Notes on Quantum Optomechanics for Quantum Optics

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Quantum optomechanics and electromechanics. Radiation pressure. Capacitive electromechanical coupling. Generic setup. Sideband resolved interactions. Cooling. Membrane-in-the-middle setup. Pulsed optomechanics. Squeezed mechanical states. Optomechanical entanglement. Quantum transducer.

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This note should be available online at `https://git.io/vbOT2`.

INTRODUCTION AND REFERENCES

There are good reviews. De-facto standard review is by Aspelmeyer, Kippenberg and Marquardt. There is a book by them. Another book by Milburn and Bowen.

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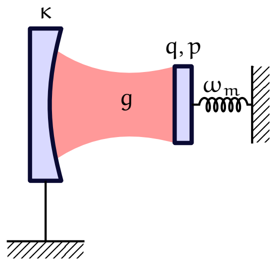

FIG. 1. Principal scheme of an optomechanical cavity.

I. QUANTUM OPTOMECHANICS AND ELECTROMECHANICS

A. Radiation Pressure

Consider a stream of photons impinging on a perfectly reflective mirror. Each photon carries momentum $p = \hbar\omega/c$, elastic reflection transfers to the mirror $\Delta p = 2p$. If there is a stream of such photons, so that N photons are reflected per interval of time $\Delta\tau$, the net force will be

$$F = \frac{N}{\Delta\tau} \times \Delta p = \frac{N}{\Delta\tau} \times \frac{2\hbar\omega}{c} = \frac{2}{c} \times \frac{N\hbar\omega}{\Delta\tau} = \frac{2P}{c}, \quad (1)$$

where $P \equiv N\hbar\omega/\Delta\tau$ is the power associated with the stream of photons. For the circulating power of 1 MW the force equals 0.3 mN.

If the mirror has finite reflectivity $R < 1$, then the force is even smaller,

$$F = \frac{P}{c}(R + 1).$$

B. Hamiltonian Formulation

The common property of different optomechanical platforms is that the radiation pressure drives the motion of the mechanical mode and in response the mechanical displacement changes the resonant frequency of the optical mode. In this sense majority of the optomechanical systems are equivalent to a Fabry-Pérot cavity with a movable mirror (see Fig. 1). The field in the cavity is described by an annihilation operator a. The movable mirror is a harmonic oscillator with eigenfrequency ω_m that is characterized by displacement from the equilibrium l.

The Hamiltonian of the system reads [?]

$$H = \hbar\omega_{cav}(l)a^\dagger a + \hbar\omega_m b^\dagger b - i\hbar E(a^\dagger e^{-i\omega_p t} - \text{h.c.}),$$

where the first term represents the own energy of the cavity mode (with a term accounting for the motion of the mirror), second — the energy of mechanical oscillations and third describes the pump at frequency ω_p with intensity proportional to E. The displacement l and momentum Φ of the mechanical mode are expressed in terms of bosonic operators $b, b^\dagger$ as follows

$$l = \sqrt{\frac{\hbar}{2m\omega_m}}(b + b^\dagger); \quad \Phi = \sqrt{\frac{\hbar m\omega_m}{2}}(b - b^\dagger)/i,$$

so that the commutator equals

$$[l, \Phi] = -i\frac{\hbar}{2}\left[b + b^\dagger, b - b^\dagger\right] = -i\hbar\left[b^\dagger, b\right] = i\hbar.$$

Zero-motion spread (variance of the displacement fluctuations in the ground state) equals

$$x_0^2 \equiv \left\langle (l - \bar{l})^2 \right\rangle_{|0\rangle} = \left\langle l^2 \right\rangle_{|0\rangle} = \frac{\hbar}{2m\omega_m}\left\langle 0\middle|(b + b^\dagger)^2\middle|0\right\rangle = \frac{\hbar}{2m\omega_m}.$$

In a similar fashion, zero-motion momentum fluctuations

$$p_0^2 \equiv \left\langle \Phi^2 \right\rangle_{|0\rangle} = \hbar m\omega_m/2.$$

Using the ground state variances x_0 and p_0 we define the more convenient dimensionless position q and momentum p of the mechanical oscillator:

$$q \equiv \frac{l}{x_0} = b + b^\dagger; \quad p \equiv \frac{\Phi}{p_0} = (b - b^\dagger)/i.$$

The dimensionless quadratures q and p have the commutator $[q, p] = 2i$.

