Managerial Regret and Inventory Pricing

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In this paper, we study a regretful seller's problem of selling a fixed number of goods over a finite and known time horizon. The seller engages in counterfactual thinking to compare her selected price with other forgone alternatives. If a forgone alternative (*ex post*) generates a better outcome than the selected one, the seller experiences regret. We characterize the pricing decision of a regretful seller and find that, although regret leads the seller to set a price that is lower than that set by an unbiased seller, the regretful seller employs decision policies whose structure is similar to those of the unbiased seller: the price decreases with the remaining inventory and increases with the time-to-go. Interestingly, we find that the seller who has a greater number of goods does not necessarily receive greater revenue.

Key words: revenue management, managerial bias, decision analysis, anticipated regret

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1. Introduction

Regret is a negative emotion experienced when a decision maker learns that an alternative action would have resulted in a more favorable outcome. As one of the most consistent and widespread emotions, regret has been explored in a wide range of fields, from psychology (Larrick 1993) to economics (Filiz-Ozbay and Ozbay 2007) to operations management (Schweitzer and Cachon 2000) as well as in both field studies and controlled experiments. In the case of markets, a price-setting seller can experience one of the two opposite types of regret, as indicated in the following excerpt.

> *In today's market, almost every seller will experience Seller's Regret. But it's their choice (sort of) which type of Seller's Regret they'll experience.*
>
> **Type 1***: "Damn! [...] We should have priced it higher! [...]"*
>
> **Type 2***: "Crap. [...] We should have [...] priced it lower. [...]"*
>
> —Jennifer Allan-Hagedorn (2008), Author & Founder of Sell With Soul.

For the first type, the seller regrets pricing too low when a customer purchases the product, which we refer to as *underpricing regret*. For the second type, the seller regrets pricing too high when a customer does not buy the product, which we refer to as *overpricing regret*. As the excerpt alludes, sellers can anticipate these two experienced regrets, which in turn affects their *ex ante* choices.

The extant literature provides abundant evidence to support the assertion that anticipated regret affects the seller's decision when she faces a revenue management problem. For example, using controlled experiments, Davis et al. (2011) observe that a seller might either regret setting a high price when she fails to sell the product or regret setting a low price when she successfully sells the product. Cesaret and Katok (2019) also employ similar opposite regrets to explain the behavior of sellers who dynamically allocate a fixed capacity among customers of different product valuations. They find that anticipated regret can provide a plausible explanation for a seller's *insufficiently demanding* behavior—that is, charging a price that is lower than the optimal one, or accepting too many customers with a low willingness-to-pay, while leaving high-value customers unserved—which has been observed in controlled experiments (Bearden et al. 2008, Davis et al. 2011) and empirical studies (Heching et al. 2002, Caro and de Tejada Cuenca 2020). Overall, this research indicates that anticipated regret can reasonably explain a seller's behavior in a revenue management context.

Against this backdrop, we analytically explore how a seller's anticipated regret affects her pricing policies and expected revenues. To this end, we first incorporate the anticipated regret into a single-period setting, where the exact value of the customer's willingness-to-pay for a particular product is unknown to the seller. The seller is biased in the sense that she engages in counterfactual thinking to compare her price with other forgone prices. If it turns out that a forgone price (*ex post*) generates better revenue than the chosen one, the seller experiences regret due to the loss of the forgone revenue. Consequently, the seller anticipates underpricing (overpricing) regret if the customer does (does not) purchase the product. The seller then decides on a price that balances both the revenue and the anticipated regret. As a validation for our model, we analytically show that the widely observed insufficiently demanding effect also exists in our context—that is, the regretful seller charges a lower price than the unbiased seller. Although the regretful seller always earns less than the unbiased seller and overpricing regret always hurts the seller, underpricing regret can actually be a positive force for the seller—that is, the higher the level of underpricing regret, the higher the seller's actual revenue. Intuitively, underpricing regret leads to a higher price, thereby providing a counterbalance to regret that would drive the price lower than the optimal price, and consequently benefiting the seller.

We also extend our analysis to the case where the seller dynamically sets her price over a finite time horizon. We develop a model that incorporates the notion of regret into the dynamic pricing model pioneered by Gallego and Van Ryzin (1994). As a validation for this model, we analytically show not only that the insufficiently demanding effect observed in the single-period setting exists in its multi-period analog but also that the biased seller deploys a decision policy of the same form as the unbiased seller. In other words, the regretful seller charges a price that decreases with the number of items left unsold, but increases in the remaining selling time period.

We then investigate the impact of a key parameter—the initial inventory level—on the seller's performance. For the unbiased seller, as intuition suggests, a higher inventory level always translates into higher revenue for the seller. However, this is not an absolute for the biased seller. Although the seller benefits from having zero to some inventory, having even more inventory is not necessarily beneficial for the seller, particularly when the inventory level is already relatively high. Intuitively speaking, a higher inventory level has two opposite effects on the biased seller. The first effect is that more inventory generates more assets for the seller, thereby benefiting the seller. The second effect is that more inventory may distort the pricing decisions of the regretful seller, thereby harming her, particularly when the inventory is relatively high. As the second effect outweighs the first, the seller can ultimately be hurt by a higher inventory level.

Previous research has employed regret bias in its various forms to explain and describe a variety of decision making behavior. For example, Bell (1982, 1983) incorporates regret into decision making and shows that this emotion can explain certain well-known behavioral anomalies. In the context of operations management, the seminal paper of Schweitzer and Cachon (2000) posits that newsvendors attempt to minimize regret in their order decisions, thereby explaining newsvendor behaviors. In revenue management, Filiz-Ozbay and Ozbay (2007) and Engelbrecht-Wiggans and Katok (2008) find two types of regret—the regret when an auction winner realizes that she would have preserved her winning position by bidding lower and the regret when the loser realizes that the winning bid was affordable—can explain the well-known phenomenon of overbidding behavior at auctions. Similarly, Nasiry and Popescu (2012) and Özer and Zheng (2015) also utilize two opposite types of regret to explain customers' purchase behavior. Moreover, Davis et al. (2011) experimentally observe that the seller experiences regret when she fails to sell her product or when she sells the product but realizes that a higher price might have yielded more revenue; they conclude that these regrets can better explain the seller's pricing behavior than other explanations like risk aversion. Cesaret and Katok (2019) show that these two opposite regrets can also reasonably explain the insufficiently demanding effect in the context of revenue management, which has been observed by Heching et al. (2002), Davis et al. (2011), and Caro and de Tejada Cuenca (2020). We complement this literature by investigating the theoretical impact of anticipated regret; we find that anticipated regret—particularly underpricing regret—can be a positive force for the seller.

Our work is also related to the extensive literature on revenue management (Gallego and Van Ryzin 1994, Hu et al. 2016, Hu and Nasiry 2018, Stamatopoulos and Tzamos 2019). A key assumption of this literature is that the seller is a rational decision maker. We extend this literature by incorporating the prevalent notion of anticipated regret into the seller's decision making to investigate its theoretical implications in dynamic pricing. We make three key contributions. First, we find that the biased seller charges a lower price than the unbiased seller. Second, we establish that as the time-to-go (remaining inventory) increases, the seller will set a higher (lower) price. Thus, we tangentially add to the body of literature on revenue management that focuses on structural property in order to facilitate managerial insight or efficient computation (Gallego and Van Ryzin 1994, Chen et al. 2017, Stamatopoulos and Tzamos 2019). Third, and interestingly, we find that the biased seller is not destined to be worse off when her initial inventory level is lower.

2. The Seller's Regret

Consider a monopolist who has a one-shot opportunity to sell one unit of a product to a customer. The customer's willingness-to-pay for the product is unknown except to the customer herself, and the seller only knows its distribution. Let $f(\cdot)$ and $F(\cdot)$ be the density function and cumulative distribution function of the customer's willingness-to-pay v. The support of v is $[a,b]$, where $a \geq 0$ and $a < b < +\infty$. An *unbiased* seller sets her price p to maximize the expected revenue, $p\bar{F}(p) + zF(p) = z + (p-z)\bar{F}(p)$, where $\bar{F}(\cdot) := 1 - F(\cdot)$ and $z \in [0,b]$ is the salvage value—that is, the seller's opportunity cost of selling one unit of inventory to customers. Thus, the seller's problem is equivalent to setting price p in order to maximize the *additional revenue* beyond the salvage value z:

$$\pi(p,z) := (p-z)\bar{F}(p). \tag{1}$$

We assume v has an increasing failure rate (IFR)—namely, $h(x) := f(x)/\bar{F}(x)$ is increasing in x, for any x—so that the seller has a unique optimal price $p^* := \arg\max_p \pi(p,z)$. Without a qualifier, increasing (decreasing) actually implies non-decreasing (non-increasing) in this paper. In fact, the optimal price $p^* = a$ if $h(a)(a-z) > 1$; otherwise, the optimal price p^* is characterized by $h(p)(p-z) = 1$. Given the optimal price, we define $m(z) := d\pi(p^*,z)/dz$. Clearly, $m(z) \leq 0$, implying that the additional revenue decreases with the opportunity cost.

Within this construct, a *regretful* seller is one who engages in counterfactual thinking. On the one hand, if the customer does not buy the product priced at p, then the seller loses her selling opportunity and has to salvage her product at value z. If the seller had hypothetically observed this, she would know that the customer's willingness-to-pay is lower than p. Based on this hindsight information, the seller expects to obtain more revenue by setting a price lower than p:

$$\max_{y<p}\left[y\frac{\bar{F}(y)-\bar{F}(p)}{F(p)} + z\left(1 - \frac{\bar{F}(y)-\bar{F}(p)}{F(p)}\right)\right] - z = \max_{y<p}\frac{\pi(y,z)-(y-z)\bar{F}(p)}{F(p)}, \tag{2}$$

where $[\bar{F}(y)-\bar{F}(p)]/F(p)$ is the customer's purchase probability for the product priced at y from the perspective of the seller who was aware that the customer's willingness-to-pay is lower than p. Consequently, if the customer does not purchase the product, the seller regrets setting a high price. On the other hand, if the customer buys the product priced at p, then the seller collects revenue

p. If she had known this information before pricing, then she would infer that the customer's willingness-to-pay is higher than p, thereby yielding more revenue:

$$\max_{y \ge p}\left[y\frac{\bar{F}(y)}{\bar{F}(p)} + z\left(1 - \frac{\bar{F}(y)}{\bar{F}(p)}\right)\right] - p = \max_{y \ge p}\frac{\pi(y,z)-\pi(p,z)}{\bar{F}(p)}, \tag{3}$$

where $\bar{F}(y)/\bar{F}(p)$ is the seller's probability of selling the product at price y, if she had known the customer's willingness-to-pay is higher than p. Consequently, if the customer purchases the product, the seller may regret the underpricing. In sum, although the regretful seller does not *ex ante* observe the exact value of the customer's willingness-to-pay, she can infer that from the customer's *ex post* decision and, consequently, experience regret.

The seller might take these two opposite regrets into consideration and behave as if she were maximizing

$$\begin{aligned}\hat{\pi}(p,z) &:= \pi(p,z) - \underbrace{\alpha F(p)\left[\max_{y<p}\frac{\pi(y,z)-(y-z)\bar{F}(p)]}{F(p)}\right]}_{\text{overpricing regret}} - \underbrace{\beta \bar{F}(p)\left[\max_{y \ge p}\frac{\pi(y,z)-\pi(p,z)}{\bar{F}(p)}\right]}_{\text{underpricing regret}} \\ &= \pi(p,z) - \underbrace{\alpha\left[\max_{y<p}\left[\pi(y,z)-(y-z)\bar{F}(p)\right]\right]}_{\text{overpricing regret}} - \underbrace{\beta\left[\max_{y\ge p}\pi(y,z)-\pi(p,z)\right]}_{\text{underpricing regret}}, \end{aligned} \tag{4}$$

where $\alpha \ge 0$ and $\beta \ge 0$, rather than only maximizing $\pi(p,z)$ defined in (1)—that is, the first term of the above objective function. Consistent with the literature (e.g., Engelbrecht-Wiggans and Katok 2008, Nasiry and Popescu 2012, Cesaret and Katok 2019), we model regret as one forgone surplus multiplied by the probability that this surplus is incurred, and the regret enters additively into the objective function. In (4), parameters α and β measure the marginal value of overpricing regret and underpricing regret in comparison to the expected revenue: the higher the parameter, the higher the level of regret bias. Thus, we refer to α and β as the *overpricing parameter* and *underpricing parameter*, respectively. We also let $\alpha \le 1$ and $\beta \le 1$ such that one unit of actual expected value is valued more by the seller than one unit of counterfactual value.

Lemma 1. *Define* $r_o(p,z) := \max_{y<p}[\pi(y,z) - (y-z)\bar{F}(p)]$ *and* $r_u(p,z) := \max_{y \ge p}[\pi(y,z) - \pi(p,z)]$. *Then,* $r_o(p,z)$ *increases in* p *while* $r_u(p,z)$ *decreases in* p*—that is,* $\partial r_o(p,z)/\partial p \ge 0$ *and* $\partial r_u(p,z)/\partial p \le 0$. *Moreover,* $r_u(p,z) = 0$ *always holds when* $p \ge p^*$.

Lemma 1 indicates that the magnitude of expected overpricing regret increases as the price increases. Intuitively, the higher the seller sets the price, the more she suffers if the product goes unsold. Conversely, the underpricing regret decreases as the price increases. This is reasonable, as a high price obviates a situation in which the seller might regret selling her product. In fact, the seller always regrets a lost sale because, if she had known beforehand that her product would not sell at the higher price, she would price it lower to gain a surplus. However, $r_u(p,z)=0$ when $p\geq p^*$. That is, the seller may not regret when she garners a sale because, even if the seller had known this in advance, she would not increase her price at the expense of lowering the likelihood of a customer making the purchase if the price was already high (higher than p^*).

Overall, regret occurs when the seller compares her revenue with the maximal revenue that can be extracted with the *partial* information on the customer's willingness-to-pay revealed by the chosen price. Such partial information—particularly, the hindsight information associated with a sale—may not necessarily alter the seller's pricing decision so that $r_u(p,z)=0$, even if she had observed the information in advance.

We need to impose conditions to ensure the regularity of $\hat{\pi}(p,z)$. To this end, let $\hat{m}(z,\alpha,\beta):=\partial\hat{\pi}(\hat{p}(\alpha,\beta),z)/\partial z$ denote the biased seller's marginal value from the opportunity cost, where $\hat{p}(\alpha,\beta)$ is the solution of (4). Henceforth in this paper, we assume that $\hat{m}(z,\alpha,\beta)\leq 0$, as in the unbiased case, and $\partial\hat{m}(z,\alpha,\beta)/\partial\alpha\geq 0$. In essence, these assumptions mean that the regret parameters are small, so that the resulted regrets are low relative to the expected revenue. These assumptions are satisfied when the customer's willingness-to-pay follows, for example, uniform and exponential distributions as well as their truncated variants; see Appendix B for details.

