HOW TO STUDY HOW TO SOLVE

Arithmetic—Calculus

H. M. DADOURIAN

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Arithmetic through Calculus

by

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PREFACE

The success of the pamphlet published two years ago under this title encouraged the author to expand it into a small book. Parts I and II constitute a revision of the pamphlet. Part III contains the bulk of the new material.

The book has been made suitable for secondary school as well as college students by replacing illustrative examples from Analytics and Calculus, in Part II, with examples from school mathematics.

In order to reassure those who, like the author, are allergic to pedagogical devices which are offered as substitutes for the exercise of intelligence, it may be stated that the directions for solving problems are designed to help the student to avoid self-created obstacles and to deepen his understanding of the principles involved in the solution of problems.

While the book is addressed to students, it is hoped that it will prove suggestive to teachers.

I am indebted to my colleague and friend, Professor Thurman L. Hood, for reading the manuscript of Part I and for making valuable suggestions.

H.M.D.

Hartford, Conn.
May, 1951.

INTRODUCTION

Students who do poorly in mathematics and mathematical sciences ascribe their troubles to the difficulty of the subjects and to their own lack of mathematical talent. It is true that these subjects are relatively difficult, and that the study of advanced mathematics requires a special type of aptitude. But forty-five years' experience in teaching mathematics, physics, and mechanics has convinced me that most of the student's troubles arise not so much from his lack of mathematical talent or from the intrinsic difficulty of the subjects as from the following sources:

1. The student's fear of mathematics, and his notion that it will be of no real value to him in his future work. These ideas create psychological barriers and blocks in the path of the student's progress.

2. Failure to associate mathematics with experience and common sense, leading to mystification and befuddlement.

3. Lack of knowledge of how to study effectively, which results in waste of time and effort.

4. Disorderly ways of attempting to solve problems, which expose the student to blunders and errors.

It is the aim of this book to help the student to remove these extraneous sources of trouble. The book is divided into the following three parts:

Part I. How to Study. This part is concerned with the most effective ways of studying mathematics, and also the reasons for studying it.

Part II. How to Solve. In this part, general yet definite directions are given for solving problems so as to avoid self-created obstacles. These directions are explained and illustrated with examples.

Part III. Arithmetic through Calculus. The purpose of this part is to elucidate and illustrate certain pivotal topics and methods of school and college mathematics, and to further illustrate the general directions for solving problems.

Parts I and II may be read with profit by secondary school, as well as college, students. Some sections of Part III may be read as review and refresher material, and other sections as supplementary material on topics currently under study.

PART I. HOW TO STUDY

MENTAL ATTITUDE

Most of the conditions for effective study are the same for all subjects—conditions such as good health and eyesight, proper lighting, freedom from distractions and worries, adequate knowledge of prerequisite subjects, and balanced allotment of time to rest, recreation, and work. In this part of the book we shall discuss the mental attitudes and habits of work which are conducive to effective study and, consequently, to economy of time, effort, and thought.

Mental attitude is one of the most important factors affecting the student's success in mathematics. When an intelligent student who does good work in other subjects fails in mathematics, the main reason is likely to be his adverse attitude toward the subject. We shall discuss mental attitude under the headings *incentive*, *interest*, *self-confidence*, *common sense*, and the *will to learn*.

Incentive. Incentive is a powerful factor in the successful accomplishment of any task. Unfortunately, many students find little or no incentive for the study of mathematics. They often ask, "What good is this course to me, since I do not intend to become a mathematician, a scientist, or an engineer?" This is a legitimate question. It should be asked; but it should be asked with the object of finding the right answer, not as a justification for avoiding the subject or as an excuse for doing poor work in it. Furthermore, similar questions should be asked about every subject the student is required to study.

It is true that the majority of high school and college graduates do not make direct use of most of the mathematics they study. But this is the case also with other subjects. Few who study literature become professional writers; few who study history become historians, revolutionists, or generals. Why, then, study a subject which one is not going to use directly in his future vocation? The answer to this question is to be found in the function and aim of general education.

The aim of general education is to help the student acquire knowledge, appreciate beauty, and develop imaginative and intellectual power. The study of mathematics plays an important role in the attainment of every one of these objectives.

Mathematics is a science, and as such it is a source of knowledge. It is the most exact, elegant, and advanced of the sciences, and for this reason it has been called the Queen of the Sciences. Nothing, not even the miracles of the physical sciences and technology, gives one a better idea of the apparently unlimited capacity of the human mind than higher mathematics.

Mathematics is also a language. It is the most perfect and the most powerful of the languages created by man. It is a universal language; even more universal than music, which differs from country to country. It is a unique language in the sense that no other language can take its place. Because, as a language, mathematics serves the sciences, it has been called the Handmaiden of the Sciences. When a chain of reasoning is carried out in the language of mathematics, the reasoning process becomes surer and swifter. When a law of Nature is expressed in the mathematical language, it not only becomes more precise, but its hidden implications come to the surface. An illustration of this characteristic of mathematics is afforded by Clerk Maxwell's discovery, which led to the development of radio, radar, and television. When Maxwell expressed the experimental laws of electromagnetism in the form of the four equations named after him, he discovered the fact that electromagnetic disturbances travel with the velocity of light. This prompted Heinrich Hertz, who experimentally proved Maxwell's discovery, to exclaim, "Mathematical equations are wiser than we!"

Mathematics is an art characterized by simplicity of form and of expression. Sculpture is the art of creating patterns of shapes. Painting is the art of creating patterns of shapes and colors. Music is the art of creating patterns of sound. Literature is the art of creating patterns of human behavior expressed in words. Mathematics is the art of creating patterns of ideas expressed in symbols.

Mathematics, like music, is a less imitative and a more imaginatively created art than either sculpture or painting; it is the finest of the fine arts. "Mathematics," wrote Bertrand Russell, "possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, sublimely pure, and capable of a stern perfection such as only the greatest art can show." Alfred North Whitehead, another philosopher-mathematician, wrote, "The science of pure mathematics, in its modern development, may claim to be the most original creation of the human spirit."

The development of imaginative faculty and intellectual power is the most important and the most difficult function of education. Mathematics is a powerful means of developing them. "We repeat," said Voltaire, "there was more imagination in the head of Archimedes than in that of Homer."

It has been recognized by wise men of all ages that mathematics is the most potent instrument that can be used to sharpen the mind; that it is the best means of training in analysis and synthesis—that is, taking apart component elements of a problem and then reintegrating them into a new whole, and weighing the pros and cons of an issue in the scales of reason. This character of mathematics makes its study valuable to anyone, whether or not he expects to make direct use of it later.

At critical moments of life man is faced with problems which call for solution; he has to make decisions which must be based on a correct analysis of the given conditions. Such is the case, for instance, when a physician has to prescribe for a patient or a judge has to render a decision. The former must make a correct diagnosis on the basis of the symptoms, and the latter must weigh the evidence. The training in analysis and synthesis which an intelligent study of mathematics affords prepares one to meet such situations.

Even in the study of such a seemingly unrelated subject as law, the value of mathematics is forcefully brought out in the following statement by William B. Munro, a distinguished Professor of Government:

"In the Harvard Law School there are more than a thousand students, all of them college graduates, drawn from every section of the country. Nearly all of them have specialized during their undergraduate years in some single subject or group of subjects—language, history, science, philosophy, economics, mathematics, and so on. Offhand one would probably say that the young man who had devoted most of his attention to the study of history, government, and economics while in college would be gaining the best preparation for the study of law, for these are the subjects which in their content come nearest to the law, but that is not what we found. On the contrary, the results of the enquiry showed that the young men who had specialized in the exact sciences, and especially in mathematics, were on the whole better equipped for the study of law and were making higher rank in it than were those who had devoted their energies to subjects more closely akin . . ." *

The authors of the *Harvard Report* have this to say about the place of mathematics in general education:

"We have already emphasized the indispensable part which mathematics plays in the study of the natural sciences. This by no means exhausts its position as a tool and as an effective mode of thought in general education. In subjects other than the sciences—notably in economics, psychology, sociology, and anthropology—frequent and increasing use is made of graphic presentation of data, of statistics, and of simple algebraic formulas. Almost all students meet one or more of these fields either in the course of formal education or later, and hence should be prepared early with the simple mathematical techniques required for their pursuit.

"This argument has particular and immediate force for the prospective college student. But the need of elementary mathematics in fact involves a much larger and a constantly growing section of the general population. The complexities of organization and technology in modern industry, in government, and in the national defense make increasing demands upon the mathematical equipment and skills of the ordinary participant and worker . . .

"Beyond this, however, mathematics has an important and increasing role in general education. It helps build

* William B. Munro; *Bulletin of the Am. Assoc. of University Professors*, **XI**, 411.

some of the skills and comprehensions that make the effective individual. Within the past fifty years mathematics and logic have been fused into a single structure. Insofar as logical thinking is rigorous, abstract, and rational, its connection with mathematics is obvious. The ability to analyze a concrete situation into its elements, to synthesize components into a related whole, to isolate and select relevant factors, defining them rigorously, meanwhile discarding the irrelevant; and the ability to combine these factors, often in novel ways, so as to reach a solution, all are important features of mathematical procedure." *

Interest. Interest is an even more potent element of effective study than incentive. A student may have incentives for the study of a mathematical subject, yet he may not find it interesting. An engineering student, for instance, has every incentive for the study of the calculus, yet he may find it dull. In that case he will be greatly handicapped, because he will have difficulty in learning and in retaining what he has learned. To become interested in mathematics one has to understand its concepts and principles. Learning by rote and grinding out answers to problems by the mere use of formulas does not promote interest in the subject; on the contrary, it destroys it.

There is a close analogy between the functioning of the mind and that of the digestive system. A point-to-point correspondence could be established between the two, from the raw materials of intellectual and physical foods, through their preparation, consumption, digestion, and assimilation. As with ordinary foods, there is a great variety of intellectual foods which have to be prepared,

* *Harvard Report on General Education in a Free Society,* p. 160, 1945.

served, and consumed in different ways. It is the job of authors of textbooks and of instructors to prepare and serve intellectual food properly; it is up to the student to consume it in a way which is conducive to enjoyment, digestion, and assimilation. Some ordinary foods, such as ice cream, need not be chewed, while others, such as steak, have to be thoroughly masticated to be enjoyed and digested. In our analogy, mathematics corresponds to the steak. Attempting to learn mathematics by reading the text perfunctorily or by merely memorizing formulas is like swallowing chunks of steak; it results in mental indigestion, revulsion, and loss of interest.

Mathematics is generally regarded as a dry subject. One reason for this view lies in the contrast between the character of mathematics and the nature of man. Mathematics has no emotional content; man is largely an emotional being. Mathematics is an abstract science; man is mainly interested in the concrete. Reasoning plays a dominant role in mathematics; reasoning comes hard to man. But while mathematics itself has no emotional content, success in it creates aesthetic and intellectual satisfaction. Success in the solution of a difficult problem, for instance, brings to one the joys that accompany creative work and discovery. While mathematics is an abstract science, it is applicable to innumerable concrete problems, not only of science and technology, but also of daily life. While to reason things out is irksome to most men, the intellectual power developed by the study of mathematics creates confidence in one's reasoning process and makes the use of that power a source of enjoyment.

There are other factors which contribute to the notion that mathematics is uninteresting. Having to learn arithmetic by memory and constant repetition of simple operations gives the impression to the child that mathematics is devoid of pleasurable ideas. Memorizing the multiplication table cannot be regarded as an exhilarating experience. Even adults often equate mathematics to arithmetic when they say, "Being a mathematician, you must be a lightning calculator." The notion that mathematics is uninteresting is not always weakened by the study of other courses in school mathematics; on the contrary, in many cases it is changed into conviction. Consequently, most students avoid, whenever possible, not only courses in mathematics but also in other sciences which are served by mathematics.

A detailed discussion of how to arouse and sustain interest in the study of mathematics is beyond the scope of this book, but a few general comments on the subject will not be out of place here. Mathematics can be made more interesting by a greater emphasis on applications, and in the case of arithmetic by the introduction of laboratory methods. For instance, instead of asking the child to memorize $4 \times 4 = 16$, he can be asked to form a square of four rows of four blocks each and to find the total number by counting. Such a procedure not only introduces an element of discovery into this simple problem, but also gives the teacher an opening to lead the child to the fact that multiplication is a short cut to adding equal numbers, and to the concept of the square.

Instead of offering different mathematical subjects in succession, and in distinct, unrelated and large doses, they could be taught more or less simultaneously and gradually, as in some European countries. The presentation of each mathematical subject could be streamlined by omitting from textbooks topics which do not belong to the main line of advance or which could be postponed. The main object of the study of geometry should be to acquaint the student with the deductive method of reasoning. This objective could be better attained by studying fewer theorems and solving some original problems, rather than by spending a whole year on hundreds of theorems. The time thus made available could be used in teaching the elements of analytic geometry and the calculus, which are far more interesting than the subjects which precede them.

Most of the interesting applications of mathematics are to the natural sciences. For this reason, as well as for the purpose of fitting education to the needs of modern times, the elements of the natural sciences could be taught continuously in primary and secondary schools, as mathematics and English are taught.

Putting some of these suggestions into effect would require a greater number of teachers and more money for education, but in these days of astronomical figures in the national budget, the addition of a billion or two would not materially affect the tax burden.

Self-confidence. Many students acquire an inferiority complex toward mathematics. "I am not a born mathematician" or "I am no good at mathematics" is a common plaint, offered as an excuse for avoiding a course in mathematics or for doing poor work in it. This would be a valid excuse if the student were required to do creative work or to take courses in the upper reaches of mathematics, where the subject becomes highly abstract and far removed from ordinary human experience. Mathematics, like a mountain trail, becomes steeper and its atmosphere increasingly rarefied as the summit is approached. But, as in the case of most mountain trails, the going is relatively easy on the lower stretches.

The average student is familiar, if vaguely, with most of the concepts introduced in elementary college mathematics. These concepts are given precision and a high degree of definition, to be sure, but even then they are well within the student's experience and intuitive knowledge. Furthermore, new concepts are thoroughly explained and profusely illustrated in textbooks and by instructors; successive courses are dovetailed and the subject matter in each course is gently graded. It is no more necessary to be a born mathematician to do reasonably well in the subject than it is to be a born poet to read poetry.

Even a weak foundation in a prerequisite course is not fatal to success. Most of the student's difficulties are due not so much to a shaky foundation as to the persistence of his adverse attitude toward the subject and his faulty methods of study and work, which have made his foundation shaky.

In a mathematics course, the more important principles, processes, and formulas of preceding courses are constantly used. While this feature of mathematics makes the subject difficult for the negligent student, to the conscientious one it affords an opportunity for improving and strengthening his foundation and for gaining confidence.

Common sense. It is not uncommon for an intelligent and sensible young man to talk, act, and work in class as if he had divested himself of his common sense before entering the room. He often states problems in a way which either makes no sense or is foreign to his usual manner of expressing himself; he does things or makes statements that are patently at variance with his knowledge, experience, and common sense. In short, he behaves in class in a manner which is detrimental to his progress in the subject and which is discouraging and disconcerting to the instructor who tries to help develop the intellectual power of his students. The following instances, drawn from actual classroom experience, illustrate the results of failure to associate mathematics with common sense.

A student, asked to state the problem which he had solved at the blackboard, began:

Student: You have a boat. . . .

Instructor: I have no boat. . . .

Student: They want . . .

Instructor: Who are *they?*

Student: The authors of the book.

Instructor: The book has one author; besides, who cares what the author of the book wants? Why not state the problem simply, without dragging in irrelevant matters?

No one would think that three heirs of an impecunious person would become millionaires, to say nothing of becoming infinitely rich, by dividing a zero inheritance among themselves. Yet some college freshmen do not hesitate to write $\frac{0}{3} = \infty$!

Every grocery boy knows that the total cost of a number of items is obtained by *adding* the costs of the individual items; yet a college student *multiplied* the expressions for the costs of three items to obtain the total cost.

Another student obtained $165,000 for the tax to be paid on an income of $5,000, under a simple and sensible income tax law stipulated in a problem; yet the absurdity of the result did not seem to bother him in the least.

After having read the proof of the formula $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$, and while this formula is staring him in the face from a page in his book, where it is printed in black type and conspicuously placed, some students show no hesitation in writing $\sin(30° + 45°) = \sin 30° + \sin 45°$.

A few years ago, I had two sections of a calculus class, one of which made better progress than the other. In a monthly test given to the faster section, one of the problems required the minimum time necessary to go from a point on one bank of a river to another point on the opposite bank, downstream. Before applying the method of the calculus to find the shortest time required for the journey, the students had to find the algebraic expression for the total time necessary to drive a distance along the river and then to go by boat to the second point. Most of the students wrote the right expression for the time and proceeded to minimize it; but a few either multiplied the distances by the velocities or divided the velocities by the distances to obtain the expression for the time.

When I commented on these errors before the slower section, the class not only saw the point but laughed heartily as if to say, "How stupid! Everyone knows that the time of travel is obtained by dividing the distance by the velocity." Yet when this class encountered the same problem in a test a week later, one-third of them multiplied the distances by the velocities, and another third divided the velocities by the distances.

Mathematics is a subject in which certain aspects of common sense are refined, made more precise, and extended. If the student looks at mathematics in this light, he is more likely to avoid the kind of absurd mistake which is the result of considering mathematics as some kind of hocus-pocus, unrelated to common sense and experience.

The will to learn. One cannot become learned by luck, or by gifts, or by bequest, as one might become rich over night by such means. One has to work for it. On the other hand, intellectual wealth cannot be lost or taken away, as can ordinary wealth.

It has been truly said that there is no royal road to learning. In fact, the way to learning is not a road; it is a trail, or at best a path, which does not admit vehicles. Progress on it can be made only through personal exertion. Books make available to us the path, and teachers show the way; but everyone has to do his or her own learning. Furthermore, progress on the path of learning requires determination and perseverance—*the will to learn.* This is true even for persons who achieve greatness in the arts and the sciences; they become great not only because they are talented, but also because they work harder than others in the same field.

Genius has been defined as *the capacity for work.* Few have the capacity of a genius, but everyone has the opportunity to work to the limit of his or her own capacity. In the words of La Rochfoucauld, "Our capacity exceeds our will power, and it is often only to excuse ourselves that we hug the belief that things are impossible." Be on guard against becoming a victim of such an excuse.

IN ONE'S STUDY

In the last analysis, true education is self-education. Lectures, recitations, instructors, classes, and the other instruments of formal education are adjuncts which facilitate self-education but cannot take its place. This is especially true of education through the study of mathematics. The very nature of mathematics and its exacting character require *learning by reasoning*, *learning by doing*, *regularity in study*, *concentration*, *carefulness*, *orderliness*, and *thoroughness*.

Learning by reasoning. We learn by observing, memorizing, reasoning things out, and doing. Every one of these means is important to learning, but their relative importance changes with the mental age of the learner and from subject to subject. In primary education memorizing plays a more important role than reasoning. Manual and other physical skills such as typewriting and swimming require a greater emphasis on doing than on any of the other means of learning.

By the time a student enters high school his memory and his powers of observation are fully developed, whereas his reasoning power is at a rudimentary stage.* Therefore the main object of secondary and higher education is, perhaps I should say should be, the development of the reasoning power. Mathematics is one of the best means of developing that power and can be studied effectively only by using it.

The student who attempts to learn mathematics by memorizing without understanding, and by mechanically applying formulas to obtain answers to problems, derives no real benefit from its study. Furthermore, the subject appears to him as a maze of formulas, equations, theorems, derivations, and problems of great diversity, too complex to comprehend and to retain in memory. On the other hand, if he learns the subject by understanding its concepts, principles, and processes, he finds that the complex appearing mass of material has a simple structure which can be comprehended and retained with relative ease.

The term "chain of reasoning" is especially pertinent to mathematics. If the principles and procedures involved in the solution of a problem are well understood, the links

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* It seems to me that primary schools do not take full advantage of the child's power of observation and inquisitiveness.

in the chain of reasoning follow one another naturally, as the links of a material chain do when it is pulled at one end.

By emphasizing understanding and reasoning things out we do not mean to imply, however, that a certain amount of memory work is not necessary. The student should make a conscious effort to memorize the simpler and the more frequently used formulas, for otherwise he will be forced to hunt for them constantly, and consequently his reasoning process will become clogged and retarded.

It is generally recognized that skill acquired in the performance of one kind of work is not transferable to an entirely different type of work. For example, skill in typing is not transferable to swimming. From this fact, some educators have concluded that skill acquired by the study of mathematics is not applicable to nonmathematical fields. This is true when mathematics is studied purely as a skill to be acquired by manipulating symbols and is memorized by imitation and constant repetition, without understanding. But when it is studied through understanding, the intellectual power that is developed thereby becomes available for application in nonmathematical fields, as illustrated by the quotation on page 4.

