Part IV

Applications

20

Triangle spread rules

This chapter introduces concepts for working over the rational and decimal number fields. It shows how to practically construct a spread ruler, how to define rays and sectors, and gives the important *Triangle spread rules* that augment rational trigonometry in these particular fields, and which are particularly useful for practical applications. The arguments and definitions are generally informal.

20.1 Spread ruler

The **spread ruler** shown in Figure 20.1 allows you to measure spreads between two lines in a similar way that a protractor measures angles between two rays.

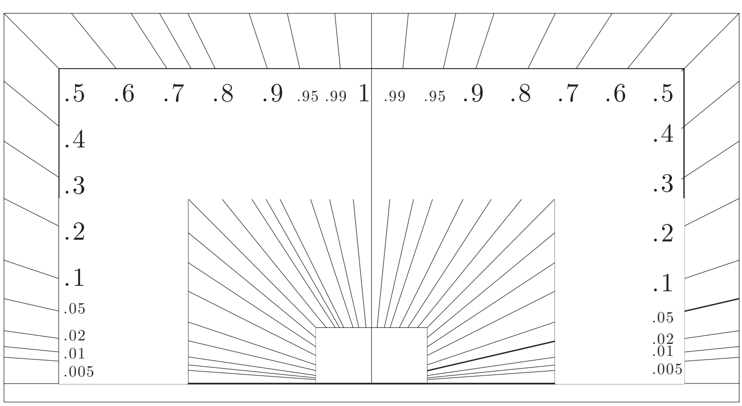
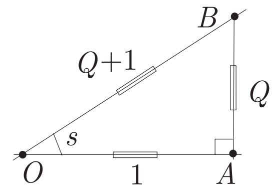


Figure 20.1: Spread ruler

Making a spread ruler is perhaps more straightforward than making a protractor. Consider the right triangle $\overline{OAB}$ with spread s at O, with $Q(O,A) \equiv 1$ and $Q(A,B) \equiv Q$. Then by Pythagoras' theorem and the Spread ratio theorem (page 77)



$$s = \frac{Q}{Q+1}$$

so that

$$Q = \frac{s}{1-s}.$$

It follows that if $O \equiv [0,0]$ and $A \equiv [1,0]$, then the position of B, given s, is

$$B = \left[1, \sqrt{s/(1-s)}\right].$$

Exercise 20.1 Show how to use the symmetry between s and $1-s$ to restrict necessary values of s to the range $[0, 1/2]$. $\diamond$

Here are some approximate values for the construction of a spread ruler.

s	0.05	0.1	0.2	0.25	0.3	0.4	0.5
$\sqrt{s/(1-s)}$	0.230	0.333	0.5	0.577	0.655	0.816	1.0

20.2 Line segments, rays and sectors

The definitions of this section hold for the decimal or rational number fields, and rely on properties of positive numbers. For these fields, the terms *side* and **line segment** will be used interchangeably. The point A **lies on** the line segment $\overline{A_1A_2}$ precisely when

$$A = \lambda_1 A_1 + \lambda_2 A_2$$

for some numbers $\lambda_1, \lambda_2 \geq 0$ satisfying $\lambda_1 + \lambda_2 = 1$. Such a point A is **interior** to the line segment precisely when $\lambda_1, \lambda_2 > 0$. The notion 'A lies on $\overline{A_1A_2}$' is more general than 'A is an element of $\overline{A_1A_2}$', since $\overline{A_1A_2} \equiv \{A_1, A_2\}$ has only two elements.

Two line segments $\overline{A_1A_2}$ and $\overline{B_1B_2}$ **overlap** precisely when there is a point which is interior to both, and are **adjacent** precisely when there is no point which is interior to both and exactly one point which is an element of both.

Suppose that three collinear points A_1, A_2 and A_3 form quadrances $Q_1 \equiv Q(A_2, A_3)$, $Q_2 \equiv Q(A_1, A_3)$ and $Q_3 \equiv Q(A_1, A_2)$. Then $\{Q_1, Q_2, Q_3\}$ is a quad triple, so that given Q_1 and Q_2, the third quadrance Q_3 is obtained from the triple quad formula

$$(Q_3 - Q_1 - Q_2)^2 = 4Q_1Q_2.$$

For a general field this is as much as one can say. However over the decimal or rational number fields one knows that quadrances are always positive, so that the solutions are

$$Q_3 = Q_1 + Q_2 \pm 2\sqrt{Q_1Q_2}.$$

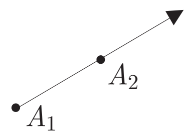
Then you can determine, in terms of the relative positions of $\overline{A_1A_3}$ and $\overline{A_2A_3}$, just which of these two possibilities occurs.

Collinear quadrance rules Suppose that $Q_1 \equiv Q(A_2, A_3)$, $Q_2 \equiv Q(A_1, A_3)$ and $Q_3 \equiv Q(A_1, A_2)$ are the quadrances formed by three collinear points A_1, A_2 and A_3. Then

1. If $\overline{A_1A_3}$ and $\overline{A_2A_3}$ are adjacent then $Q_3 = Q_1 + Q_2 + 2\sqrt{Q_1Q_2}$
2. If $\overline{A_1A_3}$ and $\overline{A_2A_3}$ are overlapping then $Q_3 = Q_1 + Q_2 - 2\sqrt{Q_1Q_2}$.

A **ray** $\overrightarrow{A_1A_2}$, also written $\overleftarrow{A_2A_1}$, is an ordered pair $[A_1, A_2]$ of distinct points, with the convention that

$$\overrightarrow{A_1A_2} = \overrightarrow{A_1A_3}$$



precisely when

$$A_3 = \lambda_1 A_1 + \lambda_2 A_2$$

for some numbers λ_1 and λ_2 satisfying $\lambda_1 + \lambda_2 = 1$ and $\lambda_2 \geq 0$.
This notion treats A_2 and A_3 symmetrically. The point A_1 is the **base point** of the ray $\overrightarrow{A_1A_2}$. A point B **lies on** the ray $\overrightarrow{A_1A_2}$ precisely when

$$B = \lambda_1 A_1 + \lambda_2 A_2$$

with $\lambda_1 + \lambda_2 = 1$ and $\lambda_2 \geq 0$. Two rays $\overrightarrow{A_1A_2}$ and $\overrightarrow{B_1B_2}$ are **parallel** precisely when A_1A_2 is parallel to B_1B_2.

A **sector** $\alpha \equiv \overleftrightarrow{A_2A_1A_3}$ is a set $\left\{\overleftarrow{A_2A_1}, \overrightarrow{A_1A_3}\right\}$ of non-parallel rays with a common base point A_1. The point B **lies on** the sector $\overleftrightarrow{A_2A_1A_3}$ precisely when

$$B = \lambda_2 A_2 + \lambda_1 A_1 + \lambda_3 A_3$$

with $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and $\lambda_2, \lambda_3 \geq 0$. Figure 20.2 shows (some of) the points B lying on $\alpha = \overleftrightarrow{A_2A_1A_3}$.

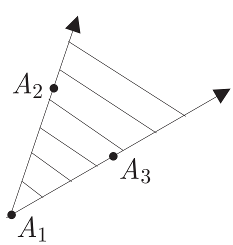


Figure 20.2: The sector $\overleftrightarrow{A_2A_1A_3} = \left\{\overleftarrow{A_2A_1}, \overrightarrow{A_1A_3}\right\}$

A sector $\alpha \equiv \overleftrightarrow{A_2A_1A_3}$ determines two rays $\overrightarrow{A_1A_2}$ and $\overrightarrow{A_1A_3}$, together with two lines A_1A_2 and A_1A_3. The **spread** $s(\alpha)$ **of the sector** α is the spread between these two lines, so that

$$s(\alpha) = s\left(\overleftrightarrow{A_2A_1A_3}\right) \equiv s(A_1A_2, A_1A_3).$$

20.3 Acute and obtuse sectors

The sector $\alpha \equiv \overleftrightarrow{A_2A_1A_3}$ is **acute type**, abbreviated as **(ac)**, precisely when

$$Q(A_1, A_2) + Q(A_1, A_3) \geq Q(A_2, A_3)$$

and **obtuse type**, abbreviated as **(ob)**, precisely when

$$Q(A_1, A_2) + Q(A_1, A_3) \leq Q(A_2, A_3).$$

The sector α is a **right sector** precisely when $s(\alpha) = 1$. By Pythagoras' theorem, a sector is a right sector precisely when it is both acute and obtuse.

Exercise 20.2 Show that these definitions are indeed well-defined. $\diamond$

A general sector determines both a spread and a type. These two pieces of information can be usefully recorded together when referring to sectors. Figure 20.3 shows two sectors, the left with an (acute) spread of $s = 0.625$ (ac) and the right with an (obtuse) spread of 0.845 (ob).

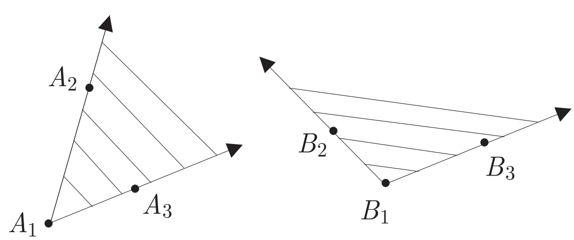


Figure 20.3: Acute and obtuse sectors

If three mutually non-parallel rays $\overrightarrow{A_0A_1}$, $\overrightarrow{A_0A_2}$ and $\overrightarrow{A_0A_3}$ have the common base point A_0, then there are two possible relations between the two sectors $\beta_3 \equiv \overleftrightarrow{A_1A_0A_2}$ and $\beta_1 \equiv \overleftrightarrow{A_2A_0A_3}$. They **overlap** precisely when there is a point B which lies on both sectors but not on any of the rays $\overrightarrow{A_0A_1}$, $\overrightarrow{A_0A_2}$ or $\overrightarrow{A_0A_3}$. They are **adjacent** precisely when the only points which lie on both sectors lie on the ray $\overrightarrow{A_0A_2}$. These two situations are respectively shown in Figure 20.4.

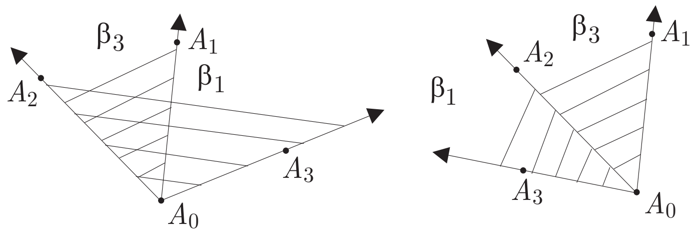


Figure 20.4: Overlapping and adjacent sectors

If $\beta_2 \equiv \overleftrightarrow{A_3A_0A_1}$ then the three sectors β_1, β_2 and β_3 may have the property that one of them overlaps with each of the other two, while those other two are adjacent, as in either of the diagrams in Figure 20.4. Another possibility is that any two of them are adjacent, as in Figure 20.5.

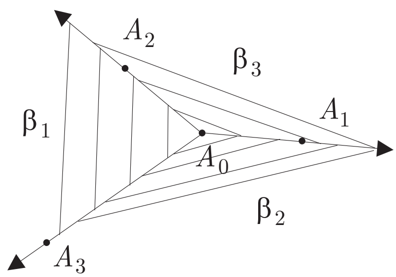


Figure 20.5: Three adjacent sectors

A triangle $\overline{A_1A_2A_3}$ gives rise to three distinguished sectors, namely $\alpha_1 \equiv \overleftrightarrow{A_2A_1A_3}$, $\alpha_2 \equiv \overleftrightarrow{A_3A_2A_1}$ and $\alpha_3 \equiv \overleftrightarrow{A_1A_3A_2}$. The spreads of these sectors are then the usual spreads of the triangle. If the spreads of the three sectors are $s_1 \equiv s(\alpha_1)$, $s_2 \equiv s(\alpha_2)$ and $s_3 \equiv s(\alpha_3)$ then the Triple spread formula asserts that $\{s_1, s_2, s_3\}$ is a spread triple, so that

$$(s_3 - (s_1 + s_2 - 2s_1s_2))^2 = 4s_1s_2(1 - s_1)(1 - s_2). \tag{20.1}$$

When viewed as a quadratic equation in s_3 in the decimal number field, the two solutions can be labelled the **little spread** $r_l = r_l(s_1, s_2)$ and the **big spread** $r_b \equiv r_b(s_1, s_2)$ where

$$\begin{aligned} r_l(s_1, s_2) &= s_1 + s_2 - 2s_1s_2 - 2\sqrt{s_1s_2(1 - s_1)(1 - s_2)} \\ r_b(s_1, s_2) &= s_1 + s_2 - 2s_1s_2 + 2\sqrt{s_1s_2(1 - s_1)(1 - s_2)}. \end{aligned}$$

20.4 Acute and obtuse triangles

The point B **lies on** the triangle $\overline{A_1A_2A_3}$ precisely when

$$B = \lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 A_3$$

for some numbers $\lambda_1, \lambda_2, \lambda_3 \geq 0$ satisfying $\lambda_1 + \lambda_2 + \lambda_3 = 1$. Such a point B is **interior** to the triangle if $\lambda_1, \lambda_2, \lambda_3 > 0$.

A triangle $\overline{A_1A_2A_3}$ is **acute** if all three of its sectors are acute. Otherwise it is **obtuse**. If the quadrances of the triangle are Q_1, Q_2 and Q_3 as usual, then the sector with base point A_1 (or just the **sector at** A_1) is acute precisely when $Q_2 + Q_3 \geq Q_1$, and similarly for the other sectors. So the triangle is acute precisely when

$$Q_1 + Q_2 \geq Q_3 \qquad Q_2 + Q_3 \geq Q_1 \qquad Q_3 + Q_1 \geq Q_2. \tag{20.2}$$

Observe that if the corresponding spreads of the triangle are s_1, s_2 and s_3, then by the Spread law and the fact that all the quadrances and spreads are positive, the sector at A_3 is acute, or alternatively the spread of the sector $s\left(\overleftrightarrow{A_1A_3A_2}\right)$ is acute, precisely when either

$$Q_1 + Q_2 \geq Q_3 \qquad \text{or} \qquad s_1 + s_2 \geq s_3.$$

Exercise 20.3 Show that if a triangle $\overline{A_1A_2A_3}$ has spreads s_1, s_2 and s_3, then any two of the following inequalities implies the third, and implies the triangle is acute.

$$s_1 \geq |s_2 - s_3| \qquad s_2 \geq |s_3 - s_1| \qquad s_3 \geq |s_1 - s_2|. \quad \diamond$$

Exercise 20.4 Show that a triangle can have at most one obtuse sector. $\diamond$

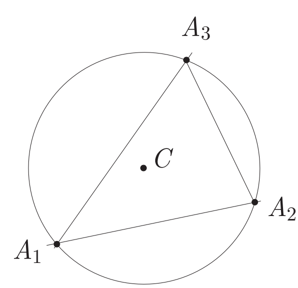
Problem 2 Show that $\overline{A_1A_2A_3}$ is acute precisely when the circumcenter C lies on the triangle.

Solution. Suppose that the quadrances of $\overline{A_1A_2A_3}$ are Q_1, Q_2 and Q_3 as usual, and that the quadrea is $\mathcal{A}$. The Affine circumcenter theorem (page 146) shows that C is the affine combination

$$C = \gamma_1 A_1 + \gamma_2 A_2 + \gamma_3 A_3$$

where

$$\begin{aligned}\gamma_1 &\equiv Q_1 (Q_2 + Q_3 - Q_1) / \mathcal{A} \\ \gamma_2 &\equiv Q_2 (Q_1 + Q_3 - Q_2) / \mathcal{A} \\ \gamma_3 &\equiv Q_3 (Q_1 + Q_2 - Q_3) / \mathcal{A}.\end{aligned}$$



So C is in the interior of $\overline{A_1A_2A_3}$ precisely when the three inequalities for acuteness are satisfied. ∎

20.5 Triangle spread rules

The following rules apply to the rational and decimal number fields, and those closely related to them. They provide an important guide to dealing with acute and obtuse sectors in practical applications of rational trigonometry.

Triangle spread rules Suppose that s_1, s_2 and s_3 are the respective spreads of the three sectors $\alpha_1 \equiv \overleftrightarrow{A_2A_1A_3}$, $\alpha_2 \equiv \overleftrightarrow{A_3A_2A_1}$ and $\alpha_3 \equiv \overleftrightarrow{A_1A_3A_2}$ of a triangle $\overline{A_1A_2A_3}$. Then

1. The spread s_3 is equal to $r_b(s_1, s_2)$ precisely when s_1 and s_2 are both acute
2. The spread s_3 is obtuse precisely when s_1 and s_2 are acute and $s_1 + s_2 \leq 1$.

This is summarized in the following table, which gives the value of s_3, depending on s_1 and s_2.

s_3	s_1 (ac) s_2 (ac)	s_1 (ac) s_2 (ob)
$s_1 + s_2 \leq 1$	$r_b(s_1, s_2)$ (ob)	$r_l(s_1, s_2)$ (ac)
$s_1 + s_2 \geq 1$	$r_b(s_1, s_2)$ (ac)	$r_l(s_1, s_2)$ (ac)

Problem 3 Demonstrate the validity of these rules.

Solution (Rule 1). Recall that the Triple spread formula, as a quadratic equation in s_3, has normal form

$$(s_3 - (s_1 + s_2 - 2s_1s_2))^2 = 4s_1s_2(1 - s_1)(1 - s_2).$$

Suppose that

$$s_3 = r_l(s_1, s_2) = s_1 + s_2 - 2s_1s_2 - 2\sqrt{s_1s_2(1 - s_1)(1 - s_2)}. \tag{20.3}$$

If s_1 is acute then $s_3 + s_2 \geq s_1$, so that

$$s_2 - s_1s_2 \geq \sqrt{s_1s_2(1 - s_1)(1 - s_2)}.$$

Use the fact that any spread s satisfies $0 \leq s \leq 1$ to see that both sides are positive, so the inequality is maintained when both sides are squared. Thus

$$s_2(1 - s_1) \geq s_1(1 - s_2)$$

which is equivalent to

$$s_2 \geq s_1.$$

Similarly if s_2 is acute then
$$s_1 \geq s_2.$$
Thus if both s_1 and s_2 are acute then $s_1 = s_2 \equiv s$, in which case by (20.3)
$$s_3 = 2s - 2s^2 - 2s(1-s) = 0.$$
This is impossible, so you may conclude that if s_1 and s_2 are acute, then
$$s_3 = s_1 + s_2 - 2s_1s_2 + 2\sqrt{s_1s_2(1-s_1)(1-s_2)} = r_b(s_1, s_2).$$
Conversely if $s_3 = r_b(s_1, s_2)$ then
$$s_3 + s_2 = s_1 + 2s_2(1-s_1) + 2\sqrt{s_1s_2(1-s_1)(1-s_2)} \geq s_1$$
so s_1 is acute, and similarly s_2 is acute. ■

Solution (Rule 2). Recall from Exercise 7.2 (page 90) that the Triple spread formula $S(s_1, s_2, s_3) = 0$ can be rewritten as the equation
$$\begin{aligned} & s_3(s_3 - (s_1+s_2))(1-(s_1+s_2)) \\ = \; & ((s_1+s_2-s_3)s_3 + (s_3 - s_1 + s_2)(s_3 - s_2 + s_1))(1-s_3). \end{aligned} \tag{20.4}$$
Now s_1 and s_2 are acute precisely when
$$s_3 - s_1 + s_2 \geq 0 \qquad \text{and} \qquad s_3 - s_2 + s_1 \geq 0$$
respectively, while s_3 is obtuse precisely when
$$s_3 - (s_1 + s_2) \geq 0.$$
Any spread s of a triangle satisfies $0 < s \leq 1$. So if s_1 and s_2 are acute and $s_1 + s_2 \leq 1$, then s_3 must be obtuse, since otherwise the right hand side of (20.4) is strictly positive while the left hand side is negative.

