Random Process: Temporal Characteristics

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Time Average

Time Average

The **time average** of a quantity is defined as

$$A[\cdot] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [\cdot]\,dt$$

A is used to denote *time average* (E is used to denote statistical average)

Mean value

$$\bar{x} = A[x(t)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,dt$$

Time Autocorrelation function

$$\mathscr{R}_{xx}(\tau) = A[x(t)x(t+\tau)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t+\tau)\,dt$$

Ergodic Process: Motivation

- **Wireless Communication** (Fading Channels): Radio signals in mobile networks undergo fading due to reflections, scattering, and Doppler shifts.
  - If the fading process is ergodic, then measuring the channel over time (say, one user walking around) is enough to estimate average channel statistics.
    - No need to have thousands of different users in different places.
  - "Your mobile internet depends on the assumption that fading is ergodic"
- **Music and Speech Signal Processing**: Pitch detection, speech compression, and voice authentication all rely on ergodic assumptions.
- **Medical Signals (ECG/EEG)**: Doctors want to measure average properties of the heart or brain signal.
  - If ECG is ergodic, one patient's long ECG signal is enough to estimate average heartbeat energy or rhythm.
  - Devices like Holter monitors rely on ergodicity to analyze your heartbeat over 24 hours.

Ergodic Process

An ergodic process is a random process where time averages are equal to ensemble/statistical averages.

$$\bar{x} = \bar{X}$$

$$\mathscr{R}_{xx}(\tau) = R_{XX}(\tau)$$

Ergodic Theorem:If a process is ergodic, then the time average of a function along one trajectory is equal to the expected value (ensemble average) of that function, almost surely.

Jointly Ergodic & Mean Ergodic

Two random process are *jointly ergodic* if they are individually ergodic and also have a time cross-correlation function that equals the statistical cross-correlation function

$$\mathscr{R}_{xy}(\tau) = A[x(t)y(t+\tau)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)y(t+\tau)dt = R_{XY}(\tau)$$

Mean-Ergodic process: A random process $X(t)$ is said to be *mean ergodic* if its statistical average $\bar{X}$ equals the time average $\bar{x}$ of a sample function $x(t)$ with probability 1 for all sample functions.

$$E[X(t)] = \bar{X} = A[x(t)] = \bar{x}$$

Correlation Ergodic Processes

A stationary continuous random process $X(t)$ with autocorrelation function $R_{XX}(t)$ is said to be *correlation ergodic* or ergodic in the autocorrelation if and only if

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} X(t)X(t+\tau)dt = R_{XX}(\tau) \qquad \text{for all } \tau$$

Correlation Functions

Autocorrelation: Motivation

Applications of Autocorrelation function

- *Speech Signal Processing*: Pitch detection and Speaker identification
  - Autocorrelation of a speech segment reveals the **pitch period** (fundamental frequency of the voice).
- *Radar & Sonar*: Detecting time delay to measure distance.
  - A good radar signal has a sharp autocorrelation peak (to improve target resolution).
- *EEG/ECG applications*:
  - In ECG (heart signals), autocorrelation can reveal hidden periodicity of heartbeats, useful in *arrhythmia detection*.
  - In EEG (brain signals), autocorrelation can detect abnormal rhythmic patterns (like in epilepsy).

Autocorrelation

The auto correlation function of a random process $X(t)$ is the correlation $E[X_1 X_2]$ of two random variables $X_1 = X(t_1)$ and $X_2 = X(t_2)$ defined by the process at times t_1 and t_2

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$

Let $t_1 = t$ and $t_2 = t + \tau$ (τ real number), the above equations is

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)]$$

If $X(t)$ is wide-sense stationary, then

$$R_{XX}(\tau) = E[X(t)X(t+\tau)]$$

Properties of Autocorrelation function

The autocorrelation function of wide-sense stationary random process exhibits the following properties