Learning by doing. Knowledge that is acquired by observing, by memorizing, and even by reasoning is likely to be soon forgotten and lost. But when *doing* is added to the other means of acquiring knowledge, the result is far more lasting. A swimmer never forgets how to swim.

Knowledge which is acquired without doing is generally known as "book learning." It is referred to contemptuously because its possessor is unable to apply it. The expression is unfortunate because it may give a wrong impression of the value of books. In general, it is far more economical of time and effort to learn from the experience of others than to learn by personal experience, and it is less likely to prove disastrous. It is not pleasant to learn by personal experience that falling off the roof is not healthy. Books afford us the opportunity to learn from the knowledge and wisdom acquired by countless generations through toil and tribulation. However, in order that book learning become of real value, it has to be transmuted into something approaching learning through personal experience, by putting it into practice.

The student may think, for instance, that he understands a definition because he is familiar with the dictionary meanings of the words and the grammatical structure. Yet he may be mistaken. One may think, for instance, that he understands the famous definition, "Energy is the capacity for doing work," but unless he has a working knowledge of problems involving energy, the definition cannot mean much to him.

Learning mathematics by doing means reproducing on paper the analytical work which is in the textbook, filling the gaps which are left in the analyses with such statements as "It may be shown that . . . ," and "Solving . . . we obtain." It means solving problems, particularly the problems which appear to be difficult.

Mathematics textbooks contain many problems under a given topic. In most cases it is not possible to solve all the problems in the time at the disposal of the student. Furthermore, it would be a waste of time to solve them all, since many of them differ only in incidental details such as numerical values. It is desirable, therefore, to proceed as follows: solve one problem of each type; read the other problems to find out whether or not you can solve them, and then try to solve only those that appear to be difficult. (More on this under Direction 4 in Part II.)

Regularity in study. Courses in mathematics and topics in a given course form a hierarchy; that is, each topic or each course is, in general, based on the preceding topics or courses. To put it another way, mathematics is built vertically, like a skyscraper whose upper stories are supported by the lower. This characteristic makes regularity of study not only desirable but absolutely necessary. Neglect of a few lessons makes it difficult, if not impossible, to understand the lesson of the day. In a descriptive subject such as history, which may be said to be built laterally, it is possible to begin in the middle of a book and get some benefit from reading it. But such is not the case in mathematics; one may not be able even to read the book, to say nothing of understanding what is in it.

While the hierarchic nature of mathematics is a source of difficulty to the negligent student, it is a boon to the conscientious, because, as has already been pointed out, while one part of mathematics is being studied, constant use is made of what has preceded it. This affords the student an opportunity to improve his knowledge of what he has learned in the past and to obtain a firmer grip on it.

Concentration. Mathematics is a highly condensed science. Because of its extensive use of symbols, ideas and processes of thought become compressed and compact when expressed in mathematical form. Economy of thought and expression is one of its most striking characteristics. The subject does not lend itself to perfunctory reading and listless listening. General and vague impressions of the meanings of its symbols are next to useless. The study of mathematics requires a high degree of concentration.

Concentration does not mean either tenseness or passive contemplation. It means, rather, reflection combined with action. In connection with solving a problem, reflection means analyzing the problem into its component parts and marshaling the facts which are relevant to the problem; action means actually carrying out the analytical work involved in the solution of the problem.

Carefulness and orderliness. Mathematics is a powerful yet a very delicate language. Its symbols stand for precise and definite concepts. It is the precision machine tool of thought. As an automobile whose front wheels are properly aligned tends to keep in a straight course, mathematics keeps the reasoning process in line if due care is exercised and the analytical work is carried out in an orderly way. If a word is misspelled or omitted in a sentence in ordinary language, the mistake can easily be discovered by rereading it. But if a mistake is made in a mathematical expression, the mistake may not be discovered by reading it over. Furthermore, an error in a sentence is not likely to be carried over into succeeding sentences, whereas a mistake in a mathematical expression is almost certain to be carried over into succeeding expressions. For these reasons mathematical work has to be carried out not only carefully, but also in an orderly manner so as to reduce the chances of error. When a student fails to solve a problem which is within his competence, it is usually due to careless and disorderly ways of working. Specific suggestions will be made in Part II on how to go about solving problems, therefore we shall conclude this section with the following suggestions of a more general character:

Study at your desk or table, leaning over it, not with your feet on it.

Provide yourself with a soft pencil and sheets of paper. Write deliberately and try to organize the material as if it were intended for the printer. Avoid scratch-paper work as much as possible. Do scratch-paper work for purposes of trial and experiment only; do not let it take the place of an orderly solution of a problem.

At every step of the analysis, be sure that you know the reason for taking the step.

Thoroughness. Thoroughness—willingness to do more than one is required to do, the habit of doing a complete job—is a potent factor in success in any walk of life. The soldier who wins the Distinguished Service Medal is the one who does more than he is required to do in the line of duty.

Cultivate the habit of being thorough by solving problems so that the analytical work is complete, orderly, and neat. Like a good artisan, take pride in your work. Do not be satisfied with merely getting answers. Thoroughness may take more time in a given instance, but it is economical of time and labor in the long run.

IN THE CLASSROOM

Effective study requires that the student take full advantage of the opportunities offered by the classroom and the instructor. For this, it is necessary that he have a clear understanding and appreciation of their functions, and his responsibility toward them.

Function of the classroom. College students are required to attend classes, are allowed a certain number of cuts, are disciplined for over-cutting, are placed on the Dean's list which gives them the privilege of unlimited cuts, etc., as if attending classes were an unpleasant duty to be performed solely for the benefit of others. Such rules and regulations tend to sustain and promote the attitude toward school acquired in childhood days. They encourage the student to behave in the sphere of education in a way that no one behaves in any other sphere of human activity, with the possible exception of the inmates of certain institutions. No one has ever seen a group of young men rushing out of a restaurant rejoicing because they have been told that the dinner has been cancelled with no refunds. Yet that is exactly what happens whenever students arrive only to learn that the class has been cancelled.

It is not our purpose here to question the need of carrying into high schools and colleges requirements designed for grammar school children. It is rather to warn the student against succumbing to the psychological effects produced by these regulations.

Attendance in class is a privilege and an opportunity. The prerogative to attend classes is the real privilege, not the right of unlimited cuts. The classroom is an extension of the student's study; it is the place where he learns the niceties of the subject which he is likely otherwise to miss, where he obtains answers to questions which he has failed to find through his own efforts. It is not the place to learn the bulk of the factual material of the subject; the place for that is his study. The classroom is not the place merely for displaying one's knowledge or exposing one's ignorance; it is the place, rather, where the student adds to his knowledge.

Function of the instructor. The relationship of the student to his instructor is similar to that of a man climbing the Matterhorn to his Swiss guide. The guide's job is to show the way, the best way, to warn the climber against dangers, to help him over difficult places, to encourage him, to call his attention to interesting views, the surrounding mountains, and so on. Similarly, the instructor's function is to show the most effective way to learn the subject, to warn the student against common blunders, to help him overcome difficulties, to emphasize and to interpret fundamental concepts and principles, to systematize and organize the subject matter in order to give the student a better idea of what it is all about, and to inspire and promote the student's interest in the subject. It is no more the function of the instructor to do the work for the student than it is the job of the guide to carry the climber up the mountain. As the mountain climber has to do his own climbing, so does the student have to do his own learning.

Student responsibility. Attendance in class is not only a privilege and an opportunity, but also a responsibility. A class is a team, of which the instructor is the coach and the captain. The degree of success of the class depends not only on the instructor but equally on the other members of the team. Therefore you, as a member of the team, have a responsibility which you are duty bound to carry out properly. Failure to do so is not simply a personal matter with you, because it adversely affects your classmates, your instructor, and even your school or college. That responsibility extends even further—to your parents, who make sacrifices for your education, and to the benefactors and the members of the community who support your school.

Suggestions. Learn as much as you can from the textbook before going into class—the development of the ability to learn from books is one of the important elements of education. After your formal education is over, you can have a library but you cannot have a corps of instructors at your service.

Cultivate the habit of asking questions. Try to find satisfactory answers to questions yourself and, if you fail, take them to the instructor.

Pay strict attention to the advice, suggestions, and directions of the instructor. The average child is told by his elders, "Do this," and "Don't do that," so often, and without being given the reasons, that he develops immunity to suggestions. If this has been your past experience, try to rise above it.

Do not take notes in class unless asked to do so. It is far more important to understand what is being explained than to spend valuable classroom time in taking notes on material which may be found in the textbook, or which could be mimeographed and passed out.

Try to solve the more difficult problems and, if you fail, instead of merely telling the instructor so, show him your working papers, so that he may explain where you have gone wrong, or why you were stuck, and point out the way to proceed. In this way you learn more than if the instructor solves the entire problem for you.

In a mathematics class, students are often asked to explain the problems they have worked out at the blackboard. Pay careful attention to the explanation by a fellow student, not only to learn from his explanation but also to be in a position to contribute to the discussion when the opportunity arises. When called on to explain your own work at the blackboard, state the problem and explain your solution as clearly as you can. While listening to the instructor or a classmate, you are in the role of student, but while explaining your problem, you are in the role of teacher.

BEFORE AND DURING EXAMINATION

Before examination. Whatever the purpose of an examination may be from the point of view of the instructor, it should be considered by the student as an opportunity for a review which will enable him to see the "forest" through which he has been walking during the term. Even if a student is interested in nothing more than improving his final grade, he can do nothing better toward that end than to make a serious effort to review the subject with the purpose of seeing it as a whole.

The most effective method of reviewing is to make a written summary. The summary should be more than a table of contents or a collection of definitions, equations, and formulas. It should contain principles and procedures, clear and concise statements on their application, and illustrative examples. For instance, the statement that if a point lies on a curve the coordinates of the point satisfy the equation of the curve should be followed (a) by an explanation of how this enables us to determine arbitrary constants in the equation of a curve when the coordinates of as many points on the curve are given as there are constants in the equation, and (b) by an example which illustrates it.

The summary should be so organized, its different parts so coordinated and correlated, that other members of the class could profit from reading it. This may entail a change of the order in which the subject is presented in the textbook. For instance, in making a summary of the chapter on conic sections, it is desirable to reverse the order of the textbook, and to begin with the statement that if an equation of the second degree in two variables has a graph, it represents a conic, and then to give the conditions under which the equation represents a circle, or a parabola, or an ellipse, or a hyperbola.

Such a summary could not be expected from the majority of students without guidance. It is desirable, therefore, that the instructor give the class an outline of the summary with suggestions on how to expand it into a satisfactory summary. One may ask, "Why not have the instructor prepare the summary, and have it mimeographed and distributed?" The reason is simply that this would not fulfill the purpose of the summary, which is to make the student review the subject thoroughly and in the process of making the summary crystallize his ideas.

The summary should be prepared well in advance of the examination. Staying up late the night before an examination for last-minute cramming is not advisable. Success in a mathematics examination depends more on the ability to reason things out than on memory work, therefore it is more important to go to the examination room with a rested body and a clear head than with a tired body and a memory crammed with formulas.

During examination. Do not worry. Failure in an examination is not fatal.

Read each problem carefully to be sure that you understand what are given and what are required.

Use a soft pencil and write clearly and in an orderly manner.

Solve the easier problems first, but write the solutions in the order in which problems appear in the examination paper. This can be done by leaving blank spaces in the examination book for problems whose solutions are postponed.

Do not spend an undue amount of time on any one problem.

Do not leave the room before the end of the examination period. Instead, use any spare time you may have for checking your work, improving it, and making a thorough job of it.

PART II. HOW TO SOLVE

Often the student's failure to solve a problem is due more to his faulty approach and disorderly work than to the difficulty of the problem. He often fails to understand the nature of the problem, particularly a verbal problem, because he does not distinguish between the important elements and the incidental details. Even when he understands the nature of the problem and knows the processes involved in its solution, he may fail to solve it because the way he works exposes him to blunders and errors. It is not at all uncommon for members of a class to obtain widely divergent answers to a relatively simple numerical problem.

These causes of failure can be almost entirely removed and the student's chances of solving problems greatly increased if the directions given in this part of the book are followed conscientiously. Unfortunately, students disregard and sometimes even resent suggestions on the proper approach to problems and on blunderproof methods of solving them. Many a time the present writer has been confronted with the retort, "What do you care how I solve a problem so long as I get the answer?"

There are at least two good answers to such a question. First, "Unless you work in an orderly way the chances of your solving the problem and obtaining the right answer are greatly reduced." Second, and this is more important, "The object of solving a problem is not to obtain the answer; it is rather to carry out a chain of reasoning which leads to the answer. The answer to the kind of problem you are asked to solve has no value by itself, and cannot be exchanged for something of value. The profit you gain from solving a problem comes from the process of solving; and the magnitude of the profit depends on the way you solve."

It is very important that the student understand and appreciate the difference between the objectives of education and training on the one hand, and the objectives of business, industry, and other extra-educational activities, on the other. In the latter, the result is important, whereas in the former the *process* of obtaining the result is the primary consideration. When a man is in danger of drowning, for instance, the answer to his problem of saving himself is of supreme importance. If he succeeds in making the shore by clinging to a log and pushing it forward or by some other unprofessional means, no one would accuse him of having used a wrong technique. But when he is in training as a member of a swimming team, his problem is not to make the farther end of the swimming pool by any means which he may find convenient, it is rather to learn the particular stroke which will make him an efficient and fast swimmer.

The solution of a worthwhile problem requires the exercise of the faculties of initiative, imagination, and reasoning. No pedagogical method or rule can take the place of these faculties. However, the way the solution of a problem is attempted may help or hinder the application of these faculties. Therefore the following directions are offered not as substitutes, but as means of facilitating the application of these powers to the solution of problems. These directions help make the reasoning process smoother and the principles involved clearer; they are conducive to saving time, reducing the chances of blunders, and increasing the chances of success.

GENERAL DIRECTIONS FOR SOLVING PROBLEMS

1. Don't be afraid of the problem.
2. Read the problem carefully and determine what are given and what are required.
3. Restate the problem in its bare outlines, leaving out incidental details.
4. Formulate a plan of action, a strategy.
5. Using an appropriate system of notation, assign a symbol to each of the given and required magnitudes.
6. If the problem admits of a figure, draw a suitable one and label its parts.
7. Make a table of the given and required magnitudes.
8. Write all of the principal equations necessary for the solution of the problem before manipulating any of them.
9. Solve the equations simultaneously for the required magnitudes, and obtain an expression for each in terms of the given, and only the given, magnitudes.
10. In case there is only one, or a principal required magnitude, start with its mathematical definition or expression whenever convenient.
11. Write successive expressions of a required magnitude in the "columnar" form. Follow the straight-course method; avoid the zigzag method.
12. Before taking a new step in the analysis, put the last expression in as simple a form as possible.
13. Discuss the literal equation which gives a required magnitude in terms of the symbols of the given magnitudes.
14. Solve numerical problems as literal problems first, and introduce the numerical data after the expres-

sions of the required magnitudes are obtained in terms of the symbols of the given magnitudes.

15. Use your common sense at every step of the analysis. At each step ask yourself if the step you have just taken is sensible and then, at the end, whether the final result is reasonable.

EXPLANATION OF THE DIRECTIONS

1. Don't be afraid of the problem.

The average student appears to be fearful not only of mathematics in general, but also of problems whose solutions are not immediately evident to him. If a problem looks difficult, he is likely to give up the attempt at solving it, saying, "I don't see it." If the student "sees" the solution of a problem on first reading, it is not worth solving so far as he is concerned. Problems which are worth solving are the very ones which appear difficult. Furthermore, if the student follows these directions conscientiously, he will find that he *can* solve most of the problems which he is required to solve, even if they appear difficult.

It is not surprising that the student is afraid to tackle a problem which appears difficult. The solution of such a problem requires intellectual courage and initiative, and the exercise of imagination and reasoning power. But man is far less courageous in facing intellectual difficulties than in facing physical dangers; he shows a greater initiative in matters physical than intellectual; he is more used to sustained physical exertion than to mental effort; he is more at home in expressing his emotions than in forging a chain of reasoning.

There is a perfectly sound biological explanation for the fact that man is clumsy in the use of his brain and that he finds reasoning irksome. The human brain, the organ of reasoning, was the last to develop in the long process of evolution. Man is not as efficient in the use of his brain as he is, for instance, in the use of his hands, because he has not used his brain as long.

The study of mathematics, particularly the solving of difficult problems, is conducive to the development of intellectual courage, initiative, and reasoning power, provided it is done with understanding of the principles and processes involved, and provided problems are solved in a way which tends to make the reasoning process smooth, streamlined, and free of blunders. A poor swimmer fears to go into deep water and finds it irksome to learn to swim because he expends too much energy in keeping himself afloat and making headway, but as he becomes more efficient he gains confidence, and swimming becomes enjoyable. Solving problems presents an analogous situation. Fortunately, failure to solve a problem is not fatal; consequently there is no reason for fearing to tackle it. "The only thing we have to fear is fear itself."—Franklin D. Roosevelt.

2. Read the problem carefully and determine what are given and what are required.

The average student often has difficulty in understanding verbal problems. This is usually ascribed to his inability to read, but since problems are stated in simple language and in one or two short sentences this explanation does not seem adequate. The following appears to be a better explanation:

Most of the student's reading has been in descriptive subjects in which words do not have the degree of precision of mathematical and scientific terms, and ideas are not compressed into few words, as is the case in the statement of a problem. Consequently the student falls into the habit of reading casually and of being satisfied with obtaining a vague and general impression of what he reads.*

Unfortunately, vague and general impressions obtained by a casual reading of a problem are of little value to the understanding of its nature. The problem must be read carefully, with the object of determining exactly what magnitudes are given and required.

Note: We have used the words "given" and "required," instead of "known" and "unknown," because the former describe more accurately the conditions and magnitudes in mathematical problems, which are purely hypothetical.

3. *Restate the problem in its bare outlines, leaving out incidental details.*

The student is often confused and distracted by incidental details in a problem, such as numerical data. This source of difficulty can be avoided by restating the problem in its bare outlines, and indicating the details in a figure and a table of data.

Example. A ship is 20 miles to the west of another ship. The first ship is moving east at the rate of 10 miles per hour; the second ship is moving north at the rate of 15 miles per hour. Find the distance between the two ships 5 hours later.

This problem, which is solved on p. 43, may be restated as follows:

The initial positions, velocities, and directions of motion of two ships are given; it is required to find their distance apart at a later time.

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* It is one of the anomalies of this so-called scientific age that most of our youth go through school and college with little contact with science, and become adults ignorant of the scientific approach to problems of life, and untouched by the scientific spirit.

Prove and find problems. Some problems contain the terms "prove that" or "show that," while others contain the word "find." We shall call these *prove-problems* and *find-problems*, respectively, and their forms of statement *prove-form* and *find-form*.

A problem which is stated in the prove-form raises two types of difficulty that have nothing to do with the intrinsic difficulty of the problem. One of these is due to the fact that most students consider prove-problems to be more difficult, in general, than find-problems. When a class is given an opportunity to ask questions during the last class before an examination, the first question almost invariably is, "Are we responsible for proofs?" This happens even when the proofs are easier than most of the problems.

The other type of difficulty arises from the fact that in the prove-form the answer is usually given in the text of the problem. Because of this, the student is apt to take the answer to be one of the conditions or one of the given magnitudes of the problem; in other words, he confuses the "required" with the "given."

Both of these sources of difficulty can be avoided by restating a prove-problem in the find-form. Furthermore, the new statement can often be so phrased as to indicate the line of reasoning to be followed in the solution of the problem.

EXAMPLES. (1) Prove that the sum of the three angles of a triangle is equal to two right angles.

Find-form: Find the sum of the three angles of any triangle.

(2) Prove analytically that the medians of a triangle meet at a point.

Find-form: Find the coordinates of the points of intersection of the medians of a triangle and compare the results to see if the points coincide. (This form of the statement indicates the way in which the problem is to be solved.)

Nonmathematical phrases. Some verbal problems contain phrases which do not admit of literal (word-to-symbol) translation into mathematics, although they stand for mathematical concepts. In restating such a problem, these phrases should be replaced by equivalent phrases which can be so translated.

EXAMPLE. Water is flowing into a cylindrical reservoir at a constant rate. It is observed that the surface of the water is rising at the rate of 2 inches per minute. Find the rate at which water is coming in, if the reservoir is 20 feet in diameter.

Note: The rate at which water flows into a reservoir equals, if there is no outflow, the rate at which the volume of the water in the reservoir increases, and the rate at which the surface rises equals the rate at which the depth increases. Therefore the problem may be restated, in its bare outlines, in the following form, which lends itself to direct translation into mathematics:

Given the rate at which the depth of the water in a reservoir is increasing, find the rate at which the volume of the water is increasing. (This form of the statement of the problem indicates the way the problem is to be solved whatever the shape of the reservoir may be.)

