Random Process: Temporal Characteristics

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Table of Contents

What are we going to study?

- In random variables we are concerned with *how frequently an event occurs*
- In *Stochastic/Random*: pay attention to the *time sequence* of the events
- We study *autocorrelation* function and *autocovariance* function of a stochastic process
  - They are the useful summaries of the time structure of a process
- We study *wide sense stationary processes* and *cross correlation* of wide sense stationary random process

Introduction to Random Process

Random Process

- The random variable approach is applied to random problems that are not functions of time.
- In engineering, many random problems are time dependent.
  - Speech signals, communication signals
- In certain random experiments, the outcome may be a function of time.
- Such time functions are called *random processes*
- Eg: In communication systems, desired deterministic signal is often accompanied by undesired random waveform varying with time
- Random process is denoted by $X(s,t)$
- Also known as *stochastic process*
  - stochastic meaning: random

A random process is a collection of random variables in a probability space indexed by time.

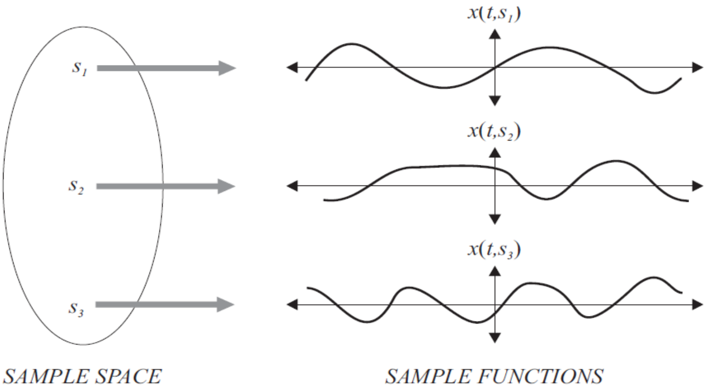
Terminology

- **Random Variable**: One measurement at one instant.
  - Example: Temperature at 12 PM today.
- **Random Process** Many such random variables, one for each time point.
  - Example: Temperature measured continuously during the day.
- **Realization (Sample Function)**: The actual curve observed in one experiment (one day's temperature profile).
- **Ensemble**: Collection of realizations from many experiments (many days).

Example

- Random variable: Toss a coin once
- Random process: Toss a coin every minute
- One realization: One student tossing every minute
- Ensemble: Whole class tossing every minute

Sample functions



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Example

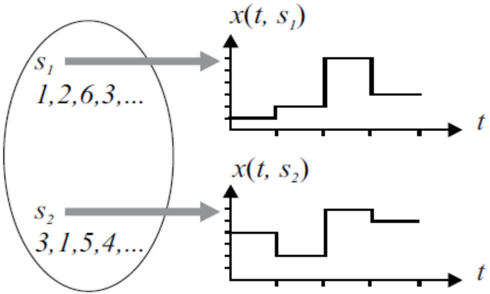

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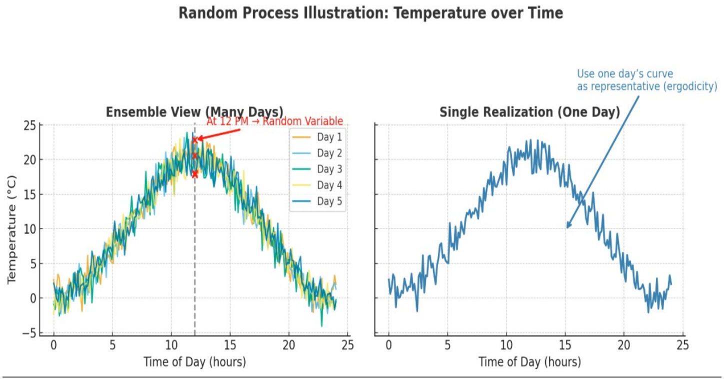
Figure: Random process example

- Roll a die at time instants $T = 0, 1, 2, ..$
- Waveform corresponding to the particular sequence of rolls.

