

Supporting materials for “Novel approaches of synchronous averaging of gear and bearing vibrations”

In this file, some extensions of “Novel approaches of synchronous averaging of gear and bearing vibrations” are presented. First, in Section 1, the synchronous averaging (SA) process is analyzed as a continuous filter. Then, in Section 2, the SA using the order domain is presented with more details. In Section 3, the reduction of interferences and random noise is analyzed based on the SA via the order. In Section 4, more examples from the gear experiment are presented. In Section 5, the proof for the delays as a filtering mechanism is discussed, and finally, in Section 6, more examples from the bearing experiments are given.

1. Moving average filter

Traditionally, SA analysis as a continuous filter is based on the Z-transform [1]. However, due to convergence properties that may not hold, we will present a different analysis that leads to the same results. Equation 1 illustrates the effect of SA on an exponential signal, namely $\exp(2\pi i \cdot order \cdot t)$. Not surprisingly, due to the property of linear time invariance of the filter, it is an eigenvector. Then, based on Equation 1, Equation 2 presents the filter magnitude in the order domain. Equation 3 develops the expressions from Equation 2, enabling refinement in Equation 4. This is the final result presented in the paper.

$$\begin{aligned} y(t) &= \frac{1}{M} \sum_{m=0}^{M-1} x(t - m \cdot N \cdot \Delta cyc) = \frac{1}{M} \sum_{m=0}^{M-1} \exp(2\pi i \cdot order \cdot (t - m \cdot 1)) = \\ &= \frac{1}{M} \exp(2\pi i \cdot order \cdot t) \sum_{m=0}^{M-1} (\exp(-2\pi i \cdot order))^m = \\ &= \frac{1}{M} \cdot \frac{(\exp(-2\pi i \cdot order))^M - 1}{\exp(-2\pi i \cdot order) - 1} \cdot \exp(2\pi i \cdot order \cdot t) \end{aligned} \quad 1$$

$$|H(order)| = \left| \frac{Y(order)}{X(order)} \right| = \frac{1}{M} \left| \frac{(\exp(-2\pi i \cdot order))^M - 1}{\exp(-2\pi i \cdot order) - 1} \right| \quad 2$$

$$\begin{aligned} |(\exp(-2\pi i \cdot order))^L - 1| &= \sqrt{(\exp(-2\pi i \cdot L \cdot order) - 1)(\exp(2\pi i \cdot L \cdot order) - 1)} \\ &= \sqrt{1 - \exp(-2\pi i \cdot L \cdot order) - (\exp(2\pi i \cdot L \cdot order) - 1) + 1} \end{aligned} \quad 3$$

$$= \sqrt{2 - 2 \cdot \cos(2\pi L \cdot order)} = \sqrt{2 \cdot 2 \sin^2(\pi L \cdot order)} = |2 \sin(\pi L \cdot order)| \quad 3$$

$$|H(order)| = \frac{1}{M} \left| \frac{(\exp(-2\pi i \cdot order))^M - 1}{\exp(-2\pi i \cdot order) - 1} \right| = \frac{1}{M} \left| \frac{2 \sin(\pi M \cdot order)}{2 \sin(\pi \cdot 1 \cdot order)} \right| = \frac{1}{M} \left| \frac{\sin(\pi M \cdot order)}{\sin(\pi \cdot order)} \right| \quad 4$$

2. SA using the order domain

SA is implemented in the cycle domain after angular resampling by averaging the segments, as explained in the paper. This section will explain how SA can be calculated using the order. The SA is calculated using Equation 5, where sa is the calculated SA with N samples, M is the number of averaged segments, sig is the signal in the cycle domain with a length of $M \cdot N$.

$$sa[n] = \frac{1}{M} \sum_{m=0}^{M-1} sig[m \cdot N + n] \quad 5$$

As known, any signal s with the length L can be written as a sum of exponent vectors where the coefficients c_k are the coordinate values after discrete Fourier transform as presented in Equation 6.

$$s[n] = \frac{1}{L} \sum_{k=0}^{L-1} c_k \cdot \exp\left(2\pi i \frac{k \cdot n}{L}\right) \quad 6$$

Henceforth, the signal sig from Equation 5 can be written using Equation 7, where a_k are its coordinate values after discrete Fourier transform.

$$sig[n] = \frac{1}{M \cdot N} \sum_{k=0}^{M \cdot N - 1} a_k \cdot \exp\left(2\pi i \frac{k \cdot n}{M \cdot N}\right) \quad 7$$

The effect of the synchronous averaging on the exponent vectors is analyzed in Equation 8, where $l \cdot M + d = k$ from Equation 7, and $d < M$.

$$\begin{aligned} \sum_{m=0}^{M-1} \exp\left(2\pi i \frac{(l \cdot M + d)(m \cdot N + n)}{M \cdot N}\right) &= \sum_{m=0}^{M-1} \exp\left(2\pi i \frac{(l \cdot M + d) \cdot m \cdot N}{M \cdot N}\right) \cdot \exp\left(2\pi i \frac{(l \cdot M + d) \cdot n}{M \cdot N}\right) = \\ &= \sum_{m=0}^{M-1} \exp\left(2\pi i \frac{l \cdot M \cdot m \cdot N}{M \cdot N}\right) \cdot \exp\left(2\pi i \frac{d \cdot m \cdot N}{M \cdot N}\right) \cdot \exp\left(2\pi i \frac{(l \cdot M + d) \cdot n}{M \cdot N}\right) = \\ &= \sum_{m=0}^{M-1} \exp(2\pi i \cdot l \cdot m) \cdot \exp\left(2\pi i \frac{d \cdot m}{M}\right) \cdot \exp\left(2\pi i \frac{(l \cdot M + d) \cdot n}{M \cdot N}\right) \\ &= \exp\left(2\pi i \frac{(l \cdot M + d) \cdot n}{M \cdot N}\right) \cdot \sum_{m=0}^{M-1} 1 \cdot \exp\left(2\pi i \frac{d \cdot m}{M}\right) \end{aligned} \quad 8$$

Now, if $d = 0$ we get Equation 9, and else we get Equation 10.

$$\begin{aligned} \exp\left(2\pi i \frac{(l \cdot M) \cdot n}{M \cdot N}\right) \cdot \sum_{m=0}^{M-1} \exp\left(2\pi i \frac{0 \cdot m}{M}\right) &= \exp\left(2\pi i \frac{(l \cdot M) \cdot n}{M \cdot N}\right) \cdot \sum_{m=0}^{M-1} 1 = M \cdot \exp\left(2\pi i \frac{k \cdot n}{M \cdot N}\right) \\ \exp\left(2\pi i \frac{(l \cdot M + d) \cdot n}{M \cdot N}\right) \cdot \sum_{m=0}^{M-1} \exp\left(2\pi i \frac{d \cdot m}{M}\right) &= \exp\left(2\pi i \frac{(l \cdot M + d) \cdot n}{M \cdot N}\right) \cdot \frac{\exp\left(2\pi i \frac{d \cdot M}{M}\right) - 1}{\exp\left(2\pi i \frac{d}{M}\right) - 1} = \\ &= \exp\left(2\pi i \frac{(l \cdot M + d) \cdot n}{M \cdot N}\right) \cdot \frac{1 - 1}{\exp\left(2\pi i \frac{d}{M}\right) - 1} = \exp\left(2\pi i \frac{(l \cdot M + d) \cdot n}{M \cdot N}\right) \cdot 0 = 0 \end{aligned} \quad 9, 10$$

So, over all we get Equation 11:

$$\begin{aligned} sa[n] &= \frac{1}{M} \sum_{m=0}^{M-1} sig[m \cdot N + n] = \frac{1}{M} \cdot \frac{1}{M \cdot N} \sum_{m=0}^{M-1} \sum_{k=0}^{M \cdot N - 1} a_k \exp\left(2\pi i \frac{k \cdot (m \cdot N + n)}{M \cdot N}\right) = \\ &= \frac{1}{M} \cdot \frac{1}{M \cdot N} \sum_{k=0}^{M \cdot N - 1} a_k \exp\left(2\pi i \frac{k \cdot n}{M \cdot N}\right) \sum_{m=0}^{M-1} \exp\left(2\pi i \frac{k \cdot m \cdot N}{M \cdot N}\right) = \frac{1}{M} \cdot \frac{1}{M \cdot N} \sum_{k=0}^{N-1} a_{k \cdot M} \exp\left(2\pi i \frac{k \cdot n}{N}\right) \cdot M = \end{aligned} \quad 11$$

$$= \frac{1}{M \cdot N} \sum_{k=0}^{N-1} a_{k,M} \exp\left(2\pi i \frac{k \cdot n}{N}\right) = \frac{1}{M} \cdot \frac{1}{N} \sum_{k=0}^{N-1} a_{k,M} \exp\left(2\pi i \frac{k \cdot n}{N}\right)$$

I.E., we can present the SA using Equation 12.

$$sa[n] = \frac{1}{M} \cdot \frac{1}{N} \sum_{k=0}^{N-1} a_{k,M} \exp\left(2\pi i \frac{k \cdot n}{N}\right) = \frac{1}{N} \sum_{k=0}^{N-1} \frac{a_{k,M}}{M} \exp\left(2\pi i \frac{k \cdot n}{N}\right) \quad 12$$

Thus, as depicted in **Fig. 1**, for calculating the SA using the order domain, the signal is converted to the order domain, and then the values of the complete orders are extracted. After division by the number of averaged segments M , the signal is converted back to the cycle domain.