In case the motion of mechanical mode is slow compared to the cavity frequency and the free spectral range of the cavity, it can be considered adiabatic. Therefore, the resonant optical frequency of the cavity can be written as

$$\omega_{cav}(l) \approx \omega_{cav}(0) + \frac{\partial\omega_{cav}}{\partial l}l \approx \omega_{cav}\cdot\left(1 - \frac{l}{L}\right) = \omega_{cav}\left(1 - q\frac{x_0}{L}\right) \equiv \omega_{cav} - g_0 q,$$

where L is the equilibrium length of the cavity and for simplicity we kept notation ω_{cav} for $\omega_{cav}(0)$. The interaction Hamiltonian, therefore, reads

$$H_{int} = -\hbar g_0 a^\dagger a(b^\dagger + b).$$

The quantity $g_0 \equiv \omega_{cav}x_0/L$ is the so-called single-photon optomechanical coupling rate. Its meaning is approximately the frequency shift per a phonon of mechanical oscillations. Its full definition reads

$$g_0 = \sqrt{\frac{\hbar}{2m\omega_m}}\frac{\omega_{cav}}{L}.$$

For parameters $m = 1\,\text{ng}$, $\omega_m = 10\,\text{MHz}$, $\omega_{cav} = 2\pi \times 281\,\text{THz}$, $L = 1\,\text{mm}$, one has $g_0 = 0.2\,\text{Hz}$.

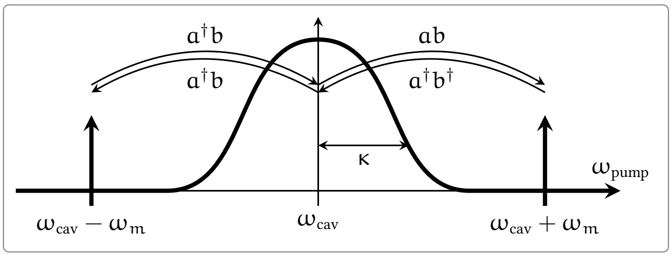

FIG. 2. Scheme of the dominant processes in sideband resolved regime.

C. Linearized Hamiltonian of Optomechanical interaction

As the single-photon coupling rate g_0 is typically very small in the state of the art experimental setups, therefore, to enhance the optomechanical coupling, a strong classical optical pump is applied. The interaction is then linearized around the mean classical values and the Hamiltonian of the linearized interaction reads in the frame rotating at the frequency of the pump

$$H = \hbar\Delta a^\dagger a + \hbar\omega_m b^\dagger b - \hbar g(a^\dagger e^{-i\theta} + ae^{i\theta})(b + b^\dagger), \quad (2)$$

where $\Delta \equiv \omega_{cav} - \omega_p$ is the detuning of the pump, $g \equiv g_0\sqrt{n_{cav}}$ is the coupling rate enhanced by average number of intracavity photons created by the pump.

We now apply a rotating frame, defined by the first two terms in Eq. (2):

$$H_{RF} = \hbar\Delta a^\dagger a + \hbar\omega_m b^\dagger b.$$

This is equivalent to replacement $a \to ae^{-i\Delta t}$ and $b \to be^{-i\omega_m t}$. Thus we obtain the Hamiltonian of the linearized optomechanical interaction:

$$H = -\hbar g\left(a^\dagger e^{-i\theta + i\Delta t} + ae^{i\theta - i\Delta t}\right) \times \left(be^{-i\omega_m t} + b^\dagger e^{i\omega_m t}\right). \quad (3)$$

II. SIDEBAND RESOLVED INTERACTIONS

A. Remarks

Optomechanical systems can be considered in the known frame of parametric systems. There is a strong pump that enables the interaction between the resonant cavity mode and the mechanical mode. The latter two modes play the role of signal and idler.

If the pump is tuned to the sum frequency ($\omega_p = \omega_{cav} + \omega_m$), the system works as a parametric amplifier, in case of difference frequency ($\omega_p = \omega_{cav} - \omega_m$), as a parametric converter.

