. We then use the premises of the theorem and the negation of the conclusion to arrive at a contradiction. (The reason that such proofs are valid rests on $p \to q \equiv (p \land \neg q) \to F$.

1. Assume p and ¬q are true.
2. Then we use steps from proof of $\neg q \to \neg p$ to show that ¬p is true.
3. This leads to the contradiction $p \land \neg p$

Proofs by Contradiction[2]

$(p \wedge \neg q) \rightarrow F$

Example 5

Give a proof by contradiction of the theorem “If 3n + 2 is odd, then n is odd.”

Solution:

Step 1: Let p be “3n + 2 is odd” and q be “n is odd.” To construct a proof by contradiction, assume that both p and ¬q are true. That is, assume that 3n + 2 is odd and that n is not odd.

Step 2: Because n is not odd, we know that it is even. Because n is even, there is an integer k such that n = 2k. This implies that 3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1). Because 3n + 2 is 2t, where t = 3k + 1, 3n + 2 is even. Note that the statement “3n + 2 is even” is equivalent to the statement ¬p, because an integer is even if and only if it is not odd.

Step 3: Because both p and ¬p are true, we have a contradiction. This completes the proof by contradiction, proving that if 3n + 2 is odd, then n is odd.

Proofs of Equivalence

Proofs of Equivalence

To prove a theorem that is a biconditional statement, that is, a statement of the form $p \leftrightarrow q$, we **show that $p \to q$ and $q \to p$ are both true**. This is based on the tautology:

$$(p \leftrightarrow q) \leftrightarrow (p \to q) \land (q \to p)$$

Example: Prove the theorem: "If n is an integer, then n is odd if and only if n^2 is odd."

Solution: $p \to q$ by "direct proof", to prove q →p, we can use "proof by contraposition". Therefore we can conclude $p \leftrightarrow q$.

Proofs of Equivalence

Sometimes a theorem states that several propositions are equivalent. Such as p_1, p_2, p_3,…, p_n are equivalent can be written as

$$p_1 \leftrightarrow p_2 \leftrightarrow \cdots \leftrightarrow p_n,$$

which states that all n propositions have the same truth values, and consequently, that for all i and j with $1 \le i \le n$ and $1 \le j \le n$, p_i and p_j are equivalent. One way to prove these are mutually equivalent is to use the tautology

$$p_1 \leftrightarrow p_2 \leftrightarrow \cdots \leftrightarrow p_n \leftrightarrow (p_1 \rightarrow p_2) \land (p_2 \rightarrow p_3) \land \cdots \land (p_n \rightarrow p_1).$$

Proofs of Equivalence

We can show that p_1, p_2, p_3 are equivalent by showing that $p_1 \rightarrow p_2$, $p_2 \rightarrow p_3$, $p_3 \rightarrow p_1$

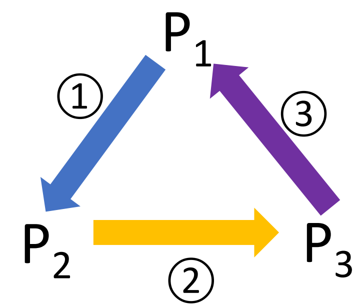
Example 6:

Show that these statements about integer n are equivalent

P_1: "n is even"

P_2:"n-1" is odd

P_3: "n^2 is even"



Solution:

① $p_1 \rightarrow p_2$: we use direct proof. Suppose that n is even. Then n = 2k for some integer k. Consequently, n − 1 = 2k − 1 = 2(k − 1) + 1. This means that n − 1 is odd because it is of the form 2m + 1, where m is the integer k − 1

② $p_2 \rightarrow p_3$: we use direct proof. Suppose n − 1 is odd. Then n − 1 = 2k + 1 for some integer k. Hence, n = 2k + 2 so that $n^2 = (2k + 2)^2 = 4k^2 + 8k + 4 =2(2k^2 + 4k + 2)$. This means that n^2 is twice the integer $2k^2 + 4k + 2$, and hence is even.

③ $p_3 \rightarrow p_1$: we use a proof by contraposition. That is, we prove that if n is not even, then n^2 is not even. (Example 1 is direct proof, it shares the same steps as here)

What is wrong with this?