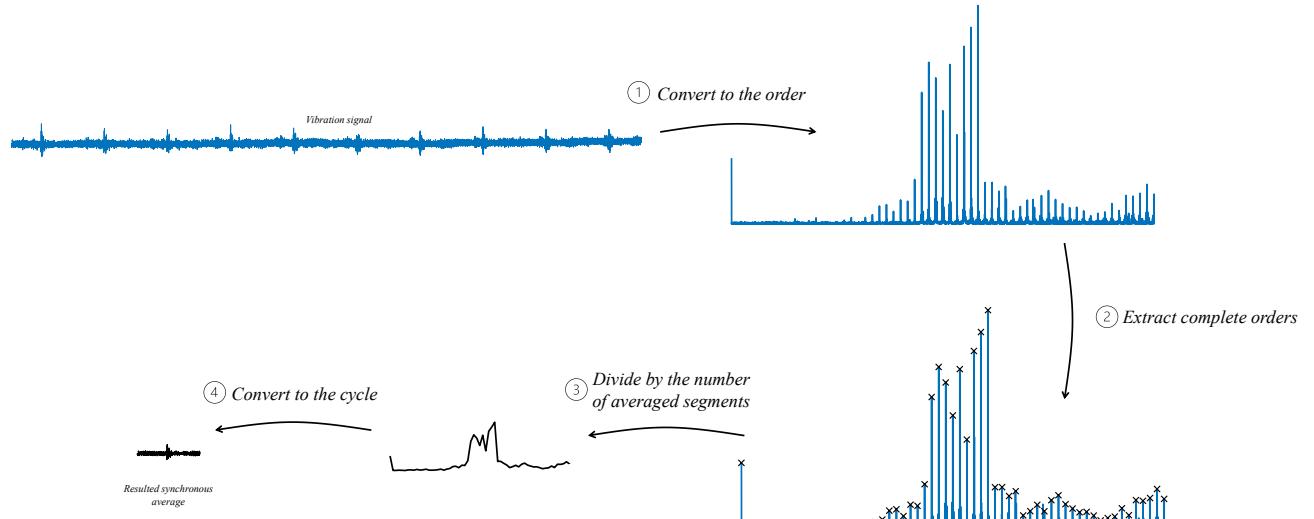


Fig. 1 – Block diagram of SA calculated using the order domain by extracting the complete orders.

3. Reducing interferences and random noise

The rate of reducing interferences can be analyzed using Equation 13, where the last equality holds due to Equation 3.

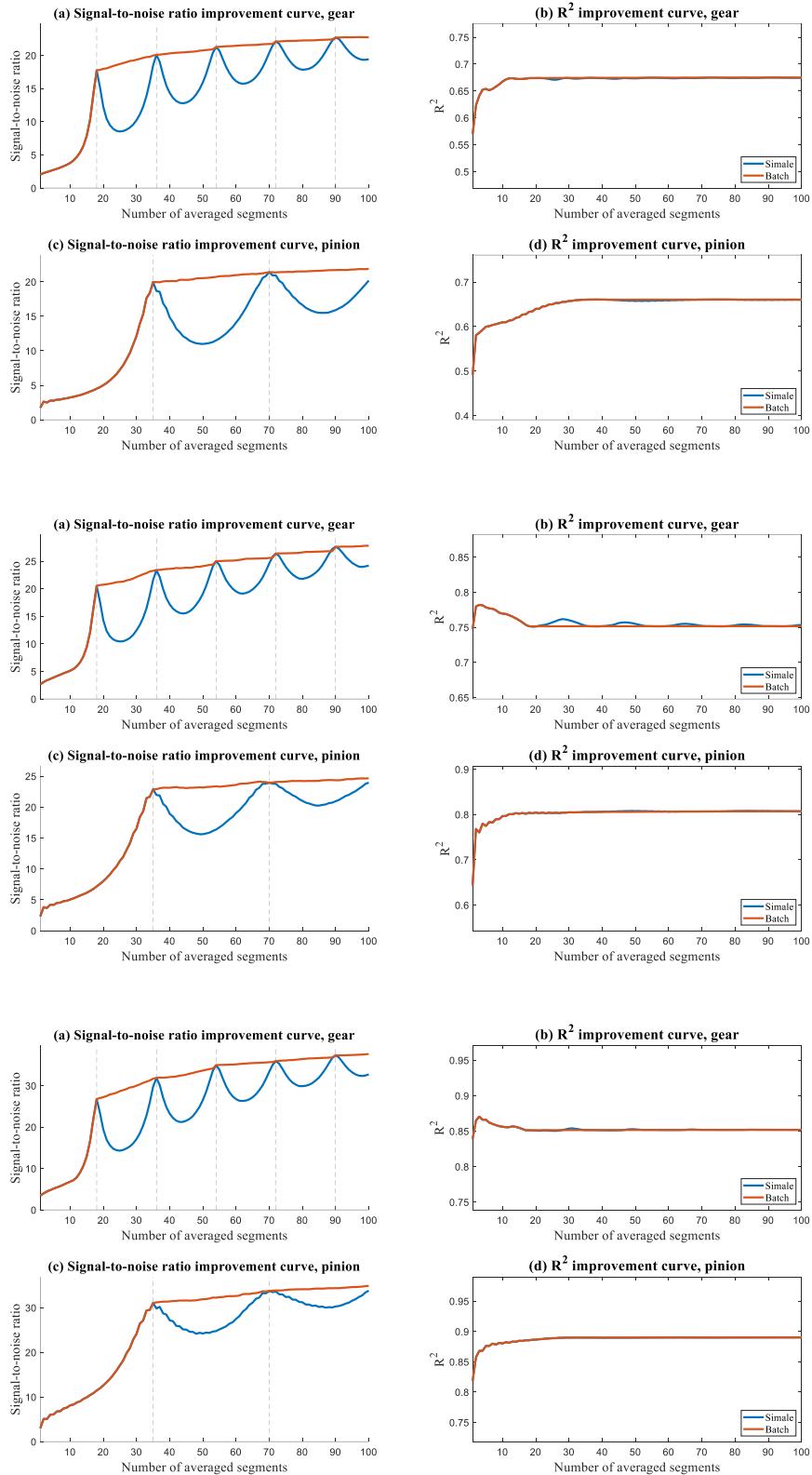
$$|leakage(M)| = \left| \frac{1}{M} \sum_{m=0}^{M-1} \exp(2\pi i \cdot order \cdot m) \right| = \frac{1}{M} \left| \frac{\exp(2\pi i \cdot M \cdot order) - 1}{\exp(2\pi i \cdot order) - 1} \right| = \frac{1}{M} \left| \frac{\sin(\pi M \cdot order)}{\sin(\pi \cdot order)} \right| \quad 13$$