Conversely suppose s_3 is obtuse. Then s_1 and s_2 are acute by Exercise 20.4. If $s_1 + s_2 > 1$ then the left hand side of (20.4) is strictly negative, so that
$$((s_1+s_2-s_3)s_3 + (s_3-s_1+s_2)(s_3-s_2+s_1)) = s_3s_1 + s_3s_2 + 2s_1s_2 - s_1^2 - s_2^2 < 0.$$
But then
$$s_3(s_1+s_2) < (s_1-s_2)^2$$
which is impossible since
$$s_3(s_1+s_2) \geq (s_1+s_2)^2 > (s_1-s_2)^2.$$
Thus $s_1 + s_2 \leq 1$. ■

21

Two dimensional problems

This chapter gives some geometrical applications in the decimal number plane.

21.1 Harmonic relation

Problem 4 Distinct points A_1, A_2 and A_3 lie on a line l, with B_1, B_2 and B_3 points on the respective altitudes from A_1, A_2 and A_3 to l, such that B_1, B_2 and A_3 are collinear, as are A_1, B_2 and B_3, as in Figure 21.1. Define $Q_1 \equiv Q(A_1, B_1)$, $Q_2 \equiv Q(A_2, B_2)$ and $Q_3 \equiv Q(A_3, B_3)$. Show that $\{1/Q_1, 1/Q_2, 1/Q_3\}$ is a quad triple.

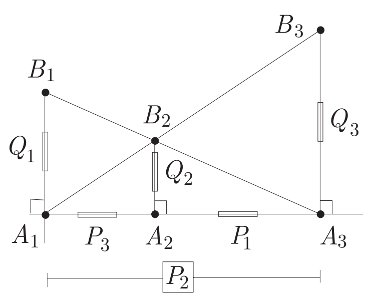


Figure 21.1: Harmonic configuration

Solution. Define $P_1 \equiv Q(A_2, A_3)$, $P_2 \equiv Q(A_1, A_3)$ and $P_3 \equiv Q(A_1, A_2)$. Then by the Twist ratio theorem (page 78)

$$t(A_3A_2, A_3B_2) = \frac{Q_2}{P_1} = \frac{Q_1}{P_2} \tag{21.1}$$
$$t(A_1A_2, A_1B_2) = \frac{Q_2}{P_3} = \frac{Q_3}{P_2}. \tag{21.2}$$

By the Triple quad formula $\{P_1, P_2, P_3\}$ forms a quad triple, so that

$$(P_1 + P_3 - P_2)^2 = 4P_1P_3.$$

Divide both sides by P_2^2 and substitute using (21.1) and (21.2) to get

$$\left(\frac{Q_2}{Q_1} + \frac{Q_2}{Q_3} - 1\right)^2 = 4\frac{Q_2^2}{Q_1Q_3}$$

or

$$\left(\frac{1}{Q_1} + \frac{1}{Q_3} - \frac{1}{Q_2}\right)^2 = 4\frac{1}{Q_1}\frac{1}{Q_3}.$$

This is the statement that $\{1/Q_1, 1/Q_2, 1/Q_3\}$ is a quad triple. ■

21.2 Overlapping triangles

Problem 5 Two triangles $\overline{ABC}$ and $\overline{ABD}$ share a side $\overline{AB}$ as shown in Figure 21.2, with quadrances as indicated. What are the quadrances $Q(A, E)$, $Q(B, E)$, $Q(C, E)$ and $Q(D, E)$?

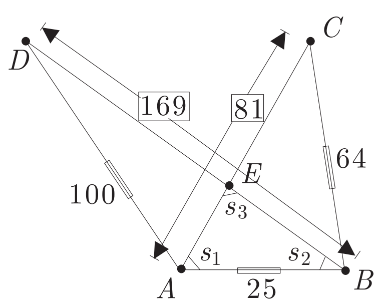


Figure 21.2: Overlapping triangles

Solution. Introduce the spreads s_1, s_2 and s_3 of the sectors of $\overline{ABE}$ as shown. Use the Cross law in $\overline{ABC}$, together with the fact that $81 + 25 > 64$, to get

$$s_1 = 1 - \frac{(81 + 25 - 64)^2}{4 \times 25 \times 81} = \frac{176}{225} \text{ (ac)}.$$

Similarly in $\overline{ABD}$, since $169 + 25 > 100$,

$$s_2 = 1 - \frac{(169 + 25 - 100)^2}{4 \times 169 \times 25} = \frac{2016}{4225} \text{ (ac)}.$$

Now use the Triple spread formula in $\overline{ABE}$ to obtain the quadratic equation

$$s_3^2 - \frac{975\,136}{950\,625} s_3 + \frac{215\,296}{2313\,441} = 0. \tag{21.3}$$

Since s_1 and s_2 are acute, and

$$s_1 + s_2 = \frac{176}{225} + \frac{2016}{4225} = \frac{47\,888}{38\,025} > 1$$

the Triangle spread rules (page 219) show that the correct solution to (21.3) is

$$\begin{aligned} s_3 &= r_b(s_1, s_2) \text{ (ac)} \\ &= \frac{487\,568}{950\,625} + \frac{10\,528}{316\,875}\sqrt{154} \text{ (ac)}. \end{aligned}$$

Then the Spread law in $\overline{ABE}$ gives

$$\frac{176/225}{Q(B,E)} = \frac{2016/4225}{Q(A,E)} = \frac{1}{25}\left(\frac{487\,568}{950\,625} + \frac{10\,528}{316\,875}\sqrt{154}\right).$$

This yields the values

$$\begin{aligned} Q(A,E) &= \frac{34\,556\,382}{525\,625} - \frac{2238\,516}{525\,625}\sqrt{154} \\ Q(B,E) &= \frac{56\,649\,307}{525\,625} - \frac{3669\,666}{525\,625}\sqrt{154}. \end{aligned}$$

Now the Collinear quadrance rules show that since $\overline{AC}$ and $\overline{AE}$ are overlapping,

$$\begin{aligned} Q(C,E) &= Q(A,C) + Q(A,E) - 2\sqrt{Q(A,C)\,Q(A,E)} \\ &= \frac{111\,662\,307}{525\,625} - \frac{7758\,666}{525\,625}\sqrt{154} \end{aligned}$$

and similarly

$$\begin{aligned} Q(D,E) &= Q(B,D) + Q(B,E) - 2\sqrt{Q(B,D)\,Q(B,E)} \\ &= \frac{18\,789\,082}{525\,625} + \frac{1476\,384}{525\,625}\sqrt{154}. \end{aligned}$$

Note, perhaps surprisingly, that the square roots involved work out pleasantly, meaning that all expressions of the form

$$\sqrt{a + b\sqrt{154}}$$

which occur turn out to be expressible in the simpler form $c + d\sqrt{154}$, with c and d rational numbers. ■

21.3 Eyeball theorem

This result is described in [Gutierrez].

Problem 6 Suppose that two circles have centers C_1 and C_2, respective quadrances K_1 and K_2, and that tangents from each center to the other circle are drawn, intersecting the two circles in points A, B and E, F respectively, as in Figure 21.3. Show that

$$Q(A, B) = Q(E, F).$$

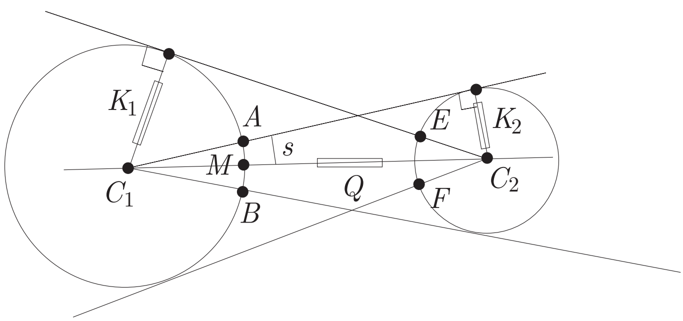


Figure 21.3: Eyeball theorem

Solution. Suppose that $Q(C_1, C_2) \equiv Q$ and define the spread $s \equiv s(C_1A, C_1C_2)$. Then use the Spread ratio theorem (page 77) to find that

$$s = \frac{K_2}{Q}.$$

Let M denote the midpoint of the side $\overline{AB}$, so that in the right triangle $\overline{AMC_1}$

$$s = \frac{Q(A, M)}{K_1}.$$

From this

$$Q(A, M) = \frac{K_1 K_2}{Q}$$

so that by the Midpoint theorem (page 60)

$$Q(A, B) = \frac{4K_1 K_2}{Q}.$$

This is symmetric in K_1 and K_2, so it also equals $Q(E, F)$. ■

21.4 Quadrilateral problem

Problem 7 A quadrilateral $\overline{A_1A_2A_3A_4}$ has quadrances $Q_{12} \equiv 65$, $Q_{34} \equiv 26$, and $Q_{14} \equiv 49$, and diagonal quadrances $Q_{13} \equiv 61$ and $Q_{24} \equiv 100$ as in Figure 21.4. Find $Q \equiv Q_{23}$.

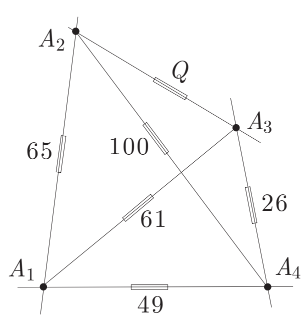


Figure 21.4: A quadrilateral problem

Solution 1: (Using Euler's function). The Four point relation (page 191) states that

$$E\,(100, 65, 49, 61, 26, Q) = 0.$$

This becomes the quadratic equation

$$(Q - 114)^2 = (80)^2$$

with solutions 34 and 194. But $65 + 49 > 100$ so $\overleftrightarrow{A_2A_1A_4}$ is acute, and then so is $\overleftrightarrow{A_2A_1A_3}$. Thus $65 + 61 > Q$, so that $Q = 34$. ■

Solution 2: (Using spreads and the Triangle spread rules). Let the intersection of the diagonal lines A_1A_3 and A_2A_4 be C. From the Cross law and the definitions of acute and obtuse,

$$s\left(\overleftrightarrow{A_4A_1A_3}\right) = 1 - \frac{(49 + 61 - 26)^2}{4 \times 49 \times 61} = \frac{25}{61} \text{ (ac)}.$$

Similarly

$$\begin{aligned} s\left(\overleftrightarrow{A_2A_1A_4}\right) &= 64/65 \text{ (ac)} \qquad & s\left(\overleftrightarrow{A_1A_4A_2}\right) &= 16/25 \text{ (ac)} \\ s\left(\overleftrightarrow{A_1A_4A_3}\right) &= 25/26 \text{ (ac)} \qquad & s\left(\overleftrightarrow{A_1A_2A_4}\right) &= 784/1625 \text{ (ac)}. \end{aligned}$$

This yields Figure 21.5, also showing the unknown spreads

$$x \equiv s\left(\overleftrightarrow{A_2A_1A_3}\right) \qquad \text{and} \qquad z \equiv s\left(\overleftrightarrow{A_1CA_4}\right).$$

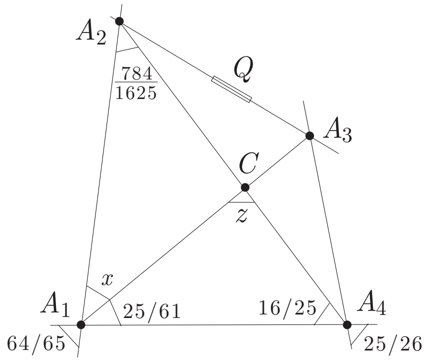


Figure 21.5: Spreads known and unknown

Then the Triple spread formula applied to the spread triple $\{25/61, 16/25, z\}$ gives the quadratic equation

$$\left(z - \frac{801}{1525}\right)^2 = \left(\frac{144}{305}\right)^2.$$

Now

$$25/61 + 16/25 = \frac{1601}{1525} > 1$$

so use the Triangle spread rules in $\overline{A_1A_4C}$ to see that

$$z = \frac{801}{1525} + \frac{144}{305} = \frac{1521}{1525} \text{ (ac)}.$$

Although you could now similarly solve for x using the Triple spread formula in $\overline{A_1A_2C}$, another approach is to apply the Two spread triples theorem (page 98). Since $\{x, 1521/1525, 784/1625\}$ and $\{x, 25/61, 64/65\}$ are both spread triples, and the sector $\overleftrightarrow{A_1CA_2}$ is obtuse,

$$\begin{aligned} x &= \frac{\left(\frac{1521}{1525} - \frac{784}{1625}\right)^2 - \left(\frac{25}{61} - \frac{64}{65}\right)^2}{2 \times \left(\frac{1521}{1525} + \frac{784}{1625} - \frac{25}{61} - \frac{64}{65} - 2 \times \frac{1521}{1525} \times \frac{784}{1625} + 2 \times \frac{25}{61} \times \frac{64}{65}\right)} \\ &= \frac{1849}{3965} \text{ (ac)}. \end{aligned}$$

Now use the Cross law in $\overline{A_1A_2A_3}$,

$$(Q - 65 - 61)^2 = 4 \times 61 \times 65 \times \left(1 - \frac{1849}{3965}\right)$$

to obtain

$$(Q - 126)^2 = (92)^2.$$

Since x is acute, $65 + 61 \geq Q$, so the solution must be

$$Q = 126 - 92 = 34. \quad \blacksquare$$

22

Three dimensional problems

This chapter illustrates applications of rational trigonometry and universal geometry to practical problems involving three-dimensional space over the decimal numbers. Giving a careful and reasonably complete introduction to three-dimensional geometry is not easy, which is one of the reasons why no-one has done it yet. Instead, the usual physical arguments and description by pictures will be adopted, which is of course logically unsatisfying.

22.1 Planes

The notions of parallel and perpendicular lines extend to three-dimensional space. Rather briefly, a **plane** Π is given by a linear equation in the coordinates $[x, y, z]$ of space, with the plane passing through the non-collinear points A, B and C denoted ABC. We'll assume that for the decimal number field most of the results of two dimensional geometry developed thus far hold in any plane in three-dimensional space.

Two planes are **parallel** if they do not intersect. A line n is **perpendicular** to a plane Π if it is perpendicular to every line lying on Π. In such a case n is a **normal** to Π. Any two lines perpendicular to a plane Π are themselves parallel.

Define the **spread** $S(\Pi_1, \Pi_2)$ between the planes Π_1 and Π_2 to be the spread $s(n_1, n_2)$ between respective normals n_1 and n_2. Two planes Π_1 and Π_2 are **perpendicular** precisely when $S(\Pi_1, \Pi_2) = 1$; this is equivalent to the condition that one of the planes contains (or passes through) a normal to the other.

The spread between a line l and a plane Π intersecting at a point A is defined to be the spread between l and the line m formed by intersecting Π with the plane through l and the normal n to Π at A.

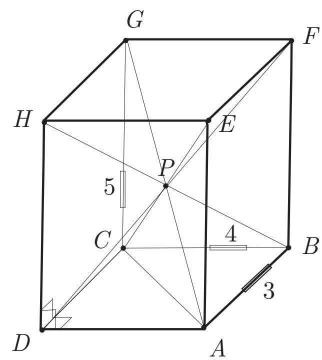
22.2 Boxes

A *box* is assumed to be **rectangular**, meaning that any two of its faces which meet are perpendicular.

Problem 8 The horizontal sides of a box have quadrances 3 and 4, while the vertical side has quadrance 5. Find the quadrances of the long diagonals, the spread that they make with the base, and the possible spreads between two long diagonals.

Solution. Label the vertices of the box as shown, with

$$Q(A,B)=3 \qquad Q(B,C)=4 \qquad Q(C,G)=5.$$



Then by Pythagoras' theorem

$$Q(A,C)=Q(A,B)+Q(B,C)=3+4=7$$

and so also

$$Q(A,G)=Q(A,C)+Q(C,G)=7+5=12.$$

Thus the quadrance of the long diagonal side $\overline{AG}$ is 12 and by symmetry the other long diagonal sides $\overline{BH}$, $\overline{DF}$ and $\overline{CE}$ also have quadrance 12. The spread that any of these long diagonals makes with the base (the plane containing A, B, C and D) is

$$s(AC,AG)=\frac{Q(C,G)}{Q(A,G)}=\frac{5}{12}.$$

If P is the center of the box then the quadrance from P to any vertex is one quarter the quadrance of a long diagonal side, hence 3. The spread between the two diagonals AG and BH, which intersect at P, is then equal to the spread $s(PA,PB)$ in the equilateral triangle $\overline{APB}$ with equal quadrances 3, which by the Equilateral triangle theorem (page 125) is $3/4$.

The spread between the two diagonals AG and DF is the spread $s(PA,PD)$ in the isosceles triangle $\overline{DPA}$ with quadrances $3,3$ and 4. By the Isosceles triangle theorem (page 122) this is

$$s(PA,PD)=\frac{4}{3}\left(1-\frac{4}{4\times 3}\right)=\frac{8}{9}.$$

Similarly

$$s(PA,PE)=\frac{5}{3}\left(1-\frac{5}{4\times 3}\right)=\frac{35}{36}.$$

The three possibilities for spreads between diagonals are $3/4$, $8/9$ and $35/36$. ■

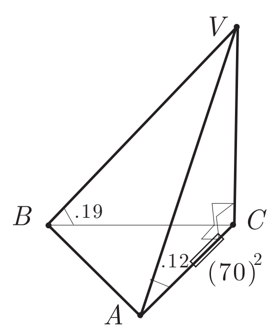
Exercise 22.1 Show more generally that if the quadrances of a box are P, Q and R then the three spreads formed by pairs of long diagonals are

$$\frac{4P(Q+R)}{(P+Q+R)^2}$$
$$\frac{4Q(R+P)}{(P+Q+R)^2}$$
$$\frac{4R(P+Q)}{(P+Q+R)^2}. \quad \diamond$$

Exercise 22.2 Show that in Problem 8 the spread between the plane ABP and the line PF is $20/27$. $\diamond$

Problem 9 The top V of a flagpole subtends a spread of 0.12 at a point A which is a distance of 70 due south, and a spread of 0.19 at a point B which is due west of the flagpole. Calculate the distance $|A,B|$ from A to B.

