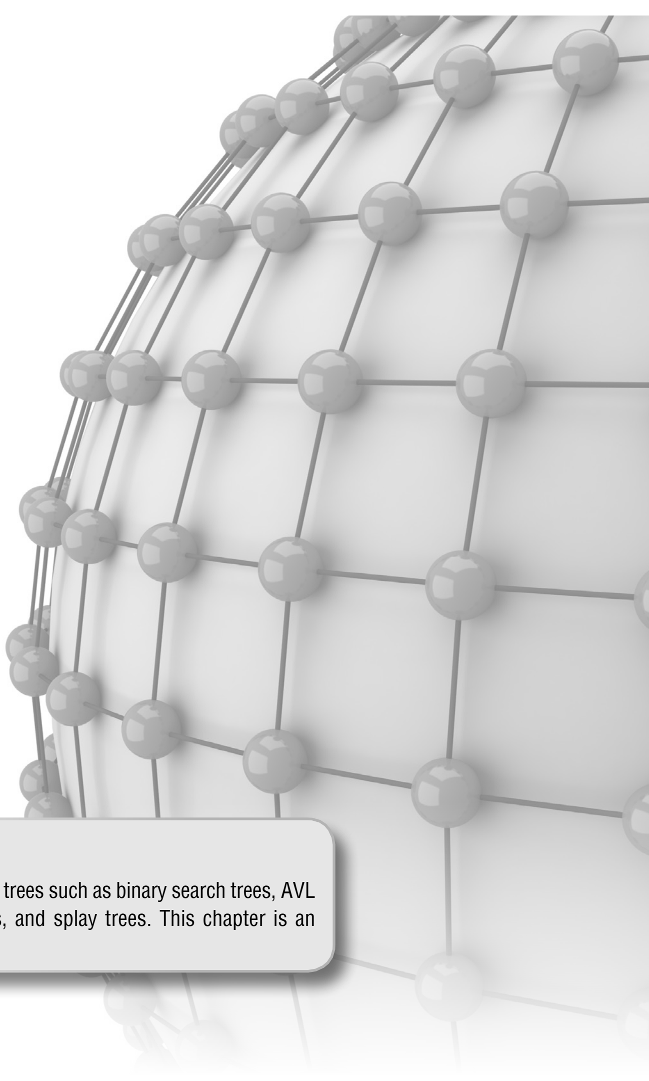
CHAPTER 10

Efficient Binary Trees



LEARNING OBJECTIVE

In this chapter, we will discuss efficient binary trees such as binary search trees, AVL trees, threaded binary trees, red-black trees, and splay trees. This chapter is an extension of binary trees.

10.1 BINARY SEARCH TREES

We have already discussed binary trees in the previous chapter. A binary search tree, also known as an ordered binary tree, is a variant of binary trees in which the nodes are arranged in an order. In a binary search tree, all the nodes in the left sub-tree have a value less than that of the root node. Correspondingly, all the nodes in the right sub-tree have a value either equal to or greater than the root node. The same rule is applicable to every sub-tree in the tree. (Note that a binary search tree may or may not contain duplicate values, depending on its implementation.)

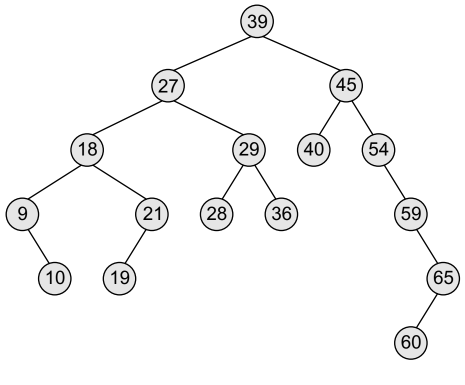


Figure 10.1 Binary search tree

Look at Fig. 10.1. The root node is 39. The left sub-tree of the root node consists of nodes 9, 10, 18, 19, 21, 27, 28, 29, and 36. All these nodes have smaller values than the root node. The right sub-tree of the root node consists of nodes 40, 45, 54, 59, 60, and 65. Recursively, each of the sub-trees also obeys the binary search tree constraint. For example, in the left sub-tree of the root node, 27 is the root and all elements in its left sub-tree (9, 10, 18, 19, 21) are smaller than 27, while all nodes in its right sub-tree (28, 29, and 36) are greater than the root node's value.

Since the nodes in a binary search tree are ordered, the time needed to search an element in the tree is greatly reduced. Whenever we search for an element, we do not need to traverse the entire tree. At every node, we get a hint regarding which sub-tree to search in. For example, in the given tree, if we have to search for 29, then we know that we have to scan only the left sub-tree. If the value is present in the tree, it will only be in the left sub-tree, as 29 is smaller than 39 (the root node's value). The left sub-tree has a root node with the value 27. Since 29 is greater than 27, we will move to the right sub-tree, where we will find the element. Thus, the average running time of a search operation is `O(log`$_2$`n)`, as at every step, we eliminate half of the sub-tree from the search process. Due to its efficiency in searching elements, binary search trees are widely used in dictionary problems where the code always inserts and searches the elements that are indexed by some key value.

Binary search trees also speed up the insertion and deletion operations. The tree has a speed advantage when the data in the structure changes rapidly.

Binary search trees are considered to be efficient data structures especially when compared with sorted linear arrays and linked lists. In a sorted array, searching can be done in `O(log`$_2$`n)` time, but insertions and deletions are quite expensive. In contrast, inserting and deleting elements in a linked list is easier, but searching for an element is done in `O(n)` time.

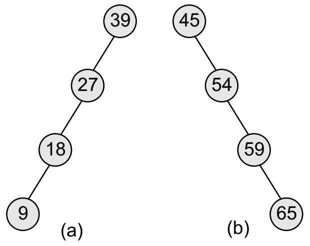


Figure 10.2 (a) Left skewed, and (b) right skewed binary search trees

However, in the worst case, a binary search tree will take `O(n)` time to search for an element. The worst case would occur when the tree is a linear chain of nodes as given in Fig. 10.2.

To summarize, a binary search tree is a binary tree with the following properties:

- The left sub-tree of a node `N` contains values that are less than `N`'s value.
- The right sub-tree of a node `N` contains values that are greater than `N`'s value.
- Both the left and the right binary trees also satisfy these properties and, thus, are binary search trees.

Example 10.1 State whether the binary trees in Fig. 10.3 are binary search trees or not.

Solution

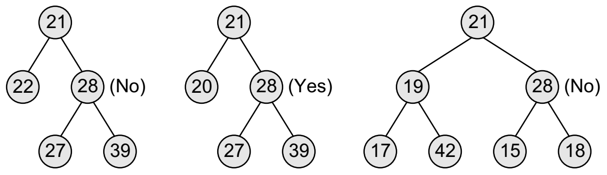
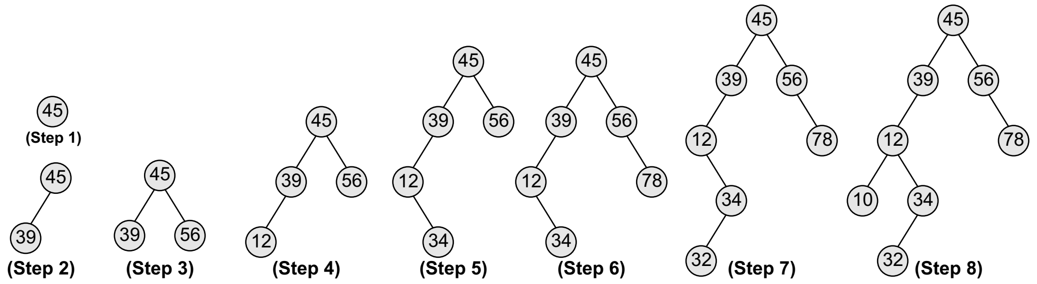


Figure 10.3 Binary trees

Example 10.2 Create a binary search tree using the following data elements:

45, 39, 56, 12, 34, 78, 32, 10, 89, 54, 67, 81

Solution



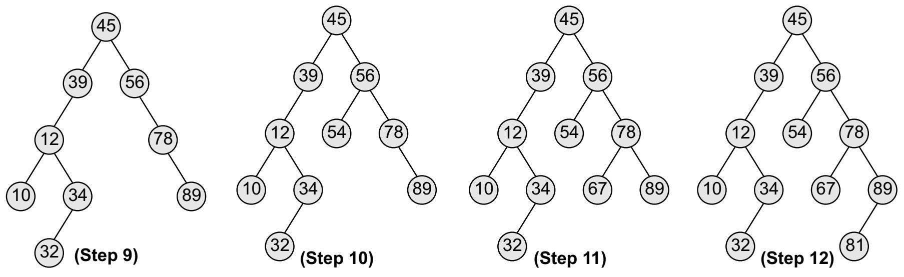


Figure 10.4 Binary search tree

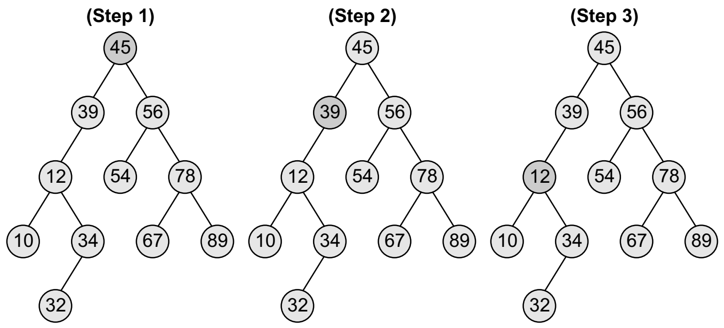


Figure 10.5 Searching a node with value 12 in the given binary search tree

10.2 OPERATIONS ON BINARY SEARCH TREES

In this section, we will discuss the different operations that are performed on a binary search tree. All these operations require comparisons to be made between the nodes.

10.2.1 Searching for a Node in a Binary Search Tree

The search function is used to find whether a given value is present in the tree or not. The searching process begins at the root node. The function first checks if the binary search tree is empty. If it is empty, then the value we are searching for is not present in the tree. So, the search algorithm terminates by displaying an appropriate message. However, if there are nodes in the tree, then the search function checks to see if the key value of the current node is equal to the value to be searched. If not, it checks if the value to be searched for is less than the value of the current node, in which case it should be recursively called on the left child node. In case the value is greater than the value of the current node, it should be recursively called on the right child node.

Look at Fig. 10.5. The figure shows how a binary tree is searched to find a specific element. First, see how the tree will be traversed to find the node with value 12. The procedure to find the node with value 67 is illustrated in Fig. 10.6.

The procedure to find the node with value 40 is shown in Fig. 10.7. The search would terminate after reaching node 39 as it does not have any right child.

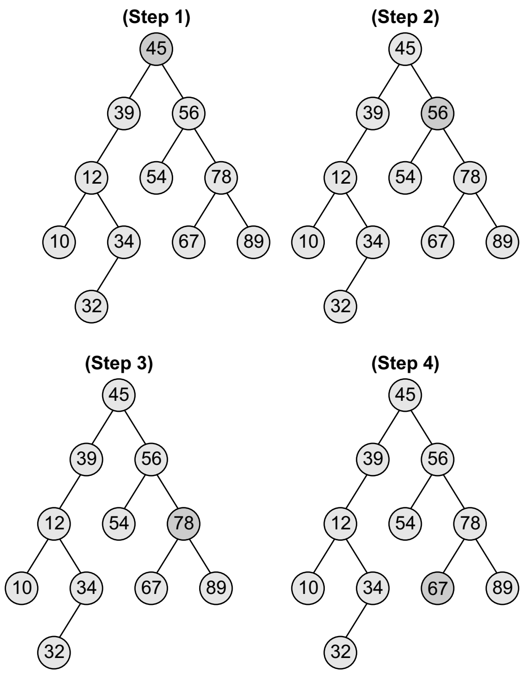


Figure 10.6 Searching a node with value 67 in the given binary search tree

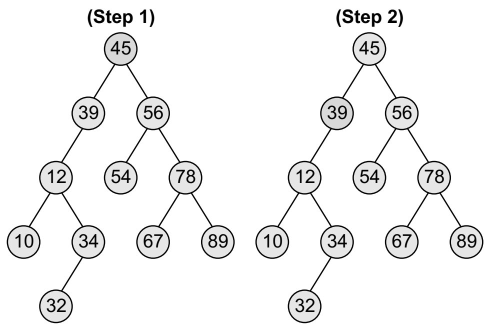


Figure 10.7 Searching a node with the value 40 in the given binary search tree

Now let us look at the algorithm to search for an element in the binary search tree as shown in Fig. 10.8. In Step 1, we check if the value stored at the current node of `TREE` is equal to `VAL` or if the current node is `NULL`, then we return the current node of `TREE`. Otherwise, if the value stored at the current node is less than `VAL`, then the algorithm is recursively called on its right sub-tree, else the algorithm is called on its left sub-tree.

10.2.2 Inserting a New Node in a Binary Search Tree

The insert function is used to add a new node with a given value at the correct position in the binary search tree. Adding the node at the correct position means that the new node should not violate the properties of the binary search tree. Figure 10.9 shows the algorithm to insert a given value in a binary search tree.

The initial code for the insert function is similar to the search function. This is because we first find the correct position where the insertion has to be done and then add the node at that position. The insertion function changes the structure of the tree. Therefore, when the insert function is called recursively, the function should return the new tree pointer.

```
searchElement (TREE, VAL)

Step 1: IF TREE -> DATA = VAL OR TREE = NULL
          Return TREE
        ELSE
         IF VAL < TREE -> DATA
          Return searchElement(TREE -> LEFT, VAL)
         ELSE
          Return searchElement(TREE -> RIGHT, VAL)
         [END OF IF]
        [END OF IF]
Step 2: END
```

Figure 10.8 Algorithm to search for a given value in a binary search tree

```
Insert (TREE, VAL)

Step 1: IF TREE = NULL
            Allocate memory for TREE
            SET TREE -> DATA = VAL
            SET TREE -> LEFT = TREE -> RIGHT = NULL
        ELSE
            IF VAL < TREE -> DATA
                  Insert(TREE -> LEFT, VAL)
            ELSE
                  Insert(TREE -> RIGHT, VAL)
            [END OF IF]
        [END OF IF]
Step 2: END
```

Figure 10.9 Algorithm to insert a given value in a binary search tree

In Step 1 of the algorithm, the insert function checks if the current node of `TREE` is `NULL`. If it is `NULL`, the algorithm simply adds the node, else it looks at the current node's value and then recurs down the left or right sub-tree.

If the current node's value is less than that of the new node, then the right sub-tree is traversed, else the left sub-tree is traversed. The insert function continues moving down the levels of a binary tree until it reaches a leaf node. The new node is added by following the rules of the binary search trees. That is, if the new node's value is greater than that of the parent node, the new node is inserted in the right sub-tree, else it is inserted in the left sub-tree. The insert function requires time proportional to the height of the tree in the worst case. It takes `O(log n)` time to execute in the average case and `O(n)` time in the worst case.

Look at Fig. 10.10 which shows insertion of values in a given tree. We will take up the case of inserting 12 and 55.

10.2.3 Deleting a Node from a Binary Search Tree

The delete function deletes a node from the binary search tree. However, utmost care should be taken that the properties of the binary search tree are not violated and nodes are not

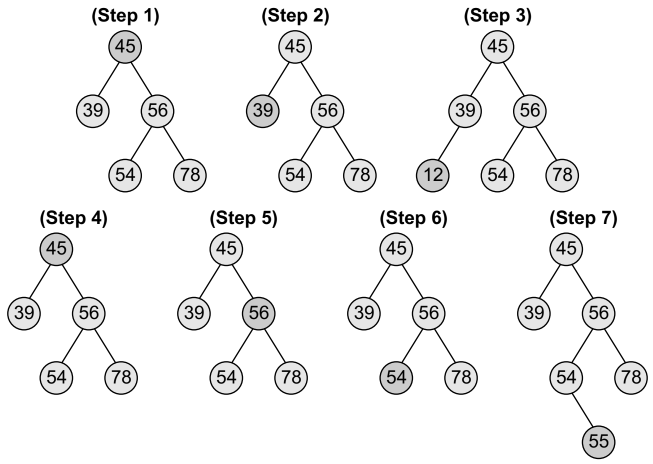


Figure 10.10 Inserting nodes with values 12 and 55 in the given binary search tree

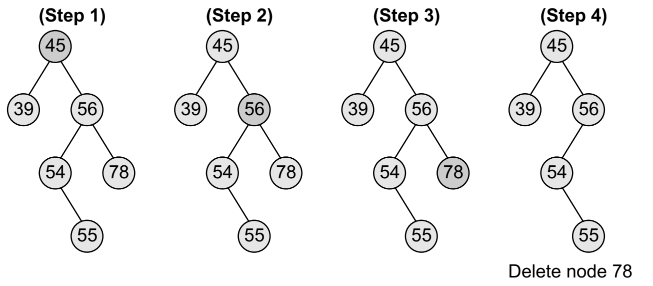


Figure 10.11 Deleting node 78 from the given binary search tree

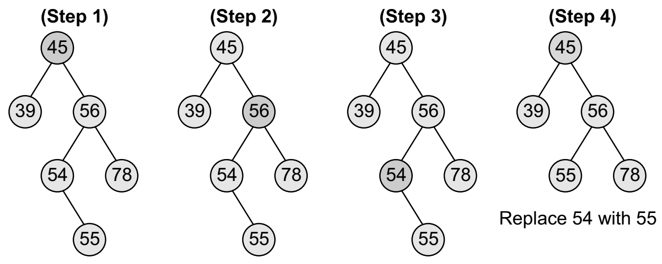


Figure 10.12 Deleting node 54 from the given binary search tree

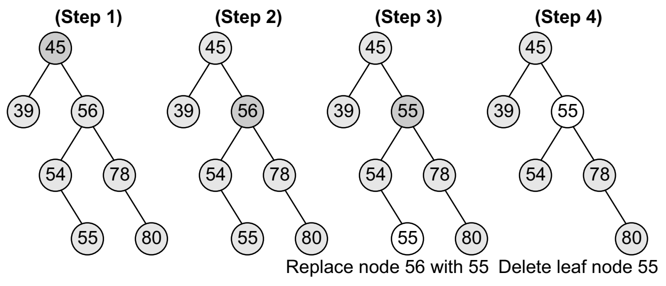


Figure 10.13 Deleting node 56 from the given binary search tree

lost in the process. We will take up three cases in this section and discuss how a node is deleted from a binary search tree.

