MA 214 - Introduction to Numerical Analysis

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Disclaimer

This document is a compilation of the notes I made while taking the course MA 214 (Introduction to Numerical Analysis) in my 4^{th} semester at IIT Bombay. It is not meant to serve as a replacement for any formal textbook or lecture on the subject, since the theory is not discussed at all.

There will probably be many instances where I use certain common symbols without explicitly mentioning what they mean. It is to be assumed that they carry their usual meanings.

If you have any suggestions and/or spot any errors, you know where to contact me.

Contents

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1 Interpolation Theory

Suppose $(n+1)$ real points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ are known. Further the set of points x_i is spread out over the interval $[a, b]$. Then the problem of approximating a function over the interval $[a, b]$ passing through these points is called interpolation.

We define the norm on $\mathcal{C}[a.b]$ as: $||f|| = \max_{x \in [a,b]} |f(x)|$. To define the 'closeness' of 2 functions formally, we consider the quantity $||f - g|| = \max_{x \in [a,b]} |f(x) - g(x)|$. The Weierstrass approximation theorem states that:

Take a function $f \in \mathcal{C}[a, b]$. Given any real number $\varepsilon > 0$, there exists a polynomial p such that

$$||f - p|| < \varepsilon \implies |f(x) - p(x)| < \varepsilon \quad \forall x \in [a, b]$$

1.1 Lagrange interpolation formula

Given $n+1$ distinct real points $x_0, x_1, \ldots, x_n$ and $n+1$ real numbers $y_0, y_1, \ldots, y_n$, there exists a unique polynomial $p_n \in \mathbb{P}_n$ such that $p(x_i) = y_i$ for $i = 0, 1, \ldots, n$. Construct n^{th} degree polynomials $L_0^n(x), L_1^n(x), \ldots, L_n^n(x)$ such that

$$L_k^n(x_i) = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases} \implies \boxed{p_n(x) = \sum_{k=0}^{n} y_k L_k^n(x)}$$

The lagrange polynomials L_k^n can be found as

$$\boxed{L_k^n(x) = \prod_{j=0, j\neq k}^{n} \frac{(x - x_j)}{(x_k - x_j)}}$$

1.2 Newton's divided differences

Let $x_0, x_1, \ldots, x_n$ be $n+1$ real distinct points in $[a, b]$. Let $f : [a, b] \to \mathbb{R}$ be a function whose values are known at these points. We want to find a polynomial $p_n(x) \in \mathbb{P}_n$ such that $p_n(x_i) = f(x_i)$ for $i = 0, 1, \ldots, n$.

We define the divided differences (independant of order of points) as follows:

$$f[x_0] := f(x_0)$$

$$f[x_0, x_1, \ldots, x_{m+1}] := \frac{f[x_1, \ldots, x_{m+1}] - f[x_0, \ldots, x_m]}{x_{m+1} - x_0}$$

Then the polynomial $p_n(x)$ can be written as:

$$p_n(x) := f[x_0] + f[x_0, x_1](x - x_0) + \cdots + f[x_0, x_1, \ldots, x_n] \prod_{k=0}^{n-1} (x - x_k)$$

1.3 Matrix representation

The problem can also be expressed as a system of linear equations and solved for the coefficients using matrix equations. A matrix similar to the Vandermonde matrix is generated.

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & & \ddots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$

1.4 Error estimation

Take $f \in \mathcal{C}^{n+1}[a,b]$. Let $x_0, x_1, \ldots, x_n$ be $n+1$ distinct points in $[a,b]$. Let $p \in \mathbb{P}_n$ such that $p(x_i) = f(x_i)$ for $i = 1, 2, \ldots, n$. Then for all $x \in [a,b]$, there exists $\xi = \xi(x) \in (a,b)$ such that

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{k=0}^{n} (x - x_k)$$

Taking maximum over $x \in [a,b]$, we can see that our choice of interpolation points influences the error significantly.

$$\max_{x \in [a,b]} |f(x) - p(x)| \leqslant \frac{1}{(n+1)!} ||f^{(n+1)}|| \max_{x \in [a,b]} \prod_{k=0}^{n} |(x - x_k)|$$