Binomial, Poisson, Gaussian, Uniform, Exponential and Rayleigh Distributions

Dr. G. Omprakash

Assistant Professor, ECE, KLEF

Table of Contents

Discrete Distributions

Binomial Random Variable

Models the number of successes in n independent trials, each with probability of success p

Applications:

- Vaccine Success Rate
  - *"A vaccine is 95% effective. In a group of 20 people, what is the probability exactly 18 are protected?"*
- Quality Control in Manufacturing
  - *"In a batch of 100 bulbs with a defect rate of 2%, what's the probability that at most 3 bulbs are defective?"*
- Marketing
  - *"If 10% of visitors make a purchase, what's the probability that 5 out of 30 visitors buy something?"*
- Warehouse Pick and Pack Errors
  - *"There's a 1% chance a packing error happens in any shipment. What's the chance that in 50 shipments, exactly 2 have errors?"*

Binomial Distribution

Binomial density can be applied to experiments having only two possible outcomes on any given trial. Let $0 < p < 1$, and $N = 1, 2, 3, ...,$ then

Binomial density function

$$f_X(x) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} \delta(x-k)$$

The quantity $\binom{N}{k}$ is the binomial coefficient defined as $\binom{N}{k} = \frac{N!}{k!(N-k)!}$.

Binomial Distribution function

$$F_X(x) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} u(x-k)$$

Binomial Distribution

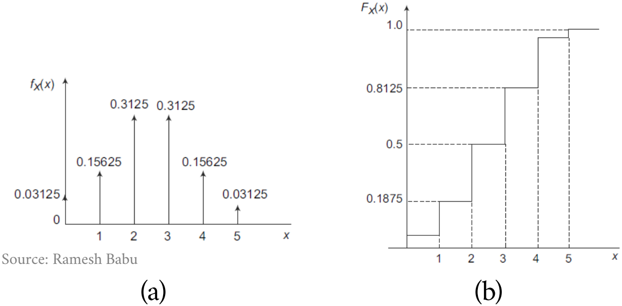


Figure: (a) Binomial density function (b) Binomial distribution function' $N = 5, p = 0.5$

Mean and Variance of Binomial RV

Mean of the Binomial random variable

$$\mu_X = E(X) = np$$

Variance of the Binomial random variable

$$\sigma_X^2 = Var(X) = np(1 - p)$$

For derivation, Refer page 62,63 in Schaum's Outline series book (Hwei P. Hsu)

Problems

The random variable X has a binomial distribution with $N = 8$ and $p = 0.5$. Determine the following probabilities. (a) $P(X = 4)$ (b) $P(X \leq 2)$ (c) $P(X \geq 7)$ (d) $P(3 \leq X < 5)$.

Problems

The random variable X has a binomial distribution with $N = 8$ and $p = 0.5$. Determine the following probabilities. (a) $P(X = 4)$ (b) $P(X \leq 2)$ (c) $P(X \geq 7)$ (d) $P(3 \leq X < 5)$.

(a) N=8,p=0.5

Problems

The random variable X has a binomial distribution with $N = 8$ and $p = 0.5$. Determine the following probabilities. (a) $P(X = 4)$ (b) $P(X \leq 2)$ (c) $P(X \geq 7)$ (d) $P(3 \leq X < 5)$.

(a) N=8,p=0.5

$$P(X = k) = \binom{N}{k} p^k (1 - p)^{N-k}$$
$$P(X = 4) = \binom{8}{4} (0.5)^4 (1 - 0.5)^{8-4}$$

(b)$P(X \leq 2) = P(X = 1) + P(X = 2)$
(c)$P(X \geq 7) = P(X = 7) + P(X = 8)$
(d)$P(3 \leq X < 5) = P(X = 3) + P(X = 4)$

Problems

During October, Chennai has rainfall on an average three days a week. Obtain the probability that (a) rain will fall on at least 2 days of a given week, (b) first three days of a given week will be rainy and the remaining 4 days will be wet.