Random variables X_i are derived from a random process $X(t)$ at times t_i

$$X_i = X(t_i, s) = X(t_i)$$

Example


Random Process Illustration: Temperature over Time
Ensemble View (Many Days)
At 12 PM → Random Variable
Day 1
Day 2
Day 3
Day 4
Day 5
Temperature (°C)
Time of Day (hours)
Single Realization (One Day)
Use one day's curve
as representative (ergodicity)
Time of Day (hours)

Stock price as a random process

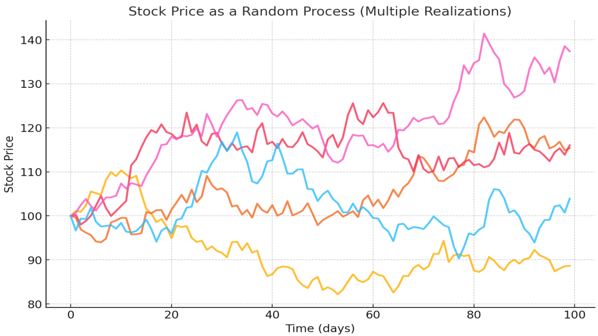


Figure: Stock price as a random process

Each colored line is a realization

ECG signals as a random process

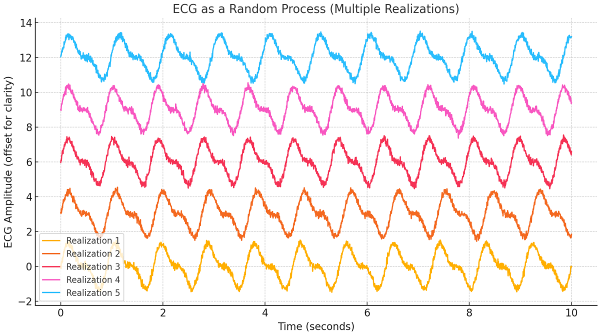


Figure: ECG signals as a random process

Each curve is a realization of the ECG signal (like different patients or different trials).

ECG signals as a random process

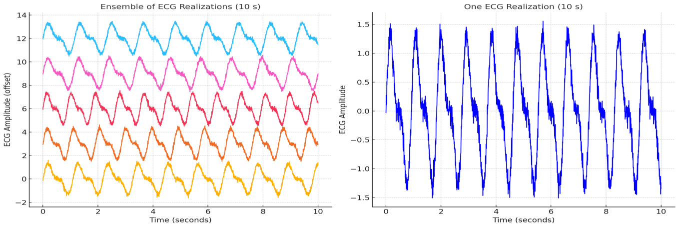


Figure: ECG signals as a random process

- Left panel: Multiple ECG realizations (ensemble view): Different trials or patients over 10 seconds.
- Right panel: One ECG realization (time evolution): How a single patient's ECG looks over the same 10 seconds

Multiple Vs Single realization

- Ensemble provides statistical view across the population at a fixed time.
- Using a single ECG trace we can compute time averages (e.g., average heart rate of one patient over time).
- What are we trying to do with multiple realizations?
  - Multiple realizations let us estimate the true statistical properties of the random process (**mean, variance, autocorrelation**).
    - Mean ECG shape across 100 patients
- What can we do with a single realization?
  - We can compute time averages like: **Average heart rate, Power spectrum**
- **If the process is ergodic (i.e., time average $\approx$ ensemble average), then a single long realization is enough to estimate the statistical properties.**

Classification of Random Process

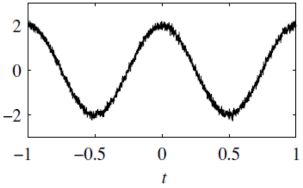
Classification

Based on the the values of t and x, the random process can be classified into

- Continuous random process
  - x and t continuous
- Discrete random process
  - x discrete, t continuous
- Continuous random sequence
  - x continuous, t discrete
- Discrete random sequence
  - x discrete, t discrete

Classification

If X is continuous and t can have any of a continuum of values, then $X(t)$ is called a ***continuous random process***



Source: Goodman

Figure: Continuous Random process

If X assumes discrete values while t is continuous, then $X(t)$ is called a ***discrete random process***

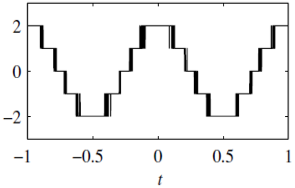


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Figure: Discrete Random process

Classification

In a ***continuous random sequence***, the random variable X is continuous but the time t has only discrete values.

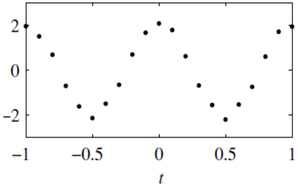

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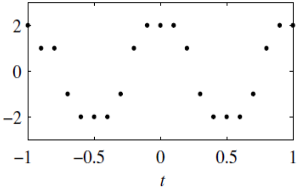
Figure: Continuous Random sequence

In a ***discrete random sequence***, both random variables X and time t are discrete.



