# ME 226 (Mechanical Measurements) - Homework 1

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Honor Pledge: I have abided by the course honor code. I have neither received solutions from my peers nor shared my solutions with them. All the solutions in this document are my own work.

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(1) We have the data given as follows:

| Input (MPa) | Increasing Output (MPa) | Decreasing Output (MPa) |
|---|---|---|
| 0 | 0.25 | 0.2 |
| 10 | 10.56 | 10.6 |
| 20 | 21.65 | 21.75 |
| 30 | 32.21 | 32.65 |
| 40 | 43.65 | 43.98 |
| 50 | 52.3 | 52.73 |

i) According to least squares method, we have the following equations:

$$m = \frac{1}{D}\left(N\sum_k x_k y_k - \sum_k x_k \sum_k y_k\right)$$
$$c = \frac{1}{D}\left(\sum_k x_k^2 \sum_k y_k - \sum_k x_k \sum_k x_k y_k\right) \quad \text{and } D = N\sum_k x_k^2 - \left(\sum_k x_k\right)^2$$

The individual quantities can be calculated as shown below:

$$\sum_k x_k = 2(0 + 10 + 20 + 30 + 40 + 50) = 300$$

$$\sum_k x_k^2 = 2(0 + 100 + 400 + 900 + 1600 + 2500) = 11000$$

$$\sum_k x_k y_k = 105.6 + 106 + 433 + 435 + 966.3 + 979.5 + 1746 + 1759.2 + 2615 + 2636.5$$

$$= 11786.1$$

$$\sum_k y_k = 0.25 + 0.2 + 10.56 + 10.6 + 21.65 + 21.75 + 32.21 + 32.65 + 43.65 + \dots$$

$$\dots 43.98 + 52.3 + 52.73 = 322.63$$

Thus we can find the value of $D$, $m$ and $c$ as follows:

$$D = 12(11000) - (300)^2 = 42000$$

$$\boxed{\therefore m = \frac{12(11786.1) - (300)(322.63)}{42000} = 1.063}$$

$$\boxed{\therefore c = \frac{(11000)(322.63) - (300)(11786.1)}{42000} = 0.312}$$

Thus the equation of best linear fit for the given data is $y = 1.063x + 0.312$

ii) First, we find the standard deviation of $y$:

$$\sigma_y^2 = \frac{1}{N-2}\sum_k (mx_k + c - y_k)^2$$

$$= \frac{1}{10}\left[\begin{array}{l} 0.062^2 + 0.112^2 + 0.382^2 + 0.342^2 + (-0.078)^2 + (-0.178)^2 + \dots \\ (-0.008)^2 + (-0.448)^2 + (-0.818)^2 + (-1.148)^2 + 1.162^2 + 0.732^2 \end{array}\right] = 0.4624$$

Thus we get $\boxed{\sigma_y = 0.68}$.

Based on this, we can calculate the remaining standard deviations as follows:

$$\therefore \sigma_x^2 = \frac{\sigma_y^2}{m^2} = 0.4092 \implies \boxed{\sigma_x = 0.64}$$

$$\therefore \sigma_m^2 = \frac{N\sigma_y^2}{D} = 1.321 \times 10^{-4} \implies \boxed{\sigma_m = 0.011}$$

$$\therefore \sigma_c^2 = \frac{\sigma_y^2 \sum_k x_k^2}{D} = 0.1211 \implies \boxed{\sigma_c = 0.35}$$

iii) We have the output value $y = 25.35$ and want to find the true input value. Using the best linear fit equation obtained earlier:

$$y = 1.063x + 0.312$$

$$\therefore x = \frac{y - 0.312}{1.063}$$

$$\therefore x = \frac{25.35 - 0.312}{1.063} = 23.55$$

The uncertainty of the input value within $\pm 3\sigma$ limits is $3 \times 0.64 = 1.92$. Thus the final value of the input is $\boxed{x = 23.55 \pm 1.92 \text{ MPa}}$.