Problems

During October, Chennai has rainfall on an average three days a week. Obtain the probability that (a) rain will fall on at least 2 days of a given week, (b) first three days of a given week will be rainy and the remaining 4 days will be wet.

Given $p = \frac{3}{7}$; $n = 7$

Problems

During October, Chennai has rainfall on an average three days a week. Obtain the probability that (a) rain will fall on at least 2 days of a given week, (b) first three days of a given week will be rainy and the remaining 4 days will be wet.

Given $p = \frac{3}{7}$; $n = 7$
(a). $P(X \geq 2) = P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6) + P(X = 7)$
(b). $(1 - p)^3 p^4 = 0.0063$

Problems

A manufacturer of light bulbs claims that 80% of its bulbs meet the quality standards. A sample of 10 light bulbs is randomly selected from the production line.
(a). What is the probability that exactly 7 bulbs in the sample meet the quality standards? (b). Calculate the probability that at least 9 bulbs in the sample meet the quality standards.

Problems

A manufacturer of light bulbs claims that 80% of its bulbs meet the quality standards. A sample of 10 light bulbs is randomly selected from the production line.

(a). What is the probability that exactly 7 bulbs in the sample meet the quality standards? (b). Calculate the probability that at least 9 bulbs in the sample meet the quality standards.

Soln: n=10, p=0.8

(a). $P(X = 7) = \binom{10}{7}(0.8)^7(1 - 0.8)^{(10-7)}$

(b). $P(X \geq 9) = P(X = 9) + P(X = 10)$

Poisson Random Variable: Applications

It models the number of times an event occurs in a fixed interval.
Applications

- Emergency Calls to a Hospital
  - *"On average, 5 emergency calls come to a hospital every hour. What's the probability of getting exactly 3 calls in the next hour?"*
- Customer Support Calls
  - *"A call center receives an average of 12 calls per 10 minutes. What's the probability that exactly 15 calls arrive in the next 10 minutes?"*
- System Failures in Engineering
  - *"An aircraft component fails on average once every 10,000 hours of operation. What is the probability that a part will fail exactly twice in 20,000 hours?"*
- Website Hits
  - *"Your website gets an average of 100 hits per hour. What's the chance you get more than 120 hits in the next hour?"*

Poisson Random Variable

Poisson Random Variable is applied to wide variety of counting applications.

- T : Duration of time interval of interest
- λ: Average rate at which events occur
- $b = \lambda T$

Poisson Density and Distribution

The Poissom random variable X has a density and distribution function given by

$$f_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} \delta(x-k)$$

$$F_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} u(x-k)$$

Mean and Variance of Poisson RV

Mean value

$$\mu_X = E(X) = b$$

Variance

$$\sigma_X^2 = Var(X) = b$$

Page 63,64 in the Schaums Reference book

Examples

People enter a club at a rate of two for every 4 minutes.
(a). What is the probability that no one enters between 10.00 PM and 10.10 PM?
(b). What is the probability that at least 4 people enter during that time?

Examples

People enter a club at a rate of two for every 4 minutes.
(a). What is the probability that no one enters between 10.00 PM and 10.10 PM?
(b). What is the probability that at least 4 people enter during that time?

Given $\lambda = \frac{2}{4}$ and $T = 10$min; $\implies b = \lambda T$=$0.5 \times 10 = 5$.
(a). $P(X = k) = \frac{e^{-b} b^k}{k!} \implies P(X = 0) = \frac{e^{-5} 5^0}{0!} = 0.6065$
(b).

$$\begin{aligned} P(X \geq 4) &= 1 - P(X \leq 3) \\ &= 1 - P(X = 0) - P(X = 1) - P(X = 2) - P(X = 3) \end{aligned}$$

Examples

A bakery receives an average of 25 customers per hour. Let X be the random variable representing the number of customers that arrive in an hour. (a) Calculate the probability that there are exactly 20 customers in an hour. (b) Find the probability that there are at most 15 customers in an hour.

Examples

A bakery receives an average of 25 customers per hour. Let X be the random variable representing the number of customers that arrive in an hour. (a) Calculate the probability that there are exactly 20 customers in an hour. (b) Find the probability that there are at most 15 customers in an hour.