Case 1: Deleting a Node that has No Children

Look at the binary search tree given in Fig. 10.11. If we have to delete node 78, we can simply remove this node without any issue. This is the simplest case of deletion.

Case 2: Deleting a Node with One Child

To handle this case, the node's child is set as the child of the node's parent. In other words, replace the node with its child. Now, if the node is the left child of its parent, the node's child becomes the left child of the node's parent. Correspondingly, if the node is the right child of its parent, the node's child becomes the right child of the node's parent. Look at the binary search tree shown in Fig. 10.12 and see how deletion of node 54 is handled.

Case 3: Deleting a Node with Two Children

To handle this case, replace the node's value with its *in-order predecessor* (largest value in the left sub-tree) or *in-order successor* (smallest value in the right sub-tree). The in-order predecessor or the successor can then be deleted using any of the above cases. Look at the binary search tree given in Fig. 10.13 and see how deletion of node with value 56 is handled.

This deletion could also be handled by replacing node 56 with its in-order successor, as shown in Fig. 10.14.

Now, let us look at Fig. 10.15 which shows the algorithm to delete a node from a binary search tree.

In Step 1 of the algorithm, we first check if `TREE=NULL`, because if it is true, then the node to be deleted is not present in the tree. However, if that is not the case, then we check if the value to be deleted is less than the current node's data. In case the value is less, we call the algorithm recursively on the node's left sub-tree, otherwise the algorithm

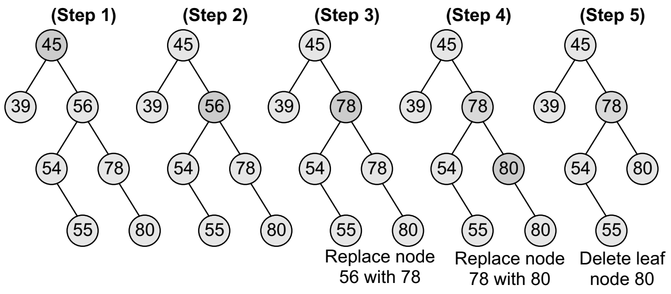


Figure 10.14 Deleting node 56 from the given binary search tree

```
Delete (TREE, VAL)

Step 1: IF TREE = NULL
          Write "VAL not found in the tree"
        ELSE IF VAL < TREE -> DATA
          Delete(TREE->LEFT, VAL)
        ELSE IF VAL > TREE -> DATA
          Delete(TREE -> RIGHT, VAL)
        ELSE IF TREE -> LEFT AND TREE -> RIGHT
          SET TEMP = findLargestNode(TREE -> LEFT)
          SET TREE -> DATA = TEMP -> DATA
          Delete(TREE -> LEFT, TEMP -> DATA)
        ELSE
          SET TEMP = TREE
          IF TREE -> LEFT = NULL AND TREE -> RIGHT = NULL
               SET TREE = NULL
          ELSE IF TREE -> LEFT != NULL
               SET TREE = TREE -> LEFT
          ELSE
               SET TREE = TREE -> RIGHT
          [END OF IF]
          FREE TEMP
        [END OF IF]
Step 2: END
```

Figure 10.15 Algorithm to delete a node from a binary search tree

is called recursively on the node's right sub-tree.

Note that if we have found the node whose value is equal to `VAL`, then we check which case of deletion it is. If the node to be deleted has both left and right children, then we find the in-order predecessor of the node by calling `findLargestNode(TREE –> LEFT)` and replace the current node's value with that of its in-order predecessor. Then, we call `Delete(TREE –> LEFT, TEMP –> DATA)` to delete the initial node of the in-order predecessor. Thus, we reduce the case 3 of deletion into either case 1 or case 2 of deletion.

If the node to be deleted does not have any child, then we simply set the node to `NULL`. Last but not the least, if the node to be deleted has either a left or a right child but not both, then the current node is replaced by its child node and the initial child node is deleted from the tree.

The delete function requires time proportional to the height of the tree in the worst case. It takes `O(log n)` time to execute in the average case and Ω`(n)` time in the worst case.

10.2.4 Determining the Height of a Binary Search Tree

In order to determine the height of a binary search tree, we calculate the height of the left sub-tree and the right sub-tree. Whichever height is greater, 1 is added to it. For example, if the height of the left sub-tree is greater than that of the right sub-tree, then 1 is added to the left sub-tree, else 1 is added to the right sub-tree.

Look at Fig. 10.16. Since the height of the right sub-tree is greater than the height of the left sub-tree, the height of the tree = height (right sub-tree) + 1= 2 + 1 = 3.

Figure 10.17 shows a recursive algorithm that determines the height of a binary search tree.

In Step 1 of the algorithm, we first check if the current node of the `TREE = NULL`. If the condition is true, then 0 is returned to the calling code. Otherwise, for every node, we recursively call the algorithm to calculate the height of its left sub-tree as well as its right sub-tree. The height of the tree at that node is given by adding 1 to the height of the left sub-tree or the height of right sub-tree, whichever is greater**.**

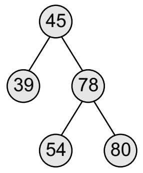


Figure 10.16 Binary search tree with height = 3

10.2.5 Determining the Number of Nodes

Determining the number of nodes in a binary search tree is similar to determining its height. To calculate the total number of elements/nodes

```
Height (TREE)

Step 1: IF TREE = NULL
            Return 0
        ELSE
         SET LeftHeight = Height(TREE -> LEFT)
         SET RightHeight = Height(TREE -> RIGHT)
          IF LeftHeight > RightHeight
               Return LeftHeight + 1
          ELSE
               Return RightHeight + 1
          [END OF IF]
        [END OF IF]
Step 2: END
```

Figure 10.17 Algorithm to determine the height of a binary search tree

in the tree, we count the number of nodes in the left sub-tree and the right sub-tree.

```
    Number of nodes = totalNodes(left sub-tree)
+ totalNodes(right sub-tree) + 1
```

Consider the tree given in Fig. 10.18. The total number of nodes in the tree can be calculated as:

```
Total nodes of left sub-tree = 1
Total nodes of left sub-tree = 5
Total nodes of tree = (1 + 5) + 1
                    = 7
```

Figure 10.19 shows a recursive algorithm to calculate the number of nodes in a binary search tree. For every node, we recursively call the algorithm on its left sub-tree as well as the right sub-tree. The total number of nodes at a given node is then returned by adding 1 to the number of nodes in its left as well as right sub-tree. However if the tree is empty, that is `TREE = NULL`, then the number of nodes will be zero.

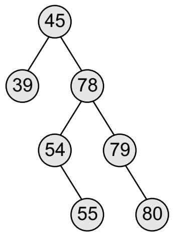


Figure 10.18 Binary search tree

Determining the Number of Internal Nodes

To calculate the total number of internal nodes or non-leaf nodes, we count the number of internal nodes in the left sub-tree and the right sub-tree and add 1 to it (1 is added for the root node).

```
Number of internal nodes =
  totalInternalNodes(left sub-tree) +
totalInternalNodes(right sub-tree) + 1
```

Consider the tree given in Fig. 10.18. The total number of internal nodes in the tree can be calculated as:

```
Total internal nodes of left sub-tree = 0
Total internal nodes of right sub-tree = 3
Total internal nodes of tree = (0 + 3) + 1
                             = 4
```

```
totalNodes(TREE)

Step 1: IF TREE = NULL
            Return 0
        ELSE
            Return totalNodes(TREE -> LEFT)
                + totalNodes(TREE -> RIGHT) + 1
        [END OF IF]
Step 2: END
```

Figure 10.19 Algorithm to calculate the number of nodes in a binary search tree

Figure 10.20 shows a recursive algorithm to calculate the total number of internal nodes in a binary search tree. For every node, we recursively call the algorithm on its left sub-tree as well as the right sub-tree. The total number of internal nodes at a given node is then returned by adding internal nodes in its left as well as right sub-tree. However, if the tree is empty, that is `TREE = NULL`, then the number of internal nodes will be zero. Also if there is only one node in the tree, then the number of internal nodes will be zero.

```
totalInternalNodes(TREE)

Step 1: IF TREE = NULL
            Return 0
        [END OF IF]
        IF TREE -> LEFT = NULL AND TREE -> RIGHT = NULL
            Return 0
        ELSE
           Return totalInternalNodes(TREE -> LEFT) +
            totalInternalNodes(TREE -> RIGHT) + 1
       [END OF IF]
Step 2: END
```

Figure 10.20 Algorithm to calculate the total number of internal nodes in a binary search tree

Determining the Number of External Nodes

To calculate the total number of external nodes or leaf nodes, we add the number of external nodes in the left sub-tree and the right sub-tree. However if the tree is empty, that is `TREE = NULL`, then the number of external nodes will be zero. But if there is only one node in the tree, then the number of external nodes will be one.

```
Number of external nodes = totalExternalNodes(left sub-tree) +
                                 totalExternalNodes (right sub-tree)
```

Consider the tree given in Fig. 10.18. The total number of external nodes in the given tree can be calculated as:

```
Total external nodes of left sub-tree = 1
Total external nodes of left sub-tree = 2
Total external nodes of tree = 1 + 2
                             = 3
```

```
totalExternalNodes(TREE)

Step 1: IF TREE = NULL
            Return 0
        ELSE IF TREE –> LEFT = NULL AND TREE –> RIGHT = NULL
            Return 1
        ELSE
            Return totalExternalNodes(TREE –> LEFT) +
            totalExternalNodes(TREE –> RIGHT)
        [END OF IF]
Step 2: END
```

Figure 10.21 Algorithm to calculate the total number of external nodes in a binary search tree

Figure 10.21 shows a recursive algorithm to calculate the total number of external nodes in a binary search tree. For every node, we recursively call the algorithm on its left sub-tree as well as the right sub-tree. The total number of external nodes at a given node is then returned by adding the external nodes in its left as well as right sub-tree. However if the tree is empty, that is `TREE = NULL`, then the number of external nodes will be zero. Also if there is only one node in the tree, then there will be only one external node (that is the root node).

10.2.6 Finding the Mirror Image of a Binary Search Tree

Mirror image of a binary search tree is obtained by interchanging the left sub-tree with the right sub-tree at every node of the tree. For example, given a tree `T`, the mirror image of `T` can be obtained as `T′`. Consider the tree T given in Fig. 10.22.

Figure 10.23 shows a recursive algorithm to obtain the mirror image of a binary search tree. In the algorithm, if `TREE != NULL`, that is if the current node in the tree has one or more nodes, then the algorithm is recursively called at every node in the tree to swap the nodes in its left and right sub-trees.

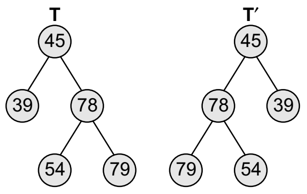


Figure 10.22 Binary search tree T and its

10.2.7 Deleting a Binary Search Tree

To delete/remove an entire binary search tree from the memory, we first delete the elements/nodes in the left sub-tree and then delete the nodes in the right sub-tree. The algorithm shown in Fig. 10.24 gives a recursive procedure to remove the binary search tree.

```
MirrorImage(TREE)

Step 1: IF TREE != NULL
            MirrorImage(TREE –> LEFT)
            MirrorImage(TREE –> RIGHT)
            SET TEMP = TREE –> LEFT
            SET TREE –> LEFT = TREE –> RIGHT
            SET TREE –> RIGHT = TEMP
        [END OF IF]
Step 2: END
```

Figure 10.23 Algorithm to obtain the mirror image mirror image T′ of a binary search tree

10.2.8 Finding the Smallest Node in a Binary Search Tree

The very basic property of the binary search tree states that the smaller value will occur in the left sub-tree. If the left sub-tree is NULL, then the value of the root node will be smallest as compared to the nodes in the right sub-tree. So, to find the node with the smallest value, we find the value of the leftmost node of the left sub-tree. The recursive algorithm to find the smallest node in a binary search tree is shown in Fig. 10.25.

```
deleteTree(TREE)

Step 1: IF TREE != NULL
            deleteTree (TREE -> LEFT)
            deleteTree (TREE -> RIGHT)
            Free (TREE)
        [END OF IF]
Step 2: END
```

Figure 10.24 Alogrithm to delete a binary search tree

```
findSmallestElement(TREE)

Step 1: IF TREE = NULL OR TREE -> LEFT = NULL
            Returen TREE
        ELSE
            Return findSmallestElement(TREE -> LEFT)
        [END OF IF]
Step 2: END
```

Figure 10.25 Algorithm to find the smallest node in a binary search tree

10.2.9 Finding the Largest Node in a Binary Search Tree

To find the node with the largest value, we find the value of the rightmost node of the right sub-tree. However, if the right sub-tree is empty, then the root node will be the largest value in the tree. The recursive algorithm to find the largest node in a binary search tree is shown in Fig. 10.26.

```
findLargestElement(TREE)

Step 1: IF TREE = NULL OR TREE -> RIGHT = NULL
            Return TREE
        ELSE
           Return findLargestElement(TREE -> RIGHT)
        [END OF IF]
Step 2: END
```

Figure 10.26 Algorithm to find the largest node in a binary search tree

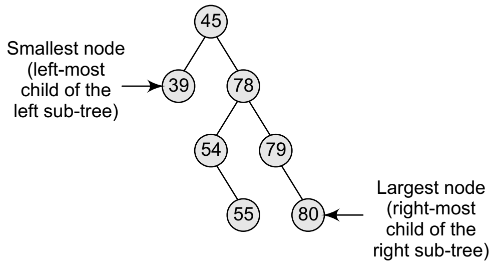

45
39
78
54
79
55
80
Smallest node (left-most child of the left sub-tree)
Largest node (right-most child of the right sub-tree)

Figure 10.27 Binary search tree

Consider the tree given in Fig. 10.27. The smallest and the largest node can be given as:

PROGRAMMING EXAMPLE

1. Write a program to create a binary search tree and perform all the operations discussed in the preceding sections.

```
#include <stdio.h>
#include <conio.h>
#include <malloc.h>
struct node
{
       int data;
       struct node *left;
       struct node *right;
};
struct node *tree;
void create_tree(struct node *);
struct node *insertElement(struct node *, int);
void preorderTraversal(struct node *);
void inorderTraversal(struct node *);
void postorderTraversal(struct node *);
struct node *findSmallestElement(struct node *);
struct node *findLargestElement(struct node *);
struct node *deleteElement(struct node *, int);
struct node *mirrorImage(struct node *);
int totalNodes(struct node *);
int totalExternalNodes(struct node *);
int totalInternalNodes(struct node *);
int Height(struct node *);
struct node *deleteTree(struct node *);
int main()
{
       int option, val;
       struct node *ptr;
       create_tree(tree);
       clrscr();
       do
       {
              printf("\n ******MAIN MENU******* \n");
              printf("\n 1. Insert Element");
              printf("\n 2. Preorder Traversal");
              printf("\n 3. Inorder Traversal");
              printf("\n 4. Postorder Traversal");
              printf("\n 5. Find the smallest element");
              printf("\n 6. Find the largest element");
              printf("\n 7. Delete an element");
              printf("\n 8. Count the total number of nodes");
              printf("\n 9. Count the total number of external nodes");
              printf("\n 10. Count the total number of internal nodes");
              printf("\n 11. Determine the height of the tree");
              printf("\n 12. Find the mirror image of the tree");
              printf("\n 13. Delete the tree");
              printf("\n 14. Exit");
              printf("\n\n Enter your option : ");
              scanf("%d", &option);
              switch(option)
              {
                     case 1:
                          printf("\n Enter the value of the new node : ");
                          scanf("%d", &val);
                          tree = insertElement(tree, val);
                          break;
                     case 2:
                          printf("\n The elements of the tree are : \n");
                          preorderTraversal(tree);
                          break;
                     case 3:
                          printf("\n The elements of the tree are : \n");
                          inorderTraversal(tree);
                          break;
                     case 4:
                          printf("\n The elements of the tree are : \n");
                          postorderTraversal(tree);
                          break;
                     case 5:
                          ptr = findSmallestElement(tree);
                          printf("\n Smallest element is :%d",ptr–>data);
                          break;
                     case 6:
                          ptr = findLargestElement(tree);
                          printf("\n Largest element is : %d", ptr–>data);
                          break;
                     case 7:
                          printf("\n Enter the element to be deleted : ");
                          scanf("%d", &val);
```

```
                                tree = deleteElement(tree, val);
                                break;
                           case 8:
                                printf("\n Total no. of nodes = %d", totalNodes(tree));
                                break;
                           case 9:
                                printf("\n Total no. of external nodes = %d",
                                          totalExternalNodes(tree));
                                break;
                           case 10:
                                printf("\n Total no. of internal nodes = %d",
                                          totalInternalNodes(tree));
                                break;
                           case 11:
                                printf("\n The height of the tree = %d",Height(tree));
                                break;
                           case 12:
                                tree = mirrorImage(tree);
                                break;
                           case 13:
                                tree = deleteTree(tree);
                                break;
                    }
             }while(option!=14);
             getch();
             return 0;
      }
      void create_tree(struct node *tree)
      {
             tree = NULL;
      }
      struct node *insertElement(struct node *tree, int val)
      {
             struct node *ptr, *nodeptr, *parentptr;
             ptr = (struct node*)malloc(sizeof(struct node));
             ptr->data = val;
             ptr->left = NULL;
             ptr->right = NULL;
             if(tree==NULL)
             {
                    tree=ptr;
                    tree->left=NULL;
                    tree->right=NULL;
             }
             else
             {
                    parentptr=NULL;
                    nodeptr=tree;
                    while(nodeptr!=NULL)
                    {
                           parentptr=nodeptr;
                           if(val<nodeptr->data)
                                  nodeptr=nodeptr->left;
                           else
                                  nodeptr = nodeptr->right;
                    }
                    if(val<parentptr->data)
                           parentptr->left = ptr;
                    else
                           parentptr->right = ptr;
             }
             return tree;
      }
      void preorderTraversal(struct node *tree)
      {
        if(tree != NULL)
        {
              printf("%d\t", tree->data);
              preorderTraversal(tree->left);
              preorderTraversal(tree->right);
        }
}
void inorderTraversal(struct node *tree)
{
        if(tree != NULL)
        {
              inorderTraversal(tree->left);
              printf("%d\t", tree->data);
              inorderTraversal(tree->right);
        }
}
void postorderTraversal(struct node *tree)
{
        if(tree != NULL)
        {
              postorderTraversal(tree->left);
              postorderTraversal(tree->right);
              printf("%d\t", tree->data);
        }
}
struct node *findSmallestElement(struct node *tree)
{
        if( (tree == NULL) || (tree->left == NULL))
              return tree;
        else
              return findSmallestElement(tree ->left);
}
struct node *findLargestElement(struct node *tree)
{
        if( (tree == NULL) || (tree->right == NULL))
              return tree;
        else
              return findLargestElement(tree->right);
}
struct node *deleteElement(struct node *tree, int val)
{
        struct node *cur, *parent, *suc, *psuc, *ptr;
        if(tree->left==NULL)
        {
              printf("\n The tree is empty ");
              return(tree);
        }
        parent = tree;
        cur = tree->left;
        while(cur!=NULL && val!= cur->data)
        {
              parent = cur;
              cur = (val<cur->data)? cur->left:cur->right;
        }
        if(cur == NULL)
        {
              printf("\n The value to be deleted is not present in the tree");
              return(tree);
        }
        if(cur->left == NULL)
              ptr = cur->right;
        else if(cur->right == NULL)
```

```
                ptr = cur->left;
        else
        {
                // Find the in-order successor and its parent
                psuc = cur;
                cur = cur->left;
                while(suc->left!=NULL)
                {
                        psuc = suc;
                        suc = suc->left;
                }
                if(cur==psuc)
                {
                        // Situation 1
                        suc->left = cur->right;
                }
                else
                {
                        // Situation 2
                        suc->left = cur->left;
                        psuc->left = suc->right;
                        suc->right = cur->right;
                }
                ptr = suc;
        }
        // Attach ptr to the parent node
        if(parent->left == cur)
                parent->left=ptr;
        else
                parent->right=ptr;
        free(cur);
        return tree;
}
int totalNodes(struct node *tree)
{
        if(tree==NULL)
                return 0;
        else
                return(totalNodes(tree->left) + totalNodes(tree->right) + 1);
}
int totalExternalNodes(struct node *tree)
{
        if(tree==NULL)
                return 0;
        else if((tree->left==NULL) && (tree->right==NULL))
                return 1;
        else
                return (totalExternalNodes(tree->left) +
                totalExternalNodes(tree->right));
}
int totalInternalNodes(struct node *tree)
{
        if( (tree==NULL) || ((tree->left==NULL) && (tree->right==NULL)))
                return 0;
        else
                return (totalInternalNodes(tree->left)
                + totalInternalNodes(tree->right) + 1);
}
int Height(struct node *tree)
{
        int leftheight, rightheight;
        if(tree==NULL)
               return 0;
        else
        {
               leftheight = Height(tree->left);
               rightheight = Height(tree->right);
               if(leftheight > rightheight)
                      return (leftheight + 1);
               else
                      return (rightheight + 1);
        }
}
struct node *mirrorImage(struct node *tree)
{
        struct node *ptr;
        if(tree!=NULL)
        {
               mirrorImage(tree->left);
               mirrorImage(tree->right);
               ptr=tree->left;
               ptr->left = ptr->right;
               tree->right = ptr;
        }
}
struct node *deleteTree(struct node *tree)
{
        if(tree!=NULL)
        {
               deleteTree(tree->left);
               deleteTree(tree->right);
               free(tree);
        }
}
```

Output

```
*******MAIN   MENU*******
1. Insert Element
2. Preorder Traversal
3. Inorder Traversal
4. Postorder Traversal
5. Find the smallest element
6. Find the largest element
7. Delete an element
8. Count the total number of nodes
9. Count the total number of external nodes
10. Count the total number of internal nodes
11. Determine the height of the tree
12. Find the mirror image of the tree
13. Delete the tree
14. Exit
Enter your option : 1
Enter the value of the new node : 1
Enter the value of the new node : 2
Enter the value of the new node : 4
Enter your option : 3
2    1    4
Enter your option : 14
```

10.3 THREADED BINARY TREES

A threaded binary tree is the same as that of a binary tree but with a difference in storing the `NULL` pointers. Consider the linked representation of a binary tree as given in Fig. 10.28.

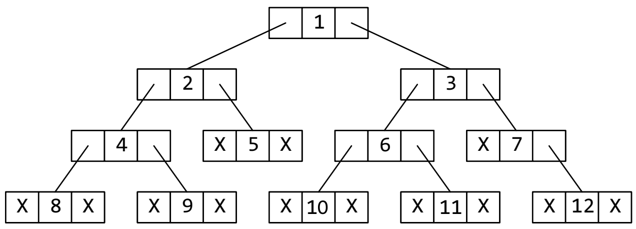


Figure 10.28 Linked representation of a binary tree

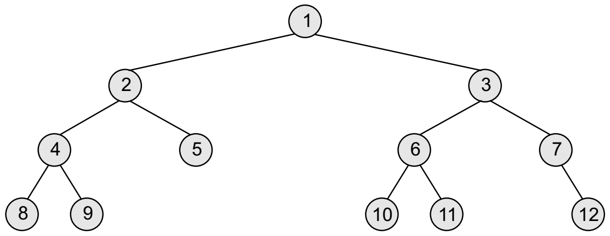


Figure 10.29 (a) Binary tree without threading

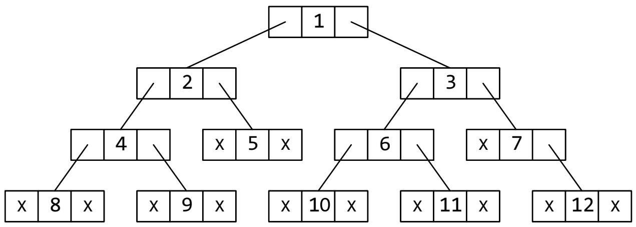


Figure 10.29 (b) Linked representation of the binary tree (without threading)

In the linked representation, a number of nodes contain a NULL pointer, either in their left or right fields or in both. This space that is wasted in storing a NULL pointer can be efficiently used to store some other useful piece of information. For example, the NULL entries can be replaced to store a pointer to the in-order predecessor or the in-order successor of the node. These special pointers are called *threads* and binary trees containing threads are called *threaded trees*. In the linked representation of a threaded binary tree, threads will be denoted using arrows.

There are many ways of threading a binary tree and each type may vary according to the way the tree is traversed. In this book, we will discuss in-order traversal of the tree. Apart from this, a threaded binary tree may correspond to one-way threading or a two-way threading.

In one-way threading, a thread will appear either in the right field or the left field of the node. A one-way threaded tree is also called a single-threaded tree. If the thread appears in the left field, then the left field will be made to point to the in-order predecessor of the node. Such a one-way threaded tree is called a left-threaded binary tree. On the contrary, if the thread appears in the right field, then it will point to the in-order successor of the node. Such a one-way threaded tree is called a right-threaded binary tree.

In a two-way threaded tree, also called a double-threaded tree, threads will appear in both the left and the right field of the node. While the left field will point to the in-order predecessor of the node, the right field will point to its successor. A two-way threaded binary tree is also called a fully threaded binary tree. One-way threading and two-way threading of binary trees are explained below. Figure 10.29 shows a binary tree without threading and its corresponding linked representation. *The in-order traversal of the tree is given as 8, 4, 9, 2, 5, 1, 10, 6, 11, 3, 7, 12*

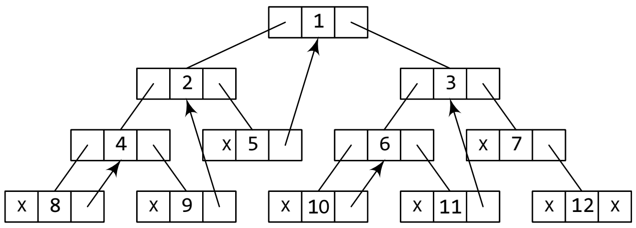


Figure 10.30 (a) Linked representation of the binary tree with one-way threading

One-way Threading

Figure 10.30 shows a binary tree with one-way threading and its corresponding linked representation.

Node 5 contains a NULL pointer in its RIGHT field, so it will be replaced to point to node 1, which is its in-order successor. Similarly, the RIGHT field of node 8 will point to node 4, the RIGHT field of node 9 will point to node 2, the RIGHT field of node 10 will point to node

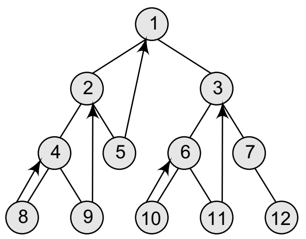


Figure 10.30 (b) Binary tree with one-way threading

6, the RIGHT field of node 11 will point to node 3, and the RIGHT field of node 12 will contain NULL because it has no in-order successor.

Two-way Threading

Figure 10.31 shows a binary tree with two-way threading and its corresponding linked representation.

Node 5 contains a NULL pointer in its LEFT field, so it will be replaced to point to node 2, which is its in-order predecessor. Similarly, the LEFT field of node 8 will contain NULL because it has no in-order predecessor, the LEFT field of node 7 will point to node 3, the LEFT field of node 9 will point to node 4, the LEFT field of node 10 will point to node 1, the LEFT field of node 11 will contain 6, and the LEFT field of node 12 will point to node 7.

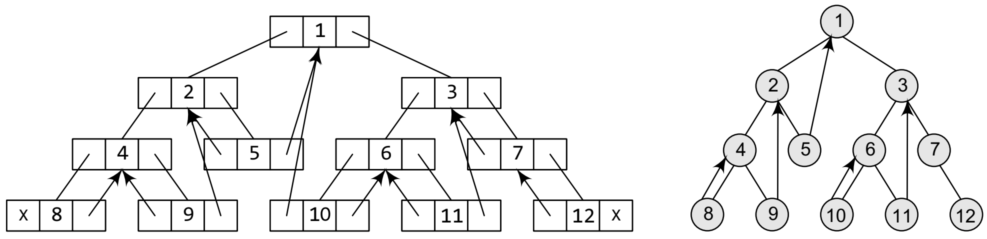
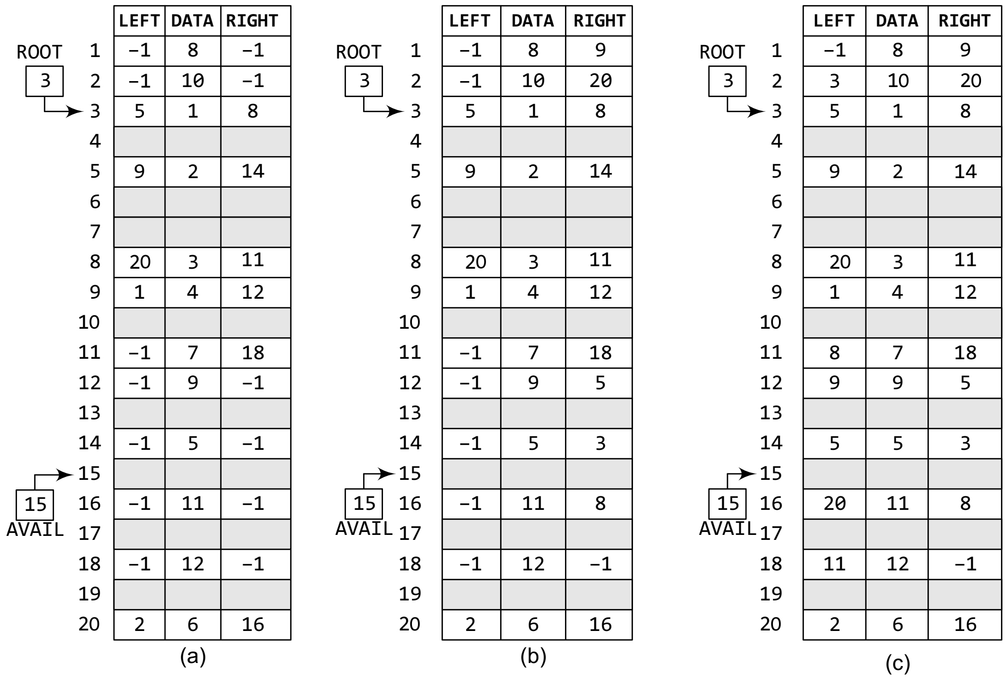


Figure 10.31 (a) Linked representation of the binary tree with threading, (b) binary tree with two-way threading

Now, let us look at the memory representation of a binary tree without threading, with one-way threading, and with two-way threading. This is illustrated in Fig. 10.32.


ROOT
3
15
AVAIL

(a)

	LEFT	DATA	RIGHT
1	–1	8	–1
2	–1	10	–1
3	5	1	8
4			
5	9	2	14
6			
7			
8	20	3	11
9	1	4	12
10			
11	–1	7	18
12	–1	9	–1
13			
14	–1	5	–1
15			
16	–1	11	–1
17			
18	–1	12	–1
19			
20	2	6	16

(b)

	LEFT	DATA	RIGHT
1	–1	8	9
2	–1	10	20
3	5	1	8
4			
5	9	2	14
6			
7			
8	20	3	11
9	1	4	12
10			
11	–1	7	18
12	–1	9	5
13			
14	–1	5	3
15			
16	–1	11	8
17			
18	–1	12	–1
19			
20	2	6	16

(c)

	LEFT	DATA	RIGHT
1	–1	8	9
2	3	10	20
3	5	1	8
4			
5	9	2	14
6			
7			
8	20	3	11
9	1	4	12
10			
11	8	7	18
12	9	9	5
13			
14	5	5	3
15			
16	20	11	8
17			
18	11	12	–1
19			
20	2	6	16

Figure 10.32 Memory representation of binary trees: (a) without threading, (b) with one-way, and (c) two-way threading

10.3.1 Traversing a Threaded Binary Tree

For every node, visit the left sub-tree first, provided if one exists and has not been visited earlier. Then the node (root) itself is followed by visiting its right sub-tree (if one exists). In case there is no right sub-tree, check for the threaded link and make the threaded node the current node in consideration. The algorithm for in-order traversal of a threaded binary tree is given in Fig. 10.33.

```
Step 1: Check if the current node has a left child that has not been visited. If a left child exists
        that has not been visited, go to Step 2, else go to Step 3.
Step 2: Add the left child in the list of visited nodes. Make it as the current node and then go
        to Step 6.
Step 3: If the current node has a right child, go to Step 4 else go to Step 5.
Step 4: Make that right child as current node and go to Step 6.
Step 5: Print the node and if there is a threaded node make it the current node.
Step 6: If all the nodes have visited then END else go to Step 1.
```

Figure 10.33 Algorithm for in-order traversal of a threaded binary tree

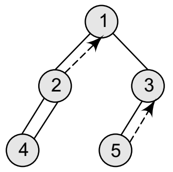


Figure 10.34 Threaded binary tree

Let's consider the threaded binary tree given in Fig. 10.34 and traverse it using the algorithm.