Let us rewrite the Hamiltonian from Eq. (3) as

$$-\frac{H_{lin}}{\hbar g} = a^\dagger b^\dagger e^{i(\Delta+\omega_m)t} + abe^{-i(\Delta+\omega_m)t} a^\dagger be^{-i(\Delta-\omega_m)t} + ab^\dagger e^{+i(\Delta-\omega_m)t} \quad (4)$$

So that it is easier to see the important particular cases of $\Delta = \pm\omega_m$, or the same, $\omega_p = \omega_{cav} \pm \omega_m$ (see Fig. 2).

Why do we need sideband resolution?

B. Red Detuning. Laser Cooling

We first consider the case of red detuning, when the pump is tuned to the lower mechanical sideband of the cavity, that is $\omega_p = \omega_{cav} - \omega_m$, or $\Delta = \omega_{cav} - \omega_p = \omega_m$.

The Hamiltonian (4) takes the form

$$H_R = -\hbar g\left(a^\dagger b + a b^\dagger\right) - \hbar g\left[a^\dagger b^\dagger e^{2i\omega_m t} + a b e^{-2i\omega_m t}\right].$$

If the interaction is not too strong, we can apply the rotating wave approximation (RWA), that is the rapid terms in the square brackets can be omitted, so the Hamiltonian reduces to

$$H_{BS} = -\hbar g(a^\dagger b + a b^\dagger),$$

which is well known as the Hamiltonian of a beam-splitter, capable to swap states between the modes a and b. In other words, this is a parametric converter, in which the dominant process (see Fig. 2) upconverts a phonon to a resonant cavity photon, combining the former with a photon of pump. This way the excitations of the mechanical mode are sucked out and the mechanical mode is cooled.

The cooling is limited by scattering from the second mechanical sideband ($\omega_{cav} - 2\omega_m$) that was described by $a^\dagger b^\dagger e^{2i\omega_m t}$ in the original Hamiltonian. The intensity of this process is $\propto (\kappa/\omega_m)^2$.

C. Blue Detuning. Squeezing

Another important case is the one of blue detuning, when the pump is tuned to the upper mechanical sideband of the cavity, that is $\omega_p = \omega_{cav} + \omega_m$, or $\Delta = \omega_{cav} - \omega_p = -\omega_m$.

The Hamiltonian (4) takes the form

$$H_B = -\hbar g\left(a^\dagger b^\dagger + a b\right) - \hbar g\left[a^\dagger b e^{2i\omega_m t} + a b^\dagger e^{-2i\omega_m t}\right].$$

In the RWA it reduces to

$$H_{AMP} = -\hbar g(a^\dagger b^\dagger + a b),$$

known as a Hamiltonian of the amplifier, or two-mode squeezer etc. This interaction was used in experiment to entangle mechanics and microwaves.

D. Modulation of the pump. Quantum Non-Demolition interaction.

If we consider zero detuning $\Delta = 0$, but instead modulate the coupling constant $g \to 2g\cos(\omega_m t)$, we will have in the Hamiltonian

$$H = -\hbar g(e^{i\omega_m t} + e^{-i\omega_m t})(a^\dagger + a)(b^\dagger e^{i\omega_m t} + b e^{-i\omega_m t}).$$

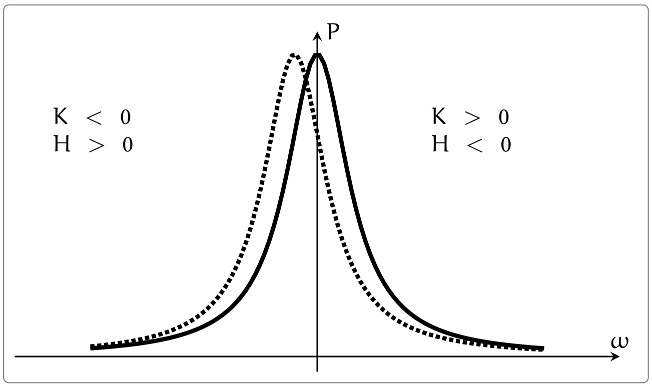

FIG. 3. How does the sign of introduced optical spring depend on the detuning.