Soln: $\lambda = 25$

(a). Probability of exactly 20 customers in an hour P(X=20)

$$P(X = 20) = \frac{25^{20} e^{-25}}{20!};$$

(b). $P(X \leq 15) = P(X = 0) + P(X = 1) + \ldots + P(X = 15)$

Poisson as a Binomial approximation

Poisson R.V can be used as an approximation to Binomial R.V with parameters (n, p), when n is large and p is small enough such that np is moderate.

For derivation: page 2.44-45 in Ramesh Babu Textbook)

Given $n, p \implies b = np$

Examples

In a lot of semiconductor diodes, 1 in 400 diodes is defective. If the diodes are packed in boxes of 100, what is the probability that any given box of diodes will contain (a) no defective, (b) 1 or more defective, and (c) less than 2 defectives diodes?.

Examples

In a lot of semiconductor diodes, 1 in 400 diodes is defective. If the diodes are packed in boxes of 100, what is the probability that any given box of diodes will contain (a) no defective, (b) 1 or more defective, and (c) less than 2 defectives diodes?.

Given $p = \frac{1}{400}$; $n = 100 \implies \underline{=} np = 0.25$. We know that

$$P(X = k) = \frac{b^k e^{-b}}{k!}; k = 0, 1, 2, ..$$

(a) $P(X = 0)$;
(b) P(1 or more defective)=$P(X \geq 1) = 1 - P(X = 0)$
(c) P(Less than 2 defective)=$P(X < 2) = P(X = 0) + P(X = 1)$

A manufacturer of cotton pins knows that 5% of his products are defective. If he sells pins in boxes of 100 and guarantees that not more than 4 pins will be defective, what is the approximate probability that a box will fail to meet the guaranteed quality?

A manufacturer of cotton pins knows that 5% of his products are defective. If he sells pins in boxes of 100 and guarantees that not more than 4 pins will be defective, what is the approximate probability that a box will fail to meet the guaranteed quality?

Given $p = \frac{5}{100}$, $n = 100 \implies \lambda = np = 5$

$$P(X = k) = \frac{b^k e^{-b}}{k!}; k = 0, 1, 2, ..$$

The box will fail to meet the quality if $X > 4$.

$$\begin{aligned} P(X > 4) &= 1 - P(X \le 4) \\ &= 1 - P(X = 0) - P(X = 1) - P(X = 2) - P(X = 3) - P(X = 4) \end{aligned}$$

Continuous Distributions

Gaussian/Normal Distribution : Applications

- *"Temperature sensors in engineering give values that fluctuate around the true value, and these errors are often Gaussian."*
- *The accuracy of missiles or artillery rounds is modeled using Gaussian errors in both X and Y directions*
    - Combined $\implies$ 2D Gaussian, resulting in a circular confidence region.
- Flight Control Systems
    - Sensor data (altitude, speed, pressure) is assumed to have Gaussian errors due to calibration and environment

Why model errors using Gaussian?

Measurement errors arise due to: Human handling, Environmental noise, Electronic fluctuation

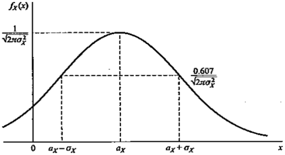
The sum (or average) of these effects tends to follow a Gaussian distribution: Central Limit Theorem

Gaussian Random Variable

Gaussian Density and Distribution

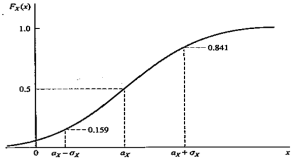
$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-(x-a_X)^2/2\sigma_X^2}$$

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \int_{-\infty}^{x} e^{-(s-a_X)^2/2\sigma_X^2} ds$$



Source: Peebles

(a)



(b)

Figure: (a) Gaussian density function (b) Gaussian distribution function

Mean and Variance of Gaussian RV

Mean value

$$\mu_X = E(X) = a_X$$

Variance

$$Var(X) = \sigma_X^2$$

Normalized Gaussian Distribution

Assume that the height of the clouds above the ground at some location is a Gaussian random variable X with $a_X = 1830m$ and $\sigma_X = 460m$. What is the probability that clouds will be higher than 2750m.

$$P(X > 2750) = 1 - P(X \leq 2750)$$
$$= 1 - \frac{1}{\sqrt{2\pi\sigma_X^2}} \int_{-\infty}^{2750} e^{-(s-a_X)^2/2\sigma_X^2} ds$$

We cannot calculate above equation directly. Use Normalized Gaussian distribution!!