(2) We have 15 sample values of $L$ and $D$ each and we know that surface area $A = \pi D L + \pi \dfrac{D^2}{2}$. We first calculate the value of surface area at each of the sample points:

$$A_1 = \pi(20.4)(100.1) + \pi\frac{(20.4)^2}{2} = 7068.96$$

$$A_2 = \pi(19.5)(99.9) + \pi\frac{(19.5)^2}{2} = 6717.275$$

$$A_3 = \pi(20.1)(99.8) + \pi\frac{(20.1)^2}{2} = 6936.589$$

$$A_4 = \pi(19.9)(100.2) + \pi\frac{(19.9)^2}{2} = 6886.324$$

$$A_5 = \pi(20.1)(100.3) + \pi\frac{(20.1)^2}{2} = 6968.162$$

$$A_6 = \pi(20.3)(99.8) + \pi\frac{(20.3)^2}{2} = 7011.988$$

$$A_7 = \pi(20.2)(100) + \pi\frac{(20.2)^2}{2} = 6986.965$$

$$A_8 = \pi(19.6)(100.4) + \pi\frac{(19.6)^2}{2} = 6785.589$$

$$A_9 = \pi(19.7)(99.7) + \pi\frac{(19.7)^2}{2} = 6779.981$$

$$A_{10} = \pi(19.8)(99.9) + \pi\frac{(19.8)^2}{2} = 6829.948$$

$$A_{11} = \pi(20)(101) + \pi\frac{(20)^2}{2} = 6974.336$$

$$A_{12} = \pi(19.2)(99) + \pi\frac{(19.2)^2}{2} = 6550.598$$

$$A_{13} = \pi(19.4)(99.5) + \pi\frac{(19.4)^2}{2} = 6655.401$$

$$A_{14} = \pi(20)(100.6) + \pi\frac{(20)^2}{2} = 6949.203$$

$$A_{15} = \pi(20.4)(100.5) + \pi\frac{(20.4)^2}{2} = 7094.596$$

From the calculated data, the mean value $\bar{A}$ is:

$$\bar{A} = \frac{\displaystyle\sum_{i=1}^{N} A_i}{N} = \frac{103195.915}{15} = 6879.728$$

From this data we can now calculate the sample variance as follows:

$$\sigma_A^2 = \sum_{i=1}^{N} \frac{(A_i - \bar{A})^2}{N-1} = 24441.0175 \implies \sigma_A = 156.336$$

Thus the final value of the area within $\pm 3\sigma$ error limits is $\boxed{A = 6879.728 \pm 469.008 \text{ mm}^2}$.

(3) Before we employ the least squares method to solve this problem, we need to prove the expressions for it first. We know that the input-output relation for the data is $q_0 = aq_i + b$. For convenience, we can rewrite this as $y = ax_b$, where $y$ represents the output and $x$ represents the input. We can then define the error $E$ as:

$$E = \sum_{i=1}^{N}(y_i - (ax_i + b))^2$$

For minimum error, we must have $\dfrac{\partial E}{\partial a} = 0$ and $\dfrac{\partial E}{\partial b} = 0$.

$$\frac{\partial E}{\partial a} = 0 \implies \sum_{i=1}^{N} 2(-x_i)(y_i - (ax_i + b)) = 0$$

$$\frac{\partial E}{\partial b} = 0 \implies \sum_{i=1}^{N} 2(y_i - (ax_i + b)) = 0$$

We can further simplify the above equations to get the following system of equations:

$$a\left(\sum_{i-1}^{N} x_i^2\right) + b\left(\sum_{i-1}^{N} x_i\right) = \left(\sum_{i-1}^{N} x_i y_i\right)$$

$$a\left(\sum_{i-1}^{N} x_i\right) + bN = \left(\sum_{i-1}^{N} y_i\right)$$

This system of linear equations can further be expressed in a matrix form as follows:

$$\begin{bmatrix} \sum_{i=1}^{N} x_i^2 & \sum_{i=1}^{N} x_i \\ \sum_{i=1}^{N} x_i & N \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} x_i y_i \\ \sum_{i=1}^{N} y_i \end{bmatrix}$$

$$\therefore \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} x_i^2 & \sum_{i=1}^{N} x_i \\ \sum_{i=1}^{N} x_i & N \end{bmatrix}^{-1} \begin{bmatrix} \sum_{i=1}^{N} x_i y_i \\ \sum_{i=1}^{N} y_i \end{bmatrix} \implies \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{D} \begin{bmatrix} N & -\sum_{i=1}^{N} x_i \\ -\sum_{i=1}^{N} x_i & \sum_{i=1}^{N} x_i^2 \end{bmatrix} \begin{bmatrix} \sum_{i=1}^{N} x_i y_i \\ \sum_{i=1}^{N} y_i \end{bmatrix}$$

where the determinant $D = N \sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2$. Thus, the final expressions for $a$ and $b$ are:

$$a = \frac{1}{D}\left(N \sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} y_i\right)$$

$$b = \frac{1}{D}\left(\sum_{i=1}^{N} x_i^2 \sum_{i=1}^{N} y_i - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} x_i y_i\right)$$