Data

$$\frac{h}{t} = 2\,\frac{\text{in.}}{\text{min}},$$

$$2r = 20 \text{ ft},$$

$$i = ?$$

Solution. Let V and V' denote, respectively, the volume of the water at one instant and at an interval of time t later, h denote the increase in depth during the interval, and i denote the rate at which water is coming in (the current of water).

Then

$$
\begin{aligned}
i &= \frac{V' - V}{t} \\
&= \frac{\pi r^2 h}{t} \\
&= \pi r^2 \cdot \frac{h}{t} \\
&= \pi \times (10 \text{ ft})^2 \times 2 \frac{\text{in.}}{\text{min}} \\
&= 100\pi \text{ ft}^2 \times \frac{2 \text{ ft}}{12 \text{ min}} \\
&= \frac{50\pi}{3} \frac{\text{ft}^3}{\text{min}} \\
&= 52.36 \frac{\text{ft}^3}{\text{min}}.
\end{aligned}
$$

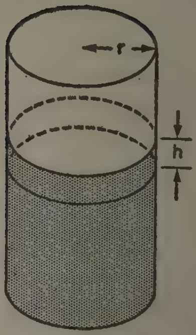


FIG. 1

4. Formulate a plan of action, a strategy.

The formulation of a plan of action is, in general, the most important part of the solution of a problem. When the correct strategy is thought out the problem is virtually solved, and the actual execution of the strategy becomes simply a matter of careful and orderly work.

Strategy has to be adapted to the circumstances in a particular case; therefore no general rule for a plan of action can be laid down. This is as true for attacking a problem as it is for attacking an enemy force. In general, the formulation of a plan of action amounts to stating, "If I do this, that, and the other, the problem will be solved."

EXAMPLES. (1) One hundred gallons of milk with a butterfat content of 5% is to be obtained by mixing cream of 35% butterfat and milk of 3% butterfat. Find the amounts of cream and milk to be mixed.

Strategy. The most important part of the strategy in this problem, which is solved on page 53, is the recognition of the fact that the amount of butterfat in each of the three liquids equals the product of the percentage of butterfat and the volume of the liquid. Bearing this in mind, we say, "The amount of butterfat in the mixture equals the sum of the amounts in the components of the mixture, and the volume of the mixture equals the volume of the sum of the components. The two equations thus obtained are sufficient to solve the problem."

(2) Prove that the points (5,5), (2,8), and (10,10) are the vertices of a right triangle.

Strategy. Before actually carrying out the solution of this problem, the student should say, "If I show that the sides satisfy the Pythagorean theorem, the problem is solved," or "If I show that the slope of one of the sides is the negative reciprocal of the slope of another side, the problem is solved."

Strategy in the use of formulas. In a problem which requires merely the application of a formula, the strategy, if it can be so dignified, consists in recognizing the form of the equation or expression to which the formula is to be applied.

When there are a number of formulas from which one must be chosen for application to an expression, the left-hand numbers of the formulas should be compared with the expression.

Sometimes the expression will have to be modified so as to put it in the form of the left-hand member of the correct formula. For instance, to apply the formula $(a + b)^2 = a^2 + 2ab + b^2$ to $(x + y + z)^2$, the latter has to be put in the form, say, $[(x + y) + z]^2$, and $(x + y)$ treated as a single symbol.

Sense of form. The sense of form plays a very important role in the study of mathematics. From the point of view of pure mathematics, the form of a formula is important, not the symbols it contains nor what the symbols stand for. If the formula $(a + b)^2 = a^2 + 2ab + b^2$ is written as $(\square + \bigcirc)^2 = \square^2 + 2\square\bigcirc + \bigcirc^2$, it would be valid whatever the squares and the circles contain, provided the contents of all the squares are the same and the contents of all the circles are the same. This characteristic of pure mathematics is brought out in the following paradoxical statement of Bertrand Russell, "Mathematics is the subject in which we never know what we are talking about, nor whether what we are saying is true."

Crucial conditions. In some applied problems a crucial condition is implied but not explicitly stated. In such a case the right plan of action cannot be formulated without recognizing that condition. In Example (1), the crux of the problem is, the fact that *the amount of butterfat equals the percentage times the volume.* The following mechanics problem affords another illustration.

EXAMPLE. Find the height of the starting point of a "loop the loop" necessary to enable a cyclist to make the "loop."

The crucial condition in this problem is that *the velocity at the top of the loop must be such that the centrifugal force equals the weight,* that is, $mg = \frac{mv^2}{r}$.

Some problems should be solved completely by carrying out the necessary detailed analysis, while others should be solved only mentally by formulating a plan of action. In this way a greater mastery of the subject can be obtained without too much expenditure of time.

5. *Using an appropriate system of notation, assign a symbol to each of the given and required magnitudes.*

A set of mathematical symbols is called a *system of notation.* Well-chosen notation is conducive to clear thinking, whereas badly chosen notation is likely to lead to confusion.

Letters of the first part of the alphabet, and letters of the last part with subscripts, are generally used to denote constants. Letters of the last part of the alphabet, without subscripts, are used to denote variables. This rule is not strictly adhered to, especially in applied problems where it is desirable to have letters suggest the magnitudes they represent. For example, the letters l, A, V, v, and F are often used to denote a length, an area, a volume, a velocity, and a force, respectively, irrespective of their positions in the alphabet and without regard to whether they represent constants or variables. Considering any required magnitude as an "unknown" and using x as its symbol, irrespective of what the magnitude may be, is not recommended.

Letters of the Greek alphabet are usually used to denote angles: α, β, γ to denote constant angles, and θ, ϕ, ψ to denote variable angles.

It is desirable to use a single symbol to denote a single magnitude. It is better, for example, to use s for a distance than AB, and to use α for an angle than BAC. In geometry, two letters are used to denote a length and three letters to denote an angle, because this type of notation is well adapted to Euclid's Geometry. Such a notation is ill-adapted, however, to algebra, trigonometry, analytic geometry, the calculus, and mechanics.

A single symbol does not necessarily mean a single letter. If a symbol stands for an operation it usually contains more than one letter. This is the case when the *operational definition** of a magnitude is important in the

* An operational definition is one that indicates the operation which is to be performed in order to obtain the magnitude defined.

solution of a problem. For instance, the operational symbol of a constant or average velocity is s/t, and that of a variable velocity is ds/dt, in the notation of the calculus.

6. *If the problem admits of a figure, draw a suitable one and label its parts.*

It is desirable to place the figure in the upper right-hand section of the paper or blackboard whenever convenient. This leaves room for the analytical work to proceed from top to bottom, and from left to right. Figures should be drawn with some regard to size, shape, and proportion.

The need for dimension signs, such as ← → and ⟿, can usually be avoided and the figure made simpler in appearance by adopting the plan whereby a letter denotes the distance between two consecutive intersection or contact points on a line. In Fig. 2, *a*, *b*, and *c* denote the sides of the large triangle, whereas *d*, *e*, and *h* denote the legs of the right triangles.

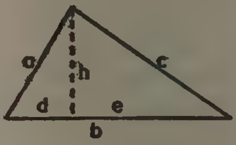


FIG. 2

Unless necessary, it is desirable not to label points, such as the vertices of the triangles in Fig. 2.

An angle should be indicated by a curved arrow when the direction as well as the magnitude is involved; but when the direction is not involved, the angle should be indicated by an arc without an arrowhead or with an arrowhead at each end.

7. *Make a table of the given and required magnitudes.*

A table of the given and required magnitudes, coupled with a figure, makes the nature of most problems immediately visible to the eye and thereby helps clear thinking.

In the case of a literal problem, the symbol of each given magnitude may be equated to a check mark in the table. In the case of a numerical problem, the symbol of each given magnitude should be equated to its numerical value and unit. In either case the symbol of each required magnitude should be equated to a question mark.

EXAMPLE. A Norman window has the shape of a rectangle surmounted by a semicircle. If the height of the rectangular part is three halves of the width, find expressions for (a) the perimeter and (b) the area in terms of the width.

Data

$a = \checkmark$

$b = \frac{3}{2}a$

$P = ?$

$A = ?$

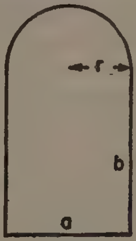


FIG. 3

Solution. Using the notation of the table of data and Fig. 3, we have

(a) $P = a + 2b + \pi r$

$$= a + 2 \times \frac{3}{2} a + \pi \frac{a}{2}$$

$$= \left(4 + \frac{\pi}{2}\right) a.$$

(b) $A = ab + \frac{1}{2} \pi r^2$

$$= \frac{3}{2} a^2 + \frac{1}{2} \pi \left(\frac{a}{2}\right)^2$$

$$= \left(\frac{3}{2} + \frac{\pi}{8}\right) a^2.$$

8. *Write all of the principal equations necessary for the solution of the problem before manipulating any of them.*

The principal equations of a problem are the mathematical definitions of the required magnitudes and the equations obtained by translating the conditions specified in the problem into the mathematical language. Other equations which may have to be brought into the solution of a problem, and which may be called *auxiliary equations*, should be introduced as they are needed.

In a verbal problem, translation into an equation is generally the most important part of the solution of a problem. It is also the most difficult part, especially when the statement of the problem contains words or phrases which stand for mathematical concepts but do not lend themselves to a direct translation into symbols. When a student fails to solve a verbal problem in calculus, the failure is almost always due to his inability to translate the conditions of the problem into the algebraic equation which must be obtained before applying the method of calculus.

When a literal (word-for-word) translation is made from one language to another the result is not, in general, in the idiom of the second language. This is the case also in a literal (word-to-symbol) translation into mathematics. Therefore if an equation obtained by literal translation is not in the mathematical idiom, it should be changed into one that is. An equation is, in general, in the mathematical idiom if it is as simple and compact as possible.

Examples. (1) An open box with a capacity of 5,000 cubic inches and a height of 10 inches is to be made from a cardboard 40 inches wide by cutting squares from the corners of the cardboard and then turning up its sides. Find the length of the cardboard.

Data

$$V = 5{,}000 \text{ in}^3, \qquad w = 40 \text{ in},$$
$$h = 10 \text{ in}, \qquad l = ?$$

Solution. Using the notation of Fig. 4 and the table of data, we have

$$V = (l - 2h)(w - 2h)h.$$

$$l = 2h + \frac{V}{(w - 2h)h}$$

$$= \left[20 + \frac{5000}{(40 - 20)10}\right] \text{in.}$$

$$= 45 \text{ in.}$$

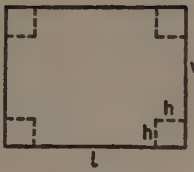


FIG. 4

(2) A bridge is to be built at a cost of \$1,000,000; if the state is to contribute twice as much as the county, and the county 50% more than the town, what is the contribution of each?

Solution. Let s, c, and t denote the contributions of the state, the county, and the town, respectively; then the principal equations of the problem are

$$s + c + t = \$1{,}000{,}000, \quad s = 2c, \quad \text{and} \quad c = \frac{3}{2}t.$$

Eliminating s and c from the first equation, we get

$$3t + \frac{3}{2}t + t = \$1{,}000{,}000;$$

hence $t = \$181{,}818.18,$
$c = \$272{,}727.27,$
and $s = \$545{,}454.55.$

9. *Solve the equations simultaneously for the required magnitudes, and obtain an expression for each in terms of the given, and only the given, magnitudes.*

Solving equations simultaneously means obtaining expressions for the required magnitudes in terms of the symbols of the given magnitudes or their values. If the expressions contain the symbols of other required magnitudes or subsidiary magnitudes which are neither given nor required, the problem is not solved in the strict sense of the term.

Because the student of algebra meets linear simultaneous equations first, which may be solved readily by the method of elimination by addition or subtraction, he sometimes attempts to solve simultaneous equations by that method even when the method is not applicable. It is advisable to use the method of elimination by substitution, even when the method of addition or subtraction is adequate, in order to become fully at ease with this more generally applicable method.

It is sometimes necessary to introduce subsidiary magnitudes which are neither given nor required. These should be eliminated; their values should not be found, unless thereby the computation involved becomes easier. Disregard of this rule is likely to waste time and to increase the chances of blunders.

It is desirable to place the symbol of a required magnitude on the left-hand side of the equation sign and its expression or value on the right-hand side. It is bad form to write, for example, $\pi r^2 = A$ and $2 = x$, instead of $A = \pi r^2$ and $x = 2$.

10. *In case there is only one, or a principal required magnitude, start with its mathematical definition or expression whenever convenient.*

This approach to the solution of a problem often gives the solution a greater directness and simplicity. The definition of the sine of an angle is made the starting point in the proof given on page 46 and the expression for s in the following example.

EXAMPLE. A ship is 20 miles to the west of another ship. The first ship is moving east at the rate of 10 miles per hour, the second ship is moving north at the rate of 15 miles per hour. Find the distance between the two ships five hours later.

Data

$a = 20\text{ m},$

$v_1 = 10\,\dfrac{\text{m}}{\text{hr}},$

$v_2 = 15\,\dfrac{\text{m}}{\text{hr}},$

$(s)_{t=5} = ?$

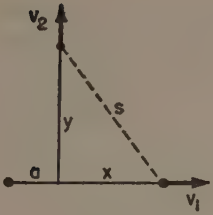


FIG. 5

Solution. Using the notation of Fig. 5 and the table of data, we have

$$
\begin{aligned}
s &= \sqrt{x^2 + y^2} \\
&= \sqrt{(v_1t - a)^2 + (v_2t)^2} \qquad \begin{bmatrix} x = v_1t - a \\ y = v_2t \end{bmatrix} \\
&= \sqrt{(10 \times 5 - 20)^2 + (15 \times 5)^2}\text{ m} \\
&= \sqrt{(30)^2 + (75)^2}\text{ m} \\
&= 15\sqrt{4 + 25}\text{ m} \\
&= 15\sqrt{29}\text{ m} \\
&= 80.78\text{ m.}
\end{aligned}
$$

Note: The way the required magnitude is indicated in the table of data shows that s and t are variables and that the value of s is required *when* $t = 5$.

11. *Write successive expressions of a required magnitude in the "columnar" form. Follow the straight-course method; avoid the zigzag method.*

Zigzag method. Students, even teachers, usually solve problems in a way which we shall call the *zigzag* method. It is not easy to describe this method, for it is anything but methodical. When a problem is solved by the zigzag method, the analysis does not follow continuously along a main line of advance. Instead, its course makes abrupt changes in direction, and the continuity of the chain of reasoning is broken by auxiliary equations with which the analysis is interlarded. Consequently the analysis becomes cumbersome, difficult to carry out, and difficult to follow.

When the zigzag method is used, an equation is found and some mathematical operations are performed on it, or the equation is modified by means of other equations. Then it is laid aside and another equation is found and similarly treated. This process is continued until a sufficient number of equations are found; then the equations are solved simultaneously to obtain the final answer.

EXAMPLE. Prove the formula $\sin(\theta + \varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi$.

Note: The following is an exact copy of the proof taken from a relatively recent trigonometry text. To follow its zigzag course requires concentration to a degree out of proportion to the intrinsic difficulty of the problem. Furthermore, after reading it, one would have difficulty retracing its course. One can satisfy himself that this is no exaggeration by following the proof step by step, reproducing it, and then following a similar procedure with the derivation of the same formula given in the next section.

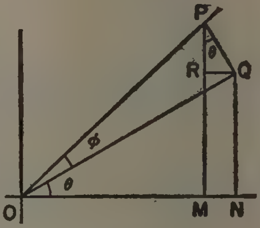


FIG. 6

After describing the construction of Fig. 6 and letting $OP = 1$, the proof proceeds as follows:

$$\sin(\theta + \phi) = \frac{MP}{OP} = MP = MR + RP.$$

But $MR = NQ$, and $\sin\theta = \dfrac{NQ}{OQ}$; hence $MR = NQ = OQ\sin\theta$.

Since $\cos \phi = \dfrac{OQ}{OP} = OQ$, we have $MR = \sin \theta \cos \phi$. Furthermore, $\cos \theta = \dfrac{RP}{QP}$, thus $RP = QP \cos \theta$. Since $QP = \dfrac{QP}{OP} = \sin \phi$, we get $RP = \cos \theta \sin \phi$. Therefore, $\sin (\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi$.

The following are some of the bad features of this derivation:

(1) The construction of the figure is described in the book, but the reasons for constructing it in the particular way are not given.

(2) Each length in the figure is specified by the labels of its end points, with the consequence that one has to move his eyes from one label to another to find out what length a particular pair of letters represents.

(3) The length OP, which is not a pure number, is equated to a pure number.

(4) The ratio of two lengths, which is a pure number, is equated to a length which is not a pure number. This is done four times.

(5) The proof follows a zigzag course with the sines and cosines of the angles as points of change in direction.

(6) No attempt is made to check the result by letting $\phi = 0$ in order to see if the formula reduces to $\sin \theta = \sin \theta$.

Straight-course method. The method used in the following derivation of the same formula is called *the straight-course method*, because the reasoning and the analysis follow a straightforward course, without breaks or changes in direction. Its form will be called the *columnar form*, because successive expressions of a required magnitude and the equation signs are placed in vertical columns.

EXAMPLE. Express $\sin (\theta + \varphi)$ as functions of θ and φ.

Solution. Since $\sin(\theta + \varphi)$ is to be expressed in terms of functions of θ and φ, we construct three interrelated right triangles, each of which has one of the angles θ, φ, and $(\theta + \varphi)$. In Fig. 7, the interrelation is achieved by making the line r' the hypotenuse of the θ-triangle as well as the adjacent side of the ϕ-triangle, and the line r the hypotenuse of both the ϕ-triangle and the $(\theta + \phi)$-triangle. Then, using the notation of the figure, we have

$$\sin(\theta + \phi) = \frac{y}{r} \qquad \text{[By definition]}$$

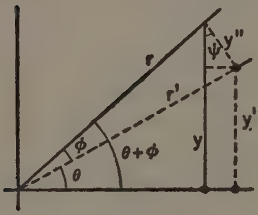


FIG. 7

$$= \frac{y' + y'' \cos\psi}{r}$$

$$= \frac{r' \sin\theta + y'' \cos\theta}{r} \qquad [\psi = \theta]$$

$$= \frac{r\cos\phi \cdot \sin\theta + r\sin\phi \cdot \cos\theta}{r}$$

$$= \sin\theta\cos\phi + \cos\theta\sin\phi.$$

Discussion. Every term in the analysis is a pure number; hence the analysis is dimensionally correct (see Direction 13, p. 48). The formula reduces to $\sin\theta = \sin\theta$ when $\phi = 0$, as it should. When $\varphi = \phi$, the formula reduces to the special case

$$\sin 2\theta = 2\sin\theta\cos\theta.$$

The following features of this derivation are worth noting:

1. The definition of the required magnitude is made the starting point of the analysis.

2. Successive expressions in the right-hand member and the equation signs are placed in vertical columns.

3. The left-hand member is not repeated because each expression (after the first) in the right-hand member follows from the preceding expression, not from the left-hand member.

4. Auxiliary equations and explanatory material are placed outside (to the right) of the main line of the analysis, consequently the chain of reasoning is not broken by interlarding the analysis with explanatory material and auxiliary equations.

5. Each expression is so written that the change from the preceding expression becomes immediately evident, thus reducing the need for explanatory material. For example, $r' \sin \theta$ in the third expression is written as $r \cos \phi \cdot \sin \theta$ in the fourth, thus indicating clearly that r' has been replaced by $r \cos \phi$.

6. In the zigzag method of the proof of the formula, it is taken for granted that the final result is known, and that it contains only the sines and cosines of θ and ϕ. In other words, the analysis follows a zigzag course toward a previously known point. In the straight-course method, on the other hand, each expression follows naturally from the preceding expression, therefore it is not necessary to know the final result in advance, beyond the fact that $\sin (\theta + \phi)$ is to be expressed as functions of θ and ϕ. This feature makes the straight-course method especially useful for solving a problem when the exact nature of the final result is not known.

12. Before taking a new step in the analysis, put the last expression in as simple a form as possible.

It is highly desirable to simplify expressions as the analysis proceeds. Simplification is accomplished by such arithmetical and algebraic operations as combining similar terms, factoring, cancelling, reducing complex (multiple-deck) fractions to simple (double-deck) fractions, etc.

Whether or not an expression is in the simplest form may depend on the operation to be performed on it. For example, $\sqrt{x}(x + 1)^2dx$ is in the simplest algebraic form, but from the point of view of the calculus operation called *integration*, $(x^{5/2} + 2x^{3/2} + x^{1/2})dx$ is simpler. However, the student who cultivates the habit of simplifying expressions will learn to judge what constitutes simplicity in a given case.