Normalized Gaussian Distribution

Normalized Gaussian Distribution is denoted by $F(x)$.
Substitute $a_X = 0$ and $\sigma_X = 1$ in the GDF. we get

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2} ds$$

for $x < 0$

$$F(-x) = 1 - F(x)$$

Determining $F_X(x)$ from $F(x)$

What is the relation between $F(x)$ and $F_X(x)$?
Consider the Gaussian Distribution

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \int_{-\infty}^{x} e^{-(s-a_X)^2/2\sigma_X^2} ds$$

Changing the variable $u = \frac{s-a_X}{\sigma_X} \implies du = \frac{ds}{\sigma_X}$.
The limits change: $s = -\infty \implies u = -\infty$; $s = x \implies u = \frac{x-a_X}{\sigma_X}$.
The above equation becomes

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \int_{-\infty}^{\frac{x-a_X}{\sigma_X}} e^{-u^2/2} du\sigma_X = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-a_X}{\sigma_X}} e^{-u^2/2} du$$

$$= F\left(\frac{x-a_X}{\sigma_X}\right)$$

$$F_X(x) = F\left(\frac{x-a_X}{\sigma_X}\right)$$

If we get back to the previous problem with $a_X = 1830$ and $\sigma_X = 460$

$$
\begin{aligned}
P(X > 2750) &= 1 - P(X \leq 2750) \\
&= 1 - F_X(2750) = 1 - F(\frac{2750 - 1830}{460}) = 1 - F(2) \\
&= 1 - 0.9773 = 0.0227
\end{aligned}
$$

TABLE B-1
Values of $F(x)$ for $0 \le x \le 3.89$ in steps of 0.01

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9773	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998	.9998
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998
3.6	.9998	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
3.7	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
3.8	.9999	.9999	.9999	.9999	.9999	.9999	.9999	1.0000	1.0000	1.0000

Problems

The lifetime of IC chips manufactured by a semiconductor manufacturer is approximately normally distributed with mean = 5×10^6 hours and standard deviation of 5×10^6 hours. A mainframe manufacturer requires that at least 95% of a batch should have a lifetime greater than 4×10^6 hours. Will the deal be made?

Problems

The lifetime of IC chips manufactured by a semiconductor manufacturer is approximately normally distributed with mean = 5×10^6 hours and standard deviation of 5×10^6 hours. A mainframe manufacturer requires that at least 95% of a batch should have a lifetime greater than 4×10^6 hours. Will the deal be made?

$\mu_X = 5 \times 10^6, \sigma_X = 5 \times 10^6$

We have to check if $P(X > 4 \times 10^6) \geq .95$

Problems

The lifetime of IC chips manufactured by a semiconductor manufacturer is approximately normally distributed with mean = 5×10^6 hours and standard deviation of 5×10^6 hours. A mainframe manufacturer requires that at least 95% of a batch should have a lifetime greater than 4×10^6 hours. Will the deal be made?

$\mu_X = 5 \times 10^6, \sigma_X = 5 \times 10^6$

We have to check if $P(X > 4 \times 10^6) \geq .95$

$$
\begin{aligned}
P(X > 4 \times 10^6) &= 1 - P(X \leq 4 \times 10^6) \\
&= 1 - F_X(4 \times 10^6) = 1 - F\left(\frac{4 \times 10^6 - 5 \times 10^6}{5 \times 10^6}\right) \\
&= 1 - F(-0.2) = 1 - (1 - F(0.2)) \\
&= F(0.2) = 0.5793
\end{aligned}
$$

The deal cannot be made!!