From the given data, we have $\sum_{i=1}^{N} x_i = 27$, $\sum_{i=1}^{N} x_i^2 = 179$, $\sum_{i=1}^{N} x_i y_i = 180.6$ and $\sum_{i=1}^{N} y_i = 27.35$.

Thus we can easily calculate the values of $a, b, D$ as follows:

$$D = N \sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2 = 7(179) - (27)^2 = 524$$

$$\therefore a = \frac{1}{D}\left(N\sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} y_i\right) = \frac{1}{524}[7(180.6) - (27)(27.35)] = 1.003$$

$$\therefore b = \frac{1}{D}\left(\sum_{i=1}^{N} x_i^2 \sum_{i=1}^{N} y_i - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} x_i y_i\right) = \frac{1}{524}[(179)(27.35) - (180.6)(27)] = 0.037$$

Thus, $\boxed{a = 1.003}$ and $\boxed{b = 0.037}$. The linear best fit expression for the following set of data is $q_o = 1.003q_i + 0.037$. To calculate the accuracy of these results, we first need to find $\sigma_y$.

$$\sigma_y^2 = \frac{1}{N-2}\sum_{i=1}^{N}(ax_i + b - y_i)^2$$

$$= \frac{1}{5}[(-0.063)^2 + (-0.06)^2 + (-0.007)^2 + (0.146)^2 + (-0.051)^2 + (0.158)^2 + (-0.133)^2]$$

$$= 0.0148$$

Thus we get $\boxed{\sigma_y = 0.122}$ and accuracy of measurement based on $\pm 3\sigma$ limits is $\pm 0.366$. Now to find the accuracy of the calculated values of $a$ and $b$:

$$\sigma_a^2 = \frac{N\sigma_y^2}{D} = 1.977 \times 10^{-4} \implies \boxed{\sigma_a = 0.014}$$

$$\sigma_b^2 = \frac{\sigma_y^2 \sum_i x_i^2}{D} = 5.056 \times 10^{-3} \implies \boxed{\sigma_b = 0.071}$$

(4) We have the relation $\kappa = q\dfrac{\ln(r_0) - \ln(r_1)}{2\pi l(T_1 - T_0)}$. The root sum square error can be found as follows:

$$E_{RSS} = \sqrt{\left(\frac{\partial \kappa}{\partial q}\Delta q\right)^2 + \left(\frac{\partial \kappa}{\partial r_0}\Delta r_0\right)^2 + \left(\frac{\partial \kappa}{\partial r_1}\Delta r_1\right)^2 + \left(\frac{\partial \kappa}{\partial l}\Delta l\right)^2 + \left(\frac{\partial \kappa}{\partial T_1}\Delta T_1\right)^2 + \left(\frac{\partial \kappa}{\partial T_0}\Delta T_0\right)^2}$$

$$\therefore E_{RSS}^2 = \left(\frac{\partial \kappa}{\partial q}\right)^2 \sigma_q^2 + \left(\frac{\partial \kappa}{\partial r_0}\right)^2 \sigma_{r_0}^2 + \left(\frac{\partial \kappa}{\partial r_1}\right)^2 \sigma_{r_1}^2 + \left(\frac{\partial \kappa}{\partial l}\right)^2 \sigma_l^2 + \left(\frac{\partial \kappa}{\partial T_1}\right)^2 \sigma_{T_1}^2 + \left(\frac{\partial \kappa}{\partial T_0}\right)^2 \sigma_{T_0}^2$$