Source: Goodman

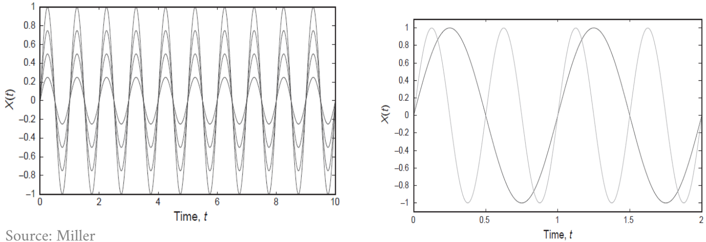
Figure: Discrete Random sequence

Deterministic

Deterministic process: A process is called *deterministic* if future values of any sample function can be predicted from the past values.

$$X(t) = A\cos(\omega_0 t + \Theta)$$

Here A, Θ or ω may be random variables.



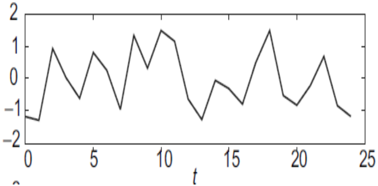
Source: Miller

(a) (b)

Figure: (a) A is random (b) ω is random

Nondeterministic

Nondeterministic process: A process is called *nondeterministic* if future values of any sample function cannot be predicted from past values



Source: Ramesh Babu

Figure: Nondeterministic Random process

Distribution and Density functions

Distribution function

Distribution function: The value of a random process $X(t)$ at a particular time t_1 is a random variable $X_1 = X(t_1)$. The *distribution function* (denoted by $F_X(x_1; t_1)$) associated with this random variable is given by

$$F_X(x_1; t_1) = P\{X(t_1) \leq x_1\} \qquad x_1 \text{ is any real number}$$

☞ $F_X(x_1; t_1)$ is the *First-order distribution* function

Second-order distribution function: For two random variables $X_1 = X(t_1)$ and $X_2 = X(t_2)$, the *second-order distribution* function is given by

$$F_X(x_1, x_2; t_1, t_2) = P\{X(t_1) \leq x_1, X(t_2) \leq x_2\}$$

Nth-**order joint distribution function**: For N random variables $X_i = X(t_i), i = 1, 2, ..., N$, the Nth-*order joint distribution* function is given by

$$F_X(x_1, ..., x_N; t_1, ..., t_N) = P\{X(t_1) \leq x_1, ..., X(t_N) \leq x_N\}$$

Density Function

The joint density function of a random process can be obtained by differentiating the corresponding distribution functions:
The *first-order* density function is

$$f_X(x_1; t_1) = \frac{d}{dx_1} F_X(x_1; t_1)$$

The *second-order* density function is

$$f_X(x_1, x_2; t_1, t_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F_X(x_1, x_2; t_1, t_2)$$

The Nth *order* density function is

$$f_X(x_1, ..., x_N; t_1, ..., t_N) = \frac{\partial^N}{\partial x_1 ... \partial x_N} F_X(x_1, ..., x_N; t_1, ..., t_N)$$

Problem

In the fair-coin experiment, a random process $X(t)$ is defined as follows: $X(t) = cos\pi t$ if heads occur, $X(t) = t$ if tails occur. (a) Find $E[X(t)]$. (b) Find $F_X(x, t)$ for $t = 0.25, 0.5, 1$.

$E[X(t)] = \sum_i p_{X_i}(x) X_i(t) = 0.5cos\pi t + 0.5t$

At each time instant we have one random variable. Since there are three time instants $\implies$ three random variables $\implies$ three CDFs.