1. Node 1 has a left child i.e., 2 which has not been visited. So, add 2 in the list of visited nodes, make it as the current node.
2. Node 2 has a left child i.e., 4 which has not been visited. So, add 4 in the list of visited nodes, make it as the current node.
3. Node 4 does not have any left or right child, so print 4 and check for its threaded link. It has a threaded link to node 2, so make node 2 the current node.
4. Node 2 has a left child which has already been visited. However, it does not have a right child. Now, print 2 and follow its threaded link to node 1. Make node 1 the current node.
5. Node 1 has a left child that has been already visited. So print 1. Node 1 has a right child 3 which has not yet been visited, so make it the current node.
6. Node 3 has a left child (node 5) which has not been visited, so make it the current node.
7. Node 5 does not have any left or right child. So print 5. However, it does have a threaded link which points to node 3. Make node 3 the current node.
8. Node 3 has a left child which has already been visited. So print 3.
9. Now there are no nodes left, so we end here. The sequence of nodes printed is—4 2 1 5 3.

Advantages of Threaded Binary Tree

- It enables linear traversal of elements in the tree.
- Linear traversal eliminates the use of stacks which in turn consume a lot of memory space and computer time.
- It enables to find the parent of a given element without explicit use of parent pointers.
- Since nodes contain pointers to in-order predecessor and successor, the threaded tree enables forward and backward traversal of the nodes as given by in-order fashion.

Thus, we see the basic difference between a binary tree and a threaded binary tree is that in binary trees a node stores a `NULL` pointer if it has no child and so there is no way to traverse back.

Programming Example

2. Write a program to implement simple right in-threaded binary trees.

```
#include <stdio.h>
#include <conio.h>
struct tree
{
        int val;
        struct tree *right;
        struct tree *left;
        int thread;
};
struct tree *root = NULL;
struct tree* insert_node(struct tree *root, struct tree *ptr, struct tree *rt)
{
        if(root == NULL)
        {
                root = ptr;
                if(rt != NULL)
                {
                        root->right = rt;
                        root->thread = 1;
                }
        }
        else if(ptr->val < root->val)
                root->left = insert_node(root->left, ptr, root);
        else
                if(root->thread == 1)
                {
                        root->right = insert_node(NULL, ptr, rt);
                        root->thread=0;
                }
                else
                        root->right = insert_node(root->right, ptr, rt);
        return root;
}
struct tree* create_threaded_tree()
{
        struct tree *ptr;
        int num;
        printf("\n Enter the elements, press –1 to terminate ");
        scanf("%d", &num);
        while(num != -1)
        {
        ptr = (struct tree*)malloc(sizeof(struct tree));
                ptr->val = num;
                ptr->left = ptr->right = NULL;
                ptr->thread = 0;
                root = insert_node(root, ptr, NULL);
                printf(" \n Enter the next element ");
                fflush(stdin);
                scanf("%d", &num);
        }
        return root;
}
void inorder(struct tree *root)
{
        struct tree *ptr = root, *prev;
        do
        {
                while(ptr != NULL)
                {
                        prev = ptr;
                        ptr = ptr->left;
```

```
                    }
                    if(prev != NULL)
                    {
                            printf(" %d", prev->val);
                            ptr = prev->right;
                            while(prev != NULL && prev->thread)
                            {
                                    printf(" %d", ptr->val);
                                    prev = ptr;
                                    ptr = ptr->right;
                            }
                    }
            }while(ptr != NULL);
    }
    void main()
    {
        //   struct tree *root=NULL;
             clrscr();
             create_threaded_tree();
             printf(" \n The in-order traversal of the tree can be given as : ");
             inorder(root);
             getch();
    }
```

Output

```
Enter the elements, press -1 to terminate 5
Enter the next element 8
Enter the next element 2
Enter the next element 3
Enter the next element 7
Enter the next element -1
The in-order traversal of the tree can be given as:
2       3       5       7       8
```

10.4 AVL TREES

AVL tree is a self-balancing binary search tree invented by G.M. Adelson-Velsky and E.M. Landis in 1962. The tree is named AVL in honour of its inventors. In an AVL tree, the heights of the two sub-trees of a node may differ by at most one. Due to this property, the AVL tree is also known as a height-balanced tree. The key advantage of using an AVL tree is that it takes `O(log n)` time to perform search, insert, and delete operations in an average case as well as the worst case because the height of the tree is limited to `O(log n)`.

The structure of an AVL tree is the same as that of a binary search tree but with a little difference. In its structure, it stores an additional variable called the `BalanceFactor`. Thus, every node has a balance factor associated with it. The balance factor of a node is calculated by subtracting the height of its right sub-tree from the height of its left sub-tree. A binary search tree in which every node has a balance factor of –1, 0, or 1 is said to be height balanced. A node with any other balance factor is considered to be unbalanced and requires rebalancing of the tree.

Balance factor = Height (left sub-tree) – Height (right sub-tree)

- If the balance factor of a node is 1, then it means that the left sub-tree of the tree is one level higher than that of the right sub-tree. Such a tree is therefore called as a *left-heavy tree*.
- If the balance factor of a node is 0, then it means that the height of the left sub-tree (longest path in the left sub-tree) is equal to the height of the right sub-tree.
- If the balance factor of a node is –1, then it means that the left sub-tree of the tree is one level lower than that of the right sub-tree. Such a tree is therefore called as a *right-heavy tree*.

Look at Fig. 10.35. Note that the nodes 18, 39, 54, and 72 have no children, so their balance factor = 0. Node 27 has one left child and zero right child. So, the height of left sub-tree = 1, whereas the height of right sub-tree = 0. Thus, its balance factor = 1. Look at node 36, it has a left sub-tree with height = 2, whereas the height of right sub-tree = 1. Thus, its balance factor = 2 – 1 = 1. Similarly, the balance factor of node 45 = 3 – 2 =1; and node 63 has a balance factor of 0 (1 – 1).

Now, look at Figs 10.35 (a) and (b) which show a right-heavy AVL tree and a balanced AVL tree.

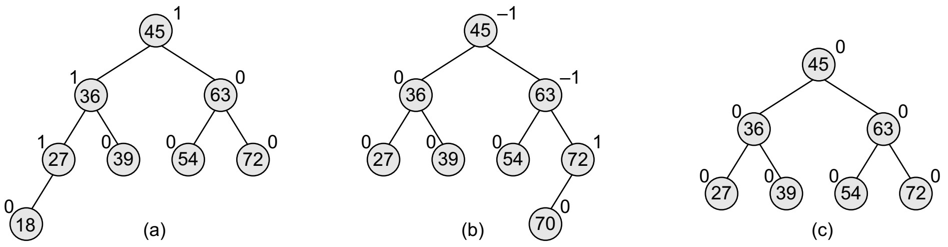


Figure 10.35 (a) Left-heavy AVL tree, (b) right-heavy tree, (c) balanced tree

The trees given in Fig. 10.35 are typical candidates of AVL trees because the balancing factor of every node is either 1, 0, or –1. However, insertions and deletions from an AVL tree may disturb the balance factor of the nodes and, thus, rebalancing of the tree may have to be done. The tree is rebalanced by performing rotation at the critical node. There are four types of rotations: LL rotation, RR rotation, LR rotation, and RL rotation. The type of rotation that has to be done will vary depending on the particular situation. In the following section, we will discuss insertion, deletion, searching, and rotations in AVL trees.

10.4.1 Operations on AVL Trees

Searching for a Node in an AVL Tree

Searching in an AVL tree is performed exactly the same way as it is performed in a binary search tree. Due to the height-balancing of the tree, the search operation takes `O(log n)` time to complete. Since the operation does not modify the structure of the tree, no special provisions are required.

Inserting a New Node in an AVL Tree

Insertion in an AVL tree is also done in the same way as it is done in a binary search tree. In the AVL tree, the new node is always inserted as the leaf node. But the step of insertion is usually followed by an additional step of rotation. Rotation is done to restore the balance of the tree. However, if insertion of the new node does not disturb the balance factor, that is, if the balance factor of every node is still –1, 0, or 1, then rotations are not required.

During insertion, the new node is inserted as the leaf node, so it will always have a balance factor equal to zero. The only nodes whose balance factors will change are those which lie in the path between the root of the tree and the newly inserted node. The possible changes which may take place in any node on the path are as follows:

- Initially, the node was either left- or right-heavy and after insertion, it becomes balanced.
- Initially, the node was balanced and after insertion, it becomes either left- or right-heavy.
- Initially, the node was heavy (either left or right) and the new node has been inserted in the heavy sub-tree, thereby creating an unbalanced sub-tree. Such a node is said to be a *critical node*.

Consider the AVL tree given in Fig. 10.36.

If we insert a new node with the value 30, then the new tree will still be balanced and no rotations will be required in this case. Look at the tree given in Fig. 10.37 which shows the tree after inserting node 30.

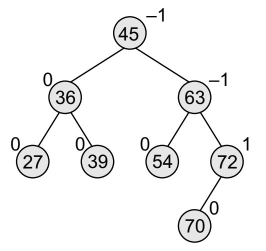


Figure 10.36 AVL tree

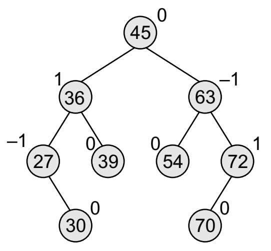


Figure 10.37 AVL tree after inserting a node with the value 30

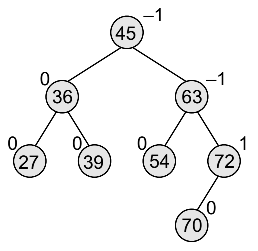


Figure 10.38 AVL tree

Let us take another example to see how insertion can disturb the balance factors of the nodes and how rotations are done to restore the AVL property of a tree. Look at the tree given in Fig. 10.38.

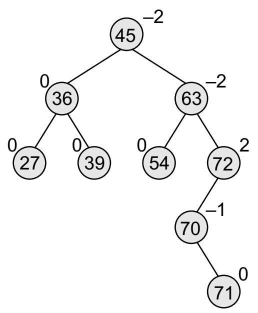


Figure 10.39 AVL tree after inserting a node with the value 71

After inserting a new node with the value 71, the new tree will be as shown in Fig. 10.39. Note that there are three nodes in the tree that have their balance factors 2, –2, and –2, thereby disturbing the *AVLness* of the tree. So, here comes the need to perform rotation. To perform rotation, our first task is to find the critical node. Critical node is the nearest ancestor node on the path from the inserted node to the root whose balance factor is neither –1, 0, nor 1. In the tree given above, the critical node is 72. The second task in rebalancing the tree is to determine which type of rotation has to be done. There are four types of rebalancing rotations and application of these rotations depends on the position of the inserted node with reference to the critical node. The four categories of rotations are:

- *LL rotation* The new node is inserted in the left sub-tree of the left sub-tree of the critical node.
- *RR rotation* The new node is inserted in the right sub-tree of the right sub-tree of the critical node.
- *LR rotation* The new node is inserted in the right sub-tree of the left sub-tree of the critical node.
- *RL rotation* The new node is inserted in the left sub-tree of the right sub-tree of the critical node.

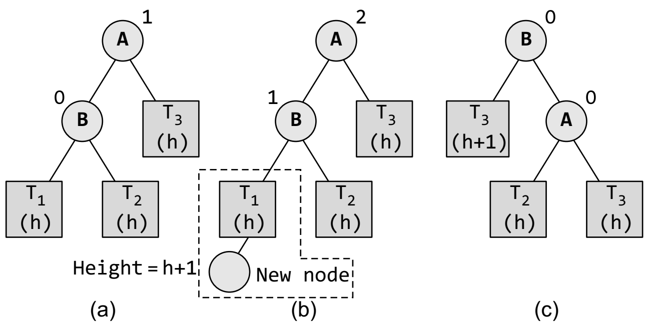


Figure 10.40 LL rotation in an AVL tree

LL Rotation

Let us study each of these rotations in detail. First, we will see where and how LL rotation is applied. Consider the tree given in Fig. 10.40 which shows an AVL tree.

Tree (a) is an AVL tree. In tree (b), a new node is inserted in the left sub-tree of the left sub-tree of the critical node `A` (node `A` is the critical node because it is the closest ancestor whose balance factor is not –1, 0, or 1), so we apply `LL` rotation as shown in tree (c).

Note The new node has now become a part of tree T_1.

While rotation, node B becomes the root, with T_1 and A as its left and right child. T_2 and T_3 become the left and right sub-trees of A.

Example 10.3 Consider the AVL tree given in Fig. 10.41 and insert 18 into it.

Solution

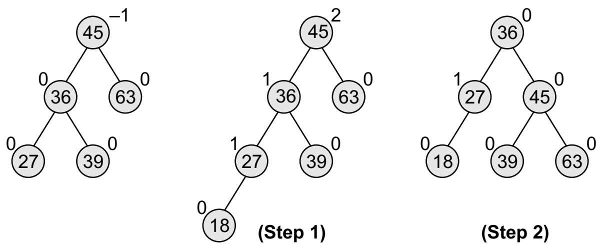


Figure 10.41 AVL tree

RR Rotation

Let us now discuss where and how RR rotation is applied. Consider the tree given in Fig. 10.42 which shows an AVL tree.

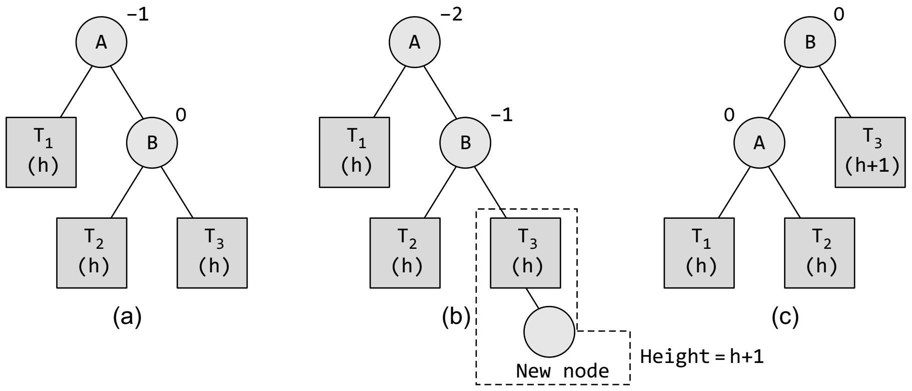


Figure 10.42 RR rotation in an AVL tree

Example 10.4 Consider the AVL tree given in Fig. 10.43 and insert 89 into it.

Solution

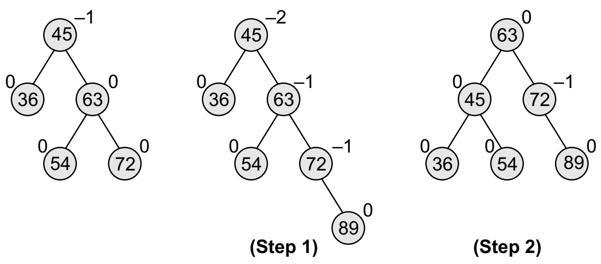


Figure 10.43 AVL tree

Tree (a) is an AVL tree. In tree (b), a new node is inserted in the right sub-tree of the right sub-tree of the critical node A (node A is the critical node because it is the closest ancestor whose balance factor is not –1, 0, or 1), so we apply RR rotation as shown in tree (c). Note that the new node has now become a part of tree T_3.

While rotation, node B becomes the root, with A and T_3 as its left and right child. T_1 and T_2 become the left and right sub-trees of A.

LR and RL Rotations

Consider the AVL tree given in Fig. 10.44 and see how LR rotation is done to rebalance the tree.