Let us first calculate the individual terms:

$$\frac{\partial \kappa}{\partial q}\sigma_q = \left(\frac{\ln(r_0) - \ln(r_1)}{2\pi l(T_1 - T_0)}\right)\sigma_q = \frac{\ln 2}{2\pi(0.5)(30)} \times \frac{1}{3} = 2.45 \times 10^{-3} \text{ W/m/K}$$

$$\frac{\partial \kappa}{\partial r_0}\sigma_{r_0} = \left(\frac{1}{r_0}\right)\frac{q}{2\pi l(T_1 - T_0)}\sigma_{r_0} = \frac{70}{2\pi(0.02)(0.5)(30)} \times \frac{0.0005}{3} = 6.19 \times 10^{-3} \text{ W/m/K}$$

$$\frac{\partial \kappa}{\partial r_1}\sigma_{r_1} = -\left(\frac{1}{r_1}\right)\frac{q}{2\pi l(T_1 - T_0)}\sigma_{r_1} = -\frac{70}{2\pi(0.01)(0.5)(30)} \times \frac{0.0005}{3} = -12.38 \times 10^{-3} \text{ W/m/K}$$

$$\frac{\partial \kappa}{\partial l}\sigma_l = -q\left(\frac{\ln(r_0) - \ln(r_1)}{2\pi l^2(T_1 - T_0)}\right)\sigma_l = -\frac{70\ln 2}{2\pi(0.5)^2(30)} \times \frac{0.01}{3} = -3.43 \times 10^{-3} \text{ W/m/K}$$

$$\frac{\partial \kappa}{\partial T_1}\sigma_{T_1} = -q\left(\frac{\ln(r_0) - \ln(r_1)}{2\pi l(T_1 - T_0)^2}\right)\sigma_{T_1} = -\frac{70\ln 2}{2\pi(0.5)(30)^2} \times \frac{1}{3} = -5.72 \times 10^{-3} \text{ W/m/K}$$

$$\frac{\partial \kappa}{\partial T_0}\sigma_{T_0} = q\left(\frac{\ln(r_0) - \ln(r_1)}{2\pi l(T_1 - T_0)^2}\right)\sigma_{T_0} = \frac{70\ln 2}{2\pi(0.5)(30)^2} \times \frac{1}{3} = 5.72 \times 10^{-3} \text{ W/m/K}$$

Thus we have, $E_{RSS} = \sqrt{(2.45)^2 + (6.19)^2 + (-12.38)^2 + (-3.43)^2 + (-5.72)^2 + (5.72)^2} \times 10^{-3}$

i.e. $\boxed{E_{RSS} = 1.657 \times 10^{-2} \text{ W/m/K}}$.

The maximum possible error in measurement of thermal conductivity is simply the absolute sum error.

$$E_{max} = \left|\frac{\partial \kappa}{\partial q}\sigma_q\right| + \left|\frac{\partial \kappa}{\partial r_0}\sigma_{r_0}\right| + \left|\frac{\partial \kappa}{\partial r_1}\sigma_{r_1}\right| + \left|\frac{\partial \kappa}{\partial l}\sigma_l\right| + \left|\frac{\partial \kappa}{\partial T_1}\sigma_{T_1}\right| + \left|\frac{\partial \kappa}{\partial T_0}\sigma_{T_0}\right|$$

Thus the maximum possible error is $\boxed{E_{max} = 3.58 \times 10^{-3} \text{ W/m/K}}$.

(5) We have the relation $F = \dfrac{\epsilon E b t^2}{3r(\sin\theta - \frac{2}{\pi})}$. We are also given that the root sum square limit of the load is 2.5 kN. We also know that

$$E_{RSS}^2 = \left(\frac{\partial F}{\partial \epsilon}\right)^2 \sigma_\epsilon^2 + \left(\frac{\partial F}{\partial E}\right)^2 \sigma_E^2 + \left(\frac{\partial F}{\partial b}\right)^2 \sigma_b^2 + \left(\frac{\partial F}{\partial t}\right)^2 \sigma_t^2 + \left(\frac{\partial F}{\partial r}\right)^2 \sigma_r^2 + \left(\frac{\partial F}{\partial \theta}\right)^2 \sigma_\theta^2$$