13. Discuss the literal equation which gives a required magnitude in terms of the symbols of the given magnitudes.

If every geometrical or physical quantity which enters into an equation is represented by a letter, the equation is said to be *homogeneous* or *dimensionally correct.* Every term in such an equation represents the same kind of quantity.

The object of discussing an equation is (a) to check it, and (b) to obtain interesting special cases. Checking consists in seeing whether or not the equation is dimensionally correct and reduces to a well-known special case.

Dimensional considerations do not constitute a check for numerical coefficients. These may be checked, however, by seeing if an equation or expression reduces to a well-known case. For example, if A denotes the area of an ellipse, the equation $A = 2\pi ab$ checks dimensionally yet is incorrect, because if we let $b = a$, we obtain $A = 2\pi a^2$, which does not give the area of a circle as it should.

Discussion does not apply to equations which are not homogeneous. But if all the equations involved in the solution of a problem are supposed to be homogeneous, discussion becomes very valuable, not only for checking the final result and for obtaining interesting special cases, but also for checking intermediate steps. This is especially true in applied problems into which a number of different kinds of quantity enter, as in physical and engineering problems. Many mistakes of a dimensional nature can be avoided if the student becomes dimension-conscious.

The equation $\sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi$ was discussed on page 46; the following examples afford further illustrations.

EXAMPLES. (1) Discuss the equation $V = \frac{\pi h^2}{3}(3r - h)$, where V is the volume of a segment of one base cut from a sphere of radius r, and h is the altitude of the segment, Fig. 8.

Discussion. (a) The equation is dimensionally correct, since h^2r and h^3 represent volumes. If $h = 0$, $V = 0$, as it should, since the segment reduces to a point as h approaches zero. (b) If $h = r$, $V = \frac{2}{3}\pi r^3$, the volume of a hemisphere. If $h = 2r$, $V = \frac{4}{3}\pi r^3$, the volume of a sphere. The equation checks.

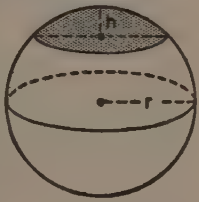


FIG. 8

(2) Discuss the equation $v = v_0 + (\sin\alpha - \mu\cos\alpha)gt$, where v is the velocity of a body which is projected down a rough inclined plane with an initial velocity v_0, α is the angle of inclination of the plane, μ is the coefficient of friction, g is the gravitational acceleration, and t is the time of motion.

Discussion. Every term in the equation represents a velocity, since the expression within the parentheses is a pure number and gt has the same dimensions as velocity; therefore the equation is dimensionally correct.

If $\alpha = 0$ and $\mu = 0$, then $v = v_0$, that is, the body moves with a constant velocity on a horizontal and smooth plane, as required by Newton's first law of motion. If $v_0 = 0$ and $\alpha = 90°$, then $v = gt$, the velocity of a body falling freely from rest. If $v_0 = 0$ and $\mu = 0$, then $v = gt\sin\alpha$, the velocity of a body sliding down a smooth plane. The equation checks.

14. *Solve numerical problems as literal problems first, and introduce the numerical data after the expressions of the required magnitudes are obtained in terms of the symbols of the given magnitudes.*

Students usually attempt to solve a numerical problem in mathematics, physics, or mechanics as if it were a problem in arithmetic. They plunge immediately into arithmetical operations and computations instead of first applying the mathematical and physical principles involved. This often results in making the solution confused and cumbersome, in unnecessary labor, in errors and blunders, and in the masking of the geometrical or physical character of the problem, if not in complete failure.

Changing a numerical problem to a literal one and then solving it according to the directions given in this book has the following advantages:

1. The solution becomes a solution of all problems of the same type.
2. The principles involved become clearer.
3. The solution becomes neater.
4. The computation becomes shorter.
5. The chances of blunders are reduced.
6. An error in the computed value of one required magnitude is not carried into the value of another.
7. Discussion of the results becomes possible.

In short, the great advantages of algebra over arithmetic are brought to bear on the solution of the problem.

15. *Use your common sense at every step of the analysis. At each step ask yourself if the step you have just taken is sensible and then, at the end, whether the final result is reasonable.*

This direction epitomizes all the others, since they are designed to help the exercise of common sense in connection with solving problems.

The following examples are given to illustrate further the way in which problems may be solved in accordance with the general directions.

EXAMPLES. (1) A boy starts to walk to a town 15 miles away at the rate of 3 miles per hour. After walking some distance, he obtains a ride from a motorist and travels the rest of the distance at an average rate of 30 miles per hour. If the boy reached the town one hour and 12 minutes after he started, how long and how far did he walk?

Data

$$s = 15\text{ m}, \qquad t = \frac{6}{5}\text{ hr},$$
$$v_1 = 3\,\frac{\text{m}}{\text{hr}}, \qquad t_1 = ?$$
$$v_2 = 30\,\frac{\text{m}}{\text{hr}}, \qquad s_1 = ?$$

Solution. Using the notation of the table of data and Fig. 9, we write $s = s_1 + s_2$ and $t = t_1 + t_2$, the principal equations of the problem, and then proceed as follows:

$$\begin{aligned} s &= s_1 + s_2 \\ &= v_1t_1 + v_2t_2 \\ &= v_1t_1 + v_2(t - t_1) \\ &= (v_1 - v_2)t_1 + v_2t. \end{aligned}$$

FIG. 9

Solving this equation for t_1 and s_1, we have

$$t_1 = \frac{s - v_2t}{v_1 - v_2} = \frac{\left(15 - 30 \times \frac{6}{5}\right)\text{ m}}{(3 - 30)\,\frac{\text{m}}{\text{hr}}} = \frac{7}{9}\text{ hr.} \qquad s_1 = v_1t_1 = 3\,\frac{\text{m}}{\text{hr}} \times \frac{7}{9}\text{ hr} = \frac{7}{3}\text{ m.}$$

(2) When an elastic cord is stretched, the stretching force is proportional to the elongation produced (Hooke's law). If a force of 15 pounds changes the length of a cord from 50 inches to 50.23 inches, find the force necessary to produce an elongation of 0.75 inch.

Data

$$F \propto l, \qquad l_2 = 0.75 \text{ in},$$
$$l_1 = (50.23 - 50) \text{ in} \qquad F_1 = 15 \text{ lb},$$
$$= 0.23 \text{ in}, \qquad F_2 = ?$$

Solution. The force is proportional to, or varies as, the elongation; therefore

$$F = kl,$$

where k is the constant of proportionality. Since this equation is valid for F_1 and F_2, and the corresponding elongations l_1 and l_2, we have

$$F_1 = kl_1, \quad \text{and} \quad F_2 = kl_2.$$

Solving these equations simultaneously for F_2, we have

$$\begin{aligned} F_2 &= \frac{l_2}{l_1} F_1 \\ &= \frac{0.75}{0.23} \times 15 \text{ lb} \\ &= 48.9 \text{ lb}. \end{aligned}$$

Note: Observe that k, which is neither given nor required, was eliminated, instead of finding its value and then using it to find the value of F_2.

(3) One hundred gallons of milk with butterfat content of 5% is to be obtained by mixing cream of 35% butterfat and milk of 3% butterfat. Find the amounts of cream and milk to be mixed.

Data

$$V = 100 \text{ gal}, \qquad p_2 = 3\%,$$
$$p = 5\%, \qquad V_1 = ?$$
$$p_1 = 35\%, \qquad V_2 = ?$$

Solution. The amount of butterfat in each liquid equals the product of the percentage of butterfat and the volume. Furthermore, the amount of butterfat in the mixture equals the sum of the amounts in the liquids mixed. Therefore

$$pV = p_1V_1 + p_2V_2, \tag{1}$$

and

$$V = V_1 + V_2. \tag{2}$$

Eliminating V_2 between (1) and (2), we get

$$pV = p_1V_1 + p_2(V - V_1). \tag{3}$$

Solving (3) for V_1 and then for V_2, we have

$$V_1 = \frac{p - p_2}{p_1 - p_2} V \quad (4) \qquad V_2 = \frac{p - p_1}{p_2 - p_1} V \quad (5)$$

$$= \frac{5 - 3}{35 - 3} \times 100 \text{ gal} \qquad = \frac{5 - 35}{3 - 35} \times 100 \text{ gal}$$

$$= 6.25 \text{ gal.} \qquad = 93.75 \text{ gal.}$$

Check: $V = V_1 + V_2 = 100$ gal.

Note: Equations (4) and (5) are applicable to other mixtures of two items, such as chemicals with different concentrations, articles with different prices, investments with different rates of return. Of course, the symbols in the equations have to be given different meanings or replaced by other symbols more appropriate to the items. Thus solving this problem as a literal problem first has resulted in obtaining valuable formulas, as well as in making the solution simpler and more understandable.

(4) A weight suspended by a string is pulled aside by a horizontal force until the string makes an angle θ with the vertical. Find the horizontal force and the force of tension of the string.

Solution. In this problem it is taken for granted that W and θ are given. Applying the condition of equilibrium* to

* If a body is in equilibrium, the algebraic sum of the components of the extenual forces in any direction equals zero. See *Analytical Mechanics*, 3rd Ed.; H. M. Dadourian; D. Van Nostrand Co.

Fig. 10 by setting the sum of the forces in the horizontal and vertical directions equal to zero, we obtain

$$F - T\sin\theta = 0,$$
$$T\cos\theta - W = 0.$$

Data
$W = \checkmark$
$\theta = \checkmark$
$T = ?$
$F = ?$

Solving these equations for T and F, we get

$$T = \frac{W}{\cos\theta},$$

$$F = W\tan\theta.$$

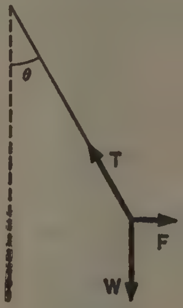


Fig. 10

Discussion. (a) All of the equations are dimensionally correct, since every term represents the same kind of quantity, namely, a force. When $\theta = 0$, $T = W$ and $F = 0$, as they should. The equations check. (b) As θ approaches 90°, that is, as the string supporting the weight becomes horizontal, both T and F increase indefinitely. This brings out the interesting fact that a flexible string cannot be made perfectly horizontal by merely pulling at its ends so long as it supports a weight, however small the weight and however great the tension.

Note: It is important to distinguish between a vector and the line in which it lies by making the vector thicker and of different length than the line. If a vector represents a force, the figure becomes more realistic if the tail end of the vector is placed at the point where the force is supposed to act if it is not a resistance or frictional force, and the arrow end if it is a resistance force.

(5) A flagstaff stands on top of a tower. From a point on the ground 500 feet from the base of the tower the angles of elevation of the bottom and top of the flagstaff are 29°15′ and 34°36′, respectively. Find the height of the flagstaff.

Data

$a = 500$ ft, $\beta = 34°36'$,
$\alpha = 29°15'$, $h = ?$

Solution. Using the notation of Fig. 11 and the table of data, we obtain

$$H + h = a \tan \beta.$$

$$\begin{aligned} h &= a \tan \beta - H \\ &= a \tan \beta - a \tan \alpha \\ &= (\tan \beta - \tan \alpha)a \\ &= (0.6898 - 0.5600)500 \text{ ft} \\ &= 64.9 \text{ ft.} \end{aligned}$$

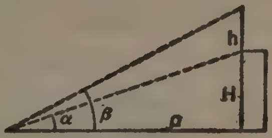


FIG. 11

Discussion. The equation $h = (\tan \beta - \tan \alpha)a$ is dimensionally correct, since both of its members represent lengths. If $\alpha = 0$, then $h = a \tan \beta$, and if $\alpha = \beta$, then $h = 0$. These results are to be expected. If we let $\beta = 0$, we get $h = -a \tan \alpha$. This result may give the impression that it is valid for any value of α, but that is not the case, because the conditions of the problem require that $\alpha = 0$ when $\beta = 0$, therefore $h = 0$ when $\beta = 0$.

If we let $a = 0$, we might write $h = (\tan \beta - \tan \alpha) \times 0 = 0$. This would be absurd, however, since we cannot reduce the height of the flagstaff to zero by looking at the top of the building and the top of the flagstaff from the base of the building. The trouble in this case lies in treating $(\tan \beta - \tan \alpha)$ as a finite number, which is not the case because both α and β become 90° when $a = 0$, and consequently the equation becomes $h = (\infty - \infty) \times 0$. The right-hand member of this equation is *indeterminate* (see p. 67), therefore the equation does not give any information about the value of h.

This discussion brings out the following important points. When we assign a value to one of the variables in discussing an equation, we must see if the assigned value is compatible with the conditions from which the equation was obtained and if the right-hand member becomes indeterminate.

PART III.

ARITHMETIC THROUGH CALCULUS

GUIDE POSTS AND WARNING SIGNS

"Oh, students, study mathematics,
and do not build without foundation."

LEONARDO DA VINCI

In this part of the book, we shall consider some sections of secondary school and college mathematics, with the following purposes in view:

1. To elucidate certain important topics.
2. To supplement the general directions and suggestions given in Part II by other directions and suggestions adapted to the solution of certain types of problems.
3. To give some of the more frequently used theorems and formulas.
4. To warn the student against common errors and bad practices.

The selection of the material is more or less arbitrary, and is based on the author's judgment as to what might profitably be included in the limited space of this book.

It is to be expected that the student will read those sections which come under subjects he has studied in the past or is currently studying; in other words, it is assumed that the student has some knowledge of the material in the sections which he reads. For this reason, and in order to save space, definitions, reasons, and proofs are not always given. It is hoped that the student will supply these through his own efforts or with the aid of textbooks.

ARITHMETIC AND ALGEBRA

"Nature's book is written in mathematical language."

GALILEO

Arithmetic is specialized algebra, and is concerned with operations with definite numbers represented by numerals. Algebra is generalized arithmetic, and is concerned with operations with arbitrary constants and variables represented by letters, as well as with definite numbers. Furthermore, the concept of number is extended in algebra to imaginary and complex numbers.

The rules of the four fundamental operations of addition, subtraction, multiplication, and division are the same in both subjects. However, the word addition is often used in algebra to denote the addition of numbers with unlike signs as well as with like signs; hence the term *algebraic sum.*

Computation. The following suggestions should be of help in simplifying arithmetical computations.

1. Change mixed numbers to fractions; for instance, $3\frac{1}{4}$ to $\frac{13}{4}$.

2. Do not replace exact numbers, such as $\sqrt{2}$ and $\frac{2}{3}$, by their approximate values in decimals until after a required magnitude is expressed in terms of the original values given in the data. This procedure often makes the computation simpler because of cancellation and rationalization.

3. Reduce large numbers in fractions and under root signs by factoring or cancellation. For instance, reduce $\frac{265}{85}$ to $\frac{53}{17}$, and $\sqrt{75^2 + 50^2}$ to $25\sqrt{13}$. Disregard this suggestion when logarithms are used, except when the

new form of the number is better adapted to the use of logarithms, as in the case of the second illustration.

4. If the final numerical value of a geometrical or physical quantity is very large or very small, change the unit so as to bring the numerical value to a reasonable figure, for instance, change 18,400 feet to 3.5 miles, and 0.024 ton to 48 pounds.

5. If the data contain different units of the same kind of quantity, reduce them to the same unit either in the table of data or after the original data are put in the expression of a required magnitude. Sometimes the latter procedure makes the computation simpler because of cancellation of units.

Significant figures. Numbers which arise from measurement and observation represent, in general, approximate values. The degree of accuracy of such a number is indicated by the number of *significant figures* it contains. When an approximate number is expressed in the decimal system, the number of the significant figures is supposed to be the number of digits it contains between and including the first and the last digits which are not zeros. Thus 0.005307, 0.5307, 5307, 530,700 are supposed to have four significant figures. However, when the last nonzero digit is followed by one or more zeros, the number of significant figures becomes ambiguous. In such a case the ambiguity may be removed by expressing the number in the so-called *scientific form*, that is, in powers of 10. Thus if 530,700 is written as 5.307×10^5, 5.3070×10^5, or 5.30700×10^5, it is made clear that the number contains 4, 5, or 6 significant figures, respectively.

When an approximate number is multiplied or divided by another number, the result does not become more exact than the original number. Therefore the resulting number should be *rounded off*, so that it contains the same number of significant figures as the original number. For instance, if 101,000, correct to four significant figures, is divided by 3, the result should be written as 33,670, not 33,666.66. . . . However, if the result is to be added to or subtracted from another number of equal or greater accuracy, it may be desirable to write it as if it had one more significant figure than the original number.

One of the advantages of computing by logarithms lies in the fact that the number of significant figures is kept within the number of the digits in the table of logarithms.

Parentheses and their equivalents. Parentheses and brackets indicate that the terms enclosed by them are to be treated as a single quantity. Many of the errors made by students of elementary mathematics are due to disregard of this rule.

Do not enclose a single term or a single symbol within parentheses. Do not use a dot as the equivalent of parentheses. For example, write the product of a and $(b + c)$ as $a(b + c)$; not as $(a)b + c$ or $a \cdot b + c$.

Division lines and root signs act as parentheses, therefore it is not necessary to enclose them within parentheses: $\frac{x + a}{x - a}y$ and $a\sqrt{x + a}$ are preferable to $\left(\frac{x + a}{x - a}\right)y$ and $a(\sqrt{x + a})$.

Terms under a root sign also have to be treated as a single quantity, and therefore it is not permissible to take the roots of the terms individually.

$$\sqrt{9 + 16} \neq 3 + 4; \quad \sqrt[3]{x^3 + 8y^3} \neq x + 2y.$$

Multiplying polynomials. In multiplying two polynomials, unnecessary labor can be avoided by multiplying each term of the shorter polynomial with the terms of the longer one in the order in which they appear, without rewriting the polynomials one under the other. Thus

$$(x - a)(2x - y + b) = 2x^2 - xy + bx - 2ax + ay - ab.$$

Order of symbols. The student is advised to note the order in which symbols appear in mathematical expressions and equations. In general, a factor which is simpler precedes a more complex one: a definite number comes before an arbitrary constant, a constant before a variable, letters which represent the same kind of magnitude in alphabetical order, the coefficient of a quantity before the symbol of the quantity, etc. The following samples illustrate the sensible way in which expressions and equations may be written:

$$2\sqrt{5}, \quad 3abxy, \quad a \sin \alpha, \quad s = \frac{1}{2} gt^2, \quad F_2 = \frac{l_2}{l_1} F_1.$$

Fractions. Slanting division lines are sometimes used in books because they save space and are more convenient in typesetting. As neither of these considerations is important to the student's work, it is desirable to avoid slanting division lines.

Negative exponents are often used in books, sometimes because they save space and are convenient in typesetting, and at other times because they are desirable. When the latter reason does not apply, the student is advised to put an expression which contains a negative exponent in the form of a fraction. For instance, $(x - a)(x + a)^{-2}$ should be expressed as $\dfrac{x - a}{(x + a)^2}$.

Do not cancel terms in the numerator and the denominator simply because they are equal.

$$\frac{4x + a}{6x - a} \neq \frac{2 + 1}{3 - 1}.$$

Before reducing fractions to a common denominator, factor out common factors. Thus

$$\frac{x - a}{x + b} - \left(\frac{x - a}{x + b}\right)^2 = \frac{x - a}{x + b}\left(1 - \frac{x - a}{x + b}\right)$$

$$= \frac{x - a}{x + b} \cdot \frac{x + b - x + a}{x + b}$$

$$= \frac{(a + b)(x - a)}{(x + b)^2}.$$

Do not expand the denominator unless it is thereby simplified.

Do not "throw away" the denominator unless the fraction equals zero.

Unless a great number of terms is involved, it is better to do the cancelling mentally instead of using cancellation signs, as the latter makes the work messy. It is better to write, for instance,

$$\frac{a(x - b)^3}{a^2(x - b)} = \frac{(x - b)^2}{a}$$

than

$$\frac{a(x - b)^3}{a^2(x - b)} = \frac{\cancel{a}(x - b)^{\cancel{3}\,2}}{a^{\cancel{2}}\cancel{(x - b)}} = \frac{(x - b)^2}{a}.$$

In writing a fraction, draw the division line first if the fraction is simple, and the main division line if the fraction is complex.

To reduce a complex fraction to a simple one, invert the expression below the main division line and multiply it by the expression above the main division line.

$$\frac{\frac{3}{8}}{\frac{4}{5}} = \frac{3}{8} \times \frac{5}{4} = \frac{15}{32}. \qquad \frac{\frac{a-x}{a+x}}{a^2+x^2} = \frac{a-x}{a+x} \times \frac{1}{a^2+x^2} = \frac{a-x}{(a+x)(a^2+x^2)}.$$

If a complex fraction contains a mixed fraction, reduce the latter to a simple fraction before simplifying the complex fraction. Thus

$$\frac{x+y}{1+\frac{y}{x}} = \frac{x+y}{\frac{x+y}{x}} = \frac{x+y}{x+y}\,x = x.$$

Factors. If the product of a number of factors equals zero, at least one of them must equal zero. Thus, if $5x(x-a)(x^2+a^2) = 0$, then $x = 0$ and $x - a = 0$, but $x^2 + a^2 \neq 0$ (unless imaginary numbers are to be considered).