Proposition 1. *a) The solution of* (4) *is* $\hat{p}(\alpha,\beta)=a$ *if* $h(a)(a-z)[1+\frac{\alpha}{1+\beta}]>1$. *Otherwise,* $\hat{p}(\alpha,\beta)$ *is uniquely determined by*

$$h(p)\left[p-z+\frac{\alpha}{1+\beta}\left[y(p,z)-z\right]\right]=1, \tag{5}$$

where $y(p,z)=a$ *if* $\bar{F}(p)+(a-z)f(a)>1$ *and otherwise* $y(p,z)$ *is characterized by* $\bar{F}(p)+(y-z)f(y)=\bar{F}(y)$.
b) Consequently, $\hat{p}(\alpha,\beta)$ *is decreasing in* α *but increasing in* β*—that is,* $\frac{\partial\hat{p}(\alpha,\beta)}{\partial\alpha}\leq 0$ *and* $\frac{\partial\hat{p}(\alpha,\beta)}{\partial\beta}\geq 0$.

Proposition 1a indicates that the effect of the regret parameters on the seller's price decision is characterized by a sufficient statistic $\alpha/(1+\beta)$. Thus, one can segment sellers described by parameters α and β into equivalence classes according to $\alpha/(1+\beta)$. Within one class, the seller charges an identical price. This also implies that the biased seller's price deviates from the optimal price only when the overpricing parameter is nonzero ($\alpha>0$). That is, $\hat{p}(\alpha=0,\beta)=p^*$. Proposition 1b further indicates that the more sensitive a seller is to overpricing regret, the lower that seller should price her product; the more sensitive a seller is to underpricing regret, the higher that seller should price her product. Intuitively speaking, the seller would price lower (higher) to avoid overpricing (underpricing) regret.

Proposition 2. *The regretful seller charges a lower price than the unbiased seller, i.e.,* $\hat{p}(\alpha,\beta)\leq p^*$.

Proposition 2 indicates that the price charged by the biased seller is always lower than the price charged by an unbiased seller, although underpricing regret motivates the seller to increase her price (Proposition 1b). To arrive at a pricing decision, the seller balances her revenue and the two anticipated regrets. The overpricing regret always increases in the price, while the underpricing regret disappears if the biased seller charges a price higher than the unbiased optimal price p^* (Lemma 1). Consequently, the biased seller is always worse off by setting a price higher than p^* so that she would never set a price higher than p^*.

Albeit counterintuitive, Proposition 2 is consistent with the extant laboratory and empirical observations. Heching et al. (2002) empirically observe that managers of an apparel retailer generally price products lower than those suggested by optimal models. In a similar vein, Caro and de Tejada Cuenca (2020) observe that managers for the apparel brand Zara generally set prices that are lower than the ones recommended by the decision support systems with an aim to maximize revenue. Further, Bearden et al. (2008) and Cesaret and Katok (2019) find that subjects in revenue management experiments tend to be insufficiently demanding in their laboratory studies; this is as if the subjects employ a pricing policy that is lower than the optimal one.

We next evaluate how regret impacts the seller's expected revenue. Given the price $\hat{p}(\alpha,\beta)$, the seller's resulting expected revenue is $\pi(\hat{p}(\alpha,\beta),z)+z$, where $\pi(\cdot,z)$ is defined in (1).

Proposition 3. *The seller's expected revenue is decreasing in the overpricing parameter α while increasing in the underpricing parameter β when $\alpha>0$.*

Proposition 3 indicates that the seller hurts from her overpricing regret, which is consistent with our intuition that decision bias results in self-harm. However, Proposition 3 also indicates that the underpricing regret can be a positive force for the seller when the overpricing parameter is nonzero ($\alpha>0$). For insights, recall that the biased seller always underprices the product. The overpricing regret lowers the price and, consequently, pushes the price further away from the unbiased price (Proposition 1b), thereby hurting the seller. Conversely, underpricing regret increases the price charged by the biased seller, thereby pulling the price toward the optimal price and, consequently, benefiting the seller. In sum, although the biased seller always earns less than the unbiased seller by charging a price that deviates from the optimal price, the underpricing regret can actually serve as a counterbalance to the price deviation, thereby benefiting the biased seller.

3. Dynamic Pricing with Regret

We first introduce the dynamic pricing model without regret. Following the typical framework in revenue management (Gallego and Van Ryzin 1994, Bitran and Mondschein 1997), at the beginning of the time horizon, the seller is endowed with $c(>0)$ units of inventory that cannot be replenished during the finite sales horizon. For ease of presentation, we adopt a discrete-time model, where the sales horizon is divided into T periods, each of which is sufficiently short to ensure that there is, at most, one customer arrival. The time periods are ordered in reverse, such that $t=T$ at the beginning of the sales horizon and $t=0$ at the end. In each time period, the customer arrival probability λ is independent of t and the salvage value is zero at the end of the selling season. Following the literature (e.g., Bitran and Mondschein 1997), we assume that λ is relatively low, i.e., $\lambda \le 1/2$.

The seller's objective is to maximize the season's total revenue by selecting an appropriate price in every period; that is, for a given pair of time-to-go and inventory (t, x), the seller chooses a price p to maximize the total expected revenue. Let $V_t(x)$ be the seller's optimal expected value when the remaining time period is t and the remaining inventory is x units. Then, we formulate the seller's dynamic pricing problem as follows:

$$V_t(x) = (1-\lambda)\ V_{t-1}(x) + \lambda\ \max_p \left\{F(p)V_{t-1}(x) + \bar{F}(p)[p + V_{t-1}(x-1)]\right\} \tag{6}$$

$$= V_{t-1}(x) + \lambda\ \max_p \pi(p, \Delta V_{t-1}(x)), \tag{7}$$

with boundary conditions $V_t(0)=0$ for $t=1,...,T$, and $V_0(x)=0$ for all $x \le c$. The right-hand side of (6) has two terms. The first term corresponds to the case without an arriving customer. The second term corresponds with the case of an arriving customer, which generalizes the single-period setting by including the value-to-go of selling the product $V_{t-1}(x-1)$ and the value-to-go of not selling the product $V_{t-1}(x)$. In (7), $\Delta V_t(x) := V_t(x) - V_t(x-1)$ represents the *marginal value of inventory*—that is, the seller's opportunity cost for selling one unit inventory—for a given pair (t, x). Accordingly, the unbiased seller's multi-period problem is equivalent to maximizing revenue $\pi(p, z)$ defined in (1), by replacing the opportunity cost z by its multi-period counterpart $\Delta V_{t-1}(x)$.

LEMMA 2 **(Bitran and Mondschein 1997)**. *a) $\Delta V_t(x)$ increases in the time-to-go t and decreases in the remaining inventory x.*
b) Let $p_t(x)$ be the optimal solution to the dynamic program in (7). Then, $p_t(x)$ increases in the time-to-go t and decreases in the remaining inventory x.

Lemma 2a indicates that the marginal value of inventory increases in the time-to-go for a given inventory level. Similarly, the marginal value of inventory would decrease if the seller has a higher inventory level for a given time-to-go. Lemma 2b further shows that, given the remaining inventory, the seller would set the price higher if the time-to-go increases. Moreover, given the time-to-go, the seller would set the price lower if the seller has a higher inventory level. Note that the marginal value of inventory is always non-negative—that is, $V_t(x+1) \ge V_t(x)$—because a seller with a higher stock can at least mimic the pricing policy of the seller with a lower stock. However, this is not absolute when the seller is biased, as shown later in Proposition 6.

We now incorporate the notion of regret into this dynamic setting. To this end, we have to make assumptions regarding regret timing and seller sophistication. For regret timing, we assume that the seller experiences regret in each period after observing the consumer's purchase choice. This is plausible, as the seller observes such a choice in each period and consequently experiences regret after the observation. For seller sophistication, following the literature (O'Donoghue and Rabin 1999, Baucells and Zhao 2019), we assume that the seller has perfect foresight of her regret behavior in future periods.

Consequently, the seller experiences regret in each period by considering the hindsight information in that period and the resulting salvage value (the value-to-go in a dynamic setting). Let $U_t(x)$ be the seller's mental value-to-go when the remaining time period is t and the remaining inventory is x units. Then, if the customer does not buy the product priced at p, then the seller loses her selling opportunity and obtains a mental value-to-go $U_{t-1}(x)$ at period $t-1$. If the seller had hypothetically observed this, she would know the customer's willingness-to-pay is lower than p, and thereby the seller's experienced (overpricing) regret is

$$\begin{aligned}
&\max_{y<p}\left[(y+U_{t-1}(x-1))\frac{\bar{F}(y)-\bar{F}(p)}{F(p)}+U_{t-1}(x)\left(1-\frac{\bar{F}(y)-\bar{F}(p)}{F(p)}\right)\right]-U_{t-1}(x)\\
&=\max_{y<p}\left[y\frac{\bar{F}(y)-\bar{F}(p)}{F(p)}+\Delta U_{t-1}(x)\left(1-\frac{\bar{F}(y)-\bar{F}(p)}{F(p)}\right)\right]-\Delta U_{t-1}(x)\\
&=\max_{y<p}\frac{\pi(y,\Delta U_{t-1}(x))-(y-\Delta U_{t-1}(x))\bar{F}(p)}{F(p)},
\end{aligned} \tag{8}$$

where $\Delta U_t(x) := U_t(x) - U_t(x-1)$. This overpricing regret is equivalent to replacing the single-period opportunity cost z by its multi-period counterpart $\Delta U_{t-1}(x)$. In a similar vein, if the consumer purchases the product, the experienced (underpricing) regret is equivalent to replacing z by $\Delta U_{t-1}(x)$:

$$\begin{aligned}
&\max_{y\geq p}\left[(y+U_{t-1}(x-1))\frac{\bar{F}(y)}{\bar{F}(p)}+U_{t-1}(x)\left(1-\frac{\bar{F}(y)}{\bar{F}(p)}\right)\right]-[p+U_{t-1}(x-1)]\\
&=\max_{y\geq p}\left[y\frac{\bar{F}(y)}{\bar{F}(p)}+\Delta U_{t-1}(x)\left(1-\frac{\bar{F}(y)}{\bar{F}(p)}\right)\right]-p\\
&=\max_{y\geq p}\frac{\pi(y,\Delta U_{t-1}(x))-\pi(p,\Delta U_{t-1}(x))}{\bar{F}(p)}.
\end{aligned} \tag{9}$$

Given these aforementioned regrets, we now formulate the seller's dynamic problem. As per our assumptions on regret timing and seller sophistication, the regretful seller behaves as if solving the following:

$$U_t(x) = (1-\lambda)\ U_{t-1}(x) + \lambda \max_p \left\{ F(p)U_{t-1}(x) - \underbrace{\alpha F(p)\left[\max_{y<p}\frac{\pi(y,\Delta U_{t-1}(x))-(y-\Delta U_{t-1}(x))\bar{F}(p)}{F(p)}\right]}_{\text{overpricing regret}}\right.$$

$$
\begin{aligned}
&+\bar{F}(p)\left[p+U_{t-1}(x-1)\right]-\underbrace{\beta\bar{F}(p)\left[\max_{y\geq p}\frac{\pi(y,\Delta U_{t-1}(x))-\pi(p,\Delta U_{t-1}(x))}{\bar{F}(p)}\right]}_{\text{underpricing regret}}\Bigg\}\\
&=(1-\lambda)\ U_{t-1}(x)+\lambda\max_{p}\Big\{F(p)U_{t-1}(x)-\alpha\ r_o(p,\Delta U_{t-1}(x))\\
&\qquad+\bar{F}(p)[p+U_{t-1}(x-1)]-\beta\ r_u(p,\Delta U_{t-1}(x))\Big\}\\
&=U_{t-1}(x)+\lambda\ \max_{p}\hat{\pi}(p,\Delta U_{t-1}(x)) \qquad (10)
\end{aligned}
$$

with boundary conditions $U_t(0)=0$ for $t=1,\cdots,T$, and $U_0(x)=0$ for all $x\leq c$. Accordingly, the biased seller behaves as if maximizing $\hat{\pi}(p,\Delta U_{t-1}(x))$ rather than the unbiased seller's profit $\pi(p,\Delta V_{t-1}(x))$. Nonetheless, similar to the unbiased seller, the biased seller's multi-period problem is equivalent to maximizing the single-period revenue $\hat{\pi}(p,z)$ defined in (4), but replacing the opportunity cost z by its multi-period counterpart $\Delta U_{t-1}(x)$.

Remark 1. We use an example to illustrate our regret notion. Suppose $T=2,\ c=1,\ \lambda=\alpha=\beta=0.5$, and $v\sim U[0,1]$. In the second period, if the inventory level $x=1$, then the regretful seller charges a price $p=0.4615$ from (5) with $z=0$. Accordingly, $\Delta U_1(1)=U_1(1)-U_1(0)=\lambda\hat{\pi}(p=0.4615,z=0)=0.1106$, where $U_1(0)=0$. Given this, in the first period, the seller's price $p=0.5211$ from (10).

One alternative regret notion is that the seller experiences regret at the ending period of the selling horizon when all information in the entire selling horizon is revealed. Although this seems to be a parsimonious notion, it actually leads to an extremely complicated hindsight information structure, thereby complicating the regret formulation. This is because, under this notion, regret depends on the consumer arrival (whether the consumer arrives) and consumer decision (whether the consumer buys the product or not) in each period so that the number of distinct regret formulations would increase in an exponential order of the number of periods T. Indeed, even in the simplest two-period case of one unit of initial inventory (as in the above illustrative example), we would have eight different regret formulations. □

4. Dynamic Pricing: Structural and Numerical Results

Next, we consider the regretful seller's dynamic pricing decision, and study how regret distorts the seller's decision.

Proposition 4. *For any bias parameters α and β,*

a) $\Delta U_t(x)$ increases in the time-to-go t and decreases in the remaining inventory x.

b) Let $\hat{p}_t(x,\alpha,\beta)$ denote the solution to (10). Then, $\hat{p}_t(x,\alpha,\beta)$ increases in the time-to-go t and decreases in the remaining inventory x.

Proposition 4 generalizes the classical structural properties in Lemma 2 to a setting with a behavioral seller that incurs both overpricing and underpricing regrets during each period. In essence, although the human decision maker's psychological burden changes the pricing policy, the basic structural properties that are well established in revenue management remain unaltered. This echoes the extant experimental observations revealing that sellers often employ decision policies that are similar to the optimal policy, although they depart from optimality (Heching et al. 2002, Bearden et al. 2008, Cesaret and Katok 2019).

