OPERATIONS ON MULTIPLE RANDOM VARIABLES

Dr. G. Omprakash

Assistant Professor, ECE, KLEF

Table of Contents

Expectation of Function of a Random Variables

When more than a single random variable is involved, expectation must be taken with respect to all variables involved. If $g(X,Y)$ is a function of two random variables then expected value is given by

$$\boxed{E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dxdy}$$

For N random variables $X_1, X_2, ..., X_N$

$$\boxed{E[g(X_1,X_2,..,X_N)] = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g(x_1,...,x_N) f_{X_1,...,X_N}(x_1,...,x_N)\,dx_1...dx_N}$$

Joint Moments

Joint Moments about origin

The joint moment m_{nk} is given by

$$\boxed{m_{nk} = E[X^n Y^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k f_{X,Y}(x,y)\,dx\,dy}$$

- $n+k$ is called the **order of the moments**
- m_{02}, m_{20}, m_{11} are the *second-order* moments
- m_{11} is called the **correlation of X and Y**

$$R_{XY} = m_{11} = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\,dx\,dy$$

Correlation of X and Y

$$R_{XY} = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{X,Y}(x,y)dxdy$$

- X and Y are said to be uncorrelated if $R_{XY} = E[XY] = E[X]E[Y]$
- If X and Y are orthogonal $\implies R_{XY} = 0$
- Independence $\implies$ Uncorrelated
- Uncorrelated $\not\Rightarrow$ Independence

Independence $\implies$ Uncorrelated

Given X and Y are independent random variables. Consider

$$R_{XY} = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\,dxdy$$

If X and Y are independent $\implies$ $f_{X,Y}(x,y) = f_X(x)f_Y(y)$. The above equation becomes

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\,dxdy &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_X(x) f_Y(y)\,dxdy \\ &= \int_{-\infty}^{\infty} x f_X(x)\,dx \int_{-\infty}^{\infty} y f_Y(y)\,dy = E[X]E[Y]\end{aligned}$$

$$\boxed{E[XY] = E[X]E[Y]; \qquad \text{Independence} \implies \text{Uncorrelated}}$$

Uncorrelated $\nRightarrow$ Independence

Consider X is a uniformly distributed random variable $U[-1,1]$. Let $Y = X^2$ (Y **depends on** X)
Consider

$$E[XY] = E[XX^2] = E[X^3] = \int_{-1}^{1} x^3 f_X(x)dx = \int_{-1}^{1} x^3 \frac{1}{2}dx = \frac{1}{2}\frac{x^4}{4}\Big|_{-1}^{1} = 0$$

$$E[X]E[Y] = E[X]E[X^2] = \int_{-1}^{1} x f_X(x)dx \int_{-1}^{1} x^2 f_X(x)dx$$

$$= \int_{-1}^{1} x\frac{1}{2}dx \int_{-1}^{1} x^2\frac{1}{2}dx = \left(\frac{1}{2}\frac{x^2}{2}\Big|_{-1}^{1}\right)\left(\frac{1}{2}\frac{x^3}{3}\Big|_{-1}^{1}\right) = 0$$

$E[XY] = E[X]E[Y]$. X and Y are uncorrelated but Y is depending on X

Problems

Q1. If X and Y are independent random variables with pdfs $f_X(x) = \frac{8}{x^3}, x > 2$ and $f_Y(y) = 2y, 0 < y < 1$. Find E[XY]

Ans: 8/3

Q2. Two statistically independent random variables X_1 and X_2 have mean value $\mu_{X_1} = 5$ and $\mu_{X_2} = 10$. Find the mean value of the following functions:

(a). $g(X_1, X_2) = X_1 + 3X_2$ (b). $g(X_1, X_2) = -X_1X_2 + 3X_1 + X_2$

Ans: (a) 35, (b) -25

Joint Central Moments

For two random variables X and Y, the joint central moment μ_{nk} is given by

$$\mu_{nk} = E[(X - \bar{X})^n (Y - \bar{Y})^k]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - \bar{X})^n (y - \bar{Y})^k f_{X,Y}(x, y)\,dxdy$$

The second-order central moments

$$\mu_{20} = E[(X - \bar{X})^2] = \sigma_X^2$$
$$\mu_{02} = E[(Y - \bar{Y})^2] = \sigma_Y^2$$