Problems

A company conducts tests for job aspirants and passes them if they achieve a score of 500. If the test scores are normally distributed with a mean of 485 and a standard deviation of 30, what percentage of the aspirants pass the test?

Problems

A company conducts tests for job aspirants and passes them if they achieve a score of 500. If the test scores are normally distributed with a mean of 485 and a standard deviation of 30, what percentage of the aspirants pass the test?

$\mu_X = 485, \sigma_X = 30$

What is $P(X > 500) \times 100$

$$
\begin{aligned}
P(X > 500) &= 1 - P(X \leq 500) \\
&= 1 - F_X(500) = 1 - F(\frac{500 - 485}{30}) \\
&= 1 - F(0.5) = 1 - 0.6915 = 0.3085
\end{aligned}
$$

Problems

The average test marks in a class is 80. The standard deviation is 6. If the marks are distributed normally, how many students in a class of 200 receives marks between 70 and 90.

$\mu_X = 80, \sigma_X = 6$
What is $P(70 < X \leq 90) \times 200$?

Problems

The average test marks in a class is 80. The standard deviation is 6. If the marks are distributed normally, how many students in a class of 200 receives marks between 70 and 90.

$\mu_X = 80, \sigma_X = 6$
What is $P(70 < X \leq 90) \times 200$?

$$\begin{aligned}
P(70 < X \leq 90) &= F_X(90) - F_X(70) \\
&= F(\frac{90-80}{6}) - F(\frac{70-80}{6}) \\
&= F(1.67) - F(-1.67) \\
&= F(1.67) - (1 - F(1.67)) \\
&= 2F(1.67) - 1 = 2(0.9525) - 1 = 0.905
\end{aligned}$$

The number of students who receive marks between 70 and 90 is
$200 \times 0.905 = 181$

Calculator for Normal Distribution

https://www.youtube.com/watch?v=uqx2dgosuN0
https://www.youtube.com/watch?v=Ugdngb1jy7s

Uniform Distribution: Applications

Applications:

- Random Number Generators
  - “When you use rand() in Python or MATLAB, it gives a number between 0 and 1 uniformly.”
- Defects in Manufacturing Along a Wire
  - If defects occur randomly along a 10-meter wire, and any point is equally likely, what’s the probability the defect is in the first 2 meters?”
- Errors introduced in the round off process are uniformly distributed (Digital Communication)

Uniform

Uniform Density and Distribution

$$f_X(x) = \begin{cases} 1/(b-a) & a \leq x \leq b \\ 0 & \text{elsewhere} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x < a \\ (x-a)/(b-a) & a \leq x < b \\ 1 & b \leq x \end{cases}$$

for real constants $-\infty < a < \infty$ and $b > a$

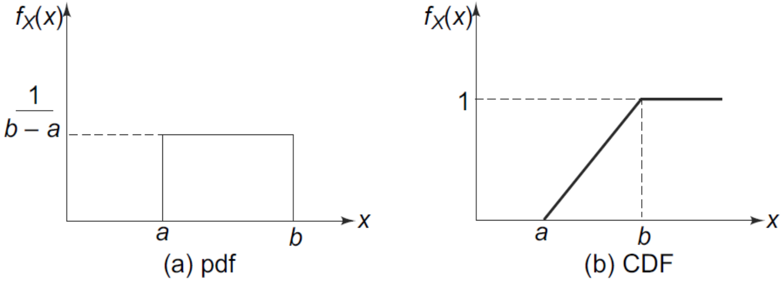
Mean and Variance of Uniform RV

Mean value

$$\mu_X = E(X) = \frac{a+b}{2}$$

Variance

$$Var(X) = \sigma_X^2 = \frac{(b-a)^2}{12}$$



Source: Ramesh Babu

Figure: PDF and CDF of a Uniform Random Variable

Example

A company manufactures a certain type of electronic component, and the time it takes to test each component is uniformly distributed between 4 seconds and 8 seconds. Find the probability that a randomly chosen component will take more than 6 seconds to test.