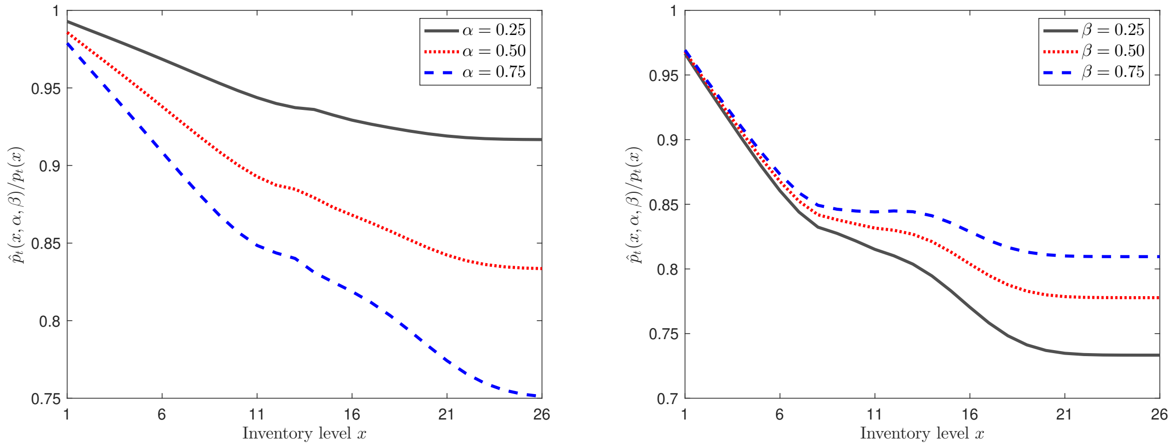
PROPOSITION 5. *For any (t,x), the biased seller charges a price lower than the unbiased seller—that is, $\hat{p}_t(x,\alpha,\beta) \leq p_t(x)$.*

Proposition 5 shows that the key insight in Section 2—the biased seller charges a lower price than the unbiased seller—continues to hold in a multi-period setting. In addition to the force that causes the regretful seller to lower the current price (Proposition 2), regret has another function in a dynamic setting. In particular, regret creates disutility for the seller, thereby lowering the value of the inventory, and consequently, incentivizing the seller to decrease the product price. Both forces affect the price in the same direction and, consequently, the biased seller deviates from the optimal price by charging less. In fact, such a deviation can become larger as the seller has more inventory. To see this, we plot the price ratio between the biased price and the unbiased price—that is, $\hat{p}_t(x,\alpha,\beta)/p_t(x)$—in Figure 1. As per this figure, this price ratio decreases as the remaining inventory level increases; see Lemma 5 in Appendix A for a more formal characterization. In other words, more inventory may exacerbate the biased seller's price distortion from the optimality. This observation helps explain the forthcoming Proposition 6.

Moreover, Figure 1 shows that, for a given inventory level, the price ratio between the biased price and the unbiased price decreases, thereby implying that the seller's price becomes more biased, as the overpricing (underpricing) parameter increases (decreases); see Figure 1a (Figure 1b). This is consistent with the intuition underlying Proposition 1b that the overpricing bias distorts the seller's price while the underpricing bias can mitigate such a distortion.

Figure 2 plots the sample paths of $\hat{p}_t(x,\alpha,\beta)$ for two levels of initial stock: $c=20$ and $c=30$. We observe that both unbiased and biased prices jump upward corresponding to sales, and then they decrease until another sale is made, at which point the price takes another jump. This behavior follows from Lemma 2 and Proposition 4: the upward jumps are due to the fact that the seller prices higher if she has fewer items to sell over a given interval t, and the decreasing price between sales follows from the fact that $\hat{p}_t(x,\alpha,\beta)$ is decreasing in t for a fixed x. Moreover, the biased price path is generally lower than the unbiased path, especially when the time-to-go is large and the initial stock level is high ($c=30$). This behavior follows from Proposition 5 and the fact that the

Figure 1 **The price ratio $\hat{p}_t(x,\alpha,\beta)/p_t(x)$ for different overpricing and underpricing parameters**



(a) $\beta = 0$ (b) $\alpha = 1$

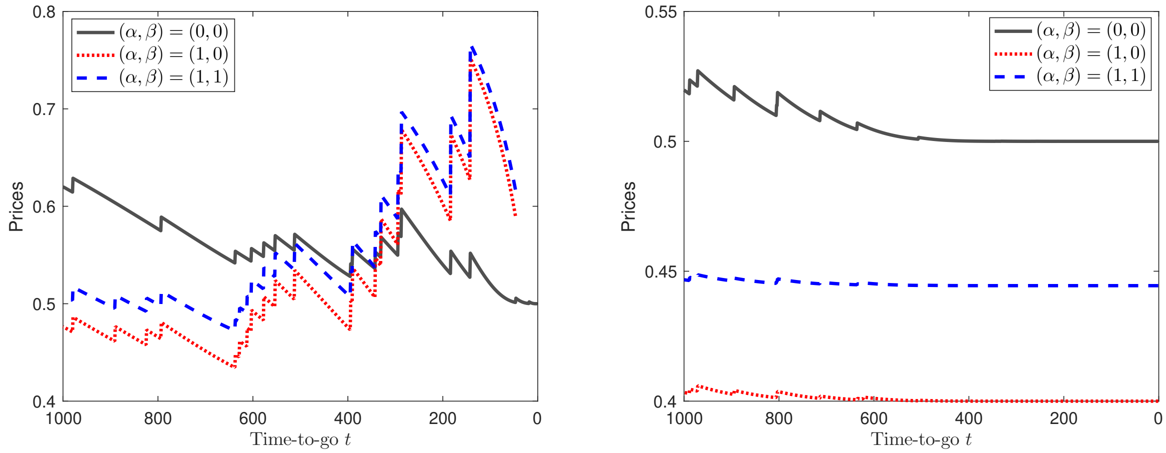
Notes. In this figure, $v \sim U[0.5, 1.5]$ and $\lambda = 0.5$. The horizontal axis corresponds to the inventory level x, and the vertical axis corresponds to the price ratio $\hat{p}_t(x,\alpha,\beta)/p_t(x)$ when $t = 40$. Panel (a) shows the price ratios for different overpricing parameters with $\beta = 0$. Panel (b) shows the price ratios for different underpricing parameters with $\alpha = 1$.

(biased and unbiased) sellers have similar stock levels just after they initiate business. Nonetheless, when the time-to-go is small, the biased price path can surpass the unbiased price path if the initial stock is relatively low ($c = 20$). This is because the biased seller's underpricing behavior leads to a lower stock; thus she prices it higher than the unbiased seller does. Consequently, in contrast to the unbiased price path that is generally decreasing over time, the biased price can increase with the passage of time. This type of "abnormal" price path is indicative of the presence of regret in the practice.

So far, we study the impact of regret on the seller's pricing decision, given the seller's inventory quantity c. To extend the scope of our study, we also consider the effect of regret on endogenous stock decisions. When the seller decides on her stock at the beginning of the selling horizon, she balances her surplus for the entire time horizon and her payments to purchase the goods. If the payment is linear in the stock quantity, we find that the regret bias leads to the seller's understocking—that is, $\hat{c}(\alpha,\beta) < c^*$, where $\hat{c}(\alpha,\beta)$ and c^* are the stock decisions for the regretful seller and unbiased seller, respectively (Proposition 7 in Appendix A). This is because regret causes a decrease in the surplus generated by the inventory (Lemma 3 in Appendix A), thereby decreasing the seller's incentive to stock.

We next numerically illustrate the impact of regret on the pricing decision and stock decision. We consider a 50-period selling season with the arrival probability $\lambda = 0.5$. The consumer's willingness-to-pay is Weibull distributed, with mean $\mu = 5$ and standard deviation σ such that the coefficient

Figure 2 **Sample paths of** $\hat{p}_t(x,\alpha,\beta)$



(a) $c=20$ (b) $c=30$

Notes. For this example, $v\sim U[0,1]$ and $(\lambda,T)=(0.05,1000)$. The horizontal axis corresponds to the time-to-go t, and the vertical axis corresponds to a sample price path. Panel (a) shows price paths $\hat{p}_t(x,\alpha,\beta)$ for different overpricing and underpricing parameters with initial inventory $c=20$. Panel (b) shows price paths $\hat{p}_t(x,\alpha,\beta)$ for different overpricing and underpricing parameters with initial inventory $c=30$.

of variation σ/μ ranges from 0.1 to 1.5. The Weibull distribution allows for flexibility to model different reservation price behaviors, and has been tested numerically (Bitran and Mondschein 1997) and empirically (Bitran et al. 1998). Both the overpricing parameter α and the underpricing parameter β range from 0.2 to 1.0. This range covers the regret levels experimentally observed in the laboratory (Davis et al. 2011). See Table 1 for a summary of the parameters. For the exogenous stock case, the inventory level ranges from 1 to 50, and hence there are $18,750$ scenarios. For the endogenous stock case, the seller's unit purchasing cost g varies from 0.1 to 4, and accordingly, there are $4,500$ scenarios.

Table 1 **Tested parameter values**

Parameters	Included values
σ/μ	{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, 1.3, 1.4, 1.5}
α	{0.2, 0.4, 0.6, 0.8, 1}
β	{0.2, 0.4, 0.6, 0.8, 1}
x	$\{1, 2, 3, \cdots, 50\}$
g	{0.1, 0.2, 0.3, 0.4, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4}

Table 2 reports the percentage of the price decrease $[1-\hat{p}_t(x,\alpha,\beta)/p_t(x)]\times 100\%$ and the percentage of the stock decrease $[1-\hat{c}(\alpha,\beta)/c^*]\times 100\%$ due to the seller's regret. In particular, we report the quantile statistics for different levels of overpricing regret. As per the left side of Table 2, the biased seller sets a significantly lower price than the optimal price; the reduction, on average, increases from 4.44% to 18.63% when α increases from 0.2 to 1.0. This is consistent with Proposition 1 and Figure 1 in that the higher the overpricing regret, the lower the price the seller sets as compared to the optimal price. The regret bias can even lower the pricing decision by 42.61%. As per the right side of Table 2, the biased seller stocks a significantly lower inventory than the optimal; the stock reduction, on average, increases from 4.24% to 19.53% when α increases from 0.2 to 1.0. The regret can even lower the stock by 50.00%.

Table 2 Percentages of price reduction and stock reduction (%)

	price reduction					stock reduction				
α	0.2	0.4	0.6	0.8	1.0	0.2	0.4	0.6	0.8	1.0
Minimum	0.28	0.55	0.83	1.11	1.38	0.00	0.00	0.00	0.00	0.00
5%	0.87	1.67	2.42	3.11	3.72	0.00	0.00	0.00	3.70	4.00
25%	2.78	5.31	7.76	10.02	12.09	0.00	4.00	6.25	10.53	12.50
50%	4.59	8.76	12.54	15.98	19.26	0.00	7.69	12.50	16.67	20.00
75%	5.88	11.24	16.21	20.79	25.03	7.69	12.50	16.67	22.22	25.00
95%	7.97	15.10	21.54	27.38	32.49	16.67	20.00	25.00	28.57	33.33
Maximum	10.84	20.34	28.62	36.05	42.61	25.00	25.00	28.57	40.00	50.00
Average	4.44	8.48	12.17	15.54	18.63	4.24	8.39	12.54	16.01	19.53

5. The Value of Inventory

In a dynamic setting, in contrast to the setting in Section 2 where the seller has a one-shot selling opportunity for a single unit of inventory, the seller has multiple selling opportunities for a fixed number of goods. Since the seller's regret can significantly distort the pricing distortion, we next explore how such a distortion can possibly impact the value of inventory.

Given $\hat{p}_t(x,\alpha,\beta)$, we evaluate the seller's resulting expected revenue. Let $\hat{V}_t(x)$ be the seller's revenue-to-go when the time-to-go is t and the remaining inventory is x units, and $\Delta\hat{V}_t(x) := \hat{V}_t(x) - \hat{V}_t(x-1)$ be the marginal value of inventory. Then,

$$\begin{aligned}\hat{V}_t(x) &= (1-\lambda)\ \hat{V}_{t-1}(x) + \lambda\left\{F(\hat{p}_t(x,\alpha,\beta))\hat{V}_{t-1}(x) + \bar{F}(\hat{p}_t(x,\alpha,\beta))\left[\hat{p}_t(x,\alpha,\beta) + \hat{V}_{t-1}(x-1)\right]\right\} \\ &= \hat{V}_{t-1}(x) + \lambda\ \pi(\hat{p}_t(x,\alpha,\beta), \Delta\hat{V}_{t-1}(x)) \end{aligned} \tag{11}$$

with boundary values $\hat{V}_t(0) = 0$ for $t = 1, ..., T$ and $\hat{V}_0(x) = 0$ for all $x \le c$. That is, although the biased seller behaves as if solving (10), her resultant revenue is characterized by (11).

Next, we investigate the impact of the seller's inventory level x on the value function $\hat{V}_t(x)$. Intuition suggests that a higher inventory level implies a higher revenue for the seller. However, we find that this is not always true for the biased seller.

Proposition 6. *For any t, $\Delta\hat{V}_t(x) \ge 0$ when $x = 1$. However, there exist parameters (α,β) and inventory level $\underline{x}$ such that the biased seller earns less revenue with more inventory, i.e., $\Delta\hat{V}_t(x) < 0$, when λ is small enough and $x \in (\underline{x}, t]$.*

Proposition 6 indicates that the marginal value of inventory can change from positive to negative as the inventory level increases from nothing to something and then to more. In particular, the seller benefits from having zero inventory to some inventory—that is, $\Delta\hat{V}_t(1) \geq 0$. However, interestingly, having even more inventory is not necessarily beneficial to the seller, particularly when the inventory level is relatively high, such that $x \in (\underline{x}, t]$. For illustration, we plot the marginal value of inventory for the non-trivial case of $x \leq t$ in Figure 3. We observe in this figure that the region of the negative marginal value of inventory grows as the overpricing (underpricing) parameter increases (decreases), which is consistent with the opposite effects of the two types of regret on the price ratio in Figure 1. More importantly, the marginal value of inventory is likely to be negative when the inventory level x is large (the time-to-go t is also large to ensure $t \geq x$).