With help of RWA we obtain

$$H_{QND} = -\hbar g(a^\dagger + b)(b^\dagger + b) \equiv -\hbar g X q,$$

where we introduced the amplitude quadrature of the intracavity field $X = a^\dagger + a$. By adjusting the phase of the pump (θ) one can couple as well to the phase quadrature $Y = (a - a^\dagger)/i$, and changing the phase of the modulation $g \to 2g\cos(\omega_m t + \phi)$ it is possible to couple to a different quadrature of the mechanical motion.

III. PULSED OPTOMECHANICS

A. Optical Spring

One more intuitive picture of the optomechanical interaction is the so-called optical spring. Since the force of the radiation pressure is determined by cirtulating power (see Eq. (1)) and the power depends on the position of the mirror, one can write

$$F(x) = \frac{2P(x)}{c} \approx \frac{2}{c}\left(P(0) + \left.\frac{\partial P}{\partial x}\right|_{x=0} \cdot x\right) = F_0 - Kx.$$

From the last equation one sees that in the expression for the radiation pressure force there is a term linear in x, as if there was a spring with rigidity K attached to the mirror.

Importantly, the equilibrium radiation pressure is determined by the position of the mirror not instantaneously, but with a certain delay of the order of the cavity decay time ($\tau_* \approx \kappa^{-1}$), therefore

$$F_{RP} = F_0 - Kx(t - \tau_*) \approx F_0 - Kx(t) + K\tau^*\dot{x}(t).$$

In this equation there is not only spring constant K, but as well viscous damping constant $H = K\tau_*$. Importantly, these two constants have opposite signs, that is if the radiation pressure creates positive optical spring ($K > 0$), it introduces negative damping ($H < 0$) and the other way round.

To illustrate let us consider a Fabry-Pérot cavity with a movable mirror. If we choose the red detuning, that is tune on the left slope $\omega_p < \omega_{cav}$, then if the cavity elongates, so that $L \to L+x$, then the resonant frequency decreases. As a result, the circulating power increases and the resulting additional force of the radiation pressure will push the mirror further away to elongate the cavity even more. This corresponds to the negative optical spring coefficient $K < 0$.

In case of detuning to the right slope $\omega_p > \omega_{cav}$, the elongation of the cavity leads to decrease of the power and the additional radiation pressure force is restoring. This is equivalent to the positive optical spring $K > 0$.

The additional spring constant and damping are added to the own dynamics of the mechanical oscillator. If we write the classical equation of motion of the mirror, which is a damped harmonic oscillator itself, it will read

$$m\ddot{x} + h\dot{x} + kx = F_s + F_0 + Kx + H\dot{x},$$

or

$$m\ddot{x} + (h - H)\dot{x} + (k - K)x = F_s + F_0.$$

Here F_s is the sum of all forces unrelated to optical springs.

The additional spring and damping (one of which is necessarily negative) are added to the intrinsic spring and damping of the oscillator. This might cause a situation where the total damping or spring is negative, which causes instabilities in steady state.

One way to get rid of instabilities is to get rid of the steady states and go to pulses.

B. Langevin equations

The missing element of our consideration has to deal with the fact that the Hamiltonians written so far concern the intracavity field, however, the quantum states of interest are defined in the field outside. To take that into account we write the Heizenberg-Langevin equations of motion for the intracavity field operators a, b.

1. State swap

First let us consider the state swap Hamiltonian $H_{BS} = -\hbar g a^\dagger b + \text{h.c.}$ The equations of motion then are

$$\dot{a} = igb - \kappa a + \sqrt{2\kappa}a^{in},$$
$$\dot{b} = iga - \frac{\gamma}{2}b + \sqrt{\gamma}b^{th}.$$

where a^{in} and b^{th} are the input fluctuations. Decay rates are. The correlations are