Example

A company manufactures a certain type of electronic component, and the time it takes to test each component is uniformly distributed between 4 seconds and 8 seconds. Find the probability that a randomly chosen component will take more than 6 seconds to test.

$$f_X(x) = \begin{cases} 1/4 & 4 \leq x \leq 8 \\ 0 & \text{elsewhere} \end{cases}$$

Example

A company manufactures a certain type of electronic component, and the time it takes to test each component is uniformly distributed between 4 seconds and 8 seconds. Find the probability that a randomly chosen component will take more than 6 seconds to test.

$$f_X(x) = \begin{cases} 1/4 & 4 \le x \le 8 \\ 0 & \text{elsewhere} \end{cases}$$

$$P(X > 6) = 1 - P(X \le 6) = 1 - \int_4^6 f_X(x)\,dx = 1 - \int_4^6 \frac{1}{4}\,dx = 1 - \frac{6-4}{4} = 0.5$$

Example

The time it takes to assemble a product in a factory is uniformly distributed between 10 minutes and 20 minutes. Find the probability that it takes less than 15 minutes to assemble a product.

Example

The time it takes to assemble a product in a factory is uniformly distributed between 10 minutes and 20 minutes. Find the probability that it takes less than 15 minutes to assemble a product.

$$f_X(x) = \begin{cases} 1/10 & 10 \leq x \leq 20 \\ 0 & \text{elsewhere} \end{cases}$$

Example

The time it takes to assemble a product in a factory is uniformly distributed between 10 minutes and 20 minutes. Find the probability that it takes less than 15 minutes to assemble a product.

$$f_X(x) = \begin{cases} 1/10 & 10 \leq x \leq 20 \\ 0 & \text{elsewhere} \end{cases}$$

$$\begin{aligned} P(X \leq 15) &= F(X = 15) \\ &= \int_{10}^{15} f_X(x)dx = \int_{10}^{15} \frac{1}{10}dx = \frac{15-10}{10} = 0.5 \end{aligned}$$

Exponential Distribution:Applications

The exponential distribution is widely used in modeling the **time until an event occurs**, particularly when the event occurs continuously and independently at a constant average rate.

- **Reliability Engineering**
    - Modeling time until failure of mechanical or electronic components.
        - Estimating the lifespan of hard drives, LED bulbs, or electric fuses.
- **Telecommunications and Queueing Theory**
    - Time between arrivals of packets in a network, or between phone calls at a call center.
        - Designing buffer sizes, load balancing in servers, call routing algorithms
- **Traffic Flow and Transportation**
    - Time between arrivals of vehicles at a toll booth or traffic signal.
        - Simulation of traffic systems for infrastructure planning and optimization

Exponential Density and Distribution

$$f_X(x) = \begin{cases} \lambda e^{-\lambda(x-a)} & x \geq a \\ 0 & x < a \end{cases}$$

For $a = 0$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

for real number $-\infty < a < \infty$

Exponential Distribution

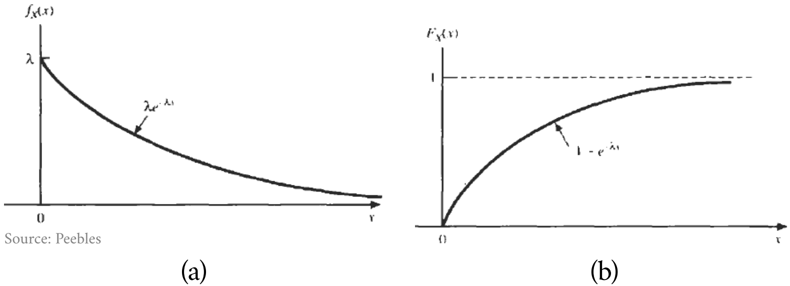


Figure: (a) Exponential density function (b) Exponential distribution function

Mean and Variance of Exponential RV

Mean

$$\mu_X = E(X) = \frac{1}{\lambda}$$

Variance

$$\sigma_X^2 = Var(X) = \frac{1}{\lambda^2}$$

Problems

The time required to complete a work is an exponential distributed random variable with $\lambda = \frac{1}{2}$. What is the probability that time to complete the work exceeds 2 hours?