Covariance

The second-order joint moment μ_{11}, denoted by C_{XY} is the covariance of X and Y

$$C_{XY} = \mu_{11} = E[(X - \bar{X})(Y - \bar{Y})]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x - \bar{X})(x - \bar{Y})f_{X,Y}(x, y)dxdy$$

After expansion and simplification

$$\boxed{C_{XY} = R_{XY} - \bar{X}\bar{Y} = R_{XY} - E[X]E[Y] = E[XY] - E[X]E[Y]}$$

- If X and Y are independent or uncorrelated: $C_{XY} = 0$
- If X and Y are orthogonal: $C_{XY} = -E[X]E[Y]$

Correlation Coefficient

The normalized second order moment is given by

$$\rho = \frac{\mu_{11}}{\sqrt{\mu_{20}\mu_{02}}} = \frac{C_{XY}}{\sigma_X \sigma_Y}$$
$$= E\left[\frac{X - \bar{X}}{\sigma_X} \frac{Y - \bar{Y}}{\sigma_Y}\right]$$

The range of correlation coefficient is

$$-1 \leq \rho \leq 1$$

Problems

Let X and Y be random variables having joint density function

$$f_{X,Y}(x,y) = \begin{cases} x+y & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & elsewhere \end{cases}$$

Find the covariance of X and Y

Problems

Let X and Y be random variables having joint density function

$$f_{X,Y}(x,y) = \begin{cases} x+y & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & elsewhere \end{cases}$$

Find the covariance of X and Y

$C_{XY} = E[XY] - E[X]E[Y]$

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{X,Y}(x,y)dxdy = \int_{0}^{1}\int_{0}^{1} xy(x+y)dxdy = \frac{1}{3}$$

$$f_X(x) = \int_0^1 f_{X,Y}(x,y)dy = x + \frac{1}{2}; f_Y(y) = \int_0^1 f_{X,Y}(x,y)dx = y + \frac{1}{2};$$

$E[X] = \frac{7}{12}$; $E[Y] = \frac{7}{12}$; $C_{XY} = -\frac{1}{144}$

Joint Characteristic Function

Joint Characteristic Function

The *joint characteristic function* of random variables X and Y is defined by

$$\boxed{\Phi_{X,Y}(\omega_1,\omega_2) = E[e^{j\omega_1 X + j\omega_2 Y}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)e^{j\omega_1 X + j\omega_2 Y}dxdy}$$

Joint moments m_{nk} can be found from the joint characteristic function as follows

$$\boxed{m_{nk} = (-j)^{n+k}\frac{\partial^{n+k}\Phi_{X,Y}(\omega_1,\omega_2)}{\partial\omega_1^n\partial\omega_2^k}\bigg|_{\omega_1=0,\omega_2=0}}$$

Problem

Two random variables X and Y have the joint characteristic function

$$\Phi_{X,Y}(\omega_1, \omega_2) = e^{-2\omega_1^2 - 8\omega_2^2}$$

Show X and Y are zero mean and are uncorrelated.

Problem

Two random variables X and Y have the joint characteristic function

$$\Phi_{X,Y}(\omega_1, \omega_2) = e^{-2\omega_1^2 - 8\omega_2^2}$$

Show X and Y are zero mean and are uncorrelated.

$$\bar{X} = E[X] = m_{10} = -j \frac{\partial \Phi_{X,Y}(\omega_1, \omega_2)}{\partial \omega_1} \bigg|_{\omega_1=0,\omega_2=0}$$

$$\bar{Y} = E[Y] = m_{01} = -j \frac{\partial \Phi_{X,Y}(\omega_1, \omega_2)}{\partial \omega_2} \bigg|_{\omega_1=0,\omega_2=0}$$

Problem

Two random variables X and Y have the joint characteristic function

$$\Phi_{X,Y}(\omega_1, \omega_2) = e^{-2\omega_1^2 - 8\omega_2^2}$$

Show X and Y are zero mean and are uncorrelated.

$$\bar{X} = E[X] = m_{10} = -j \left. \frac{\partial \Phi_{X,Y}(\omega_1, \omega_2)}{\partial \omega_1} \right|_{\omega_1=0, \omega_2=0}$$

$$\bar{Y} = E[Y] = m_{01} = -j \left. \frac{\partial \Phi_{X,Y}(\omega_1, \omega_2)}{\partial \omega_2} \right|_{\omega_1=0, \omega_2=0}$$

