2.14: EIGENVECTORS AND EIGENVALUES

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In Sections 11-13, we have been considering some aspects of the moments of inertia of plane laminas, and we have discussed such matters as rotation of axes, and such concepts as product moments of inertia, principal axes, principal moments of inertia and the momental ellipse. We next need to develop the same concepts with respect to three-dimensional solid bodies. In doing so, we shall need to make use of the algebraic concepts of eigenvectors and eigenvalues. If you are already familiar with such matters, you may want to skip this section and move on to the next. If the ideas of eigenvalues and eigenvectors are new to you, or if you are a bit rusty with them, this section may be helpful. I do assume that the reader is at least familiar with the elementary rules of matrix multiplication.

EXAMPLE 2.14.1

Consider what happens when you multiply a vector, for example the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

by a square matrix, for example the matrix $\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$ we obtain:

$$\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

The result of the operation is another vector that is in quite a different direction from the original one.

However, now let us multiply the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ by the same matrix. The result is $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$. The result of the multiplication is merely to multiply the vector by 3 without changing its direction. The vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a very special one, and it is called an *eigenvector* of the matrix, and the multiplier 3 is called the corresponding *eigenvalue*. "Eigen" is German for "own" in the sense of "my own book". There is one other eigenvector of the matrix; it is the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Try it; you should that the corresponding eigenvalue is 2.

In short, given a square matrix **A**, if you can find a vector **a** such that **Aa** = λ **a**, where λ is merely a scalar multiplier that does not change the direction of the vector **a**, then **a** is an eigenvector and λ is the corresponding eigenvalue.

In the above, I told you what the two eigenvectors were, and you were able to verify that they were indeed eigenvectors and you were able to find their eigenvalues by straightforward arithmetic. But, what if I hadn't told you the eigenvectors? How would you find them?

Let $\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ and let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be an eigenvector with corresponding eigenvalue λ. Then we must have

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}.$$

That is,

$$(A_{11} - \lambda)x_1 + A_{12}x_2 = 0$$

and

$$A_{21}x_1 + (A_{22} - \lambda)x_2 = 0.$$

These two equations are consistent only if the determinant of the coefficients is zero. That is,

$$\begin{bmatrix} A_{11} - \lambda & A_{12} \\ A_{21} & A_{22} - \lambda \end{bmatrix} = 0$$

This equation is a quadratic equation in λ, known as the *characteristic equation*, and its two roots, the *characteristic* or *latent roots* are the eigenvalues of the matrix. Once the eigenvalues are found the ratio of x_1 to x_2 is easily found, and hence the eigenvectors.

Similarly, if **A** is a 3×3 matrix, the characteristic equation is

$$\begin{bmatrix} A_{11} - \lambda & A_{12} & A_{13} \\ A_{21} & A_{22} - \lambda & A_{23} \\ A_{31} & A_{32} & A_{33} - \lambda \end{bmatrix} = 0$$

This is a cubic equation in λ, the three roots being the eigenvalues. For each eigenvalue, the ratio $x_1 : x_2 : x_3$ can easily be found and hence the eigenvectors. The characteristic equation is a cubic equation, and is best solved numerically, not by algebraic formula. The cubic equation can be written in the form

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0,$$

and the solutions can be checked from the following results from the theory of equations:

$$\lambda_1 + \lambda_2 + \lambda_3 = -a_2,$$

$$\lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2 = a_1,$$

$$\lambda_1\lambda_2\lambda_3 = -a_0.$$

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