Solution. This problem is given in terms of distance, so first convert the information into rational trigonometry. If the base of the flagpole is C then $Q(A,C) = (70)^2 = 4900$. In the right triangle $\overline{ACV}$ the spread at A is 0.12, so the spread at V is $1 - 0.12 = 0.88$, and the Spread law gives



$$Q(C,V) = \frac{0.12}{0.88} \times Q(A,C) = \frac{3}{22} \times 4900 = \frac{7350}{11}.$$

In the right triangle $\overline{BCV}$ the spread at B is 0.19, so the spread at V is $1 - 0.19 = 0.81$, and the Spread law gives

$$\begin{aligned} Q(C,B) &= \frac{0.81}{0.19} \times Q(C,V) \\ &= \frac{81}{19} \times \frac{7350}{11} = \frac{595\,350}{209}. \end{aligned}$$

Use Pythagoras' theorem to get

$$Q(A,B) = 4900 + \frac{595\,350}{209} = \frac{1\,619\,450}{209}.$$

So far no approximations have been introduced. To calculate the distance from A to B, take the square root of the quadrance, to get

$$|A,B| = \frac{35\sqrt{276\,298}}{209}$$

which is approximately 88.02. $\blacksquare$

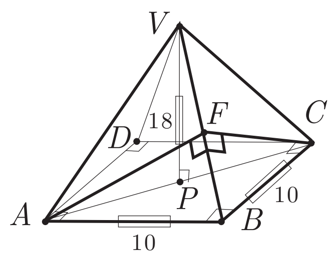
22.3 Pyramids

A *pyramid* consists of a rectangular base with an apex directly above the center of the base.

Problem 10 A square $\overline{ABCD}$ with quadrance 10 is the base of a pyramid. The quadrance from the center P of the base to the apex V, directly above it, is 18. Find the spread $s(VA, VC)$, and the spread between the planes ABV and BCV.

Solution. The triangle $\overline{AVC}$ is isosceles with P the midpoint of the side $\overline{AC}$, and VP bisects the vertex at V. In the right triangle $\overline{ABC}$, use Pythagoras' theorem to see that $Q(A, C) = 20$, so that

$$Q(A, P) = Q(A, C)/4 = 5.$$



Use Pythagoras' theorem in the right triangle $\overline{APV}$ to see that $Q(A, V) = 23$, so that

$$s(VA, VP) = 5/23.$$

By symmetry $s(VC, VP) = 5/23$, so now use the Equal spreads theorem (page 94) to get

$$s(VA, VC) = 4 \times \frac{5}{23} \times \left(1 - \frac{5}{23}\right) = \frac{360}{529}.$$

To determine the spread between the planes ABV and BCV, find the foot F of the altitude from A to VB, which by symmetry is also the foot of the altitude from C to VB. Then the plane AFC is perpendicular to VB, so that the spread S between the planes ABV and BCV is equal to the spread r between the lines AF and FC. The isosceles triangle $\overline{ABV}$ has quadrances $23, 23$ and 10, so by the Isosceles triangle theorem (page 122)

$$s(VA, VB) = \frac{10}{23}\left(1 - \frac{10}{4 \times 23}\right) = \frac{205}{(23)^2}$$

and thus

$$Q(A, F) = Q(A, V)\, s(VA, VF) = 205/23.$$

Similarly

$$Q(C, F) = 205/23.$$

So in the isosceles triangle $\overline{AFC}$

$$r = s(FA, FC) = \frac{20}{205/23}\left(1 - \frac{20}{4 \times (205/23)}\right) = \frac{1656}{1681} = S. \blacksquare$$

22.4 Wedges

A *wedge* is formed by two intersecting planes, often with one of the planes horizontal.

Problem 11 Suppose an inclined plane has a spread of S with the horizontal plane. An insect climbing up the plane walks on a straight line which makes a spread of r with the line of greatest slope. At what spread to the horizontal does the insect climb on this path?

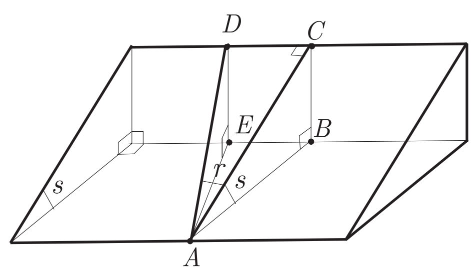


Figure 22.1: Path on a wedge

Solution. Denote by AC a line making the maximum possible spread $s \equiv S$ with the horizontal, and AD the path of the insect, as in Figure 22.1. The spread between AC and AD is r, and you need to find the spread $s\,(AE, AD)$. There are no units in the problem, so assume that $Q\,(B, C) = Q\,(E, D) = 1$. From the right triangle $\overline{ABC}$

$$Q\,(A, C) = \frac{Q\,(B, C)}{s} = \frac{1}{s}.$$

From the right triangle $\overline{ACD}$, with right vertex at C,

$$Q\,(A, D) = \frac{Q\,(A, C)}{1 - r} = \frac{1}{s\,(1 - r)}.$$

Thus the right triangle $\overline{ADE}$ gives

$$s\,(AE, AD) = \frac{Q\,(D, E)}{Q\,(A, D)} = s\,(1 - r) = S\,(1 - r)\,. \quad \blacksquare$$

22.5 Three dimensional Pythagoras' theorem

Problem 12 Suppose that three points B_1, B_2 and B_3 in space are distinct from a point C and that the three lines CB_1, CB_2 and CB_3 are mutually perpendicular. Let $\mathcal{A}$ be the quadrea of the triangle $\overline{B_1B_2B_3}$, and $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ the quadreas of the triangles $\overline{CB_2B_3}, \overline{CB_1B_3}$ and $\overline{CB_1B_2}$ respectively. Show that

$$\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3.$$

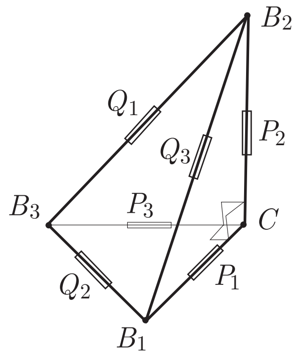


Figure 22.2: Three-dimensional Pythagoras

Solution. Let Q_1, Q_2 and Q_3 denote the quadrances of the triangle $\overline{B_1B_2B_3}$, with $P_1 = Q(B_1, C)$, $P_2 = Q(B_2, C)$ and $P_3 = Q(B_1, C)$. Since the triangles $\overline{CB_2B_3}, \overline{CB_1B_3}$ and $\overline{CB_1B_2}$ are right triangles,

$$Q_1 = P_2 + P_3 \qquad Q_2 = P_1 + P_3 \qquad Q_3 = P_1 + P_2.$$

The quadrea $\mathcal{A}$ of $\overline{B_1B_2B_3}$ is

$$\begin{aligned}\mathcal{A} &= 4Q_1Q_2 - (Q_1 + Q_2 - Q_3)^2 \\ &= 4(P_2 + P_3)(P_1 + P_3) - 4P_3^2 \\ &= 4(P_2P_3 + P_1P_3 + P_1P_2).\end{aligned}$$

But by the Right quadrea theorem (page 68)

$$\mathcal{A}_1 = 4P_2P_3 \qquad \mathcal{A}_2 = 4P_1P_3 \qquad \mathcal{A}_3 = 4P_1P_2.$$

Thus $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3$. ■

Exercise 22.3 Show that any triangle $\overline{B_1B_2B_3}$ forming part of such a right tetrahedron is acute, and given such a triangle there are in general exactly two such tetrahedra. ◇

22.6 Pagoda and seven-fold symmetry

Problem 13 A retired engineer decides to build the roof of a pagoda with a base of a regular 7-gon, with the quadrance of each side 2, and the apex V above the center C of the regular 7-gon at a quadrance of 1 from the base. The roof then consists of seven identical isosceles triangles. What should the quadrances and spreads of these triangles be?

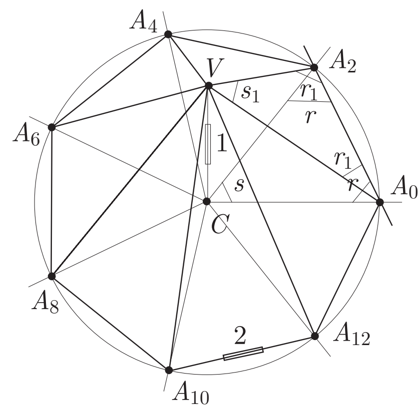


Figure 22.3: A seven-sided pagoda

Solution. Suppose the regular 7-gon is $\overline{A_0A_2A_4A_6A_8A_{10}A_{12}}$ as in Figure 22.3. The lines $A_0C, A_2C, \cdots, A_{12}C$ form a regular star of order seven, so the spread $s \equiv s\left(\overleftrightarrow{A_0CA_2}\right)$ must satisfy

$$S_7(s) = s\left(7 - 56s + 112s^2 - 64s^3\right)^2 = 0.$$

Of the three approximate solutions,

$$0.188\,255 \qquad 0.611\,260 \qquad 0.950\,484$$

the relevant one is

$$s \approx 0.611\,260 \text{ (ac)}.$$

Define the spreads of the sectors

$$\begin{aligned} r &\equiv s\left(\overleftrightarrow{CA_0A_2}\right) = s\left(\overleftrightarrow{CA_2A_0}\right) \\ r_1 &\equiv s\left(\overleftrightarrow{VA_0A_2}\right) = s\left(\overleftrightarrow{VA_2A_0}\right) \\ s_1 &\equiv s\left(\overleftrightarrow{A_0VA_2}\right). \end{aligned}$$

Use the Isosceles triangle theorem (page 122) with $\overline{A_0A_2C}$ to get

$$s = S_2(r) = 4r(1-r)$$

so that

$$r = \frac{1 \pm \sqrt{1-s}}{2}.$$

This gives the possibilities

$$r \approx 0.188\,25 \qquad r \approx 0.811\,75$$

and the Triangle spread rules show that the relevant one is

$$r \approx 0.811\,75 \text{ (ac)}.$$

Use the Spread law in $\overline{A_0A_2C}$ to see that

$$\begin{aligned} Q(A_0, C) = Q(A_2, C) &= \frac{rQ(A_0, A_2)}{s} \\ &\approx \frac{0.811\,745}{0.611\,260} \times 2 \approx 2.655\,9. \end{aligned}$$

Then use Pythagoras' theorem in $\overline{A_0CV}$ to obtain

$$Q(A_0, V) \approx 2.655\,9 + 1 = 3.655\,9.$$

Apply the Isosceles triangle theorem to $\overline{A_0A_2V}$ to get

$$4Q(A_0, V)(1-r_1) = Q(A_0, A_2) = 2$$

so that

$$r_1 \approx 0.863\,24$$

and

$$s_1 = S_2(r_1) = 4r_1(1-r_1) \approx 0.472\,24.$$

The triangle $\overline{A_0A_2V}$ thus has approximate quadrances $2, 3.655\,9$ and $3.655\,9$, and respective approximate spreads $0.472\,2, 0.863\,2$ and $0.863\,2$. ■

Exercise 22.4 Show that the spread S between the planes VA_0A_2 and VA_2A_4 is approximately $0.224\,4$. ◇

23

Physics applications

Some applications to physics are given, including maximizing the trajectory of a projectile, a derivation of Snell's law, and a rational formulation of Lorentzian addition of velocities in Einstein's special theory of relativity. An example of algebraic dynamics over a finite field is discussed. Some basic calculus will be assumed here.

23.1 Projectile motion

The motion of a projectile is a parabola, and if the projectile begins at the origin with velocity $\overrightarrow{v} \equiv [a, b]$ as in Figure 23.1, then its position at time t is given by

$$\left[at, bt - \frac{gt^2}{2}\right]$$

where g is the acceleration due to gravity.

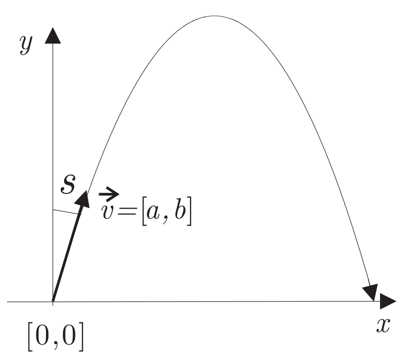


Figure 23.1: Projectile motion

Problem 14 Given that the initial speed $v \equiv \sqrt{a^2+b^2}$ is fixed, what spread s from the vertical results in the projectile traveling the farthest horizontally before it comes to ground again at a point $[x, 0]$ for some x?

Solution. The projectile comes to ground at time t, where

$$bt - \frac{gt^2}{2} = 0$$

so that either $t = 0$ or $t = 2b/g$. Using rational trigonometry, quadrance is preferred over distance, so the question is what value of $A \equiv a^2$ and $B \equiv b^2$, subject to the condition $A + B = v^2 \equiv V$, results in the horizontal quadrance

$$x^2 = (at)^2 = \frac{4AB}{g^2}$$

being maximized? This is then the problem of maximizing the product AB of two numbers A and B given their sum V. The maximum occurs when $A = B = V/2$, giving a maximum horizontal quadrance of

$$x^2 = \frac{V^2}{g^2}.$$

So the projectile should be fired at a spread of $s = 1/2$ from the vertical. ∎

Problem 15 Suppose that the projectile is fired from the origin on a hill represented by the line l through the origin making a spread of r with the vertical as in Figure 23.2. Given that the initial speed v is fixed, what spread s from the vertical results in a maximal horizontal displacement after landing?

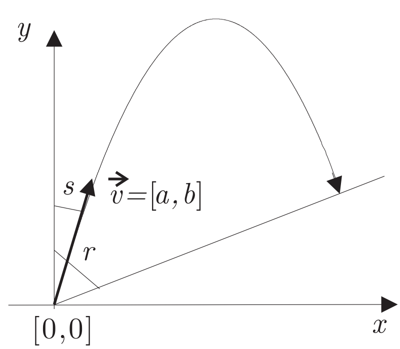


Figure 23.2: Projectile fired on a hill

Solution. The hill is determined by the equation $x^2 = r\left(x^2 + y^2\right)$ and so the projectile intercepts the hill when

$$(at)^2 = r\left((at)^2 + \left(bt - gt^2/2\right)^2\right).$$

This yields that $t = 0$, or t satisfies the quadratic equation

$$\left(t - \frac{2b}{g}\right)^2 = \frac{4a^2\left(1-r\right)}{g^2 r}.$$

Thus

$$t = \frac{2b}{g} \pm \frac{2a}{g}\sqrt{\frac{1-r}{r}}.$$

To maximize the horizontal displacement, you need to maximize at, or equivalently

$$f\left(a, b\right) \equiv ab \pm a^2 c$$

by choosing a and b subject to the constraint

$$g\left(a, b\right) \equiv a^2 + b^2 = V \tag{23.1}$$

and where c is the constant

$$c \equiv \sqrt{\frac{1-r}{r}}.$$

This can now be done by converting it to a one-dimensional calculus problem, but it is also interesting to apply the method of Lagrange. At a relative maximum the gradients

$$\nabla f = \left(b \pm 2ac, a\right) \qquad \nabla g = \left(2a, 2b\right)$$

should be proportional, implying that

$$\left(b \pm 2ac\right)b - a^2 = 0.$$

Rearrange and square to eliminate the ambiguity of the sign

$$4a^2b^2c^2 = \left(a^2 - b^2\right)^2$$

and substitute using (23.1) to get

$$4a^2\left(V - a^2\right)c^2 = \left(2a^2 - V\right)^2.$$

This quadratic equation in a^2 can be written

$$\left(a^2 - \frac{V}{2}\right)^2 = \frac{V^2c^2}{4\left(1+c^2\right)} = \frac{V^2\left(1-r\right)}{4}.$$

Thus

$$a^2 = \frac{V}{2}\left(1 \pm \sqrt{1-r}\right)$$

and the spread s between the initial direction and the vertical is

$$s = \frac{a^2}{V} = \frac{1 \pm \sqrt{1-r}}{2}.$$

But this is equivalent to

$$r = 4s\left(1-s\right) = S_2\left(s\right)$$

so that the projectile's initial direction should bisect the vertex formed by the hill and the vertical. Note that there are two solutions, one downhill and the other uphill. ■

23.2 Algebraic dynamics

Recently mathematicians have begun investigating dynamics in finite fields. Here is a particularly simple case modelled on the usual projectile motion under constant negative acceleration due to gravity. Whether such an example has any possible physical significance is unclear, but it seems interesting from a mathematical perspective.

Example 23.1 Suppose that in $\mathbb{F}_{11}$ a particle starts at time $t = 0$ with position $p_0 \equiv [0, 0]$, velocity $v_0 \equiv [1, 3]$ and has constant acceleration $a_t \equiv [0, -1]$ for times $t = 0, 1, 2, 3, \cdots$. Suppose that subsequent positions and velocities are determined for future times by the equations

$$\begin{aligned} p_{t+1} &\equiv p_t + v_t \\ v_{t+1} &\equiv v_t + a_t. \end{aligned}$$

This results in the following positions and velocities, which then repeat.

Time	0	1	2	3	4	5
Position	$[0, 0]$	$[1, 3]$	$[2, 5]$	$[3, 6]$	$[4, 6]$	$[5, 5]$
Velocity	$[1, 3]$	$[1, 2]$	$[1, 1]$	$[1, 0]$	$[1, 10]$	$[1, 9]$

Time	6	7	8	9	10	11
Position	$[6, 3]$	$[7, 0]$	$[8, 7]$	$[9, 2]$	$[10, 7]$	$[0, 0]$
Velocity	$[1, 8]$	$[1, 7]$	$[1, 6]$	$[1, 5]$	$[1, 4]$	$[1, 3]$

The position at time t is $\left[t, 5t^2 - 2t\right]$. The trajectory contains exactly those points lying on the curve with equation $x^2 + 4x + 2y = 0$, which turns out to be a parabola (black circles) in the sense of Chapter 15. The directrix is the line $l \equiv \langle 0 : 1 : 3 \rangle$ (gray boxes) and the focus is $F \equiv [9, 7]$ (open box) as shown in Figure 23.3. Notice, perhaps surprisingly, that the vertex of this parabola is the point $[9, 2]$.

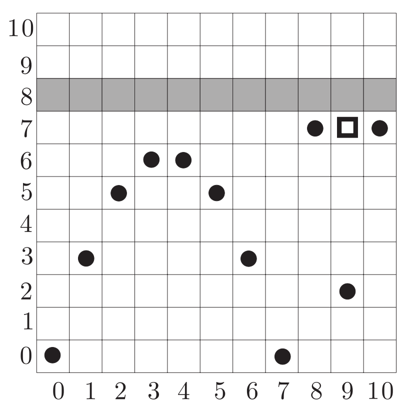


Figure 23.3: Trajectory in $\mathbb{F}_{11}$ $\diamond$

23.3 Snell's law

Problem 16 Suppose a particle travels from the point $A \equiv [0, a]$ to the point $B \equiv [c, -b]$, where $a, b > 0$, via some variable point $D \equiv [x, 0]$ on the horizontal axis as in Figure 23.4. If the particle has speed v_1 in the region $y \geq 0$, and speed v_2 in the region $y < 0$, what choice of D minimizes the total time taken?

