Stabilization of Rigid Body Dynamics by Internal and External Torques

Bloch, Krishnaprasad, Marsden, Sanchez de Alvarez

SC 618 Paper Review Presentation
Group 1

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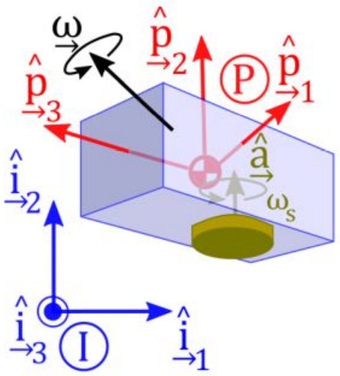
Contents

Introduction

- Stabilizing the angular momentum & attitude equations of the rigid body with $n \geq 2$ torques
- Plays a pivotal role in areas such as incremental motion planning, space vehicles, etc
- Previous works:
    - Arnold and Holm: stability of rigid body with flexible attachment
    - Bloch and Marsden: stabilization of rigid body angular momentum equations about middle axis of inertia by external torque (minor or major axis) by energy-Casimir method
- Problem: analyze a rigid body with internal rotors with quadratic feedbacks such that the system is Hamiltonian wrt. a Lie-Poisson bracket

The Dual Spin Satellite



- Analyzed by Krishnaprasad (1985) and Sanchez de Alvarez (1986)
- Direction of a spinning body is fixed unless external torque is applied
- Need to be stabilized so that perturbations (internal due to motor thrust, or external) do not significantly affect the nominal direction of incremental velocity
- Drawbacks of single-spin stabilization wrt. communication efficiency
- Attitude spin stabilization - stabilize the axis along the thruster
- Gyrostat - asymmetric rotor & platform used to stabilize angular momentum about the longitudinal spin axis

Rigid Body With An External Torque

- Rigid body equations:

$$\dot{\omega}_1 = \frac{I_2 - I_3}{I_1}\,\omega_2\omega_3,$$
$$\dot{\omega}_2 = \frac{I_3 - I_1}{I_2}\,\omega_3\omega_1,$$
$$\dot{\omega}_3 = \frac{I_1 - I_2}{I_3}\,\omega_1\omega_2 + u,$$

$$u = -\varepsilon\frac{I_1 - I_2}{I_3}\,\omega_1\omega_2.$$

- Equations of motion:

$$\dot{m}_1 = a_1 m_2 m_3,$$
$$\dot{m}_2 = a_2 m_3 m_1,$$
$$\dot{m} = a_3(1-\varepsilon)m_1 m_2,$$

$$a_1 = \frac{I_2 - I_3}{I_2 I_3}, \quad a_2 = \frac{I_3 - I_1}{I_1 I_3}, \quad \text{and} \quad a_3 = \frac{I_1 - I_2}{I_1 I_2}.$$

- This system is Hamiltonian wrt. the Lie-Poisson structure $\{F, G\}_F = -\nabla M_F^2 \cdot (\nabla F \times \nabla G).$

Rigid Body With An External Torque (contd.)

The obtained equations of motion are the generalized Euler equations for the Lie group:

$$G_\Sigma = \{A | A \in SL(3), A^T \Sigma A = \Sigma\}$$

The corresponding Eulerian form is: $\dot{m} = \Lambda_\Sigma \nabla H_F(m)$, where $\mathbf{\Lambda}_\Sigma$ is the Poisson tensor. Stability is proved by determining a suitable Lyapunov function using energy-Casimir method.

- Energy-Casimir method: a technique gives a sufficient condition for stability of Hamiltonian systems through minimization of energy
- Casimir functions: functions that commute with every other dynamic variable
- Rigid body system with feedback $u = -\epsilon a_3 m_1 m_2$ can be proved to be stable about the relative equilibrium $(0, M, 0)$

Recovery of Externally Torqued System And The Heavy Rigid Body

The fact that the system with external torque feedback has **conserved quantities** and **a Lie-Poisson structure**, leads one to ask if there is a mechanical extension of the system to larger system <u>where the closed loop dynamics is realized by an internal torque feedback</u>.

This section shows that such a mechanical extension does indeed exist-

-a rigid body carrying three symmetric rotors with associated internal torques.

Configuration Space $SO(3) \times S^1 \times S^1 \times S^1$

Locked inertia tensor $\mathbb{I}_{\mathrm{lock}} = \mathbb{I}_{\mathrm{body}} + \mathbb{I}_{\mathrm{rotor}} + \mathbb{I}'_{\mathrm{rotor}}$

The Lagrangian (kinetic energy) of the free system is the total kinetic energy of the body plus the total kinetic energy of the rotor,

$$
\begin{aligned}
L &= \tfrac{1}{2}(\Omega \cdot \mathbb{I}_{\mathrm{body}}\Omega) + \tfrac{1}{2}\Omega \cdot \mathbb{I}'_{\mathrm{rotor}}\Omega \\
&\quad + \tfrac{1}{2}(\Omega + \Omega_r) \cdot \mathbb{I}_{\mathrm{rotor}}(\Omega + \Omega_r) \qquad (3.1) \\
&= \tfrac{1}{2}(\Omega \cdot \mathbb{I}_{\mathrm{lock}} - \mathbb{I}_{\mathrm{rotor}})\Omega) \\
&\quad + \tfrac{1}{2}(\Omega + \Omega_r) \cdot \mathbb{I}_{\mathrm{rotor}}(\Omega + \Omega_r), \qquad (3.2)
\end{aligned}
$$