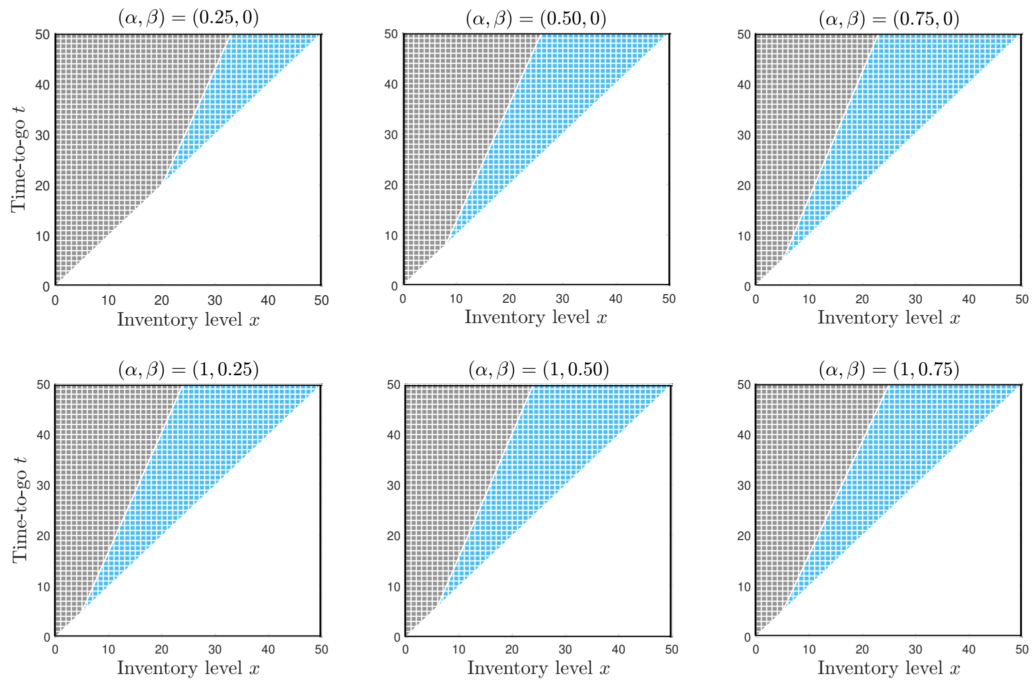
Overall, Proposition 6 shows that the biased seller can earn less by having more inventory, especially when the inventory level is already high. For insights, inventory has two effects on the seller's revenue. The first effect is a direct effect that inventory is an asset and more inventory means more revenue, which is consistent with the case of the unbiased seller in Lemma 2a and the case of $x = 1$ in Proposition 6. Moreover, this direct effect is weak when the inventory level is high, a case in which additional inventory has only a small chance of being sold and translated into revenue for the seller. The second effect is an indirect effect: regret biases the seller's pricing decision, and more inventory may lead to an even more biased decision—that is, an even lower selling price. Moreover, as alluded by Figure 1, this indirect effect is strong when the inventory level is high because additional inventory exacerbates the seller's price distortion from the optimality. Thus, when the inventory level is relatively high, the indirect effect can outweigh the direct effect such that the seller ultimately suffers from having more inventory.

6. Conclusions

We study the effects and implications of regret, defined as a cognitive bias that essentially describes a seller who behaves as if she were maximizing utility by taking into account the anticipated regret arising from a pricing decision rather than only the expected revenue. We find that the regretful seller always underprices her product, and depending on the type of regret, a more biased seller is not necessarily destined to receive a smaller expected revenue. Although regret biases the seller's pricing, it does not change the structure of the pricing policy for revenue management. Interestingly, in a dynamic setting, we find that the seller with a lower inventory level does not necessarily receive a lower expected revenue.

Several research directions can usefully be pursued in the future. First, in our model, the seller is a price-setter. It would be interesting to consider a price-taker who is confronted with a sequence of prices and must decide at each point how much of the inventory should be sold. Second, our study

Figure 3 **The marginal value of inventory for different overpricing and underpricing parameters**



Notes. This figure shows the sign of $\Delta\hat{V}_t(x)$ for various (t,x) and (α,β). For the gray area, $\Delta\hat{V}_t(x)>0$, while for the blue area, $\Delta\hat{V}_t(x)<0$. In this figure, $v\sim U[0.5,1.5]$ and $\lambda=0.5$. In the top three subfigures, we vary α such that $\alpha=0.25,0.50,0.75$ while fixing $\beta=0$. In the bottom three subfigures, we vary β such that $\beta=0.25,0.50,0.75$ while fixing $\alpha=1$.

could be extended to experimental or empirical contexts. For example, future research can follow approaches provided in the literature (Cesaret and Katok 2019) to test different regret biases and then accordingly determine the impact of regret on pricing.

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Appendix A: Proofs

In our proofs, we will use the equivalent notations $\hat{p}(z)=\hat{p}(\alpha,\beta)$ and $p^*(z)=p^*$, respectively, by making the dependence on z explicit when necessary.

Proof of Lemma 1: Given $r_o(p,z)=\max_{y<p}[\pi(y,z)-(y-z)\bar{F}(p)]=\max_{y<p}(y-z)[\bar{F}(y)-\bar{F}(p)]$, from the envelop theorem, we have $\partial r_o(p,z)/\partial p=(y(p,z)-z)f(p)\geq 0$, where $y(p,z)=\arg\max_{y<p}(y-z)[\bar{F}(y)-\bar{F}(p)]\geq z$. As $F(x)$ is IFR and $p^*=\arg\max_p \pi(p,z)$, $\pi(p,z)$ increases in p for $p<p^*$ and decreases in p otherwise. Given $r_u(p,z)=\max_{y\geq p}[\pi(y,z)-\pi(p,z)]=\max_{y\geq p}\pi(y,z)-\pi(p,z)$, then $r_u(p,z)=\pi(p^*,z)-\pi(p,z)$ if $p<p^*$ and $r_u(p,z)=0$ otherwise. Accordingly, $\partial r_u(p,z)/\partial p=-\partial\pi(p,z)/\partial p<0$ if $p<p^*$ and $\partial r_u(p,z)/\partial p=0$ otherwise. □

Proof of Proposition 1: (a) To characterize the biased seller's decision, we first define $s(y,p,z):=\pi(y,z)-(y-z)\bar{F}(p)=(y-z)[\bar{F}(y)-\bar{F}(p)]$. Then,

$$\frac{\partial s(y,p,z)}{\partial y}=\bar{F}(y)-\bar{F}(p)-(y-z)f(y)=\underbrace{f(y)}_{\geq 0}\underbrace{\left[\frac{1-\bar{F}(p)/\bar{F}(y)}{h(y)}-(y-z)\right]}_{\text{decreases in } y \text{ from } h'(\cdot)>0}$$

and $\frac{\partial s(y,p,z)}{\partial y}|_{y=p}=-f(p)(p-z)<0$. Thus, $s(y,p,z)$ is decreasing in y if $\frac{\partial s(y,p,z)}{\partial y}|_{y=a}<0$, equivalently, $\bar{F}(p)+(a-z)f(a)>1$, and is first increasing and then decreasing in y otherwise. For either case, $y(p,z):=\arg\max_{y<p}s(y,p,z)$ is uniquely determined. In particular, if $\bar{F}(p)+(a-z)f(a)>1$, then $y(p,z)=a$. Otherwise, $y(p,z)$ satisfies $\frac{\partial s(y,p,z)}{\partial y}=0\Longleftrightarrow\bar{F}(p)+(y-z)f(y)=\bar{F}(y)$. Furthermore, from $\frac{\partial^2 s(y,p,z)}{\partial y\partial p}=f(p)\geq 0$,

$$\frac{\partial y(p,z)}{\partial p}\geq 0. \tag{12}$$

We now characterize the seller's pricing decision. If $p<p^*$, from (4),

$$\begin{aligned}\frac{\partial\hat{\pi}(p,z)}{\partial p}&\overset{\text{(a)}}{=}\frac{\partial\pi(p,z)}{\partial p}-\alpha f(p)\left[y(p,z)-z\right]+\beta\frac{\partial\pi(p,z)}{\partial p}\\&\overset{\text{(b)}}{=}(1+\beta)\left[\bar{F}(p)-f(p)(p-z)-\frac{\alpha}{1+\beta}f(p)\left[y(p,z)-z\right]\right]\\&\overset{\text{(c)}}{=}\underbrace{(1+\beta)\bar{F}(p)}_{\geq 0}\underbrace{\left[1-h(p)(p-z)-\frac{\alpha}{1+\beta}h(p)\left[y(p,z)-z\right]\right]}_{\text{decreases in } p \text{ from } h'(\cdot)>0,\ (12),\text{ and } y(p,z)\geq z},\end{aligned} \tag{13}$$

where (a) is from the envelop theorem, (b) is from the algebra, and (c) is from the definition of $h(\cdot)$. If $p \geq p^*$, $r_u(p,z)=0$ from Lemma 1. Similarly,

$$\begin{aligned}
\frac{\partial \hat{\pi}(p,z)}{\partial p} &= \frac{\partial \pi(p,z)}{\partial p} - \alpha f(p)\left[y(p,z)-z\right] \\
&= \bar{F}(p)\left[1-h(p)(p-z)-\alpha h(p)\left[y(p,z)-z\right]\right] \\
&\leq \bar{F}(p)\left[1-h(p^*)(p^*-z)-\alpha h(p^*)\left[y(p^*,z)-z\right]\right] \\
&\leq 0,
\end{aligned}$$

where the first inequality is because $1-h(p)(p-z)-\alpha h(p)\left[y(p,z)-z\right]$ decreases in p and $p \geq p^*$, and the second inequality is because $h(p^*)(p^*-z) \geq 1$ and the fact that $y(p,z) \geq z$ always holds. Thus, $\hat{\pi}(p,z)$ is decreasing in p so that $\hat{p}(\alpha,\beta)=a$ if $\frac{\partial \hat{\pi}(p,z)}{\partial p}|_{p=a} < 0$, or equivalently, $h(a)(a-z)[1+\frac{\alpha}{1+\beta}] > 1$. Otherwise, $\hat{\pi}(p,z)$ is first increasing and then decreasing in y; thus, $\hat{p}(\alpha,\beta)$ is uniquely determined by

$$\frac{\partial \hat{\pi}(p,z)}{\partial p} = 0 \Longleftrightarrow h(p)(p-z) + h(p)\frac{\alpha}{1+\beta}\left[y(p,z)-z\right] = 1 \Longleftrightarrow (5).$$

(b) From part (a), $\frac{\partial^2 \hat{\pi}(p,z)}{\partial p \partial \alpha} = -f(p)\left[y(p,z)-z\right] \leq 0 \Longrightarrow \frac{\partial \hat{p}(\alpha,\beta)}{\partial \alpha} \leq 0$—that is, $\hat{p}(\alpha,\beta)$ is decreasing in α. Furthermore, if $h(a)(a-z)[1+\frac{\alpha}{1+\beta}] \leq 1$, Equation (13) implies that

$$\left.\frac{\partial^2 \hat{\pi}(p,z)}{\partial p \partial \beta}\right|_{p=\hat{p}(\alpha,\beta)} = \left[\bar{F}(p) - f(p)(p-z)\right]_{p=\hat{p}(\alpha,\beta)} = \left[\frac{\alpha}{1+\beta} f(p)[y(p,z)-z]\right]_{p=\hat{p}(\alpha,\beta)} \geq 0.$$

As $\hat{p}(\alpha,\beta)$ is unique, one can argue that $\hat{p}(\alpha,\beta)$ is increasing in β—that is, $\frac{\partial \hat{p}(\alpha,\beta)}{\partial \beta} \geq 0$ for $h(a)(a-z)[1+\frac{\alpha}{1+\beta}] \leq 1$. Note that $\hat{p}(\alpha,\beta)=a$ if $h(a)(a-z)[1+\frac{\alpha}{1+\beta}] \geq 1$ and accordingly $\hat{p}(\alpha,\beta)$ is continuous in β when $h(a)(a-z)[1+\frac{\alpha}{1+\beta}]=1$. Thus, $\hat{p}(\alpha,\beta)$ is always increasing in β. □

Proof of Proposition 2: We prove this by contradiction. Suppose that the biased seller charges a price $\hat{p}$ such that $\hat{p} > p^*$. Then,

$$\begin{aligned}
\hat{\pi}(p^*,z) - \hat{\pi}(\hat{p},z) &\overset{(a)}{=} \pi(p^*,z) - \pi(\hat{p},z) - \alpha\left[r_o(p^*,z) - r_o(\hat{p},z)\right] - \beta\left[r_u(p^*,z) - r_u(\hat{p},z)\right] \\
&\overset{(b)}{=} \pi(p^*,z) - \pi(\hat{p},z) - \alpha\left[r_o(p^*,z) - r_o(\hat{p},z)\right] \\
&\overset{(c)}{>} 0,
\end{aligned}$$

where (a) is from (4), (b) is because $r_u(\hat{p},z) = r_u(p^*,z) = 0$ that follows from Lemma 1 and $\hat{p} > p^*$, and (c) is from $\pi(p^*,z) > \pi(\hat{p},z)$ and $\frac{\partial r_o(p,z)}{\partial p} = (y(p,z)-z)f(p) \geq 0$. This leads a contradiction and the result then follows. □

Proof of Proposition 3: Given $\hat{p}(\alpha,\beta) \leq p^*$ (see Proposition 2) and the uni-modularity of $\pi(p,z)$ in p, it is easy to see $\frac{\partial \pi(p,z)}{\partial p}|_{p=\hat{p}(\alpha,\beta)} \geq 0$. Then,

$$\frac{\partial \pi(\hat{p}(\alpha,\beta),z)}{\partial \alpha} = \left.\frac{\partial \pi(p,z)}{\partial p}\right|_{p=\hat{p}(\alpha,\beta)} \frac{\partial \hat{p}(\alpha,\beta)}{\partial \alpha} \leq 0$$

and

$$\frac{\partial \pi(\hat{p}(\alpha,\beta),z)}{\partial \beta} = \left.\frac{\partial \pi(p,z)}{\partial p}\right|_{p=\hat{p}(\alpha,\beta)} \frac{\partial \hat{p}(\alpha,\beta)}{\partial \beta} \geq 0,$$

where the inequalities are from Proposition 1b. Thus, we conclude this proof. □

Proof of Proposition 4: (a) To facilitate the proof, we define $\pi^*(z) = \max_p (p-z)\bar{F}(p)$. Consequently,

$$\frac{d\pi^*(z)}{dz} = -\bar{F}(p^*(z)) \le 0. \tag{14}$$

Next, we complete the proof by following a two-step procedure: we first establish that $\Delta U_t(x)$ decreases in x and then establish that $\Delta U_t(x)$ increases in t.