Problems

The time required to complete a work is an exponential distributed random variable with $\lambda = \frac{1}{2}$. What is the probability that time to complete the work exceeds 2 hours?

$P(X > 2) = 1 - P(X \leq 2)$
We know that $F_X(x) = P(X \leq x) = 1 - e^{-\lambda x}$
$P(X > 2) = 1 - P(X \leq 2) = 1 - (1 - e^{-\lambda x}) = e^{-\lambda x}$

Problems

The number of years a washing machine functions is exponentially distributed with $\lambda = \frac{1}{10}$. What is the probability that it will be working after an additional 10 years?

Problems

The number of years a washing machine functions is exponentially distributed with $\lambda = \frac{1}{10}$. What is the probability that it will be working after an additional 10 years?
Find $P(X > 10)$, $F_X(x) = 1 - e^{-\lambda x}$

Problems

The number of years a washing machine functions is exponentially distributed with $\lambda = \frac{1}{10}$. What is the probability that it will be working after an additional 10 years?
Find $P(X > 10)$, $F_X(x) = 1 - e^{-\lambda x}$

$$\begin{aligned} P(X > 10) &= 1 - P(X \leq 10) \\ &= 1 - F(10) = 1 - (1 - e^{-\lambda 10}) \\ &= e^{-1} \end{aligned}$$

Problem

The mileage which car owners get with certain kinds of radial tyres is a random variable having an exponential distribution with a mean of 40000 km. Find the probability that one of these tyres will last (a) at least 20,000 km, and (b) at most 30,000 km

Problem

The mileage which car owners get with certain kinds of radial tyres is a random variable having an exponential distribution with a mean of 40000 km. Find the probability that one of these tyres will last (a) at least 20,000 km, and (b) at most 30,000 km

Given $\frac{1}{\lambda} = 40000 \implies \lambda = \frac{1}{40000}$. Find $P(X \geq 20000)$, $P(X \leq 30000)$

$P(X \geq 20000) = 1 - P(X \leq 20000)$

$$P(X \leq 20000) = \int_0^{20000} e^{-\frac{x}{40000}} dx$$

Change in variable: $\frac{x}{40000} = t \implies dx = 40000 dt$; Limits change: when $x = 20000$, $t = 0.5$

$$P(X \leq 20000) = \int_0^{0.5} e^{-t} dt = 0.3935$$

$P(X > 20000) = 1 - P(X \leq 20000) = 1 - 0.3935 = 0.6065$;

$P(X \leq 30000) = 0.5270$

Rayleigh Distribution: Applicatons

Rayleigh Distribution used to model the magnitude of a two-dimensional vector whose components are independent and normally distributed

- **Wireless Communication**
  - Modeling multipath fading of radio signals.
    - When there is no line-of-sight (NLOS) path, the signal amplitude follows a Rayleigh distribution.
- **Radar Systems**
  - Modeling the radar echo amplitude from rough surfaces or sea clutter.
    - Radar signal processing and target detection algorithms
- **Oceanography and Wave Height Modeling**
  - The height of random waves (measured from trough to crest) under certain sea states follows a Rayleigh distribution.
    - Ship and offshore platform design, predicting extreme wave events.

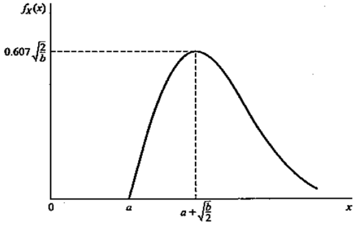
Rayleigh Distribution

Rayleigh Density and Distribution

$$f_X(x) = \begin{cases} \frac{2}{b}(x-a)e^{-(x-a)^2/b} & x \geq a \\ 0 & x < a \end{cases}$$