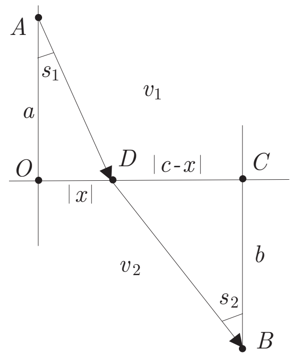


Figure 23.4: Snell's Law

Solution. The basic formula relating distance d, time t and speed v is

$$v = d/t. \tag{23.2}$$

This is not a formula involving universal geometry, as distance is involved. Nevertheless, let's proceed some way in the classical framework before switching over to rational trigonometry. With $|O, D| = |x|$ and $|D, C| = |c - x|$, the times t_1 and t_2 taken to travel from A to D (in a straight line) and from D to B respectively are

$$\begin{aligned} t_1 &= \frac{|A, D|}{v_1} = \frac{\sqrt{a^2 + x^2}}{v_1} \\ t_2 &= \frac{|D, B|}{v_2} = \frac{\sqrt{(c - x)^2 + b^2}}{v_2}. \end{aligned}$$

The total time t taken is then

$$t = \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{(c - x)^2 + b^2}}{v_2}.$$

This is a function of x, since a, b, v_1 and v_2 are constants. You could now use calculus to find the value of x at which this function attains a maximum or minimum.

To do so, the derivative of $\sqrt{x}$ is required. Instead, let's reconsider the problem from the viewpoint of rational trigonometry.

Since $Q \equiv d^2$ is a rational concept, it makes sense to square (23.2), obtaining

$$V = Q/T$$

where $V \equiv v^2$ and $T \equiv t^2$. Since $Q(A, D) = a^2 + x^2$ and $Q(D, B) = (c - x)^2 + b^2$, the squared times T_1 and T_2 taken to travel from A to D and from D to B respectively are

$$T_1 = \frac{a^2 + x^2}{V_1} \tag{23.3}$$

$$T_2 = \frac{(c - x)^2 + b^2}{V_2}. \tag{23.4}$$

Now $t = t_1 + t_2$, so Exercise 5.8 shows that $\{T, T_1, T_2\}$ is a quad triple. All three quantities depend on a variable x and the aim is to choose x so as to minimize T. The following argument deals with this general situation.

Suppose that $\{T, T_1, T_2\}$ is a quad triple, so that

$$(T_1 + T_2 - T)^2 = 4T_1T_2 \tag{23.5}$$

and that all three quantities T, T_1 and T_2 depend on a variable x. Take differentials to obtain

$$2(T_1 + T_2 - T)\left(\frac{dT_1}{dx} + \frac{dT_2}{dx} - \frac{dT}{dx}\right) = 4\frac{d(T_1T_2)}{dx}. \tag{23.6}$$

To maximize or minimize T, set

$$\frac{dT}{dx} = 0.$$

Square (23.6) to get

$$(T_1 + T_2 - T)^2 \left(\frac{dT_1}{dx} + \frac{dT_2}{dx}\right)^2 = 4\left(T_2\frac{dT_1}{dx} + T_1\frac{dT_2}{dx}\right)^2.$$

Now substitute (23.5) so that

$$T_1T_2\left(\frac{dT_1}{dx} + \frac{dT_2}{dx}\right)^2 = \left(T_2\frac{dT_1}{dx} + T_1\frac{dT_2}{dx}\right)^2.$$

Upon expansion, rearrangement and cancellation of an extraneous factor $T_1 - T_2$, this becomes the following general formula for a maximum or minimum

$$T_2\left(\frac{dT_1}{dx}\right)^2 = T_1\left(\frac{dT_2}{dx}\right)^2. \tag{23.7}$$

Now to return to the case at hand, apply (23.7) to (23.3) and (23.4) where

$$\begin{aligned}\frac{dT_1}{dx} &= \frac{2x}{V_1} \\ \frac{dT_2}{dx} &= \frac{2(x-c)}{V_2}.\end{aligned}$$

You get

$$\frac{\left((c-x)^2+b^2\right)}{V_2} \times \frac{4x^2}{V_1^2} = \frac{\left(a^2+x^2\right)}{V_1} \times \frac{4(c-x)^2}{V_2^2}$$

or

$$\frac{V_2}{V_1} = \frac{(c-x)^2}{(c-x)^2+b^2} \times \frac{\left(a^2+x^2\right)}{x^2}.$$

But the spreads s_1 and s_2 made by the lines AD and DB respectively with the vertical are

$$s_1 = \frac{x^2}{a^2+x^2}$$

and

$$s_2 = \frac{(c-x)^2}{(c-x)^2+b^2}.$$

This yields *Snell's Law—The time taken is minimized when*

$$\frac{V_2}{V_1} = \frac{s_2}{s_1}. \quad \blacksquare$$

The rational solution presented here avoids differentiation of the square root function and uses only derivatives of linear and quadratic functions.

This analysis also suggests a view of physics in which not only the square of distance, but also the squares of speed and time play a larger role. Such ideas were introduced in Einstein's theory of relativity in 1905. In fact Einstein showed that neither the square of distance nor the square of time was ultimately of significance, but in suitable units only the *difference* between them. The square of mass also figures prominently.

In retrospect one can speculate that if rational trigonometry had been developed prior to the twentieth century, then the value of Einstein's revolutionary ideas would have been recognized more readily, and indeed they might have been anticipated earlier. Universal geometry and relativity theory naturally have common aspects.

Perhaps there is the potential to take this further, as current formulations of special (and general) relativity rely on square root functions, and from the point of view of universal geometry this is not optimal. The next section shows how to eliminate this dependence in one special situation.

23.4 Lorentzian addition of velocities

If a train travels along a track with speed v_1 and a bullet is fired from the train in the same direction with speed v_2 with respect to the train, then in Newtonian mechanics the speed v of the bullet with respect to the ground is the sum of the two speeds

$$v = v_1 + v_2. \tag{23.8}$$

Thus the respective squares V, V_1 and V_2 of the speeds v, v_1 and v_2 form a quad triple, in other words

$$(V_1 + V_2 - V)^2 = 4V_1V_2. \tag{23.9}$$

In Einstein's special theory of relativity, (23.8) needs to be modified to

$$v = \frac{v_1 + v_2}{1 + v_1v_2} \tag{23.10}$$

where units have been chosen so that the speed of light is $c = 1$. Square both sides of (23.10) and rearrange to get

$$v^2\left(1 + 2v_1v_2 + v_1^2v_2^2\right) = v_1^2 + 2v_1v_2 + v_2^2$$

or

$$v^2 - v_1^2 - v_2^2 + v^2v_1^2v_2^2 = 2v_1v_2\left(1 - v^2\right).$$

Then square both sides again to get

$$(V_1 + V_2 - V - VV_1V_2)^2 = 4V_1V_2(1 - V)^2. \tag{23.11}$$

Note that for small values of V, V_1 and V_2 this is approximated by (23.9). Furthermore (23.11) can be rewritten as the symmetric expression

$$(V + V_1 + V_2 - VV_1V_2)^2 = 4(VV_1 + VV_2 + V_1V_2 - 2VV_1V_2)$$

which is a form quite close to the Triple twist formula (page 93).

24

Surveying

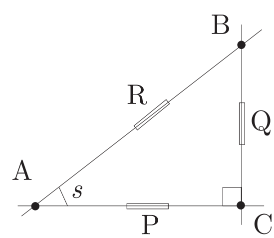
In this chapter classical problems in surveying are solved using rational trigonometry, such as finding heights of objects from a variety of measurements, and Regiomontanus' problem of determining the maximum spread subtended by a window. Some of the examples are parallel to ones from [Shepherd], allowing a comparison between rational and classical methods. As an application of one of the formulas obtained, the important spherical analogue of Pythagoras' theorem is derived.

24.1 Height of object with vertical face

Problem 17 An observer at A measures the vertical spread s to the point B directly above C. The quadrance $Q(A, C) \equiv P$ is known. What is the vertical quadrance $Q \equiv Q(B, C)$?

Solution. The Complementary spreads theorem (page 79) shows that the spread at B is $1 - s$, so the Spread law gives

$$Q = \frac{sP}{1-s}. \quad \blacksquare$$



Example 24.1 Suppose the quadrance from A to C is 100 and the spread at A is measured with a theodolite to be $s \equiv 0.587$. Then

$$Q = \left(\frac{0.587}{0.413}\right) \times 100 = 142.131. \quad \diamond$$

24.2 Height of object with inaccessible base

Problem 18 The points A_1, A_2 and C are horizontal and in a line, and the point A_3 is vertically above C, as in either of the diagrams in Figure 24.1. The spreads s_1 and s_2 in $\overline{A_1A_2A_3}$ are measured, and the quadrance $Q_3 \equiv Q(A_1, A_2)$ is known. What is the vertical quadrance $Q \equiv Q(A_3, C)$?

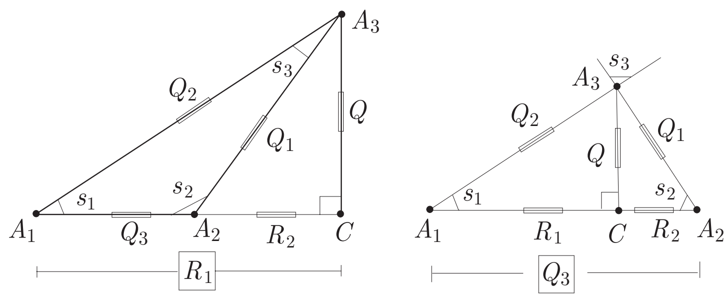


Figure 24.1: Height from two spread readings

Solution. Let $s_3 \equiv s(A_3A_1, A_3A_2)$. The Triple spread formula as a quadratic equation in s_3 is

$$(s_3 - (s_1 + s_2 - 2s_1s_2))^2 = 4s_1s_2(1 - s_1)(1 - s_2).$$

For each of the two solutions the triangle $\overline{A_1A_2A_3}$ may be solved using the Spread law for the quadrance Q_1, since Q_3 is known. Then the right triangle $\overline{A_2A_3C}$ with right vertex C may be solved, using s_2 and Q_1, to obtain Q. ■

Example 24.2 Suppose that $Q_3 \equiv 25$ and that $s_1 \equiv s\left(\overleftrightarrow{A_2A_1A_3}\right) \equiv 0.2352$ (ac) and $s_2 \equiv s\left(\overleftrightarrow{A_1A_2A_3}\right) \equiv 0.3897$ (ob) as in the first of the diagrams in Figure 24.1. The Triple spread formula becomes $(s_3 - 0.4416)^2 = 0.1711$. Use the Triangle spread rules, and the fact that $s_1 + s_2 \leq 1$, to get

$$s_3 = r_l(s_1, s_2)\ \text{(ac)} = 0.4416 - \sqrt{0.171\,1}\ \text{(ac)} \approx 0.0280\ \text{(ac)}.$$

Then apply the Spread law in $\overline{A_1A_2A_3}$ to get

$$0.0280/25 \approx 0.2352/Q_1$$

from which $Q_1 \approx 211.5$. Then in the right triangle $\overline{A_2A_3C}$

$$Q = s_2Q_1 \approx 0.3897 \times 211.5 \approx 82.4. \quad \diamond$$

24.3 Height of a raised object

Problem 19 The points A_1, A_2 and C are horizontal and in a line. There are two points D and A_3 vertically above the point C as in Figure 24.2. The spreads s_1 and s_2 in triangle $\overline{A_1A_2A_3}$ are measured, as is the spread $r \equiv s(A_2C, A_2D)$. The quadrance Q_3 between A_1 and A_2 is known. What is the vertical quadrance $R \equiv Q(A_3, D)$?

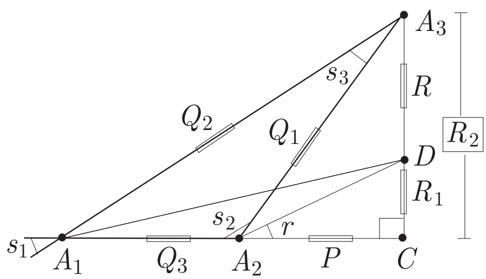


Figure 24.2: Height of a raised object

Solution. Define the quadrances $R_1 \equiv Q(C, D)$, $R_2 \equiv Q(C, A_3)$ and $P \equiv Q(A_2, C)$. Use the Triple spread formula and the Triangle spread rules in $\overline{A_1A_2A_3}$ to find s_3. Then the Spread law in $\overline{A_1A_2A_3}$ gives

$$Q_1 = \frac{s_1 Q_3}{s_3}.$$

In the right triangle $\overline{A_2A_3C}$

$$R_2 = s_2 Q_1 = \frac{s_1 s_2 Q_3}{s_3} \tag{24.1}$$

and

$$P = (1 - s_2) Q_1.$$

Then in the right triangle $\overline{A_2DC}$

$$R_1 = \frac{rP}{1-r} = \frac{(1-s_2) r Q_1}{1-r}$$

so that also

$$R_1 = \frac{s_1 (1-s_2) r Q_3}{s_3 (1-r)}. \tag{24.2}$$

Now $\{R, R_1, R_2\}$ is a quad triple so solve

$$(R - R_1 - R_2)^2 = 4 R_1 R_2$$

with the Collinear quadrance rules (page 215) to obtain R. ■

24.4 Regiomontanus' problem

Regiomontanus, whose name was Johann Müller, lived from 1436 to 1476, and published mathematical and astronomical books. In his most famous work *On Triangles of Every Kind,* he mentions the following extremal problem.

Problem 20 (Regiomontanus' problem) In Figure 24.3, what value of the quadrance P will maximize the spread s subtended by the window $\overline{BD}$? The positions of the points B, D and C on the vertical line are known and fixed, so the quadrances Q, Q_1 and Q_2 can be taken as given.

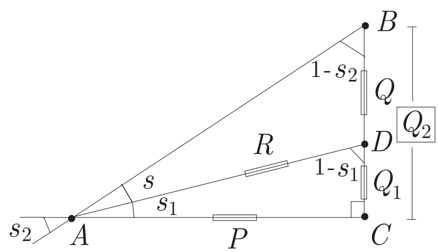


Figure 24.3: Regiomontanus' problem

Solution. Pythagoras' theorem gives $R = P + Q_1$, while from the Spread ratio theorem

$$s_2 = \frac{Q_2}{P + Q_2}.$$

From the Spread law in $\overline{ABD}$

$$\frac{s}{Q} = \frac{1 - s_2}{R}.$$

Combine these equations to get

$$\begin{aligned} s &= \frac{QP}{(P + Q_1)(P + Q_2)} \\ &= \frac{Q}{Q_1 + Q_2 + P + (Q_1 Q_2 / P)}. \end{aligned}$$

Now choose P so that this expression is maximized, or equivalently so that

$$P + \frac{Q_1 Q_2}{P}$$

is minimized. With the product of two summands constant, the sum is minimum when the summands are equal, so that $P^2 = Q_1 Q_2$. Thus P must be the *geometric mean* of Q_1 and Q_2. ■

24.5 Height from three spreads

Problem 21 A triangle $\overline{A_1A_2A_3}$ is horizontal, the point B is directly above A_3, and D is a third point lying on A_1A_2, as in Figure 24.4. The vertical spreads

$$r_1 \equiv s(A_1A_3, A_1B) \qquad r_2 \equiv s(A_2A_3, A_2B) \qquad r_3 \equiv s(DA_3, DB)$$

are known, as are the quadrances $P_1 \equiv Q(A_1, D)$ and $P_2 \equiv Q(A_2, D)$. Find the vertical quadrance $H \equiv Q(A_3, B)$.

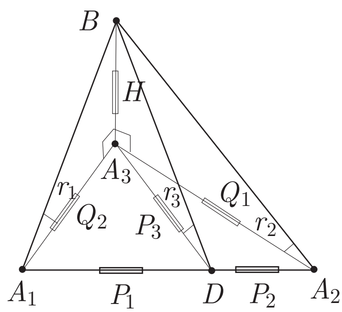


Figure 24.4: Height from three spreads

Solution. Suppose the quadrances of $\overline{A_1A_2A_3}$ are Q_1, Q_2 and Q_3 as usual. Let $P_3 \equiv Q(A_3, D)$. From the right triangle $\overline{A_1A_3B}$

$$r_1 = \frac{H}{H + Q_2}$$

so that

$$Q_2 = \frac{(1 - r_1)H}{r_1}.$$

Similarly from the right triangles $\overline{A_2A_3B}$ and $\overline{DA_3B}$

$$Q_1 = (1 - r_2)H/r_2 \qquad \text{and} \qquad P_3 = (1 - r_3)H/r_3.$$

Now in the triangle $\overline{A_1A_2A_3}$ use Stewart's theorem (page 136) to get

$$P_2(P_3 + P_1 - Q_2)^2 = P_1(P_3 + P_2 - Q_1)^2.$$

Substitute for Q_2, Q_1 and P_3, to get for H the quadratic equation

$$P_2\left(H\left(\frac{1}{r_3} - \frac{1}{r_1}\right) + P_1\right)^2 = P_1\left(H\left(\frac{1}{r_3} - \frac{1}{r_2}\right) + P_2\right)^2. \quad \blacksquare$$

24.6 Vertical and horizontal spreads

Problem 22 The points A_1, A_2 and A_3 form a horizontal triangle with quadrances Q_1, Q_2 and Q_3, and spreads s_1, s_2 and s_3 as usual. The point B is directly above the point A_3. What is the relationship between the vertical spreads $r_1 \equiv s(A_1A_3, A_1B)$ and $r_2 \equiv s(A_2A_3, A_2B)$?

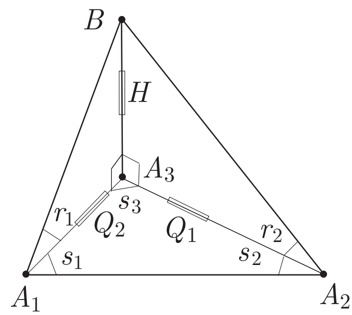


Figure 24.5: Vertical and horizontal spreads

Solution. Suppose that $H \equiv Q(A_3, B)$ as in Figure 24.5. From the right triangle $\overline{A_1A_3B}$

$$\frac{H}{Q_2} = \frac{r_1}{1 - r_1}$$

and similarly from the right triangle $\overline{A_2A_3B}$

$$\frac{H}{Q_1} = \frac{r_2}{1 - r_2}.$$

Use the Spread law in the triangle $\overline{A_1A_2A_3}$ and the previous equations to get

$$\frac{s_1}{s_2} = \frac{Q_1}{Q_2} = \frac{r_1}{(1 - r_1)} \frac{(1 - r_2)}{r_2}.$$

This can also be written as either

$$\frac{s_1(1 - r_1)}{r_1} = \frac{s_2(1 - r_2)}{r_2}$$

or

$$\frac{s_1}{r_1} - \frac{s_2}{r_2} = s_1 - s_2. \quad \blacksquare$$

24.7 Spreads over a right triangle

Problem 23 (Spreads over a right triangle) Suppose that the points A_1, A_2 and A_3 form a horizontal right triangle with right vertex at A_3, and that B is directly above the point A_3 as in Figure 24.6. What is the relationship between the spreads $s \equiv s(BA_1, BA_2)$, $r_1 \equiv s(A_1A_3, A_1B)$ and $r_2 \equiv s(A_2A_3, A_2B)$?