$\underline{\Delta U_t(x) \text{ decreases in } x.}$ We show this by induction. For $t=0$, $\Delta U_t(x) = U_t(x) - U_t(x-1) = 0$. Thus, $\Delta U_t(x)$ decreases in x for $t=0$. Suppose that $\Delta U_t(x)$ decreases in x for t—that is, $\Delta U_t(x) \le \Delta U_t(x-1)$. Then, for $t+1$,

$$\begin{aligned}
&\Delta U_{t+1}(x) - \Delta U_{t+1}(x+1) \\
\overset{(a)}{=}\ & U_{t+1}(x) - U_{t+1}(x-1) - [U_{t+1}(x+1) - U_{t+1}(x)] \\
\overset{(b)}{=}\ & U_t(x) - U_t(x-1) - [U_t(x+1) - U_t(x)] \\
& + \lambda\left[2\max_p \hat{\pi}(p, \Delta U_t(x)) - \max_p \hat{\pi}(p, \Delta U_t(x-1)) - \max_p \hat{\pi}(p, \Delta U_t(x+1))\right] \\
\overset{(c)}{=}\ & \Delta U_t(x) - \Delta U_t(x+1) + \lambda\left[2\max_p \hat{\pi}(p, \Delta U_t(x)) - \max_p \hat{\pi}(p, \Delta U_t(x-1)) - \max_p \hat{\pi}(p, \Delta U_t(x+1))\right] \\
\overset{(d)}{\ge}\ & \Delta U_t(x) - \Delta U_t(x+1) + \lambda\left[\max_p \hat{\pi}(p, \Delta U_t(x)) - \max_p \hat{\pi}(p, \Delta U_t(x+1))\right] \\
\overset{(e)}{=}\ & \Delta U_t(x) - \Delta U_t(x+1) + \lambda \max_p [(1+\beta)\pi(p, \Delta U_t(x)) - \alpha r_o(p, \Delta U_t(x)) - \beta\pi^*(\Delta U_t(x))] \\
& - \lambda \max_p [(1+\beta)\pi(p, \Delta U_t(x+1)) - \alpha r_o(p, \Delta U_t(x+1)) - \beta\pi^*(\Delta U_t(x+1))] \\
\overset{(f)}{\ge}\ & \Delta U_t(x) - \Delta U_t(x+1) + \lambda [(1+\beta)\pi(p, \Delta U_t(x)) - \alpha r_o(p, \Delta U_t(x)) - \beta\pi^*(\Delta U_t(x))]_{p=\hat{p}(\Delta U_t(x+1))} \\
& - \lambda [(1+\beta)\pi(p, \Delta U_t(x+1)) - \alpha r_o(p, \Delta U_t(x+1)) - \beta\pi^*(\Delta U_t(x+1))]_{p=\hat{p}(\Delta U_t(x+1))} \\
\overset{(g)}{\ge}\ & \Delta U_t(x) - \Delta U_t(x+1) + \lambda [(1+\beta)\pi(p, \Delta U_t(x)) - \beta\pi^*(\Delta U_t(x))]_{p=\hat{p}(\Delta U_t(x+1))} \\
& - \lambda [(1+\beta)\pi(p, \Delta U_t(x+1)) - \beta\pi^*(\Delta U_t(x+1))]_{p=\hat{p}(\Delta U_t(x+1))} \\
\overset{(h)}{=}\ & [\Delta U_t(x) - \Delta U_t(x+1)][1 - \lambda(1+\beta)\bar{F}(\hat{p}(\Delta U_t(x+1)))] + \lambda\beta[\pi^*(\Delta U_t(x+1)) - \pi^*(\Delta U_t(x))] \\
\overset{(i)}{\ge}\ & 0,
\end{aligned}$$

where (a) and (c) are from the definition of $\Delta U_t(\cdot)$, (b) is from Equation (10), (d) is from $\hat{\pi}(p, \Delta U_t(x)) \ge \hat{\pi}(p, \Delta U_t(x-1))$—which follows from the assumption that $\hat{m}(z, \alpha, \beta) \le 0$ and the assumption that $\Delta U_t(x) \le \Delta U_t(x-1)$, (e) is because

$$\hat{\pi}(p, z) = (1+\beta)\pi(p, z) - \alpha r_o(p, z) - \beta\pi^*(z), \tag{15}$$

which follows from (4) and Proposition 2, (f) is by definition of maximization, (g) is because of the fact that $\frac{\partial r_o(p,z)}{\partial z} = -[F(p) - F(y(p,z))] \le 0$ and the assumption that $\Delta U_t(x+1) \le \Delta U_t(x)$, (h) is from the algebra, and (i) is because $\lambda(1+\beta)\bar{F}(\hat{p}(\Delta U_t(x+1))) \le \lambda(1+\beta) \le 1$—which follows from our assumption that $\lambda \le 0.5$, and Equation (14).

$\underline{\Delta U_t(x) \text{ increases in } t.}$ By the definition of $\Delta U_{t+1}(x)$,

$$\Delta U_{t+1}(x) = U_{t+1}(x) - U_{t+1}(x-1) = \Delta U_t(x) + \lambda\left[\max_p \hat{\pi}(p, \Delta U_t(x)) - \max_p \hat{\pi}(p, \Delta U_t(x-1))\right] \ge \Delta U_t(x),$$

where the inequality is because $\hat{\pi}(p,z)$ decreases in z (our assumption that $\hat{m}(z,\alpha,\beta)\le 0$) and $\Delta U_t(x)\le \Delta U_t(x-1)$, as shown in the first step of this proof.

(b) From Equation (10), $\hat{p}_t(x,\alpha,\beta)=\hat{p}(\Delta U_{t-1}(x))$. To prove part (b), given the results of part (a), it suffices to establish that $\hat{p}(z)$ increases in z. We first note that Proposition 1 implies that $y(p,z)=a$ if $\bar{F}(p)+(a-z)f(a)>1$, and otherwise $y(p,z)$ is characterized by $\bar{F}(p)+(y-z)f(y)=\bar{F}(y)\Longleftrightarrow z=y-\frac{1-\bar{F}(p)/\bar{F}(y)}{h(y)}$. Because $\frac{1-\bar{F}(p)/\bar{F}(y)}{h(y)}$ decreases in y, we have

$$\frac{\partial y(p,z)}{\partial z}=\left[1-\frac{\partial \frac{1-\bar{F}(p)/\bar{F}(y)}{h(y)}}{\partial y}\right]^{-1}_{y=y(p,z)}<1. \tag{16}$$

Then,

$$\begin{aligned}\frac{\partial^2\hat{\pi}(p,z)}{\partial p\partial z}&\overset{\text{(a)}}{=}(1+\beta)\bar{F}(p)h(p)\left[1-\frac{\alpha}{1+\beta}\left(\frac{\partial y(p,z)}{\partial z}-1\right)\right]\\&\overset{\text{(b)}}{\ge}0,\end{aligned}$$

where (a) is from (13) and (b) is from (16). Consequently, $\hat{p}(z)$ increases in z. □

Proof of Proposition 5: We prove the result in the following two steps. First, we show that $\hat{p}_t(x,\alpha=0,\beta)=p_t(x)$ for any β. If $\alpha=0$, from (15),

$$\max_p\hat{\pi}(p,z)=\max_p(1+\beta)\pi(p,z)-\beta\pi^*(z)=\max_p\pi(p,z).$$

In other words, when $\alpha=0$, the biased retailer's problem is equivalent to the unbiased retailer's problem. Based on this property, one can easily show that $\hat{p}_t(x,\alpha=0,\beta)=p_t(x)$ for any pair (t,x) by induction.

Second, we show that $\hat{p}_t(x,\alpha,\beta)$ decreases in α. To facilitate the proof, we rewrite $\hat{p}(z)$ as $\hat{p}(z,\alpha,\beta)$. To show that $\hat{p}_t(x,\alpha,\beta)$ decreases in α, we note that

$$\frac{\partial\hat{p}_t(x,\alpha,\beta)}{\partial\alpha}=\frac{\partial\hat{p}(\Delta U_{t-1}(x),\alpha,\beta)}{\partial\alpha}=\underbrace{\left.\frac{\partial\hat{p}(z,\alpha,\beta)}{\partial z}\right|_{z=\Delta U_{t-1}(x)}}_{\ge 0\text{ (proof of Proposition 4b)}}\frac{\partial\Delta U_{t-1}(x)}{\partial\alpha}+\underbrace{\left.\frac{\partial\hat{p}(z,\alpha,\beta)}{\partial\alpha}\right|_{z=\Delta U_{t-1}(x)}}_{\le 0\text{ (Proposition 1b)}}.$$

Thus, it suffices to show that $\frac{\partial\Delta U_t(x)}{\partial\alpha}\le 0$, which is established below.

Lemma 3 *For any (t,x), $\partial\Delta U_t(x)/\partial\alpha\le 0$.*

Proof of Lemma 3: We prove it by induction. For $t=0$, $\Delta U_t(x)=0$. Thus, $\frac{\partial\Delta U_t(x)}{\partial\alpha}\le 0$. Suppose that $\frac{\partial\Delta U_t(x)}{\partial\alpha}\le 0$ holds for t. For $t+1$, from (10),

$$\Delta U_{t+1}(x)=U_{t+1}(x)-U_{t+1}(x-1)=\Delta U_t(x)+\lambda\hat{\pi}(\hat{p}(\Delta U_t(x)),\Delta U_t(x))-\lambda\hat{\pi}(\hat{p}(\Delta U_t(x-1)),\Delta U_t(x-1)),$$

where both $\hat{\pi}(\hat{p}(z),z)$ and $\Delta U_t(x)$ depend on α. Thus,

$$\begin{aligned}\frac{\partial\Delta U_{t+1}(x)}{\partial\alpha}&\overset{\text{(a)}}{=}\frac{\partial\Delta U_t(x)}{\partial\alpha}+\lambda\left[\frac{\partial\hat{\pi}(\hat{p}(z),z)}{\partial z}\frac{\partial\Delta U_t(x)}{\partial\alpha}+\frac{\partial\hat{\pi}(\hat{p}(z),z)}{\partial\alpha}\right]_{z=\Delta U_t(x)}\\&\qquad-\lambda\left[\frac{\partial\hat{\pi}(\hat{p}(z),z)}{\partial z}\frac{\partial\Delta U_t(x-1)}{\partial\alpha}+\frac{\partial\hat{\pi}(\hat{p}(z),z)}{\partial\alpha}\right]_{z=\Delta U_t(x-1)}\end{aligned}$$

$$
\begin{aligned}
&\overset{\text{(b)}}{\leq} \frac{\partial \Delta U_t(x)}{\partial \alpha} + \lambda \left[\frac{\partial \hat{\pi}(\hat{p}(z),z)}{\partial z} \frac{\partial \Delta U_t(x)}{\partial \alpha} + \frac{\partial \hat{\pi}(\hat{p}(z),z)}{\partial \alpha} \right]_{z=\Delta U_t(x)} - \lambda \left. \frac{\partial \hat{\pi}(\hat{p}(z),z)}{\partial \alpha} \right|_{z=\Delta U_t(x-1)} \\
&\overset{\text{(c)}}{=} \left[1 + \lambda \left. \frac{\partial \hat{\pi}(\hat{p}(z),z)}{\partial z} \right|_{z=\Delta U_t(x)} \right] \frac{\partial \Delta U_t(x)}{\partial \alpha} + \lambda \left[\left. \frac{\partial \hat{\pi}(\hat{p}(z),z)}{\partial \alpha} \right|_{z=\Delta U_t(x)} - \left. \frac{\partial \hat{\pi}(\hat{p}(z),z)}{\partial \alpha} \right|_{z=\Delta U_t(x-1)} \right] \\
&\overset{\text{(d)}}{\leq} \left[1 + \lambda \left. \frac{\partial \hat{\pi}(\hat{p}(z),z)}{\partial z} \right|_{z=\Delta U_t(x)} \right] \frac{\partial \Delta U_t(x)}{\partial \alpha} \\
&\overset{\text{(e)}}{\leq} [1 - \lambda(1+\beta)\bar{F}(\hat{p}(\Delta U_t(x)))] \frac{\partial \Delta U_t(x)}{\partial \alpha} \\
&\overset{\text{(f)}}{\leq} 0,
\end{aligned}
$$

where (a) and (c) are from the algebra, (b) is from $\frac{\partial \hat{\pi}(\hat{p}(z),z)}{\partial z} \leq 0$ that follows our assumption $\hat{m}(z,\alpha,\beta) \leq 0$ and the assumption that $\frac{\partial \Delta U_t(x-1)}{\partial \alpha} \leq 0$, (d) is from $\frac{\partial^2 \hat{\pi}(\hat{p}(z),z)}{\partial z \partial \alpha} \geq 0$ that follows from our assumption $\partial \hat{m}(z,\alpha,\beta)/\partial \alpha \geq 0$ and $\Delta U_t(x) \leq \Delta U_t(x-1)$ that follows from Proposition 4a, (e) is from

$$
\frac{\partial \hat{\pi}(\hat{p}(z),z)}{\partial z} = -(1+\beta)\bar{F}(\hat{p}(z)) + \alpha[F(\hat{p}(z)) - F(y(\hat{p}(z),z))] + \beta \bar{F}(p^*(z)) \geq -(1+\beta)\bar{F}(\hat{p}(z)) \tag{17}
$$

that follows from (15) and the envelope theorem, and (f) is from $1 - \lambda(1+\beta)\bar{F}(\hat{p}(\Delta U_t(x))) \geq 1 - \lambda(1+\beta) \geq 0$ that follows from our assumption $\lambda \leq 0.5$, and the assumption that $\frac{\partial \Delta U_t(x)}{\partial \alpha} \leq 0$. This completes our proof in the second step. □

Proof of Proposition 6: We first show $\Delta \hat{V}_t(x) > 0$ for $x = 1$. From (4), $\max_p \hat{\pi}(p,z) \geq \hat{\pi}(p^*,z) = \pi(p^*,z) - \alpha[\pi(y(p^*,z),z) - (y(p^*,z) - z)\bar{F}(p^*)] > 0$ for any $0 \leq \alpha \leq 1$ and $z = 0$. Thus, $U_1(1) > 0$. From Proposition 4a, then $U_t(1) > 0$ for any t. To show that $\Delta \hat{V}_t(1) > 0$, it then suffices to show that $U_t(x) \leq \hat{V}_t(x)$. We prove this by induction. It is trivial for $t = 0$. Suppose that it holds for t. Then, for $t+1$,

$$
\begin{aligned}
U_{t+1}(x) &\overset{\text{(a)}}{=} U_t(x) + \lambda \hat{\pi}(\hat{p}_{t+1}(x,\alpha,\beta), \Delta U_t(x)) \\
&\overset{\text{(b)}}{=} U_t(x) + \lambda[\pi(\hat{p}_{t+1}(x,\alpha,\beta), \Delta U_t(x)) - \alpha r_o(\hat{p}_{t+1}(x,\alpha,\beta), \Delta U_t(x)) - \beta r_u(\hat{p}_{t+1}(x,\alpha,\beta), \Delta U_t(x))] \\
&\overset{\text{(c)}}{\leq} U_t(x) + \lambda \pi(\hat{p}_{t+1}(x,\alpha,\beta), \Delta U_t(x)) \\
&\overset{\text{(d)}}{=} \lambda \bar{F}(\hat{p}_{t+1}(x,\alpha,\beta))\hat{p}_{t+1}(x,\alpha,\beta) + [1 - \lambda \bar{F}(\hat{p}_{t+1}(x,\alpha,\beta))]U_t(x) + \lambda \bar{F}(\hat{p}_{t+1}(x,\alpha,\beta))U_t(x-1) \\
&\overset{\text{(e)}}{\leq} \lambda \bar{F}(\hat{p}_{t+1}(x,\alpha,\beta))\hat{p}_{t+1}(x,\alpha,\beta) + [1 - \lambda \bar{F}(\hat{p}_{t+1}(x,\alpha,\beta))]\hat{V}_t(x) + \lambda \bar{F}(\hat{p}_{t+1}(x,\alpha,\beta))\hat{V}_t(x-1) \\
&\overset{\text{(f)}}{=} \hat{V}_{t+1}(x),
\end{aligned}
$$

where (a) is from (10), (b) is from (4), (c) and (d) are from the algebra, (e) is from the assumption that $U_t(x) \leq \hat{V}_t(x)$ for any x, and (f) is from (11).