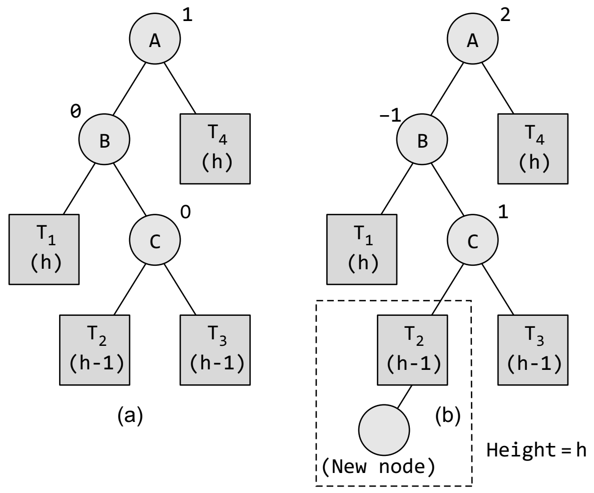


Figure 10.44 LR rotation in an AVL tree

Example 10.5 Consider the AVL tree given in Fig. 10.45 and insert 37 into it.

Solution

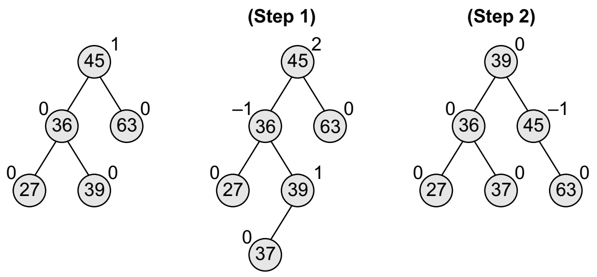


Figure 10.45 AVL tree

Tree (a) is an AVL tree. In tree (b), a new node is inserted in the right sub-tree of the left sub-tree of the critical node A (node A is the critical node because it is the closest ancestor whose balance factor is not –1, 0 or 1), so we apply LR rotation as shown in tree (c). Note that the new node has now become a part of tree T_2.

While rotation, node C becomes the root, with B and A as its left and right children. Node B has T_1 and T_2 as its left and right sub-trees and T_3 and T_4 become the left and right sub-trees of node A.

Now, consider the AVL tree given in Fig. 10.46 and see how RL rotation is done to rebalance the tree.

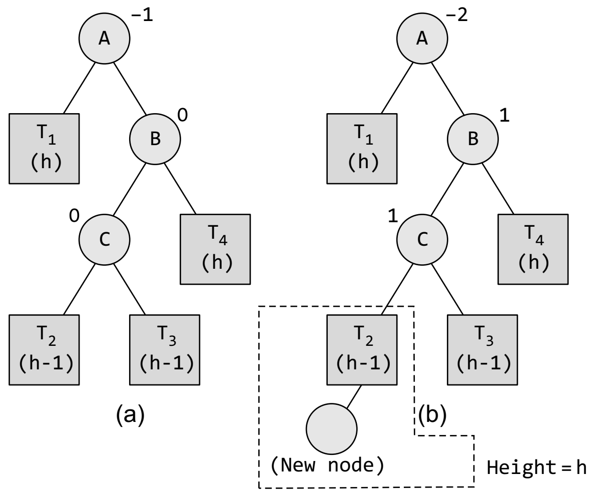


Figure 10.46 RL rotation in an AVL tree

Tree (a) is an AVL tree. In tree (b), a new node is inserted in the left sub-tree of the right sub-tree of the critical node A (node A is the critical node because it is the closest ancestor whose balance factor is not –1, 0, or 1), so we apply RL rotation as shown in tree (c). Note that the new node has now become a part of tree T_2.

While rotation, node C becomes the root, with A and B as its left and right children. Node A has T_1 and T_2 as its left and right sub-trees and T_3 and T_4 become the left and right sub-trees of node B.

Example 10.6 Construct an AVL tree by inserting the following elements in the given order. 63, 9, 19, 27, 18, 108, 99, 81.

Solution

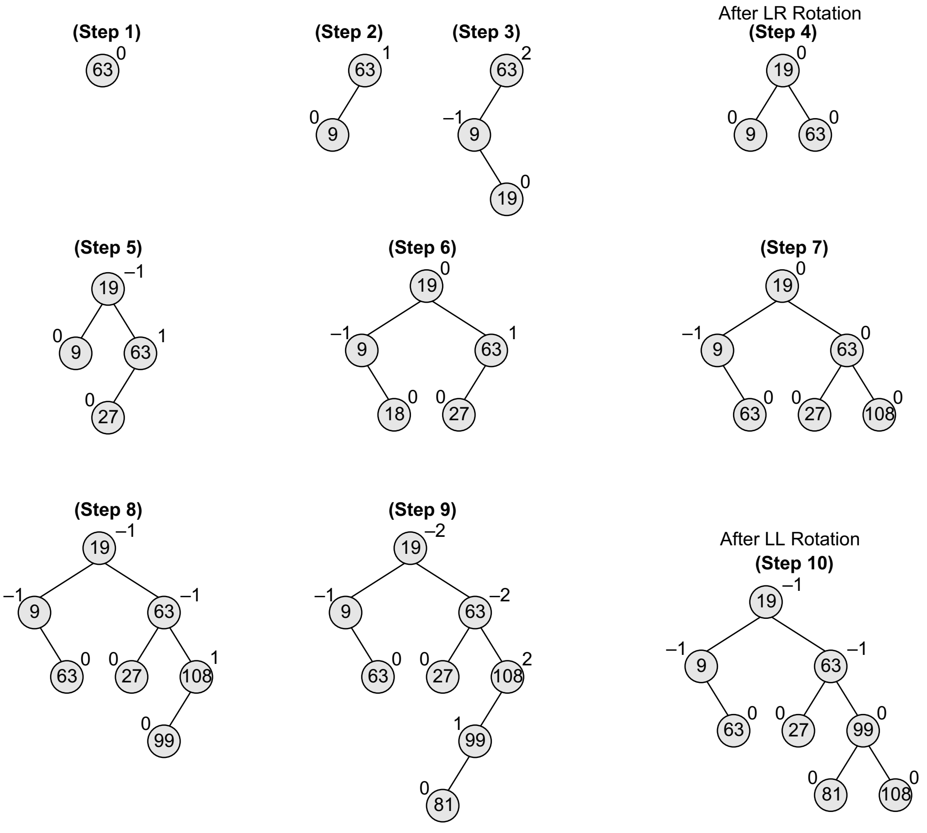


Figure 10.47 AVL tree

Deleting a Node from an AVL Tree

Deletion of a node in an AVL tree is similar to that of binary search trees. But it goes one step ahead. Deletion may disturb the AVLness of the tree, so to rebalance the AVL tree, we need to perform rotations. There are two classes of rotations that can be performed on an AVL tree after deleting a given node. These rotations are R rotation and L rotation.

On deletion of node X from the AVL tree, if node A becomes the critical node (closest ancestor node on the path from X to the root node that does not have its balance factor as 1, 0, or –1), then the type of rotation depends on whether X is in the left sub-tree of A or in its right sub-tree. If the node to be deleted is present in the left sub-tree of A, then L rotation is applied, else if X is in the right sub-tree, R rotation is performed.

Further, there are three categories of L and R rotations. The variations of L rotation are L-1, L0, and L1 rotation. Correspondingly for R rotation, there are R0, R-1, and R1 rotations. In this section, we will discuss only R rotation. L rotations are the mirror images of R rotations.

R0 Rotation

Let B be the root of the left or right sub-tree of A (critical node). R0 rotation is applied if the balance factor of B is 0. This is illustrated in Fig. 10.48.

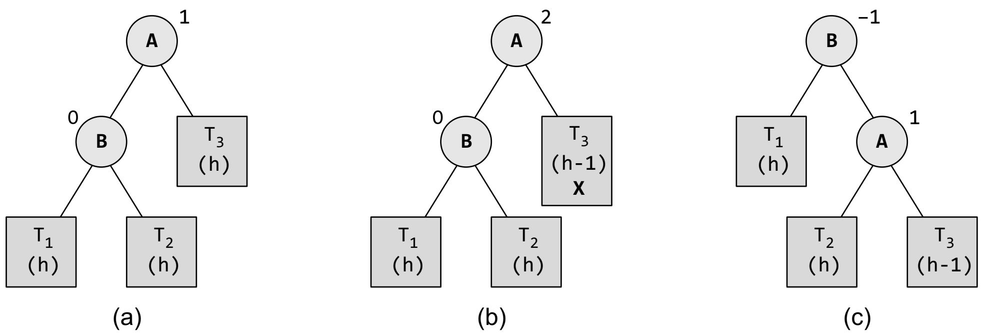


Figure 10.48 R0 rotation in an AVL tree

Tree (a) is an AVL tree. In tree (b), the node X is to be deleted from the right sub-tree of the critical node A (node A is the critical node because it is the closest ancestor whose balance factor is not –1, 0, or 1). Since the balance factor of node B is 0, we apply R0 rotation as shown in tree (c).

During the process of rotation, node B becomes the root, with T_1 and A as its left and right child. T_2 and T_3 become the left and right sub-trees of A.

R1 Rotation

Let B be the root of the left or right sub-tree of A (critical node). R1 rotation is applied if the balance factor of B is 1. Observe that R0 and R1 rotations are similar to LL rotations; the only difference is that R0 and R1 rotations yield different balance factors. This is illustrated in Fig. 10.50.

Tree (a) is an AVL tree. In tree (b), the node X is to be deleted from the right sub-tree of the critical node A (node A is the critical node because it is the closest ancestor whose balance factor is not –1, 0, or 1). Since the balance factor of node B is 1, we apply R1 rotation as shown in tree (c).

During the process of rotation, node B becomes the root, with T_1 and A as its left and right children. T_2 and T_3 become the left and right sub-trees of A.

Example 10.7 Consider the AVL tree given in Fig. 10.49 and delete 72 from it.

Solution

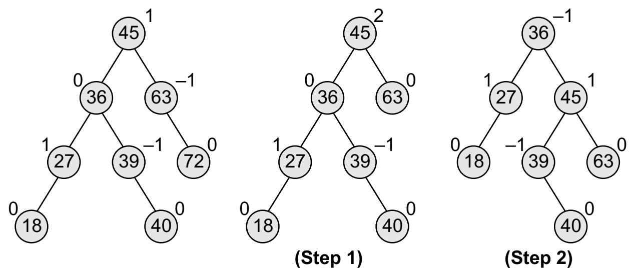


Figure 10.49 AVL tree

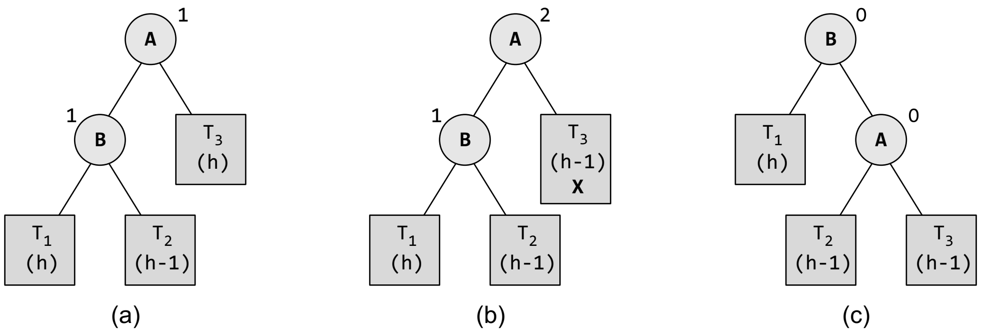


Figure 10.50 R1 rotation in an AVL tree

Example 10.8 Consider the AVL tree given in Fig. 10.51 and delete 72 from it.

Solution

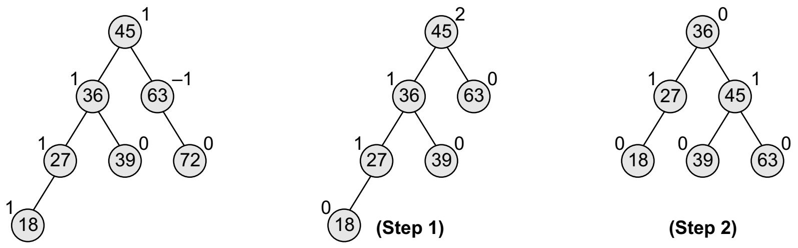


Figure 10.51 AVL tree

R–1 Rotation

Let B be the root of the left or right sub-tree of A (critical node). R-1 rotation is applied if the balance factor of B is –1. Observe that R-1 rotation is similar to LR rotation. This is illustrated in Fig. 10.52.

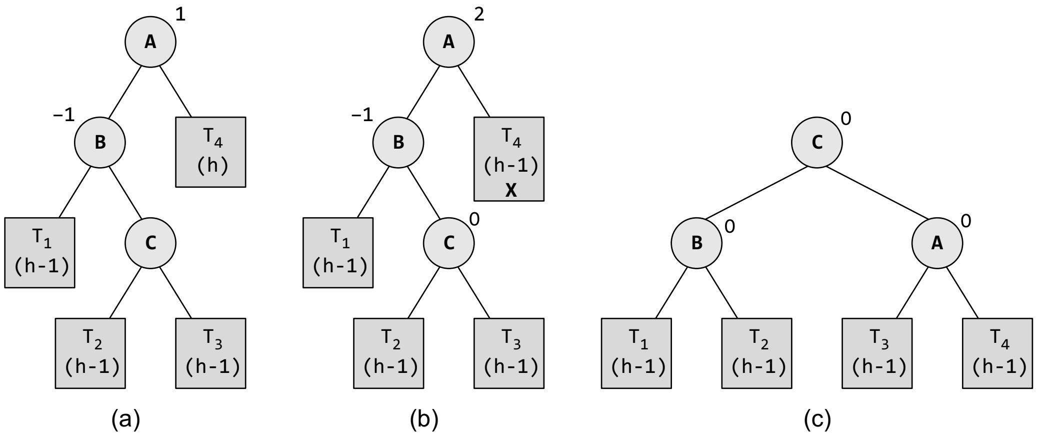


Figure 10.52 R1 Rotation in an AVL tree

Tree (a) is an AVL tree. In tree (b), the node X is to be deleted from the right sub-tree of the critical node A (node A is the critical node because it is the closest ancestor whose balance factor is not –1, 0 or 1). Since the balance factor of node B is –1, we apply R-1 rotation as shown in tree (c).

While rotation, node C becomes the root, with T_1 and A as its left and right child. T_2 and T_3 become the left and right sub-trees of A.

Example 10.9 Consider the AVL tree given in Fig. 10.53 and delete 72 from it.

Solution

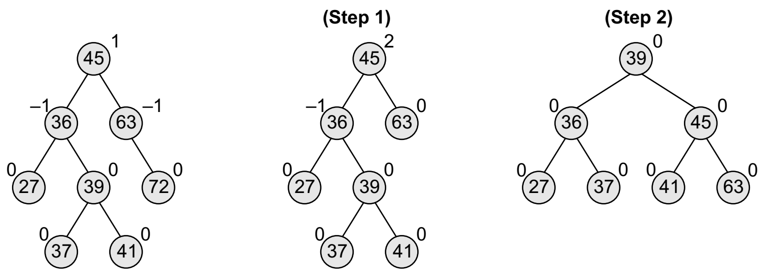


Figure 10.53 AVL tree

Example 10.10 Delete nodes 52, 36, and 61 from the AVL tree given in Fig. 10.54.