By the Legendre transform, the conjugate momenta are:

$$
\begin{aligned}
m &= \frac{\partial L}{\partial \Omega} = (\mathbb{I}_{\mathrm{lock}} - \mathbb{I}_{\mathrm{rotor}})\Omega + \mathbb{I}_{\mathrm{rotor}}(\Omega + \Omega_r) \\
&= \mathbb{I}_{\mathrm{lock}}\Omega + \mathbb{I}_{\mathrm{rotor}}\Omega_r, \qquad (3.3) \\
l &= \frac{\partial L}{\partial \Omega_r} = \mathbb{I}_{\mathrm{rotor}}(\Omega + \Omega_r), \qquad (3.4)
\end{aligned}
$$

the equations of motion with internal torques (controls) u in the rotors are

$$
\begin{aligned}
\dot{m} &= m \times \Omega = m \times (\mathbb{I}_{\mathrm{lock}} - \mathbb{I}_{\mathrm{rotor}})^{-1}(m - l), \qquad (3.5) \\
\dot{l} &= u. \qquad (3.6)
\end{aligned}
$$

Change of Variables

$\pi = m - l$ and l $\quad \mathbb{I} = \mathbb{I}_{\mathrm{lock}} - \mathbb{I}_{\mathrm{rotor}}$

$$
\begin{aligned}
\dot{\pi} &= (\pi + l) \times \mathbb{I}^{-1}\pi - u, \qquad (3.7) \\
\dot{l} &= u. \qquad (3.8)
\end{aligned}
$$

Theorem 3.1. There is a choice of internal torque feedback u(pi, l) such that the body dynamics in the system (3.7)-(3.8) are precisely those of system (2.3) (with external torque feedback).

Consider now the following interesting case.

Let m = pi + I as before and take as controls u'(pi, m).

$$\dot{\pi} = \pi \times \mathbb{I}^{-1}\pi + u'(\pi, m),$$
$$\dot{m} = m \times \mathbb{I}^{-1}\pi. \tag{3.14}$$

Proposition 3.2. For $u'(\pi, m) = -C\chi \times m$ where C is a constant, and χ is a constant (body fixed) vector, the equation (3.14) are the equations for the heavy top. Hence these admit the conserved Hamiltonian $H = \frac{1}{2}\pi \cdot \mathbb{I}^{-1}\pi + C\chi \cdot m$ and kinematic conserved quantities (Casimir functions), $C_1 = \|m\|^2$ and $C_2 = \pi \cdot m$.

Heavy Top

THE HEAVY TOP

We now turn our attention to the heavy top, i.e., a rigid body moving above a fixed point and under the influence of gravity. We let A be a given rotation in $SO(3)$ with corresponding Euler angles denoted (ϕ,ψ,θ). The conjugate momenta are denoted p_ϕ, p_ψ, p_θ so that $(\phi,\psi,\theta,p_\phi,p_\psi,p_\theta)$ coordinatize $T^*SO(3)$. We let m denote the body angular momentum and let $v = A^{-1}k$ where k is the unit vector along the spatial z-axis. We assume that the center of mass is at $(0,0,\ell)$ when A is identity. Coordinates for the vectors (m,v) are most conveniently expressed in the body coordinate system; see Figure 2.

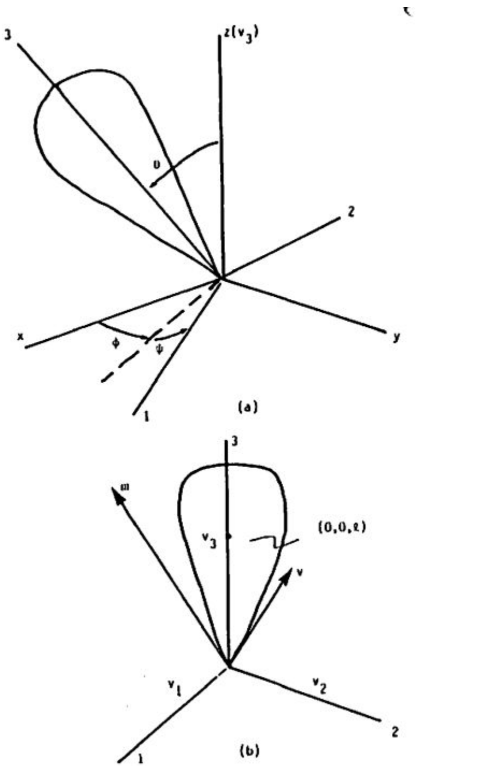


Figure 2. The heavy rigid body, illustration space (x,y,z) and body $(1,2,3)$ coordinates, and the Euler angles (ϕ,ψ,θ).

