Expected Value of a Random Variable, Function of a Random Variable, Moments, Characteristic function, Moment Generating Function

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Expectation

Expectation of a continuous random variable is given by

$$\boxed{E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx}$$

Expectation of a discrete random variable (with N possible values x_i having probabilities $P(x_i)$ is given by

$$\boxed{E[X] = \sum_{i=1}^{N} x_i P(x_i)}$$

Properties of Expectation

- Expectation of a constant is equal to the constant itself.
    - $E[a] = \int_{-\infty}^{\infty} a f_X(x)dx = a \int_{-\infty}^{\infty} f_X(x)dx = a$
- The expectation of constant times the random variable is equal to the constant times expectation of the random variable.
    - $E[CX] = CE[X]$
- If a and b are constants then
    - $E[aX + b] = aE[X] + b$
- If $X \geq 0$ then $E[X] \geq 0$
- Expectation of the sum of random variables is equal to the sum of expectations.
    - $E[g_1(x) + g_2(x)] = E[g_1(x)] + E[g_2(x)]$

Problems

If X is a random variable with the pdf $f_X(x)$, Find $E[X]$

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Problems

If X is a random variable with the pdf $f_X(x)$, Find $E[X]$

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_{0}^{1} x(2x) dx = \frac{2}{3}x^3 \Big|_0^1 = \frac{2}{3}$$

Function of a Random Variable

Expectation of a function of random variable is given by

$$\boxed{E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx}$$

Problem: Let X be a continuous random variable with pdf $f_X(x) = \frac{8}{x^3}, x > 2$. Find $E[W]$ where $W = \frac{X}{3}$.

Function of a Random Variable

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Problem: Let X be a continuous random variable with pdf $f_X(x) = \frac{8}{x^3}, x > 2$. Find $E[W]$ where $W = \frac{X}{3}$.

$$E[W] = E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx = \int_{2}^{\infty} \frac{x}{3} \frac{8}{x^3}\, dx = \frac{8}{3} \int_{2}^{\infty} x^{-2}\, dx$$

$$= \frac{-8}{3} \frac{1}{x}\Bigg|_{2}^{\infty} = \frac{4}{3}$$

Moments

Applications of Moments

Moments and central moments are used in many practical domains to capture **shape characteristics** of a distribution.

- **Engineering & Signal Processing**
  - Image processing – Raw moments help in object recognition, pattern matching
  - Acoustics – Moments of a power spectrum can characterize audio features like spectral centroid
  - Central moment: Feature extraction in computer vision, invariant to translation
- **Finance and Risk analysis**
  - 1st moment $\longrightarrow$ expected return
  - 2nd central moment $\longrightarrow$ volatility (risk)
  - 3rd central moment $\longrightarrow$ skewness (bias toward gains or losses)
  - 4th central moment $\longrightarrow$ kurtosis (tail risk / extreme events)
- **Communication Systems**
  - Moments of received signal constellation distinguish between modulation schemes.
  - Central moments are Robust to signal offset—important when carrier

Moments about origin

Let the function $g(X) = X^n \qquad n = 0, 1, 2, \ldots$
The nth moment for continuous random variable X is given by

$$\boxed{m_n = E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) dx}$$

$$m_0 = E[X^0] = E[1] = \int_{-\infty}^{\infty} f_X(x) dx = 1$$

$$m_1 = E[X] = \bar{X} = \int_{-\infty}^{\infty} x f_X(x) dx$$

The nth moment for discrete random variable X is given by

$$m_n = E[X^n] = \sum_i x_i^n p_X(x_i)$$

Central Moments

Moments about the mean value of X are called *central moments* (denoted by μ_n).
$g(X) = (X - \bar{X})^n \qquad n = 0, 1, 2, \ldots$

$$\boxed{\mu_n = E[(X - \bar{X})^n] = \int_{-\infty}^{\infty} (x - \bar{X})^n f_X(x)\,dx}$$