Solution

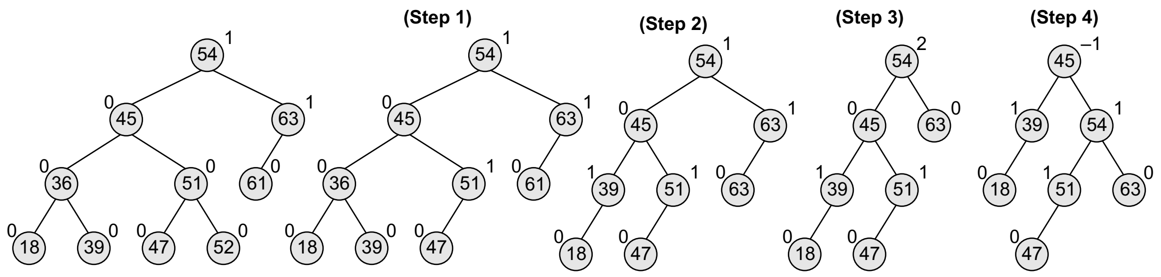


Figure 10.54 AVL tree

Programming Example

3. Write a program that shows insertion operation in an AVL tree.

```
#include <stdio.h>
typedef enum { FALSE ,TRUE } bool;
struct node
{
        int val;
        int balance;
        struct node *left_child;
        struct node *right_child;
};
struct node* search(struct node *ptr, int data)
{
        if(ptr!=NULL)
                if(data < ptr –> val)
                        ptr = search(ptr –> left_child,data);
                else if( data > ptr –> val)
                        ptr = search(ptr –> right_child, data);
        return(ptr);
}
struct node *insert (int data, struct node *ptr, int *ht_inc)
{
        struct node *aptr;
        struct node *bptr;
        if(ptr==NULL)
        {
                ptr = (struct node *) malloc(sizeof(struct node));
                ptr –> val = data;
                ptr –> left_child = NULL;
                ptr –> right_child = NULL;
                ptr –> balance = 0;
                *ht_inc = TRUE;
                return (ptr);
        }
        if(data < ptr –> val)
        {
                ptr –> left_child = insert(data, ptr –> left_child, ht_inc);
                if(*ht_inc==TRUE)
                {
                      switch(ptr –> balance)
                      {
                            case -1: /* Right heavy */
                            ptr –> balance = 0;
                            *ht_inc = FALSE;
                            break;
                            case 0: /* Balanced */
                                  ptr –> balance = 1;
                                  break;
                            case 1: /* Left heavy */
                                  aptr = ptr –> left_child;
                                  if(aptr –> balance == 1)
                                  {
                                        printf("Left to Left Rotation\n");
                                        ptr –> left_child= aptr –> right_child;
                                        aptr –> right_child = ptr;
                                        ptr –> balance = 0;
                                        aptr –> balance=0;
                                        ptr = aptr;
                                  }
                                  else
                                  {
                                        printf("Left to right rotation\n");
                                        bptr = aptr –> right_child;
                                        aptr –> right_child = bptr –> left_child;
                                        bptr –> left_child = aptr;
                                        ptr –> left_child = bptr –> right_child;
                                        bptr –> right_child = ptr;
                                        if(bptr –> balance == 1 )
                                              ptr –> balance = -1;
                                        else
                                              ptr –> balance = 0;
                                        if(bptr –> balance == -1)
                                              aptr –> balance = 1;
                                        else
                                              aptr –> balance = 0;
                                        bptr –> balance=0;
                                        ptr = bptr;
                                  }
                                  *ht_inc = FALSE;
                      }
                }
        }
        if(data > ptr –> val)
        {
                ptr -> right_child = insert(info, ptr -> right_child, ht_inc);
                if(*ht_inc==TRUE)
                {
                      switch(ptr -> balance)
                      {
                            case 1: /* Left heavy */
                                  ptr -> balance = 0;
                                  *ht_inc = FALSE;
                                  break;
                            case 0: /* Balanced */
                                  ptr -> balance = -1;
                                  break;
                            case -1: /* Right heavy */
                                  aptr = ptr -> right_child;
                                  if(aptr -> balance == -1)
                                  {
                                        printf("Right to Right Rotation\n");
                                        ptr -> right_child= aptr -> left_child;
                                        aptr -> left_child = ptr;
                                        ptr -> balance = 0;
                                        aptr -> balance=0;
                                        ptr = aptr;
                                  }
                                  else
                                  {
                                        printf("Right to Left Rotation\n");
                                        bptr = aptr -> left_child;
                                        aptr -> left_child = bptr -> right_child;
                                        bptr -> right_child = aptr;
                                        ptr -> right_child = bptr -> left_child;
                                        bptr -> left_child = pptr;
                                        if(bptr -> balance == -1)
                                              ptr -> balance = 1;
                                        else
                                              ptr -> balance = 0;
                                        if(bptr -> balance == 1)
                                              aptr -> balance = -1;
                                        else
                                              aptr -> balance = 0;
                                        bptr -> balance=0;
                                        ptr = bptr;
                                  }/*End of else*/
                                  *ht_inc = FALSE;
                      }
                }
        }
        return(ptr);
}
void display(struct node *ptr, int level)
{
        int i;
        if ( ptr!=NULL )
        {
                display(ptr -> right_child, level+1);
                printf("\n");
                for (i = 0; i < level; i++)
                      printf(" ");
                printf("%d", ptr -> val);
                display(ptr -> left_child, level+1);
        }
}
void inorder(struct node *ptr)
{
        if(ptr!=NULL)
        {
                inorder(ptr–> left_child);
                printf("%d ",ptr–> val);
                inorder(ptr–> right_child);
        }
}
main()
{
        bool ht_inc;
        int data ;
        int option;
        struct node *root = (struct node *)malloc(sizeof(struct node));
        root = NULL;
        while(1)
        {
                printf("1.Insert\n");
                printf("2.Display\n");
                printf("3.Quit\n");
                printf("Enter your option : ");
                scanf("%d",&option);
                switch(choice)
                {
                      case 1:
                           printf("Enter the value to be inserted : ");
                           scanf("%d", &data);
                                  if( search(root,data) == NULL )
                                        root = insert(data, root, &ht_inc);
                                        else
                                        printf("Duplicate value ignored\n");
                                    break;
                      case 2:
                           if(root==NULL)
                           {
                                  printf("Tree is empty\n");
                                  continue;
                           }
                           printf("Tree is :\n");
                           display(root, 1);
                           printf("\n\n");
                           printf("Inorder Traversal is: ");
                           inorder(root);
                           printf("\n");
                           break;
                      case 3:
                           exit(1);
                           default:
                                  printf("Wrong option\n");
                           }
                }
        }
```

10.5 RED-BLACK TREES

A red-black tree is a self-balancing binary search tree that was invented in 1972 by Rudolf Bayer who called it the 'symmetric binary B-tree'. Although a red-black tree is complex, it has good worst-case running time for its operations and is efficient to use as searching, insertion, and deletion can all be done in `O(log n)` time, where `n` is the number of nodes in the tree. Practically, a red-black tree is a binary search tree which inserts and removes intelligently, to keep the tree reasonably balanced. A special point to note about the red-black tree is that in this tree, no data is stored in the leaf nodes.

10.5.1 Properties of Red-Black Trees

A red-black tree is a binary search tree in which every node has a colour which is either red or black. Apart from the other restrictions of a binary search tree, the red-black tree has the following additional requirements:

1. The colour of a node is either red or black.
2. The colour of the root node is always black.
3. All leaf nodes are black.
4. Every red node has both the children coloured in black.
5. Every simple path from a given node to any of its leaf nodes has an equal number of black nodes.

Look at Fig. 10.55 which shows a red-black tree.

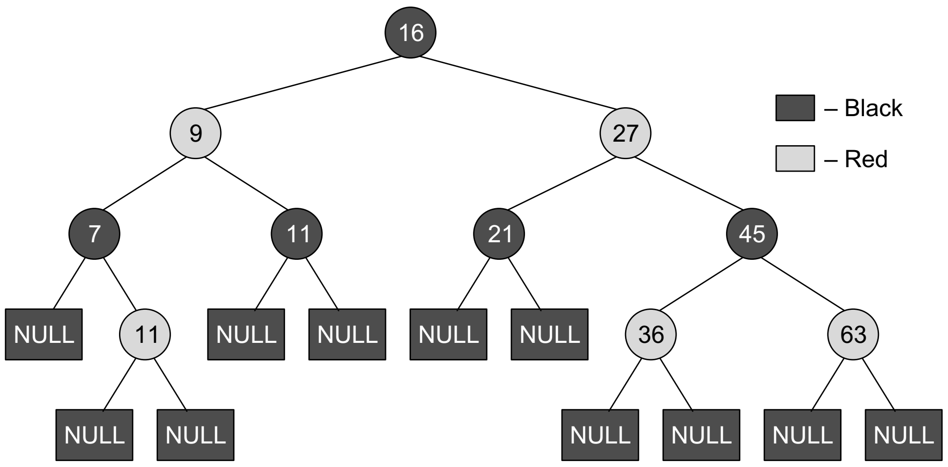


Figure 10.55 Red-black tree

These constraints enforce a critical property of red-black trees. *The longest path from the root node to any leaf node is no more than twice as long as the shortest path from the root to any other leaf in that tree.*

This results in a roughly balanced tree. Since operations such as insertion, deletion, and searching require worst-case times proportional to the height of the tree, this theoretical upper bound on the height allows red-black trees to be efficient in the worst case, unlike ordinary binary search trees.

To understand the importance of these properties, it suffices to note that according to property 4, no path can have two red nodes in a row. The shortest possible path will have all black nodes, and the longest possible path would alternately have a red and a black node. Since all maximal paths have the same number of black nodes (property 5), *no path is more than twice as long as any other path.*

Figure 10.56 shows some binary search trees that are not red-black trees.

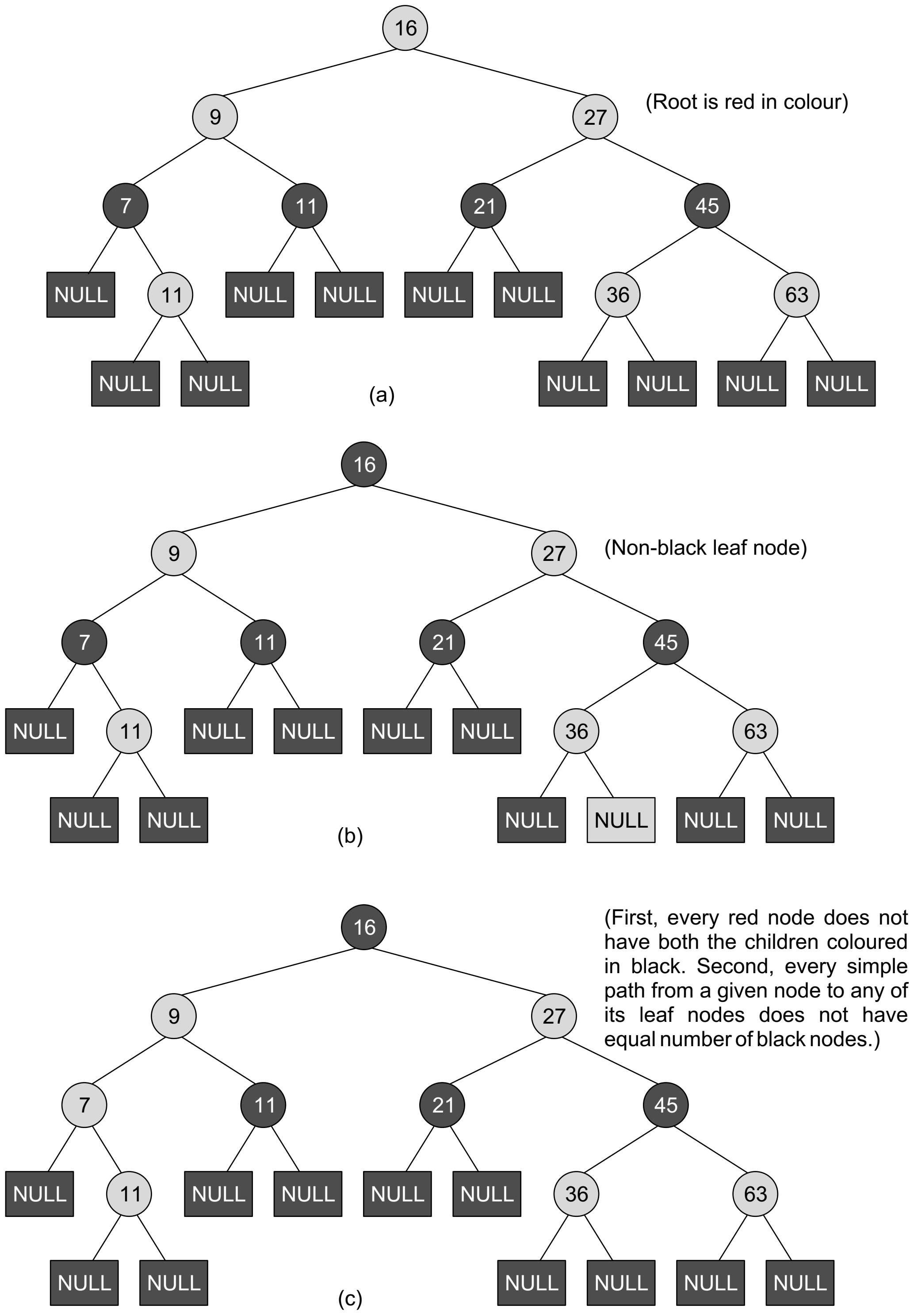

(Root is red in colour)
(a)
(Non-black leaf node)
(b)
(First, every red node does not have both the children coloured in black. Second, every simple path from a given node to any of its leaf nodes does not have equal number of black nodes.)
(c)

Figure 10.56 Trees

10.5.2 Operations on Red-Black Trees

Preforming a read-only operation (like traversing the nodes in a tree) on a red-black tree requires no modification from those used for binary search trees. Remember that every red-black tree is a special case of a binary search tree. However, insertion and deletion operations may violate the properties of a red-black tree. Therefore, these operations may create a need to restore the red-black properties that may require a small number of (`O(log n)` or amortized `O(1)`) colour changes.

Inserting a Node in a Red-Black Tree

The insertion operation starts in the same way as we add a new node in the binary search tree. However, in a binary search tree, we always add the new node as a leaf, while in a red-black tree, leaf nodes contain no data. So instead of adding the new node as a leaf node, we add a red interior node that has two black leaf nodes. Note that the colour of the new node is red and its leaf nodes are coloured in black.

Once a new node is added, it may violate some properties of the red-black tree. So in order to restore their property, we check for certain cases and restore the property depending on the case that turns up after insertion. But before learning these cases in detail, first let us discuss certain important terms that will be used.

Grandparent node (`G`) of a node (`N`) refers to the parent of `N`'s parent (`P`), as in human family trees. The C code to find a node's grandparent can be given as follows:

```
struct node * grand_parent(struct node *n)
{
      // No parent means no grandparent
      if ((n != NULL) && (n -> parent != NULL))
            return n -> parent -> parent;
      else
            return NULL;
}
```

Uncle node (`U`) of a node (`N`) refers to the sibling of `N`'s parent (`P`), as in human family trees. The C code to find a node's uncle can be given as follows:

```
struct node *uncle(struct node *n)
{
      struct node *g;
      g = grand_parent(n);
      //With no grandparent, there cannot be any uncle
      if (g == NULL)
            return NULL;
      if (n -> parent == g -> left)
            return g -> right;
      else
            return g -> left;
}
```

When we insert a new node in a red-black tree, note the following:

- All leaf nodes are always black. So property 3 always holds true.
- Property 4 (both children of every red node are black) is threatened only by adding a red node, repainting a black node red, or a rotation.
- Property 5 (all paths from any given node to its leaf nodes has equal number of black nodes) is threatened only by adding a black node, repainting a red node black, or a rotation.

Case 1: The New Node N is Added as the Root of the Tree

In this case, N is repainted black, as the root of the tree is always black. Since N adds one black node to every path at once, Property 5 is not violated. The C code for case 1 can be given as follows:

```
void case1(struct node *n)
{
      if (n -> parent == NULL) // Root node
            n -> colour = BLACK;
      else
            case2(n);
}
```

Case 2: The New Node's Parent P is Black

In this case, both children of every red node are black, so Property 4 is not invalidated. Property 5 is also not threatened. This is because the new node N has two black leaf children, but because N is red, the paths through each of its children have the same number of black nodes. The C code to check for case 2 can be given as follows:

```
void case2(struct node *n)
{
      if (n -> parent -> colour == BLACK)
            return; /* Red black tree property is not violated*/
      else
            case3(n);
}
```

In the following cases, it is assumed that N *has a grandparent node* G, *because its parent* P *is red, and if it were the root, it would be black. Thus,* N *also has an uncle node* U *(irrespective of whether* U *is a leaf node or an internal node).*

Case 3: If Both the Parent (P) and the Uncle (U) are Red

In this case, Property 5 which says all paths from any given node to its leaf nodes have an equal number of black nodes is violated. Insertion in the third case is illustrated in Fig. 10.57.

In order to restore Property 5, both nodes (P and U) are repainted black and the grandparent G is repainted red. Now, the new red node N has a black parent. Since any path through the parent or uncle must pass through the grandparent, the number of black nodes on these paths has not changed.

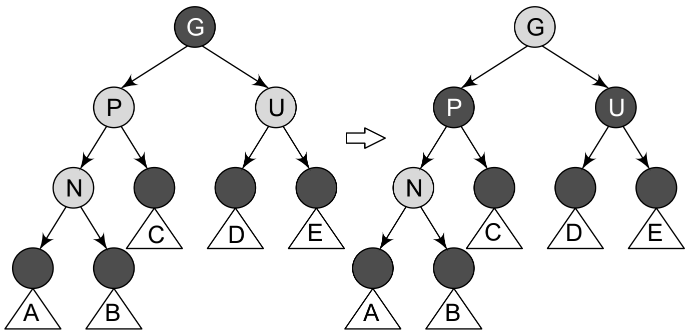


Figure 10.57 Insertion in Case 3

However, the grandparent G may now violate Property 2 which says that the root node is always black or Property 4 which states that both children of every red node are black. Property 4 will be violated when G has a red parent. In order to fix this problem, this entire procedure is recursively performed on G from Case 1. The C code to deal with Case 3 insertion is as follows:

```
void case3(struct node *n)
{
      struct node *u, *g;
      u = uncle (n);
      g = grand_parent(n);
      if ((u != NULL) && (u -> colour == RED)) {
            n -> parent -> colour = BLACK;
            u -> colour = BLACK;
            g -> colour = RED;
            case1(g);
      }
      else {
            insert_case4(n);
      }
}
```

Note In the remaining cases, we assume that the parent node P is the left child of its parent. If it is the right child, then interchange *left* and *right* in cases 4 and 5.