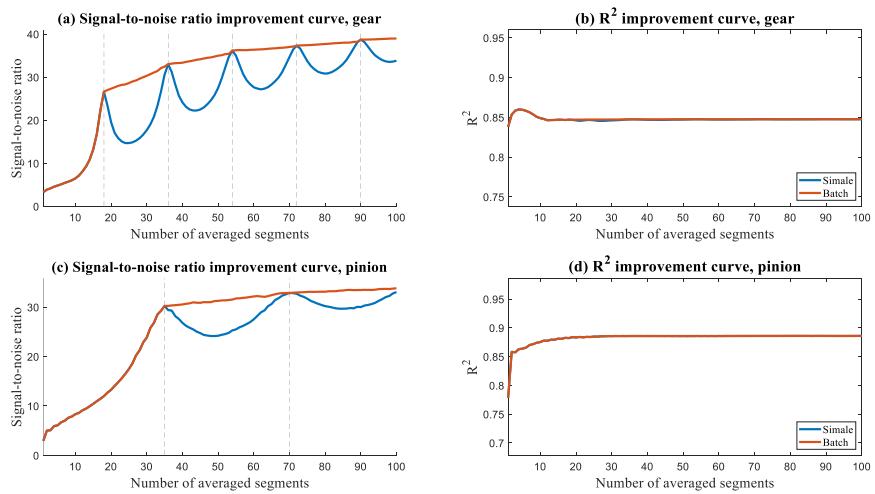
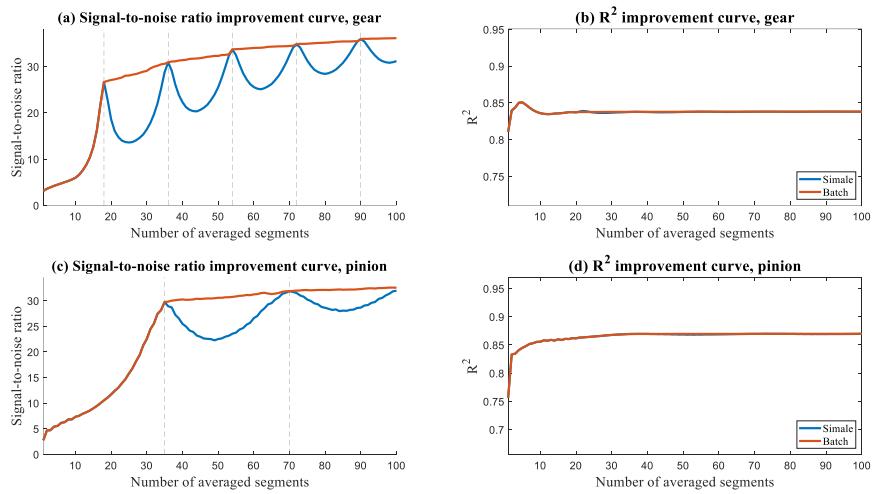
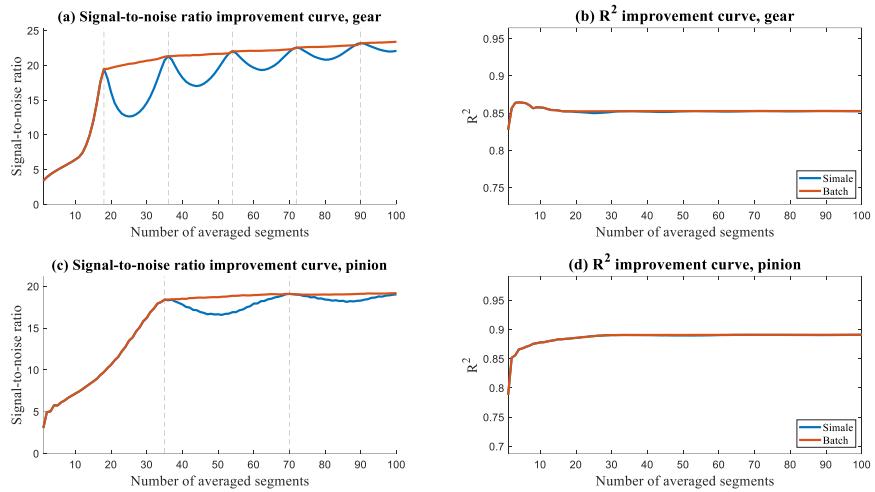
The random noise reduction can be analyzed using Equation 14.

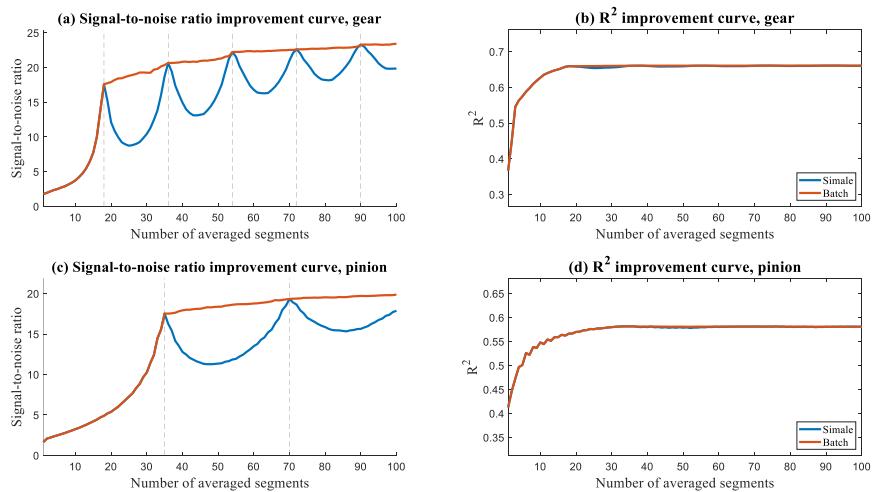
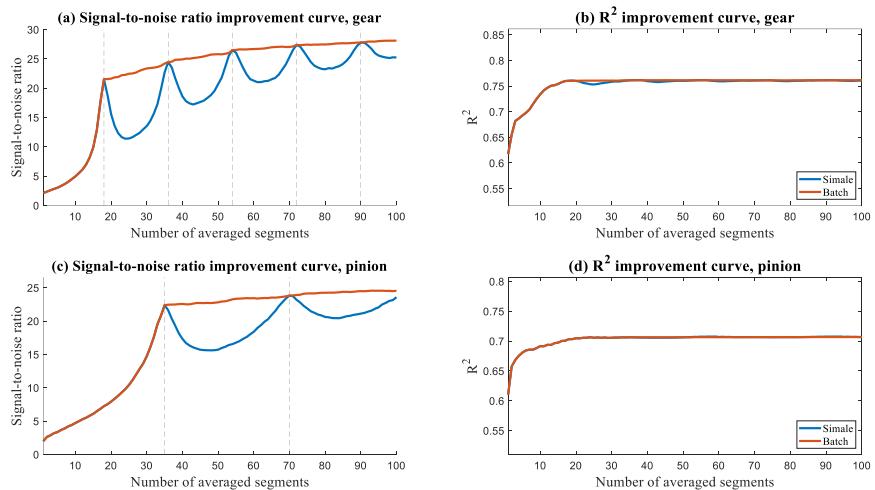
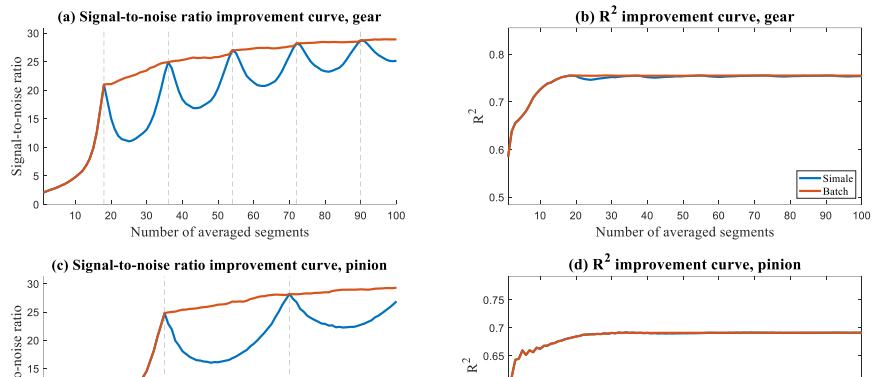
$$\begin{aligned} var(sa[n]) &= var(sa[0]) = var\left(\frac{1}{N} \sum_{k=0}^{N-1} \frac{a_{k \cdot M}}{M} \exp\left(2\pi i \frac{k \cdot 0}{N}\right)\right) = \frac{1}{N^2} \cdot \frac{1}{M^2} \cdot var\left(\sum_{k=0}^{N-1} a_{k \cdot M}\right) = \\ &= \frac{1}{N^2} \cdot \frac{1}{M^2} \cdot var\left(\sum_{k=0}^{N-1} \sum_{l=0}^{N \cdot M - 1} \exp\left(-2\pi i \frac{k \cdot M \cdot l}{N \cdot M}\right) \cdot sig[l]\right) = \frac{1}{N^2} \cdot \frac{1}{M^2} \cdot var\left(\sum_{l=0}^{N \cdot M - 1} \sum_{k=0}^{N-1} \exp\left(-2\pi i \frac{k \cdot l}{N}\right) \cdot sig[l]\right) = \\ &= \frac{1}{N^2} \cdot \frac{1}{M^2} \cdot \sum_{l=0}^{N \cdot M - 1} var\left(\sum_{k=0}^{N-1} \exp\left(-2\pi i \frac{k \cdot l}{N}\right) \cdot sig[l]\right) = \frac{1}{N^2} \cdot \frac{1}{M^2} \cdot \sum_{l=0}^{N \cdot M - 1} \left| \sum_{k=0}^{N-1} \exp\left(-2\pi i \frac{k \cdot l}{N}\right) \right|^2 \cdot var(sig[l]) = \\ &= \frac{1}{N^2} \cdot \frac{1}{M^2} \cdot \sigma^2 \cdot \sum_{l=0}^{N \cdot M - 1} \begin{cases} N^2, & l = 0, N, 2N, \dots, N(M-1) \\ 0, & \text{else} \end{cases} = \frac{1}{N^2} \cdot \frac{1}{M^2} \cdot \sigma^2 \cdot M \cdot N^2 = \frac{\sigma^2}{M} \end{aligned} \quad 14$$