"Proof" that *1 = 2*

Step	Reason
1. $a = b$	Premise
2. $a^2 = a \times b$	Multiply both sides of (1) by a
3. $a^2 - b^2 = a \times b - b^2$	Subtract b^2 from both sides of (2)
4. $(a - b)(a + b) = b(a - b)$	Algebra on (3)
5. $a + b = b$	Divide both sides by $a - b$
6. $2b = b$	Replace a by b in (5) because $a = b$
7. $2 = 1$	Divide both sides of (6) by b

Solution: Step 5. a - b = 0 by the premise and division by 0 is undefined.

Mathematical Induction

Section 5.1

Section Summary

Mathematical Induction

Examples of Proof by Mathematical Induction

Mistaken Proofs by Mathematical Induction

Guidelines for Proofs by Mathematical Induction

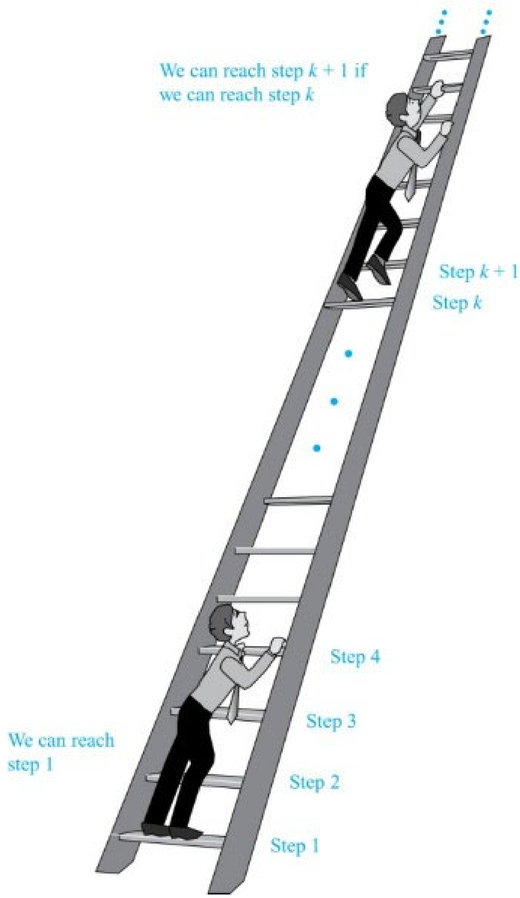
Climbing an Infinite Ladder

Suppose we have an infinite ladder:

1. We can reach the first rung of the ladder.
2. If we can reach a particular rung of the ladder, then we can reach the next rung.

From (1), we can reach the first rung. Then by applying (2), we can reach the second rung. Applying (2) again, the third rung. And so on. We can apply (2) any number of times to reach any particular rung, no matter how high up.

This example motivates proof by mathematical induction.



Principle of Mathematical Induction

Principle of Mathematical Induction: To prove that $P(n)$ is true for all positive integers n, we complete these steps:

- *Basis Step*: Show that $P(1)$ is true.
- *Inductive Step*: Show that $P(k) \to P(k + 1)$ is true for all positive integers k.

To complete the inductive step, assuming the *inductive hypothesis* that $P(k)$ holds for an arbitrary integer k, show that $P(k + 1)$ must be true.

Climbing an Infinite Ladder Example:

- BASIS STEP: By (1), we can reach rung 1.
- INDUCTIVE STEP: Assume the inductive hypothesis that we can reach rung k. Then by (2), we can reach rung $k + 1$.

Hence, $P(k) \to P(k + 1)$ is true for all positive integers k. We can reach every rung on the ladder.

Important Points About Using Mathematical Induction

Mathematical induction can be expressed as the rule of inference

$$\left(P(1) \wedge \forall k\left(P(k) \rightarrow P(k+1)\right)\right) \rightarrow \forall n\, P(n),$$

where the domain is the set of positive integers.