It is desirable to know the following formulas:

(1) $a^2 - b^2 = (a+b)(a-b)$.

(2) $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$.

(3) $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.

Note: In case you are not sure of the signs, you can check them by letting $b = a$ and $b = -a$ to see if the right-hand member equals the left-hand member.

(4) $(a+b)^2 = a^2 + 2ab + b^2$.

(5) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

(6) $$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}a^{n-4}b^4 + \cdots\cdots + b^n.$$

The last formula, called the *binomial theorem*, is valid only when n is a positive integer.

Quadratic equation. The equation $ax^2 + bx + c = 0$ may be solved by the following three methods:

1. By the formula $$x = \frac{-b \pm \sqrt{b - 4ac}}{2a}.$$
2. By factoring.
3. By completing the square.

The last method is preferable to the first because, being a process, it is easier to remember. It is preferable to the second method also, because it not only requires less ingenuity, but also is useful in putting the equations of conic sections in standard forms.

The following derivation of the formula illustrates the way a quadratic equation is solved by the method of completing the square.

Dividing the equation $ax^2 + bx + c = 0$ by the coefficient of x^2, and transferring the constant term to the right-hand member, we have

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

In order to make the left-hand member of this equation a perfect square, we add the square of half the coefficient of x to both members, and then solve the equation for x, as follows:

$$x^2 + \frac{b}{a}x + \left(\frac{1}{2}\frac{b}{a}\right)^2 = \left(\frac{1}{2}\frac{b}{a}\right)^2 - \frac{c}{a},$$

$$\left(x + \frac{1}{2}\frac{b}{a}\right)^2 = \frac{b^2 - 4ac}{4a^2},$$

$$x + \frac{b}{2a} = \pm\frac{1}{2a}\sqrt{b^2 - 4ac},$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Laws of exponents. With the understanding that a^{-n} means $1/a^n$, that $a^{1/n}$ means $\sqrt[n]{a}$, and that the denominators are not zero, we have the following laws of exponents.

$$\text{I. } a^m \cdot a^n = a^{n+m},$$

$$\text{II. } \frac{a^m}{a^n} = a^{m-n},$$

$$\text{III. } (a^m)^n = a^{mn},$$

$$\text{IV. } a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m},$$

$$\text{V. } a^n b^n = (ab)^n,$$

$$\text{VI. } \frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n.$$

It follows from Law II that any number raised to the zero power equals 1, provided the number is not zero; thus

$$a^0 = a^{1-1} = \frac{a}{a} = 1, \qquad a \neq 0.$$

Logarithms. Whenever a new kind of mathematical operation becomes necessary, a new symbol is adopted to indicate the operation, and it is given a name. For instance, the operation by means of which the equation $y = x^3$ is solved for x is indicated by the symbol $\sqrt[3]{\ }$, called the cube root, and the resulting equation is written as $x = \sqrt[3]{y}$.

When the exponential equation

$$N = b^x \tag{1}$$

is solved for x, the result is written in the form

$$x = \log_b N. \tag{2}$$

The right-hand member of Equation (2) is read, and x is called, the *logarithm of N to the base b.* Equations (1) and (2) are identical except in form, as $y = x^3$ and $x = \sqrt[3]{y}$ are identical. Equation (1) states, "N is the number obtained when b is raised to the power x." Equation (2) states, "x is that power to which b must be raised in order to obtain N." Thus Equation (2) amounts to reading Equation (1) backward. Hence the definition, *the logarithm of N to the base b is the power to which b must be raised in order to obtain N.*

The following table illustrates the meaning of a logarithm.

N	b	$\log_b N$	check	N	b	$\log_b N$	check
4	2	2	$2^2 = 4$	100	10	2	$10^2 = 100$
8	2	3	$2^3 = 8$	1000	10	3	$10^3 = 1000$
16	2	4	$2^4 = 16$	0.01	10	-2	$10^{-2} = 0.01$
9	3	2	$3^2 = 9$	0.001	10	-3	$10^{-3} = 0.001$
27	3	3	$3^3 = 27$	10^n	10	n	$10^n = 10^n$
81	3	4	$3^4 = 81$	1	$b > 0$	0	$b^0 = 1$
25	5	2	$5^2 = 25$	b	b	1	$b^1 = b$
125	5	3	$5^3 = 125$	$N < 0$	b	Undefined except in complex numbers	

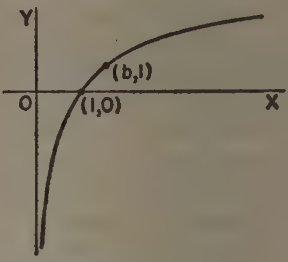
Important points about logarithms:

The base may be any positive number, that is, any number greater than zero.

The logarithm of the base equals 1.

The logarithm of a negative number is not defined in elementary mathematics.

The logarithm is negative if the number lies between 0 and 1.



The logarithm of 1 is zero.

The logarithm is positive if the number is greater than 1.

As a number increases, its logarithm increases less and less rapidly.

As a proper fraction approaches zero its logarithm increases negatively more and more rapidly.

These points are brought out by the curve in the figure.

Laws of logarithms. The following laws are obtained by applying the definition of a logarithm to the first three laws of exponents:

$$\text{I.}\ \log MN = \log M + \log N.$$

$$\text{II.}\ \log \frac{M}{N} = \log M - \log N.$$

$$\text{III.}\ \log N^p = p \log N.$$

The base is not indicated in these laws, because they are valid for any permissible base.

$$\text{Log}\ (\text{M} \pm \text{N}) \neq \log \text{M} \pm \log \text{N}.$$

It is a waste of time to write the logarithm of a numerical expression in the expanded form. For instance, writing $\log \frac{2.57 \times 876.4}{389.7} = \log 2.57 + \log 876.4 - \log 389.7$, before actually adding and subtracting the logarithms, is like telling oneself what to do before doing it. There is no need for it.

Whenever a logarithm appears, unsought, in pure or applied mathematics, it is always to the *natural base* e $(= 2.7182818\ldots)$. But since logarithmic tables which are used in computation are generally to the base 10, it is desirable to know the conversion formula

$$\log_e N = 2.3026 \log_{10} N,$$

where 2.3026 is an approximate number correct to five significant figures.

Zero and infinity. Zero is a definite number, as any other number; it is not "nothing." Zero may enter into any mathematical operation except as a denominator. However, although division by zero is not permissible, a variable denominator may be allowed to approach zero. If, as the denominator of a fraction approaches zero the numerator does not approach zero, the quotient increases or decreases indefinitely, that is, becomes positively or negatively infinite, depending on whether the quotient becomes and remains positive or negative.

The mathematical infinite may be defined as a variable which increases or decreases indefinitely. Many of the laws of finite numbers do not apply to the infinite, as indicated by the following equations: $\infty + n = \infty$ and $\infty - n = \infty$, where n is any number; $n\infty = \infty$, $\infty/n = \infty$, $\infty^n = \infty$, and $\sqrt[n]{\infty} = \infty$, where n is any number other than zero.

If both the numerator and the denominator of a quotient approach zero or become infinite, also if one factor of a product approaches zero while the other factor becomes infinite, the result may be a definite value or may become infinite, depending on the character of the quotient or product. Such cases, called *indeterminate forms*, are studied in calculus.

GEOMETRY

"Before you enter the study of law sufficient ground must be laid . . . Mathematical reasoning and deductions are, therefore, a fine preparation for investigating the abstruse speculations of law."

THOMAS JEFFERSON

Geometry is an entirely deductive science in which the subject is developed from a few assumptions by strictly logical reasoning. The assumptions used to be called *axioms*, and were supposed to be self-evident truths. Furthermore, the Euclidian geometry was considered to be the only conceivable geometry. But the development of non-Euclidian geometries in the nineteenth century, and their application to our space by the theory of relativity, showed that the set of axioms on which geometry is based cannot be said to be true or false; they are pure assumptions to which questions of truth do not apply. But a theorem of geometry may be true or false depending on whether it is consistent or inconsistent with the axioms adopted. For these reasons, the word *axiom* is generally abandoned in favor of the word *postulate.* These developments have modified our concept of truth, and have introduced two scientific criteria of truth, namely, consistency with postulates and experimental verification.

Under the heading *Geometry*, we shall include a few of the more frequently used theorems and mensuration formulas.

Theorems.

1. An exterior angle of a triangle equals the sum of the opposite interior angles:

$$\theta = \alpha + \beta. \qquad \text{(Fig. 12)}$$

2. Two angles are equal if the sides of one are perpendicular to the sides of the other:

$$\varphi = \theta. \qquad \text{(Fig. 13)}$$

3. An angle inscribed in a circle equals one-half the angle which the intercepted arc subtends at the center:

$$\varphi = \tfrac{1}{2}\theta. \qquad \text{(Fig. 14)}$$

4. An angle inscribed in a semicircle is a right angle:

$$\varphi = 90°. \qquad \text{(Fig. 15)}$$

5. An angle included between a tangent to a circle and a chord drawn from the point of tangency equals one-half the angle which the intercepted arc subtends at the center:

$$\varphi = \tfrac{1}{2}\theta. \qquad \text{(Fig. 16)}$$

FIG. 12

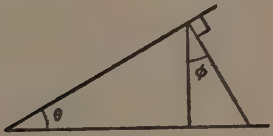


FIG. 13

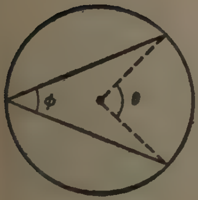


FIG. 14

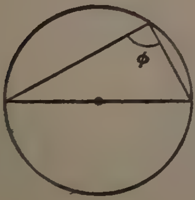


FIG. 15

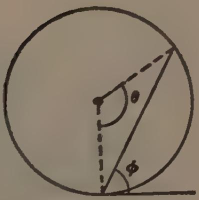


FIG. 16

6. The line which joins the mid-points of two sides of a triangle is parallel to, and equals one-half the length of, the third side:

$$a = \tfrac{1}{2}b. \qquad \text{(Fig. 17)}$$

7. The medians of a triangle meet at a point which is twice as far from each vertex as from the mid-point of the opposite side:

$$d_1 = 2d_2. \qquad \text{(Fig. 18)}$$

8. The line which joins the mid-points of the nonparallel sides of a trapezoid is parallel to, and equals one-half the sum of, the two bases:

$$c = \tfrac{1}{2}(a + b). \qquad \text{(Fig. 19)}$$

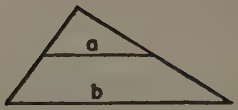


FIG. 17

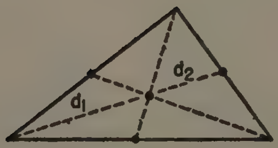


FIG. 18

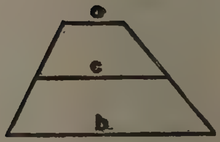


FIG. 19

Mensuration formulas.

Notation

C = circumference. B = area of base.
P = perimeter. S = surface or lateral area.
A = area. V = volume.

1. **Pythagorean theorem:** $c^2 = a^2 + b^2$. (Fig. 20)

2. **Similar triangles:** $\dfrac{a'}{a} = \dfrac{b'}{b} = \dfrac{c'}{c}$. (Fig. 21)

3. **Triangle:** $A = \dfrac{1}{2}\,bh$. (Fig. 22)

4. **Parallelogram:** $A = bh$. (Fig. 23)

5. **Trapezoid:** $A = \dfrac{1}{2}\,(a + b)h$. (Fig. 24)

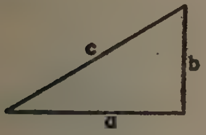


FIG. 20

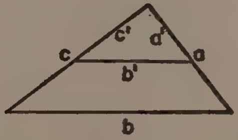


FIG. 21

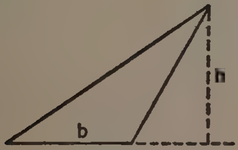


FIG. 22

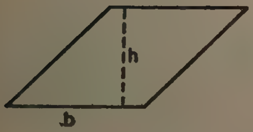


FIG. 23

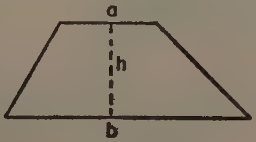


FIG. 24

6. Circle: $C = 2\pi r$, $A = \pi r^2$. (Fig. 25)

7. Sphere: $S = 4\pi r^2$, $V = \frac{4}{3}\pi r^3$. (Fig. 26)

8. Spherical segment of one base:

$$V = \frac{1}{3}\pi h^2(3r - h)$$

$$= \frac{1}{6}\pi h(3b^2 + h^2). \quad \text{(Fig. 27)}$$

9. Spherical segment of two bases:

$$V = \frac{1}{6}\pi h(3a^2 + 3b^2 + h^2) \quad \text{(Fig. 28)}$$

10. Ellipse: $A = \pi ab$. (Fig. 29)

11. Ellipsoid: $V = \frac{4}{3}\pi abc$. (Fig. 30)

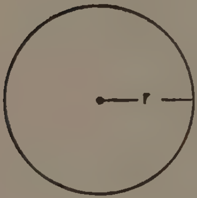


Fig. 25

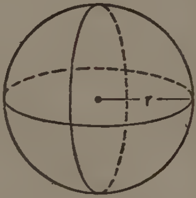


Fig. 26

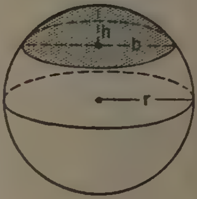


Fig. 27

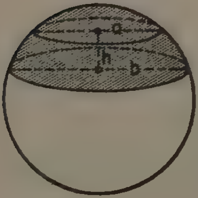


Fig. 28

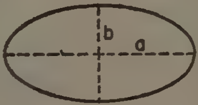


Fig. 29

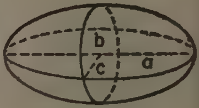


Fig. 30

12. Any prism or cylinder with parallel bases:

$$V = Bh. \qquad \text{(Figs. 31, 32)}$$

13. Regular pyramid: $S = \frac{1}{2}Ps.$ (Fig. 33)

14. Right circular cone: $S = \frac{1}{2}Cs = \pi rs.$ (Fig. 34)

15. Any pyramid or cone: $V = \frac{1}{3}Bh.$ (Figs. 35, 36)

16. Frustum of any pyramid or cone:

$$V = \frac{1}{3}h(B + B' + \sqrt{BB'}). \qquad \text{(Figs. 37, 38)}$$

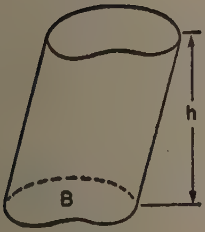


FIG. 31

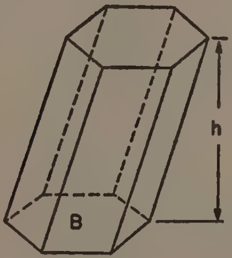


FIG. 32

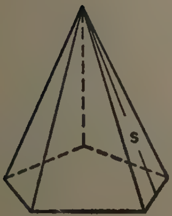


FIG. 33

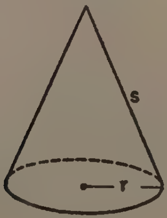


FIG. 34

17. Frustum of regular pyramid: $S = \frac{1}{2}(P + P')s.$ (Fig. 39)

18. Frustum of right circular cone:

$$S = \frac{1}{2}(C + C')s = \pi(r + r')s. \qquad \text{(Fig. 40)}$$

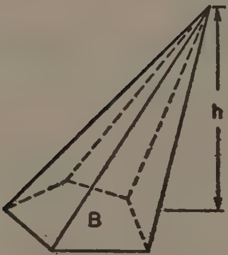


Fig. 35

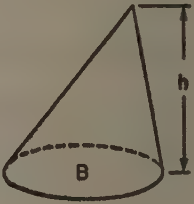


Fig. 36

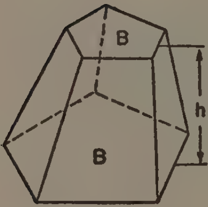


Fig. 37

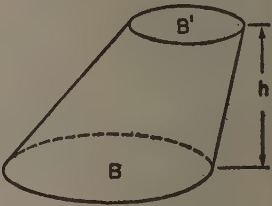


Fig. 38

Fig. 39

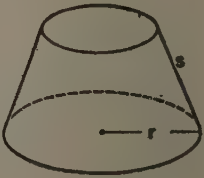


Fig. 40

TRIGONOMETRY

"The strength of Mathematics lies in the avoidance of all unnecessary thought, in the utmost economy of thought-operation."

ERNEST MACH

Trigonometry is the study and application of the characteristic properties of angles called *trigonometric functions*. It is indispensable to navigation, surveying, engineering, the physical sciences, and higher mathematics.

Reduction rules. The following simple rules enable us to reduce functions of any large positive angle or any negative angle to functions of a positive acute angle, and obviate the need for six reduction formulas for each of the angles $(90° + \theta)$, $(180° + \theta)$, $(180° - \theta)$, $(270° + \theta)$, $(270° - \theta)$, $(360° + \theta)$, and $(360° - \theta)$.

1. Place the angle in the standard position, Figs. 41-43.
2. Form a right triangle by dropping a perpendicular on the x-axis from a point (x,y) on the terminal side of the angle, and let φ denote the acute angle of the triangle at the origin of the coordinate axes. The angle φ, called the *reference angle*, is considered to be positive.
3. Place arrowheads at the termini of the sides of the triangle to indicate the signs of x and y; r is always positive.
4. Equate each function of the angle to the corresponding function of φ, and assign a negative sign to the latter in case x is negative but not y, or vice versa.

EXAMPLES. (1) Reduce functions of $(-476°)$ to functions of its reference angle.

Solution. Applying the reduction rules, we obtain Fig. 41 and the following results.

$$\varphi = (360° + 180°) - 476°$$
$$= 64°.$$

$\sin(-476°) = -\sin 64°$, $\cot(-476°) = \cot 64°$,
$\cos(-476°) = -\cos 64°$, $\sec(-476°) = -\sec 64°$,
$\tan(-476°) = \tan 64°$, $\csc(-476°) = -\csc 64°$.

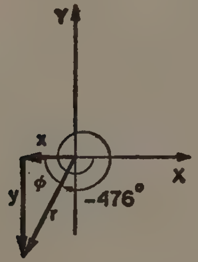


FIG. 41

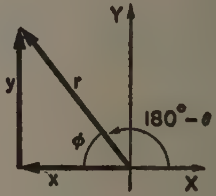


FIG. 42

(2) Reduce functions of $(180° - \theta)$ to functions of θ.

Solution. Evidently the reference angle is θ, Fig. 42; therefore

$\sin(180° - \theta) = \sin\theta$, $\cot(180° - \theta) = -\cot\theta$,
$\cos(180° - \theta) = -\cos\theta$, $\sec(180° - \theta) = -\sec\theta$,
$\tan(180° - \theta) = -\tan\theta$, $\csc(180° - \theta) = \csc\theta$.

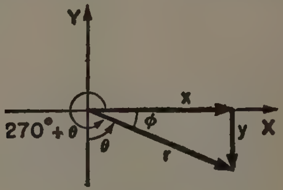


FIG. 43

(3) Reduce functions of $(270° + \theta)$ to functions of θ.

Solution. From Fig. 43, where $\varphi = 90° - \theta$, we obtain

$$\begin{aligned}
\sin(270^\circ + \theta) &= -\sin\varphi = -\cos\theta,\\
\cos(270^\circ + \theta) &= \cos\varphi = \sin\theta,\\
\tan(270^\circ + \theta) &= -\tan\varphi = -\cot\theta,\\
\cot(270^\circ + \theta) &= -\cot\varphi = -\tan\theta,\\
\sec(270^\circ + \theta) &= \sec\varphi = \csc\theta,\\
\csc(270^\circ + \theta) &= -\csc\varphi = -\sec\theta.
\end{aligned}$$

In Figs. 41-43 θ is taken to be acute, but the results are valid even if it is not acute.

Variation, range, and period of sin x. If the radius of the circle in Fig. 44 is taken to be of unit length, then the sine of an angle x equals the numerical value of the length of the corresponding arrow y. As the radius which forms the terminal side of x starts from $x = 0°$ and makes a complete rotation, sin x changes as indicated by the lengths of the arrows. The continuous arrows represent positive values of sin x, and the broken arrows negative values.

