Random Process: Spectral Characteristics

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Motivation

Spectral characteristics: Why do we need them?

- **Biomedical Signals**: They are random but have clear frequency content.
  - Take ECG data $\longrightarrow$ show PSD $\longrightarrow$ most energy lies below 40 Hz
  - Helps in designing filters (e.g., removing high-frequency noise or 50 Hz powerline interference).
- **Music vs Noise**: Why does music sound “structured” but noise sounds “random”?
  - Plot PSD of music vs white noise.
  - Music has spectral peaks (harmonics), noise has a flat spectrum.
- **Stock Market Returns**: Finance also treats returns as random processes.
  - Spectral analysis helps identify these dominant frequencies, which correspond to economic cycles or investor behaviors.
  - Time-domain analysis (like simple trend analysis) often misses these periodic components.

Power Density Spectrum

Why Power Density? Why not the Fourier transform?

- The spectral properties of a *deterministic* signal $x(t)$ are contained in its *Fourier transform*: $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$
- For a Fourier transform to exist: *deterministic* signal $x(t)$ must satisfy Dirichlet's conditions: **Absolutely integrable, Finite number of discontinuities, Finite number of maxima/minima**
- We cannot apply Fourier transform to *random process* $x(t)$
  - Dirichlet condition: **Absolute Integrability (Finite Energy) is not satisfied for random process** $x(t)$
- A stationary random process (like white noise, financial returns, etc.) is defined over all time and **never decays**.
  - Even if you take the Fourier Transform of one realization, that transform will differ for every realization — completely random and not useful.

Power Spectral Density (PSD)

- What we actually care about is the average frequency content of the process across all realizations — its *spectral characteristics*.
- Instead of taking the FT of the random process itself, we take the FT of its autocorrelation function $R_X(\tau) = E[X(t)X(t+\tau)]$.

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

- $R_{XX}(\tau)$ is deterministic and integrable

Power Density spectrum: Derivation

Consider a random process $X(t)$. Let us take one sample function $x(t)$. Let $x_T(t)$ be portion of sample function that exists between $-T$ and T

$$x_T(t) = \begin{cases} x(t) & -T < t < T \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Since $x_T(t)$ is finite duration, it is absolutely integrable and the Fourier transform can be defined

$$X_T(\omega) = \int_{-T}^{T} x_T(t) e^{-j\omega t} dt = \int_{-T}^{T} x(t) e^{-j\omega t} dt \tag{2}$$

Power Density spectrum: Derivation

We are interested in Power..First calculate energy and divide it by time duration to get power. The energy contained in $x(t)$ in the interval $(-T, T)$ is

$$E(T) = \int_{-T}^{T} x^2(t)dt \tag{3}$$

Using Parseval's theorem, the relation between $x(t)$ and its Fourier transform $X_T(\omega)$ is given by

$$E(T) = \int_{-T}^{T} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_T(\omega)|^2 d\omega \tag{4}$$

The average power $P(T)$ is given by

$$P(T) = \frac{1}{2T}\int_{-T}^{T} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|X_T(\omega)|^2}{2T} d\omega \tag{5}$$

$\frac{|X_T(\omega)|^2}{2T}$ is the power density spectrum for a finite duration sample function $x(t)$. It varies for different sample functions of $X(t)$. So $P(T)$ is a random quantity. Take the expected value of $P(T)$ (with $T \to \infty$) to get the average power of the random process

$$P_{XX} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E[X^2(t)]dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{E|X_T(\omega)|^2}{2T} d\omega \tag{6}$$

Two observation from the previous equation:(1). Average power of the random process $X(t)$ is given by the time average of its second moment

$$\boxed{P_{XX} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E[X^2(t)] = A[E[X^2(t)]]}$$

P_{XX} can also be obtained from the frequency domain using the equation

$$P_{XX} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) d\omega = \int_{-\infty}^{\infty} S_{XX}(f) df \tag{7}$$

where $S_{XX}(\omega)$ is the PSD given by

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T} \tag{8}$$

Power Density Spectrum

The power density spectrum for the random process is given by

$$\boxed{S_{XX}(\omega) = \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T}}$$

Properties of PSD

- The power density spectrum $S_{XX}(\omega)$ is a real value and non-negative function: $S_{XX}(\omega) \geq 0$
- The power density spectrum is an even function in ω: $S_{XX}(\omega) = S_{XX}(-\omega)$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega\tau}d\tau \implies S_{XX}(-\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{j\omega\tau}d\tau$$