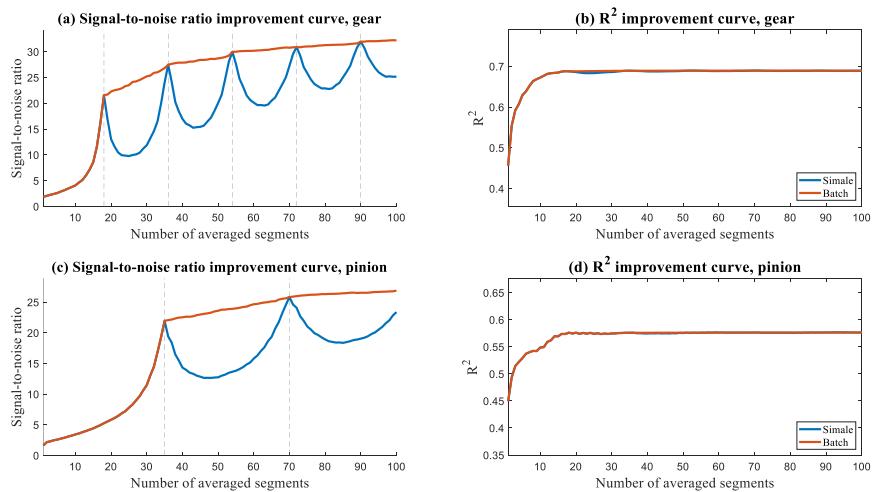
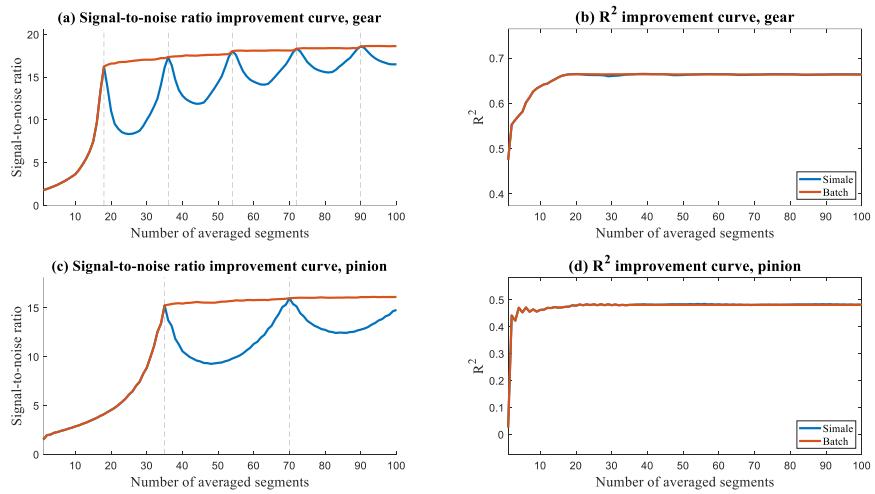
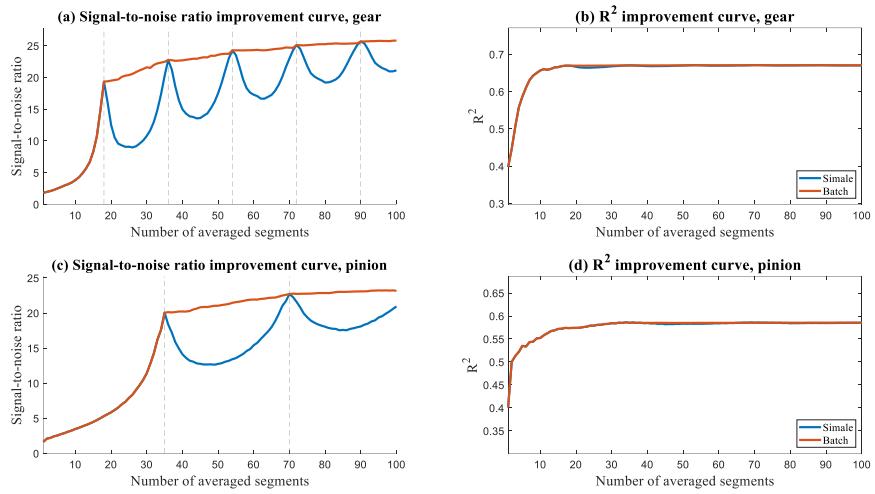
4. More examples from the gear experiment

The results of Simple SA and Batch SA on the gear experiment are provided in **Fig. 2**.









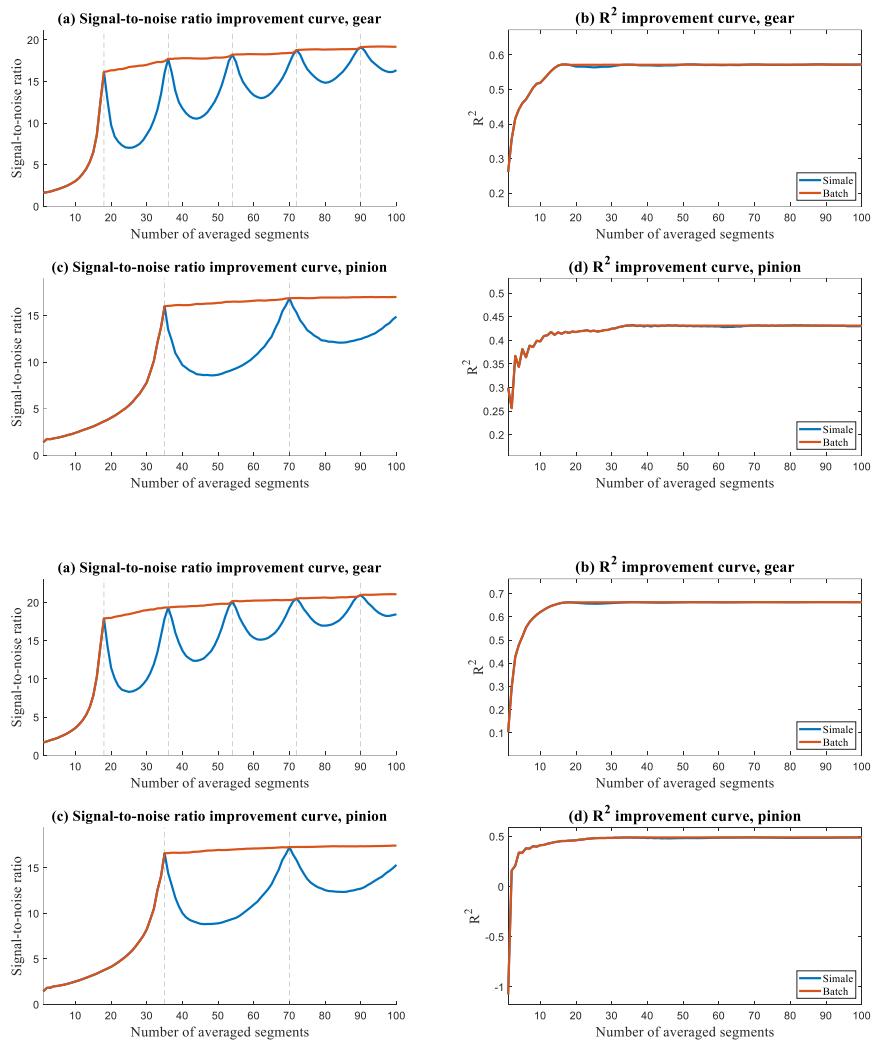


Fig. 2 – Results of Simple SA and Batch SA on the 14 records of the gear experiment.