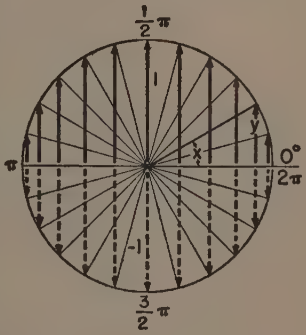


FIG. 44

As the terminal side of the angle continues to rotate, the same sequence in the values of sin x is repeated during every complete rotation. If the direction of rotation is reversed, then sin x changes as indicated by the same arrows but in the reverse order.

From these considerations and from Fig. 45, where ordinates represent values of sin x and where the pattern is repeated at intervals of x equal to 2π, we conclude that

$$\begin{aligned} -1 \leqslant \sin x &\leqslant 1, \\ \sin(-x) &= -\sin x, \\ \sin(x + 2n\pi) &= \sin x, \\ P &= 2\pi, \end{aligned}$$

where n is any integer, and P is called the *period*.

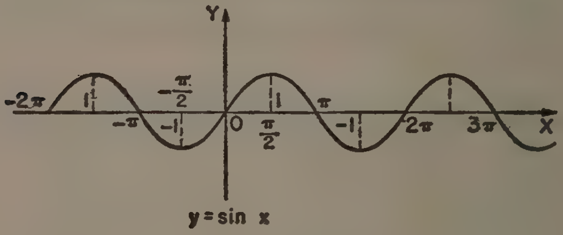


FIG. 45

Variation, range and period of cos x. From Figs. 46 and 47, we obtain

$$\begin{aligned} -1 \leqslant \cos x &\leqslant 1, \\ \cos(-x) &= \cos x, \\ \cos(x + 2n\pi) &= \cos x, \\ P &= 2\pi. \end{aligned}$$

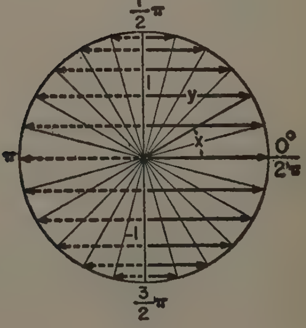


FIG. 46

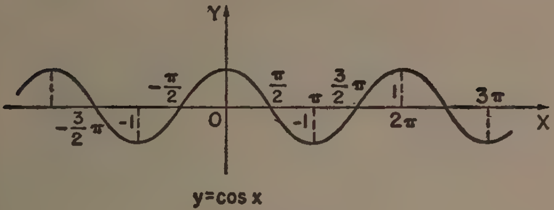


FIG. 47

Variation, range, and period of tan x. From Fig. 48, and from Fig. 49, where y_0, y_1, etc. represent tan x, we obtain

$$-\infty < \tan x < \infty,$$
$$\tan(-x) = -\tan x,$$
$$\tan(x + n\pi) = \tan x,$$
$$P = \pi.$$

The period is π in this case, because the same sequence of values of tan x is repeated every time the terminal side of the angle x rotates through 180° in Fig. 48, and the pattern is repeated at intervals of x equal to π in Fig. 49.

Equations. Equations may be classified into the following types:

Equations of condition. A conditional equation is valid only for certain values of the required magnitude it contains. For example, $(x + 1)^2 = 1$ is valid only for $x = 1$ and -2, and $\sin \theta = 1$ is valid, if θ is restricted to the interval between 0° and 360°, only for 30° and 150°.

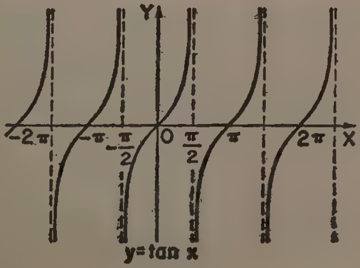
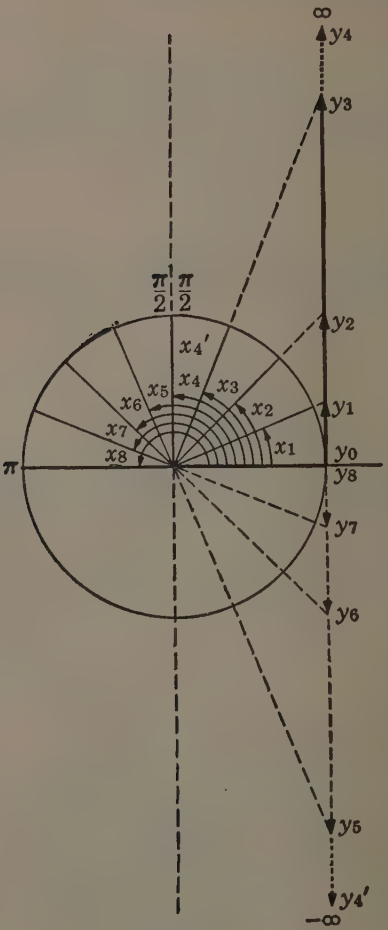


FIG. 48



∞
y_4
y_3
$\frac{\pi}{2}$
$\frac{\pi}{2}$
y_2
x_4'
x_4
x_3
x_5
x_6
x_2
y_1
x_7
x_1
y_0
x_8
π
y_8
y_7
y_6
y_5
y_4'
−∞

FIG. 49

Identities. An identity is valid for all values of the variables for which the equation is defined. For example, $(x + y)^2 = x^2 + 2xy + y^2$ is valid for all values of x and y; and $\sin^2 \theta + \cos^2 \theta = 1$ is valid for all values of θ.

Definitions. An equation which defines a quantity shows how the value of the quantity may be found. Such a definition has been called an *operational definition* by Professor P. W. Bridgman, in order to distinguish it from a dictionary definition, which usually gives synonyms or near synonyms but no information about the physical or mental operation to be performed to obtain the thing defined.

Formulas. A formula is either an equation of identity which is important because of its relative simplicity and frequency of application, or an equation which defines an important quantity.

Solution of trigonometric equations. The method of solving an equation depends on the form of the equation, not on the symbols it contains or the quantities which the symbols represent. Therefore if a trigonometric equation of condition has the same form as a particular type of algebraic equation, it is solved in the same way as the algebraic equation.

If a trigonometric equation contains more than one kind of function or angle, it has to be reduced, with the aid of other equations, to an equation or equations containing one kind of function and angle, just as an algebraic equation which contains more than one unknown has to be reduced to an equation or equations containing one unknown.

EXAMPLES. (1) Find the angles between 0° and 360° which satisfy the equation $2 \sin^2 \theta + \sin \theta - 1 = 0$.

Solution. This is a quadratic equation in $\sin\theta$, and has the form $2x^2 + x - 1 = 0$. It can be solved by the use of the quadratic formula, or by factoring, or by the method of completing the square. Using the last method, we write

$$\sin^2\theta + \tfrac{1}{2}\sin\theta = \tfrac{1}{2},$$
$$\sin^2\theta + \tfrac{1}{2}\sin\theta + \tfrac{1}{16} = \tfrac{9}{16},$$
$$(\sin\theta + \tfrac{1}{4})^2 = \tfrac{9}{16},$$
$$\sin\theta = -\tfrac{1}{4} \pm \tfrac{3}{4}.$$

Hence $\qquad \sin\theta = \frac{1}{2}, \quad \text{or} \quad \sin\theta = -1.$

From $\sin\theta = \frac{1}{2}$, we get $\theta = 30°$ or $150°$.
From $\sin\theta = -1$, we get $\theta = 270°$.

(2) Find the angles between 0° and 360° which satisfy the equation $\sin 2\theta - \cos\theta = 0$.

Solution. The equation contains two different angles, namely θ and 2θ, therefore we change it to an equation which contains only θ or 2θ. Changing $\sin 2\theta$ to functions of θ, we obtain

$$2\sin\theta\cos\theta - \cos\theta = 0.$$

This equation contains two kinds of functions. We may either change it into an equation which contains only one kind of function or resolve it into two equations each of which contains one kind of function. The latter is more convenient in this particular problem.

$$\cos\theta(2\sin\theta - 1) = 0.$$

Therefore $\cos\theta = 0$, and $\sin\theta = \frac{1}{2}$. From the first, we get $\theta = 90°, 270°$; and from the second, $\theta = 30°, 150°$.

Inverse functions. If an equation in two variables is solved for one and then for the other variable, the right-hand members of the two resulting equations are said to be the *inverse* of each other. The simpler of the two is called the *direct function* or simply the *function*, and the other the *inverse function*.

In writing a function and its inverse, it is general practice to use the same symbols for the independent and dependent variables in both functions. For example, the inverse of $y = \frac{1}{8}x^3$ is obtained by solving it for x and then interchanging the positions of x and y. Thus $y = 2x^{1/3}$ is the inverse of $y = \frac{1}{8}x^3$.

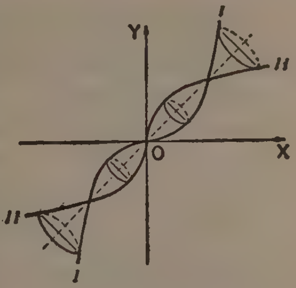


FIG. 50

The graphs of a function and its inverse are identical curves but are differently located on the xy-plane. They are symmetrical with respect to the 45° line; hence if one of the graphs is rotated through 180° about the 45°-line, it coincides with the other.

In Fig. 50, Curve I is the graph of $y = \frac{1}{8}x^3$, and Curve II the graph of $y = 2x^{1/3}$.

Inverse trigonometric functions. If the equation $y = \sin x$ is solved for x and then the positions of x and y are interchanged, the result (the inverse of $y = \sin x$) is written in the form $y = \text{arc} \sin x$ or $y = \sin^{-1} x$.* In either form the equation states that "y is that angle whose sin is x," and is read "y equals arc sin of x" or "y equals inverse sin of x." The inverse of each of the other five trigonometric functions is similarly expressed.

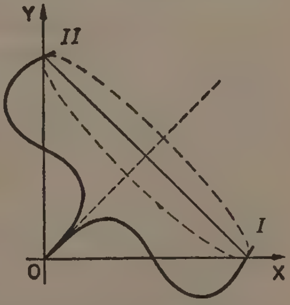


FIG. 51

* The symbol $\sin^{-1} x$ has no relation to the reciprocal of $\sin x$; the latter should be written either as $1/\sin x$ or $(\sin x)^{-1}$.

In Fig. 51, Curve I is the graph of $y = \sin x$, and Curve II the graph of $y = \text{arc} \sin x$.

Formulas. The following are some of the most frequently used trigonometric formulas, or identities.

1. $\sin^2\theta + \cos^2\theta = 1.$
2. $\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$
3. $\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta.$
4. $\tan (\alpha + \beta) = \dfrac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}.$

The sine, the cosine, and the tangent of $(\alpha - \beta)$ can be obtained by replacing β with $-\beta$ and remembering that $\sin (-\beta) = -\sin \beta$, $\cos (-\beta) = \cos \beta$, and $\tan (-\beta) = -\tan \beta$. The functions of 2α can be obtained by letting $\beta = \alpha$.

5. $\dfrac{a}{\sin \alpha} = \dfrac{b}{\sin \beta} = \dfrac{c}{\sin \gamma}.$ (Law of sines)

6. $c^2 = a^2 - 2ab \cos \gamma + b^2.$ (Law of cosines)

If we let $\gamma = 90°$, the law of cosine reduces to $c^2 = a^2 + b^2$. Therefore the Pythagorean theorem may be regarded as a special case of the law of cosines.

ANALYTIC GEOMETRY

"Mathematicians study not objects but relations between objects; the replacement of these objects is therefore indifferent to them, provided the relations (functions) do not change."

HENRI POINCARÉ

Analytic geometry is the result of a union of algebra and geometry, brought about by Descartes (1596-1650). The fundamental concept in analytic geometry is that of *function*, that is, the relationship of two or more variables, or the dependence of the value of a variable on the values of other variables.

The functional relationship between variables may be stated in words, or may be given in the form of an equation, or may be presented by a graph if the number of variables does not exceed three. The chief aim of analytic geometry is to translate functions which are given in one of the three forms into the other two forms in order to bring out fully their properties.

The importance of analytic geometry to mathematics derives from the fact that mathematics, with the exception of arithmetic, is the study of functions. Its importance to science arises from the fact that the principal object of science is the discovery and the application of the functional relationships among variables of Nature.

Plotting curves. Rectangular coordinates. The following observations should prove helpful in plotting curves on the xy-plane.

1. A first degree equation in x and y represents a straight line. As two points determine a straight line,

only two points, preferably the intercepts, need be plotted in order to draw the line.

2. A second degree equation in x and y represents, if it has a graph, a conic section or its limiting case.

3. If an equation contains even powers or even roots, x or y or both become imaginary for values greater or less than a certain limiting value or values. It is desirable to find these values in order to determine the region in which the curve lies.

4. If y equals a polynomial in x with positive exponents, the term which contains the highest power *dominates* the other terms for large values of x, and determines the trend of the curve in regions distant from the origin. As x increases, the curve becomes steeper if the highest power of x is greater than 1, and flattens out if the highest power is a proper fraction.

Plotting curves. Polar coordinates. The following are important points to remember in plotting curves in polar coordinates.

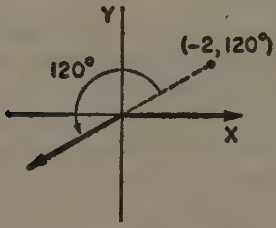


Fig. 52

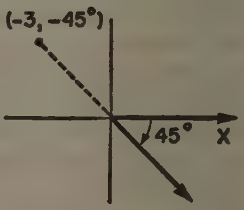


Fig. 53

1. If the radius vector ρ is negative for a given value of the angle θ, the point lies on the extension of the terminal side of θ, drawn in the opposite direction, as shown in Figs. 52 and 53.

2. If the equation contains multiple angles, θ should be given values which make consecutive values of θ differ by a smaller amount than when the equation contains a single angle. For instance, in plotting points to obtain the graph of $\rho = \sin\theta$, the value of θ may be increased by 30° at a time. But in the case of $\rho = \sin 2\theta$, it is better to make the interval 15°, for otherwise the loops in the curve may not be discovered.

3. It is desirable to connect points as they are plotted. This is especially important if the curve has loops.

Arbitrary constants. If an equation in x and y contains one arbitrary constant the equation represents *a system of curves* which have a common property. The equation $y = x + b$, for instance, represents a system of parallel lines which make an angle of 45° with the x-axis; and $x^2 + y^2 = a^2$ represents a system of concentric circles which have the origin for their common center.

The values of x and y change from point to point on a curve in the system; the value of the arbitrary constant changes from curve to curve.

We may arbitrarily assign any permissible value to the constant, and thereby obtain the equation of a particular member of the system of curves. We can also find the equation of the particular member of the system which passes through a given point by replacing x and y in the equation with the coordinates of the point, determining the value of the constant, and then substituting the value of the constant in the original equation.

The meaning of an arbitrary constant in an equation is usually determined by giving one of the variables its initial value. For instance, by letting $x = 0$ in $y = mx + b$, we find that b represents the y-intercepts of the system of lines. Again, if we let $t = 0$ in $v = gt + c$, the equation for the velocity of a falling body, we find that c is the initial velocity.

Geometric theorems. To prove analytically a theorem of plane geometry, it is permissible and advisable to place the figure on the xy-plane in such a way as to reduce the number of arbitrary constants involved to a minimum.

EXAMPLE. Prove that the diagonals of a rectangle bisect each other.

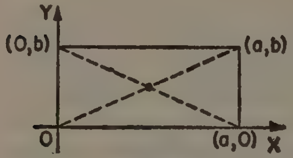


FIG. 54

Proof. The number of arbitrary constants involved in specifying the coordinates of the four vertices of a rectangle may be as high as 8 and as low as 2, depending on the way the rectangle is placed on the xy-plane. The following proof is perfectly general, yet it involves only two arbitrary constants.

Placing the rectangle as in Fig. 54, we find the equations of the lines on which the diagonals lie,

$$y = \frac{b}{a}x, \qquad \text{[by point-slope form]}$$

$$\frac{x}{a} + \frac{y}{b} = 1. \qquad \text{[by intercept form]}$$

Solving these equations simultaneously, we obtain $x = \frac{1}{2}a$ and $y = \frac{1}{2}b$ for the coordinates of the intersection point. But since $(\frac{1}{2}a, \frac{1}{2}b)$ is the mid-point of each diagonal, the diagonals bisect each other.

Note: The letters x and y represent variables in the equations of the lines, yet they appear as constants after the equations are solved simultaneously. This apparently contradictory result is due to the omission of the following details:

Let (x_1, y_1) be the intersection point; then since this point lies on both lines, its coordinates satisfy the equations of the lines. Therefore

$$y_1 = \frac{b}{a}x_1 \quad \text{and} \quad \frac{x_1}{a} + \frac{y_1}{b} = 1.$$

Solving these equations simultaneously, we get

$$x_1 = \tfrac{1}{2}a \quad \text{and} \quad y_1 = \tfrac{1}{2}b.$$

Representative point. Whenever a curve is referred to rectangular coordinates, it is very important to reserve x and y to denote exclusively the coordinates of the point which represents any and every one of the infinity of points on the curve, and to use (x,y) as the symbol of that point. We shall call the point (x,y) the *representative point.*

Derivation. The process by means of which the equation of a curve is found, by translating into mathematics the verbal statement of the properties of the curve, is called a *derivation.* A derivation problem is sometimes called a *locus problem.*

To derive the equation of a curve, proceed as follows:

1. Draw a system of axes, indicate the given points and lines, take a point to be the representative point, and label the latter (x,y).

It is very important to place the representative point at a representative position. If it is placed on one of the coordinate axes, or on a given line, or on a horizontal or vertical line through a given point, the result is bound to be wrong. Attempting to derive the equation of a curve while placing (x,y) in a favored position is like trying to think clearly on an issue while clinging to a preconceived and favored idea.

The representative point may be placed in any quadrant, but when it is placed in a quadrant other than the first, the signs of x and y have to be taken into consideration in translating the conditions of the problem into an equation.

2. Draw the necessary construction lines and label them.

3. Make a literal (word-to-symbol) translation of the statement of the problem so as to obtain an equation.

4. Simplify the equation, or put it in the standard form, in order to have it in the idiom of the mathematical language.

5. Plot or trace the curve.

It is more convenient to trace a curve, after finding its important points and other features, than to plot it.

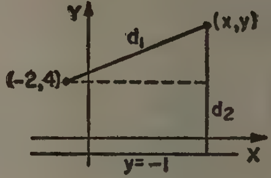


FIG. 55

EXAMPLES. (1) A point moves so that its distance from the point $(-2,4)$ always equals its distance from the line $y = -1$; derive the equation of its path.

Solution. The problem states that $d_1 = d_2$, Fig. 55. A literal translation of this condition in terms of the coordinates of the representative point yields

$$\sqrt{(x+2)^2 + (y-4)^2} = y + 1. \tag{1}$$

Simplifying (1) we obtain

$$x^2 + 4x - 10y + 19 = 0 \tag{2}$$

for the general form of the equation of the parabola. To put the equation in the standard form, we either solve (1) for $(x + 1)^2$ or complete the square in (2), and obtain

$$(x+2)^2 = 10(y - \tfrac{3}{2}). \tag{3}$$

Comparing (3) with $(x - h)^2 = 4p(y - k)$, we find that the vertex of the parabola is at $(-2,\frac{3}{2})$, $p = \frac{5}{2}$, the focus is at $(-2,4)$, and the ends of the focal chord are at $(3,4)$ and $(-7,4)$. We make use of these data to trace the curve, Fig. 56.

(2) Find the locus of the middle points of the ordinates of the circle $x^2 + y^2 = a^2$.

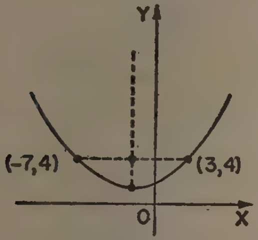


FIG. 56

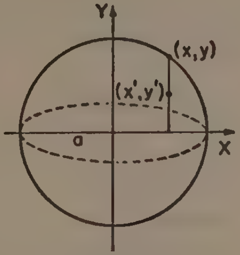


FIG. 57

Solution. We draw the circle, Fig. 57, take a representative ordinate, and label its middle point (x',y') in order to distinguish it from the corresponding representative point (x,y) on the circle. By the conditions of the problem, $x = x'$ and $y = 2y'$. Making these substitutions in the equation of the circle and then dropping the primes, we get

$$4x^2 + y^2 = 4a^2,$$

which is the equation of an ellipse.

(3) The cables of a suspension bridge form parabolic curves. The lowest points of the cables are at the same level as the road-bed and the highest points are at the tops of pillars, 140 feet above the roadbed and 1600 feet apart. Find the length of a vertical rod which connects the roadbed to one of the cables and which is 450 feet from the middle of the bridge.