All of the variances are unknown to us. We ignore the error in $E$ (given in the question). However, we can calculate the values of the derivatives:

$$\frac{\partial F}{\partial \epsilon} = \frac{Ebt^2}{3r(\sin\theta - \frac{2}{\pi})} = \frac{(210 \times 10^9)(25 \times 10^{-3})(36 \times 10^{-6})}{3(75 \times 10^{-3})(\frac{1}{\sqrt{2}} - \frac{2}{\pi})} = 1.19 \times 10^7$$

$$\frac{\partial F}{\partial b} = \frac{\epsilon E t^2}{3r(\sin\theta - \frac{2}{\pi})} = \frac{(210 \times 10^9)(3400 \times 10^{-6})(36 \times 10^{-6})}{3(75 \times 10^{-3})(\frac{1}{\sqrt{2}} - \frac{2}{\pi})} = 1.62 \times 10^6$$

$$\frac{\partial F}{\partial t} = \frac{2\epsilon E b t}{3r(\sin\theta - \frac{2}{\pi})} = \frac{2(3400 \times 10^{-6})(210 \times 10^9)(25 \times 10^{-3})(6 \times 10^{-3})}{3(75 \times 10^{-3})(\frac{1}{\sqrt{2}} - \frac{2}{\pi})} = 1.35 \times 10^7$$

$$\frac{\partial F}{\partial r} = -\frac{\epsilon E b t^2}{3r^2(\sin\theta - \frac{2}{\pi})} = -\frac{(3400 \times 10^{-6})(210 \times 10^9)(25 \times 10^{-3})(36 \times 10^{-6})}{3(75 \times 10^{-3})^2(\frac{1}{\sqrt{2}} - \frac{2}{\pi})} = -5.41 \times 10^5$$

$$\frac{\partial F}{\partial \theta} = -\frac{\epsilon E b t^2 \cos\theta}{3r(\sin\theta - \frac{2}{\pi})^2} = -\frac{(3400 \times 10^{-6})(210 \times 10^9)(25 \times 10^{-3})(36 \times 10^{-6})(\frac{1}{\sqrt{2}})}{3(75 \times 10^{-3})(\frac{1}{\sqrt{2}} - \frac{2}{\pi})^2} = 4.07 \times 10^5$$

By method of equal effects, each of the quantities on RHS will be $\dfrac{2500^2}{5} = 1.25 \times 10^6 \text{ N}^2$. Thus we have:

$$\left(\frac{\partial F}{\partial \epsilon}\right)^2 \sigma_\epsilon^2 = 1.25 \times 10^6 \implies \boxed{\sigma_\epsilon = 9.39 \times 10^{-5}}$$

$$\left(\frac{\partial F}{\partial b}\right)^2 \sigma_b^2 = 1.25 \times 10^6 \implies \boxed{\sigma_b = 6.90 \times 10^{-4}}$$

$$\left(\frac{\partial F}{\partial t}\right)^2 \sigma_t^2 = 1.25 \times 10^6 \implies \boxed{\sigma_t = 8.28 \times 10^{-5}}$$

$$\left(\frac{\partial F}{\partial r}\right)^2 \sigma_r^2 = 1.25 \times 10^6 \implies \boxed{\sigma_r = 2.07 \times 10^{-3}}$$

$$\left(\frac{\partial F}{\partial \theta}\right)^2 \sigma_\theta^2 = 1.25 \times 10^6 \implies \boxed{\sigma_\theta = 2.75 \times 10^{-3}}$$

If uncertainty is based on absolute limits, then each of the quantities on RHS is $\dfrac{2500}{5} = 500 \text{ N}$.

$$\left|\frac{\partial F}{\partial \epsilon}\right| \sigma_\epsilon = 500 \implies \boxed{\sigma_\epsilon = 4.20 \times 10^{-5}}$$

$$\left|\frac{\partial F}{\partial b}\right| \sigma_b = 500 \implies \boxed{\sigma_b = 3.09 \times 10^{-4}}$$

$$\left|\frac{\partial F}{\partial t}\right| \sigma_t = 500 \implies \boxed{\sigma_t = 3.70 \times 10^{-5}}$$

$$\left|\frac{\partial F}{\partial r}\right| \sigma_r = 500 \implies \boxed{\sigma_r = 9.24 \times 10^{-4}}$$

$$\left|\frac{\partial F}{\partial \theta}\right| \sigma_\theta = 500 \implies \boxed{\sigma_\theta = 1.23 \times 10^{-3}}$$