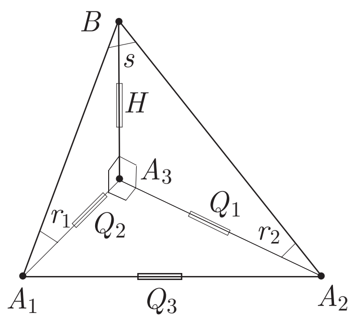


Figure 24.6: Spreads over a right triangle

Solution. Let the quadrances of $\overline{A_1A_2A_3}$ be Q_1, Q_2 and Q_3, and let $H \equiv Q(A_3, B)$. By Pythagoras' theorem

$$\begin{aligned} Q(A_1, A_2) &= Q_3 = Q_1 + Q_2 \\ Q(A_1, B) &= Q_2 + H \\ Q(A_2, B) &= Q_1 + H. \end{aligned}$$

From the Cross law in $\overline{A_1A_2B}$

$$((Q_1 + H) + (Q_2 + H) - (Q_1 + Q_2))^2 = 4(Q_1 + H)(Q_2 + H)(1 - s).$$

Thus ultimately independent of the triangle $\overline{A_1A_2A_3}$,

$$1 - s = \left(\frac{H}{Q_1 + H}\right)\left(\frac{H}{Q_2 + H}\right) = r_1 r_2. \quad \blacksquare \tag{24.3}$$

Exercise 24.1 (Harder) Suppose $\overline{A_1A_2A_3}$ is an equilateral triangle, and that B is directly above the circumcenter C of $\overline{A_1A_2A_3}$. Show that if

$$q \equiv s(BA_1, BA_2) = s(BA_2, BA_3) = s(BA_1, BA_3)$$

and S is the spread between any two of the planes A_1A_2B, A_2A_3B and A_1A_3B, then

$$(1 - Sq)^2 = 4(1 - S)(1 - q). \quad \diamond$$

24.8 Spherical analogue of Pythagoras' theorem

From (24.3) follows a remarkable and important formula. Suppose that the points A_1, A_2 and A_3 form a horizontal right triangle with right vertex at A_3, and that O is directly above the point A_3. Define the spreads $q_1 \equiv s(OA_1, OA_3)$, $q_2 \equiv s(OA_2, OA_3)$ and $q \equiv s(OA_1, OA_2)$ as in Figure 24.7. Then q_1 and q_2 are complementary to the spreads r_1 and r_2 in Figure 24.6.

The use of the small letter q here and in the previous exercise anticipates projective trigonometry, where the quadrance between two 'projective points' is defined to be the spread between the associated lines through the origin.

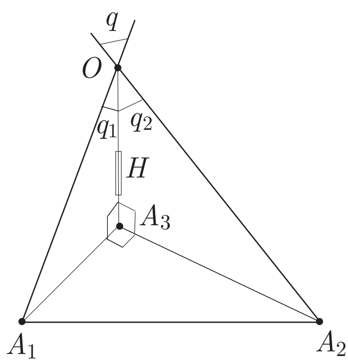


Figure 24.7: Spherical Pythagoras' theorem

Using (24.3),

$$\begin{aligned} q &= 1 - r_1 r_2 \\ &= 1 - (1 - q_1)(1 - q_2). \end{aligned}$$

So

$$q = q_1 + q_2 - q_1 q_2.$$

This is the *spherical or (elliptic) analogue of Pythagoras' theorem.* Its pivotal role in projective trigonometry will be explained more fully in a subsequent volume. Note that if q_1 and q_2 are small then this is approximated by the usual planar form of Pythagoras' theorem.

25

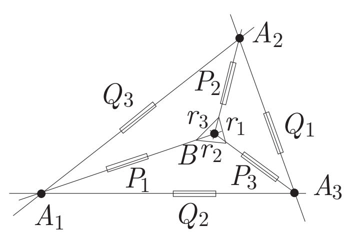
Resection and Hansen's problem

The problems of Snellius-Pothenot and Hansen are among the most famous of surveying problems, and are also of importance in navigation. The Snellius-Pothenot, or resection, problem has a number of solutions, and the one presented here uses Euler's Four point relation. Hansen's problem is illustrated with a specific example, and an exercise shows its connection with a somewhat notorious problem of elementary Euclidean geometry.

25.1 Snellius-Pothenot problem

The problem of resection was originally stated and solved by Snellius (1617) and then by Pothenot (1692).

Problem 24
The quadrances Q_1, Q_2 and Q_3 of $\overline{A_1A_2A_3}$ are known. The spreads $r_1 \equiv s(BA_2, BA_3)$, $r_2 \equiv s(BA_1, BA_3)$ and $r_3 \equiv s(BA_1, BA_2)$ are measured. Find $P_1 \equiv Q(B, A_1)$, $P_2 \equiv Q(B, A_2)$ and $P_3 \equiv Q(B, A_3)$.



The problem cannot be solved if B lies on the circumcircle c of $\overline{A_1A_2A_3}$, since in that case the Subtended spread theorem (page 178) shows that any point on c yields the same values for r_1, r_2 and r_3. Here is a procedure to find P_1 and P_2, which works provided B is not on c, using the Four point relation (page 191).

Solution. Take the circumcircle c_3 of $\overline{A_1A_2B}$ and let H, called **Collin's point**, be the intersection of c_3 with A_3B which is distinct from B.

Define the quadrances $R_1 \equiv Q(H, A_1)$, $R_2 \equiv Q(H, A_2)$ and $R_3 \equiv Q(H, A_3)$. By the Subtended spread theorem, the spreads $s(A_1H, A_1A_2)$, $s(A_2H, A_2A_1)$ and $s(HA_1, HA_2)$ are respectively r_1, r_2 and r_3. Let $v_1 \equiv s(HA_1, HA_3)$ and $v_2 \equiv s(HA_2, HA_3)$. This is shown in Figure 25.1.

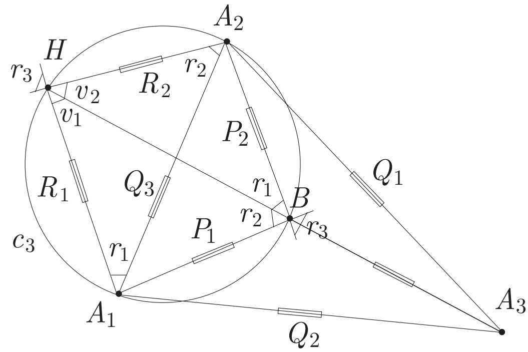


Figure 25.1: Snellius-Pothenot problem

Use the Spread law in $\overline{A_1A_2H}$ to get

$$R_1 = r_2Q_3/r_3 \qquad \text{and} \qquad R_2 = r_1Q_3/r_3. \tag{25.1}$$

The Four point relation applied to the triangle $\overline{A_1A_2A_3}$ with the additional point H is

$$E(Q_1, Q_2, Q_3, R_1, R_2, R_3) = 0.$$

By Exercise 17.5, this is the quadratic equation in R_3 given by

$$\left(R_3 - R_1 - R_2 + Q_3 - Q_1 - Q_2 + \frac{(Q_1 - Q_2)(R_2 - R_1)}{Q_3}\right)^2$$
$$= \frac{A(Q_1, Q_2, Q_3)\, A(R_1, R_2, Q_3)}{4Q_3^2}$$

where A is Archimedes' function. After substituting for the values of R_1 and R_2 from (25.1), this becomes the equation

$$(R_3 - C)^2 = D$$

where

$$C = \frac{(Q_1 + Q_2 + Q_3)(r_1 + r_2 + r_3) - 2(Q_1r_1 + Q_2r_2 + Q_3r_3)}{2r_3}$$

and

$$D = \frac{r_1r_2A(Q_1, Q_2, Q_3)}{r_3}.$$

For either of the two solutions to this equation, the Cross law in $\overline{A_1A_3H}$ gives

$$v_1 = 1 - \frac{(R_1 + R_3 - Q_2)^2}{4R_1R_3}$$

while the Cross law in $\overline{A_2A_3H}$ gives

$$v_2 = 1 - \frac{(R_2 + R_3 - Q_1)^2}{4R_2R_3}.$$

Then the Spread laws in $\overline{A_1BH}$ and $\overline{A_2BH}$ give the required values

$$P_1 = \frac{v_1R_1}{r_2} = \frac{v_1Q_3}{r_3}$$
$$P_2 = \frac{v_2R_2}{r_1} = \frac{v_2Q_3}{r_3}. \quad \blacksquare$$

Example 25.1 Suppose that the triangle $\overline{A_1A_2A_3}$ has points

$$A_1 \equiv [1,1] \qquad A_2 \equiv [5,2] \qquad A_3 \equiv [3,-1]$$

with quadrances

$$Q_1 = 13 \qquad Q_2 = 8 \qquad Q_3 = 17.$$

If B is taken to be the point $[4,5]$ then

$$r_1 = 81/370 \qquad r_2 = 196/925 \qquad r_3 = 169/250.$$

The three values r_1, r_2 and r_3 will be taken as measurements, and the location of B otherwise considered unknown. Then from (25.1)

$$R_1 = r_2Q_3/r_3 \qquad \text{and} \qquad R_2 = r_1Q_3/r_3$$

gives

$$R_1 = 33320/6253 \qquad \text{and} \qquad R_2 = 34425/6253.$$

Now use the Four point relation

$$E(Q_1, Q_2, Q_3, R_1, R_2, R_3) = 0$$

to get the quadratic equation

$$\left(R_3 - \frac{46\,216}{6\,253}\right)^2 = \left(\frac{2\,520}{481}\right)^2$$

with solutions

$$\text{i) } R_3 = 13456/6253 \qquad \text{or} \qquad \text{ii) } R_3 = 78976/6253.$$

i) If $R_3 = 13456/6253$ then

$$v_1 = 1 - \frac{(R_1 + R_3 - Q_2)^2}{4R_1R_3} = \frac{169}{170}$$
$$v_2 = 1 - \frac{(R_2 + R_3 - Q_1)^2}{4R_2R_3} = \frac{169}{425}$$

so that

$$P_1 = \frac{v_1R_1}{r_2} = \frac{v_1Q_3}{r_3} = 25$$
$$P_2 = \frac{v_2R_2}{r_1} = \frac{v_2Q_3}{r_3} = 10.$$

ii) If $R_3 = 78976/6253$ then

$$v_1 = 1 - \frac{(R_1 + R_3 - Q_2)^2}{4R_1R_3} = \frac{33\,124}{52\,445}$$
$$v_2 = 1 - \frac{(R_2 + R_3 - Q_1)^2}{4R_2R_3} = \frac{474\,721}{524\,450}$$

so that

$$P_1 = \frac{v_1R_1}{r_2} = \frac{v_1Q_3}{r_3} = \frac{9\,800}{617}$$
$$P_2 = \frac{v_2R_2}{r_1} = \frac{v_2Q_3}{r_3} = \frac{14\,045}{617}.$$

The first of these cases correctly yields the quadrances to the initial point $B \equiv [4, 5]$. $\diamond$

The two solutions obtained in the previous Example correspond to the two points B and B' that make the same spreads r_1, r_2 and r_3 with the reference triangle $\overline{A_1A_2A_3}$. The relation between these two points may be described by the following known result (see [Wells, page 258]).

Let B be a point not on the lines A_1A_2, A_2A_3 and A_1A_3, and let B_3, B_1 and B_2 be the reflections of B in the lines A_1A_2, A_2A_3 and A_3A_1 respectively. Let c_1, c_2 and c_3 be the respective circumcircles of the triangles $\overline{A_2A_3B_1}, \overline{A_1A_3B_2}$ and $\overline{A_1A_2B_3}$. Then c_1, c_2 and c_3 intersect in a unique point B'.

The map that sends B to B' in the above result is not a bijection. If B is any point on the circumcircle of $\overline{A_1A_2A_3}$, then it turns out that B' is always the orthocenter of $\overline{A_1A_2A_3}$.

Exercise 25.1 Use the Triangle spread rules to identify the correct choice of R_3 in the previous Example. $\diamond$

Exercise 25.2 Find another solution to the resection problem, not using the Four point relation. $\diamond$

25.2 Hansen's problem

Problem 25 (Hansen's problem) Two known points A and B with known quadrance $Q \equiv Q(A,B)$ are sighted from two variable points C and D. The four spreads $s(DA,DB)$, $s(DB,DC)$, $s(CA,CB)$ and $s(CA,CD)$ are measured from the points C and D. The positions of C and D are to be determined, in the sense that the quadrances $Q(A,C)$, $Q(B,C)$, $Q(A,D)$ and $Q(B,D)$ are to be found.

This problem was solved by Hansen (1795-1884), a German astronomer, but according to [Dorrie] also by others before him. The treatment presented here will be illustrated by a particular example. The general case follows the same lines. Assume the quadrance between the fixed points A and B is $Q(A,B) \equiv 26$.

Suppose that the following spreads are known

$$s\left(\overleftrightarrow{ADB}\right) = 361/425 \text{ (ac)} \qquad s\left(\overleftrightarrow{BDC}\right) = 169/250 \text{ (ac)}$$

$$s\left(\overleftrightarrow{BCA}\right) = 441/697 \text{ (ac)} \qquad s\left(\overleftrightarrow{ACD}\right) = 121/410 \text{ (ac)}.$$

This information is shown to scale in Figure 25.2, along with the intersection E of AC and BD.

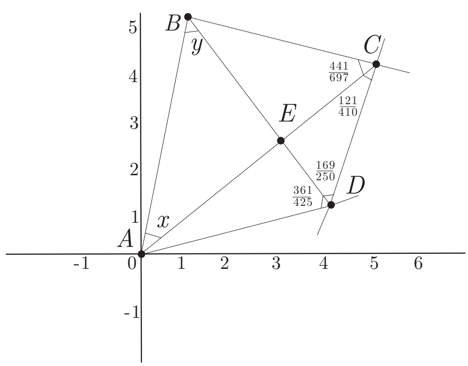


Figure 25.2: Hansen's problem I

Solution. Apply the Triangle spread rules to the three sectors with base D. Note that this new application *inverts the type.* Since

$$361/425 + 169/250 = 6483/4250 \geq 1$$

and both spreads are acute,

$$\begin{aligned} s\left(\overleftrightarrow{ADC}\right) &= r_b\left(\frac{361}{425}, \frac{169}{250}\right) \text{ (ob)} \\ &= \frac{361}{425} + \frac{169}{250} - 2 \times \frac{361}{425} \times \frac{169}{250} \\ &\quad +2\sqrt{\frac{361}{425} \times \frac{169}{250} \times \frac{64}{425} \times \frac{81}{250}} \text{ (ob)} = \frac{121}{170} \text{ (ob)}. \end{aligned}$$

Similarly apply the Triangle spread rules to the sectors with base C. Since

$$441/697 + 121/410 = 6467/6970 \leq 1$$

and both spreads are acute,

$$\begin{aligned} s\left(\overleftrightarrow{BCD}\right) &= r_b\left(\frac{441}{697}, \frac{121}{410}\right) \text{ (ac)} \\ &= \frac{441}{697} + \frac{121}{410} - 2 \times \frac{441}{697} \times \frac{121}{410} \\ &\quad +2\sqrt{\frac{441}{697} \times \frac{121}{410} \times \frac{256}{697} \times \frac{289}{410}} \text{ (ac)} = \frac{169}{170} \text{ (ac)}. \end{aligned}$$

Now apply the Triangle spread rules to $\overline{CDE}$. In this case, no inversion of type takes place. Since

$$169/250 + 121/410 = 4977/5125 \leq 1$$

and both spreads are acute,

$$\begin{aligned} s\left(\overleftrightarrow{DEC}\right) &= r_b\left(\frac{169}{250}, \frac{121}{410}\right) \text{ (ob)} \\ &= \frac{169}{250} + \frac{121}{410} - 2 \times \frac{169}{250} \times \frac{121}{410} \\ &\quad +2\sqrt{\frac{169}{250} \times \frac{121}{410} \times \frac{81}{250} \times \frac{289}{410}} \text{ (ob)} = \frac{1024}{1025} \text{ (ob)}. \end{aligned}$$

The spreads $s\left(\overleftrightarrow{DAC}\right)$ and $s\left(\overleftrightarrow{DBC}\right)$ may now be determined using the same procedure, but an alternative is to use the Two spread triples theorem (page 98) and the function

$$P(a,b,c,d) \equiv \frac{(a-b)^2 - (c-d)^2}{2(a+b-c-d-2ab+2cd)}.$$

Then

$$s\left(\overleftrightarrow{DAC}\right) = P\left(\frac{1024}{1025}, \frac{361}{425}, \frac{121}{170}, \frac{121}{410}\right) = \frac{121}{697}$$

and

$$s\left(\overleftrightarrow{DBC}\right) = P\left(\frac{1024}{1025}, \frac{441}{697}, \frac{169}{250}, \frac{169}{170}\right) = \frac{169}{425}.$$

This information is now summarized in Figure 25.3, with the unknown spreads x and y to be determined.

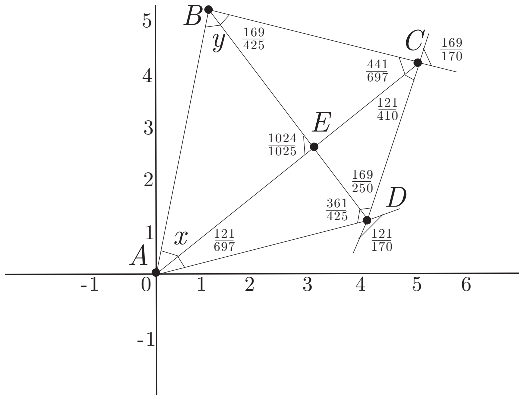


Figure 25.3: Hansen's problem II

The Alternate spreads theorem, extended to a quadrilateral as in Exercise 11.9, gives the formula

$$\frac{169}{425} \times \frac{121}{410} \times \frac{361}{425} \times x = \frac{441}{697} \times \frac{169}{250} \times \frac{121}{697} \times y$$

so that

$$a \equiv \frac{x}{y} = \frac{11\,025}{14\,801}.$$

Now in the notation of the Spread from ratio theorem (page 91), with $s = 1024/1025$,

$$a\,(1-s) = 441/606841 = (21/779)^2$$

so set

$$r \equiv 21/779.$$

Then

$$y = s/\,(a + 1 \pm 2r)$$

and

$$x = ya.$$

Substitute to get the possibilities

$$[x, y] = \left[\frac{441}{1066}, \frac{361}{650}\right]$$

or

$$[x, y] = \left[\frac{112\,896}{256\,537}, \frac{92\,416}{156\,425}\right].$$

The first of these corresponds to the picture above. The Spread law in $\overline{ABD}$ gives

$$\frac{361/425}{26} = \frac{361/650}{Q\,(A, D)}.$$

Thus $Q(A,D) = 17$. The Spread law in $\overline{ABC}$ gives

$$\frac{441/697}{26} = \frac{441/1066}{Q(B,C)}$$

so that $Q(B,C) = 17$. The Spread law in $\overline{ADC}$ gives

$$\frac{121/410}{17} = \frac{121/170}{Q(A,C)}$$

so that $Q(A,C) = 41$. The Spread law in $\overline{BCD}$ gives

$$\frac{169/250}{17} = \frac{169/170}{Q(B,D)}$$

so that $Q(B,D) = 25$. This establishes the four required quantities. ∎

The example was chosen with $A \equiv [0,0]$, $B \equiv [1,5]$, $C \equiv [5,4]$ and $D \equiv [4,1]$, and the validity of each of these computations may thereby be checked.