$$\mu_0 = E[(X - \bar{X})^0] = E[1] = \int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$\mu_1 = E[(X - \bar{X})] = \int_{-\infty}^{\infty} (x - \bar{X}) f_X(x)\,dx = \underbrace{\int_{-\infty}^{\infty} x f_X(x)\,dx}_{\bar{X}} - \bar{X} \underbrace{\int_{-\infty}^{\infty} f_X(x)\,dx}_{1}$$

$$= \bar{X} - \bar{X} = 0$$

Variance

The second central moment is the *variance* (denoted by σ_X^2)

$$\sigma_X^2 = \mu_2 = E[(X - \bar{X})^2] = \int_{-\infty}^{\infty} (x - \bar{X})^2 f_X(x) dx$$

$$\begin{aligned}\sigma_X^2 &= E[X^2 + \bar{X}^2 - 2X\bar{X}] = E[X^2] - 2\bar{X}E[X] + \bar{X}^2 \\ &= E[X^2] - \bar{X}^2 = m_2 - m_1^2\end{aligned}$$

Properties of Variance

- The variance of a constant is 0
  - $Var(a) = E[(X-a)^2] = E[(a-a)^2] = 0$
- Variance of a nonrandom constant time random variable plus a constant is equal to the first constant square times the variance of the random variable.
  - $Var(CX) = C^2 Var(X)$
- If X is a random variable and a, b are real constant, then
  - $Var(aX + b) = a^2 Var(X)$

Problem

The pmf of a discrete variable is given by

$$p_X(x) = \begin{cases} \frac{1}{4} & x = 0, 2, 4, 6 \\ 0 & \text{elsewhere} \end{cases}$$

Find mean and variance.

Problem

The pmf of a discrete variable is given by

$$p_X(x) = \begin{cases} \frac{1}{4} & x = 0, 2, 4, 6 \\ 0 & \text{elsewhere} \end{cases}$$

Find mean and variance.

$$\mu_X = E[X] = \sum_i x_i p_X(x_i) = 0(\frac{1}{4}) + 2(\frac{1}{4}) + 4(\frac{1}{4}) + 6(\frac{1}{4}) = 3$$

$$E[X^2] = \sum_i x_i^2 p_X(x_i) = 0^2(\frac{1}{4}) + 2^2(\frac{1}{4}) + 4^2(\frac{1}{4}) + 6^2(\frac{1}{4}) = 14$$

$$\sigma_X^2 = E[X^2] - (E[X])^2 = 14 - 3^2 = 5$$

Problem

Find the mean value of the continuous, exponentially distributed random variable

$$f_X(x) = \begin{cases} \frac{1}{b} e^{-\frac{(x-a)}{b}} & x \geq a \\ 0 & x < a \end{cases}$$

Problem

Find the mean value of the continuous, exponentially distributed random variable

$$f_X(x) = \begin{cases} \frac{1}{b} e^{-\frac{(x-a)}{b}} & x \geq a \\ 0 & x < a \end{cases}$$

$$E[X] = \int_a^\infty \frac{x}{b} e^{-\frac{(x-a)}{b}} dx = a + b$$

Problem-Assignment

Q. A random variable with Laplace distribution has pdf given by

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda |x|}$$

(i) Find the CDF. (ii) Find the mean and variance.(3.22 in Ramesh Babu)

Q. For the following probability density function

$$f_X(x) = \begin{cases} kx & 0 \leq x < 2 \\ k(4-x) & 2 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

(a).Find the value of k for which $f_X(x)$ is a valid pdf.(b) Find the mean and variance of X. (c) Find the CDF (3.24 in Ramesh Babu)

Skew

Skew is defined as the **measure of the asymmetry of $f_X(x)$** about its mean $\bar{X}$. It is the *third central moment* and is given by

$$\boxed{\mu_3 = E[(X - \bar{X})^3]}$$

Skewness coefficient: The normalized third central moment is known as the skewness coefficient.

$$\text{Skewness coefficient} = \frac{\mu_3}{\sigma_X^3} = \frac{E[(X-\mu_X)^3]}{E[(X-\mu_X)^2]^{3/2}}$$

Chebychev's Inequality

Chebychev's Inequality for a random variable X with mean $\bar{X}$ and variance σ_X^2 is given by

$$\boxed{P\{|X - \bar{X}| \geq \epsilon\} \leq \frac{\sigma_X^2}{\epsilon^2}}$$

Problem

Find the largest probability that any random variable's values are smaller than its mean by 3 standard deviations or larger than its mean by the same amount.

