Random Process: Spectral Characteristics

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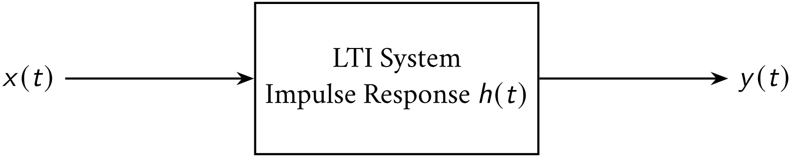
LTI System Block Diagram



$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

The General Linear System

Consider a linear system operating on $x(t)$ to cause $y(t)$ and write

$$y(t) = L[x(t)] \tag{1}$$

Here L is an *operator* representing the action of the system on $x(t)$.
A system is said to be linear if

$$y(t) = L\left[\sum_{n=1}^{N} \alpha_n x_n(t)\right] = \sum_{n=1}^{N} \alpha_n L[x_n(t)] = \sum_{n=1}^{N} \alpha_n y_n(t) \tag{8.1-2}$$

where the α_n are arbitrary constants and N is a finite number.
From the properties of the impulse function we may write

$$x(t) = \int_{-\infty}^{\infty} x(\xi)\, \delta(t-\xi)\, d\xi \tag{2}$$

The General Linear System

By substituting $x(t)$ into eqn (1) and observing that the operator acts on the time function, we obtain

$$y(t) = L[x(t)] = L\left[\int_{-\infty}^{\infty} x(\xi)\delta(t-\xi)\,d\xi\right] = \int_{-\infty}^{\infty} x(\xi)L[\delta(t-\xi)]\,d\xi \tag{8.1-4}$$

Let us define a new function $h(t,\xi)$ as the **impulse response** of the linear system; that is,

$$L[\delta(t-\xi)] = h(t,\xi) \tag{8.1-5}$$

$y(t)$ becomes

$$y(t) = \int_{-\infty}^{\infty} x(\xi)h(t,\xi)\,d\xi$$

If linear System is time-invariant

A general linear system is said to be time-invariant if the impulse response is time-invariant

$$h(t,\xi) = h(t-\xi) \tag{3}$$

now the output $y(t)$ becomes

$$y(t) = \int_{-\infty}^{\infty} x(\xi)h(t-\xi)\, d\xi$$

The above equation is known as *convolution integral* of $x(t)$ and $h(t)$. The convolution is given by

$$y(t) = x(t) * h(t)$$

Random Signal response of Linear Systems

System response-Convolution

Even if $x(t)$ is a random signal, the network's response $y(t)$ is given by the convolution integral

$$y(t) = \int_{-\infty}^{\infty} x(\xi)h(t-\xi)\, d\xi \tag{4}$$

$$= \int_{-\infty}^{\infty} h(\xi)x(t-\xi)\, d\xi \tag{5}$$

Since $x(t)$ is a sample function of random process $X(t)$, the convolution results in a new random process $Y(t)$

$$Y(t) = \int_{-\infty}^{\infty} h(\xi)X(t-\xi)\, d\xi \tag{6}$$

Mean value of System response

We know the output $Y(t)$ is given by

$$Y(t) = \int_{-\infty}^{\infty} h(\xi) X(t-\xi)\, d\xi$$

Let us assume $X(t)$ is wide-sense stationary

$$E[Y(t)] = E\left[\int_{-\infty}^{\infty} h(\xi) X(t-\xi)\, d\xi\right] = \int_{-\infty}^{\infty} h(\xi)\, \underbrace{E[X(t-\xi)]}_{\bar{X}}\, d\xi$$

$$\boxed{E[Y(t)] = \bar{X} \int_{-\infty}^{\infty} h(\xi)\, d\xi}$$

Mean-Squared value of system response

$$E[Y^2(t)] = E\left[\int_{-\infty}^{\infty} h(\xi_1)X(t-\xi_1)\,d\xi_1 \int_{-\infty}^{\infty} h(\xi_2)X(t-\xi_2)\,d\xi_2\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(t-\xi_1)X(t-\xi_2)]h(\xi_1)h(\xi_2)\,d\xi_1\,d\xi_2$$

If $X(t)$ is wide-sense stationary

$$R_{XX}(\xi_1-\xi_2) = E[X(t-\xi_1)X(t-\xi_2)]$$

the mean-squared value of the output becomes independent of t

$$\bar{Y}^2 = E[Y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\xi_1-\xi_2)h(\xi_1)h(\xi_2)\,d\xi_1\,d\xi_2$$

Autocorrelation of Response

The autocorrelation function of $Y(t)$ is given by

$$
\begin{aligned}
R_{YY}(t, t+\tau) &= E[Y(t)Y(t+\tau)] \\
&= E\left[\int_{-\infty}^{\infty} h(\xi_1)X(t-\xi_1)\,d\xi_1 \int_{-\infty}^{\infty} h(\xi_2)X(t+\tau-\xi_2)\,d\xi_2\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(t-\xi_1)X(t+\tau-\xi_2)]h(\xi_1)h(\xi_2)\,d\xi_1\,d\xi_2
\end{aligned}
$$

If $X(t)$ is wide-sense stationary

$$
R_{YY}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\xi_1-\xi_2)h(\xi_1)h(\xi_2)\,d\xi_1\,d\xi_2
$$

Autocorrelation of Response

Two important conclusions

- If $X(t)$ is wide-sense stationary, then $Y(t)$ is also wide-sense stationary ($R_{YY}(\tau)$ does not depend on t, $E[Y(t)]$ is constant)
- $R_{YY}(\tau)$ is the twofold convolution of the input autocorrelation function with the network's impulse response

$$R_{YY}(\tau) = R_{XX}(\tau) * h(-\tau) * h(\tau) \qquad (7)$$

Cross-Correlation functions of Input and Output

The cross-correlation function of $X(t)$ and $Y(t)$ is

$$R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)] = E\left[X(t)\int_{-\infty}^{\infty} h(\xi)X(t+\tau-\xi)\,d\xi\right]$$

$$= \int_{-\infty}^{\infty} E[X(t)X(t+\tau-\xi)]h(\xi)d\xi$$

If $X(t)$ is wide-sense stationary, the above equation reduces to

$$\boxed{R_{XY}(\tau) = \int_{-\infty}^{\infty} R_{XX}(\tau-\xi)h(\xi)d\xi = R_{XX}(\tau) * h(\tau)}$$

Cross-Correlation functions of Input and Output

Similarly we can show

$$\boxed{R_{YX}(\tau) = \int_{-\infty}^{\infty} R_{XX}(\tau - \xi)h(-\xi)d\xi = R_{XX}(\tau) * h(-\tau)}$$

Conclusion: Since cross-correlation functions are independent of t, $X(t)$ and $Y(t)$ are jointly wide-sense stationary.

Consider

$$R_{YY}(\tau) = \underbrace{R_{XX}(\tau) * h(\tau)}_{R_{XY}(\tau)} * h(-\tau) \tag{8}$$

$$= R_{XY}(\tau) * h(-\tau) \tag{9}$$

$$R_{YY}(\tau) = \underbrace{R_{XX}(\tau) * h(-\tau)}_{R_{YX}(\tau)} * h(\tau) \tag{10}$$

$$= R_{YX}(\tau) * h(\tau) \tag{11}$$

Acknowledge various sources for the images.

Thankyou