Exercise 25.3 (Rational version of a notorious problem) The triangle $\overline{A_1A_2A_3}$ represented to scale in Figure 25.4 is isosceles with $Q(A_1,A_3) = Q(A_2,A_3) \equiv 58$ and $Q(A_1,A_2) \equiv 36$. Also known are the spreads

$$s\left(\overleftrightarrow{A_1A_2B_2}\right) \equiv 49/170 \text{ (ac)} \qquad s\left(\overleftrightarrow{A_2A_1B_1}\right) \equiv 64/185 \text{ (ac)}$$

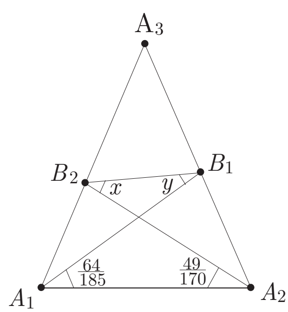


Figure 25.4: A notorious problem

Use the analysis of this section to determine the spreads

$$x = s\left(\overleftrightarrow{A_2B_2B_1}\right) \qquad y = s\left(\overleftrightarrow{A_1B_1B_2}\right).$$

[The answer is

$$\begin{aligned} x &= \frac{9834\,496}{25\,778\,545} \text{ (ac)} \\ y &= \frac{28\,654\,609}{112\,212\,490} \text{ (ac).]} \end{aligned}$$ ◇

26

Platonic solids

Rational trigonometry can be used to understand aspects of the five *Platonic solids*: the (regular) tetrahedron, cube, octahedron, icosahedron and dodecahedron. A more complete investigation involves projective trigonometry, the rational analogue of spherical trigonometry which will be explained in a future volume.

This chapter computes the *face spread* S of each Platonic solid, namely the spread between adjacent faces, as well as some related results. Curiously, the face spreads turn out to be rational numbers in all five cases.

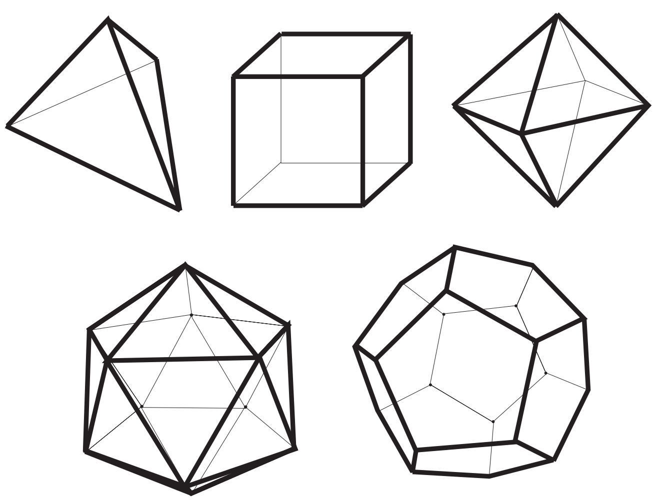


Figure 26.1: The five Platonic solids

26.1 Tetrahedron

The *tetrahedron* has four points, six sides and four faces, each an equilateral triangle. To determine the face spread, suppose that each side of a tetrahedron $\overline{ABCD}$ has quadrance Q, with M the midpoint of the side $\overline{AB}$ as in Figure 26.2. Then by Pythagoras' theorem $Q(C,M) = Q(D,M) = 3Q/4$.

The isosceles triangle $\overline{CMD}$ therefore has quadrances $3Q/4, 3Q/4$ and Q, so the Isosceles triangle theorem (page 122) shows that

$$s \equiv s(MC, MD) = \frac{Q}{3Q/4}\left(1 - \frac{Q}{3Q}\right) = \frac{8}{9}.$$

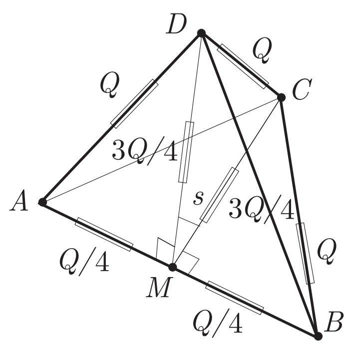


Figure 26.2: Tetrahedron

This is equal to the face spread $S \equiv S(ABC, ABD)$, as can be seen by applying the Perpendicular spreads theorem (page 79) to the plane DCM. The 'three-dimensional sector' formed by these faces towards the interior of the tetrahedron is in a natural sense acute.

Exercise 26.1 Show that the quadrance from one point of the tetrahedron to the centroid of the opposite face is $2Q/3$. Show that the quadrance from one point of the tetrahedron to the center P of the tetrahedron is $3Q/8$. $\diamond$

Exercise 26.2 Show that the face spread $S = 8/9$ is the same as the spread $s(PA, PB)$, where P is the center of the tetrahedron. $\diamond$

Exercise 26.3 By comparing the spread $s = 8/9$ with the appropriate zero of $S_5(s)$, show that it is possible to arrange five solid tetrahedrons sharing a common side. Show that it is not possible to arrange six solid tetrahedrons sharing a common side. $\diamond$

26.2 Cube

The *cube* has eight points, twelve sides and six faces, each a square. Clearly the spread made by adjacent faces is $S = 1$.

Let's consider the problem of determining the possible spreads made by two lines from the center of a cube to two points of the cube.

Suppose that a cube has each side of quadrance Q, and center P with points labelled as in Figure 26.3.

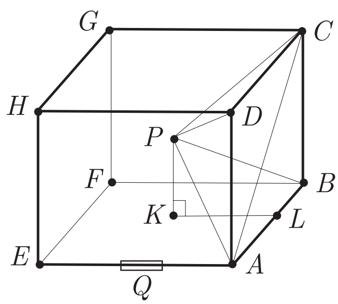


Figure 26.3: Cube

Since $Q(P,K)$, $Q(K,L)$ and $Q(A,L)$ are all equal to $Q/4$, use Pythagoras' theorem to get

$$\begin{aligned} Q(P,L) &= Q/4 + Q/4 = Q/2 \\ Q(P,A) &= Q(P,L) + Q(A,L) = 3Q/4 \\ Q(A,C) &= Q(A,B) + Q(B,C) = 2Q. \end{aligned}$$

The Isosceles triangle theorem applied to $\overline{APC}$, with quadrances $3Q/4$, $3Q/4$ and $2Q$, shows that

$$\begin{aligned} s(AP,AC) &= 1 - \frac{2Q}{4(3Q/4)} = \frac{1}{3} \\ s(PA,PC) &= \frac{2Q}{3Q/4}\left(1 - \frac{2Q}{4(3Q/4)}\right) \\ &= \frac{8}{3} \times \frac{1}{3} \\ &= \frac{8}{9}. \end{aligned}$$

Note that since $\overline{ACFH}$ is a tetrahedron, this latter formula recovers the result of Exercise 26.2.

26.3 Octahedron

The *octahedron* has six points, twelve sides and eight faces, each an equilateral triangle. To determine the face spread S, suppose that the common quadrance of a side is Q, and let M be the midpoint of the side $\overline{BE}$ as in Figure 26.4, so that CM and AM are both perpendicular to BE, and $Q(A, M) = Q(M, C) = 3Q/4$.

Then the isosceles triangle $\overline{ACM}$ has quadrances $3Q/4, 3Q/4$ and $2Q$, so using the Isosceles triangle theorem

$$S \equiv s(MA, MC) = \frac{2Q}{3Q/4}\left(1 - \frac{2Q}{3Q}\right) = \frac{8}{9}.$$

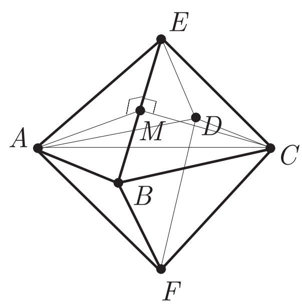


Figure 26.4: Octahedron

While the spread between adjacent faces of the tetrahedron and octahedron thus agree, the former is acute, while the latter is obtuse.

To see the equality directly, observe that the six midpoints of the sides of a tetrahedron form an octahedron. This octahedron can also be obtained by slicing off at each vertex of the tetrahedron a smaller corner tetrahedron as in Figure 26.5. The corner tetrahedron so sliced off shares adjacent faces with the central octahedron, so the face spreads are the same.

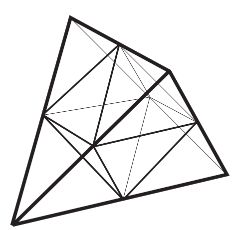


Figure 26.5: Slicing corners off a tetrahedron

26.4 Icosahedron

The *icosahedron* has twelve points, thirty sides and twenty faces, each an equilateral triangle. To determine the face spread S, suppose that V is a point of the icosahedron with adjacent points A, B, C, D and E forming a regular pentagon as in Figure 26.6. Suppose that the common quadrance of a side is Q, and that M is the midpoint of the side $\overline{VE}$, so that DM and MA are perpendicular to VE, and that $Q(D,M) = Q(M,A) = 3Q/4$.

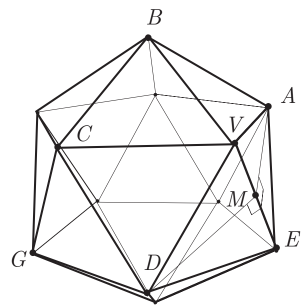


Figure 26.6: Icosahedron

Recall from Exercise 14.3 that $Q(A,D) = \beta Q/\alpha$ where

$$\alpha \equiv \left(5-\sqrt{5}\right)/8 \approx 0.345\,491\ldots \qquad \text{and} \qquad \beta \equiv \left(5+\sqrt{5}\right)/8 \approx 0.904\,508\ldots.$$

Apply the Isosceles triangle theorem to $\overline{ADM}$ with sides $3Q/4, 3Q/4$ and $\beta Q/\alpha$ and some pleasant simplification to get

$$\begin{aligned} S \quad &\equiv \quad s(MD, MA) = \frac{\beta Q/\alpha}{3Q/4}\left(1 - \frac{\beta Q/\alpha}{3Q}\right) \\ &= \quad \frac{4\beta}{3\alpha}\left(1 - \frac{\beta}{3\alpha}\right) = \frac{4}{9}. \end{aligned}$$

Exercise 26.4 Using the same diagram, show that

$$s(MD, MG) = \frac{10 - 2\sqrt{5}}{15}$$

and

$$s(MA, MG) = \frac{10 + 2\sqrt{5}}{15}.$$

Hence deduce that

$$Q(A,G) = \left(\frac{5+\sqrt{5}}{2}\right) Q. \quad \diamond$$

26.5 Dodecahedron

The *dodecahedron* has twenty points, thirty sides and twelve faces, each a regular pentagon. To determine the face spread S, suppose that each side of the dodecahedron has quadrance Q. Three sides meet at every point.

If the point V has adjacent points A, B and C then $\overline{ABC}$ is an equilateral triangle with quadrances $\beta Q/\alpha$, since this is the quadrance of a diagonal side of a regular pentagon of quadrance Q, as in Exercise 14.3. Furthermore the spread $r \equiv s(VA, VB)$ is equal to β, since this is the spread between adjacent lines of a regular pentagon. This is shown in Figure 26.7.

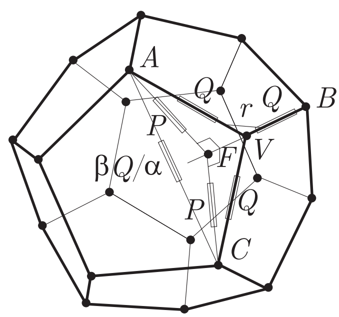


Figure 26.7: Dodecahedron

Now suppose that F is the foot of the altitude from A to BV, and so by symmetry also the foot of the altitude from C to BV. Then using the right triangle $\overline{AFV}$, the quadrance $P \equiv Q(A, F) = Q(C, F)$ is

$$P = rQ(A, V) = \beta Q.$$

In the isosceles triangle $\overline{AFC}$ the quadrances are then $\beta Q, \beta Q$ and $\beta Q/\alpha$. Use the Isosceles triangle theorem and some pleasant simplification to obtain the face spread

$$\begin{aligned} S &\equiv s(FA, FC) = \frac{\beta Q/\alpha}{\beta Q}\left(1 - \frac{\beta Q/\alpha}{4\beta Q}\right) \\ &= \frac{4\alpha - 1}{4\alpha^2} = \frac{4}{5}. \end{aligned}$$

To summarize: the face spreads of the regular tetrahedron, cube, octahedron, icosahedron and dodecahedron are respectively

$$8/9 \qquad 1 \qquad 8/9 \qquad 4/9 \qquad 4/5\ .$$

27

Rational spherical coordinates

One of the important traditional uses of angles and the transcendental trigonometric functions $\cos\theta$ and $\sin\theta$ is to establish polar coordinates in the plane, and spherical and cylindrical coordinates in three-dimensional space. This simplifies problems with rotational symmetry in advanced calculus, mechanics and engineering.

This chapter shows how to employ rational analogues to accomplish the same tasks, with examples chosen from some famous problems in the subject. The rational approach employs conventions that generalize well to higher dimensions.

27.1 Polar spread and quadrance

For a point $A \equiv [x, y]$ in Cartesian coordinates, introduce the **polar spread** s and the **quadrance** Q by

$$\begin{aligned} s &\equiv x^2/\left(x^2+y^2\right) \\ Q &\equiv x^2+y^2. \end{aligned}$$

Then $[s, Q]$ are the **rational polar coordinates** of the point $A \equiv [x, y]$. The spread s is defined between OA and the y *axis*. This convention

- corresponds to the usual practice in surveying and navigation
- integrates more smoothly with higher dimensional generalizations
- is natural for human beings, for whom *up* is more interesting than *right*.

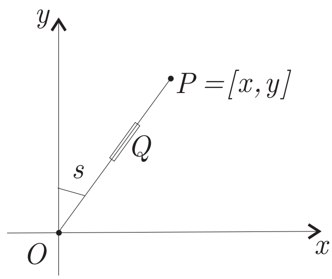


Figure 27.1: Rational polar coordinates

The rational polar coordinates s and Q determine x and y up to sign, so determine A uniquely in the first quadrant. This quadrant is better described by the respective signs of x and y, so call it also the (++)**-quadrant**.

To specify a general point A, the rational coordinates s and Q need to be augmented with two additional bits of information—the signs of x and y respectively. Now

$$\begin{aligned} x^2 &= sQ \\ y^2 &= (1-s)\,Q. \end{aligned} \qquad (27.1)$$

Take differentials of these two relations to obtain

$$\begin{aligned} 2x\,dx &= Q\,ds + s\,dQ \\ 2y\,dy &= -Q\,ds + (1-s)\ dQ. \end{aligned}$$

Thus in the (++)-quadrant

$$\begin{aligned} 4xy\,dx\,dy &= \begin{vmatrix} Q & s \\ -Q & 1-s \end{vmatrix} ds\,dQ \\ &= \begin{vmatrix} Q & s \\ 0 & 1 \end{vmatrix} ds\,dQ = Q\,ds\,dQ. \end{aligned} \qquad (27.2)$$

For future reference, note that the determinant is evaluated by adding the first row to the second to get a diagonal matrix. In the (++)-quadrant, use (27.1) to obtain

$$xy = \sqrt{s\,(1-s)}\,Q$$

so the element of area is

$$dx\,dy = \frac{1}{4\sqrt{s\,(1-s)}}\,ds\,dQ. \qquad (27.3)$$

Example 27.1 The area a of the central circle of quadrance K is, by symmetry,

$$a = 4\int_0^K \int_0^1 \frac{1}{4\sqrt{s(1-s)}}\, ds\, dQ = K\int_0^1 \frac{1}{\sqrt{s(1-s)}}\, ds.$$

This is not an integral which can be evaluated explicitly using basic calculus, motivating the definition of the number

$$\pi = \int_0^1 \frac{1}{\sqrt{s(1-s)}}\, ds. \tag{27.4}$$

So the area of the central circle of quadrance K is πK. $\diamond$

Exercise 27.1 Use the substitutions $s \equiv r^2$ and $s \equiv 1/t$ to show that

$$\pi = 2\int_0^1 \frac{dr}{\sqrt{1-r^2}} = \int_1^\infty \frac{dt}{t\sqrt{t-1}}.$$

Then use the substitutions $r \equiv 2u/\left(1+u^2\right)$ and $v \equiv 1/u$ to show that

$$\pi = 4\int_0^1 \frac{du}{1+u^2} = 4\int_1^\infty \frac{dv}{1+v^2}. \quad \diamond$$

Example 27.2 A lemniscate of Bernoulli has Cartesian equation

$$\left(x^2+y^2\right)^2 = x^2 - y^2 \tag{27.5}$$

and polar equation

$$r^2 = \cos 2\theta.$$

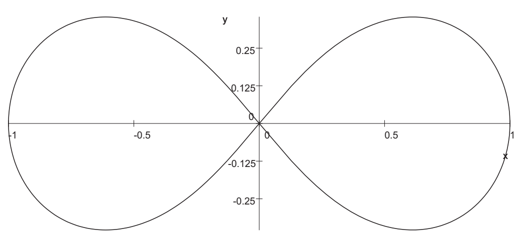


Figure 27.2: Lemniscate of Bernoulli

Replace x^2 and y^2 in (27.5) using (27.1) to get

$$\begin{aligned} Q^2 &= sQ - (1-s)\,Q \\ &= (2s-1)\,Q. \end{aligned}$$

So either of

$$Q = 2s - 1 \qquad \text{or} \qquad s = (Q+1)/2$$

is a **rational polar equation** of the lemniscate. Rational polar equations of some other classical curves are described in Appendix 1. For the lemniscate the polar spread varies in the range $1/2 \leq s \leq 1$, so the area is

$$\begin{aligned} a &= 4\int_{1/2}^{1}\int_{0}^{2s-1} \frac{1}{4\sqrt{s(1-s)}}\, dQ\, ds \\ &= \int_{1/2}^{1} \frac{2s-1}{\sqrt{s(1-s)}}\, ds = \int_{0}^{1/4} \frac{1}{\sqrt{u}}\, du = 1. \ \diamond \end{aligned}$$

Example 27.3 The integral $I = \int_0^\infty e^{-x^2}\, dx$ is difficult to evaluate using only the calculus of one variable. Using rational polar coordinates, the idea is as follows, where the integral is over the (++)-quadrant.

$$\begin{aligned} I^2 &= \int_0^\infty e^{-x^2}\, dx \int_0^\infty e^{-y^2}\, dy \\ &= \int_0^\infty \int_0^\infty e^{-\left(x^2+y^2\right)}\, dx\, dy \\ &= \int_0^\infty \int_0^1 \frac{e^{-Q}}{4\sqrt{s(1-s)}}\, ds\, dQ \\ &= \int_0^\infty e^{-Q}\, dQ \int_0^1 \frac{1}{4\sqrt{s(1-s)}}\, ds \\ &= \left[-e^{-Q}\right]_{Q=0}^{\infty} \times \pi/4 \\ &= \pi/4 \end{aligned}$$

so that $I = \sqrt{\pi}/2$. $\diamond$

The rotationally invariant measure $d\mu$ on the circle of quadrance $Q = r^2$ is, since $dQ = 2r\, dr$, determined by the equation

$$dx\, dy = d\mu\, dr = \frac{d\mu\, dQ}{2r}.$$

Compare this with (27.3) to see that

$$d\mu = \frac{r}{2\sqrt{s(1-s)}}\, ds.$$

It follows that the quarter of the central circle of radius r in the (++)-quadrant has measure $\pi r/2$, and the full circle has measure $2\pi r$.