Given

$$2x_1 = 1600 \text{ ft}, \qquad x_2 = 450 \text{ ft},$$
$$y_1 = 140 \text{ ft}, \qquad y_2 = ?$$

Solution. Referring the parabola to coordinate axes, as in Fig. 58, we have $x^2 = 4py$ for its equation. Since (x_1,y_1) and (x_2,y_2) are on the curve, $x_1^2 = 4py_1$ and $x_2 = 4py_2$. Hence

$$\begin{aligned} y_2 &= \left(\frac{x_2}{x_1}\right)^2 y_1 \\ &= \left(\frac{450}{800}\right)^2 \times 140 \text{ ft} \\ &= \left(\frac{9}{16}\right)^2 \times 140 \text{ ft} \\ &= 44.3 \text{ ft.} \end{aligned}$$

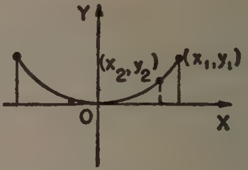


FIG. 58

Observe that unnecessary labor was avoided by eliminating p, instead of finding its value and then making use of it to find y_2.

Conics. The most general equation of the second degree in x and y is of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where A to F are constants. If such an equation has a graph it is a conic or a limiting case of a conic. In what follows we shall assume that the equation represents a conic, and that, unless otherwise stated, the xy-term is absent.

1. If $A = C$, the conic is a circle.
2. If A and C are unequal but have the same sign, the conic is an ellipse whose axes are parallel to the coordinate axes.
3. If A and C have opposite signs, the conic is a hyperbola whose axes are parallel to the coordinate axes.
4. If one of the square terms is absent, the conic is a parabola whose axis is parallel to one of the coordinate axes.

5. If both square terms are absent but the xy-term is present, the conic is an equilateral hyperbola whose asymptotes are parallel to the coordinate axes.

6. If one or both the square terms and the xy-term are present, the axes of the conic (not a circle) are inclined to the coordinate axes.

DIFFERENTIAL CALCULUS

"When we look at Nature with a philosophic eye, what do we behold? . . . We see that all things are in a state of flux; nothing remains unaltered for an instant. It is a moving, changing universe with which we have to do."

HUGH ELLIOT

In the changing universe with which we have to do, a variable quantity and its rate of change are related to (a function of) other variable quantities and their rates of change. The main task of science is the discovery of these relationships. As the calculus is concerned with the study of such relationships, it is not surprising that scientific discovery was given a tremendous impetus by the introduction of the method of the calculus by Newton (1642-1727) and Leibnitz (1646-1716), and that the degree of development of the calculus has become the true measure of scientific progress.

Calculus is the most interesting branch of mathematics and, after arithmetic, the most useful. It would be no exaggeration to say that the great men of Ancient Greece would have given years of their lives for the opportunity to attend a calculus class and if they had found a calculus book, would have been as overjoyed as a miser who finds a pot of gold.

The advanced parts of calculus, which includes most of the ever-growing body of mathematical literature, are very abstract, far removed from ordinary human experience, and consequently difficult. But elementary calculus is within the competence of anyone who knows his high school algebra. Professor E. T. Bell says, in his *Queen of Sciences*, "Any normal boy or girl of sixteen could master the calculus in half the time often devoted to stumbling through Book One of Caesar's Gallic War." Yet a graduate of an eastern university once said to me, "I got the second highest grade in my calculus class, but I did not know what it was about then, and I don't know now." That man's predicament, which is not uncommon among those who have taken a course in the subject, was probably due to a failure to understand the reasons for introducing the notation of the calculus, a notation that is radically different from that of school mathematics. If a student does not understand the reasons for introducing this novel notation, he is likely to be mystified by it. When this happens, he may learn how to apply the formulas to get answers to problems, but the why and the wherefor of the subject remain a mystery to him. It is not enough for the student to learn the rigorously stated formal definitions of such concepts as *derivative*, *limit*, and *continuity;* they have to become intuitively reasonable to him through association with his experience.

Increments. Consider the average velocity of an automobile during a trip. Let s denote the distance covered in the interval of time t; then the average velocity is given by

$$\bar{v} = \frac{s}{t}. \tag{1}$$

While this is a valid equation it is not a wholly satisfactory definition of $\bar{v}$, because it does not indicate the way in which s and t are actually obtained. In other words, equation (1) is not a strictly operational definition of average velocity. The following equation *is* an operational definition of $\bar{v}$.

$$\bar{v} = \frac{s_2 - s_1}{t_2 - t_1}, \tag{2}$$

where s_1 and s_2 denote the readings of the odometer (mileage meter) of the automobile at the beginning and at the end of the trip, and t_1 and t_2 the corresponding readings of a watch. This equation is not quite satisfactory, however, because it gives the impression that each term in the right-hand member is in itself essential, which is not the case, as will be seen from the following considerations. If the readings for s_1 and s_2 were taken from the trip-meter as well as from the odometer, and the readings for t_1 and t_2 were taken from two watches, one on standard time and the other on daylight time, the two sets of readings would have been different, yet the same value for $\bar{v}$ would have been obtained. Therefore the really important items in equation (2) are the changes in the readings of the instruments, not the readings themselves. This is clearly brought out by the equation

$$\bar{v} = \frac{\Delta s}{\Delta t}, \tag{3}$$

where Δs and Δt denote, respectively, the changes in s and t, and are called the *increments* of s and t. Equation (1) does not tell the whole truth; equation (2) tells more than the truth; equation (3), on the other hand, tells the truth, the whole truth, nothing but the truth.

Derivatives. Had the notation of the calculus stopped at the increment stage the advantage gained over the notation of algebra would have been insignificant, because average velocity, or any other average rate of change, is a purely mathematical concept which does not correspond to any physical reality. When an automobile hits a stone wall, the result depends on the velocity at the instant of impact, not on the average velocity for a period preceding the time of impact. But how is *velocity at an* *instant* to be defined? In order to find an answer to this question, consider the motion of a body in a straight line and let P, Fig. 59, be the position of the body at any instant t, s be its distance from a fixed point O, and Δs ($=PP'$) be the distance moved in the interval of time Δt. Then $\Delta s/\Delta t$ is the average velocity for the distance Δs and for the interval of time Δt. This average velocity has, in general, a value different from the velocity at P. But by making Δt, and consequently Δs, sufficiently small the average velocity can be made as close to the velocity at P as may be desired, because the velocity of a body cannot change materially in a very short time and distance, except in a collision with another body. Hence the definition: *Velocity is the limit which $\Delta s/\Delta t$ approaches as Δt approaches zero*, that is,

FIG. 59

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}.$$

It may be objected that near the limiting state, Δt and Δs become too small to be measured. True, but a similar objection may be made against the definition of any mathematical concept which involves an operation, for instance, against the definition of a straight line as the shortest distance between two points. It is not only impossible to measure a length with absolute exactness, but there are no such objects in Nature as mathematical points and straight lines. The operation involved in the definition of a mathematical concept is carried out mentally beyond the limits of the corresponding physical operation. It is surprising indeed, therefore, that mathematics is applicable to the study of Nature.

The foregoing operational symbol of the definition of velocity is too long to be convenient for use in equations, therefore the symbol $\frac{ds}{dt}$ is introduced to denote the operation indicated by $\lim\limits_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$, and is called the *derivative of s with respect to t.*

Velocity is only one of an indefinite number of magnitudes, each of which is the rate of change of a variable with respect to another variable. Therefore x and y, the generic symbols for variables, are used in the general definition of the derivative, thus

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

As Δx and, consequently, Δy approach zero, the quotient may approach a definite value, as in the case of the velocity of a body, or it may increase or decrease indefinitely depending on the functional relationship of the variables and their particular values.

It is shown in calculus books that the derivative, or the rate of change, of a function may be represented graphically by the slope of a curve. *Derivative, slope,* and *rate of change* are, respectively, the analytical, the geometrical, and the physical designations of the same concept.

The derivative of a function may be found by actually carrying out the operation indicated by its definition, that is, by the increment method, or by the use of a formula. In the latter case the operation is carried out by the formula.*

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* Formulas play a role in mathematics analogous to the role played by automatic machines in industry. They carry out readily operations that are difficult to accomplish by hand labor.

Suggestions: Do not expand the denominator in using the increment method.

Bear in mind that Δx is the increment of x not of x^2, for instance.

Distinguish between the general expression of a derivative and its special value, as in (c) of the example which follows.

The following suggestions apply to the use of formulas:

Use the formula whose *left-hand* member has the same form as the expression to be differentiated.

If the numerator of a fraction is constant, change the fraction to the power form, for instance, $\dfrac{5a}{(x+1)^2}$ to $5a(x+1)^{-2}$.

If the denominator is constant, change it to a factor, for instance, $\dfrac{(x+1)^2}{5a}$ to $\dfrac{1}{5a}(x+1)^2$.

If both the numerator and the denominator have the same power, change the fraction to the power form, for instance, $\dfrac{\sqrt{a+x}}{\sqrt{a-x}}$ to $\left(\dfrac{a+x}{a-x}\right)^{\frac{1}{2}}$.

EXAMPLE. Given $y = \dfrac{3}{x^2}$; find the derivative (a) by the increment method and (b) by formula, and (c) find the values for $x = -2$ and $x = 0$.

(a)
$$y = \frac{3}{x^2};$$

$$y + \Delta y = \frac{3}{(x+\Delta x)^2},$$

$$\Delta y = \frac{3}{(x+\Delta x)^2} - \frac{3}{x^2}$$

$$= 3\,\frac{x^2 - x^2 - 2x\Delta x - (\Delta x)^2}{(x+\Delta x)^2 x^2}$$

$$= -3\,\frac{2x + \Delta x}{(x + \Delta x)^2x^2}\,\Delta x;$$

$$\frac{\Delta y}{\Delta x} = -3\,\frac{2x + \Delta x}{(x + \Delta x)^2x^2};$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[-3\frac{2x + \Delta x}{(x + \Delta x)^2x^2}\right]$$

$$= -3\,\frac{2x}{x^4}$$

$$= -\frac{6}{x^3}.$$

(b) Putting the right-hand member of $y = \frac{3}{x^2}$ in the power form, we have

$$y = 3x^{-2};$$

$$\frac{dy}{dx} = 3(-2)x^{-3}$$

$$= -\frac{6}{x^3}.$$

(c) $$\left(\frac{dy}{dx}\right)_{x=-2} = \frac{3}{4}; \qquad \left(\frac{dy}{dx}\right)_{x \to 0} = \pm\infty.$$

The sign of the right-hand member of the last equation depends on whether x approaches zero from the negative or the positive side of zero.

Maxima and minima. The student's difficulty in solving a verbal problem on maxima and minima is invariably due to his inability to do the algebraic work which precedes the calculus part of the solution. The following directions are designed to help overcome that difficulty.

1. Read the problem carefully in order to determine what quantity is to be maximized or minimized, and assign to it a symbol that is suggestive of the quantity. If the quantity is implied but not specifically stated,

change the wording in a restatement of the problem so as to bring out clearly what quantity is to be maximized or minimized. For instance, change "most economical" to "minimum cost," and "capacity" to "volume." No knowledge of mathematics is needed to follow this direction.

2. Draw a figure whenever the problem permits, and label its parts.

If the figure contains a curve, refer it to coordinate axes, use x and y to denote the coordinates of the representative point on the curve, and make use of the equation of the curve.

3. Find an algebraic expression for the quantity to be maximized or minimized.

If the expression of the quantity contains more than one variable, it means that some conditions of the problem, such as the equation of a curve and the fact that some quantity remains constant, have to be considered in order to obtain the equations necessary for expressing the quantity as a function of one variable.

No knowledge of calculus is necessary to carry out the foregoing directions.

4. If the original expression of the quantity contains only one variable, find its derivative with respect to the variable, set it equal to zero, find the value of the variable, and then put that value in the expression to find the maximum or the minimum value of the quantity.

5. If the expression of the quantity contains more than one variable use one of the following methods:

Method A: Eliminate all variables but one from the expression, and then proceed as in the preceding direction.

Method B: Differentiate all the equations involved with respect to one of the variables, and solve the resulting equations simultaneously to obtain the desired information. The derivative of the quantity to be maximized or minimized equals zero, of course.

EXAMPLES. (1) A building with a rectangular foundation is to be built on a triangular lot; find the maximum area of the foundation.

Solution. Let the triangle in Fig. 60 represent the lot, and the rectangle the building. Then, in the notation of the figure, the area is given by

$$A = xy.$$

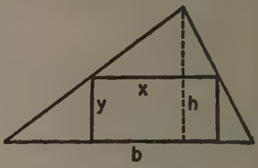

FIG. 60

Since the right-hand member contains two variables, we must find another equation in x and y by making use of the condition of the problem, that is, the fact that the rectangle is inscribed in the triangle; we are not dealing here with any arbitrary rectangle. The main triangle and the one above the rectangle are similar, therefore their bases and altitudes are proportional; that is,

$$\frac{x}{b} = \frac{h - y}{h}.$$

Using Method A, we have

$$A = xy \qquad \frac{dA}{dy} = \frac{b}{h}(h - 2y) = 0,$$

$$= b\,\frac{h - y}{h}\,y \qquad y = \frac{1}{2}h,$$

$$= \frac{b}{h}(hy - y^2). \qquad x = \frac{1}{2}b.$$

$$A = \frac{1}{4}bh.$$

(2) The frictional resistance offered by the sides of a channel to the flow of water varies as the area of contact. Find the relation between the width and the depth of an open channel with a rectangular cross section which will offer the least resistance.

Solution. Since a channel is designed to carry a definite current of water between two fixed points, the area of the cross section and the length of the channel must be considered constant. These conditions, which are only implied in the problem, have to be taken into account.

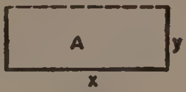

FIG. 61

Let R denote the resistance, and l, x, and y denote, respectively, the length, the width, and the depth of the channel, Fig. 61. Then

$$R = kl(x + 2y),$$

and

$$xy = A,$$

where k is the constant of proportionality and depends on the frictional qualities of the surfaces, and A is the constant area of the cross section of the channel.

Method A:

$$\begin{aligned} R &= kl(x + 2y) \\ &= kl\left(x + \frac{2A}{x}\right) \\ &= kl(x + 2Ax^{-1}). \\ \frac{dR}{dx} &= kl(1 - 2Ax^{-2}) = 0. \end{aligned}$$

Hence

$$1 - \frac{2A}{x^2} = 0,$$

$$1 - 2\frac{xy}{x^2} = 0,$$

and

$$x = 2y.$$

Thus the width of the channel must be twice the depth to minimize the resistance.

Method B.

$$R = kl(x + 2y). \qquad xy = A.$$

$$\frac{dR}{dx} = kl\left(1 + 2\frac{dy}{dx}\right) = 0, \qquad x\frac{dy}{dx} + y = 0,$$

$$2\frac{dy}{dx} - 1 = 0. \quad (1) \qquad \frac{dy}{dx} = -\frac{y}{x}. \quad (2)$$

Eliminating $\frac{dy}{dx}$ between equations (1) and (2), we have

$$x = 2y.$$

Related rates. As in the case of maxima and minima, the student's difficulty in solving a problem on related rates lies almost entirely in the algebraic part of the solution. The following directions, which are almost identical with those given in the preceding section, should prove helpful.

1. Read the problem carefully in order to determine the quantities whose rates are given or required. If these quantities are not specifically mentioned in the problem, interpretation of some of the terms becomes necessary, as in the case of the examples under this section.

2. Draw a figure, if the problem permits, and label its parts.

3. Form a table of the given and the required magnitudes.

4. Find an expression for the quantity whose rate is required. If the expression contains variables whose rates are not given, eliminate them with the help of equations obtained from the conditions of the problem.

5. Differentiate the equation which expresses the quantity whose rate is required.

EXAMPLES. (1) A man is walking away from a lamppost at the rate of 3 miles per hour. If the man is 6 feet tall and the lamppost is 15 feet high, find the rates at which (a) the end of his shadow is moving and (b) the length of his shadow is increasing.

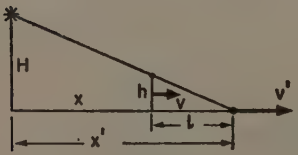

FIG. 62

Solution. The velocity of a body is the rate at which its distance from a fixed point on the path is changing, the distance being measured along the path. The fixed point may be any point on the path, but in this case the base of the lamppost is the most convenient point. Using the notation of Fig. 62, we form a table of the given and required magnitudes and then proceed with the solution.

Data

$$H = 15 \text{ ft}, \qquad v' = \frac{dx'}{dt} = ?$$

$$h = 6 \text{ ft},$$

$$v = \frac{dx}{dt} = 3\,\frac{\text{m}}{\text{hr}}. \qquad \frac{dl}{dt} = ?$$

(a) $\dfrac{x'}{H} = \dfrac{x' - x}{h}$, $\left[\begin{array}{c}\text{by similar}\\ \text{triangles}\end{array}\right]$

$$x' = \frac{H}{H-h}\,x.$$

$$\frac{dx'}{dt} = \frac{H}{H-h}\,\frac{dx}{dt}$$

$$= \frac{15}{15-6} \times 3\,\frac{\text{m}}{\text{hr}}$$

$$= 5\,\frac{\text{m}}{\text{hr}}.$$

(b) $l = x' - x$.

$$\frac{dl}{dt} = \frac{dx'}{dt} - \frac{dx}{dt}$$

$$= 5\,\frac{\text{m}}{\text{hr}} - 3\,\frac{\text{m}}{\text{hr}}$$

$$= 2\,\frac{\text{m}}{\text{hr}}.$$

Discussion: All the equations are dimensionally correct. If we let $h = 0$, thus reducing the moving body (the man) to a particle, $v' = v$ and $dl/dt = 0$, as they should.

Since all the rates involved are constants, this problem can be solved without the use of calculus.

(2) Water is flowing into a conical reservoir at a constant rate. It is observed that the surface of the water is rising at the rate of 2 inches per minute when the water is 6 feet deep. If the reservoir is 20 feet in diameter at the top and 15 feet deep, find the rate at which water is coming in.

Note: In order to be able to solve this problem we must recognize that (a) the rate at which water flows into the reservoir equals the rate at which the volume of the water in the reservoir increases; (b) the rate at which the surface rises equals the rate at which the depth of the water in the reservoir increases (see nonmathematical phrases, p. 33), and (c) the phrase, "when the water is 6 feet deep," does not mean, "the water *is* 6 feet deep." This last point is made clear by the way the rate of increase of the depth is given in the table of data which follows. If we wrote $y = 6$ ft we would be stating that y is a constant, which would be incompatible with the statement that the surface of the water is rising.

Solution. Using the notation of Fig. 63, and taking into account the points raised in the preceding paragraph, we form a table of the given and required magnitudes, and then proceed with the solution as follows:

Data

$$2a = 20 \text{ ft},$$
$$h = 15 \text{ ft}.$$
$$\left(\frac{dy}{dt}\right)_{y=6} = 2\,\frac{\text{in}}{\text{min}},$$
$$\frac{dV}{dt} = ?$$

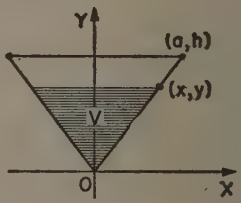

FIG. 63

$$V = \frac{1}{3}\pi x^2 y$$

$$= \frac{\pi}{3}\left(\frac{a}{h}\right)^2 y^3. \qquad \left[\frac{x}{y} = \frac{a}{h}\right]$$

$$\frac{dV}{dt} = \pi\left(\frac{a}{h}\right)^2 y^2\,\frac{dy}{dt}$$

$$= \pi\left(\frac{10}{15}\right)^2 (6 \text{ ft})^2 \times 2\,\frac{\text{in}}{\text{min}}$$

$$= \frac{22}{7}\left(\frac{2}{3}\right)^2 (6)^2\,\frac{1}{6}\,\frac{\text{ft}^3}{\text{min}}$$

$$= \frac{22 \times 8}{7 \times 3}\,\frac{\text{ft}^3}{\text{min}}$$

$$= 8.3\,\frac{\text{ft}^3}{\text{min}}.$$

INTEGRAL CALCULUS

"The method of fluxions (the calculus) is probably one of the greatest, most subtle, and sublime discoveries of any age: it opens a new world to our view, and extends our knowledge as it were, to infinity; carrying us beyond the bounds that seemed to have been prescribed to the human mind, at least infinitely beyond those to which the ancient geometry was confined."

CHARLES HUTTON

Whenever a law of Nature is formulated in mathematics it takes the form of a differential equation, that is, an equation which contains derivatives. Sometimes such a law is written in a way which masks its differential form. For instance, the law of gravitation when written as $F = k\frac{mm'}{r^2}$ appears to contain no derivative; but since F is merely a symbol for $m\frac{dv}{dt}$, the law is in the form of a differential equation. We may say therefore that Nature speaks in the language of differential equations.