Replace τ by $-\tau$

$$S_{XX}(-\omega) = \int_{\infty}^{-\infty} R_{XX}(-\tau)e^{-j\omega\tau}(-d\tau) = \int_{-\infty}^{\infty} R_{XX}(-\tau)e^{-j\omega\tau}d\tau$$

Since $R_{XX}(-\tau) = R_{XX}(\tau)$

$$S_{XX}(-\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega\tau}d\tau = S_{XX}(\omega)$$

Properties of PSD

- The power density spectrum is a real function
  - From the definition of PSD eqn (8) $|X_T(\omega)|^2$ is real so PSD is also real
- Average power is equal to the integral of $S_{XX}(\omega)$

$$P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\,d\omega$$

- $S_{\dot{X}\dot{X}}(\omega) = \omega^2 S_{XX}(\omega)$
  - The power density spectrum of the derivative $\dot{X} = dX(t)/dt$ is ω^2 times the power spectrum of $X(t)$
- Power spectrum and the autocorrelation function form a Fourier transform pair

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega\tau}\,d\tau; \qquad R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{j\omega\tau}\,d\omega;$$

Relationship b/w power spectrum and autocorrelation

Q: Derive the relationship between the power density spectrum and the autocorrelation function

We know that

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{E\left[|X_T(\omega)|\right]^2}{2T}; \text{ where } X_T(\omega) = \int_{-T}^{T} x_T(t)e^{-j\omega t}dt$$

Substitute $|X_T(\omega)|^2 = X_T^*(\omega)X_T(\omega)$ in the power density equation

$$S_{XX}(\omega) = \lim_{T\to\infty} E\left[\frac{1}{2T}\int_{-T}^{T} X(t_1)e^{j\omega t_1}dt_1 \int_{-T}^{T} X(t_2)e^{-j\omega t_2}dt_2\right]$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} E[X(t_1)X(t_2)]e^{-j\omega(t_2-t_1)}dt_2dt_1$$

Relationship b/w power spectrum and autocorrelation

We know that $E[X(t_1)X(t_2)] = R_{XX}(t_1, t_2)$. The PSD becomes

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} R_{XX}(t_1, t_2) e^{-j\omega(t_2 - t_1)} dt_2 dt_1 \tag{9}$$

Take the inverse Fourier transform of $S_{XX}(\omega)$

$$F^{-1}[S_{XX}(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega \tag{10}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} R_{XX}(t_1, t_2) e^{-j\omega(t_2 - t_1 - \tau)} dt_2 dt_1 d\omega \tag{11}$$

Relationship b/w power spectrum and autocorrelation

$$F^{-1}[S_{XX}(\omega)] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2)\frac{1}{2\pi}\left[\int_{-\infty}^{\infty} e^{-j\omega(t_2-t_1-\tau)}d\omega\right]dt_2dt_1$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2)\frac{1}{2\pi}[2\pi\delta(t_2-t_1-\tau)]dt_2dt_1$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_{XX}(t_1,t_1+\tau)dt_1 = A[R_{XX}(t,t+\tau)]$$

For a wide-sense stationary process $A[R_{XX}(t,t+\tau)] = R_{XX}(\tau)$.

Relationship b/w power spectrum and autocorrelation

Therefore

$$F^{-1}[S_{XX}(\omega)] = R_{XX}(\tau) \implies S_{XX}(\omega) = F[R_{XX}(\tau)] \tag{12}$$

The power spectral density is Fourier transform of autocorrelation function

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau; \tag{13}$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega; \tag{14}$$

The above two equations are called the *Weiner-Khintchine relations*

Problems

If $R_{XX}(\tau) = ae^{-b|\tau|}$, find the spectral density function, where a and b are constants

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$$S_{XX}(\omega) = F[R_{XX}(\tau)] = a\left[\int_{-\infty}^{0} e^{(b-j\omega)\tau} d\tau + \int_{0}^{\infty} e^{-(b+j\omega)\tau} d\tau\right] \tag{15}$$

$$= \frac{2ab}{b^2 + \omega^2} \tag{16}$$

Problems

If $R_{XX}(\tau) = (A^2/2)cos(\omega_0\tau)$, find the spectral density, where A and ω_0 are constants

Problems

If $R_{XX}(\tau) = (A^2/2)cos(\omega_0\tau)$, find the spectral density, where A and ω_0 are constants

$$S_{XX}(\omega) = \frac{A^2\pi}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] \tag{17}$$

Acknowledge various sources for the images.

Thankyou