27.2 Evaluating $\pi^2/16$

The unit quarter circle has area $\pi/4$, so a squared area of

$$\pi^2/16 \approx 0.616\,850\,275\,068\ldots.$$

To evaluate this constant, we follow ideas of Archimedes. Approximate a quarter circle successively by first one, then two, then four isosceles triangles, and so on, each time subdividing each triangle into two by a vertex bisector, as shown in Figure 27.3.

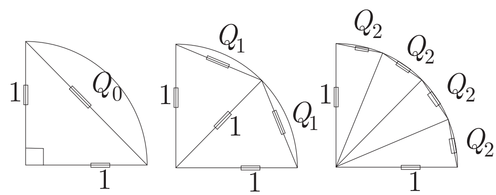


Figure 27.3: Approximations to a quarter circle

By the Quadrea spread theorem (page 82), the quadrea of an isosceles triangle with $Q_1 = Q_2 \equiv 1$ and spread $s_3 \equiv s$ is $\mathcal{A} = 4s$. After n divisions there are 2^n congruent isosceles triangles, each with spread s_n at the common point, and hence each with quadrea $4s_n$. This gives for the resulting $(2^n + 2)$-gon a total quadrea of $\mathcal{A}_n = (2^n)^2 \times 4s_n$, and so a squared area of $a_n^2 = \mathcal{A}_n/16 = 2^{2n-2}s_n$. Now since

$$s_{n+1} = \frac{1 - \sqrt{1 - s_n}}{2}$$

it follows that

$$\begin{aligned} a_{n+1}^2 &= 2^{2n}s_{n+1} = 2^{2n-1}\left(1 - \sqrt{1 - s_n}\right) \\ &= 2^{2n-1}\left(1 - 2^{-n+1}\sqrt{2^{2n-2} - a_n^2}\right) \\ &= 2^{2n-1} - 2^n\sqrt{2^{2n-2} - a_n^2}. \end{aligned}$$

Surprisingly, this recurrence relation yields a pleasant form for the general term a_n^2, as indicated by the following computations.

$$\begin{aligned} a_0^2 &= 2^{-2} = 0.25 \\ a_1^2 &= 2^{-1} - 2^0\sqrt{2^{-2} - 2^{-2}} = 2^{-1} = 0.5 \\ a_2^2 &= 2^1 - 2^1\sqrt{2^0 - 2^{-1}} = 2 - \sqrt{2} \approx 0.585\,786 \\ a_3^2 &= 2^3 - 2^2\sqrt{2^2 - 2 + \sqrt{2}} = 8 - 4\sqrt{2 + \sqrt{2}} \approx 0.608\,964 \\ a_4^2 &= 2^5 - 2^3\sqrt{2^4 - \left(8 - 4\sqrt{2 + \sqrt{2}}\right)} = 32 - 16\sqrt{2 + \sqrt{2 + \sqrt{2}}} \approx 0.614\,871 \end{aligned}$$

Exercise 27.2 Show that this pattern continues, giving a closed expression for a_n^2. $\diamond$

27.3 Beta function

Following Euler, for decimal numbers $p > 0$ and $q > 0$ define the **Beta function,** or **Beta integral**,

$$B(p,q) \equiv \int_0^1 s^{p-1}(1-s)^{q-1}\,ds.$$

There is a standard expression for the Beta function in terms of the **Gamma function** defined for $t > 0$ by

$$\Gamma(t) \equiv \int_0^\infty e^{-u}u^{t-1}du = 2\int_0^\infty e^{-x^2}x^{2t-1}dx.$$

Integration by parts and direct calculation shows that

$$\begin{aligned}\Gamma(t+1) &= t\,\Gamma(t)\\ \Gamma(1) &= 1.\end{aligned}$$

This implies that

$$\Gamma(n) = (n-1)!$$

for any positive integer $n \geq 1$.

Use rational polar coordinates to rewrite the following integral over the (++)-quadrant

$$\begin{aligned}\Gamma(p)\,\Gamma(q) &= 4\int_0^\infty e^{-x^2}x^{2p-1}dx\int_0^\infty e^{-y^2}y^{2q-1}dy\\ &= \int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}x^{2(p-1)}y^{2(q-1)}\,4xy\,dx\,dy\\ &= \int_0^\infty\int_0^1 e^{-Q}(sQ)^{p-1}((1-s)\,Q)^{q-1}\,Q\,ds\,dQ\\ &= \int_0^\infty e^{-Q}Q^{p+q-1}dQ\int_0^1 s^{p-1}(1-s)^{q-1}\,ds\\ &= \Gamma(p+q)\,B(p,q)\end{aligned}$$

where (27.2) was used to go from the second to the third line. Thus

$$B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}. \tag{27.6}$$

Values of the Beta function are particularly useful in calculations involving rational polar or spherical coordinates. Note that in particular

$$B(1/2,1/2) = \pi = (\Gamma(1/2))^2$$

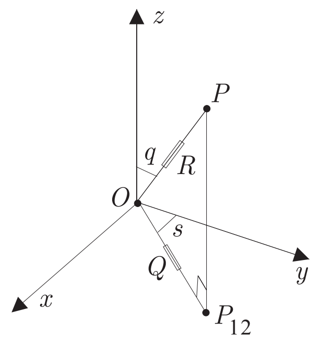
so that, recovering the computation of Example 27.3,

$$\Gamma(1/2) = 2\int_0^\infty e^{-x^2}dx = \sqrt{\pi}.$$

27.4 Rational spherical coordinates

Represent a point in three-dimensional space by $A \equiv [x, y, z]$, and define the **rational spherical coordinates** $[s, q, R]$ of A by

$$\begin{aligned} s &\equiv x^2 / \left(x^2 + y^2\right) \\ q &\equiv \left(x^2 + y^2\right) / \left(x^2 + y^2 + z^2\right) \\ R &\equiv x^2 + y^2 + z^2. \end{aligned}$$



Geometrically if $A_{12} = [x, y, 0]$ is the perpendicular projection of A onto the $x - y$ plane, then s is the polar spread between OA_{12} and the y axis, while the **second polar spread** q is the spread between OA and the z axis. Then R is the **three-dimensional quadrance**, and $Q \equiv x^2 + y^2 = qR$.

Then

$$x^2 = sqR \qquad y^2 = (1 - s)\, qR \qquad z^2 = (1 - q)\, R \tag{27.7}$$

so that x, y and z are determined, up to sign, by $[s, q, R]$. Take differentials to obtain

$$\begin{aligned} 2x\, dx &= qR\, ds + sR\, dq + sq\, dR \\ 2y\, dy &= -qR\, ds + (1 - s)\, R\, dq + (1 - s)\, q\, dR \\ 2z\, dz &= 0\, ds - R\, dq + (1 - q)\ dR. \end{aligned}$$

Thus in the $(+++)$-octant, where the signs of x, y and z are all positive,

$$\begin{aligned} 8xyz\, dx\, dy\, dz &= \begin{vmatrix} qR & sR & sq \\ -qR & (1 - s)\, R & (1 - s)\, q \\ 0 & -R & 1 - q \end{vmatrix} ds\, dq\, dR \\ &= \begin{vmatrix} qR & sR & sq \\ 0 & R & q \\ 0 & 0 & 1 \end{vmatrix} ds\, dq\, dR = qR^2\, ds\, dq\, dR \end{aligned}$$

where the determinant is evaluated by adding the first row to the second, and then the second row to the third, to obtain a diagonal matrix.

In the $(+++)$-octant, combine the equations of (27.7) to obtain

$$xyz = R^{3/2} q \sqrt{s\,(1 - s)\,(1 - q)}$$

so the element of volume is

$$dx\, dy\, dz = \frac{\sqrt{R}}{8\sqrt{s\,(1 - s)\,(1 - q)}}\, ds\, dq\, dR. \tag{27.8}$$

Example 27.4 The volume v of the central sphere of quadrance $K \equiv k^2$ $(k \geq 0)$ is eight times the volume in the $(+++)$-octant. It is thus

$$\begin{aligned} v &= 8 \int_0^K \int_0^1 \int_0^1 \frac{\sqrt{R}}{8\sqrt{s(1-s)(1-q)}}\, ds\, dq\, dR \\ &= \int_0^1 \frac{ds}{\sqrt{s(1-s)}} \int_0^1 \frac{dq}{\sqrt{1-q}} \int_0^K \sqrt{R}\, dR \\ &= \pi \left[-2\sqrt{1-q}\right]_{q=0}^1 \left[\frac{2R^{\frac{3}{2}}}{3}\right]_{R=0}^K = \frac{4\pi K^{\frac{3}{2}}}{3} = \frac{4\pi k^3}{3}. \end{aligned}$$

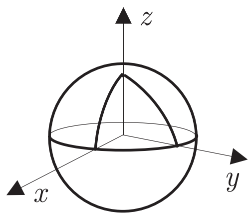


Figure 27.4: Volume of a sphere $\diamond$

Example 27.5 An 'ice cream cone' lies above the cone $z^2 = x^2 + y^2$, and inside the projective sphere $x^2 + y^2 + z^2 = z$ centered at $[0, 0, 1/2]$ with quadrance $1/4$. Write the cone as $q = 1/2$ and the sphere as $R = 1 - q$, so that the volume v is

$$\begin{aligned} v &= 4 \int_0^1 \int_0^{1/2} \int_0^{1-q} \frac{\sqrt{R}}{8\sqrt{s(1-s)(1-q)}}\, dR\, dq\, ds \\ &= \frac{\pi}{2} \int_0^{1/2} \frac{1}{\sqrt{1-q}} \left[\frac{2R^{3/2}}{3}\right]_{R=0}^{1-q} dq \\ &= \frac{\pi}{3} \int_0^{1/2} (1-q)\, dq = \frac{\pi}{3}\left[q - \frac{q^2}{2}\right]_{q=0}^{1/2} = \frac{\pi}{8}. \end{aligned}$$

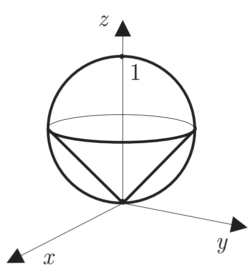


Figure 27.5: Volume of an ice cream cone $\diamond$

Example 27.6 To find the volume v of the spherical cap inside the sphere $x^2+y^2+z^2=K\equiv k^2$ $(k\geq 0)$ and lying above the plane $z=d\geq 0$, where $d\leq k$, use **rational cylindrical coordinates** $[s,Q,z]$

$$\begin{aligned}
v &= 4\int_0^1\int_0^{K-d^2}\int_d^{\sqrt{K-Q}}\frac{1}{4\sqrt{s\,(1-s)}}\,dz\,dQ\,ds \\
&= \pi\int_0^{K-d^2}\left(\sqrt{K-Q}-d\right)dQ=\pi\left[-2\,(K-Q)^{3/2}/3-Qd\right]_{Q=0}^{K-d^2} \\
&= \frac{\pi}{3}\left(d^3-3dk^2+2k^3\right)=\frac{\pi}{3}\,(k-d)^2\,(2k+d)\,.
\end{aligned}$$

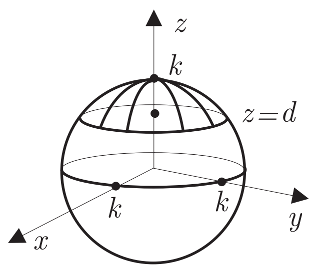


Figure 27.6: Volume of spherical cap $\diamond$

Example 27.7 The volume of the spherical ring remaining when a cylinder with axis the z-axis is removed from the central sphere of quadrance $K\equiv k^2$ $(k\geq 0)$, leaving a solid bounded by the planes $z=d$ and $z=-d$, where $d\leq k$, is

$$\begin{aligned}
v &= 8\int_0^1\int_{K-d^2}^{K}\int_0^{\sqrt{K-Q}}\frac{1}{4\sqrt{s\,(1-s)}}\,dz\,dQ\,ds=2\pi\int_{K-d^2}^{K}\sqrt{K-Q}\,dQ \\
&= 2\pi\left[-2\,(K-Q)^{3/2}/3\right]_{Q=K-d^2}^{K}=\frac{4\pi}{3}d^3.
\end{aligned}$$

Curiously, this is independent of the quadrance K of the sphere.

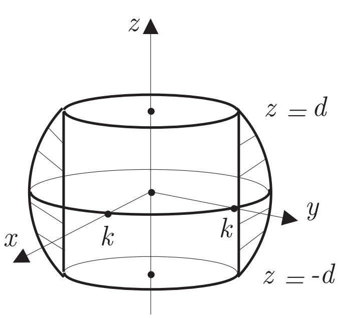


Figure 27.7: Volume of spherical ring $\diamond$

Example 27.8 To find the volume v above the paraboloid $z = x^2 + y^2$ and below the plane $z = r \geq 0$

$$v = 4 \int_0^1 \int_0^r \int_R^r \frac{1}{4\sqrt{s(1-s)}}\, dz\, dR\, ds = \pi \int_0^r (r - R)\; dR = \frac{\pi r^2}{2}.$$

As discovered by Archimedes, this is one half of the volume of the cylinder of height r and radius $\sqrt{r}$. $\diamond$

Example 27.9 The moment M_{xy} of the upper hemisphere of the unit sphere of density 1 and mass $M \equiv 2\pi/3$ with respect to the $xy-$plane is

$$M_{xy} = 4 \int_0^1 \int_0^1 \int_0^1 \frac{z\sqrt{R}}{8\sqrt{s(1-s)(1-q)}}\, ds\, dq\, dR$$

where $z = \sqrt{(1-q)\,R}$. Thus

$$M_{xy} = \frac{\pi}{2} \times 1 \times \int_0^1 R\, dR = \frac{\pi}{4}$$

and the centroid has z coordinate $\overline{z} \equiv M_{xy}/M = 3/8$, so is $[0, 0, 3/8]$.

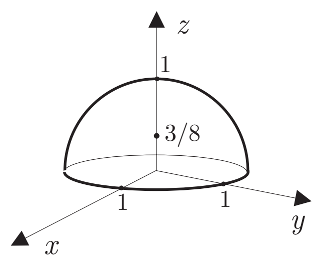


Figure 27.8: Center of mass of upper hemisphere $\diamond$

Example 27.10 The moment of inertia of the solid unit ball around the z axis is

$$\begin{aligned} I_z &= 8 \int_0^1 \int_0^1 \int_0^1 \frac{qR\sqrt{R}}{8\sqrt{s(1-s)(1-q)}}\, ds\, dq\, dR \\ &= \pi \int_0^1 R^{3/2} dR \int_0^1 \frac{q}{\sqrt{1-q}}\, dq \\ &= \pi \times \frac{2}{5} \times B\left(2, \frac{1}{2}\right) = \pi \times \frac{2}{5} \times \frac{4}{3} = \frac{8\pi}{15} \end{aligned}$$

since from (27.6)

$$B\left(2, \frac{1}{2}\right) = \frac{\Gamma(2)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{2}\right)} = \frac{\Gamma\left(\frac{1}{2}\right)}{\frac{3}{2} \times \Gamma\left(\frac{3}{2}\right)} = \frac{\Gamma\left(\frac{1}{2}\right)}{\frac{3}{2} \times \frac{1}{2} \times \Gamma\left(\frac{1}{2}\right)} = \frac{4}{3}. \quad \diamond$$

Example 27.11 The volume v of the hyperbolic cap shown in Figure 27.9, above the top sheet of the hyperboloid $z^2 - x^2 - y^2 = K \equiv k^2$ $(k \geq 0)$ and below the plane $z = d \geq 0$, where $d \geq k$, is

$$\begin{aligned} v &= 4\int_0^1 \int_0^{d^2-K} \int_{\sqrt{Q+K}}^{d} \frac{1}{4\sqrt{s(1-s)}}\, dz\, dQ\, ds = \pi \int_0^{d^2-K} \left(d - \sqrt{Q+K}\right) dQ \\ &= \pi \left[Qd - 2(Q+K)^{3/2}/3\right]_{Q=0}^{d^2-K} = \frac{\pi}{3}(k-d)^2(2k+d). \end{aligned}$$

This is the same formula as the volume of a spherical cap in Example 27.6!

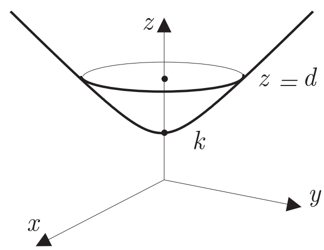


Figure 27.9: Volume of hyperbolic cap $\diamond$

Example 27.12 The volume of the hyperbolic ring shown in Figure 27.10 inside a cylinder with axis the z-axis and outside the hyperboloid of one sheet $x^2 + y^2 - z^2 = K \equiv k^2$ $(k \geq 0)$ bounded by the planes $z = d$ and $z = -d$ is

$$\begin{aligned} v &= 8\int_0^1 \int_K^{d^2+K} \int_0^{\sqrt{Q-K}} \frac{1}{4\sqrt{s(1-s)}}\, dz\, dQ\, ds = 2\pi \int_K^{d^2+K} \sqrt{Q-K}\, dQ \\ &= 2\pi \left[2(Q-K)^{3/2}/3\right]_{Q=K}^{d^2+K} = \frac{4\pi}{3} d^3. \end{aligned}$$

Curiously, this is independent of K, and is the same as the volume of the spherical ring in Example 27.7!

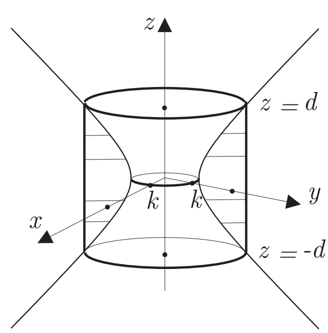


Figure 27.10: Volume of hyperbolic ring $\diamond$

27.5 Surface measure on a sphere

For a fixed value K of R, the polar spreads s and q parametrize that part of the surface of the sphere of quadrance K contained in the $(+++)$-octant. To describe the full sphere these two spreads must be augmented by three additional bits of information, namely the signs of x, y and z.