We now show that there exist parameters (α, β) and inventory level $\underline{x}$ such that the biased seller earns less revenue with more inventory, i.e., $\Delta \hat{V}_t(x) < 0$, when λ is small enough and $x \in (\underline{x}, t]$. To this end, we first present two lemmas and then follow a two-step procedure to complete the proof.

Lemma 4(a) *Let $\hat{y}(z) := y(\hat{p}(z),z)$, where $\hat{p}(z)$ and $y(p,z)$ are defined in Proposition 1. Let $\tilde{p}(\alpha)$ uniquely solve $h(p)\left[p - a + (1+\alpha)\frac{F(p)}{f(a)}\right] = 1$. If $\beta = 0$, $\hat{p}(z)$ and $\hat{y}(z)$ are uniquely characterized as follows. (i) If $z < a - \frac{1}{f(a)(1+\alpha)}$, then $(\hat{p}(z), \hat{y}(z)) = (a,a)$. (ii) If $a - \frac{1}{f(a)(1+\alpha)} \leq z < a - \frac{F(\tilde{p}(\alpha))}{f(a)}$, then $\hat{y}(z) = a$ and $\hat{p}(z)$ solves $p - z + \alpha(a - z) = \frac{1}{h(p)}$. (iii) If $z \geq a - \frac{F(\tilde{p}(\alpha))}{f(a)}$, $(\hat{p}(z), \hat{y}(z))$ solves $(p - z) + \alpha(y - z) = \frac{1}{h(p)}$ and $y - z = \frac{F(p) - F(y)}{f(y)}$.*

Lemma 4(b) *Let $w_0 = \hat{\pi}(\hat{p}(0), 0)$ and $w_1 = \bar{F}(\hat{p}(0)) - \alpha F(\hat{p}(0))$. Given that $\beta = 0$, $af(a)(1+\alpha) \leq 1$ and $af(a) > F(\tilde{p}(\alpha))$. (i) Then, $\hat{\pi}(\hat{p}(z), z) = w_0 - w_1 z + o(z)$. (ii) If $x < t$, $U_t(x) = tw_0\lambda - C_t^{x+1} w_0 w_1^x \lambda^{x+1} + o(\lambda^{x+1})$, where C_n^m is a m-combination of a set with $n(\geq m)$ elements—that is, $C_n^m = \frac{n!}{m!(n-m)!}$. Otherwise, $U_t(x) = tw_0\lambda$.*

Proof of Lemma 4(a): From Proposition 1, $\hat{p}(z)$ and $\hat{y}(z)$ are uniquely determined.

(i) Since $z < a - \frac{1}{f(a)(1+\alpha)} \Longleftrightarrow h(a)(a-z)(1+\alpha) > 1$, then $\hat{p}(z) = a$ and hence $\hat{y}(z) = a$.

(ii) It suffices to show that $\bar{F}(\hat{p}(z)) + (a-z)f(a) > 1$. We show this by contradiction. Suppose $\bar{F}(\hat{p}(z)) + (a-z)f(a) \leq 1$. From $z < a - \frac{F(\tilde{p}(\alpha))}{f(a)}$, then $F(\tilde{p}(\alpha)) < (a-z)f(a) \leq F(\hat{p}(z))$—namely, $\tilde{p}(\alpha) < \hat{p}(z)$. Thus,

$$
\begin{aligned}
1 &\overset{(a)}{=} h(\hat{p}(z))\left[\hat{p}(z) - z + \alpha\left(y(\hat{p}(z), z) - z\right)\right] \\
&\overset{(b)}{\geq} h(\hat{p}(z))\left[\hat{p}(z) - z + \alpha(a-z)\right] \\
&\overset{(c)}{\geq} h(\tilde{p}(\alpha))\left[\tilde{p}(\alpha) - z + \alpha(a-z)\right] \\
&\overset{(d)}{>} h(\tilde{p}(\alpha))\left[\tilde{p}(\alpha) - \left(a - \frac{F(\tilde{p}(\alpha))}{f(a)}\right) + \alpha\frac{F(\tilde{p}(\alpha))}{f(a)}\right] \\
&\overset{(e)}{=} h(\tilde{p}(\alpha))\left[\tilde{p}(\alpha) - a + (1+\alpha)\frac{F(\tilde{p}(\alpha))}{f(a)}\right],
\end{aligned}
$$

where (a) is from $\hat{p}(z)$ that is characterized in Proposition 1a, (b) is from $y(\hat{p}(z), z) \geq a$, (c) is from $\tilde{p}(\alpha) < \hat{p}(z)$, and $h'(\cdot) \geq 0$, (d) is from our assumption that $z < a - \frac{F(\tilde{p}(\alpha))}{f(a)}$, and (e) is from the algebra. Thus, $h(\tilde{p}(\alpha))[\tilde{p}(\alpha) - a + (1+\alpha)\frac{F(\tilde{p}(\alpha))}{f(a)}] < 1$, which contradicts the definition of $\tilde{p}(\alpha)$. Then, the result follows.

(iii) The proof is similar to the proof of (ii).

Proof of Lemma 4(b): (i) If $af(a)(1+\alpha) \leq 1$ and $af(a) > F(\tilde{p}(\alpha))$, we focus on z such that $0 \leq z < a - \frac{F(\tilde{p}(\alpha))}{f(a)}$. Given $\beta = 0$, from (ii) of Lemma 4(a), $\hat{y}(z) = a$ and $\hat{p}(z) - z + \alpha(a-z) = \frac{1}{h(\hat{p}(z))}$. Consequently,

$$
\frac{d\hat{p}(z)}{dz} - 1 - \alpha = \left[\frac{1}{h(p)}\right]'_{p=\hat{p}(z)} \frac{d\hat{p}(z)}{dz} \Longrightarrow \left[1 - \left[\frac{1}{h(p)}\right]'_{p=\hat{p}(z)}\right] \frac{d\hat{p}(z)}{dz} = 1 + \alpha. \tag{18}
$$

Moreover,

$$
\begin{aligned}
\hat{\pi}(\hat{p}(z), z) &\overset{(a)}{=} [\hat{p}(z) - z]\bar{F}(\hat{p}(z)) - \alpha(a-z)[F(\hat{p}(z)) - F(a)] \\
&\overset{(b)}{=} \left[\frac{1}{h(\hat{p}(z))} - \alpha(a-z)\right]\bar{F}(\hat{p}(z)) - \alpha(a-z)F(\hat{p}(z)) \\
&\overset{(c)}{=} \frac{\bar{F}(\hat{p}(z))}{h(\hat{p}(z))} - \alpha(a-z),
\end{aligned}
$$

where (a) is from (4) and $\hat{y}(z) = a$, (b) is from $\hat{p}(z) - z + \alpha(a-z) = \frac{1}{h(\hat{p}(z))}$, and (c) is from the algebra. Accordingly,

$$
\begin{aligned}
\frac{d\hat{\pi}(\hat{p}(z), z)}{dz} &\overset{(a)}{=} \left[\frac{\bar{F}(p)}{h(p)}\right]'_{p=\hat{p}(z)} \frac{d\hat{p}(z)}{dz} + \alpha \\
&\overset{(b)}{=} \left[-f(p)\frac{1}{h(p)} + \bar{F}(p)\left[\frac{1}{h(p)}\right]'\right]_{p=\hat{p}(z)} \frac{d\hat{p}(z)}{dz} + \alpha \\
&\overset{(c)}{=} -\bar{F}(\hat{p}(z))\left[1 - \left[\frac{1}{h(p)}\right]'_{p=\hat{p}(z)}\right]\frac{d\hat{p}(z)}{dz} + \alpha \\
&\overset{(d)}{=} -\bar{F}(\hat{p}(z))(1+\alpha) + \alpha,
\end{aligned}
$$

where (a)–(c) are from the algebra and (d) is from (18). Consequently,

$$\hat{\pi}(\hat{p}(z),z)=\hat{\pi}(\hat{p}(0),0)+[-\bar{F}(\hat{p}(0))(1+\alpha)+\alpha]z+o(z)=\hat{\pi}(\hat{p}(0),0)-[\bar{F}(\hat{p}(0))-\alpha F(\hat{p}(0))]z+o(z).$$

(ii) We prove this by induction. For $t=1$, it holds by the definition of w_0. Suppose that it holds for t. Then, we prove it for $t+1$. We have three cases: $x=1$, $1<x<t+1$, and $x\geq t+1$.

For the first case where $x=1$,

$$\begin{aligned}U_{t+1}(1) &\overset{(a)}{=} U_t(1)+\lambda\hat{\pi}(\hat{p}(U_t(1)),U_t(1))\\ &\overset{(b)}{=} tw_0\lambda-C_t^2w_0w_1\lambda^2+o(\lambda^2)+\lambda\,[w_0-w_1tw_0\lambda+o(\lambda)]\\ &\overset{(c)}{=} (t+1)w_0\lambda-[C_t^2+t]w_0w_1\lambda^2+o(\lambda^2)\\ &\overset{(d)}{=} (t+1)w_0\lambda-C_{t+1}^2w_0w_1\lambda^2+o(\lambda^2),\end{aligned}$$

where (a) is from (10), (b) is from our assumption that $U_t(1)=tw_0\lambda-C_t^2w_0w_1\lambda^2+o(\lambda^2)$ and, hence, $\hat{\pi}(\hat{p}(U_t(1)),U_t(1))=w_0-w_1tw_0\lambda+o(\lambda)$ around $\lambda=0$ (see part (i)), and (c) and (d) are from the algebra.

For the second case where $1<x<t+1$, $\Delta U_t(x)=U_t(x)-U_t(x-1)=C_t^xa\lambda^x+o(\lambda^x)$. Similarly,

$$\begin{aligned}U_{t+1}(x) &= U_t(x)+\lambda\hat{\pi}(\hat{p}(\Delta U_t(x)),\Delta U_t(x))\\ &= tw_0\lambda-C_t^{x+1}w_0w_1^x\lambda^{x+1}+o(\lambda^{x+1})+\lambda[w_0-w_1C_t^xw_0w_1^{x-1}\lambda^x+o(\lambda^x)]\\ &= (t+1)w_0\lambda-[C_t^{x+1}+C_t^x]w_0w_1^x\lambda^{x+1}+o(\lambda^{x+1})\\ &= (t+1)w_0\lambda-C_{t+1}^{x+1}w_0w_1^x\lambda^{x+1}+o(\lambda^{x+1}),\end{aligned}$$

where the last equality is because

$$\begin{aligned}C_t^{x+1}+C_t^x &= \frac{t!}{(x+1)!(t-x-1)!}+\frac{t!}{x!(t-x)!}\\ &= \frac{t!}{(x+1)!(t-x)!}[(t-x)+(x+1)]\\ &= \frac{(t+1)!}{(x+1)!(t-x)!}\\ &= C_{t+1}^{x+1}.\end{aligned}$$

If $x\geq t+1$, $U_{t+1}(x)=(t+1)w_0\lambda$. This completes the proof of Lemma 4(b).

Two-step procedure. In the first step, we derive the Taylor expansion of $\pi(\hat{p}(z),z)$ when $af(a)(1+\alpha)=1$ and $\beta=0$. Define $k:=[1-(\frac{1}{h(p)})'_{p=a}]^{-1}$. If $af(a)(1+\alpha)=1$ and $\beta=0$, then $(\hat{p}(0),\hat{y}(0))=(a,a)$ from Lemma 4(a). Thus,

$$\begin{aligned}\pi(\hat{p}(z),z) &\overset{(a)}{=} \hat{\pi}(\hat{p}(z),z)+\alpha(a-z)F(\hat{p}(z))\\ &\overset{(b)}{=} a-z+\alpha\left.\frac{d(a-z)F(\hat{p}(z))}{dz}\right|_{z=0}z+o(z)\\ &\overset{(c)}{=} a-z+\alpha\left[-F(\hat{p}(z))+(a-z)f(\hat{p}(z))\frac{d\hat{p}(z)}{dz}\right]_{z=0}z+o(z)\\ &\overset{(d)}{=} a-z+\alpha af(a)\frac{1+\alpha}{1-(\frac{1}{h(p)})'_{p=a}}z+o(z)\\ &\overset{(e)}{=} a-[1-k\alpha]\,z+o(z),\end{aligned} \tag{19}$$

where (a) is from (4) and $\hat{y}(z) = a$ around $z = 0$, (b) is from Lemma 4(b) and $\hat{p}(0) = a$, (c) is from the algebra, (d) is from $\hat{p}(0) = a$ and (18), and (e) is from $af(a)(1+\alpha) = 1$.

In the second step, we derive the Taylor expansion of $\hat{V}_t(x)$ when $af(a)(1+\alpha) = 1$ and $\beta = 0$. We show the value function's Taylor expansion at $\lambda = 0$ is $\hat{V}_t(x) = ta\lambda - (1-k\alpha x)C_t^{x+1}a\lambda^{x+1} + o(\lambda^{x+1})$ if $x < t$, and $\hat{V}_t(x) = ta\lambda$ if $x \geq t$ by induction. For $t = 1$, $\hat{V}_t(x) = a\lambda$ as $\hat{p}(0) = a$. Suppose that the said Taylor expansion holds for t, we then prove for $t+1$. We have three cases: $x = 1$, $1 < x < t+1$, and $x \geq t+1$.