The a^{in} carries the quantum state of interest. Let it be defined in a pulse $0 < t < \tau$. If the pulse is short $\gamma n_{th}\tau \ll 1$, we can ignore the mechanical decoherence. If then the optical decay sufficiently exceeds the optomechanical coupling rate, we can eliminate the cavity mode, which yields

$$0 = igb - \kappa a + \sqrt{2\kappa}a^{in},$$
$$\dot{b} = iga.$$

From where we get

$$a = i\frac{g}{\kappa}b + \sqrt{\frac{2}{\kappa}}a^{in}, \qquad (5)$$

and

$$\dot{b} = -Gb + i\sqrt{2G}a^{in},$$

where we defined $G \equiv g^2/\kappa$. We immediately obtain the solution for b:

$$b(\tau) = b(0)e^{-G\tau} + i\sqrt{2G}e^{-G\tau}\int_0^\tau d\xi\, a^{in}(\xi)e^{G\xi}. \qquad (6)$$

From this equation one sees that there is one temporal mode of the input field a^{in} that is naturally coupled to the mechanical mode. This is the exponentially growing mode with the annihilation operator defined as

$$A^{in} = \sqrt{\frac{2G}{e^{2G\tau} - 1}}\int_0^\tau d\xi\, a^{in}(\xi)e^{G\xi},$$

where the normalization factor in front of the integral secures the proper commutations:

$$\begin{aligned}&\left[A^{in}, A^{in\dagger}\right] \\ &= \frac{2G}{e^{2G\tau}-1}\iint_0^\tau d\xi\, d\xi\,'\left[a^{in}(\xi), a^{in\dagger}(\xi')\right]e^{G(\xi+\xi')} \\ &= \frac{2G}{e^{2G\tau}-1}\int_0^\tau d\xi\, e^{2G\xi} = 1.\end{aligned}$$

The state of the mechanical mode after the interaction therefore reads

$$b(\tau) = b(0)e^{-G\tau} + i\sqrt{1 - e^{-2G\tau}}A^{in}.$$

We have to solve for the leaking field a^{out}, for this we use the input-output relations

$$a^{out}(t) = -a^{in}(t) + \sqrt{2\kappa}a(t). \qquad (7)$$

By advantage of Eqs. (5) and (7) we obtain

$$\begin{aligned}&a^{out}(t) = a^{in}(t) + i\sqrt{2G}b(t) \\ &= a^{in}(t) + i\sqrt{2G}\underline{b(0)e^{-Gt}} - 2Ge^{-Gt}\int_0^t d\xi\, a^{in}(\xi)e^{G\xi}.\end{aligned}$$

In a full analogy with the input state that we defined in a certain temporal mode, we select one output mode that is coupled to the mechanical motion. The underlined term shows that the mechanical motion couples best to the exponentially decaying mode

$$A^{out} = \sqrt{\frac{2G}{1 - e^{-2G\tau}}}\int_0^\tau d\xi\, a^{out}(\xi)e^{-G\xi}.$$

Furthermore,

$$\begin{aligned}&\int_0^\tau dt\, a^{out}(t)e^{-Gt}\\&\quad=\int_0^\tau dt\, a^{in}(t)e^{-Gt}+i\sqrt{2G}b(0)\int_0^\tau dt\, e^{-2Gt}\\&\qquad\qquad-2G\int_0^\tau dt\, e^{-2Gt}\int_0^t d\xi\, a^{in}(\xi)e^{G\xi}\end{aligned}$$

The last term can be simplified

$$\begin{aligned}&-2G\int_0^\tau dt\, e^{-2Gt}\int_0^t d\xi\, a^{in}(\xi)e^{G\xi}\\&\quad=-2G\int_0^\tau d\xi\, a^{in}(\xi)e^{G\xi}\int_\xi^\tau dt\, e^{-2Gt}\\&\quad=-2G\int_0^\tau d\xi\, a^{in}(\xi)e^{G\xi}\frac{e^{-2G\xi}-e^{-2G\tau}}{2G}\\&\quad=e^{-2G\tau}\int_0^\tau dt\, a^{in}(t)e^{Gt}-\int_0^\tau dt\, a^{in}(t)e^{-Gt}.\end{aligned}$$

