Lecture 5:

Note Title 11/24/2021

Power series: A power series is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n, \quad \text{where } x_0 \in \mathbb{R}, \quad a_n \in \mathbb{R} \ \forall\, n \geq 0 \text{ and } x \in \mathbb{R}.$$

↑ around x_0.

Remark: Since the transformation $X = x - x_0$ reduces a power series around x_0 to a power series around 0, it is sufficient to consider the series $\sum_{n=0}^{\infty} a_n x^n$.

Example: $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\quad \sum_{n=0}^{\infty} n!\, x^n$, $\quad \sum_{n=0}^{\infty} x^n$.

$a_n = \frac{1}{n!}$ $\qquad a_n = n!$ $\qquad a_n = 1.$

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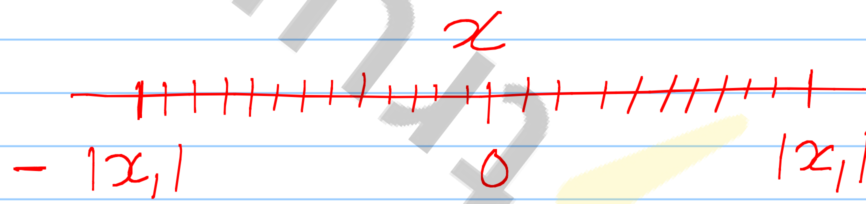
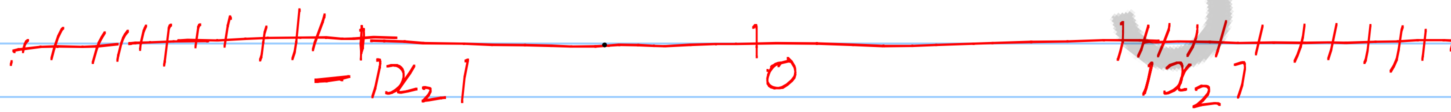
Theorem: We have

(a) If $\sum_{n=0}^{\infty} a_n x^n$ converges for some $x = x_1 \neq 0$, then it converges absolutely for all $x \in \mathbb{R}$ satisfying $|x| < |x_1|$.



(b) If $\sum_{n=0}^{\infty} a_n x^n$ diverges for $x = x_2$, then it diverges for all $x \in \mathbb{R}$ satisfying $|x| > |x_2|$

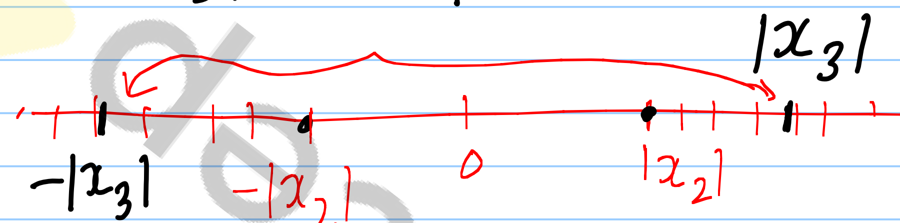


Proof: We first show that the statements (a) and (b) are equivalent. That is, (a) $\iff$ (b).

Assume (a), and let $x = x_2$ be such that $\sum_{n=0}^{\infty} a_n x_2^n$ diverges.

Claim: $\sum_{n=0}^{\infty} a_n x^n$ diverges for all x satisfying $|x| > |x_2|$.

Suppose that $\exists\, x_3$ such that $|x_3| > |x_2|$ and $\sum_{n=0}^{\infty} a_n x_3^n$ converges.



By (a), $\sum_{n=0}^{\infty} a_n x^n$ converges for all x satisfying $|x| < |x_3|$, which is a contradiction

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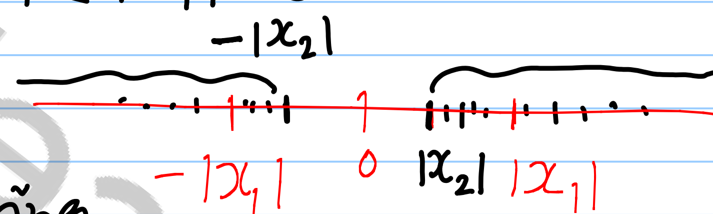
Since $\sum_{n=0}^{\infty} a_n x^n$ diverges at x_2. Hence, (a) $\Rightarrow$ (b).

Next we prove that (b) $\Rightarrow$ (a): Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges for some $x = x_1 \neq 0$.

Claim: $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for all $x \in \mathbb{R}$ satisfying $|x| < |x_1|$.

Suppose that $\exists\, x_2$ satisfying $|x_2| < |x_1|$ such that $\sum_{n=0}^{\infty} a_n x_2^n$ diverges. By (b), $\sum_{n=0}^{\infty} a_n x^n$ diverges for all x satisfying $|x| > |x_2|$.



This is a contradiction to the fact that $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = x_1$.

$\therefore$ (b) $\Rightarrow$ (a).

Since (a) and (b) are equivalent, so we prove part (a) only.

Proof of (a): Let $x_1 \neq 0$ and suppose that $\sum_{n=0}^{\infty} a_n x_1^n$ converges. Put $y_n = a_n x_1^n \;\; \forall\, n \geq 0$.

Then, we have $\sum_{n=0}^{\infty} y_n$ is convergent.

$y_n \to 0$ as $n \to \infty$

$\Rightarrow \{y_n\}$ is bounded

$\therefore \exists\, M > 0$ such that

$$|y_n| = |a_n x_1^n| < M \quad \forall\, n \geq 0.$$

Let $x \in \mathbb{R}$ such that $|x| < |x_1|$.

