Random Variables, Distribution and Density functions

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Random Variable

Random Variable

- ☞ Every random event has a story — a random variable tells it with numbers
- ✠ A Random Variable is a numerical description of an uncertain outcome
- ✧ From coin tosses to commute times — random variables make uncertainty measurable.

Random variable is a function mapping a probability space into the real line $\mathbb{R}$

Random Variable

Random variable is a *real* function that maps all elements of the sample space into points on the real line.

- Random variable is represented by a capital letter (X, Y or Z)
- Any particular value of a random variable by a lowercase letter (x,y or w)

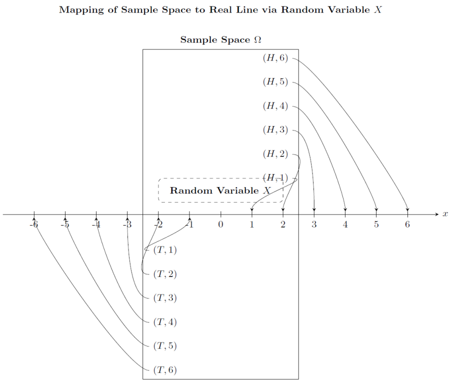
Example 1

Toss a fair coin twice

- Sample space: $S = \{HH, TT, TH, HT\}$
- Random Variable X: Number of heads
  - X(TT)=0; X(TH)=1; X(HT)=1, X(HH)=1;
- **This mapping enables us to analyze the outcomes numerically**
  - Assigning probability to the random variable
  - $P(X = 1) = P(\{TH, HT\}) = \frac{2}{4}$
  - $P(X = 0) = P(\{TT\}) = \frac{1}{4}$
  - $P(X = 2) = P(\{HH\}) = \frac{1}{4}$

Example 2

An experiment consists of rolling a die and flipping a coin. Let the random variable be a function X such that Head corresponds to positive value equal to numbers show up on the die, and Tails correspond to negative value whose magnitude is equal to numbers that show up on die.



Source: Ramesh Babu

Conditions for a function to be a random variable

- Every point in S must correspond to only one value of the random variable
  - One outcome can't be mapped to two real values
- The set $\{X \leq x\}$ shall be an event for any real number x and $P\{X \leq x\}$ equals sum of probabilities of all elementary events corresponding to $\{X \leq x\}$
  - In Eg 1: $P(X \leq 1) = P(X = 0) + P(X = 1)$
- $P(X = -\infty) = 0, P(X = \infty) = 0$

Examples

The sample space for an experiment is S = (0, 1, 2, 3). List all possible values of the following random variables:

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- $X = s^2 - 1$

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- $X = \{0, 2, 4, 6\}$
- $X = s^2 - 1$
- $X = \{-1, 0, 3, 8\}$

Countable and uncountable

Countable: A set is said to be *countable* if its elements can be put in **one-to-one correspondence with the natural numbers**.

- Number of students in college: Countable and finite
- $A = \{2, 3\}$: Countable and finite
- integers >0: Countable and infinite

If a set is not countable, it is called **uncountable**.

- $B = \{0 < x \leq 10\}$: uncountable and infinite
- $E = \{-30 \leq x \leq -1\}$: uncountable and infinite

Classification of Random Variables

- A **continuous random variable** is one having a continuous range of values
- If X can take uncountable number of values in the given range then it is continuous RV.
  - $X = \{-5 < x < 10\}$ is a continuous random variable
  - Collection of heights of students in a class
  - Collection of temperature values in a day
- A **discrete random variable** is one having only discrete values.
- If X takes countable number of values in a given range it is a Discrete RV.
  - $X = \{2, 4, 6, 8, 9, 10\}$ is a discrete random variable.
  - Tossing a coin, Rolling a dice
- A **mixed random variable** is one with both discrete and continuous values.
  - $X = \{5, 6, 10 < x < 15, 19, 20\}$ is a mixed random variable

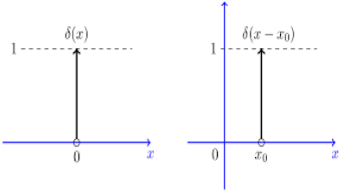
Distribution and Density functions

"In real life, we rarely care about exact outcomes. We care about likelihoods over ranges"

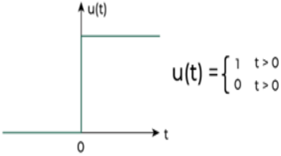
- The chance it rains before noon
- The chance your grade is above 80
- The chance your train arrives in the next 5 minutes

Distribution and density functions are the language of such reasoning.