Find

$$P\{X \geq \bar{X} + 3\sigma_X\} + P\{X \leq \bar{X} - 3\sigma_X\} = P\{|X - \bar{X}| \geq 3\sigma_X\}$$

Problem

Find the largest probability that any random variable's values are smaller than its mean by 3 standard deviations or larger than its mean by the same amount.

Find

$$P\{X \geq \bar{X} + 3\sigma_X\} + P\{X \leq \bar{X} - 3\sigma_X\} = P\{|X - \bar{X}| \geq 3\sigma_X\}$$

observe $\epsilon = 3\sigma_X$

$$P\{|X - \bar{X}| \geq 3\sigma_X\} \leq \frac{\sigma_X^2}{(3\sigma_X)^2} = \frac{1}{9}$$

Problem

Suppose the population of a city has an expected value of age 30 and a standard deviation of 5. What is the chance that population age is outside the interval (20,40)?

Given $\mu = 30$ and $\sigma_X = 5$. Find

Problem

Suppose the population of a city has an expected value of age 30 and a standard deviation of 5. What is the chance that population age is outside the interval (20,40)?

Given $\mu = 30$ and $\sigma_X = 5$. Find

$$\begin{aligned} P\{X \leq 20\} + P\{X \geq 40\} &= P\{X \leq 30 - 10\} + P\{X \geq 30 + 10\} \\ &= P\{X - 30 \leq -10\} + P\{X - 30 \geq 10\} \\ &= P\{|X - 30| \geq 10\} \leq \frac{\sigma_X^2}{10^2} = \frac{25}{100} = 0.25 \end{aligned}$$

Characteristic Function

Characteristic function of a random variable X is defined by

$$\boxed{\Phi_X(\omega) = E[e^{j\omega X}] = \int_{-\infty}^{\infty} f_X(x) e^{j\omega x} dx}$$

where $j = \sqrt{-1}$ and $-\infty < \omega < \infty$.
To find the relation between m_n and $\Phi_X(\omega)$, consider

$$\begin{aligned} E[e^{j\omega X}] &= E[1 + (j\omega X) + \frac{(j\omega X)^2}{2!} + \frac{(j\omega X)^3}{3!} + ...] \\ &= 1 + j\omega E[X] + \frac{(j\omega)^2}{2!} E[X^2] + \frac{(j\omega)^3}{3!} E[X^3]... \end{aligned}$$

$m_0 = E[X^0] = \Phi_X(0) = 1$

Relation between Characteristic Function and moments

$$\Phi_X(\omega) = 1 + j\omega E[X] + \frac{(j\omega)^2}{2!}E[X^2] + \frac{(j\omega)^3}{3!}E[X^3]...$$

Differentiate $\Phi_X(\omega)$ w.r.to ω

$$\frac{d\Phi_X}{d\omega} = jE[X] + j^2\omega E[X^2] + j^3\frac{\omega^2}{2!}E[X^3] + ..$$

$$\left.\frac{d\Phi_X}{d\omega}\right|_{\omega=0} = jE[X] \implies E[X] = m_1 = -j\left.\frac{d\Phi_X}{d\omega}\right|_{\omega=0}$$

$$\left.\frac{d^2\Phi_X}{d\omega^2}\right|_{\omega=0} = j^2E[X^2] \implies E[X^2] = m_2 = (-j)^2\left.\frac{d^2\Phi_X}{d\omega^2}\right|_{\omega=0}$$

$$\boxed{E[X^n] = m_n = (-j)^n\left.\frac{d^n\Phi_X}{d\omega^n}\right|_{\omega=0}}$$