5. Random delays as a low-pass filter

Equations 15-17 provides the analysis of random delays as a low-pass filter. $SA[k]$ is the k coordinate of the SA in the order domain. d is uniform distributed, $d \sim U[-a, a]$.

$$E(SA[k]) = \frac{1}{M} SA[k] \sum_{m=0}^{M-1} E\left(e^{2\pi i \frac{kd}{N}}\right) \quad 15$$

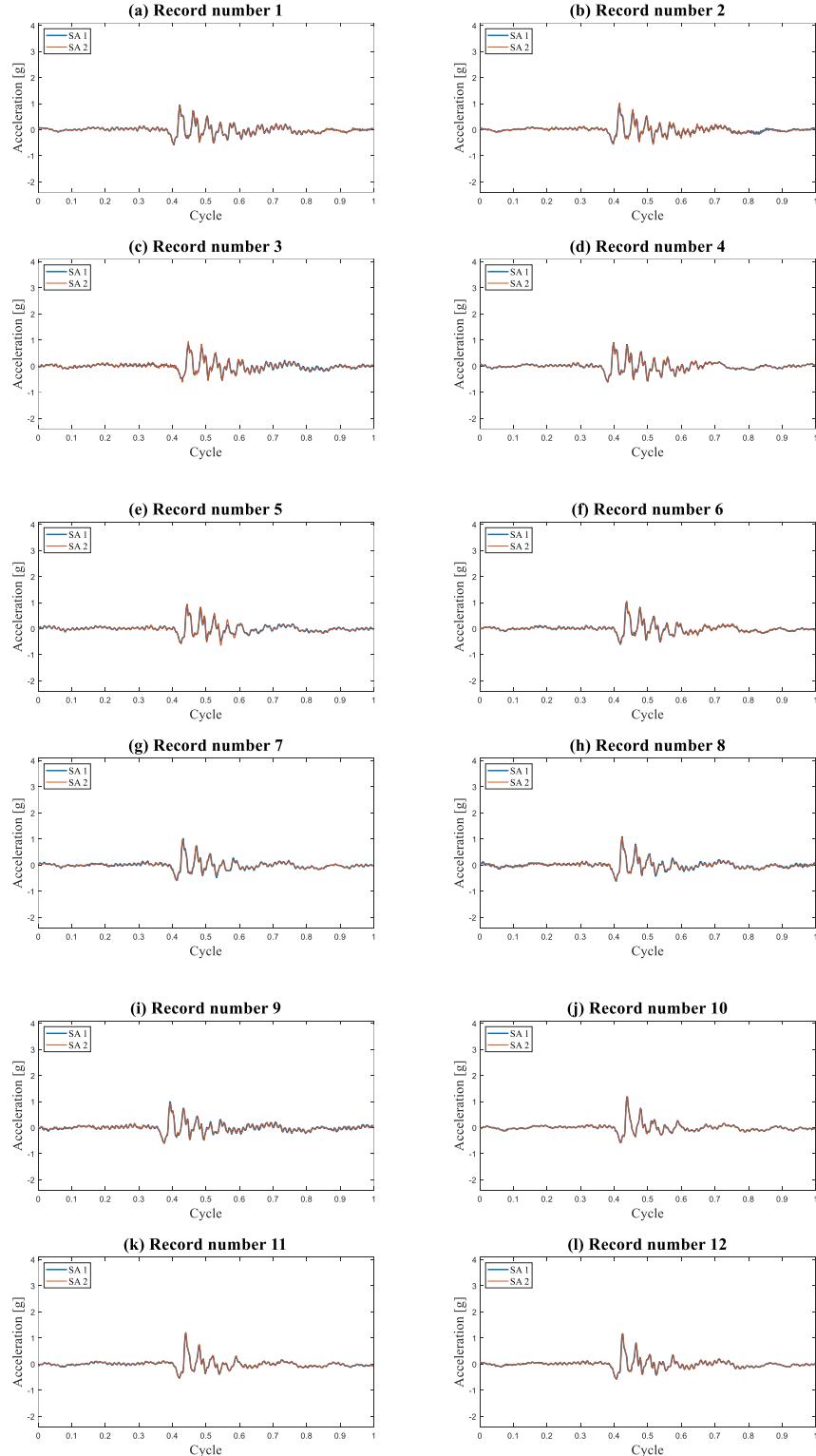
$$\begin{aligned} E\left(e^{2\pi i \frac{kd}{N}}\right) &= \int_{-a}^a \frac{1}{2a} \cdot e^{2\pi i \frac{kx}{N}} \cdot dx = \int_{-a}^a \frac{1}{2a} \cdot \left[\cos\left(2\pi \cdot \frac{kx}{N}\right) + i \cdot \sin\left(2\pi \cdot \frac{kx}{N}\right) \right] \cdot dx = \\ &= \int_{-a}^a \frac{1}{2a} \cdot \cos\left(2\pi \cdot \frac{kx}{N}\right) \cdot dx + \int_{-a}^a \frac{1}{2a} \cdot i \cdot \sin\left(2\pi \cdot \frac{kx}{N}\right) \cdot dx = \\ &= \int_{-a}^a \frac{1}{2a} \cdot \cos\left(2\pi \cdot \frac{kx}{N}\right) \cdot dx + 0 = \frac{1}{2a} \cdot \frac{1}{2\pi \cdot \frac{k}{N}} \cdot \sin\left(2\pi \cdot \frac{kx}{N}\right) \Big|_{-a}^a = \end{aligned} \quad 16$$

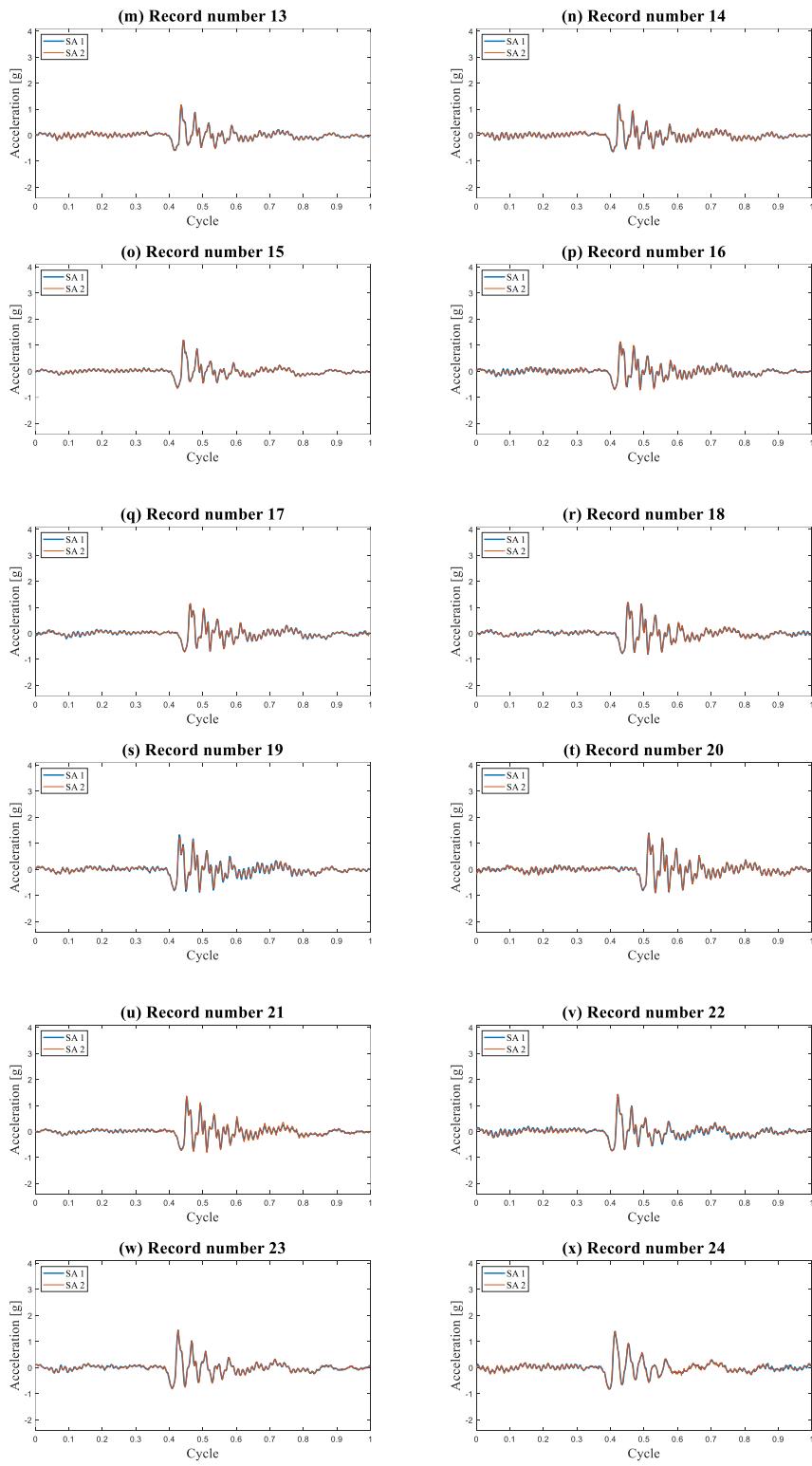
$$= \frac{1}{2a} \cdot \frac{1}{2\pi \cdot \frac{k}{N}} \cdot \left[\sin\left(2\pi \cdot \frac{ka}{N}\right) - \sin\left(-2\pi \cdot \frac{ka}{N}\right) \right] = \frac{1}{2a} \cdot \frac{1}{2\pi \cdot \frac{k}{N}} \cdot 2 \cdot \sin\left(2\pi \cdot \frac{ka}{N}\right) = \frac{\sin\left(2\pi \cdot \frac{ka}{N}\right)}{2\pi \cdot \frac{ka}{N}}$$