Uncorrelated $\implies$ Prove: $E[XY] = E[X]E[Y]$

$$E[XY] = m_{11} = (-j)^2 \left. \frac{\partial^2 \Phi_{X,Y}(\omega_1, \omega_2)}{\partial \omega_1 \partial \omega_2} \right|_{\omega_1=0, \omega_2=0}$$

Jointly Gaussian Random Variables

Jointly Gaussian Random Variables

N random variables X_1,X_2,...,X_N are said to be *jointly gaussian* if their joint density function is of the form

$$f_{X_1,X_2...,X_N}(x_1,x_2,...,x_N) = \frac{1}{\sqrt{(2\pi)^N|C_X|}} exp\left[-\frac{1}{2}(x-\bar{X})^T C_X^{-1}(x-\bar{X})\right]$$

where

$$[x-\bar{X}] = \begin{bmatrix} x_1-\bar{X}_1 \\ x_2-\bar{X}_2 \\ \vdots \\ x_N-\bar{X}_N \end{bmatrix}; \quad [C_X] = \begin{bmatrix} C_{11} & C_{12}\cdots C_{1N} \\ C_{21} & C_{22}\cdots C_{2N} \\ \vdots & \vdots \\ C_{N1} & C_{N2}\cdots C_{NN} \end{bmatrix}$$

$$C_{ij} = E[(X_i-\mu_{X_i})(X_j-\mu_{X_j})]$$

Jointly Gaussian Random Variables

For $N = 2$

$$[C_X] = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} \sigma_{X_1}^2 & Cov(X_1, X_2) \\ Cov(X_1, X_2) & \sigma_{X_2}^2 \end{bmatrix}$$

$C_{ij} = E[(X_i - \mu_{X_i})(X_j - \mu_{X_j})]$; we know that $\rho = \frac{Cov(X_1,X_2)}{\sigma_{X_1}\sigma_{X_2}}$

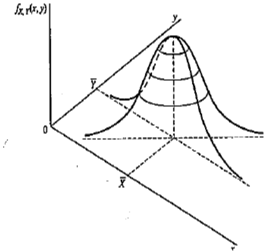
$$[C_X] = \begin{bmatrix} \sigma_{X_1}^2 & \rho\sigma_{X_1}\sigma_{X_2} \\ \rho\sigma_{X_1}\sigma_{X_2} & \sigma_{X_2}^2 \end{bmatrix} \implies |C_X| = (1-\rho^2)(\sigma_{X_1}^2\sigma_{X_2}^2)$$

$$C_X^{-1} = \frac{1}{(1-\rho^2)} \begin{bmatrix} 1/\sigma_{X_1}^2 & -\rho/\sigma_{X_1}\sigma_{X_2} \\ -\rho/\sigma_{X_1}\sigma_{X_2} & 1/\sigma_{X_2}^2 \end{bmatrix}$$

Jointly Gaussian Random Variables

Two random variables X and Y are said to be *jointly Gaussian* if their pdf is of the form

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} exp\left[-\frac{1}{2(1-\rho^2)}\left\{\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} + \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right\}\right]$$



Source: Peyton Z. Peebles,Jr.,2001

Maximum is located at the point $(\bar{X}, \bar{Y})$
$f_{X,Y}(x,y) \leq f_{X,Y}(\bar{X}, \bar{Y})$

where $f_{X,Y}(\bar{X}, \bar{Y}) = \dfrac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}$

Figure: Joint pdf of two gaussian random variables

If $\rho = 0 \implies X$ and Y are uncorrelated. Substitute $\rho = 0$ in the equation

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} exp\left[-\frac{1}{2(1-\rho^2)}\left\{\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} + \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right\}\right]$$

we get

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y} exp\left[-\frac{1}{2}\left\{\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right\}\right]$$

we can write

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

where $f_X(x)$ and $f_Y(y)$ are the marginal density functions of X and Y given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} exp\left[-\frac{(x-\bar{X})^2}{2\sigma_X^2}\right]; \qquad f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} exp\left[-\frac{(y-\bar{Y})^2}{2\sigma_Y^2}\right]$$