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At $t = 0.25$:

$X(0.25, S) : X(0.25, H) = cos\pi/4 = \frac{1}{\sqrt{2}} = 0.7 = x_1;\ X(0.25, T) = t = 0.25 = x_2$

$F_X(x, t = 0.25) = \sum_i p(x_i)U(x - x_i) = 0.5U(x - 0.25) + 0.5U(x - 0.7)$

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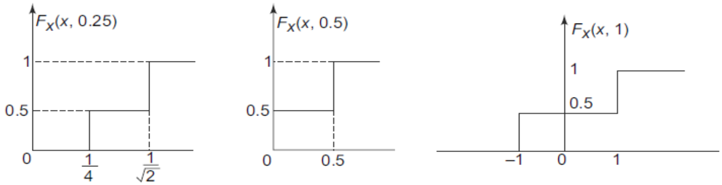
At $t = 0.5$: $X(0.5, S) : X(0.5, H) = cos\pi/2 = 0 = x_1;\ X(0.5, T) = t = 0.5 = x_2$

$F_X(x, t = 0.5) = \sum_i p(x_i)U(x - x_i) = 0.5U(x - 0) + 0.5U(x - 0.5)$

Problem

At $t = 1$: $X(1, S) : X(1, H) = cos\pi = -1 = x_1;\ X(1, T) = t = 1 = x_2$

$$F_X(x, t = 0.5) = \sum_i p(x_i) U(x - x_i) = 0.5U(x + 1) + 0.5U(x - 1)$$



Source: Ramesh Babu

Statistical Independence

Two process $X(t)$ and $Y(t)$ are *statistically independent* if the random variable group $X(t_1), X(t_2), ..., X(t_N)$ is independent of the group $Y(t_1'), Y(t_2'), ..., Y(t_M')$ for any choice of times $t_1, ..., t_N, t_1', ..., t_M'$.
The joint density function must be factorable

$$f_{X,Y}(x_1, .., x_N, y_1, .., y_M; t_1, .., t_N, t_1', .., t_M') = f_X(x_1, .., x_N; t_1, .., t_N) f_Y(y_1, .., y_M; t_1', .., t_M')$$

Stationarity

First-order Stationary Process

First-order Stationary Process A random process is called *stationary to order one* if its first-order density function does not change with a shift in time origin

$$f_X(x_1;t_1) = f_X(x_1;t_1+\Delta)$$

Consequence: $f_X(x_1;t_1)$ is independent of t_1

$$\boxed{E[X(t)] = \bar{X} = constant}$$

$$E[X_1] = E[X(t_1)]] = \int_{-\infty}^{\infty} x f_X(x;t_1)dx = \int_{-\infty}^{\infty} x f_X(x;\underbrace{t_1+\Delta}_{t_2})dx = \int_{-\infty}^{\infty} x f_X(x;t_2)dx$$

$$= E[X(t_2)] = E[X_2]$$

Since $E[X(t_1)] = E[X(t_2)]] \implies E[X(t)] = constant$

Second-order Stationary Process

Second-order Stationary Process A random process is called *second order stationary* if its density function satisfies

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta, t_2 + \Delta) \tag{1}$$

Autocorrelation function of the random process X(t) is given by

$$\boxed{R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]}$$

Autocorrelation function of a second-order stationary process is a function of time differences (τ) and not absolute time (t_1, t_2). Let $\tau = t_2 - t_1$

$$\boxed{R_{XX}(t_1, t_1 + \tau) = E[X(t_1)X(t_1 + \tau)] = R_{XX}(\tau)}$$

Wide-sense stationary process

- Many practical problems require *autocorrelation function* and *mean value* of the random process
- Problem solutions are *greatly simplified* if these quantities are not dependent on absolute time $(t_1, t_2, ..)$

A process is **wide-sense stationary** if it satisfies the following two conditions

$$\boxed{\begin{aligned} E[X(t)] &= \bar{X} = constant \\ E[X(t)X(t+\tau)] &= R_{XX}(\tau) \end{aligned}}$$

Jointly wide-sense stationary process

Two random process $X(t)$ and $Y(t)$ are **jointly wide-sense stationary** if they satisfy the following conditions

$$\boxed{\begin{aligned} E[X(t)] &= \bar{X} = constant \\ E[X(t)X(t+\tau)] &= R_{XX}(\tau) \\ E[Y(t)] &= \bar{Y} = constant \\ E[Y(t)Y(t+\tau)] &= R_{YY}(\tau) \\ E[X(t)Y(t+\tau)] &= R_{XY}(\tau) \end{aligned}}$$

Cross correlation of two random process $X(t)$ and $Y(t)$ is given by

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$$

Strict-Sense stationary (N-Order)

Consider N random variables $X_i = X(t_i)$. A random process is *stationary to order N* if its Nth order density function is invariant to a time origin shift.

$$f_X(x_1, ..., x_N; t_1, ..., t_N) = f_X(x_1, ..., x_N; t_1 + \Delta, ..., t_N + \Delta)$$