The system, which consists of a rigid body with a fixed point moving in a gravitational field, is of sufficiently low dimension that a reasonably thorough analysis is tractable, yet rich enough to possess many interesting features in common with other, more complicated systems.

The Hamiltonian Structure Of The Rigid Body With Three Rotors Under Feedback

$$\dot{m} = m \times \Omega = m \times (\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}})^{-1}(m - l), \quad (4.1)$$

$$\dot{l} = u. \quad (4.2)$$

Theorem 4.1. For the feedback

$$u = \mathbf{k}(m \times (\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}})^{-1}(m - l)), \quad (4.3)$$

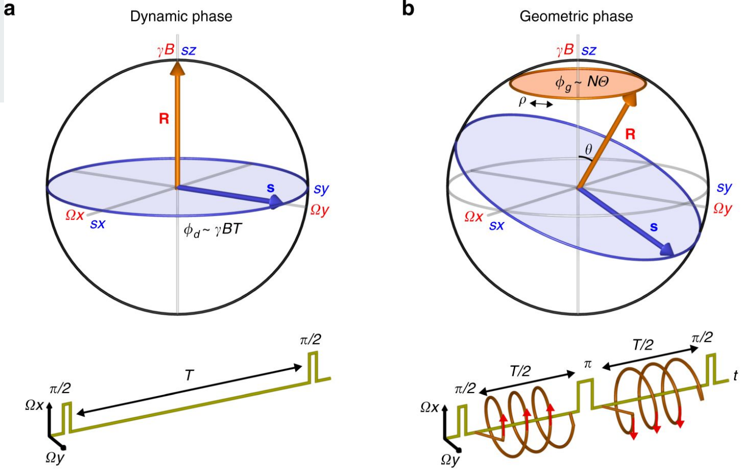
where $\mathbf{k}$ is a constant real matrix such that the matrix $\mathbb{J} = (\mathbb{1} - \mathbf{k})^{-1}(\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}})$ is symmetric, the system (3.5) reduces to a Hamiltonian system on $so(3)^*$ with respect to the standard Lie–Poisson structure $\{F, G\}(m) = -m \cdot (\nabla F \times \nabla G)$.

It is possible to have more complex feedback mechanisms where the system still reduces to a Hamiltonian system on $so(3)$. If we set $u = \mathbf{k}(m, l)\dot{m}$, the key to reduction is that $\mathbf{k}(m, l)\dot{m} - \dot{l} = 0$ is integrable. The natural case to consider is $\mathbf{k} = \mathbf{k}(m)$.

Theorem 4.2. The system (3.5), (3.6) with $u = \mathbf{k}(m)\dot{m}$, where

$$\mathbf{k}(m) = \begin{bmatrix} \mathbf{k}_1(m)^T \\ \mathbf{k}_2(m)^T \\ \mathbf{k}_3(m)^T \end{bmatrix},$$

and $\mathbf{k}_i(m) = \nabla\phi_i(m)$ for some smooth ϕ_i, and such that $(\mathbb{I}_{\text{lock}} - \mathbb{I}_{\text{rotor}})^{-1}\mathbf{k}(m)$ is symmetric, is Hamiltonian.



Phase Shifts

Definition: shift in angular motion of the rigid body system in response to external disturbances

- Perturbation of the system from stable equilibrium results in periodic motion
- Attitude equation: $\dot{A} = A(\mathbb{J}^{-1}m_b)^\wedge$, ^ is the canonical map from real vectors to elements of *so(3)*
- For periodic motion of rigid body, it can be shown that the body undergoes a certain amount of net rotation about the angular momentum vector - an example of attitude drift

This is an example of *geometric phase shift*, since the shift in phase is acquired over the course of cycles of periodic motion

Correcting Attitude Drift

In a zero total angular momentum case, the attitude drift can be compensated for by using two rotors

- Total spatial angular momentum: $\mu = A(\mathbb{I}_{\text{lock}}\Omega + b_1\dot{\alpha}_1 + b_2\dot{\alpha}_2)$
- Attitude matrix as a reverse path-ordered exponential:

$$A(t) = A(0) \cdot \tilde{\mathbb{P}}\left[-\int_0^t \left\{(\mathbb{I}_{\text{lock}}^{-1} b_1)^\wedge \dot{\alpha}_1(\sigma) + \mathbb{I}_{\text{lock}}^{-1} b_2)^\wedge \dot{\alpha}_2(\sigma)\right\} \mathrm{d}\sigma\right]$$

RHS depends only on traversed path in the space T^2 of rotor angles and not previous angular velocities

Conclusions & Key Takeaways

From this paper, we learnt the following:

- Study of rigid bodies using external torques (via gas jets) and internal torques (via rotors/wheels)
- Natural mechanical systems subjected to external forces can be modelled as Hamiltonian systems
- Various methods of geometric mechanics such as energy-momentum algorithms and energy-Casimir method
- Analysis of attitude drift due to perturbations from equilibrium, and how to correct it

Thank You!