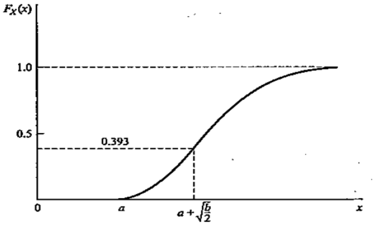
$$F_X(x) = \begin{cases} 1 - e^{-(x-a)^2/b} & x \geq a \\ 0 & x < a \end{cases}$$

Rayleigh Distribution



Source: Peebles

(a)



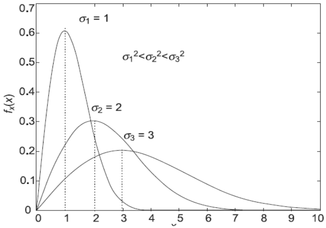
(b)

Figure: (a) Rayleigh density function (b) Rayleigh distribution function

Rayleigh Density and Distribution

$$f_X(x) = \begin{cases} \frac{x}{\sigma^2} e^{-x^2/2\sigma^2} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$F_X(x) = \begin{cases} 1 - e^{-x^2/2\sigma^2} & x \geq 0 \\ 0 & x < 0 \end{cases}$$



Source: Ramesh Babu

Problem

The radial miss distance [in meters (m)] of the landing point of a parachuting sky diver from the center of the target area is known to be a Rayleigh r.v. X with parameter $\sigma^2 = 100$. (a) Find the probability that the sky diver will land within a radius of 10 m from the center of the target area (b) Find r such that $P(X > r) = e^{-1}$

Problem

The radial miss distance [in meters (m)] of the landing point of a parachuting sky diver from the center of the target area is known to be a Rayleigh r.v. X with parameter $\sigma^2 = 100$. (a) Find the probability that the sky diver will land within a radius of 10 m from the center of the target area (b) Find r such that $P(X > r) = e^{-1}$

(a) $P(X \leq 10) = F_X(10) = 1 - e^{-100/200} = 1 - e^{-0.5} =\approx 0.393$

Problem

The radial miss distance [in meters (m)] of the landing point of a parachuting sky diver from the center of the target area is known to be a Rayleigh r.v. X with parameter $\sigma^2 = 100$. (a) Find the probability that the sky diver will land within a radius of 10 m from the center of the target area (b) Find r such that $P(X > r) = e^{-1}$

(a) $P(X \leq 10) = F_X(10) = 1 - e^{-100/200} = 1 - e^{-0.5} =\approx 0.393$

(b)

$$P(X > r) = 1 - P(X \leq r) = 1 - F_X(r)$$
$$= 1 - (1 - e^{-r^2/200}) = e^{-r^2/200})$$

$e^{-r^2/200}) = e^{-1} \implies r^2 = 200 \implies r = \sqrt{200}$

Appendix

Exponential Distribution (Peebles Textbook)

Exponential Density and Distribution

$$f_X(x) = \begin{cases} \frac{1}{b} e^{-(x-a)/b} & x \geq a \\ 0 & x < a \end{cases}$$

$$F_X(x) = \begin{cases} 1 - e^{-(x-a)/b} & x \geq a \\ 0 & x < a \end{cases}$$

for real numbers $-\infty < a < \infty$ and $b > 0$

Exponential Distribution

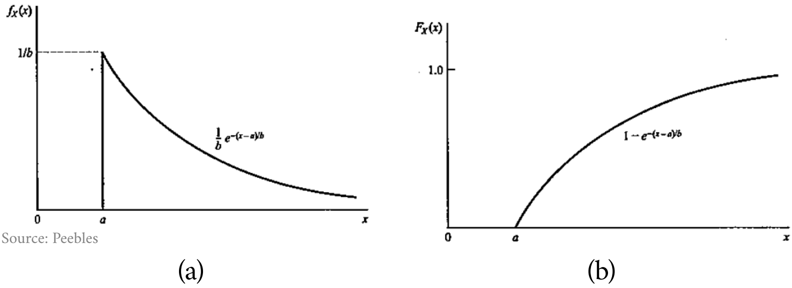


Figure: (a) Exponential density function (b) Exponential distribution function

Acknowledge various sources for the images.
Thankyou