Properties of Gaussian Random Variables

1. Gaussian random variables are completely characterized by their means, variances and covariances
2. If the Gaussian random variables are uncorrelated, they are also statistically independent
3. Random variables produced by a linear transformation of $X_1, X_2, ..., X_N$ will also be gaussian
4. Any $k-$dimensional marginal density function obtained from the N-dimensional density function by integrating out $N-k$ random variables will be gaussian
5. The conditional density $f_{X_1,...,X_k}(x_1, ..., x_k \setminus X_{k+1} = x_{x+1}, ..., X_N = x_N)$ is gaussian

Transformations of multiple random variables

Transformations of multiple random variables

Consider a random variable $Y = g(X_1, X_2, ..., X_N)$. It is necessary to determine the density of Y to calculate expectation value.
Let us define the distribution function

$$F_Y(y) = P\{Y \leq y\} = P\{g(X_1, X_2, ..., X_N) \leq y\}$$
$$= \iiint_{\{g(X_1,X_2,...,X_N)\leq y\}} f_{X_1,X_2,...,X_N}(x_1, x_2, ..., x_N)\, dx_1\, dx_2 ... dx_N$$

Density function follows differentiation

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{d}{dy} \iiint_{\{g(X_1,X_2,...,X_N)\leq y\}} f_{X_1,X_2,...,X_N}(x_1, x_2, ..., x_N)\, dx_1\, dx_2 ... dx_N$$

Problems

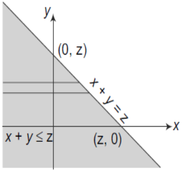
Consider the random variables X and Y with pdfs $f_X(x)$ and $f_Y(y)$. Find pdf of Z where $Z = X + Y$
We know $F_Z(z) = P(Z \leq z) = P(X + Y \leq z)$

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_{X,Y}(x, y)\, dxdy$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(z - y, y)\, dy$$

If X and Y are independent : $f_{X,Y}(x, y) = f_X(x) f_Y(y)$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z - y) f_Y(y)\, dy$$



Source: Ramesh Babu

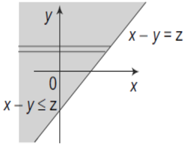
Figure: $F_Z(z)$ is integration of the shaded region

Problems

Consider the random variables X and Y with pdfs $f_X(x)$ and $f_Y(y)$. Find pdf of Z where $Z = X - Y$
We know $F_Z(z) = P(Z \leq z) = P(X - Y \leq z)$

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z+y} f_{X,Y}(x, y)\,dxdy$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(z + y, y)\,dy$$



Source: Ramesh Babu

Figure: $F_Z(z)$ is integration of the shaded region

If X and Y are independent : $f_{X,Y}(x, y) = f_X(x) f_Y(y)$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z + y) f_Y(y)\,dy$$

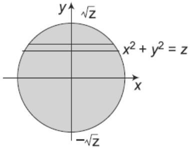
Problems

$Z = X^2 + Y^2$. Find $f_Z(z)$

$$F_Z(z) = P(x^2 + y^2 \le z) = \iint\limits_{x^2+y^2\le z} f_{X,Y}(x,y)\,dxdy$$

$$= \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} f_{X,Y}(x,y)\,dxdy$$

$$f_Z(z) = \int_{-\sqrt{z}}^{\sqrt{z}} \frac{f_{X,Y}(\sqrt{z-y^2}, y) + f_{X,Y}(-\sqrt{z-y^2}, y)}{2\sqrt{z-y^2}}\,dy$$



Source: Ramesh Babu

Figure: $F_Z(z)$ is integration of the shaded region

x limits are $-\sqrt{z-y^2}$ to $\sqrt{z-y^2}$, y limits $-\sqrt{z}$ to $\sqrt{z}$

Multiple Function-Example

Let the transformations be linear and given by

$$Y_1 = T_1(X_1, X_2) = aX_1 + bX_2$$
$$Y_2 = T_1(X_1, X_2) = cX_1 + dX_2$$

Solving for X_1 and X_2 we get

$$X_1 = T_1^{-1}(Y_1, Y_2) = (dY_1 - bY_2)/(ad - bc)$$
$$X_2 = T_2^{-1}(Y_1, Y_2) = (-cY_1 + aY_2)/(ad - bc)$$

$$[J] = \begin{bmatrix} \frac{\partial T_1^{-1}}{\partial Y_1} & \frac{\partial T_1^{-1}}{\partial Y_2} \\ \frac{\partial T_2^{-1}}{\partial Y_1} & \frac{\partial T_2^{-1}}{\partial Y_2} \end{bmatrix}$$

$$[J] = \begin{bmatrix} d/(ad - bc) & -b/(ad - bc) \\ -c/(ad - bc) & a/(ad - bc) \end{bmatrix}$$

$$|J| = \frac{1}{|ad - bc|}$$

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(x_1 = T_1^{-1}, x_2 = T_2^{-1})|J|$$
$$= \frac{f_{X_1,X_2}\left(\frac{dY_1 - bY_2}{ad - bc}, \frac{-cY_1 + aY_2}{ad - bc}\right)}{|ad - bc|}$$