- $|R_{XX}(\tau)| \leq R_{XX}(0)$
  - $R_{XX}(\tau)$ is bounded by its value at the origin
- $R_{XX}(-\tau) = R_{XX}(\tau)$
  - Autocorrelation function has even symmetry
- $R_{XX}(0) = E[X^2(t)]$
  - The bound is equal to the mean-squared value called **the power** in the process
- If $E[X(t)] = \bar{X} \neq 0$ and $X(t)$ is ergodic with no periodic components then

$$\lim_{|\tau|\to\infty} R_{XX}(\tau) = \bar{X}^2$$

Properties of Autocorrelation function

- If $X(t)$ has a periodic component, then $R_{XX}(\tau)$ will have a periodic component with the same period
- If $X(t)$ is ergodic, zero-mean, and has no periodic component, then

$$\lim_{|\tau|\to\infty} R_{XX}(\tau) = 0$$

- $R_{XX}(\tau)$ cannot have an arbitrary shape

Problem

Given the auto correlation for a stationary ergodic process with no periodic components, is

$$R_{XX}(\tau) = 25 + \frac{4}{1 + 6\tau^2}$$

Find mean and variance of $X(t)$

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Find mean and variance of $X(t)$

We know that

$$\lim_{|\tau|\to\infty} R_{XX}(\tau) = \bar{X}^2 = 25$$

$\bar{X} = \sqrt{25}$; $R_{XX}(0) = E[X^2(t)] = 29$

$$\sigma_X^2 = E[X^2(t)] - \bar{X}^2 = 29 - 25 = 4$$

Cross-Correlation function: Motivation

- **Audio & Speech Processing**: Echo cancellation, Source localization
  - Cross-correlate the microphone signal with the far-end signal to estimate echo delay.
  - Used in voice-controlled assistants (Alexa, Google Home) to locate speaker direction.
- **Medical Applications**:
  - EEG: Cross-correlation between different EEG channels reveals brain connectivity.
  - ECG: Correlation between heart signals and respiration can detect abnormalities.
- **Image Processing**: Object detection, pattern recognition

Cross-Correlation function

The cross-correlation function of two random process $X(t)$ and $Y(t)$ is defined as

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$$

let $t_1 = t$ and $t_2 = t + \tau$

$$R_{XY}(t, t + \tau) = E[X(t)Y(t + \tau)]$$

If $X(t)$ and $Y(t)$ are at least wide-sense stationary, $R_{XY}(t, t + \tau)$ is independent of time $'t'$ and we can write

$$R_{XY}(\tau) = E[X(t)Y(t + \tau)]$$

Cross-Correlation function

Special cases:

- $X(t)$ and $Y(t)$ are called **orthogonal processes** if

$$R_{XY}(t, t+\tau) = 0$$

- If the two processes are *statistically independent*, the cross-correlation function becomes

$$R_{XY}(t, t+\tau) = E[X(t)]E[Y(t+\tau)]$$

  - If $X(t)$ and $Y(t)$ are independent and wide-sense stationary, then

$$R_{XY}(\tau) = \bar{X}\bar{Y}$$

Properties of Cross-correlation function

The cross-correlation function of wide-sense stationary random process exhibits the following properties

- The symmetry of function $R_{XY}(\tau)$ is described by

$$R_{XY}(-\tau) = R_{YX}(\tau)$$

- The upper bound of the function is given by

$$|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0) R_{YY}(0)} \tag{1}$$

- The cross-correlation function is bounded by

$$|R_{XY}(\tau)| \leq \frac{1}{2}[R_{XX}(0) + R_{YY}(0)] \tag{2}$$

Eqn(1) is a tighter bound than Eqn (2), since

$$\sqrt{R_{XX}(0) R_{YY}(0)} \leq \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$$

Problem

A random process $X(t) = At$, where A is a continuous random variable uniformly distributed in $(0, 1)$. Find (a) $E[X(t)]$, (b) $R_{XX}(t, t+\tau)$ (c) Is the process stationary in any sense?