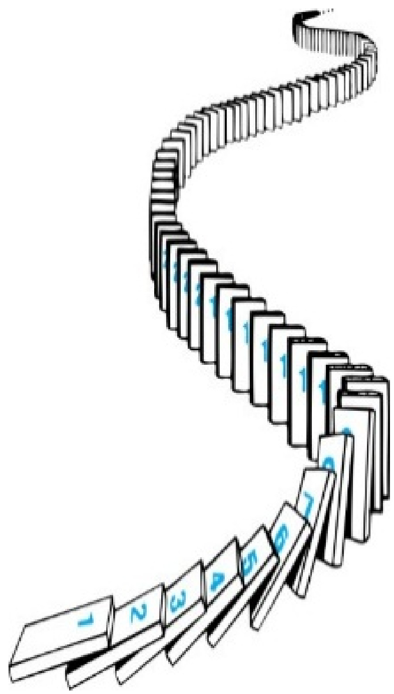
In a proof by mathematical induction, we don't assume that $P(k)$ is true for all positive integers! We show that if we assume that $P(k)$ is true, then $P(k + 1)$ must also be true.

Proofs by mathematical induction do not always start at the integer 1. In such a case, the basis step begins at a starting point b where b is an integer. We will see examples of this soon.

Remembering How Mathematical Induction Works

Consider an infinite sequence of dominoes, labeled 1,2,3, …, where each domino is standing.

Let $P(n)$ be the proposition that the nth domino is knocked over.



We know that the first domino is knocked down, i.e., $P(1)$ is true .

We also know that if whenever the kth domino is knocked over, it knocks over the $(k + 1)$st domino, i.e, $P(k) \rightarrow P(k + 1)$ is true for all positive integers k.

Hence, all dominos are knocked over.

$P(n)$ is true for all positive integers n.

Proving a Summation Formula by Mathematical Induction

Example: n is a positive integer, Show that: $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$

Note: Once we have this conjecture, mathematical induction can be used to prove it correct.

Solution:

- BASIS STEP: $P(1)$ is true since $1(1 + 1)/2 = 1$.
- INDUCTIVE STEP: Assume true for $P(k)$.

The inductive hypothesis is $1 + 2 + 3 + \cdots + k = \dfrac{k(k+1)}{2}$

Under this assumption,

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

This shows that $p(k+1)$ is true under assumption of $p(k)$ is true. And it completes the inductive step.

We have completed the basic step and inductive step, so by mathematical induction we know that $p(k)$ is true for all positive integers n. That is we proved that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for all positive integers n.

Conjecturing and Proving Correct a Summation Formula

Example: Conjecture and prove correct a formula for the sum of the first *n* positive odd integers. Then prove your conjecture.

Solution: We have:

$$1 = 1,\ 1 + 3 = 4,\ 1 + 3 + 5 = 9,\ 1 + 3 + 5 + 7 = 16,\ 1 + 3 + 5 + 7 + 9 = 25.$$

- We can conjecture that the sum of the first n positive odd integers is n^2,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

- We prove the conjecture is proved correct with mathematical induction.
- BASIS STEP: $P(1)$ is true since $1^2 = 1$.
- INDUCTIVE STEP: $P(k) \rightarrow P(k + 1)$ for every positive integer k.
  Assume the inductive hypothesis holds and then show that $P(k + 1)$ holds has well.

Inductive Hypothesis: $1 + 3 + 5 + \cdots + (2k - 1) = k^2$

- So, assuming $P(k)$, it follows that:

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) &= [1 + 3 + 5 + \cdots + (2k - 1)] + (2k + 1) \\ &= k^2 + (2k + 1) (\textit{by the inductive hypothesis}) \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2 \end{aligned}$$

- Hence, we have shown that $P(k + 1)$ follows from $P(k)$. Therefore the sum of the first n positive odd integers is n^2.

Proving Inequalities

Example: Use mathematical induction to prove that $n < 2^n$ for all positive integers n.

Solution: Let $P(n)$ be the proposition that $n < 2^n$.

BASIS STEP: $P(1)$ is true since $1 < 2^1 = 2$.

INDUCTIVE STEP: Assume $P(k)$ holds, i.e., $k < 2^k$, for an arbitrary positive integer k.

Must show that $P(k + 1)$ holds. Since by the inductive hypothesis, $k < 2^k$, it follows that:

$$k + 1 < 2^k + 1 \leq 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$$

Therefore $n < 2^n$ holds for all positive integers n.