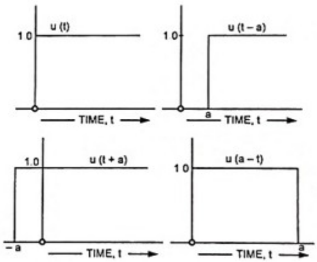
Unit Step and Impulse functions



Impulse Function



Unit-Step Function



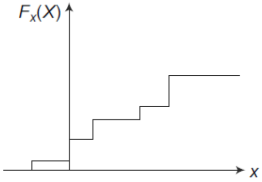
Shifted Unit-Step function

Distribution Function

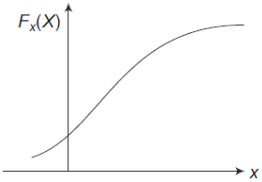
The cumulative distribution function of a random variable X is given by

$$F_X(x) = P\{X \leq x\}$$

where $P\{X \leq x\}$ is the probability of the event $\{X \leq x\}$



(a)



(b)

Figure: (a) CDF of Discrete RV, (b) CDF of Continuous RV

Distribution Function

Properties

- $F_X(-\infty) = 0$
- $F_X(\infty) = 1$
- $0 \leq F_X(x) \leq 1$
- $F_X(x_1) \leq F_X(x_2)$ if $x_1 \leq x_2$
  - $F_X(x)$ is a nondecreasing function of x
- $P\{x_1 < X \leq x_2\} = F_X(x_2) - F_X(x_1)$
- $F_X(x^+) = F_X(x)$
  - $F_X(x)$ is a function continuous from the right

Distribution function

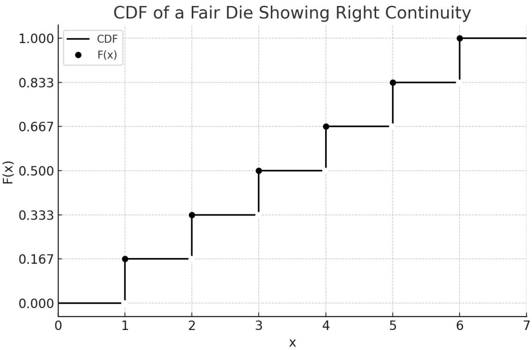
Consider the experiment: Throwing of a Die. The Sample space, $S = \{1, 2, 3, 4, 5, 6\}$.

Consider the Random variable X defined as $X = 2s$.

The values X takes $\{2, 4, 6, 8, 10, 12\}$

Event	X Values	Outcomes	$P(X \leq x)$
$X \leq 4$	$\{2, 4\}$	$\{1, 2\}$	$P(X \leq 4) = \frac{2}{6}$
$X \leq 11$	$\{2, 4, 6, 8, 10\}$	$\{1, 2, 3, 4, 5\}$	$P(X \leq 11) = \frac{5}{6}$
$X \leq -\infty$	$\{\phi\}$ Null set	$\{\phi\}$ Null set	$P(X \leq -\infty) = 0$
$X \leq \infty$	$\{2, 4, 6, 8, 10, 12\}$	$\{1, 2, 3, 4, 5, 6\}$ Sample space	$P(X \leq \infty) = 1$

$$F_X(x) = P(X \leq x)$$
$$F_X(\infty) = P(X \leq \infty) = 1$$
$$F_X(-\infty) = P(X \leq -\infty) = 0$$


CDF of a Fair Die Showing Right Continuity
CDF
F(x)
1.000
0.833
0.667
0.500
0.333
0.167
0.000
F(x)
0
1
2
3
4
5
6
7
x

CDF Example

Consider the experiment of tossing four fair coins. The random variable X is associated with the number of tails showing. Compute and sketch the CDF of X.

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Consider the experiment of tossing four fair coins. The random variable X is associated with the number of tails showing. Compute and sketch the CDF of X.
Soln:
Sample space:
$\{HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT\}$.

CDF Example

Consider the experiment of tossing four fair coins. The random variable X is associated with the number of tails showing. Compute and sketch the CDF of X.

Soln:

Sample space:

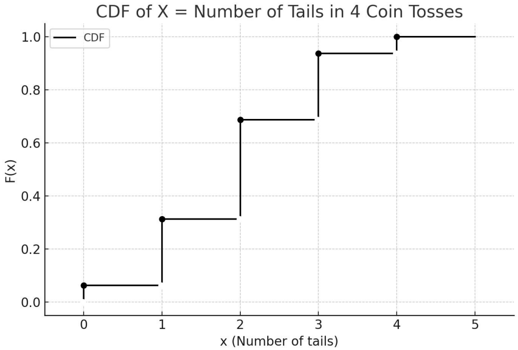
$\{HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT\}$.