Problem

A random process $X(t) = At$, where A is a continuous random variable uniformly distributed in $(0, 1)$. Find (a) $E[X(t)]$, (b) $R_{XX}(t, t+\tau)$ (c) Is the process stationary in any sense?

$$E[X(t)] = E[At] = \int_0^1 At f_A(a)da = \frac{t}{2}$$

$$R_{XX}(t, t+\tau) = E(X(t)X(t+\tau)] = E[AtA(t+\tau)] = t(t+\tau)E[A^2] = \frac{1}{3}t(t+\tau)$$

Since mean and autocorrelation functions are function of t, the random process $X(t)$ is not a stationary random process

Problem

Let two random processes $X(t)$ and $Y(t)$ be defined by

$$X(t) = Acos(\omega_0 t) + Bsin(\omega_0 t)$$
$$Y(t) = Bcos(\omega_0 t) - Asin(\omega_0 t)$$

A and B are zero mean uncorrelated random variables with variance σ^2. Prove $X(t)$ and $Y(t)$ are jointly WSS.

Covariance Function

Autocovariance & Cross-covariance

The **autocovariance function** is defined by

$$\begin{aligned} C_{XX}(t, t+\tau) &= E[(X(t) - E[X(t)])(X(t+\tau) - E[X(t+\tau)])] \\ &= R_{XX}(t, t+\tau) - E[X(t)]E[X(t+\tau)] \end{aligned}$$

The **cross-covariance function** of two random processes $X(t)$ and $Y(t)$ is defined by

$$\begin{aligned} C_{XY}(t, t+\tau) &= E[(X(t) - E[X(t)])(Y(t+\tau) - E[Y(t+\tau)])] \\ &= R_{XY}(t, t+\tau) - E[X(t)]E[Y(t+\tau)] \end{aligned}$$

Autocovariance & Cross-covariance

If $X(t)$ is wide-sense stationary , then autocovariance is given by

$$C_{XX}(\tau) = \underbrace{R_{XX}(t, t+\tau)}_{R_{XX}(\tau)} - \underbrace{E[X(t)]}_{\bar{X}}\,\underbrace{E[X(t+\tau)]}_{\bar{X}}$$
$$= R_{XX}(\tau) - \bar{X}^2$$

If $X(t)$ and $Y(t)$ are atleast jointly wide-sense stationary , then cross-covariance is given by

$$C_{XY}(\tau) = \underbrace{R_{XY}(t, t+\tau)}_{R_{XY}(\tau)} - \underbrace{E[X(t)]}_{\bar{X}}\,\underbrace{E[Y(t+\tau)]}_{\bar{Y}}$$
$$= R_{XY}(\tau) - \bar{X}\bar{Y}$$

Covariance

The *variance* of a random process is given by

$$C_{XX}(t, t+\tau) = E[(X(t) - E[X(t)])(X(t+\tau) - E[X(t+\tau)])]$$

If $\tau = 0$ and $X(t)$ is wide-sense stationary process, the variance becomes

$$\sigma_X^2 = E[\{X(t) - E[X(t)]\}^2] = R_{XX}(0) - \bar{X}^2$$

Two random processes are said to be uncorrelated if

$$C_{XY}(t, t+\tau) = 0 \implies R_{XY}(t, t+\tau) = E[X(t)]E[Y(t+\tau)]$$

Problem

A random variable process $X(t) = sin(\omega t + \phi)$ where ϕ is a random variable uniformly distributed in the interval $(0, 2\pi)$. Prove that $Cov(t, t+\tau) = R_{XX}(t, t+\tau) = \frac{cos\omega\tau}{2}$