Let $r = \dfrac{|x|}{|x_1|}$ and clearly, $r < 1$.

Now, $|a_n x^n| = |a_n| r^n |x_1|^n = r^n |a_n x_1^n| < r^n M$ $\forall n$.

Since $0 < r < 1$, so $\sum_{n=0}^{\infty} r^n$ converges.

By comparison test, $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely.

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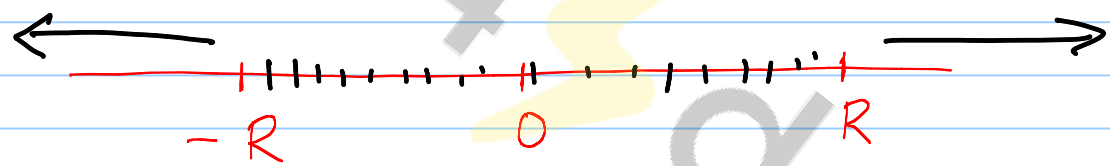
Definition (Radius of convergence): For every power series $\sum_{n=0}^{\infty} a_n x^n$, there exists a unique R satisfying $0 \le R \le \infty$ such that the series converges absolutely if $|x| < R$ and diverges if $|x| > R$.

If $R = \infty$, then $\sum_{n=0}^{\infty} a_n x^n$ converges $\forall x \in \mathbb{R}$



If $R = 0$, then $\sum_{n=0}^{\infty} a_n x^n$ converges only at $x = 0$.

Theorem: Consider the power series $\sum_{n=0}^{\infty} a_n x^n$.

Let $\beta = \limsup \sqrt[n]{|a_n|}$ and let $R = \frac{1}{\beta}$.

(We define $R=0$ if $\beta=\infty$ and $R=\infty$ if $\beta=0$).

Then ⓐ $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$

ⓑ $\sum_{n=0}^{\infty} a_n x^n$ diverges for $|x| > R$

ⓒ No conclusion if $|x| = R$.

Proof: Let $x_n = a_n x^n$. Then, $\sqrt[n]{|x_n|} = |x| \sqrt[n]{|a_n|}$

By Root test, $\sum_{n=0}^{\infty} x_n = \sum_{n=0}^{\infty} a_n x^n$ converges absolutely if $\limsup \sqrt[n]{|x_n|} < 1$.

But $\limsup \sqrt[n]{|x_n|} = \limsup |x| \cdot \sqrt[n]{|a_n|} = |x| \beta$.

$\therefore \sum_{n=0}^{\infty} a_n x^n$ converges absolutely if $|x|\beta < 1$,

that is, if $|x| < 1/\beta = R$.

Again, by Root test $\sum_{n=0}^{\infty} a_n x^n$ diverges for $|x|\beta > 1$,

that is, for $|x| > R$.

Also, by Root test, there is no conclusion if $|x|\beta = 1$,

that is, if $|x| = \frac{1}{\beta} = R$.

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Thm: Consider the power series $\sum_{n=0}^{\infty} a_n x^n$.

Suppose that $\beta = \lim \left| \frac{a_{n+1}}{a_n} \right|$ and $R = \frac{1}{\beta}$.

(We define $R=0$ if $\beta=\infty$ and $R=\infty$ if $\beta=0$).

Then, (a) $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < R$

(b) $\sum_{n=0}^{\infty} a_n x^n$ diverges for $|x| > R$

(c) No conclusion if $|x| = R$.

Proof: The proof readily follows using the Ratio test.

Example: The power series $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$ converges if and only if $x \in [-1, 1]$.

Solution: If $x = 0$, then the given series converges.

Let $x \neq 0$, $x \in \mathbb{R}$. Let $a_n = \frac{1}{n^2}$.

Now, $\beta = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$.

$\therefore$ The radius of convergence of $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$ is $R = 1$.

$\therefore$ $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$ converges $\forall\, x \in (-1, 1)$ $\boxed{|x| < 1}$

and it diverges $\forall\, x \in (-\infty, -1) \cup (1, \infty)$.

Now, we check the convergence of $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$ at the end points $x=-1$ and $x=1$.

If $|x|=1$, then $\sum_{n=0}^{\infty} \left|\frac{x^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which is convergent.

$\therefore \sum_{n=0}^{\infty} \frac{x^n}{n^2}$ is convergent if and only if $x \in [-1, 1]$.

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<u>Example:</u> For the power series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 4^n} (x-1)^n$, the radius of convergence is 4 and the interval of convergence is $(-3, 5]$.

<u>Solution:</u> $a_n = \frac{(-1)^n}{n \cdot 4^n}$, $\forall\ n \geq 1$.

$$X = x - 1.$$
$$|X| < R$$
$$|x-1| < R$$

$$\beta = \lim \left| \frac{a_{n+1}}{a_n} \right| = \lim \left| \frac{(-1)^{n+1}\ n \cdot 4^n}{(n+1)\ 4^{n+1} (-1)^n} \right| = \frac{1}{4}$$

$\therefore$ Radius of convergence, $R = \frac{1}{\beta} = 4$.

$\therefore$ The series converges absolutely if $|x-1| < 4$, that is, if $x \in (-3, 5)$.

Also, the series diverges if $x \in (-\infty, -3) \cup (5, \infty)$.

Next, we need to check the convergence at the end points $x=-3$ and $x=5$.

If $x=-3$, then $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 4^n}(x-1)^n = \sum_{n=1}^{\infty} \frac{1}{n}$, which diverges.

If $x=5$, then $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 4^n}(x-1)^n = \sum_{n=1}^{\infty} (-1)^n/n$

which converges.

$\therefore$ The inverval of convergence is $[-3, 5]$. #