- Outcomes associated with $X = 0 \implies \{HHHH\}$;
  - So $P(X = 0) = \frac{1}{16}$
- Outcomes for $X = 1 \implies \{HHHT, HHTH, HTHH, THHH\}$.
  - $P(X = 1) = 4 \cdot \frac{1}{16}$
- Outcomes for $X = 2 \implies \{HHTT, HTHT, HTTH, THHT, THTH, TTHH\}$
  - $P(X = 2) = 6 \cdot \frac{1}{16}$

CDF Example

- Outcomes for $X = 3 \implies \{HTTT, THTT, TTHT, TTTH\}$.
  - $P(X = 3) = 4 \cdot \frac{1}{16}$
- Outcomes for $X = 4 \implies \{TTTT\}$
  - $P(X = 4) = \frac{1}{16}$
- $F_X(0) = P(X \leq 0) = P(X = 0) = \frac{1}{16}$;
  $F_X(1) = P(X \leq 1) = P(X = 0) + P(X = 1) = \frac{5}{16}$;
- $F_X(2) = P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = \frac{11}{16}$;
- $F_X(3) = P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) = \frac{15}{16}$;
- $F_X(4) = P(X \leq 4) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) = 1$

CDF of X


CDF of X = Number of Tails in 4 Coin Tosses
CDF
1.0
0.8
0.6
0.4
0.2
0.0
F(x)
0
1
2
3
4
5
x (Number of tails)

Density Function

The probability density function, denoted by $f_X(x)$, is defined as the derivative of the distribution function

$$f_X(x) = \frac{dF_X(x)}{dx} \tag{1}$$

Properties

1. $f_X(x) \geq 0 \qquad$ for all x
2. $\int_{-\infty}^{\infty} f_X(x)dx = 1$
3. $F_X(x) = \int_{-\infty}^{x} f_X(\xi)d\xi$
4. $P\{x_1 < X \leq x_2\} = \int_{x_1}^{x_2} f_X(x)$

Problems

Question

Which of the following are valid probability density functions?

A.

$$f_X(x) = \begin{cases} \frac{1}{5} & 0 \le x \le 5 \\ 0 & \text{elsewhere} \end{cases}$$

B.

$$f_X(x) = \begin{cases} e^{-x} & 0 \le x \le \infty \\ 0 & \text{elsewhere} \end{cases}$$

C.

$$f_X(x) = \begin{cases} \frac{1}{4}(x^2 - 1) & |x| < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Problems

X is a continuous random variable with the pdf

$$f_X(x) = \begin{cases} A(3x - x^2) & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

(a) What is the value of A (b) Find $P(X > 1)$

Problems

X is a continuous random variable with the pdf

$$f_X(x) = \begin{cases} A(3x - x^2) & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

(a) What is the value of A (b) Find $P(X > 1)$

$A = \frac{3}{10}, P(X > 1) = \frac{13}{20}$

A probability mass function is given in the table

X	0	1	2	3	4	5	6
$P(X)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

Find (a) $P(X < 4)$ (b) $P(X \geq 5)$ and (c) $P(3 < X \leq 6)$ (d) Calculate the minimum value of k such that $P(X \leq 2) > 0.3$.

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$k = \frac{1}{49}$, (a) $P(X < 4) = \frac{16}{49}$ (b) $P(X \geq 5) = \frac{24}{49}$. (c) $P(3 < X \leq 6) = \frac{33}{49}$. (d) $k > \frac{1}{30}$

Mean and Variance

The *mean* or *expected value* of a random variable X is defined by

$$\mu_X = E(X) = \begin{cases} \sum_k x_k p_X(x_k) & \text{X: Discrete} \\ \int_{-\infty}^{\infty} x f_X(x) & \text{X: Continuous} \end{cases}$$

The *variance* of a random variable X, denoted by σ_X^2, is defined by

$$\sigma_X^2 = Var(X) = E\{[X - E(X)]^2\}$$

$$\sigma_X^2 = \begin{cases} \sum_k (x_k - \mu_k)^2 p_X(x_k) & \text{X: Discrete} \\ \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x) & \text{X: Continuous} \end{cases}$$

Acknowledge various sources for the images.

Thankyou