Problem

A random variable process $X(t) = sin(\omega t + \phi)$ where ϕ is a random variable uniformly distributed in the interval $(0, 2\pi)$. Prove that $Cov(t, t+\tau) = R_{XX}(t, t+\tau) = \frac{cos\omega\tau}{2}$

$$R_{XX}(t, t+\tau) = E(X(t)X(t+\tau)] = E[sin(\omega t + \phi)sin(\omega(t+\tau) + \phi)]$$
$$= \frac{1}{2}E[cos\omega\tau - cos(2\omega t + \omega\tau + 2\phi)]$$
$$= \frac{1}{2}cos\omega\tau - \frac{1}{2}E[cos(2\omega t + \omega\tau + 2\phi)]$$

$$E[cos(2\omega t + \omega\tau + 2\phi)] = \frac{1}{2\pi}\int_0^{2\pi} cos(2\omega t + \omega\tau + 2\phi)d\phi = 0$$

$$R_{XX}(t, t+\tau) = \frac{1}{2}cos\omega\tau = R_{XX}(\tau)$$

Problem-Contd..

$$Cov(t, t+\tau) = R_{XX}(t, t+\tau) - E[X(t)]E[X(t+\tau)]$$

$$E[X(t)] = E[sin(\omega t + \phi)] = \frac{1}{2\pi}\int_{0}^{2\pi} sin(\omega t + \phi)d\phi = 0$$

$$E[X(t+\tau)] = E[sin(\omega t + \omega\tau + \phi)] = \frac{1}{2\pi}\int_{0}^{2\pi} sin(\omega t + \omega\tau + \phi)d\phi = 0$$

$$Cov(t, t+\tau) = R_{XX}(t, t+\tau) = \frac{1}{2}cos\omega\tau$$

Problem

Find the mean, autocorrelation, variance and autocovariance of the random process $X(t) = tU$ where $U \approx U(0,1)$

Problem

Find the mean, autocorrelation, variance and autocovariance of the random process $X(t) = tU$ where $U \approx U(0,1)$

Mean value is given by:

$$\mu_X(t) = E[X(t)] = \int_{-\infty}^{\infty} tuf_U(u)du = \int_0^1 tudu = t\frac{u^2}{2}\Big|_0^1 = \frac{t}{2}$$

$$E[X^2(t)] = \int_{-\infty}^{\infty} t^2u^2f_U(u)du = \int_0^1 t^2u^2du = t^2\frac{u^3}{3}\Big|_0^1 = \frac{t^2}{3}$$

Variance of $X(t)$ is given by

$$\sigma_X^2 = E[X^2(t)] - (\mu_X(t))^2 = \frac{t^2}{3} - \frac{t^2}{4} = \frac{t^2}{12}$$

Problem

Autocorrelation

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)] = E[tu(t+\tau)u] = E[u^2]E[t(t+\tau)] = t(t+\tau)$$

Covariance of $X(t)$ is given by

$$\begin{aligned} C_{XX}(t, t+\tau) &= R_{XX}(t, t+\tau) - \mu_X(t)\mu_X(t+\tau) \\ &= t(t+\tau) - \frac{1}{2}t\frac{t+\tau}{2} = \frac{3}{4}t(t+\tau) \end{aligned}$$

Gaussian Random Processes

Gaussian Random Process

Consider a continuous random process $X(t)$ and define N random variables $X_1 = X(t_1), ..., X_N = X(t_N)$ corresponding to N time instants $t_1, t_2, ..., t_N$. If these random variables are jointly gaussian, the process is called gaussian

$$f_X(x_1, ..., x_N; t_1, ..., t_N) = \frac{1}{\sqrt{(2\pi)^N |C_X|}} exp\left[-\frac{1}{2}(\boldsymbol{x} - \bar{\boldsymbol{X}})^T C_X^{-1}(\boldsymbol{x} - \bar{\boldsymbol{X}})\right]$$

where

$$[x - \bar{X}] = \begin{bmatrix} x_1 - \bar{X}_1 \\ x_2 - \bar{X}_2 \\ \vdots \\ x_N - \bar{X}_N \end{bmatrix}; \quad [C_X] = \begin{bmatrix} C_{11} & C_{12} \cdots C_{1N} \\ C_{21} & C_{22} \cdots C_{2N} \\ \vdots & \vdots \\ C_{N1} & C_{N2} \cdots C_{NN} \end{bmatrix}$$