Strict-sense stationary $\implies$ wide-sense stationary

wide-sense stationary $\not\Rightarrow$ strict-sense stationary

Problems

Show that the random process $X(t) = Acos(\omega_0 t + \Theta)$ is wide-sense stationary if it is assumed A and ω_0 are constants and Θ is uniformly distributed random variable on the interval $(0, 2\pi)$

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$$E[X(t)] = \int_0^{2\pi} Acos(\omega_0 t + \Theta) = 0$$

Autocorrelation function is given by

$$\begin{aligned} E[X(t_1)X(t_1 + \tau)] &= E[Acos(\omega_0 t + \Theta)Acos(\omega_0(t + \tau) + \Theta)] \\ &= \frac{A^2}{2} E[cos(\omega_0 \tau) + cos(2\omega_0 t + \omega_0 \tau + 2\Theta)] \\ &= \frac{A^2}{2} cos(\omega_0 \tau) + \frac{A^2}{2} E[cos(2\omega_0 t + \omega_0 \tau + 2\Theta)] \end{aligned}$$

Second term is zero. Autocorrelation function depends only on τ and mean values is constant. Hence $X(t)$ is wide-sense stationary

Problems

Given $X(t) = Acos\omega t + Bsin\omega t$. A and B are uncorrelated zero mean random variables having variance σ^2. Find the autocorrelation of $X(t)$ and show that $X(t)$ is wide-sense stationary

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Given $E[A] = E[B] = 0, \sigma^2 = E[A^2] - (E[A])^2 = E[A^2]$;
$\sigma^2 = E[B^2] - (E[B])^2 = E[B^2]$; If A and B are uncorrelated so
$E[AB] = E[A]E[B] = 0$

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$E[AB] = E[A]E[B] = 0$
$E[X(t)] = E[Acos\omega t + Bsin\omega t] = E[A]cos\omega t + E[B]sin\omega t = 0$
Autocorrelation $R_{XX}(t, t + \tau) = E[X(t)X(t + \tau)]$

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Autocorrelation $R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)]$

$$E[X(t)X(t+\tau)] = E[(Acos\omega t + Bsin\omega t)(Acos\omega(t+\tau) + Bsin\omega(t+\tau))]$$

$$= E[A^2]cos\omega t cos\omega(t+\tau) + E[AB]cos\omega t sin\omega(t+\tau) +$$

$$E[BA]sin\omega t cos\omega(t+\tau) + E[B^2]sin\omega t sin\omega(t+\tau)$$

$$= \sigma^2(cos\omega t cos\omega(t+\tau) + sin\omega t sin\omega(t+\tau)) = \sigma^2(cos\omega\tau)$$

Autocorrelation is function of τ and mean of $X(t)$ is constant. Hence $X(t)$ is WSS

Problem

Prove that a random telegraph signal process $Y(t) = \alpha X(t)$ is a wide sense stationary process where α is a random variable which is independent of $X(t)$, assume value -1 and +1 with equal probability and $R_{XX}(t_1, t_2) = e^{-2\lambda|t_2 - t_1|}$

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$P(\alpha = -1) = \frac{1}{2}; P(\alpha = 1) = \frac{1}{2}$

$$E[\alpha] = -1\frac{1}{2} + 1\frac{1}{2} = 0; \qquad E[\alpha^2] = 1$$

Mean of $Y(t)$

$$E[Y(t)] = E[\alpha X(t)] = E[\alpha]E[X(t)] = 0$$

Autocorrelation of $Y(t)$

$$\begin{aligned} R_{YY}(t_1, t_2) &= E[Y(t_1)Y(t_2)] = E[\alpha X(t_1)\alpha X(t_2)] = E[\alpha^2]E[X(t_1)X(t_2)] \\ &= E[X(t_1)X(t_2)] = e^{-2\lambda|t_2 - t_1|} = e^{-2\lambda|\tau|} \end{aligned}$$

Since $E[Y(t)]$ is constant and R_{YY} depends on τ, $Y(t)$ is wide sense stationary

Homework

If $X(t) = Ycost + Zsint$ for all t where Y and Z are independent binary random variables, each of which assumes the values –1 and 2 with probabilities $\frac{2}{3}$ and $\frac{1}{3}$ respectively. Prove that $X(t)$ is widesense stationary. (Eg 6.6 in Ramesh Babu book)

Acknowledge various sources for the images.

Thankyou