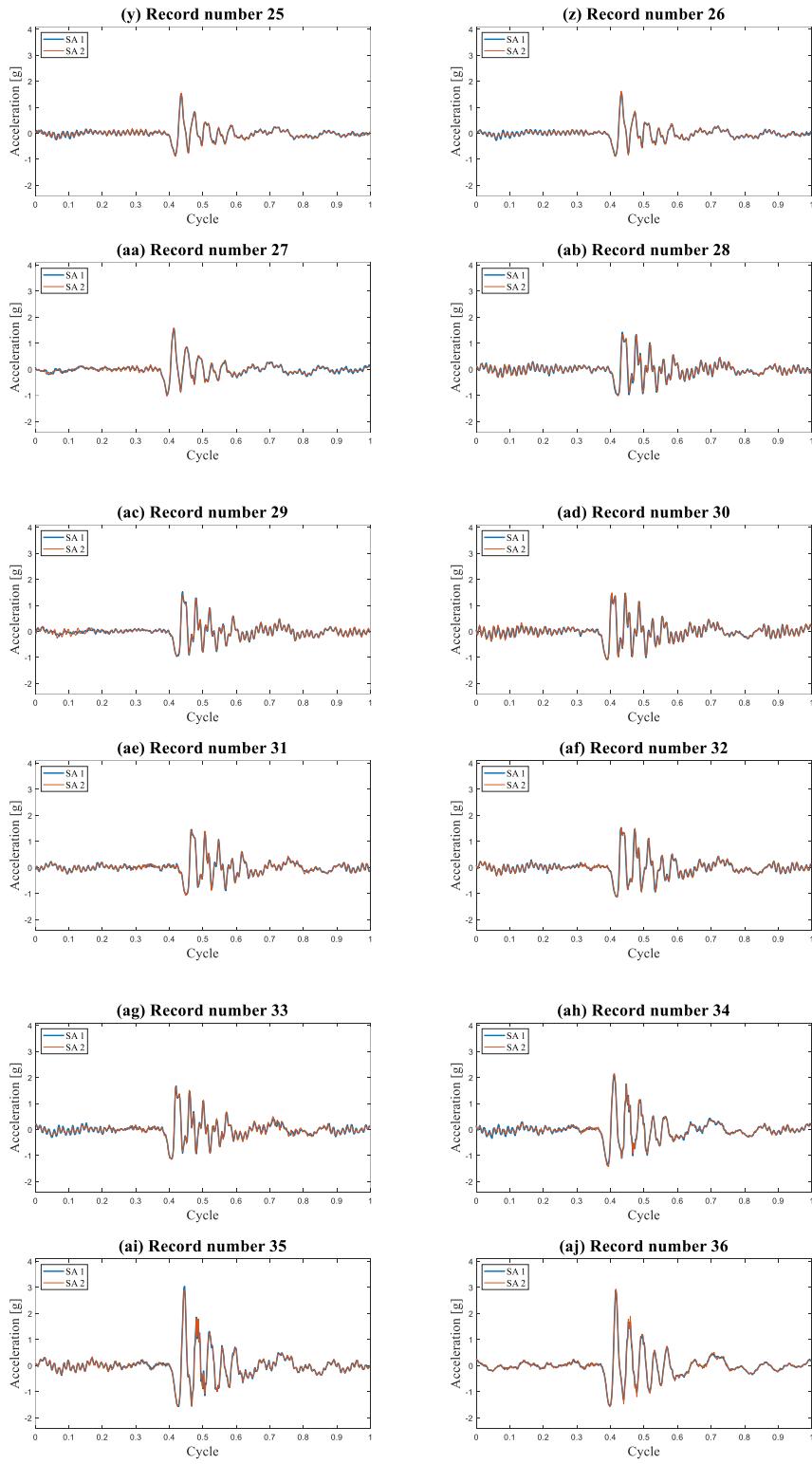
$$|E(SA_c[k])| = \left| \frac{1}{M} SA[k] \sum_{m=0}^{M-1} E\left(e^{2\pi i \frac{kd}{N}}\right) \right| = \frac{1}{M} \cdot |SA_{de}[k]| \cdot M \cdot \left| \frac{\sin\left(2\pi \cdot \frac{ka}{N}\right)}{2\pi \cdot \frac{ka}{N}} \right| = \left| \frac{\sin\left(2\pi \cdot \frac{ka}{N}\right)}{2\pi \cdot \frac{ka}{N}} \right| \cdot |SA_{de}[k]| \quad 17$$

6. More examples from the gear experiment

In this section, the calculated SAs by Angular Synchronization SA are presented for all tested cases. The results of the Z-axis of the endurance test are shown in **Fig. 3**, the Y-axis in **Fig. 4**, the Z-axis of the monitored test in **Fig. 5**, the Y-axis in **Fig. 6**, and the MFPT dataset in **Fig. 7**.