Gaussian Random Processes

The mean values $\bar{X}_i$ of $X(t_i)$ are

$$\bar{X}_i = E[X_i] = E[X(t_i)]$$

The elements of the covariance matrix C_X are

$$\begin{aligned} C_{ik} = C_{X_i X_k} &= E[(X_i - \bar{X}_i)(X_k - \bar{X}_k)] \\ &= E[\{X(t_i) - E[X(t_i)]\}\{X(t_k) - E[X(t_k)]\}] \\ &= C_{XX}(t_i, t_k) \end{aligned}$$

$C_{XX}(t_i, t_k)$ is the autocovariance of $X(t_i)$ and $X(t_k)$

$$C_{XX}(t_i, t_k) = R_{XX}(t_i, t_k) - E[X(t_i)]E[X(t_k)]$$

Gaussian Random Processes: WSS

If $X(t)$ is wide-sense stationary gaussian random process, then

$$\bar{X}_i = E[X(t_i)] = \bar{X} \qquad \textbf{constant}$$

The autocovariance and autocorrelation functions depend only on time differences

$$C_{XX}(t_i, t_k) = C_{XX}(t_k - t_i)$$
$$R_{XX}(t_i, t_k) = R_{XX}(t_k - t_i)$$

Problem

A gaussian random process is known to be WSS with a mean of $\bar{X} = 4$ and autocorrelation function $R_{XX}(\tau) = 25e^{-3|\tau|}$. Find the joint density function for three random variables $X(t_i), i = 1, 2, 3$ defined at times $t_i = t_0 + (i - 1)/2$, with t_0 a constant.

$$R_{XX}(t_k - t_i) = 25e^{-3|k-i|/2}; \qquad C_{XX}(t_k - t_i) = 25e^{-3|k-i|/2} - 16$$

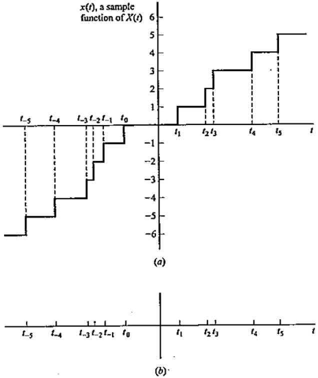
The covariance matrix is given by

$$[C_X] = \begin{bmatrix} 25 - 16 & 25e^{-3/2} - 16 & 25e^{-6/2} - 16 \\ 25e^{-3/2} - 16 & 25 - 16 & 25e^{-3/2} - 16 \\ 25e^{-6/2} - 16 & 25e^{-3/2} - 16 & 25 - 16 \end{bmatrix}$$

Poisson Random Process

Poisson Random Process

- *Poisson process* describes the number of times that some event has occurred as a function of time
  - Events occur at random times
- Eg: Arrival of a customer at a bank/super market
- $X(t)$ represents number of event occurrences with time
- $X(t)$ has integer valued, nondecreasing sample functions
- Also known as **Poisson counting process**



Source: Peebles

Figure: (a) Sample function of a Poisson discrete random process (b) The random times of occurrence of events

Poisson Random Process

The **probability** of exactly k occurrences over a time interval $(0, t)$ is

$$p[X(t) = k] = \frac{(\lambda t)^k e^{-\lambda t}}{k!}; k = 0, 1, 2, ..$$

The **probability density** of the number of occurrences is

$$f_X(x) = \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x - k)$$

The **mean** and **second moment** is given by

$$E[X(t)] = \lambda t; \qquad E[X^2(t)] = \lambda t + \lambda^2 t^2$$

Acknowledge various sources for the images.

Thankyou