Case 4: The Parent P is Red but the Uncle U is Black and N is the Right Child of P and P is the Left Child of G

In order to fix this problem, a left rotation is done to switch the roles of the new node N and its parent P. After the rotation, note that in the C code, we have re-labelled N and P and then, case 5 is called to deal with the new node's parent. This is done because Property 4 which says both children of every red node should be black is still violated. Figure 10.58 illustrates Case 4 insertion.

Note that in case N is the left child of P and P is the right child of G, we have to perform a right rotation. In the C code that handles Case 4, we check for P and N and then, perform either a left or a right rotation.

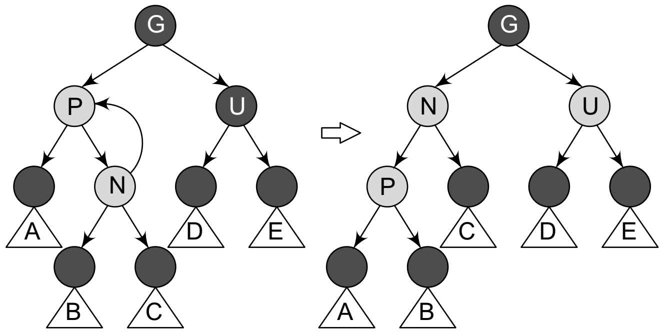


Figure 10.58 Insertion in Case 4

```
void case4(struct node *n)
{
      struct node *g = grand_
parent(n);
      if ((n == n -> parent -> right)
&& (n -> parent == g -> left))
      {
            rotate_left(n -> parent);
            n = n -> left;
      }
      else if ((n == n -> parent -> left) && (n -> parent == g -> right))
      {
            rotate_right(n -> parent);
            n = n -> right;
      }
      case5(n);
}
```

Case 5: The Parent P is Red but the Uncle U is Black and the New Node N is the Left Child of P, and P is the Left Child of its Parent G.

In order to fix this problem, a right rotation on G (the grandparent of N) is performed. After this rotation, the former parent P is now the parent of both the new node N and the former grandparent G.

We know that the colour of G is black (because otherwise its former child P could not have been red), so now switch the colours of P and G so that the resulting tree satisfies Property 4 which states that both children of a red node are black. Case 5 insertion is illustrated in Fig. 10.59.

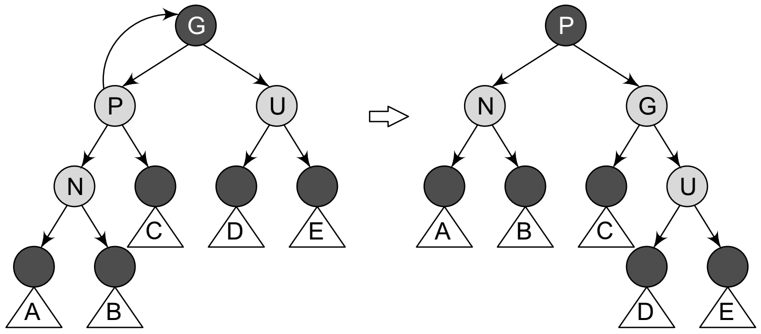


Figure 10.59 Insertion in case 5

Note that in case `N` is the right child of `P` and `P` is the right child of `G`, we perform a left rotation. In the `C` code that handles Case 5, we check for `P` and `N` and then, perform either a left or a right rotation.

```
void case5(struct node *n)
{
      struct node *g;
      g = grandparent(n);
      if ((n == n -> parent -> left) && (n -> parent == g -> left))
            rotate_right(g);
      else if ((n == n -> parent -> right) && (n -> parent == g -> right))
            rotate_left(g);
      n -> parent -> colour = BLACK;
      g -> colour = RED;
}
```

Deleting a Node from a Red-Black Tree

We start deleting a node from a red-black tree in the same way as we do in case of a binary search tree. In a binary search tree, when we delete a node with two non-leaf children, we find either the maximum element in its left sub-tree of the node or the minimum element in its right sub-tree, and move its value into the node being deleted. After that, we delete the node from which we had copied the value. Note that this node must have less than two non-leaf children. Therefore, merely copying a value does not violate any red-black properties, but it just reduces the problem of deleting to the problem of deleting a node with at most one non-leaf child.

In this section, we will assume that we are deleting a node with at most one non-leaf child, which we will call its child. In case this node has both leaf children, then let one of them be its child.

While deleting a node, if its colour is red, then we can simply replace it with its child, which must be black. All paths through the deleted node will simply pass through one less red node, and both the deleted node's parent and child must be black, so none of the properties will be violated.

Another simple case is when we delete a black node that has a red child. In this case, property 4 and property 5 could be violated, so to restore them, just repaint the deleted node's child with black.

However, a complex situation arises when both the node to be deleted as well as its child is black. In this case, we begin by replacing the node to be deleted with its child. In the C code, we label the child node as (in its new position) `N`, and its sibling (its new parent's other child) as `S`. The C code to find the sibling of a node can be given as follows:

```
struct node *sibling(struct node *n)
{
      if (n == n -> parent -> left)
            return n -> parent -> right;
      else
```

```
            return n -> parent -> left;
}
```

We can start the deletion process by using the following code, where the function `replace_node` substitutes the `child` into `N's` place in the tree. For convenience, we assume that null leaves are represented by actual node objects, rather than `NULL`.

```
void delete_child(struct node *n)
{
      /* If N has at most one non-null child */
      struct node *child;
      if (is_leaf(n -> right))
            child = n -> left;
      else
            child = n -> right;
      replace_node(n, child);
      if (n -> colour == BLACK) {
            if (child -> colour == RED)
                    child -> colour = BLACK;
            else
                    del_case1(child);
      }
      free(n);
}
```

When both `N` and its parent `P` are black, then deleting `P` will cause paths which precede through `N` to have one fewer black nodes than the other paths. This will violate Property 5. Therefore, the tree needs to be rebalanced. There are several cases to consider, which are discussed below.

Case 1: N is the New Root

In this case, we have removed one black node from every path, and the new root is black, so none of the properties are violated.

```
void del_case1(struct node *n)
{
      if (n -> parent != NULL)
            del_case2(n);
}
```

Note In cases 2, 5, and 6, we assume N is the left child of its parent P. If it is the right child, left and right should be interchanged throughout these three cases.

Case 2: Sibling S is Red

In this case, interchange the colours of `P` and `S`, and then rotate left at `P`. In the resultant tree, `S` will become `N`'s grandparent. Figure 10.60 illustrates Case 2 deletion.

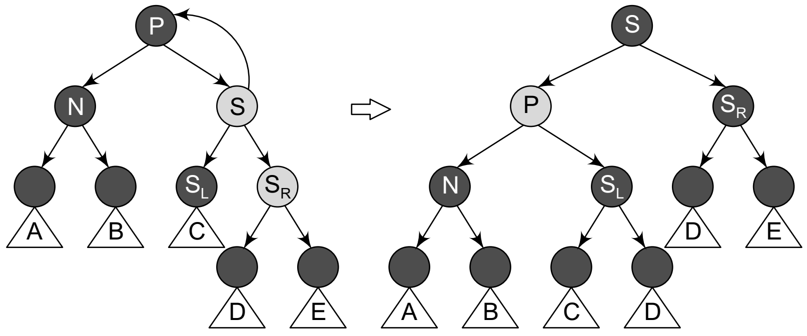


Figure 10.60 Deletion in case 2

The C code that handles case 2 deletion can be given as follows:

```
void del_case2(struct node *n)
{
      struct node *s;
      s = sibling(n);
      if (s –> colour == RED)
      {
            if (n == n –> parent –> left)
                  rotate_left(n –> parent);
            else
                  rotate_right(n –> parent);
            n –> parent –> colour = RED;
            s –> colour = BLACK;
      }
      del_case3(n);
}
```

Case 3: P, S, and S's Children are Black

In this case, simply repaint `S` with red. In the resultant tree, all the paths passing through `S` will have one less black node. Therefore, all the paths that pass through `P` now have one fewer black nodes than the paths that do not pass through `P`, so Property 5 is still violated. To fix this problem, we perform the rebalancing procedure on `P`, starting at Case 1. Case 3 is illustrated in Fig. 10.61.

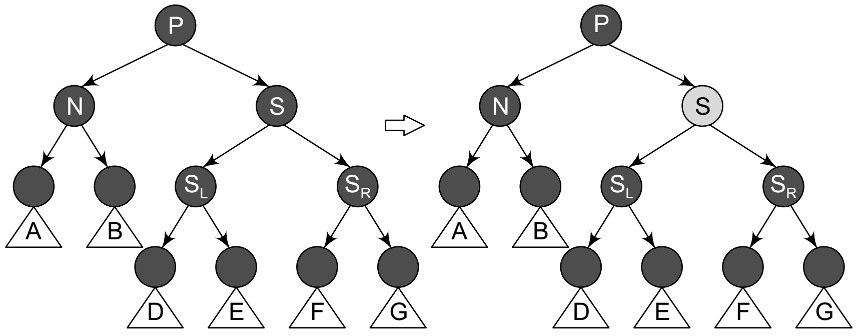


Figure 10.61 Insertion in case 3

The C code for Case 3 can be given as follows:

```
void del_case3(struct node *n)
{
      struct node *s;
      s = sibling(n);
      if ((n–>parent–>colour == BLACK) && (s–>colour == BLACK) && (s–>left–>colour == BLACK) &&
(s –> right –> colour == BLACK))
      {
            s –> colour = RED;
            del_case1(n –> parent);
      } else
            del_case4(n);
}
```

Case 4: S and S's Children are Black, but P is Red

In this case, we interchange the colours of `S` and `P`. Although this will not affect the number of black nodes on the paths going through `S`, it will add one black node to the paths going through `N`, making up for the deleted black node on those paths. Figure 10.62 illustrates this case.

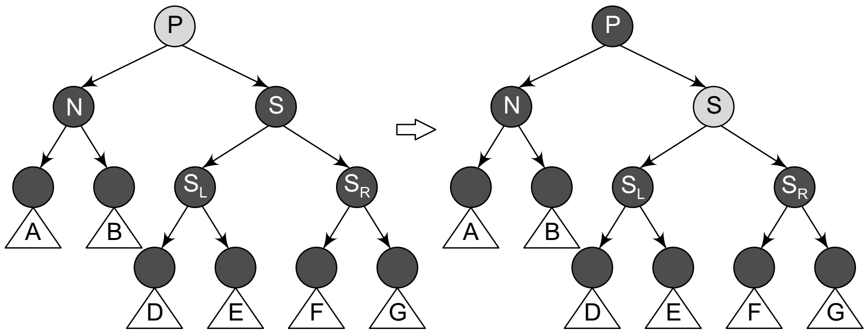


Figure 10.62 Insertion in case 4

The C code to handle Case 4 is as follows:

```
    void del_case4(struct node *n)
    {
          struct node *s;
          s = sibling(n);
          if ((n -> parent -> color == RED) && (s -> colour == BLACK) &&
(s -> left -> colour == BLACK) && (s -> right -> colour == BLACK))
          {
                s -> colour = RED;
                n -> parent -> colour = BLACK;
          } else
                del_case5(n);
    }
```

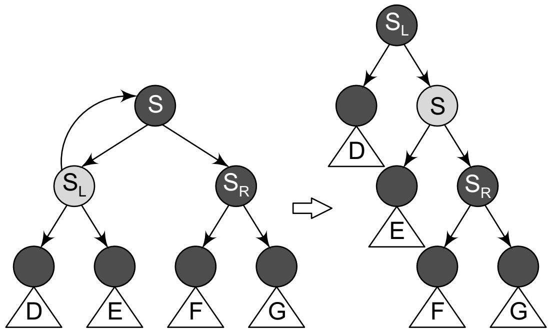


Figure 10.63 Insertion in case 5

Case 5: N is the Left Child of P and S is Black, S's Left Child is Red, S's Right Child is Black.

In this case, perform a right rotation at S. After the rotation, S's left child becomes S's parent and N's new sibling. Also, interchange the colours of S and its new parent.

Note that now all paths still have equal number of black nodes, but N has a black sibling whose right child is red, so we fall into Case 6. Refer Fig. 10.63.

The C code to handle case 5 is given as follows:

```
    void del_case5(struct node *n)
    {
          struct node *s;
          s = sibling(n);
          if (s -> colour == BLACK)
          {
          /* the following code forces the red to be on the left of the left of the parent,
or right of the right, to be correctly operated in case 6. */
                if ((n == n -> parent -> left) && (s -> right -> colour == BLACK) && (s -> left
-> colour == RED))
                            rotate_right(s);
                else if ((n == n -> parent -> right) && (s -> left -> colour == BLACK) && (s ->
right -> colour == RED))
                      rotate_left(s);
                s -> colour = RED;
                s -> right -> colour = BLACK;
        }
        del_case6(n);
}
```

Case 6: S is Black, S's Right Child is Red, and N is the Left Child of its Parent P

In this case, a left rotation is done at `P` to make `S` the parent of `P` and `S`'s right child. After the rotation, the colours of `P` and `S` are interchanged and `S`'s right child is coloured black. Once these steps are followed, you will observe that property 4 and property 5 remain valid. Case 6 is explained in Fig. 10.64.

The C code to fix Case 6 can be given as follows:

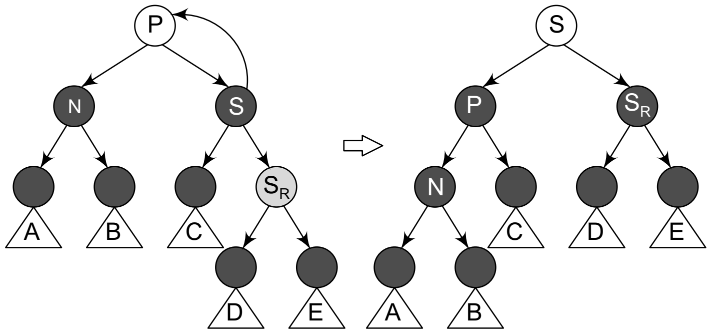


Figure 10.64 Insertion in case 6

```
Void del_case6(struct node *n)
{
        struct node *s;
        s = sibling(n);
        s -> colour = n -> parent -> colour;
        n -> parent -> colour = BLACK;
        if (n == n -> parent -> left) {
                s -> right -> colour = BLACK;
                rotate_left(n -> parent);
        } else {
                s -> left -> colour = BLACK;
                rotate_right(n -> parent);
        }
}
```

10.5.3 Applications of Red-Black Trees

Red-black trees are efficient binary search trees, as they offer worst case time guarantee for insertion, deletion, and search operations. Red-black trees are not only valuable in time-sensitive applications such as real-time applications, but are also preferred to be used as a building block in other data structures which provide worst-case guarantee.

AVL trees also support `O(log n)` search, insertion, and deletion operations, but they are more rigidly balanced than red-black trees, thereby resulting in slower insertion and removal but faster retrieval of data.

10.6 SPLAY TREES

Splay trees were invented by Daniel Sleator and Robert Tarjan. A splay tree is a self-balancing binary search tree with an additional property that recently accessed elements can be re-accessed fast. It is said to be an efficient binary tree because it performs basic operations such as insertion, search, and deletion in `O(log(n))` amortized time. For many non-uniform sequences of operations, splay trees perform better than other search trees, even when the specific pattern of the sequence is unknown.

A splay tree consists of a binary tree, with no additional fields. When a node in a splay tree is accessed, it is rotated or 'splayed' to the root, thereby changing the structure of the tree. Since the most frequently accessed node is always moved closer to the starting point of the search (or the root node), these nodes are therefore located faster. A simple idea behind it is that if an element is accessed, it is likely that it will be accessed again.

In a splay tree, operations such as insertion, search, and deletion are combined with one basic operation called *splaying*. Splaying the tree for a particular node rearranges the tree to place that node at the root. A technique to do this is to first perform a standard binary tree search for that node and then use rotations in a specific order to bring the node on top.

10.6.1 Operations on Splay Trees

In this section, we will discuss the four main operations that are performed on a splay tree. These include splaying, insertion, search, and deletion.

Splaying

When we access a node `N`, splaying is performed on `N` to move it to the root. To perform a splay operation, certain *splay steps* are performed where each step moves `N` closer to the root. Splaying a particular node of interest after every access ensures that the recently accessed nodes are kept closer to the root and the tree remains roughly balanced, so that the desired amortized time bounds can be achieved.

