Lecture 6:

Note Title 11/28/2021

Power series and radius of convergence of power series.

Theorem: (Term by term differentiation of power series): A power series can be differentiated term by term within the interval of convergence. In fact, if $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, then the radius of convergence of $\sum_{n=1}^{\infty} n a_n x^{n-1}$ is R.

Also, if $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $|x|<R$, then $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ for $|x|<R$.

<u>Proof:</u> ① $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=1}^{\infty} n a_n x^{n-1}$ have same radius of convergence.

We have $\limsup \sqrt[n]{|a_n|} = \beta$.

Also, $\limsup \sqrt[n]{|n a_n|} = \limsup \left(\sqrt[n]{n}\ \sqrt[n]{|a_n|}\right)$

$= \limsup \sqrt[n]{|a_n|} = \beta.$

However, to prove that $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges to $f'(x)$ whenever $|x| < R$ requires another concept called "uniform convergence" which is beyond the scope of this course. #

Taylor's Series: If a function f has derivatives of all orders at a point $c \in \mathbb{R}$, then we can calculate the Taylor coefficients by

$$a_0 = f(c), \quad a_n = \frac{f^{(n)}(c)}{n!} \text{ for all } n \in \mathbb{N}.$$

$$\boxed{f^{(0)}(c) = f(c)}$$

In this way, we obtain a power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

This series is called the Taylor's series of f at c.

The Taylor series of a function f at $c=0$ is known as Maclaurin's series.

Convergence of Taylor series: Let the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

have radius of convergence R.

Then. $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$ converges to $f(x)$ for $|x-c|<R$ if and only if the sequence $(R_n(x))$ of remainders converges to 0 for each x in $|x-c|<R$.

Proof: f has a derivatives of all orders at c.

$$f(x) = T_1(x) + R_1(x)$$
$$= T_2(x) + R_2(x)$$
$$\vdots$$
$$= T_n(x) + R_n(x)$$
$$\vdots$$

$$\lim_{n\to\infty} T_n(x) \xlongequal{n \to \infty} \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

$$f(x) = \lim_{n\to\infty} T_n(x) \iff \lim_{n\to\infty} R_n(x) = 0$$

$\forall x$ s.t. $|x-c| < R$.

Example: The Maclaurin series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!}$ of $\sin x$ converges to $\sin x$ $\forall x \in \mathbb{R}$.

Soln: Let $f(x) = \sin x$.

$f: \mathbb{R} \to \mathbb{R}$ is infinitely differentiable.

$f^{(2n-1)}(0) = (-1)^{n+1}$

$f^{(2n)}(0) = 0$.

$$f^{(2n-1)}(x) = (-1)^{n+1} \cos x, \quad f^{(2n)}(x) = (-1)^n \sin x \quad \forall x \in \mathbb{R}. \quad \forall n \in \mathbb{N}.$$

Hence, the Maclaurin series of $\sin x$ is the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!}.$$

Radius of convergence of $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!}$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$a_n = \frac{(-1)^{n+1}}{(2n-1)!}, \text{ so } \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(2n-1)!}{(2n+1)!} = 0.$$

$\therefore\ \beta = 0 \Rightarrow$ Radius of convergence, $R = \frac{1}{\beta} = \infty$.

$\therefore\ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!}$ converges for all $x \in \mathbb{R}$.

[This conclusion holds even though the series is $\sum_{n=1}^{\infty} a_n x^{2n-1}$ instead of $\sum_{n=1}^{\infty} a_n x^n$.]

Now we prove that $\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{x^{2n-1}}{(2n-1)!}$ converges to $\sin x$ $\forall\, x \in \mathbb{R}$.

To prove this, we have to prove that the sequence $(R_n(x))$ of remainders converges to 0 $\forall\, x \in \mathbb{R}$.

For $x=0$, the Maclaurin series of $\sin x$ becomes $0-0+\cdots$, which clearly converges to $\sin 0 = 0$.

Let $x \in \mathbb{R}$, $x \neq 0$. The remainder term in Taylor expansion of $\sin x$ about the point $c=0$ is given by $R_n(x) = \dfrac{x^{n+1}}{(n+1)!} f^{n+1}(c_n)$, where c_n lies between 0 and x.

Since $|\sin c_n| \le 1$ and $|\cos c_n| \le 1$, so $|f^{(n+1)}(c_n)| \le 1$.

$$\Rightarrow |R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!}$$

$$\Rightarrow \lim_{n\to\infty} R_n(x) = 0 \quad \forall\, x \in \mathbb{R}.$$

$\therefore$ The Maclaurin series of $\sin x$ converges to $\sin x$. #

$$b_n = \frac{|x|^{n+1}}{(n+1)!}$$

$$\frac{b_{n+1}}{b_n} = \frac{|x|^{n+2}}{(n+2)!} \frac{(n+1)!}{|x|^{n+1}}$$

$$= \frac{|x|}{n+2}$$

$$\Rightarrow \lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = 0 < 1$$

$$\Rightarrow \lim_{n\to\infty} b_n = 0.$$

Example: Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then, f is differentiable any number of times, and that $f^{(n)}(0) = 0$ for every $n \in \mathbb{N}$. Hence, in a neighborhood of 0, the Maclaurin series of f is identically zero. But, the function takes non-zero value at $x \neq 0$. Thus, the Maclaurin series of f does not converge to f for $x \neq 0$.

The reason is that the sequence of remainder terms $(R_n(x))$ does not converge to 0 for any $x \neq 0$.

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