Unfortunately, the language of differential equations is a cryptic language written in code; it has to be decoded and interpreted before its meaning becomes clear to the human mind. The differential equation which gives the law of gravitation, for instance, does not tell us directly that a planet moves in an elliptical orbit and that the line joining the planet and the sun describes equal areas in equal intervals of time. These and other details of the motion are disclosed only after the differential equation is translated into equations which do not contain derivatives. The process of such a translation is called *integration* in simple cases, and *solving the differential equation* in complex cases. The integral calculus is the study of integration and its applications.

Motion in a straight line. The motion of a body in a straight line is given by Newton's second law of motion,

$$m\frac{dv}{dt} = F,$$

where m is the mass and v is the velocity of the body, and F is the resultant of the forces which act on the body. When F is given as a constant or as a function of one or more variables, and when this differential equation is solved, the motion is said to be *discussed.*

EXAMPLES. (1) Discuss the motion of a body under the action of a constant force.

Solution. If F is constant, the acceleration also is constant, since m is considered to be constant for velocities small compared with the velocity of light (186,000 miles per second).

Denoting the constant acceleration by a, we have

$$\frac{dv}{dt} = a,$$

$$dv = adt,$$

$$v = at + c,$$

where c is the constant of integration. Let v_0 denote the initial velocity; then $v = v_0$ when $t = 0$, hence $c = v_0$, and

$$v = v_0 + at. \qquad (1)$$

But since $$v = \frac{ds}{dt},$$

we get $$ds = v_0\,dt + atdt,$$

and $$s = v_0 t + \tfrac{1}{2}at^2 + c',$$

where c' is the integration constant. Let $s = 0$ when $t = 0$, then $c' = 0$, and

$$s = v_0 t + \tfrac{1}{2}at^2. \tag{2}$$

Eliminating t between (1) and (2), we get

$$v^2 = v_0^2 + 2as. \tag{3}$$

Equations (1) to (3) are valid whatever value the acceleration may have, provided it is a constant. In a particular case, therefore, if the acceleration is found to be constant all we need to do is to replace a in equations (1) to (3) by the value of that acceleration, for instance, by g in the case of a falling body and by $g \sin \alpha$ in the case of a body moving down a smooth inclined plane.

(2) Discuss the motion of a parachute jumper, assuming the air resistance to be proportional to the velocity.

Solution. The resultant downward force is the weight mg minus the resisting force $k'v$. Therefore

$$m\frac{dv}{dt} = mg - k'v,$$

$$\frac{dv}{dt} = g - kv, \qquad \left[k = \frac{k'}{m}\right]$$

$$\frac{dv}{v - \frac{g}{k}} = -kdt.$$

$$\log\left(v - \frac{g}{k}\right) = -kt + c. \qquad \left[\int \frac{du}{u} = \log u + c\right]$$

To determine the constant of integration, let $v = v_0$ when $t = 0$, then

$$\log\left(v_0 - \frac{g}{k}\right) = c,$$

and

$$\log\left(v - \frac{g}{k}\right) - \log\left(v_0 - \frac{g}{k}\right) = -kt,$$

$$\log\frac{v - g/k}{v_0 - g/k} = -kt,$$

$$v - \frac{g}{k} = \left(v_0 - \frac{g}{k}\right) e^{-kt},$$

$$v = \frac{g}{k} + \left(v_0 - \frac{g}{k}\right) e^{-kt}.$$

If the parachute jumper has sufficient time to fall, e^{-kt} becomes very small; therefore the velocity of landing approaches the *limiting velocity* g/k, where k depends on the falling weight and the size of the parachute.

Fundamental theorem. The equation

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \int_a^b f(x)dx,$$

called the *fundamental theorem* of the integral calculus, is usually derived by considering the area under a curve whose equation is given in the symbolic form $y = f(x)$. But since $y = f(x)$ may be interpreted as the equation of a curve whatever geometrical or physical quantities x and y may represent, the theorem is far more general than one might suppose from its derivation.

The great importance of the theorem is due to the following: It is easy to consider a quantity to be the limit of a sum, but it is, in general, very difficult to evaluate the sum. On the other hand, considering the quantity as the limit of a sum enables us to set up the corresponding definite integral, which is easy to evaluate when the necessary integration formula is available.*

* Archimedes (287-212 B.C.) used the method of summation to find the areas and volumes of a few geometrical figures, such as the area of a circle and the volume of a sphere. In his delightful book *Men of Mathematics*, Professor E. T. Bell says that Archimedes was one of the three greatest mathematicians of all time, the other two being Newton (1642-1727) and Gauss (1777-1855). Yet many problems which Archimedes could not solve by the method of summation, students in freshman calculus solve by the application of the fundamental theorem.

Application to geometry. To apply the fundamental theorem to the evaluation of a geometrical quantity, proceed as follows:

1. Draw a figure of the quantity and imagine it to be divided into, or to consist of, a number of parts, called *elements*, each of which approaches zero as the number of the elements increases indefinitely.

2. Draw, in the figure, an element to represent any and every element. Such an element, which we shall call the *representative element*, should be placed in a representative position in the figure, that is, in a position that is not a special or favored position.

3. Indicate in the figure the *principal part* of the representative element, whenever it is convenient to do so. We shall call the principal part of the representative element the *representative differential element.*

4. Find the expression for the value of the representative differential element in terms of one variable.

5. Integrate the expression of the representative differential element between the limits of the variable, that is, between the values of the variable which correspond to the extreme positions of the elements.

The elements into which the quantity is imagined to be divided must be similar elements, that is, they must be such that the expression for the representative differential element is applicable to the entire quantity. If the quantity is such that it can not be divided into similar elements of a desired type, it has to be sectioned into two or more parts, each of which can be divided into similar elements.

Note: The elements are infinitesimals, since they are variables which approach zero.*

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* The word *infinitesimal* should not be confused with the word *small.* The former stands for a well-defined concept; the latter is indefinite and comparative. The Earth is small on

The representative element is an increment of the quantity. But this fact cannot be used to find its value unless the expression of the quantity is known, which is not supposed to be the case.

The representative differential element is the differential of the expression of quantity. But this fact cannot be used to find its value unless the expression of the quantity is known, which is not supposed to be the case. However the value of the representative differential element *can* be found by considering it as the principal part of the representative element. For instance, in example (1) below, the expression for dA is found by considering it to be the area of the shaded rectangle in Fig. 64.

The chain of reasoning. The chain of reasoning involved in the application of the fundamental theorem is as follows:

1. A quantity equals the sum of n elements into which it is conceived to be divided, however large n may be. This follows from the axiom, *the whole equals the sum of its parts*.

2. As n increases indefinitely, the limit of the sum of the elements equals the limit of the sum of the principal parts of the elements.

This follows from the theorem called Duhamel's Theorem, which states: *the limit of the sum of n positive infinitesimals equals, as n increases indefinitely, the limit of the sum of the principal parts of the infinitesimals.*

3. The limit of the sum of the principal parts of the elements equals the integral of the expression of the repre-

---

the astronomical scale, but is very large on the atomic scale; an atom is small on the corporeal scale, but is very large on the nuclear scale.

sentative differential element between the limits of the integration.

This follows from the fundamental theorem.

EXAMPLES. (1) Find the area bounded by $y^2 = 9x$, $2x - 3y = 0$, and $x = 5$.

Solution. The given boundaries enclose two areas, one to the left of the line $x = 5$ and the other to the right. We shall find the first area in two ways: (a) by vertical elements, and (b) by horizontal elements.

(a) Considering the area, Fig. 64, to be divided into vertical elements, we draw the representative element (the vertical strip) and the representative differential element (the shaded rectangle). Then we obtain the following expression for the latter.

$$\begin{aligned} dA &= l\,dx \\ &= (y - y')dx \\ &= (3x^{1/2} - \tfrac{2}{3}x)dx. \end{aligned}$$

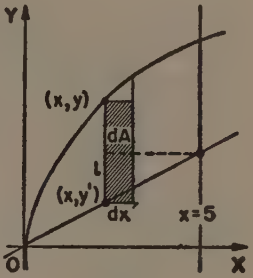

FIG. 64

The extreme left-hand element is at $x = 0$, and the extreme right-hand element is at $x = 5$; hence the limits of integration are 0 and 5. Therefore

$$\begin{aligned} A &= \int_0^5 (3x^{1/2} - \tfrac{2}{3}x)dx \\ &= \left[2x^{3/2} - \tfrac{1}{3}x^2\right]_0^5 \\ &= 10\sqrt{5} - \tfrac{25}{3}. \end{aligned}$$

(b) If we choose horizontal elements, the right-hand sides of the elements end on different curves below and above the dotted line, Fig. 65. In other words, the elements in one section of the area are not similar to those in the other section. Therefore we draw a representative differential element in each section and obtain

$$\begin{aligned} dA_1 &= l_1 dy \\ &= (x - x')dy \\ &= (\tfrac{3}{2}y - \tfrac{1}{9}y^2)dx. \end{aligned} \qquad \begin{aligned} dA_2 &= l_2 dy \\ &= (5 - x')dy \\ &= (5 - \tfrac{1}{9}y^2)dy. \end{aligned}$$

$$A = A_1 + A_2$$
$$= \int_0^{10/3} (\tfrac{3}{2}y - \tfrac{1}{9}y^2)dy + \int_{10/3}^{3\sqrt{5}} (5 - \tfrac{1}{9}y^2)dy$$
$$= 10\sqrt{5} - \tfrac{25}{3}.$$

Note: If the student pictures the area to be found as a window to be boarded with horizontal or vertical boards, he will have no more difficulty in determining the limits of integration than he would in knowing where to begin placing the boards and where to stop.

(2) Find the volume generated by rotating the area bounded by $x^2 = 4py$, $x = a$, and $y = 0$, about the line $x = a$.

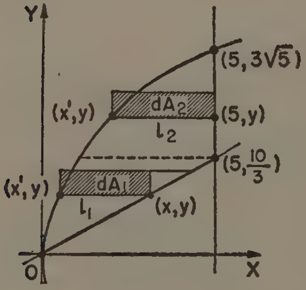

FIG. 65

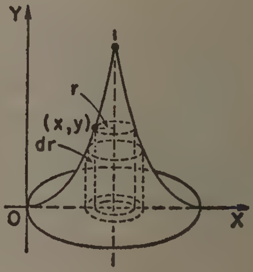

FIG. 66

Solution. Taking the elements to be coaxial circular cylindrical shells* (tubes), Fig. 66, and remembering that the principal part of the volume of a shell equals the product of the area of one of its surfaces and the thickness, we have

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* A shell is a solid, one of whose dimensions approaches zero. In a problem of physics a body may be considered as a shell if one of its dimensions is negligibly small compared with the other two dimensions.

$$
\begin{aligned}
dV &= 2\pi r l dr \\
&= 2\pi(a - x)y d(a - x) \\
&= 2\pi(a - x)\frac{x^2}{4p}(-dx) \\
&= \frac{\pi}{2p}(x^3 - ax^2)dx. \\
V &= \frac{\pi}{2p}\int_a^0 (x^3 - ax^2)dx \\
&= \frac{\pi a^4}{24p}.
\end{aligned}
$$

Note: The reason for taking a as the lower limit is not to make the volume come out positive. The reason lies in the fact that the lower limit of r is 0, and that $r = 0$ when $x = a$. To put it in another way, the integral represents the process by which the volume is generated by accretion, starting with the shell of zero radius and ending with the shell of the largest radius. This process may be compared to the growth of a tree trunk by the addition of an outer ring each succeeding year.

Application to physics. In applying the fundamental theorem to a physical quantity, the quantity is divided into elements by dividing a length or an area or a volume into elements. Then the expression of the representative differential element of the geometrical quantity is multiplied by a magnitude (or magnitudes) in order to obtain the expression for the corresponding physical quantity. We shall call the multiplier the *auxiliary magnitude.*

The fundamental theorem is applied to a physical quantity in the same way as to a geometrical quantity, but greater care must be exercised in choosing elements because the elements have to satisfy the following *condition for the choice of elements:*

The elements must be such that the value of the auxiliary magnitude does not vary within each element by more than an infinitesimal.

EXAMPLES. (1) A right circular cylindrical tank is half full of oil which weighs 60 pounds per cubic foot. If the tank is 12 feet in diameter and its axis is horizontal, find the force at one end of the tank.

Solution. In this type of problem, the auxiliary magnitude is the pressure. Since hydrostatic pressure at a point within a liquid varies with the distance of the point *from* the free surface of the liquid, the elements of area must be such that in each element the distances of the points from the free surface do not differ by more than an infinitesimal. In other words, the representative element of area must lie between two horizontal lines, for otherwise we cannot properly speak of *the* pressure on the representative differential element of area and write $dF = pdA$.

Using the notation of Fig. 67, where the circle represents one end of the tank, we have

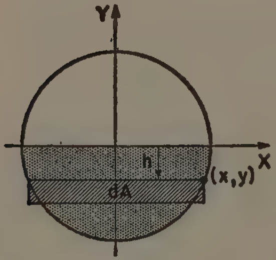

FIG. 67

$$\begin{aligned} dF &= pdA \\ &= wh \times 2xdy && [p = wh] \\ &= w(-y)2xdy && [h = -y] \\ &= w\sqrt{a^2 - y^2}(-2ydy). \end{aligned}$$

$$\begin{aligned} F &= w\int_{-a}^{0}(a^2 - y^2)^{1/2}\, d(a^2 - y^2) \\ &= \tfrac{2}{3} w\left[(a^2 - y^2)^{3/2}\right]_{-a}^{0} \\ &= \tfrac{2}{3} wa^3 \\ &= \frac{2}{3} \times 60\,\frac{\text{lb}}{\text{ft}^3} \times (6\text{ ft})^3 \\ &= 8640\text{ lb} \\ &= 4.32\text{ tons}. \end{aligned}$$

(2) Find the center of mass of a body of uniform density which has the shape of a hemisphere.

Solution. We place the hemisphere (which represents the body) as in Fig. 68 in order to make the equation of the circle on the plane of the paper as simple as possible. It is evident from symmetry that the center of mass lies on the x-axis, therefore we need find only its x-coordinate, defined by

$$\bar{x} = \frac{1}{m}\int_0^m x dm.$$

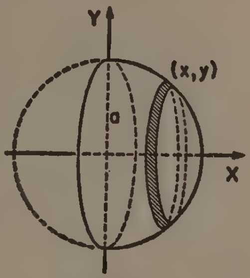

FIG. 68

In order that x, the auxiliary magnitude, have the same value (except for infinitesimal differences) for all the points within each element, we imagine the volume to be divided into elements by planes perpendicular to the x-axis.

Letting ρ denote the density, we have

$$\begin{aligned}
\bar{x} &= \frac{1}{m}\int_0^m x dm \\
&= \frac{1}{m}\int_0^V x\rho dV \\
&= \frac{1}{m}\int_0^a x\rho\pi y^2 dx \\
&= \frac{\rho\pi}{m}\int_0^a x(a^2 - x^2)dx \\
&= \frac{\rho\pi a^4}{4m} \\
&= \frac{1}{m}\cdot\rho\frac{2}{3}\pi a^3\cdot\frac{3}{8}a \qquad \left[m = \rho\frac{2}{3}\pi a^3\right] \\
&= \frac{3}{8}a.
\end{aligned}$$

(3) A chain is hanging with its upper end at one edge of a table. Assuming the effect of friction to be negligible, find the work done in pulling the chain to the top of the table.

Note: In applying the fundamental theorem to a problem on work, we may imagine (a) the distance moved by the force to be divided into elements, or (b) the body moved to be divided into elements. In the first case, the representative differential element of the work done is given by

$$dU = Fds, \tag{a}$$

and in the second case by

$$dU = sdF. \tag{b}$$

Equation (a) is convenient to use when all parts of the body move through the same distance, and equation (b) when different parts of the body move through different distances. We shall solve this problem both ways.

Solution. (a) If we consider the body moved to be the upper end of the chain, Fig. 69, then all parts of the body (the upper end) move through the same distance. Therefore we divide the distance moved by the force F into elements and use

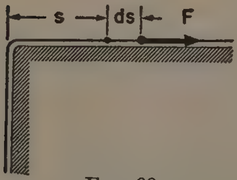

FIG. 69

$$dU = Fds$$

for the principal part of the work done by F in moving through the distance ds. In this case the chain enters into the picture only through the effect of the weight of the vertical part of the chain on F. Therefore we have

$$\begin{aligned} dU &= Fds \\ &= w(l - s)ds, \end{aligned}$$

where w is the weight per unit length, l the length, and $l - s$ the length of the vertical part of the chain.

$$\begin{aligned} U &= w\int_0^l (l - s)ds \\ &= \tfrac{1}{2}wl^2 \\ &= \tfrac{1}{2}Wl, \end{aligned}$$

where W is the weight of the chain. This result agrees with the well-known law of physics which states that the work done in lifting a body equals the weight times the vertical distance through which the center of gravity of the body is raised.

(b) Considering the work to be done in lifting the chain link by link, we divide the chain into elements, Fig. 70, and obtain

$$\begin{aligned} dU &= sdF \\ &= swds. \end{aligned}$$

$$\begin{aligned} U &= w\int_0^l sds \\ &= \tfrac{1}{2}wl^2 \\ &= \tfrac{1}{2}Wl. \end{aligned}$$

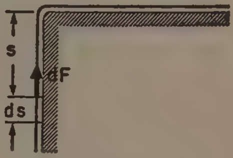

FIG. 70

(4) Find the moment of inertia of a right circular cylindrical rod with respect to a transverse axis through the mid-point of the axis of the rod.

Solution. If the elements are so chosen as to satisfy the condition for the choice of elements, double or triple integration becomes necessary. But the problem can be solved by single integration as follows.

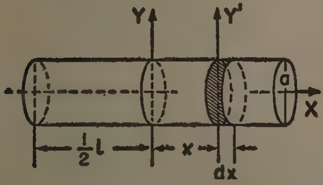

FIG. 71

We divide the cylinder, Fig. 71, into circular elements, and make use of the formula $I = \frac{1}{4}ma^2$ (the moment of inertia of a circular lamina with respect to a diameter) and the transfer theorem: *The moment of inertia with respect to any axis equals the moment of inertia with re-*

*spect to a parallel axis through the center of mass plus the product of the mass by the square of the distance between the two axes.**

Applying the formula for the moment of inertia of a circular lamina about a diameter to the representative element in Fig. 71, we obtain the following expression for the moment of inertia of the element with respect to the y'-axis.

$$\begin{aligned} dI_c &= \tfrac{1}{4}a^2dm \\ &= \tfrac{1}{4}a^2\rho dV \\ &= \tfrac{1}{4}a^2\rho\pi a^2dx, \end{aligned}$$

where the subscript c is introduced in order to emphasize the fact that the y'-axis passes through the center of mass of the element. By the transfer theorem, the moment of inertia of the element with respect to the y-axis is given by

$$\begin{aligned} dI &= dI_c + x^2dm \\ &= \frac{\pi}{4}\rho a^4dx + x^2\rho\pi a^2dx \\ &= \pi a^2\rho\left(\frac{a^2}{4} + x^2\right)dx. \end{aligned}$$

Therefore

$$\begin{aligned} I &= \pi a^2\rho\int_{-\frac{1}{2}l}^{\frac{1}{2}l}\left(\frac{a^2}{4} + x^2\right)dx \\ &= \pi a^2\rho\left(\frac{a^2l}{4} + \frac{l^3}{12}\right) \\ &= \rho\pi a^2l\left(\frac{a^2}{4} + \frac{l^2}{12}\right) \\ &= m\left(\frac{a^2}{4} + \frac{l^2}{12}\right). \end{aligned}$$

* It is important to note that the elements do not satisfy the condition for the choice of elements, and that this method is equivalent to double or triple integration, since the assumed moment of inertia of a circular lamina is found by a single or double integration.

Range of application of the fundamental theorem. What the fundamental theorem does, amounts to modifying the formulas of elementary geometry and physics, which are applicable only to very special cases, and making them generally applicable. The following partial list gives some idea of the great power and the vast range of application of the theorem.

The formula of $A = lw$, which applies only to the area of a rectangle, is changed into $dA = ldx$, or $dA = ldy$, which is applicable to any plane area.

The formula $V = Ah$, which applies only to the volume of a prism or cylinder with parallel bases, is changed to $dV = Adx$, or $dV = Ady$, which is applicable to the volume of any solid.

The formula $F = pA$, which applies only to force due to uniform pressure over an area, is changed to $dF = pdA$, which is applicable to force due to nonuniform as well as uniform pressure.

The formula $U = Fs$, which applies only to the work done by a constant force, is changed to $dU = Fds$, or $dU = sdF$, which is applicable to the work done by a variable as well as a constant force.

The formula $I = r^2m$, which applies only to the moment of a particle, is changed to $dI = r^2dm$, which is applicable to the moment of inertia of any body.