For the first case where $x = 1$,

$$\begin{aligned}
\hat{V}_{t+1}(1) &\overset{(a)}{=} \hat{V}_t(1) + \lambda\pi(\hat{p}(U_t(1)), \hat{V}_t(1)) \\
&\overset{(b)}{=} \hat{V}_t(1) + \lambda\left[\pi(\hat{p}(U_t(1)), U_t(1)) - [\hat{V}_t(1) - U_t(1)]\bar{F}(\hat{p}(U_t(1)))\right] \\
&\overset{(c)}{=} ta\lambda - (1-k\alpha)C_t^2 a\lambda^2 + o(\lambda^2) + \lambda\left[a - (1-k\alpha)ta\lambda + o(\lambda)\right] \\
&\overset{(d)}{=} (t+1)a\lambda - (1-k\alpha)[C_t^2 + t]a\lambda^2 + o(\lambda^2) \\
&\overset{(e)}{=} (t+1)a\lambda - (1-k\alpha)C_{t+1}^2 a\lambda^2 + o(\lambda^2),
\end{aligned}$$

where (a) is from (11), (b) is from the fact that

$$\begin{aligned}
\pi(\hat{p}(z_1), z_2) &= [\hat{p}(z_1) - z_2]\bar{F}(\hat{p}(z_1)) \\
&= [\hat{p}(z_1) - z_1]\bar{F}(\hat{p}(z_1)) - (z_2 - z_1)\bar{F}(\hat{p}(z_1)) \\
&= \pi(\hat{p}(z_1), z_1) - (z_2 - z_1)\bar{F}(\hat{p}(z_1)),
\end{aligned}$$

(c) is from the assumption that $\hat{V}_t(1) = ta\lambda - (1-k\alpha)C_t^2 a\lambda^2 + o(\lambda^2)$, $U_t(1) = ta\lambda - C_t^2 a\lambda^2 + o(\lambda^2)$ (Lemma 4(b)), (19) and $\hat{V}_t(1) - U_t(1) = k\alpha C_t^2 a\lambda^2 + o(\lambda^2) = o(\lambda)$, and (d) and (e) are from the algebra.

For the second case where $1 < x < t+1$, from Lemma 4(b), $\Delta U_t(x) = U_t(x) - U_t(x-1) = C_t^x a\lambda^x + o(\lambda^x)$. By assumption that $\hat{V}_t(x) = ta\lambda - (1-k\alpha x)C_t^{x+1}a\lambda^{x+1} + o(\lambda^{x+1})$, then $\Delta\hat{V}_t(x) = [1 - k\alpha(x-1)]C_t^x a\lambda^x + o(\lambda^x)$. Thus, $[\Delta\hat{V}_t(x) - \Delta U_t(x)]\bar{F}(\hat{p}(\Delta U_t(x))) = -k\alpha(x-1)C_t^x a\lambda^x + o(\lambda^x)$, which is because $\bar{F}(\hat{p}(\Delta U_t(x))) = \bar{F}(\hat{p}(0)) + o(1) = 1 + o(1)$. Following similar reasons as the case where $x = 1$,

$$\begin{aligned}
\hat{V}_{t+1}(x) &= \hat{V}_t(x) + \lambda\ \pi(\hat{p}(\Delta U_t(x)), \Delta\hat{V}_t(x)) \\
&= \hat{V}_t(x) + \lambda\left[\pi(\hat{p}(\Delta U_t(x)), \Delta U_t(x)) - (\Delta\hat{V}_t(x) - \Delta U_t(x))\bar{F}(\hat{p}(\Delta U_t(x)))\right] \\
&= ta\lambda - (1-k\alpha x)C_t^{x+1}a\lambda^{x+1} + o(\lambda^{x+1}) + \lambda\left[a - (1-k\alpha)C_t^x a\lambda^x + k\alpha(x-1)C_t^x a\lambda^x + o(\lambda^x)\right] \\
&= (t+1)a\lambda - (1-k\alpha x)(C_t^{x+1} + C_t^x)a\lambda^{x+1} + o(\lambda^{x+1}) \\
&= (t+1)a\lambda - (1-k\alpha x)C_{t+1}^{x+1}a\lambda^{x+1} + o(\lambda^{x+1}),
\end{aligned}$$

where the last equality is because $C_t^{x+1} + C_t^x = C_{t+1}^{x+1}$.

For the third case where $x \geq t+1$, $\hat{V}_{t+1}(x) = (t+1)a\lambda$.

Accordingly, if $af(a)(1+\alpha) = 1$ and $\beta = 0$, $\Delta\hat{V}_t(x) = [1 - k\alpha(x-1)]C_t^x a\lambda^x + o(\lambda^x)$ for $1 \leq x \leq t$. Consequently, for any $x \leq t$, $\Delta\hat{V}_t(x) < 0 \Longleftrightarrow 1 - k\alpha(x-1) < 0 \Longleftrightarrow x > 1 + \frac{1}{k\alpha}$, around $\lambda = 0$. □

PROPOSITION 7. *Suppose $g(\geq 0)$ is the unit cost of the product. Let $\hat{c}(\alpha,\beta) := \max_{c\in\{0,1,2,3,\cdots\}}[U_T(c) - gc]$, and $c^* := \max_{c\in\{0,1,2,3,\cdots\}}[V_T(c) - gc]$. The biased seller orders a stock level lower than the unbiased seller—that is, $\hat{c}(\alpha,\beta) \leq c^*$.*

Proof of Proposition 7: From Proposition 4a, $\Delta U_T(c)$ is decreasing in c. Thus, $\hat{c}(\alpha,\beta) = \max_{c\in\{0,1,2,3,\cdots\}}[U_T(c)-gc] = \max_{c\in\{0,1,2,3,\cdots\}}\{c:\Delta U_T(c)\geq g\}$. Similarly, $c^* = \max_{c\in\{0,1,2,3,\cdots\}}\{c:\Delta V_T(c)\geq g\}$. From the proof of Proposition 5, $U_T(c)=V_T(c)$ if $\alpha=0$. From Lemma 3, $\partial\Delta U_T(c)/\partial\alpha\leq 0$. Thus, $\Delta V_T(c)\geq\Delta U_T(c)$ for any (α,β) and the result then follows. $\square$

LEMMA 5. *If $af(a)(1+\alpha)=1$, $\beta=0$, $(1+\alpha)F(p^*(0))<1$, $(\frac{1}{h(p)})''\leq 0$, and λ is small enough, then $\hat{p}_t(x,\alpha,\beta)/p_t(x)$ is decreasing in x for any t.*

Proof of Lemma 5: For ease of presentation, we rewrite $\hat{p}_t(x,\alpha,\beta=0)$ as $\hat{p}_t(x,\alpha)$. We prove the result following two steps. In the first step, we derive the expressions of $\hat{p}_t(x,\alpha)$ and $p_t(x)$. In the second step, we consider the ratio.

<u>The expressions of $\hat{p}_t(x,\alpha)$ and $p_t(x)$.</u> We first show that $af(a)>F(\tilde{p}(\hat{\alpha}))$ for any $\hat{\alpha}\geq 0$. To that aim, we first show $af(a)>F(\tilde{p}(0))$ by contradiction and then show that $\tilde{p}(\hat{\alpha})$ decreases in $\hat{\alpha}$. Suppose that $af(a)\leq F(\tilde{p}(0))$. From the definition of $\tilde{p}(0)$ in Lemma 4(a),

$$\begin{aligned}\tilde{p}(0)-\frac{1}{h(\tilde{p}(0))}=a-\frac{F(\tilde{p}(0))}{f(a)} &\overset{(a)}{\Longrightarrow} \tilde{p}(0)-\frac{1}{h(\tilde{p}(0))}\leq 0\\ &\overset{(b)}{\Longrightarrow} \tilde{p}(0)h(\tilde{p}(0))\leq 1\\ &\overset{(c)}{\Longrightarrow} p^*(0)\geq\tilde{p}(0)\\ &\overset{(d)}{\Longrightarrow} af(a)>F(\tilde{p}(0)),\end{aligned}$$

where (a) is from $af(a)\leq F(\tilde{p}(0))$, (b) is from the algebra, (c) is from the definition and uniqueness of $p^*(0)$, and (d) is from $af(a)>F(p^*(0))$ that follows from the assumptions that $af(a)(1+\alpha)=1$ and $(1+\alpha)F(p^*(0))<1$. This leads to a contradiction and, hence, $af(a)>F(\tilde{p}(0))$. To show that $\tilde{p}(\hat{\alpha})$ decreases in $\hat{\alpha}$, note that $\tilde{p}(\hat{\alpha})-a+(1+\hat{\alpha})\frac{F(\tilde{p}(\hat{\alpha}))}{f(a)}=\frac{1}{h(\tilde{p}(\hat{\alpha}))}$,

$$\begin{aligned}&\tilde{p}'(\hat{\alpha})+\frac{F(p)}{f(a)}+(1+\hat{\alpha})\frac{f(p)}{f(a)}\tilde{p}'(\hat{\alpha})=\left[\frac{1}{h(p)}\right]'_{p=\tilde{p}(\hat{\alpha})}\tilde{p}'(\hat{\alpha})\\ \Longrightarrow\;&\left[1+(1+\hat{\alpha})\frac{f(p)}{f(a)}-\left[\frac{1}{h(p)}\right]'_{p=\tilde{p}(\hat{\alpha})}\right]\tilde{p}'(\hat{\alpha})=-\frac{F(p)}{f(a)}\\ \Longrightarrow\;&\tilde{p}'(\hat{\alpha})\leq 0.\end{aligned}$$

Next, we derive the general expression of $\hat{p}_t(x,\hat{\alpha})$. Given $af(a)(1+\alpha)=1$, then $af(a)(1+\hat{\alpha})\leq 1$ and $af(a)>F(\tilde{p}(\hat{\alpha}))$ for any $0\leq\hat{\alpha}\leq\alpha$. Let $\kappa(p):=[1-(\frac{1}{h(p)})']^{-1}$. Thus, for any $x\leq t$,

$$\begin{aligned}\hat{p}_{t+1}(x,\hat{\alpha}) &\overset{(a)}{=} \hat{p}(\Delta U_t(x))\\ &\overset{(b)}{=} \hat{p}(0)+\frac{d\hat{p}(z)}{dz}|_{z=0}\Delta U_t(x)+o(\Delta U_t(x))\\ &\overset{(c)}{=} \hat{p}(0)+(1+\hat{\alpha})\kappa(\hat{p}(0))C_t^x w_0 w_1^{x-1}\lambda^x+o(\lambda^x),\end{aligned}\tag{20}$$

where (a) is by definition, (b) is from the Taylor expansion, (c) is from (18) and $\Delta U_t(x)=C_t^x w_0 w_1^{x-1}\lambda^x+o(\lambda^x)$ for $x\leq t$—that follows from Lemma 4(b). Moreover, $\hat{p}_{t+1}(x,\hat{\alpha})=\hat{p}(0)$ for $x>t$. Note that $\hat{p}(0)$ and, accordingly, $w_0=\hat{\pi}(\hat{p}(0),0)$ and $w_1=\bar{F}(\hat{p}(0))-\hat{\alpha}F(\hat{p}(0))$ all depend on $\hat{\alpha}$.

We now derive the specific expression of $\hat{p}_{t+1}(x,\hat{\alpha})$ for $\hat{\alpha}=\alpha$ and $\hat{\alpha}=0$. For $\hat{\alpha}=\alpha$ such that $af(a)(1+\alpha)=1$, from Lemma 4(a), $\hat{p}(0)=a$, then $\kappa(\hat{p}(0))=k$, $w_0=a$, and $w_1=1$. Accordingly, from (20), $\hat{p}_{t+1}(x,\alpha)=a+(1+\alpha)kaC_t^x\lambda^x+o(\lambda^x)$ for $x\le t$ and $\hat{p}_{t+1}(x,\alpha)=a$ otherwise. Note that $p_t(x)=\hat{p}_t(x,\hat{\alpha}=0)$. For $\hat{\alpha}=0$, $\hat{p}(0)=p^*(0)$, then $\kappa(\hat{p}(0))=\kappa(p^*(0))$, $w_0=\hat{\pi}(p^*(0),0)=p^*(0)\bar{F}(p^*(0))$, and $w_1=\bar{F}(p^*(0))$. Accordingly, from (20), $p_{t+1}(x)=p^*(0)+\kappa(p^*(0))C_t^xp^*(0)[\bar{F}(p^*(0))]^x\lambda^x+o(\lambda^x)$ for $x\le t$ and $p_{t+1}(x)=p^*(0)$ otherwise.

The ratio of $\hat{p}_t(x,\alpha)$ and $p_t(x)$. For $x\le t$,

$$\begin{aligned}\frac{\hat{p}_{t+1}(x,\alpha)}{p_{t+1}(x)} &= \frac{a+(1+\alpha)kaC_t^x\lambda^x+o(\lambda^x)}{p^*(0)+\kappa(p^*(0))C_t^xp^*(0)[\bar{F}(p^*(0))]^x\lambda^x+o(\lambda^x)}\\ &= \frac{a}{p^*(0)}\frac{1+(1+\alpha)kC_t^x\lambda^x}{1+\kappa(p^*(0))C_t^x[\bar{F}(p^*(0))]^x\lambda^x}+o(\lambda^x).\end{aligned}$$

Consequently,

$$\begin{aligned}&\frac{\hat{p}_{t+1}(x+1,\alpha)}{p_{t+1}(x+1)}-\frac{\hat{p}_{t+1}(x,\alpha)}{p_{t+1}(x)}\\ \overset{(a)}{=}\ &\frac{a}{p^*(0)}\left[\frac{1+(1+\alpha)kC_t^{x+1}\lambda^{x+1}}{1+\kappa(p^*(0))C_t^{x+1}[\bar{F}(p^*(0))]^{x+1}\lambda^{x+1}}-\frac{1+(1+\alpha)kC_t^x\lambda^x}{1+\kappa(p^*(0))C_t^x[\bar{F}(p^*(0))]^x\lambda^x}\right]+o(\lambda^x)\\ \overset{(b)}{=}\ &\frac{a}{p^*(0)}\frac{\kappa(p^*(0))C_t^x[\bar{F}(p^*(0))]^x\lambda^x-k(1+\alpha)C_t^x\lambda^x}{\left[1+\kappa(p^*(0))C_t^{x+1}[\bar{F}(p^*(0))]^{x+1}\lambda^{x+1}\right]\left[1+\kappa(p^*(0))C_t^x[\bar{F}(p^*(0))]^x\lambda^x\right]}+o(\lambda^x)\\ \overset{(c)}{=}\ &\frac{a}{p^*(0)}\frac{\left[\kappa(p^*(0))[\bar{F}(p^*(0))]^x-k(1+\alpha)\right]C_t^x\lambda^x}{\left[1+\kappa(p^*(0))C_t^{x+1}[\bar{F}(p^*(0))]^{x+1}\lambda^{x+1}\right]\left[1+\kappa(p^*(0))C_t^x[\bar{F}(p^*(0))]^x\lambda^x\right]}+o(\lambda^x)\\ \overset{(d)}{\le}\ &0,\end{aligned}$$

where (a) and (c) are from the algebra, (b) is from

$$[1+(1+\alpha)kC_t^{x+1}\lambda^{x+1}][1+\kappa(p^*(0))C_t^x[\bar{F}(p^*(0))]^x\lambda^x]=1+\kappa(p^*(0))C_t^x[\bar{F}(p^*(0))]^x\lambda^x+o(\lambda^x),$$

and

$$[1+(1+\alpha)kC_t^x\lambda^x][1+\kappa(p^*(0))C_t^{x+1}[\bar{F}(p^*(0))]^{x+1}\lambda^{x+1}]=1+(1+\alpha)kC_t^x\lambda^x+o(\lambda^x),$$

and (d) is from $\kappa(p^*(0))\le k$—which follows from $p^*(0)\ge\hat{p}(0)$ and the assumption $[1/h(p)]''\le 0\Rightarrow\kappa'(p)\le 0$. Moreover, for $x>t$, $\frac{\hat{p}_{t+1}(x,\alpha)}{p_{t+1}(x)}=\frac{a}{p^*(0)}=\frac{\hat{p}_{t+1}(t+1,\alpha)}{p_{t+1}(t+1)}$. The result then follows. □

Appendix B: Assumption Verification: Uniform and Exponential Distributions

Below, we show the regularity assumptions in Section 2 are satisfied when the customer's willingness-to-pay follows uniform distribution, i.e., $f(\cdot)=1/(b-a)$, and exponential distribution, i.e., $f(\cdot)=\gamma e^{-\gamma(\cdot-a)}$, where $\gamma>0$.