Each splay step depends on three factors:

- Whether `N` is the left or right child of its parent `P`,
- Whether `P` is the root or not, and if not,
- Whether `P` is the left or right child of its parent, `G` (`N`'s grandparent).

Depending on these three factors, we have one splay step based on each factor.

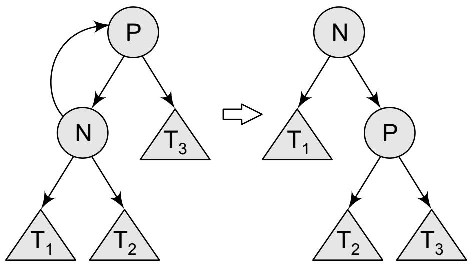


Figure 10.65 Zig step

Zig step The `zig` operation is done when `P` (the parent of `N`) is the root of the splay tree. In the `zig` step, the tree is rotated on the edge between `N` and `P`. `Zig` step is usually performed as the last step in a splay operation and only when `N` has an odd depth at the beginning of the operation. Refer Fig. 10.65.

Zig-zig step The `zig-zig` operation is performed when `P` is not the root. In addition to this, `N` and `P` are either both right or left children of their parents. Figure 10.66 shows the case where `N` and `P` are the left children. During the `zig-zig` step, first the tree is rotated on the edge joining `P` and its parent `G`, and then again rotated on the edge joining `N` and `P`.

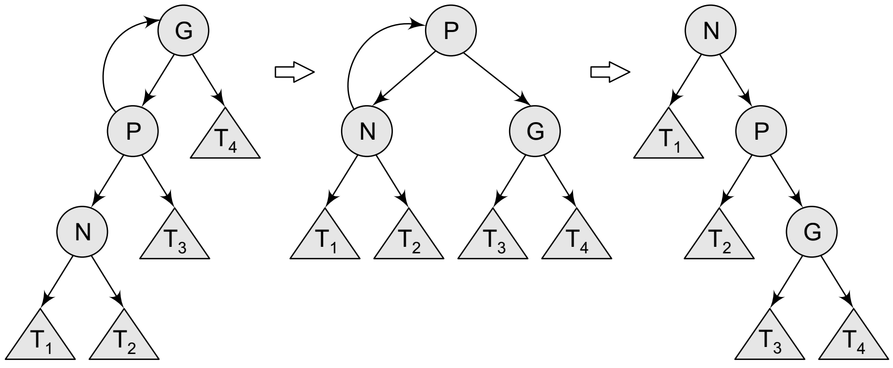


Figure 10.66 Zig-zig step

Zig-zag step The `zig-zag` operation is performed when `P` is not the root. In addition to this, `N` is the right child of `P` and `P` is the left child of `G` or vice versa. In `zig-zag` step, the tree is first rotated on the edge between `N` and `P`, and then rotated on the edge between `N` and `G`. Refer Fig.10.67.

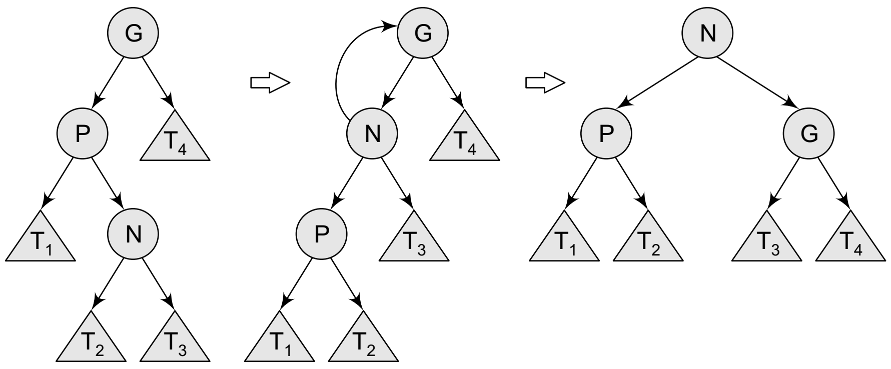


Figure 10.67 Zig-zag step

Inserting a Node in a Splay Tree

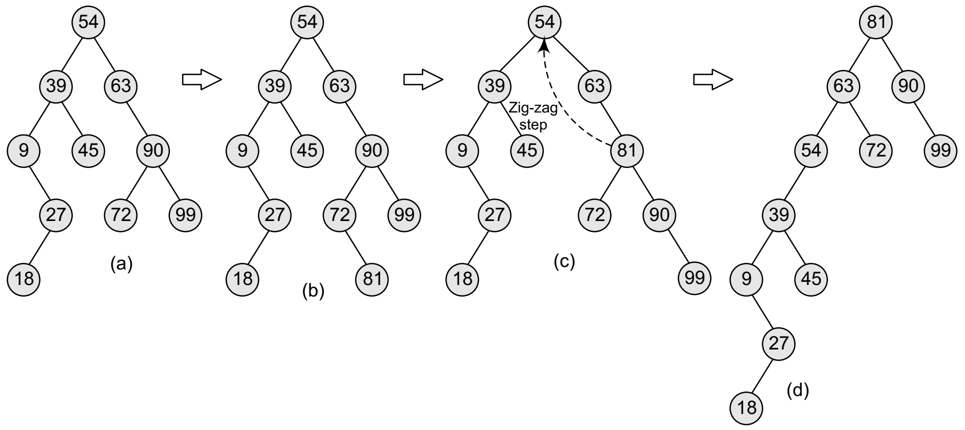
Although the process of inserting a new node `N` into a splay tree begins in the same way as we insert a node in a binary search tree, but after the insertion, `N` is made the new root of the splay tree. The steps performed to insert a new node `N` in a splay tree can be given as follows:

Step 1 Search `N` in the splay tree. If the search is successful, splay at the node `N`.

Step 2 If the search is unsuccessful, add the new node `N` in such a way that it replaces the `NULL` pointer reached during the search by a pointer to a new node `N`. Splay the tree at `N`.

Example 10.11 Consider the splay tree given on the left. Observe the change in its structure when 81 is added to it.

Solution



Note To get the final splay tree, first apply zig-zag step on 81. Then apply zig-zag step to make 81 the root node.

Example 10.12 Consider the splay tree given in Fig. 10.68. Observe the change in its structure when a node containing 81 is searched in the tree.

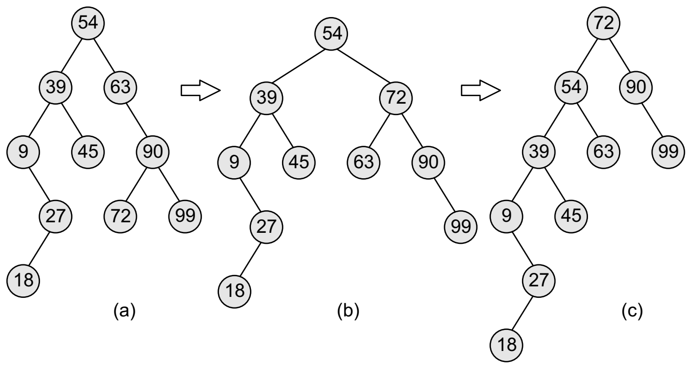


Figure 10.69 Splay tree

Searching for a Node in a Splay Tree

If a particular node N is present in the splay tree, then a pointer to N is returned; otherwise a pointer to the null node is returned. The steps performed to search a node N in a splay tree include:

- Search down the root of the splay tree looking for N.
- If the search is successful, and we reach N, then splay the tree at N and return a pointer to N.
- If the search is unsuccessful, i.e., the splay tree does not contain N, then we reach a null node. Splay the tree at the last non-null node reached during the search and return a pointer to null.

Deleting a Node from a Splay Tree

To delete a node N from a splay tree, we perform the following steps:

- Search for N that has to be deleted. If the search is unsuccessful, splay the tree at the last non-null node encountered during the search.
- If the search is successful and N is not the root node, then let P be the parent of N. Replace N by an appropriate descendent of P (as we do in binary search tree). Finally splay the tree at P.

Example 10.13 Consider the splay tree at the left. When we delete node 39 from it, the new structure of the tree can be given as shown in the right side of Fig. 10.70(a).

After splaying the tree at P, the resultant tree will be as shown in Fig. 10.70(b):

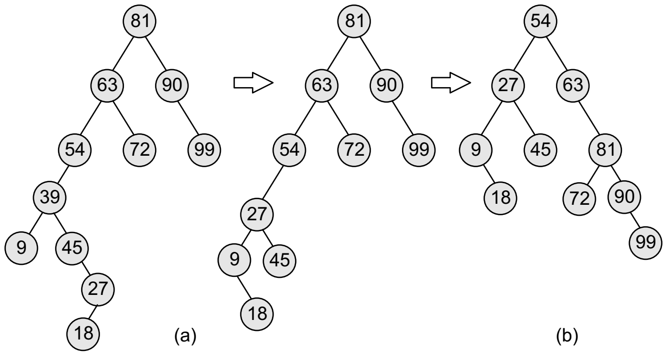


Figure 10.70 (a) Splay tree (b) Splay tree

10.6.2 Advantages and Disadvantages of Splay Trees

The advantages of using a splay tree are:

- A splay tree gives good performance for search, insertion, and deletion operations. This advantage centres on the fact that the splay tree is a self-balancing and a self-optimizing data structure in which the frequently accessed nodes are moved closer to the root so that they can be accessed quickly. This advantage is particularly useful for implementing caches and garbage collection algorithms.
- Splay trees are considerably simpler to implement than the other self-balancing binary search trees, such as red-black trees or AVL trees, while their average-case performance is just as efficient.
- Splay trees minimize memory requirements as they do not store any book-keeping data.
- Unlike other types of self-balancing trees, splay trees provide good performance (amortized `O(log n)`) with nodes containing identical keys.

However, the demerits of splay trees include:

- While sequentially accessing all the nodes of a tree in a sorted order, the resultant tree becomes completely unbalanced. This takes `n` accesses of the tree in which each access takes `O(log n)` time. For example, re-accessing the first node triggers an operation that in turn takes `O(n)` operations to rebalance the tree before returning the first node. Although this creates a significant delay for the final operation, the amortized performance over the entire sequence is still `O(log n)`.
- For uniform access, the performance of a splay tree will be considerably worse than a somewhat balanced simple binary search tree. For uniform access, unlike splay trees, these other data structures provide worst-case time guarantees and can be more efficient to use.

Points to Remember

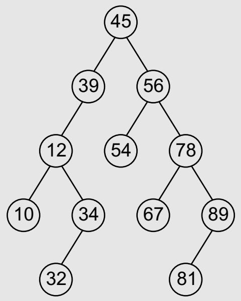
- A binary search tree, also known as an ordered binary tree, is a variant of binary tree in which all the nodes in the left sub-tree have a value less than that of the root node and all the nodes in the right sub-tree have a value either equal to or greater than the root node.
- The average running time of a search operation is $O(\log_2 n)$. However, in the worst case, a binary search tree will take `O(n)` time to search an element from the tree.
- Mirror image of a binary search tree is obtained by interchanging the left sub-tree with the right sub-tree at every node of the tree.
- In a threaded binary tree, null entries can be replaced to store a pointer to either the in-order predecessor or in-order successor of a node.
- A one-way threaded tree is also called a single threaded tree. In a two-way threaded tree, also called a double threaded tree, threads will appear in both the left and the right field of the node.
- An AVL tree is a self-balancing tree which is also known as a height-balanced tree. Each node has a balance factor associated with it, which is calculated by subtracting the height of the right sub-tree from the height of the left sub-tree. In a height balanced tree, every node has a balance factor of either 0, 1, or –1.
- A red-black tree is a self-balancing binary search tree which is also called as a 'symmetric binary B-tree'. Although a red-black tree is complex, it has good worst case running time for its operations and is efficient to use, as searching, insertion, and deletion can all be done in `O(log n)` time.
- A splay tree is a self-balancing binary search tree with an additional property that recently accessed elements can be re-accessed fast.

Exercises

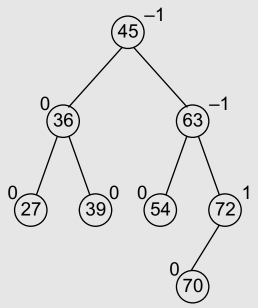
Review Questions

1. Explain the concept of binary search trees.
2. Explain the operations on binary search trees.
3. How does the height of a binary search tree affect its performance?
4. How many nodes will a complete binary tree with 27 nodes have in the last level? What will be the height of the tree?
5. Write a short note on threaded binary trees.
6. Why are threaded binary trees called efficient binary trees? Give the merits of using a threaded binary tree.
7. Discuss the advantages of an AVL tree.
8. How is an AVL tree better than a binary search tree?
9. How does a red-black tree perform better than a binary search tree?
10. List the merits and demerits of a splay tree.
11. Create a binary search tree with the input given below:
    98, 2, 48, 12, 56, 32, 4, 67, 23, 87, 23, 55, 46
    (a) Insert 21, 39, 45, 54, and 63 into the tree
    (b) Delete values 23, 56, 2, and 45 from the tree

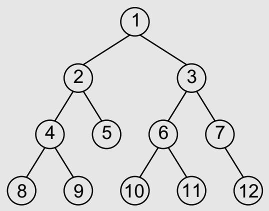
12. Consider the binary search tree given below. Now do the following operations:
    - Find the result of in-order, pre-order, and post-order traversals.
    - Show the deletion of the root node
    - Insert 11, 22, 33, 44, 55, 66, and 77 in the tree



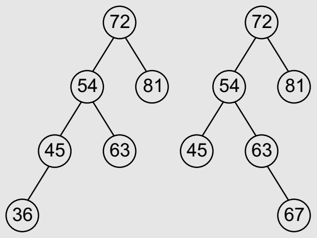
13. Consider the AVL tree given below and insert 18, 81, 29, 15, 19, 25, 26, and 1 in it.
    Delete nodes 39, 63, 15, and 1 from the AVL tree formed after solving the above question.



14. Discuss the properties of a red-black tree. Explain the insertion cases.
15. Explain splay trees in detail with relevant examples.
16. Provide the memory representation of the binary tree given below:
    - Find the result of one-way in-order, one-way pre-order, and two-way in-order threading of the tree.
    - In each case, draw the tree and also give its memory representation.



17. Balance the AVL trees given below.



18. Create an AVL tree using the following sequence of data: 16, 27, 9, 11, 36, 54, 81, 63, 72.
19. Draw all possible binary search trees of 7, 9, and 11.

Programming Exercises

1. Write a program to insert and delete values from a binary search tree.
2. Write a program to count the number of nodes in a binary search tree.

Multiple-choice Questions

1. In the worst case, a binary search tree will take how much time to search an element?
   (a) `O(n)` (b) `O(log n)`
   (c) $O(n^2)$ (d) `O(n log n)`
2. How much time does an AVL tree take to perform search, insert, and delete operations in the average case as well as the worst case?
   (a) `O(n)` (b) `O(log n)`
   (c) $O(n^2)$ (d) `O(n log n)`
3. When the left sub-tree of the tree is one level higher than that of the right sub-tree, then the balance factor is
   (a) 0 (b) 1
   (c) –1 (d) 2
4. Which rotation is done when the new node is inserted in the right sub-tree of the right sub-tree of the critical node?
   (a) LL (b) LR
   (c) RL (d) RR
5. When a node N is accessed it is splayed to make it the
   (a) Root node (b) Parent node
   (c) Child node (d) Sibling node

True or False

1. In a binary search tree, all the nodes in the left sub-tree have a value less than that of the root node.

2. If we take two empty binary search trees and insert the same elements but in a different order, then the resultant trees will be the same.
3. When we insert a new node in a binary search tree, it will be added as an internal node.
4. Mirror image of a binary search tree is obtained by interchanging the left sub-tree with the right sub-tree at every node of the tree.
5. If the thread appears in the right field, then it will point to the in-order successor of the node.
6. If the node to be deleted is present in the left sub-tree of `A`, then `R` rotation is applied.
7. Height of an AVL tree is limited to `O(log n)`.
8. Critical node is the nearest ancestor node on the path from the root to the inserted node whose balance factor is –1, 0, or 1.
9. RL rotation is done when the new node is inserted in the right sub-tree of the right sub-tree of the critical node.
10. In a red-black tree, some leaf nodes can be red.

Fill in the Blanks

1. ______ is also called a fully threaded binary tree.
2. To find the node with the largest value, we will find the value of the rightmost node of the ______.
3. If the thread appears in the right field, then it will point to the ______ of the node.
4. The balance factor of a node is calculated by ______.
5. Balance factor –1 means ______.
6. Searching an AVL tree takes ______ time.
7. ______ rotation is done when the new node is inserted in the left sub-tree of the left sub-tree of the critical node.
8. In a red-black tree, the colour of the root node is ______ and the colour of leaf node is ______.
9. The zig operation is done when ______.
10. In splay trees, rotation is analogous to _____ operation.