Linear Transformation of Gaussian RV

Gaussian random variables $X_1, X_2, ..., X_N$ are transformed to $Y_1, Y_2, ..., Y_N$

$$
\begin{aligned}
Y_1 &= a_{11}X_1 + a_{12}X_2 + \cdots + a_{1N}X_N \\
Y_2 &= a_{21}X_1 + a_{22}X_2 + \cdots + a_{2N}X_N \\
&\vdots \\
Y_N &= a_{N1}X_1 + a_{N2}X_2 + \cdots + a_{NN}X_N \\
\boldsymbol{Y} &= T\boldsymbol{X}
\end{aligned}
$$

Mean and variance/Covariance of Y is given by

$$
\bar{Y}_j = \sum_{k=1}^{N} a_{jk}\bar{X}_k; \qquad [C_Y] = [T][C_X][T]'
$$

Problem

Two gaussian random variables X_1 and X_2 have zero means and variances $\sigma^2_{X_1} = 4$ and $\sigma^2_{X_2} = 9$. Their covariance $C_{X_1X_2} = 3$. If X_1 and X_2 are transformed to new variables Y_1 and Y_2 according to

$$Y_1 = X_1 - 2X_2$$
$$Y_2 = 3X_1 + 4X_2$$

$$[T] = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}; [C_X] = \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix}$$

;

Since X_1 and X_2 are zero mean and gaussian, Y_1 and Y_2 will also be zero mean and gaussian.

$$[C_Y] = [T][C_X][T]^t = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 28 & -66 \\ -66 & 252 \end{bmatrix}$$

Hence $\sigma^2_{Y_1} = 28; \sigma^2_{Y_1} = 252; C_{Y_1Y_2} = -66;$

Problem

Let X_1 and X_2 be two independent standard Gaussian random variables. Define random variables $Y_1 = 2X_1 + X_2$ and $Y_2 = X_1 - X_2$. Find $E(Y_1)$, $E(Y_2)$, $Cov(X_1, X_2)$, correlation coefficient and joint PDF of Y_1 and Y_2.

X_1, X_2 are standard gaussian $\implies E[X_1] = 0, E[X_2] = 0, \sigma_{X_1}^2 = 1, \sigma_{X_2}^2 = 1$

X_1, X_2 are independent $\implies$ uncorrelated $C_{X_1X_2} = 0$

$$Y_1 = 2X_1 + X_2$$
$$Y_2 = X_1 - X_2$$

$$[T] = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}; [C_X] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

;

$$[C_Y] = [T][C_X][T]^t = \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}$$

;

$$\rho = \frac{Cov(Y_1, Y_2)}{\sigma_{Y_1}\sigma_{Y_2}} = \frac{1}{\sqrt{5}\sqrt{2}}$$

$$[C_Y^{-1}] = \frac{1}{9}\begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix}$$

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{\sqrt{(2\pi)^2|C_Y|}} exp\left[-\frac{1}{2}(\boldsymbol{y} - \bar{\boldsymbol{Y}})^T C_Y^{-1}(\boldsymbol{y} - \bar{\boldsymbol{Y}})\right]$$

$$|C_Y| = 9$$

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{\sqrt{(2\pi)^2 9}} exp\left[-\frac{2y_1^2 - 2y_1 y_2 + 5y_2^2}{18}\right]$$

$$[\boldsymbol{y} - \bar{\boldsymbol{Y}}] = \begin{bmatrix} y_1 - \bar{Y}_1 \\ y_2 - \bar{Y}_2 \end{bmatrix} = \begin{bmatrix} y_1 - 0 \\ y_2 - 0 \end{bmatrix}$$

Acknowledge various sources for the images.

Thankyou