Proving Inequalities

Example: Use mathematical induction to prove that $2^n < n!$, for every integer $n \geq 4$.

Solution: Let $P(n)$ be the proposition that $2^n < n!$.

BASIS STEP: $P(4)$ is true since $2^4 = 16 < 4! = 24$.

INDUCTIVE STEP: Assume $P(k)$ holds, i.e., $2^k < k!$ for an arbitrary integer $k \geq 4$. To show that $P(k + 1)$ holds:

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &< 2 \cdot k! \qquad \textit{(by the inductive hypothesis)} \\ &< (k+1)k! \\ &= (k+1)! \end{aligned}$$

Therefore, $2^n < n!$ holds, for every integer $n \geq 4$.

Note that here the basis step is $P(4)$, since $P(0)$, $P(1)$, $P(2)$, and $P(3)$ are all false.

Proving Divisibility Results

Example: Use mathematical induction to prove that $n^3 - n$ is divisible by 3, for every positive integer n.

Solution: Let $P(n)$ be the proposition that $n^3 - n$ is divisible by 3.

BASIS STEP: $P(1)$ is true since $1^3 - 1 = 0$, which is divisible by 3.

INDUCTIVE STEP: Assume $P(k)$ holds, i.e., $k^3 - k$ is divisible by 3, for an arbitrary positive integer k. To show that $P(k + 1)$ follows:

$$\begin{aligned}(k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) - (k+1) \\ &= (k^3 - k) + 3(k^2 + k)\end{aligned}$$

By the inductive hypothesis, the first term $(k^3 - k)$ is divisible by 3 and the second term is divisible by 3 since it is an integer multiplied by 3. So by part (i) of Theorem 1 in Section 4.1 , $(k + 1)^3 - (k + 1)$ is divisible by 3.

Therefore, $n^3 - n$ is divisible by 3, for every integer positive integer n.

Guidelines: Mathematical Induction Proofs

Steps:

1. $p(n)$: Express the statement that is to be proved
2. Basis Step: show that $p(b)$ is true, taking care the correct value of b. e.g., $p(1)$, ...
3. Inductive Step:

   (a) Assume $p(k)$ is true for an arbitrary integer k with $k \geq b$

   (b) By using assumption of $p(k)$ is true in (a) to prove that if $p(k+1)$ is true

4. Conclusion:

   (a) We can state that "this completes the inductive Step"

   (b) If $p(b)$ in the "Basis Step" and $p(k+1)$ in the "Inductive Step" are all true, then we can state that "By mathematical induction, P(n) is true for all integers n with n ≥ b"

Proving Results about Sets

Example: Use mathematical induction to show that if S is a finite set with n elements, where n is a nonnegative integer, then S has 2^n subsets.

Solution: Let $P(n)$ be the proposition that a set with n elements has 2^n subsets.

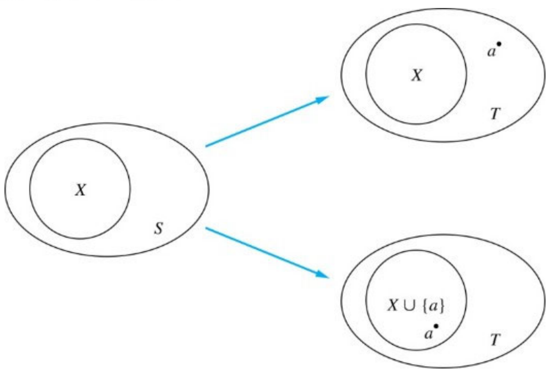
Basis Step: $P(0)$ is true, because the empty set has only itself as a subset and $2^0 = 1$.

Inductive Step: Assume $P(k)$: "a set S with k elements has 2^k subsets." is true for an arbitrary nonnegative integer k.

Let T be a set with $k + 1$ elements and $T = S \cup \{a\}$, where $a \in T$ and $S = T - \{a\}$. Hence $|S| = k$.

For each subset X of S, there are exactly two subsets of T, i.e., X and $X \cup \{a\}$.

By the inductive hypothesis S has 2^k subsets. Since there are two subsets of T for each subset of S, the number of subsets of T is $2 \cdot 2^k = 2^{k+1}$.