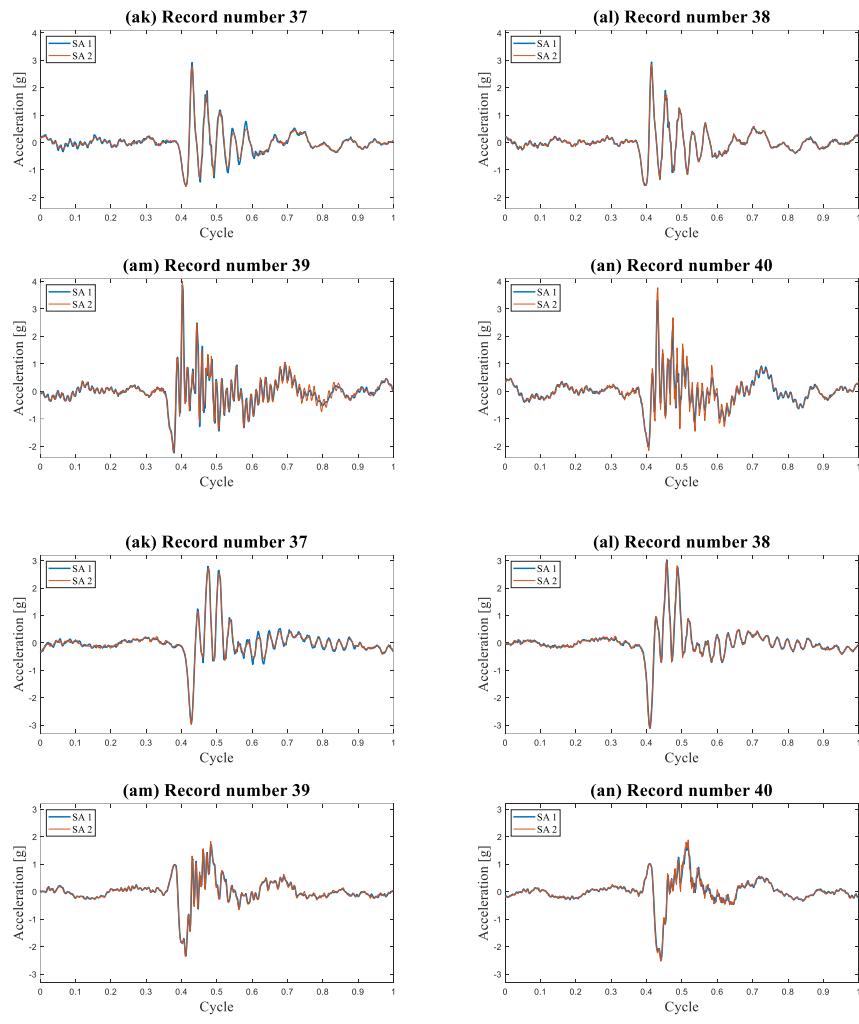
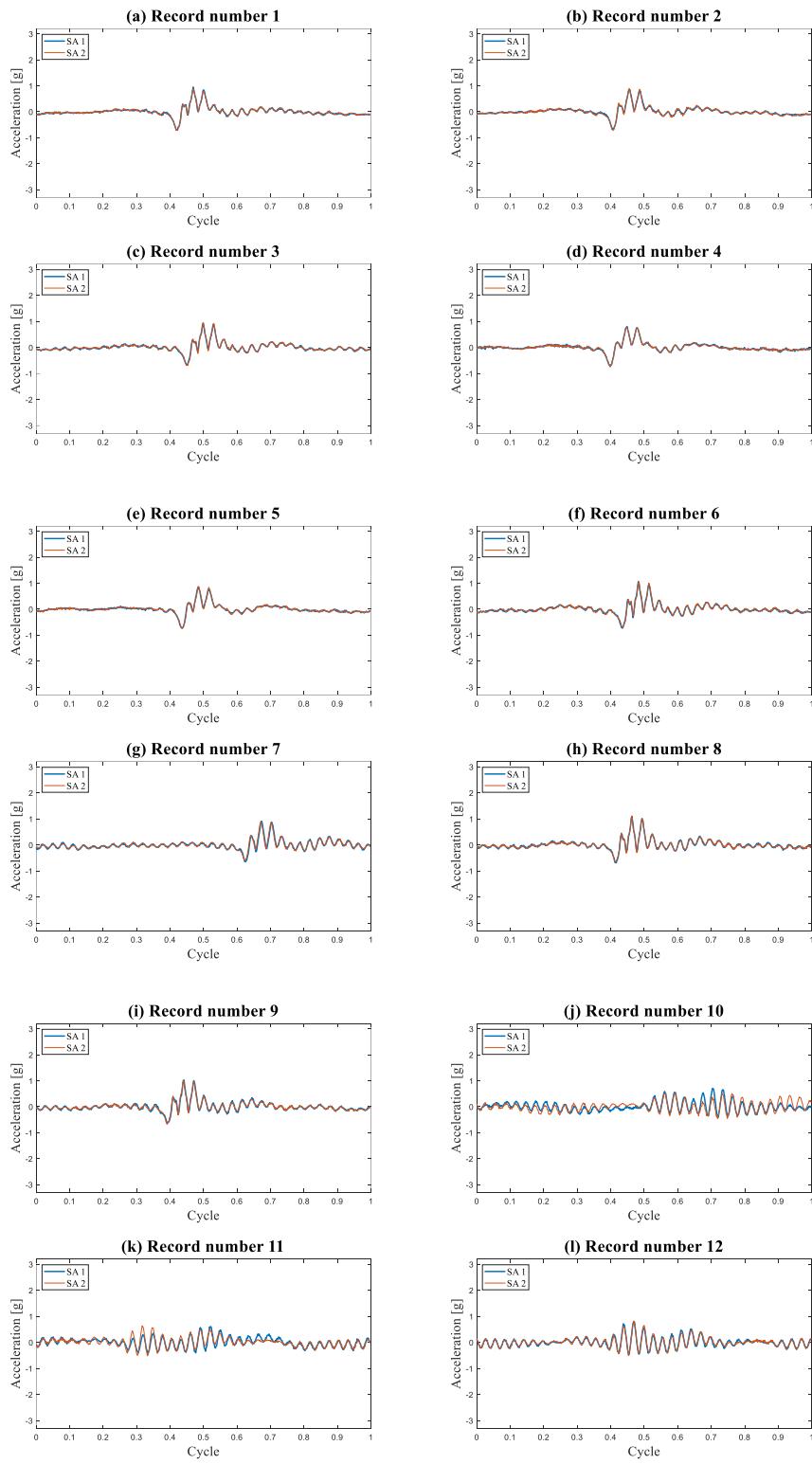
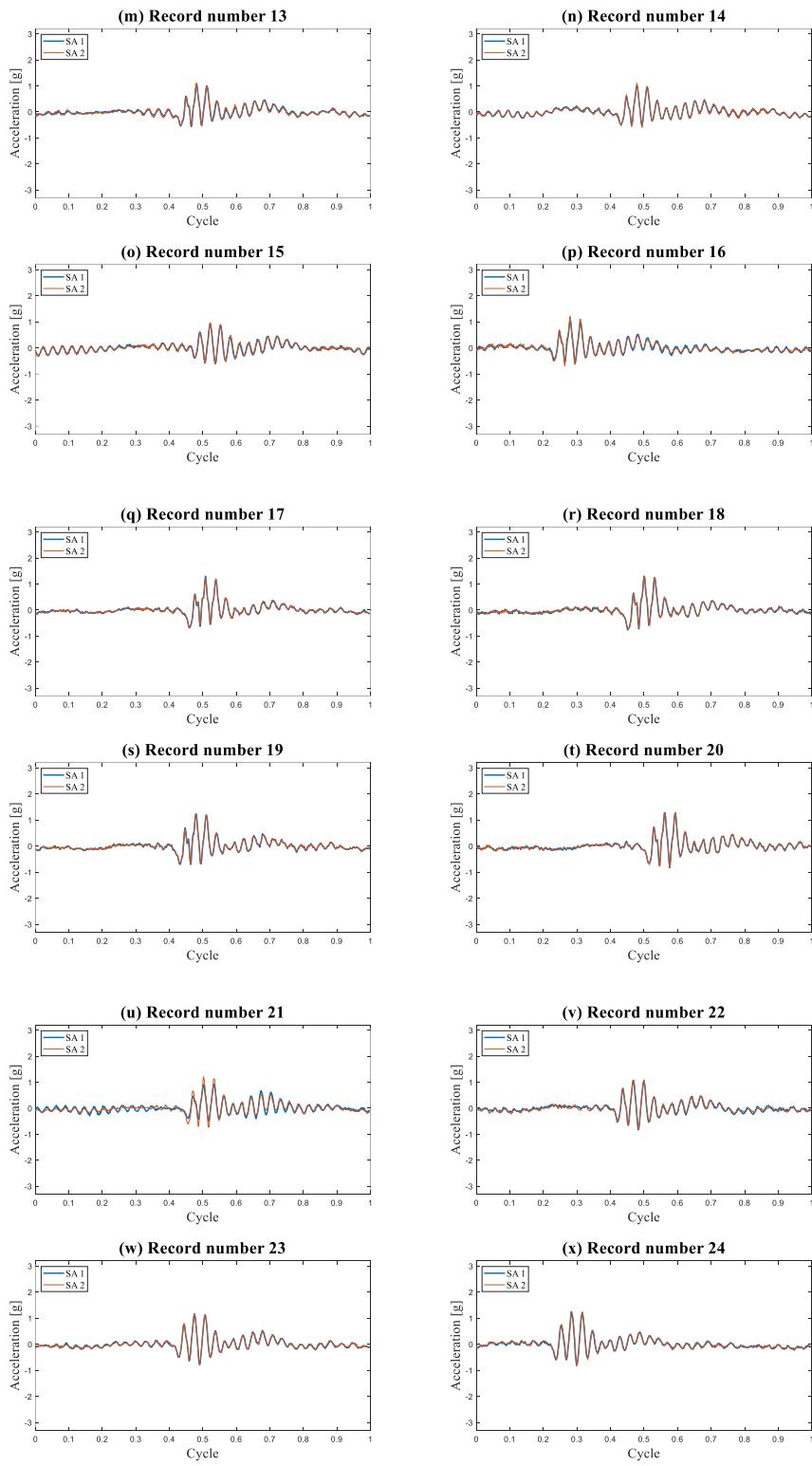


Fig. 3 – Results of Angular Synchronization SA on the endurance bearing test, Z-axis.





