Random Process: Spectral Characteristics

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- The **Cross Power Spectrum** quantifies *how much energy two signals share at each frequency*.
- **Radar**: The transmitted pulse and the received echo are related
- **Neuroscience**: Stimulus and brain response signals are related through the sensory pathway.
- **Communications**: The input and output of a noisy channel are related through the system's transfer function.
- **Audio processing**: Estimate time delays of microphone array signals for source localization

Cross Power density spectrum

Consider a real random process $W(t)$ given by the sum of two other real processes $X(t)$ and $Y(t)$:

$$W(t) = X(t) + Y(t)$$

The autocorrelation function of $W(t)$ is

$$\begin{aligned} R_{WW}(t, t+\tau) &= E[W(t)W(t+\tau)] \\ &= E[(X(t) + Y(t))(X(t+\tau) + Y(t+\tau))] \\ &= R_{XX}(t, t+\tau) + R_{YY}(t, t+\tau) + R_{XY}(t, t+\tau) + R_{YX}(t, t+\tau) \end{aligned}$$

Take the time average of both sides and take Fourier transform (denoted by $F\{\cdot\}$) of the resulting expression, we have

$$S_{WW}(\omega) = S_{XX}(\omega) + S_{YY}(\omega) + F\{A[R_{XY}(t, t+\tau)]\} + F\{A[R_{YX}(t, t+\tau)]\}$$

$F\{A[R_{XY}(t, t+\tau)]\}, F\{A[R_{YX}(t, t+\tau)]\}$ are the *cross power density spectrums*

Cross Power density spectrum-Derivation

Consider two real random processes $X(t)$ and $Y(t)$. Let $x_T(t)$ and $y_T(t)$ be truncated ensemble members that exists between $-T$ and T

$$x_T(t) = \begin{cases} x(t) & -T < t < T \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$y_T(t) = \begin{cases} y(t) & -T < t < T \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$x_T(t)$ and $y_T(t)$ are integrable and Fourier transforms are denoted as $X_T(\omega)$ and $Y_T(\omega)$ respectively. Define the *cross power* $P_{XY}(T)$ in the two processes within the interval $(-T, T)$ by

$$P_{XY}(T) = \frac{1}{2T}\int_{-T}^{T} x_T(t)y_T(t)dt = \frac{1}{2T}\int_{-T}^{T} x(t)y(t)dt$$

Cross Power density spectrum-Derivation

Using Parseval's theorem we have

$$P_{XY}(T) = \frac{1}{2T}\int_{-T}^{T} x(t)y(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{X_T^*(\omega)Y_T(\omega)}{2T}d\omega$$

The average cross power is obtained by taking the expected value and letting $T \rightarrow \infty$

$$P_{XY} = \lim_{T\rightarrow\infty}\frac{1}{2T}\int_{-T}^{T} R_{XY}(t,t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lim_{T\rightarrow\infty}\frac{X_T^*(\omega)Y_T(\omega)}{2T}d\omega$$

The *cross-power density spectrum* is defined as

$$\boxed{S_{XY}(\omega) = \lim_{T\rightarrow\infty}\frac{X_T^*(\omega)Y_T(\omega)}{2T}}$$

cross-power formula is given by

$$\boxed{P_{XY} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)d\omega}$$

Similarly we can define the other *cross-power density spectrum* as

$$\boxed{S_{YX}(\omega) = \lim_{T\to\infty} \frac{Y_T^*(\omega)X_T(\omega)}{2T}}$$

cross-power is given by

$$\boxed{P_{YX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YX}(\omega)d\omega}$$

Properties of the cross-power density spectrum

- $S_{XY}(\omega) = S_{YX}(-\omega) = S_{YX}^{*}(\omega)$
- Real parts of $S_{XY}(\omega)$ and $S_{YX}(\omega)$ are even functions of ω
- Imaginary parts of $S_{XY}(\omega)$ and $S_{YX}(\omega)$ are odd functions of ω
- $S_{XY}(\omega) = 0$ and $S_{YX}(\omega) = 0$, if $X(t)$ and $Y(t)$ are orthogonal
- If $X(t)$ and $Y(t)$ are uncorrelated and have constant means $\bar{X}$ and $\bar{Y}$, then

$$S_{XY}(\omega) = S_{YX}(\omega) = 2\pi \bar{X}\bar{Y}\delta(\omega)$$

- Cross power density spectrum and the time average of the cross-correlation function are a Fourier transform pair

$$A[R_{XY}(t, t+\tau)] \leftrightarrow S_{XY}(\omega)$$
$$A[R_{YX}(t, t+\tau)] \leftrightarrow S_{YX}(\omega)$$

Problems

The cross-correlation function of two processes $X(t)$ and $Y(t)$ is given in the equation, where A, B and ω_0 are constants. Find the cross power spectrum.

$$R_{XY}(t, t+\tau) = \frac{AB}{2}[sin(\omega_0\tau) + cos[\omega_0(2t+\tau)]]$$

Problems

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$$R_{XY}(t, t+\tau) = \frac{AB}{2}[sin(\omega_0 \tau) + cos[\omega_0(2t+\tau)]]$$

Step 1: Find the time average:

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{XY}(t, t+\tau)dt = \frac{AB}{2} sin(\omega_0\tau) + \frac{AB}{2} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} cos[\omega_0(2t+\tau)]dt$$

The second term is zero. **Step 2**: Now take the Fourier transform of the time average

$$S_{XY}(\omega) = F\left\{\frac{AB}{2} sin(\omega_0\tau)\right\} = \frac{-j\pi AB}{2}[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)]$$

Problems

Given $R_{XY}(\tau) = 4u(\tau)e^{-\alpha\tau}$. Find $S_{XY}(\omega)$

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Given $R_{XY}(\tau) = 4u(\tau)e^{-\alpha\tau}$. Find $S_{XY}(\omega)$

We know that $S_{XY}(\omega) = F[A[R_{XY}(t, t+\tau)]]$.

Given $R_{XY}(t, t+\tau)$ is independent of t, so no need to take time average. Take Fourier transform directly

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} 4u(\tau)e^{-\alpha\tau}e^{-j\omega\tau}d\tau = \int_{0}^{\infty} 4e^{-(\alpha+j\omega)\tau}d\tau$$

$$= \frac{4}{\alpha + j\omega}$$

Problems

Find the cross-spectral density function of two random processes $X(t) = Acos(\omega_0 t) + Bsin(\omega_0 t)$ and $Y(t) = -Asin(\omega_0 t) + Bcos(\omega_0 t)$. The mean values $\mu_X = \mu_Y = 0$. The variance of these processes are σ^2

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$$R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)] = -\sigma^2 sin(\omega_0 \tau)$$

$$S_{XY}(\omega) = -\sigma^2 j\pi[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]$$

Acknowledge various sources for the images.

Thankyou