$\Phi_X(\omega)$ is the *Fourier transform* (with sign of ω reversed) of $f_X(x)$

$$\boxed{\Phi_X(\omega) = \int_{-\infty}^{\infty} f_X(x) e^{j\omega x} dx}$$

Similarly if $\Phi_X(\omega)$ is known, $f_X(x)$ can be found from the *inverse Fourier transform* (with sign of x reversed)

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_X(\omega) e^{-j\omega x} dx$$

The maximum magnitude of a characteristic function is unity

$$|\Phi_X(\omega)| \leq \Phi_X(0) = 1$$

$\Phi_X(\omega)$ always exists $\implies$ *moments* can always be found if $\Phi_X(\omega)$ is known

Moment Generating Function (MGF)

Moment generating function is defined as

$$\boxed{M_X(v) = E[e^{vX}] = \int_{-\infty}^{\infty} f_X(x) e^{vx} dx}$$

where v is a real number $-\infty < v < \infty$
To find the relation between m_n and $M_X(v)$, consider

$$\begin{aligned} E[e^{vX}] &= E[1 + (vX) + \frac{(vX)^2}{2!} + \frac{(vX)^3}{3!} + ...] \\ &= 1 + vE[X] + \frac{v^2}{2!}E[X^2] + \frac{v^3}{3!}E[X^3]... \end{aligned}$$

$m_0 = E[X^0] = M_X(0) = 1$

Moment Generating Function

$$M_X(v) = E[e^{vX}] = 1 + vE[X] + \frac{v^2}{2!}E[X^2] + \frac{v^3}{3!}E[X^3]...$$

Differentiate $M_X(v)$ w.r.to v

$$\frac{dM_X(v)}{dv} = E[X] + vE[X^2] + \frac{v^2}{2!}E[X^3] + ...$$

$$\left.\frac{dM_X}{dv}\right|_{v=0} = E[X] \implies m_1 = \left.\frac{dM_X}{dv}\right|_{v=0}$$

similarly

$$\boxed{E[X^n] = m_n = \left.\frac{d^n M_X(v)}{dv^n}\right|_{v=0}}$$

Main disadvantage of MGF is that it may not exist for all random variables and all values of v

Problems

Q1. Fing MGF and m_1 for $f_X(x) = e^{-2x} \qquad x \geq 0$
Q2. Given $M_X(v) = \frac{1}{1+v}$. Find mean and variance
Q3. Find the moment generating function and characteristic function of the random variable X which has uniform distribution
Q4. The MGF of a uniform distribution for a random variable X is $\frac{1}{u}(e^{5u} - e^{4u})$. Find $E[X]$.

Transformations of a Random Variable

We wish to transform random variable X to Y by

$$Y = T(X)$$

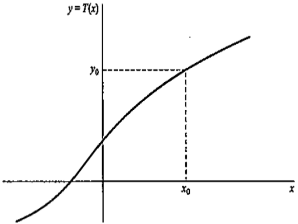
Discuss only for Continuous R.V

There are two kinds of transformations

- Monotonic Transformation
  - Increasing and decreasing
- Nonmonotonic Transformation

Monotonic Transformation-Increasing

Transformation T is monotonically increasing if $T(x_1) < T(x_2)$ for $x_1 < x_2$



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Since there is one-to-one correspondence, probability of event $\{Y \le y_0\}$ is equal to probability of event $\{X \le x_0\}$

$$F_Y(y_0) = P\{Y \le y_0\}$$
$$= P\{X \le x_0\} = F_X(x_0)$$

Two numbers are related by $y_0 = T(x_0)$ or $x_0 = T^{-1}(y_0)$

Monotonic Transformation-Increasing

$$\int_{-\infty}^{y_0} f_Y(y)\,dy = \int_{-\infty}^{x_0=T^{-1}(y_0)} f_X(x)\,dx$$