The rotationally invariant surface measure $d\nu$ on the sphere $R \equiv r^2$ is, since $dR = 2r\,dr$, determined by

$$dx\,dy\,dz = d\nu\,dr = d\nu\,dR\,/2r.$$

Compare this with (27.8) to get

$$\frac{\sqrt{R}}{8\sqrt{s\,(1-s)\,(1-q)}}\,ds\,dq\,dR = \frac{1}{2\sqrt{R}}\,d\nu\,dR.$$

Thus

$$d\nu = \frac{R\,ds\,dq}{4\sqrt{s\,(1-s)\,(1-q)}}.$$

Example 27.13 The total surface area a of the sphere of quadrance $K \equiv k^2$ is

$$\begin{aligned} a &= 8\int_0^1\int_0^1 \frac{K}{4\sqrt{s\,(1-s)\,(1-q)}}\,ds\,dq \\ &= 2K\int_0^1 \frac{ds}{\sqrt{s\,(1-s)}}\int_0^1 \frac{dq}{\sqrt{1-q}} \\ &= 2K \times \pi \times 2 = 4\pi K = 4\pi k^2.\ \diamond \end{aligned}$$

Example 27.14 The surface area a of the spherical cap of the sphere $x^2+y^2+z^2 = K \equiv k^2$ $(k \geq 0)$ lying above the plane $z = d \geq 0$, where $d \leq k$, as shown in Figure 27.6, is

$$\begin{aligned} a &= 4\int_0^{(K-d^2)/K}\int_0^1 \frac{K}{4\sqrt{s\,(1-s)\,(1-q)}}\,ds\,dq \\ &= \pi K\left[-2\,(1-q)^{1/2}\right]_{q=0}^{(K-d^2)/K} \\ &= 2\pi k^2\left(1-\frac{d}{k}\right). \end{aligned}$$

The linear dependence of this expression on d is one of the most remarkable properties of a sphere, and is responsible for the fact that an egg slicer subdivides a sphere into strips of constant surface area. This fact is also important for harmonic analysis on a sphere, and for the representation theory of the rotation group. $\diamond$

27.6 Four dimensional rational spherical coordinates

For a point $A \equiv [x, y, z, w]$ in four dimensional space define

$$\begin{aligned} s &\equiv x^2/\left(x^2+y^2\right) \\ q &\equiv \left(x^2+y^2\right)/\left(x^2+y^2+z^2\right) \\ r &\equiv \left(x^2+y^2+z^2\right)/\left(x^2+y^2+z^2+w^2\right) \\ T &\equiv x^2+y^2+z^2+w^2. \end{aligned}$$

Then T is the four-dimensional quadrance, and r is the **third polar spread** between OA and the new (fourth) $w-$axis. Then

$$\begin{aligned} x^2 &= sqrT \\ y^2 &= (1-s)\,qrT \\ z^2 &= (1-q)\,rT \\ w^2 &= (1-r)\,T. \end{aligned} \tag{27.9}$$

Take differentials and follow the established pattern to get

$$\begin{aligned} 16\,xyzw\,dx\,dy\,dz\,dw &= \begin{vmatrix} qrT & srT & sqT & sqr \\ -qrT & (1-s)\,rT & (1-s)\,qT & (1-s)\,qr \\ 0 & -rT & (1-q)\,T & (1-q)\,r \\ 0 & 0 & -T & 1-r \end{vmatrix} ds\,dq\,dr\,dT \\ &= \begin{vmatrix} qrT & srT & sqT & sqr \\ 0 & rT & qT & qr \\ 0 & 0 & T & r \\ 0 & 0 & 0 & 1 \end{vmatrix} ds\,dq\,dr\,dT \\ &= qr^2T^3\,ds\,dq\,dr\,dT. \end{aligned}$$

In the $(++++)$-octant, (27.9) yields

$$xyzw = s^{1/2}q\,r^{3/2}T^2\sqrt{(1-s)\,(1-q)\,(1-r)}$$

so the element of content (four dimensional version of volume) is

$$dx\,dy\,dz\,dw = \frac{\sqrt{r}\,T}{16\sqrt{s\,(1-s)\,(1-q)\,(1-r)}}\,ds\,dq\,dr\,dT.$$

Example 27.15 The central sphere of quadrance $K \equiv k^2$ has content

$$\begin{aligned} c &= 16\int_0^K\int_0^1\int_0^1\int_0^1 \frac{\sqrt{r}\,T}{16\sqrt{s\,(1-s)\,(1-q)\,(1-r)}}\,ds\,dq\,dr\,dT \\ &= \int_0^1 \frac{ds}{\sqrt{s\,(1-s)}}\int_0^1 \frac{dq}{\sqrt{1-q}}\int_0^1 \frac{\sqrt{r}\,dq}{\sqrt{1-r}}\int_0^K T\,dT \\ &= \pi\times 2\times B\left(\frac{3}{2},\frac{1}{2}\right)\times\frac{K^2}{2}. \end{aligned}$$

But

$$B\left(\frac{3}{2},\frac{1}{2}\right)=\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(2\right)}=\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{1}=\frac{\pi}{2}$$

so that

$$c=\frac{\pi^2K^2}{2}=\frac{\pi^2k^4}{2}. \quad \diamond$$

If $d\nu$ denotes spherical surface measure on the unit 3-sphere determined by

$$dx\,dy\,dz\,dw=d\nu\,dT\,/2$$

then (since $T=1$)

$$d\nu=\frac{\sqrt{r}}{8\sqrt{s\left(1-s\right)\left(1-q\right)\left(1-r\right)}}\,ds\,dq\,dr.$$

Exercise 27.3 Use this to show that the surface volume of the unit 3-sphere is $2\pi^2$. $\diamond$

It should now be clear how to extend rational spherical coordinates to higher dimensions. In n-dimensional space, rational spherical coordinates involve $(n-1)$ polar spreads, and one quadrance. The basic relations are algebraic, and so do not require an understanding and visualization of projections.

27.7 Conclusion

Congratulations on having made it this far—hopefully without too much cheating!

This book is only a beginning, and much remains to be done. Hundreds of classical results of Euclidean geometry may be generalized to the universal setting. A coherent and precise framework for three-dimensional geometry should be created. The number theoretical and combinatorial implications of metrical geometry over finite fields requires investigation, as do the spread polynomials along with other related special functions. Researchers should ponder the opportunities in regarding algebraic geometry as an essentially metrical theory. Many additional applications should be developed and tested, both in applied and pure mathematics. Physicists might enjoy speculating about the implications for their subject.

Rational analogues of spherical and hyperbolic geometries will be described in a future book, along with the remarkable synthesis of Euclidean and non-Euclidean geometries called *chromogeometry*.

But perhaps the most exciting possibility of all is to re-evaluate the mathematics taught (and not taught) in schools and colleges, and to think about ways of presenting to young people this simpler and more logical approach to trigonometry and geometry.

Appendix A

Rational polar equations of curves

Recall that the relationships between the Cartesian coordinates $[x, y]$ and the rational polar coordinates $[s, Q]$ of spread and quadrance are given by

$$s = x^2 / \left(x^2 + y^2\right) \qquad Q = x^2 + y^2$$
$$x^2 = sQ \qquad y^2 = (1 - s)\, Q.$$

In this Appendix, some well known curves are listed, together with the usual Cartesian and polar forms, as well as new rational polar forms involving s and Q. As in the case of both Cartesian and polar coordinates, rational polar coordinates will have the most pleasant form only when the position of the curve is suitably chosen. For example, both the Cartesian and polar equations of the ellipse become more complicated if the ellipse is rotated and/or translated.

With rational polar coordinates it often becomes convenient to express s as a function of Q, not the other way around as the usual polar situation might suggest. Surprisingly, many diverse curves seem to have rational polar equations of a somewhat similar form, typically a quadratic equation in s. This occurs particularly frequently when the equation of the curve is even. This phenomenon should be explained.

It is important to note that the rational polar forms of these curves have an enormous advantage over the usual polar forms for pure mathematics—they allow extensions of these curves to general fields. In the examples below, we adopt the notational convention that $A \equiv a^2$ and $B \equiv b^2$. The Cartesian and polar forms for these classical curves are taken from *A catalog of special plane curves* [Lawrence] and *A book of curves* [Lockwood].

The derivations of the rational polar equations are left to the reader; they are often interesting. Of course there are many additional curves to investigate.

Line The *line* has Cartesian equation $y = ax$ and polar equation $\tan\theta = a$. Using rational polar coordinates its equation is

$$s = \frac{1}{1+A}.$$

Ellipse The *ellipse* has Cartesian equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and polar equation

$$r = \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}.$$

Using rational polar coordinates its equation is

$$s = \frac{A(Q-B)}{Q(A-B)}.$$

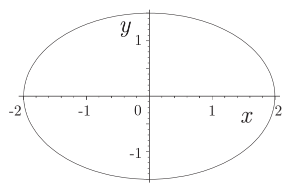


Figure A.1: Ellipse

Hyperbola The *hyperbola* has Cartesian equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

and polar equation

$$r = \frac{ab}{\sqrt{b^2\cos^2\theta - a^2\sin^2\theta}}.$$

Using rational polar coordinates its equation is

$$s = \frac{A(Q+B)}{Q(A+B)}.$$

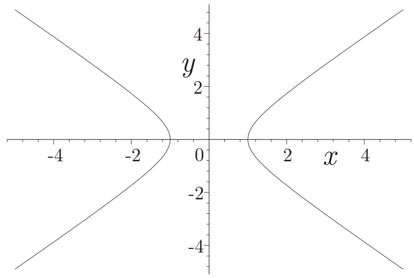


Figure A.2: Hyperbola

Parabola The *parabola* has Cartesian equation

$$y^2 = 4ax$$

and polar equation

$$\frac{2a}{r} = 1 - \cos\theta.$$

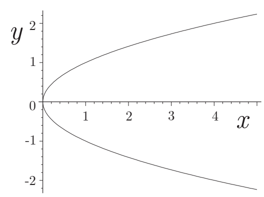


Figure A.3: Parabola

Using rational polar coordinates its equation is

$$(1-s)^2\, Q = 16As$$

or

$$\left(s - \frac{Q+8A}{Q}\right)^2 = \frac{16A\,(Q+4A)}{Q^2}.$$

Cardioid The *cardioid* has polar equations of the form

$$r = 2a\,(1+\cos\theta)$$
$$r = 2a\,(1-\cos\theta)$$

with the following respective graphs.

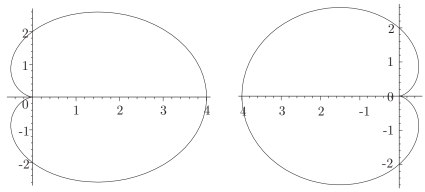


Figure A.4: Two cardioids

Both cases are covered by the rational polar equation

$$\left(s - \frac{Q+4A}{4A}\right)^2 = \frac{Q}{A}.$$

This results in the 'symmetric cardioid' shown in Figure A.5.

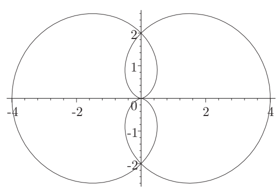


Figure A.5: Symmetric cardioid

Limacon The *limacon* is a generalization of the cardioid, and has polar equation

$$r = 2a\cos\theta + b.$$

Using rational polar coordinates its equation is

$$\left(s - \frac{Q+B}{4A}\right)^2 = \frac{BQ}{4A^2}.$$

If $B = 4A$ then this reduces to a cardioid. If $B = A$ then this is the *trisectrix* with rational polar equation

$$\left(s+\frac{(Q+A)}{4A}\right)^2=\frac{Q}{4A}.$$

Figure A.6 shows a graph of a trisectrix.

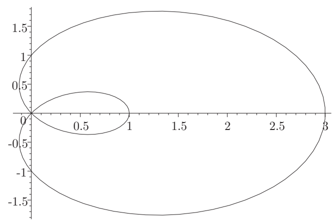


Figure A.6: Trisectrix

Astroid The *astroid* has Cartesian equation

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$$

or

$$\left(x^2+y^2-a^2\right)^3+27a^2x^2y^2=0.$$

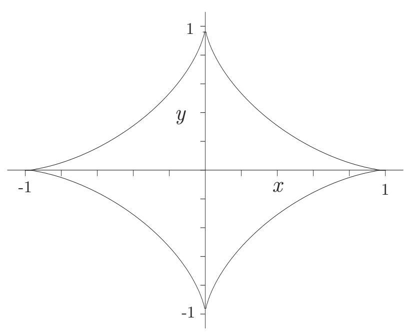


Figure A.7: Astroid

Using rational polar coordinates its equation is

$$\left(s-\frac{1}{2}\right)^2=\frac{1}{4}-\frac{(A-Q)^3}{27AQ^2}.$$

Eight curve The *eight curve*, or *lemniscate of Gerono*, has Cartesian equation

$$x^4 = a^2\left(x^2 - y^2\right)$$

and polar equation

$$r^2 = a^2 \sec^4\theta \cos 2\theta.$$

Using rational polar coordinates its equation is

$$\left(s - \frac{A}{Q}\right)^2 = \frac{(A-Q)\,A}{Q^2}.$$

Bullet nose The *bullet nose* has Cartesian equation

$$\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$$

and polar equation

$$r^2 \sin^2\theta \cos^2\theta = a^2 \sin^2\theta - b^2\cos^2\theta.$$

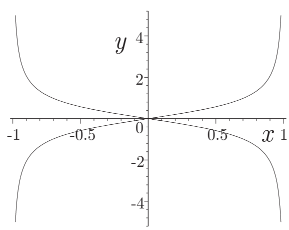


Figure A.8: Bullet nose

Using rational polar coordinates its equation is

$$\left(s - \frac{A+B+Q}{2Q}\right)^2 = \frac{\left(A+B+Q\right)^2 - 4AQ}{Q^2}.$$

Deltoid The *deltoid* has the Cartesian equation

$$\left(x^2+y^2\right)^2 - 8ax\left(x^2-3y^2\right) + 18a^2\left(x^2+y^2\right) = 27a^4.$$

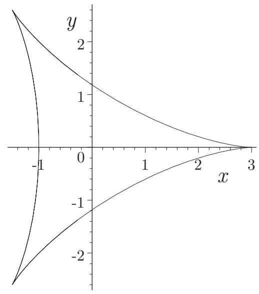


Figure A.9: Deltoid

Using rational polar coordinates its equation is

$$s\,(3-4s)^2=\frac{\left(Q^2+18AQ-27A^2\right)^2}{64Q^3A}.$$

The three-fold symmetry of the curve is reflected in the appearance of the third spread polynomial $S_3\,(s)=s\,(3-4s)^2$.

Hippopede The *hippopede,* or *horse fetter,* (Proclus, 75 BC) has Cartesian equation

$$\left(x^2+y^2\right)^2+4b^2\left(b^2-a^2\right)\left(x^2+y^2\right)=4b^4x^2$$

and polar equation

$$r^2=4b^2\left(a^2-b^2\sin^2\theta\right).$$

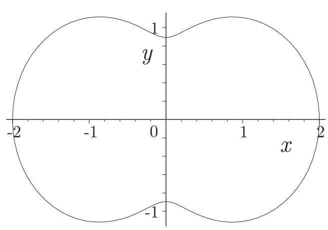


Figure A.10: Horse fetter

Using rational polar coordinates its equation is

$$s=\frac{Q}{4B}-A+B.$$

Lemniscate of Bernoulli The *lemniscate of Bernoulli* has Cartesian equation

$$\left(x^2+y^2\right)^2 = 2\left(x^2-y^2\right)$$

and polar equation

$$r^2 = 2\cos 2\theta.$$

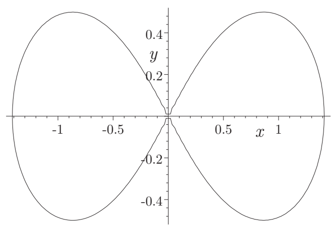


Figure A.11: Lemniscate of Bernoulli

Using rational polar coordinates its equation is

$$s = \frac{Q+2}{4}.$$

Folium of Descartes The *folium of Descartes*, has Cartesian equation

$$x^3+y^3+3xy = 0.$$

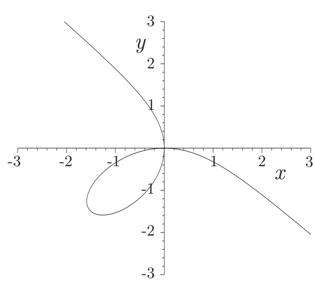


Figure A.12: Folium of Descartes

This curve has a more complicated rational polar equation; it is

$$(2s-1)^2\left(s^2-s+1\right)^2 Q^2 - 18s(1-s)\left(1-3s+3s^2\right)Q + 81s^2(1-s)^2 = 0.$$

Appendix B

Ellipson

The **ellipson** consists of all points $[x, y, z]$ inside, or on, the unit cube $0 \leq x, y, z \leq 1$ satisfying the Triple spread formula

$$(x + y + z)^2 = 2\left(x^2 + y^2 + z^2\right) + 4xyz.$$

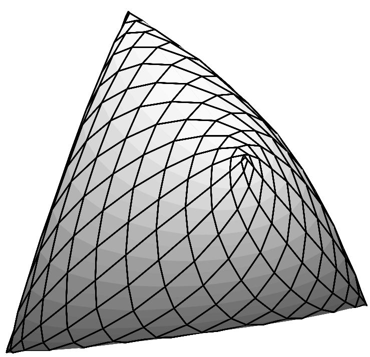


Figure B.1: The ellipson

This surface resembles an inflated tetrahedron, and indeed intersects the unit cube at precisely a tetrahedron, with points $[0, 0, 0], [1, 1, 0], [1, 0, 1]$ and $[0, 1, 1]$, and volume $1/3$. Its cross sections in any of the coordinate plane directions is otherwise always an ellipse, tangent to the unit square.

For example, the cross sections corresponding to $z = 0.1, 0.2, 0.4$ and 0.8 are shown in the $x-y$ plane in Figure B.2.

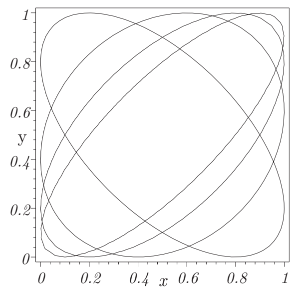


Figure B.2: Elliptical cross sections

On the other hand, a cross section parallel to a face of the tetrahedron, such as the plane $x + y + z = 2$, yields the interesting curve shown in Figure B.3.

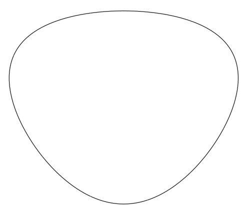


Figure B.3: Oblique cross section

In suitable coordinates, such a curve has an equation of the form

$$3b^2 a\sqrt{6} - 6a^2 - a^3\sqrt{6} - 6b^2 + 5/6 = 0.$$

In rational polar coordinates this becomes

$$s\left(3-4s\right)^2 = \frac{\left(36Q-5\right)^2}{\left(6Q\right)^3}$$

which bears some similarity to the equation defining the deltoid. The three-fold symmetry is reflected by the fact that the left hand side is $S_3\left(s\right)$.

Exercise B.1 (Harder, requires calculus) Show that the ellipson has volume

$$\pi^2/16 \approx 0.616\,850\,275\cdots. \quad \diamond$$

Theorems with pages and Important Functions

Important Functions

$$A(a,b,c) = (a+b+c)^2 - 2\left(a^2+b^2+c^2\right).$$

$$S(a,b,c) = (a+b+c)^2 - 2\left(a^2+b^2+c^2\right) - 4abc.$$

$$Q(a,b,c,d) = \left((a+b+c+d)^2 - 2\left(a^2+b^2+c^2+d^2\right)\right)^2 - 64abcd.$$

$$\begin{aligned} R(a,b,c,d) &= \begin{pmatrix} (a+b+c+d)^2 - 2\left(a^2+b^2+c^2+d^2\right) \\ -4(abc+abd+acd+bcd) + 8abcd \end{pmatrix}^2 \\ &\quad - 64abcd(1-a)(1-b)(1-c)(1-d). \end{aligned}$$

$$\begin{aligned} &E(Q_1,Q_2,Q_3,P_1,P_2,P_3) \\ &= 2\begin{pmatrix} 4P_1P_2P_3 + (P_2+P_1-Q_3)(P_2+P_3-Q_1)(P_1+P_3-Q_2) \\ -P_1(P_2+P_3-Q_1)^2 - P_2(P_1+P_3-Q_2)^2 - P_3(P_2+P_1-Q_3)^2 \end{pmatrix}. \end{aligned}$$