Proving Results about Sets

Example: Use mathematical induction to prove the following generalization of one of De Morgan's laws

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$$

Where $A_1, A_2, \dots, A_n$ are subsets of a universal set U and $n \geq 2$.

Solution: Let $P(n)$ be the identity for n sets.

Basic Step: The statement $P(2)$ asserts that $\overline{A_1 \cap A_2} = \overline{A_1} \cap \overline{A_2}$, true

Inductive Step: Let's assume $P(k)$:" $\overline{\bigcap_{j=1}^{k} A_j} = \bigcup_{j=1}^{k} \overline{A_j}$" is true, where k is an arbitrary integer with $k \geq 2$

Then Let's prove $P(k+1)$:" $\overline{\bigcap_{j=1}^{k+1} A_j} = \bigcup_{j=1}^{k+1} \overline{A_j}$" is true

$\overline{\bigcap_{j=1}^{k+1} A_j} = \overline{(\bigcap_{j=1}^{k} A_j) \cap A_{k+1}} = \overline{(\bigcap_{j=1}^{k} A_j)} \cup \overline{A_{k+1}} = (\bigcup_{j=1}^{k} \overline{A_j}) \cup \overline{A_{k+1}} = \bigcup_{j=1}^{k+1} \overline{A_j}$, then we complete the inductive step.

Because we complete both the basis step and the inductive step, by mathematical induction we know that $P(n)$ is true whenever n is a positive integer and $n \geq 2$.

An Incorrect “Proof” by Mathematical Induction

Example: Let $P(n)$ be the statement that every set of n lines in the plane, no two of which are parallel, meet in a common point. Here is a “proof” that $P(n)$ is true for all positive integers $n \geq 2$.

- BASIS STEP: The statement $P(2)$ is true because any two lines in the plane that are not parallel meet in a common point.
- INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true for the positive integer $k \geq 2$, i.e., every set of k lines in the plane, no two of which are parallel, meet in a common point.
- We must show that if $P(k)$ holds, then $P(k + 1)$ holds, i.e., if every set of k lines in the plane, no two of which are parallel, $k \geq 2$, meet in a common point, then every set of k + 1 lines in the plane, no two of which are parallel, meet in a common point.

An Incorrect "Proof" by Mathematical Induction[2]

Inductive Hypothesis: Every set of k lines in the plane, where $k \geq 2$, no two of which are parallel, meet in a common point.

Consider a set of $k + 1$ distinct lines in the plane, no two parallel. By the inductive hypothesis, the first k of these lines must meet in a common point p_1. By the inductive hypothesis, the last k of these lines meet in a common point p_2.

If p_1 and p_2 are different points, all lines containing both of them must be the same line since two points determine a line. This contradicts the assumption that the lines are distinct. Hence, $p_1 = p_2$ lies on all $k + 1$ distinct lines, and therefore $P(k + 1)$ holds. Assuming that $k \geq 2$, distinct lines meet in a common point, then every $k + 1$ lines meet in a common point.

There must be an error in this proof since the conclusion is absurd. But where is the error?

- **Answer**: It is not the case that $P(2)$ (basic step) implies $P(3)$. The first two lines must meet in a common point p_1 and the second two must meet in a common point p_2. They do not have to be the same point since only the second line is common to both sets of lines.

Guidelines: Mathematical Induction Proofs

Template for Proofs by Mathematical Induction

1. Express the statement that is to be proved in the form "for all $n \geq b$, $P(n)$" for a fixed integer b.
2. Write out the words "Basis Step." Then show that $P(b)$ is true, taking care that the correct value of b is used. This completes the first part of the proof.
3. Write out the words "Inductive Step".
4. State, and clearly identify, the inductive hypothesis, in the form "assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$."
5. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what $P(k + 1)$ says.
6. Prove the statement $P(k + 1)$ making use the assumption $P(k)$. Be sure that your proof is valid for all integers k with $k \geq b$, taking care that the proof works for small values of k, including $k = b$.
7. Clearly identify the conclusion of the inductive step, such as by saying "this completes the inductive step."
8. After completing the basis step and the inductive step, state the conclusion, namely, by mathematical induction, $P(n)$ is true for all integers n with $n \geq b$.