Therefore,

$$\begin{aligned}A^{out}&=\sqrt{\frac{2G}{1-e^{-2G\tau}}}\int_0^\tau dt\, a^{out}(t)e^{-Gt}\\&=i\sqrt{\frac{2G}{1-e^{-2G\tau}}}\sqrt{2G}b(0)\frac{1-e^{-2G\tau}}{2G}\\&\quad+\sqrt{\frac{2G}{1-e^{-2G\tau}}}e^{-2G\tau}\int_0^\tau dt\, a^{in}(t)e^{Gt}\\&=i\sqrt{1-e^{-2G\tau}}b(0)+e^{-G\tau}A^{in}.\end{aligned}\tag{8}$$

Finally, defining the input mechanical state as $B^{in}=b(0)$ and final as $B^{out}=b(\tau)$, and effective transmittivity of the pulsed protocol as

$$T=e^{-2G\tau},$$

we can rewrite the Eqs. (6) and (8) in form

$$\begin{aligned}B^{out}&=\sqrt{T}B^{in}+i\sqrt{1-T}A^{in},\\A^{out}&=\sqrt{T}A^{in}+i\sqrt{1-T}B^{in}.\end{aligned}$$

2. *Entanglement*

For the Hamiltonian $H_{AMP}=-\hbar g a^\dagger b^\dagger+$h.c. the equations of motion read

$$\begin{aligned}\dot a&=igb^\dagger-\kappa a+\sqrt{2\kappa}a^{in},\\\dot b&=iga^\dagger-\frac{\gamma}{2}b+\sqrt{\gamma}b^{th}.\end{aligned}$$

Using the very same approach of eliminating mechanical decoherence and the cavity mode, one can show that these equations lead to input-output transformations

$$\begin{aligned}A^{out}&=\sqrt{K}A^{in}+i\sqrt{K-1}B^{in\dagger},\\B^{out}&=\sqrt{K}B^{in}+i\sqrt{K-1}A^{in\dagger},\end{aligned}$$

where the optical modes are defined as

$$\begin{aligned}A^{in}&=\sqrt{\frac{2G}{1-e^{-2G\tau}}}\int_0^\tau dt\, a^{in}(t)e^{-Gt},\\A^{out}&=\sqrt{\frac{2G}{e^{2G\tau}-1}}\int_0^\tau dt\, a^{out}(t)e^{Gt}.\end{aligned}$$

C. Experiments

Important experiments were performed with the pulsed opto(electromechanics).

In microwaves: cooling the mechanical motion, entanglement between microwaves and mechanical motion, capture of a photon.

In optics: non-classical correlations between photon and phonon, Hanbury-Brown and Twiss interferometry. Entanglement between distant mechanical oscillators.

IV. NONDEMOLITION INTERACTION

To obtain nondemolition interaction from the Hamiltonian (3) we assume resonant pump ($\Delta=0$) and modulation of the pump strength ($g\to g\cos\omega_m t+\phi$). Gathering this altogether and dropping the rapid terms, we obtain

$$H_{QND}=-\hbar\frac{g}{2}(ae^{i\theta}+a^\dagger e^{-i\theta})(be^{i\phi}+b^\dagger e^{-i\phi}),$$

which is a Hamitonian of a Quantum Nondemolition interaction, we will rewrite it in terms of quadratures of cavity $X_c=a^\dagger+a$ and $P_c=(a-a^\dagger)/i$ (defined so that $[X_c,P_c]=2i$):

$$H_{QND}=-\hbar\frac{g}{2}(X_c\cos\theta-P_c\sin\theta)(q\cos\phi-p\sin\phi).$$

In the particular case of $\theta=\phi=0$,

$$H_{QND}=-\hbar\frac{g}{2}X_c q.\tag{9}$$

This Hamiltonian couples amplitude quadratures of cavity and mechanical oscillator. A proper choice of phase of the pump θ or modulation ϕ allows to couple different quadratures. This interaction was intensively investigated for the gravity-wave detection, since it does not disturb ("demolish") a quadrature of each of the oscillators. Indeed, the equations of motion read

$$\begin{aligned}\dot X_c&=0 &\qquad \dot P_c&=gq,\\\dot q&=0 &\qquad \dot p&=gX_c,\end{aligned}$$

Due to the fact that the Hamiltonian (9) represents the full Hamiltonian in the rotating frame (there are no other terms), the quadratures X_c and q stay unperturbed for the entire time of the evolution.