Homework Set 6

Yeoh Jia Er
1003633
Class 2

Q1

$X_1, X_2, \ldots X_{10}, Y_1, Y_2, \ldots, Y_{15} \sim N(20, 30)$

(i) Joint pdf of $X_1, X_2, \ldots, X_{10}$:

$$f(x_1, x_2, \ldots, x_{10}) = \prod_{i=1}^{10} \frac{1}{\sigma_i\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2} \quad \because X_1, X_2, \ldots, X_{10} \text{ are iid}$$

$$= \prod_{i=1}^{10} \frac{1}{3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i-20}{3}\right)^2}$$

$$= \left(\frac{1}{3\sqrt{2\pi}}\right)^{10} e^{-\frac{1}{2}\left[\left(\frac{x_1-20}{3}\right)^2+\left(\frac{x_2-20}{3}\right)^2+\ldots+\left(\frac{x_{10}-20}{3}\right)^2\right]}$$

ii) $\bar{X} = \frac{1}{10}(X_1+X_2+\ldots+X_{10})$

$E(\bar{X}) = E\left(\frac{1}{10}(X_1+\ldots+X_{10})\right)$

$= \frac{1}{10}[E(X_1)+\ldots E(X_{10})]$

$= \frac{1}{10} \times 10\,(E(X))$

$= 20$

$Var(\bar{X}) = Var\left(\frac{1}{10}(X_1+\ldots+X_{10})\right)$

$= \left(\frac{1}{10}\right)^2 (10)\,Var(X) \quad \because$ iid

$= \frac{3}{10}$

$\therefore \bar{X} \sim N(20, \frac{3}{10})$

iii) $\bar{Y} = \frac{1}{15}\sum_{i=1}^{15} Y_i$

Let $A = \bar{X} - \bar{Y}$

$E(A) = E(\bar{X} - \bar{Y})$

$= E(\bar{X}) - E(\bar{Y})$

$= 20 - \frac{1}{15}(15\,E(Y))$

$= 20 - 20$

$= 0$

$Var(A) = Var(\bar{X} - \bar{Y})$

$= Var(\bar{X}) + Var(\bar{Y}) \quad \because$ iid

$= \frac{3}{10} + \left(\frac{1}{15}\right)^2(15\,Var(Y)) \quad \because$ iid

$= \frac{3}{10} + \frac{3}{15}$

$= 0.5$

$\therefore A \sim N(0, 0.5)$

$Pr(|\bar{X}-\bar{Y}| > 0.3) = Pr(|A| > 0.3)$

$= Pr(A < -0.3) + Pr(A > 0.3)$

$= Pr\left(\frac{Z-0}{\sqrt{0.5}} < -0.3\right) + Pr\left(\frac{Z-0}{\sqrt{0.5}} > 0.3\right)$

$\approx Pr(Z < -0.212) + Pr(Z > 0.2121)$

$= 1 - Pr(Z < 0.212) + 1 - Pr(Z < 0.2121)$

$= 2(1 - Pr(Z < 0.212))$

$= 2(1 - 0.5832)$ (from standard normal table)

$= 0.8336$

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Q2

θ is continuous, X is discrete

i) $$\xi(\theta|\vec{x}) = \frac{P_{10}(\vec{x}|\theta)\,\xi(\theta)}{\int_{\theta=0}^{1} P_{10}(\vec{x}|\theta)\,\xi(\theta')\,d\theta'}$$

$$p(x_i|\theta) = \begin{cases} \theta^{x_i}(1-\theta)^{1-x_i}, & \text{if } x_i = 0 \text{ or } 1 \\ 0, & \text{otherwise} \end{cases}$$

$$P_{10}(\vec{x}|\theta) = \prod_{i=1}^{10} p(x_i|\theta) \quad \because \text{independent}$$

$$= \begin{cases} \theta^{(x_1+x_2+\dots+x_{10})}(1-\theta)^{10-(x_1+x_2+\dots+x_{10})}, & \text{if every } x_i = 0 \text{ or } 1; \\ 0, & \text{otherwise}; \end{cases}$$

$$\xi(\theta) = \begin{cases} \frac{1}{10}, & \text{for } \theta \in [0,1]; \\ 0, & \text{otherwise}; \end{cases}$$

for $\theta \subset [0,1]$:

$$\int_0^1 \theta^8(1-\theta)^2\left(\tfrac{1}{10}\right) d\theta$$

$$= \frac{1}{10}\int_0^1 \theta^8(1-\theta)^2\, d\theta$$

$$= \frac{1}{10}\int_0^1 \theta^8(1-2\theta+\theta^2)\, d\theta$$

$$= \frac{1}{10}\int_0^1 \theta^{10} - 2\theta^9 + \theta^8\, d\theta$$

$$= \frac{1}{10}\left[\frac{1}{11}\theta^{11} - \frac{2}{10}\theta^{10} + \frac{1}{9}\theta^9\right]_0^1$$

$$= \frac{1}{10}\left[\frac{1}{11} - \frac{2}{10} + \frac{1}{9}\right] = \frac{1}{4950}$$

$$\therefore \xi(\theta|\vec{x}) = \begin{cases} 495(\theta^8(1-\theta)^2), & \theta \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

ii) $$\int_{0.85}^{1} 495(\theta^8(1-\theta)^2)\, d\theta$$

$$= 495\left[\frac{1}{11}\theta^{11} - \frac{2}{10}\theta^{10} + \frac{1}{9}\theta^9\right]_{0.85}^{1} \quad \text{(from part i)}$$