Differentiate both sides and use Leibniz's rule

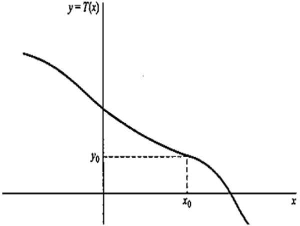
$$f_Y(y_0) = f_X[T^{-1}(y_0)]\frac{dT^{-1}(y_0)}{dy_0}$$

In compact notation

$$\boxed{f_Y(y)\,dy = f_X(x)\frac{dx}{dy}}$$

Monotonic Transformation-Decreasing

Transformation T is monotonically decreasing if $T(x_1) > T(x_2)$ for $x_1 > x_2$



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$$
\begin{aligned}
F_Y(y_0) &= P\{Y \leq y_0\} \\
&= P\{X \geq x_0\} = 1 - F_X(x_0)
\end{aligned}
$$

Repeat the previous steps

$$f_Y(y_0) = f_X[T^{-1}(y_0)]\left(-\frac{dT^{-1}(y_0)}{dy_0}\right)$$

For monotonic transformation (increasing or decreasing)

$$f_Y(y_0)dy = f_X[T^{-1}(y_0)]\left|\frac{dT^{-1}(y_0)}{dy_0}\right| \implies f_Y(y)dy = f_X(x)\left|\frac{dx}{dy}\right|$$

Problem

If X is a Gaussian with mean a_X and variance σ_X^2. For the transformation $Y = aX + b$ (a and b are constants). Find the pdf of Y.

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If X is a Gaussian with mean a_X and variance σ_X^2. For the transformation $Y = aX + b$ (a and b are constants). Find the pdf of Y.

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right)\left|\frac{1}{a}\right|$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-[(y-b)/a - a_X]^2/2\sigma_X^2}\left|\frac{1}{a}\right| = \frac{1}{\sqrt{2\pi a^2\sigma_X^2}} e^{-[y-(aa_X+b)]^2/2a^2\sigma_X^2}$$

where the mean and variance of Y is given by

$$a_Y = aa_X + b \qquad \sigma_Y^2 = a^2\sigma_X^2$$

Problem

$X \sim U(0,1)$, $Y = e^X$ (increasing). Find pdf of Y and $E[Y]$

Problem

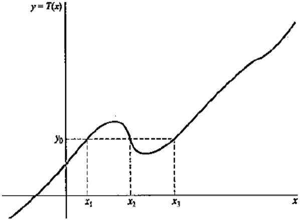
$X \sim U(0,1)$, $Y = e^X$ (increasing). Find pdf of Y and $E[Y]$
Then $x = \ln y$, $y \in (1, e)$:

$$f_Y(y) = 1 \cdot \left| \frac{d}{dy} \ln y \right| = \frac{1}{y}, \quad y \in (1, e).$$

$E[Y] = \int_1^e y \cdot \frac{1}{y}\, dy = e - 1.$

Nonmonotonic Transformations

There are more than one value of X that corresponds to event $\{Y \leq y_0\}$



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Probability of event $\{Y \leq y\}$ now equals to probability of event $\{x$ values yielding $Y \leq y_0\}$

$$F_Y(y_0) = P\{Y \leq y_0\} = P\{x \setminus Y \leq y_0\}$$
$$= \int_{\{x \setminus Y \leq y_0\}} f_X(x)\,dx$$

Event $\{Y \leq y\}$ corresponds to $\{X \leq x_1\}$ and $\{x_2 \leq X \leq x_3\}$

Differentiate $F_Y(y)$ to get density $f_Y(y)$

$$\boxed{f_Y(y) = \sum_n \frac{f_X(x_n)}{\left|\frac{dT(x)}{dx}\right|_{x=x_n}}}$$

Home assignment

Find $f_Y(y)$ from the square law transformation

$$Y = cX^2$$

Refer Peebles book Page 91

Acknowledge various sources for the images.
Thankyou