We first establish $\hat{m}(z,\alpha,\beta)\le 0$. Equation (17) implies $\hat{m}(z,\alpha,\beta)=-(1+\beta)\bar{F}(\hat{p}(z))+\alpha[F(\hat{p}(z))-F(y(\hat{p}(z),z)]+\beta\bar{F}(p^*(z))\le\alpha[F(\hat{p}(z))-F(y(\hat{p}(z),z)]-\bar{F}(\hat{p}(z))$, where the inequality is from $\hat{p}(z)\le p^*(z)$ in Proposition 2. Next, we establish

$$\alpha[F(\hat{p}(z))-F(y(\hat{p}(z),z))]\le\bar{F}(\hat{p}(z)). \tag{21}$$

Let $\tilde{\alpha} := \frac{\alpha}{1+\beta}$. According to Lemma 4(a), we have three cases. If $z < a - \frac{1}{f(a)(1+\tilde{\alpha})}$, $(\hat{p}(z), \hat{y}(z)) = (a, a)$ and hence $\alpha[F(\hat{p}(z)) - F(y(\hat{p}(z), z))] - \bar{F}(\hat{p}(z)) = -1$ which implies that (21). If $a - \frac{1}{f(a)(1+\tilde{\alpha})} \le z < a - \frac{F(\tilde{p}(\tilde{\alpha}))}{f(a)}$. Then, $\hat{p}(z)$ and $\hat{y}(z)$ are such that

$$p - z + \tilde{\alpha}(a - z) = \frac{1}{h(p)} \text{ and } \hat{y}(z) = a. \tag{22}$$

Thus, $\alpha[F(\hat{p}(z)) - F(y(\hat{p}(z), z))] - \bar{F}(\hat{p}(z)) = \alpha F(\hat{p}(z)) - \bar{F}(\hat{p}(z)) = (1+\alpha)F(\hat{p}(z)) - 1$. It is easily verified that $\hat{p}(z)$ is continuous and increasing in z. Hence, it suffices to consider the third case: $z \ge a - \frac{F(\tilde{p}(\tilde{\alpha}))}{f(a)}$, where $(\hat{p}(z), \hat{y}(z))$ uniquely solves

$$(p - z) + \tilde{\alpha}(y - z) = \frac{1}{h(p)} \text{ and } y - z = \frac{F(p) - F(y)}{f(y)}. \tag{23}$$

For the uniform willingness-to-pay, (23) reduces to $(p-z) + \tilde{\alpha}(y-z) = b - p$ and $y - z = p - y$. Accordingly, $\hat{p}(z) = [b + (1 + \frac{\tilde{\alpha}}{2})z]/[2 + \frac{\tilde{\alpha}}{2}]$ and $\hat{y}(z) = \frac{\hat{p}(z)+z}{2}$. Moreover,

$$\begin{aligned} F(\hat{p}(z)) - F(y(\hat{p}(z), z)) \le \bar{F}(\hat{p}(z)) &\overset{(a)}{\iff} \hat{p}(z) - y(\hat{p}(z), z) \le b - \hat{p}(z) \\ &\overset{(b)}{\iff} \hat{p}(z) \le \frac{2b+z}{3}, \end{aligned}$$

where (a) is from the algebra and (b) is from $\hat{y}(z) = \frac{\hat{p}(z)+z}{2}$. Thus, $\alpha[F(\hat{p}(z)) - F(y(\hat{p}(z), z))] - \bar{F}(\hat{p}(z)) \le [F(\hat{p}(z)) - F(y(\hat{p}(z), z))] - \bar{F}(\hat{p}(z)) \le 0$, where the first inequality is because $\alpha \le 1$ and the second inequality follows from $\hat{p}(z) \le (b+z)/2 \le (2b+z)/3$ by noting that $z \le b$. Thus, we establish (21).

For the exponential willingness-to-pay, (23) reduces to $(p-z) + \tilde{\alpha}(y-z) = \frac{1}{\gamma}$ and $y - z = \frac{1}{\gamma}(1 - e^{\gamma(y-p)})$. Accordingly, $\hat{p}(z) = \frac{1}{\gamma} + z - \tilde{\alpha}\frac{\tau(\tilde{\alpha})}{\gamma}$ and $\hat{y}(z) = z + \frac{\tau(\tilde{\alpha})}{\gamma}$, where $\tau(\alpha)$ is the unique solution to $e^{(1+\alpha)\tau - 1} + \tau - 1 = 0$. Thus,

$$\begin{aligned} \alpha[F(\hat{p}(z)) - F(y(\hat{p}(z), z))] \le \bar{F}(\hat{p}(z)) &\overset{(a)}{\iff} \alpha[e^{-\gamma(\hat{y}(z)-a)} - e^{-\gamma(\hat{p}(z)-a)}] \le e^{-\gamma(\hat{p}(z)-a)} \\ &\overset{(b)}{\iff} \alpha e^{-\gamma(\hat{y}(z)-a)} \le (1+\alpha)e^{-\gamma(\hat{p}(z)-a)} \\ &\overset{(c)}{\iff} \alpha \le (1+\alpha)e^{-\gamma(\hat{p}(z)-\hat{y}(z))} \\ &\overset{(d)}{\iff} \alpha \le (1+\alpha)e^{-\gamma\left(\frac{1}{\gamma} - \tilde{\alpha}\frac{\tau(\tilde{\alpha})}{\gamma} - \frac{\tau(\tilde{\alpha})}{\gamma}\right)} \\ &\overset{(e)}{\iff} \alpha \le (1+\alpha)e^{(1+\tilde{\alpha})\tau(\tilde{\alpha})-1} \\ &\overset{(f)}{\iff} \alpha \le (1+\alpha)[1 - \tau(\tilde{\alpha})] \\ &\overset{(g)}{\iff} (1+\alpha)\tau(\tilde{\alpha}) - 1 \le 0, \end{aligned}$$

where (a), (b), (c), (e), and (g) are from the algebra, (d) is because $(\hat{p}(z), \hat{y}(z)) = (\frac{1}{\gamma} + z - \tilde{\alpha}\frac{\tau(\tilde{\alpha})}{\gamma}, z + \frac{\tau(\tilde{\alpha})}{\gamma})$, and (f) is from $e^{(1+\tilde{\alpha})\tau(\tilde{\alpha})-1} + \tau(\tilde{\alpha}) - 1 = 0$. Thus, to show (21), it suffices to show $(1+\alpha)\tau(\tilde{\alpha}) - 1 \le 0$, which follows from

$$(1+\alpha)\tau(\tilde{\alpha}) \overset{(a)}{\le} (1+\alpha)\tau(\frac{\alpha}{2}) \overset{(b)}{\le} 2\tau(0.5) \overset{(c)}{\le} 1.$$

For (a), $e^{(1+\alpha)\tau(\alpha)-1} + \tau(\alpha) - 1 = 0$ implies that

$$\begin{aligned} e^{(1+\alpha)\tau(\alpha)-1}[\tau(\alpha) + (1+\alpha)\tau'(\alpha)] + \tau'(\alpha) = 0 &\Longrightarrow [1 - \tau(\alpha)][\tau(\alpha) + (1+\alpha)\tau'(\alpha)] + \tau'(\alpha) = 0 \\ &\Longrightarrow \tau'(\alpha) = -\frac{[1-\tau(\alpha)]\tau(\alpha)}{1 + [1-\tau(\alpha)](1+\alpha)}, \end{aligned} \tag{24}$$

i.e., $\tau(\alpha)$ is decreasing in α because $\tau(\alpha)<1$. Thus, $(1+\alpha)\tau(\tilde{\alpha})=(1+\alpha)\tau(\frac{\alpha}{1+\beta})\le(1+\alpha)\tau(\frac{\alpha}{2})$, where the inequality is from $\beta\le 1$. For (b), it suffices to show $(1+\alpha)\tau(\frac{\alpha}{2})$ is increasing in α because $\alpha\in[0,1]$. Note that

$$\begin{aligned}\Big[(1+\alpha)\tau(\frac{\alpha}{2})\Big]'\ge 0 &\iff \tau(\frac{\alpha}{2})+(1+\alpha)\tau'(\frac{\alpha}{2})\frac{1}{2}\ge 0\\ &\iff \tau(\frac{\alpha}{2})\ge\frac{1+\alpha}{2}\frac{[1-\tau(\frac{\alpha}{2})]\tau(\frac{\alpha}{2})}{1+[1-\tau(\frac{\alpha}{2})](1+\frac{\alpha}{2})}\\ &\iff 3\ge\tau(\frac{\alpha}{2}).\end{aligned}$$

where the second equivalence is from (24). Then $\big[(1+\alpha)\tau(\frac{\alpha}{2})\big]'\ge 0$ follows by noting $\tau(\alpha)<1$ for any α. For (c), since $e^{1.5\tau-1}+\tau-1$ is increasing in τ, and $e^{1.5*0.5-1}+0.5-1\ge 0\iff 16\ge e$, we have $\tau(0.5)\le 0.5$.

We now establish $\frac{\partial\hat{m}(z,\alpha,\beta)}{\partial\alpha}\ge 0$. By the definition of $\hat{m}(z,\alpha,\beta)$,

$$\frac{\partial\hat{m}(z,\alpha,\beta)}{\partial\alpha}=\frac{\partial^2\hat{\pi}(\hat{p}(z),z)}{\partial\alpha\partial z}=\frac{\partial}{\partial z}\frac{\partial\hat{\pi}(\hat{p}(z),z)}{\partial\alpha}=-\frac{dr_o(\hat{p}(z),z)}{dz},$$

where the last equality is from the envelope theorem, $\hat{\pi}(p,z)$ in (4), and $r_o(\hat{p}(z),z)=(\hat{y}(z)-z)[F(\hat{p}(z))-F(\hat{y}(z))]$. Next, we establish $\frac{dr_o(\hat{p}(z),z)}{dz}\le 0$ for the three cases in Lemma 4(a).

If $z<a-\frac{1}{f(a)(1+\tilde{\alpha})}$, then $(\hat{p}(z),\hat{y}(z))=(a,a)$. Thus, $r_o(\hat{p}(z),z)=0$, and the result follows.

If $a-\frac{1}{f(a)(1+\tilde{\alpha})}\le z<a-\frac{F(\tilde{p}(\tilde{\alpha}))}{f(a)}$, $\hat{p}(z)$ and $\hat{y}(z)$ are shown in (22). When v is uniform, $\hat{p}(z)=\frac{1}{2}[b+z+\tilde{\alpha}(a-z)]$. Accordingly, $r_o(\hat{p}(z),z)=\frac{a-z}{b-a}\left[\frac{1}{2}[b+z+\tilde{\alpha}(a-z)]-a\right]$, and $\frac{d^2r_o(\hat{p}(z),z)}{dz^2}=-\frac{1-\tilde{\alpha}}{b-a}\le 0$. When v is exponential, then $\hat{p}(z)=\frac{1}{\gamma}+z-\tilde{\alpha}(a-z)$. Accordingly, $r_o(\hat{p}(z),z)=(a-z)[1-e^{-(1-\gamma(1+\tilde{\alpha})(a-z))}]$, and $\frac{d^2r_o(\hat{p}(z),z)}{dz^2}=-2\gamma(1+\tilde{\alpha})e^{-(1-\gamma(1+\tilde{\alpha})(a-z))}\le 0$. For either distribution, it suffices to show that $\frac{dr_o(\hat{p}(z),z)}{dz}|_{z=a-\frac{1}{f(a)(1+\tilde{\alpha})}}\le 0$ which follows from the continuity of $r_o(\hat{p}(z),z)$ and the proof in the first case.

If $z\ge a-\frac{F(\tilde{p}(\tilde{\alpha}))}{f(a)}$, $\hat{p}(z)$ and $\hat{y}(z)$ are shown in (23). When v is uniform, $\hat{p}(z)=[b+(1+\frac{\tilde{\alpha}}{2})z]/[2+\frac{\tilde{\alpha}}{2}]$ and $\hat{y}(z)=\frac{\hat{p}(z)+z}{2}$. Thus, $r_o(\hat{p}(z),z)=\frac{[\hat{p}(z)-z]^2}{4(b-a)}=\frac{(b-z)^2}{4(b-a)[2+\frac{\tilde{\alpha}}{2}]^2}$ and $\frac{dr_o(\hat{p}(z),z)}{dz}=-\frac{2(b-z)}{4(b-a)[2+\frac{\tilde{\alpha}}{2}]^2}\le 0$. When v is exponential, $(\hat{p}(z),\hat{y}(z))=(\frac{1}{\gamma}+z-\tilde{\alpha}\frac{\tau(\tilde{\alpha})}{\gamma},z+\frac{\tau(\tilde{\alpha})}{\gamma})$, where $\tau(\alpha)$ is the unique solution to $e^{(1+\alpha)\tau-1}+\tau-1=0$. Thus, $r_o(\hat{p}(z),z)=\frac{\tau(\tilde{\alpha})}{\gamma}[e^{-\gamma(\hat{y}(z)-a)}-e^{-\gamma(\hat{p}(z)-a)}]$, and $\frac{dr_o(\hat{p}(z),z)}{dz}=-\tau(\tilde{\alpha})[e^{-\gamma(\hat{y}(z)-a)}-e^{-\gamma(\hat{p}(z)-a)}]\le 0$, where the inequality is from $\hat{p}(z)\ge\hat{y}(z)$.