$$= 495\left[\frac{1}{495} - 0.00157\right)$$

$$= 0.221$$

Q3

Bayes Estimator: $E[L(\theta,a)|\vec{x}] = \int_\Omega L(\theta,a)\,\xi(\theta|\vec{x})\,d\theta$

$$L(\theta,a) = (\theta-a)^2$$

$$\xi(\theta|\vec{x}) \sim N(\mu_1, v_1^2)$$

where $$\mu_1 = \frac{(15)^2(50) + (10)^2(50\times 54.5)}{15^2 + 50(10)^2} = 54.31,$$

$$v_1^2 = \frac{(15^2)(10^2)}{15^2+50(10^2)} = 4.31$$

} family of conjugate priors

$$\therefore \delta^*(\vec{X}) = 54.31$$

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Q4

i) $f_{\vec{X}}(\vec{x}|\theta) = \prod_{i=1}^{n} 2\theta x_i^{2\theta-1}$

$$= \begin{cases} (2\theta)^n (x_1+x_2+\dots+x_n)^{(2\theta-1)n} & \text{, if every } x_i \in [0,1]; \\ 0 & \text{, otherwise;} \end{cases}$$

ii) $\log(f_{\vec{X}}(\vec{x}|\theta)) = \log[(2\theta)^n (x_1+x_2+\dots+x_n)^{n(2\theta-1)}]$

$$= n\log(2\theta) + n(2\theta-1)\log(x_1+x_2+\dots+x_n)$$

iii) $\frac{d}{d\theta}\log(f_{\vec{X}}(\vec{x}|\theta)) = n\left(\frac{2}{2\theta}\right) + n(2)\log(x_1+x_2+\dots+x_n) = 0$

$$\frac{1}{\theta} = -\log(x_1+\dots+x_n)$$

$$\theta = \frac{-1}{\log(x_1+x_2+\dots+x_n)}$$

$$\frac{d^2}{d\theta^2}(\log(f_{\vec{X}}(\vec{x}|\theta))) = -2\frac{1}{\theta^2} + 0$$

Since Bernoulli, $0 \le \theta \le 1$, $\frac{d^2}{d\theta^2}(\log(f_{\vec{X}}(\vec{x}|\theta))) < 0$

$\therefore \theta = -\frac{1}{\log(x_1+x_2+\dots+x_n)}$ is maximum θ is negative if $\Sigma x > 1$ tho hmm

Q5

$$p_{X_i}(x_i|\lambda) = \begin{cases} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} & \text{, if } x_i = 0,1,2,3\dots \\ 0 & \text{, otherwise} \end{cases}$$

i) $p(\vec{x}|\lambda) = \prod_{i=1}^{n}\left(\frac{1}{x_i!}\right)(\lambda^{x_i} e^{-\lambda})$

$$= (e^{-\lambda})^n \prod_{i=1}^{n}\left(\frac{\lambda^{x_i}}{x_i!}\right)$$

$$\therefore p(\vec{x}|\lambda) = \begin{cases} (e^{-\lambda})^n \prod_{i=1}^{n}\left(\frac{\lambda^{x_i}}{x_i!}\right) & \text{, for every } x_i \in \mathbb{N}_0 \\ 0 & \text{, otherwise} \end{cases}$$

ii) Let $\ln(p(\vec{x}|\lambda))$ be $g(\lambda)$

$$g(\lambda) = \ln\left[e^{-n\lambda} \prod_{i=1}^{n}\left(\frac{\lambda^{x_i}}{x_i!}\right)\right]$$

$$= -n\lambda\ln(e) + \left(\sum_{i=1}^{n} x_i\right)\ln(\lambda) - \sum_{j=1}^{n}\ln(x_j!)$$

iii) $g'(\lambda) = -n + \frac{1}{\lambda}\left(\sum_{i=1}^{n} x_i\right) - 0$

$$g'(\lambda) = 0 \Rightarrow n = \frac{1}{\lambda}\left(\sum_{i=1}^{n} x_i\right)$$

$$\lambda = \frac{\sum_{i=1}^{n} x_i}{n}$$

$g''(\lambda) = -\frac{1}{\lambda^2}\left(\sum_{i=1}^{n} x_i\right) < 0 \quad \because x_i > 0, \lambda > 0,$

$\therefore \lambda = \sum_{i=1}^{n} x_i / n$

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Q6

$$f(\vec{x}|a,b) = \begin{cases} \left(\frac{1}{b-a}\right)^n, & \text{if } a \le x_i \le b \text{ for every } x_i; \\ 0, & \text{otherwise}; \end{cases}$$

$\uparrow$
Likelihood function

Let $Y = X - a$, $\theta = b - a$

$$\therefore f(\vec{y}|\theta) = \begin{cases} \frac{1}{\theta^n}, & \text{if } y_i \in [0,\theta] \text{ for every } y_i; \\ 0, & \text{otherwise}; \end{cases}$$

The MLE of θ must be a value of θ which $\theta \ge y_i$ for $i = 1, \dots, n$ that maximises $\frac{1}{\theta^n}$

Since $\frac{1}{\theta^n}$ is a decreasing function, the estimate will be the smallest value of θ such that $\theta \ge y_i$ for $i = 1, \dots, n$

$\therefore$ to minimise θ,

$\alpha = \min\{X_1, X_2, \dots, X_n\}$

$\beta = \max\{X_1, X_2, \dots, X_n\}$