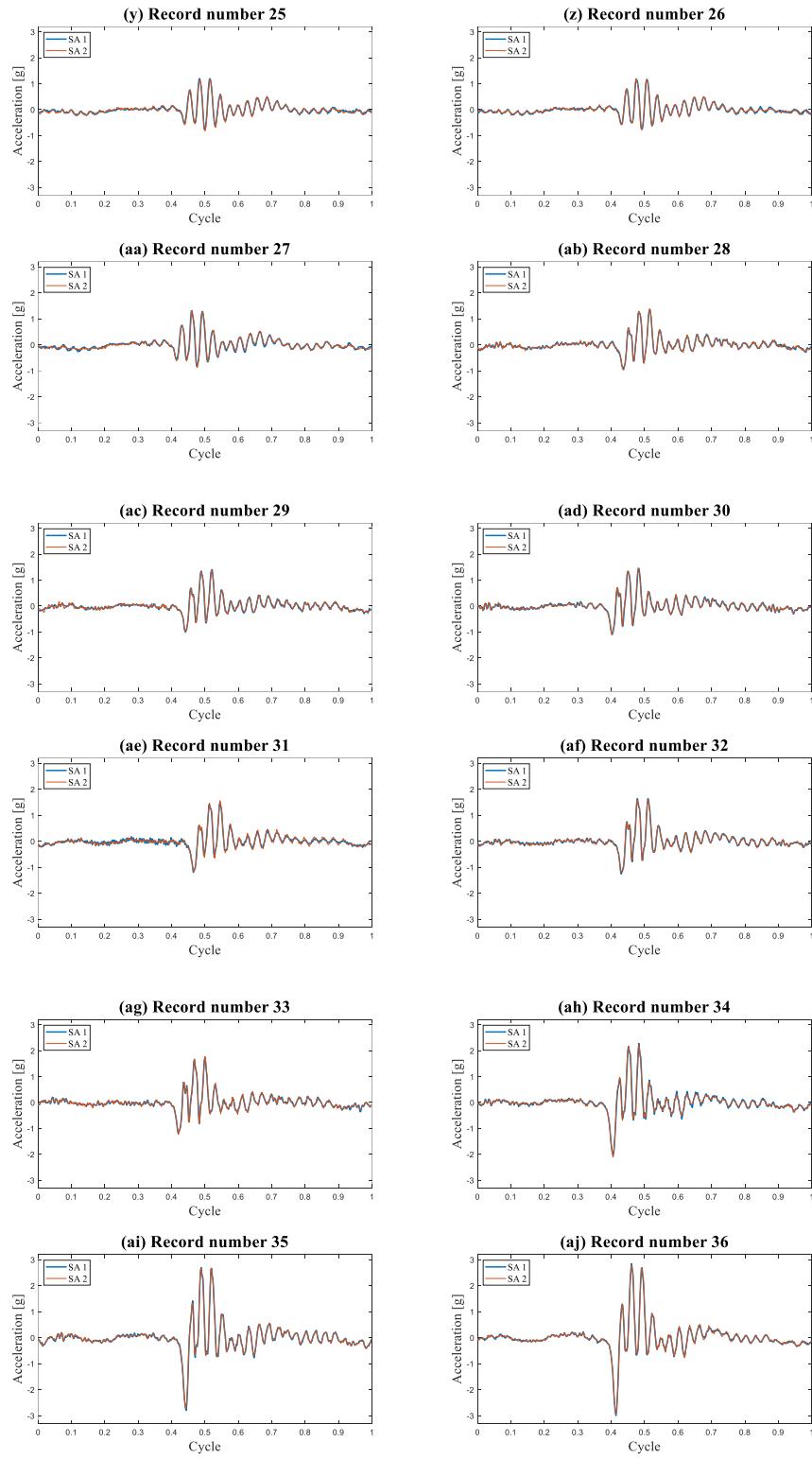
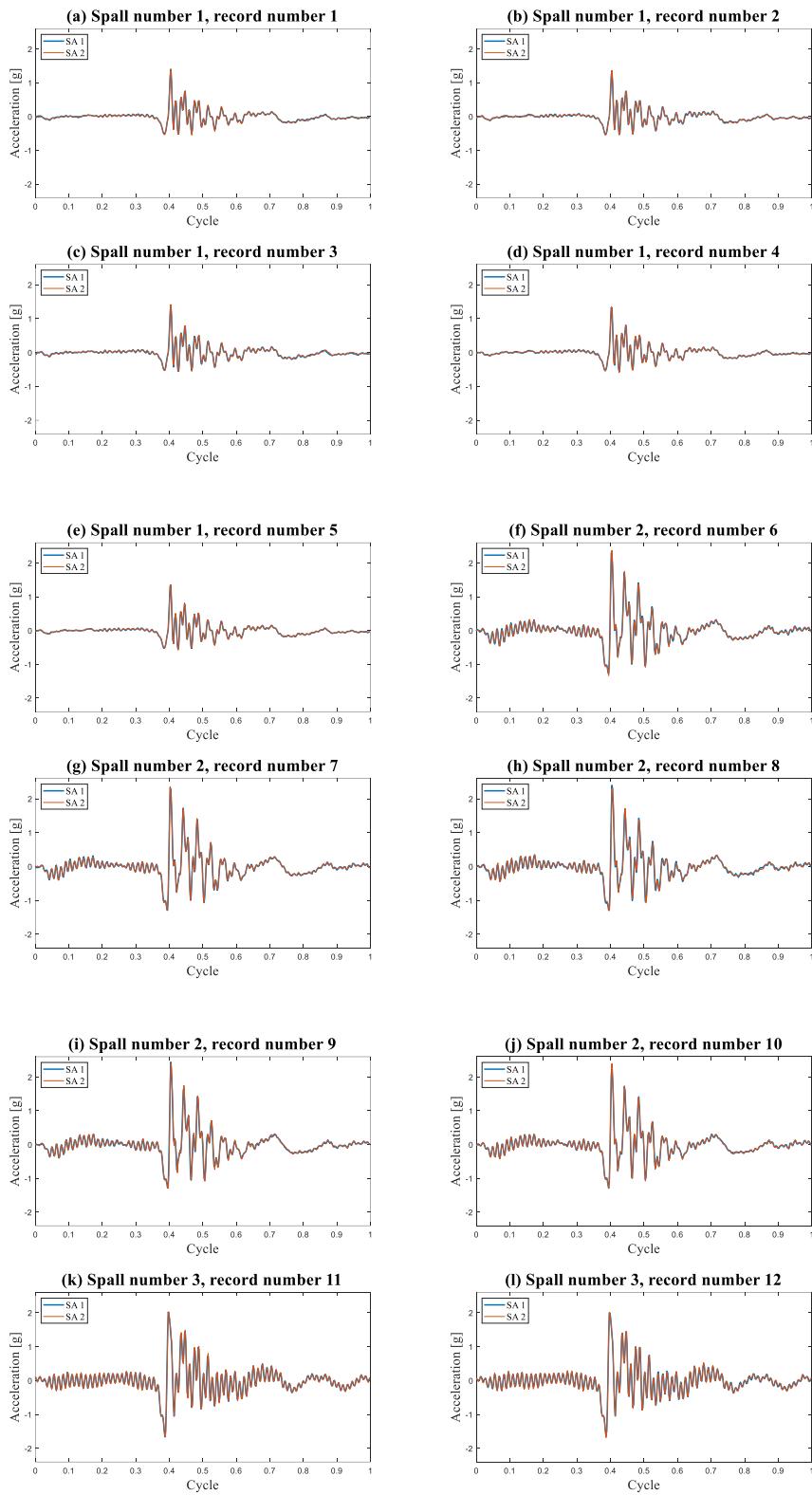
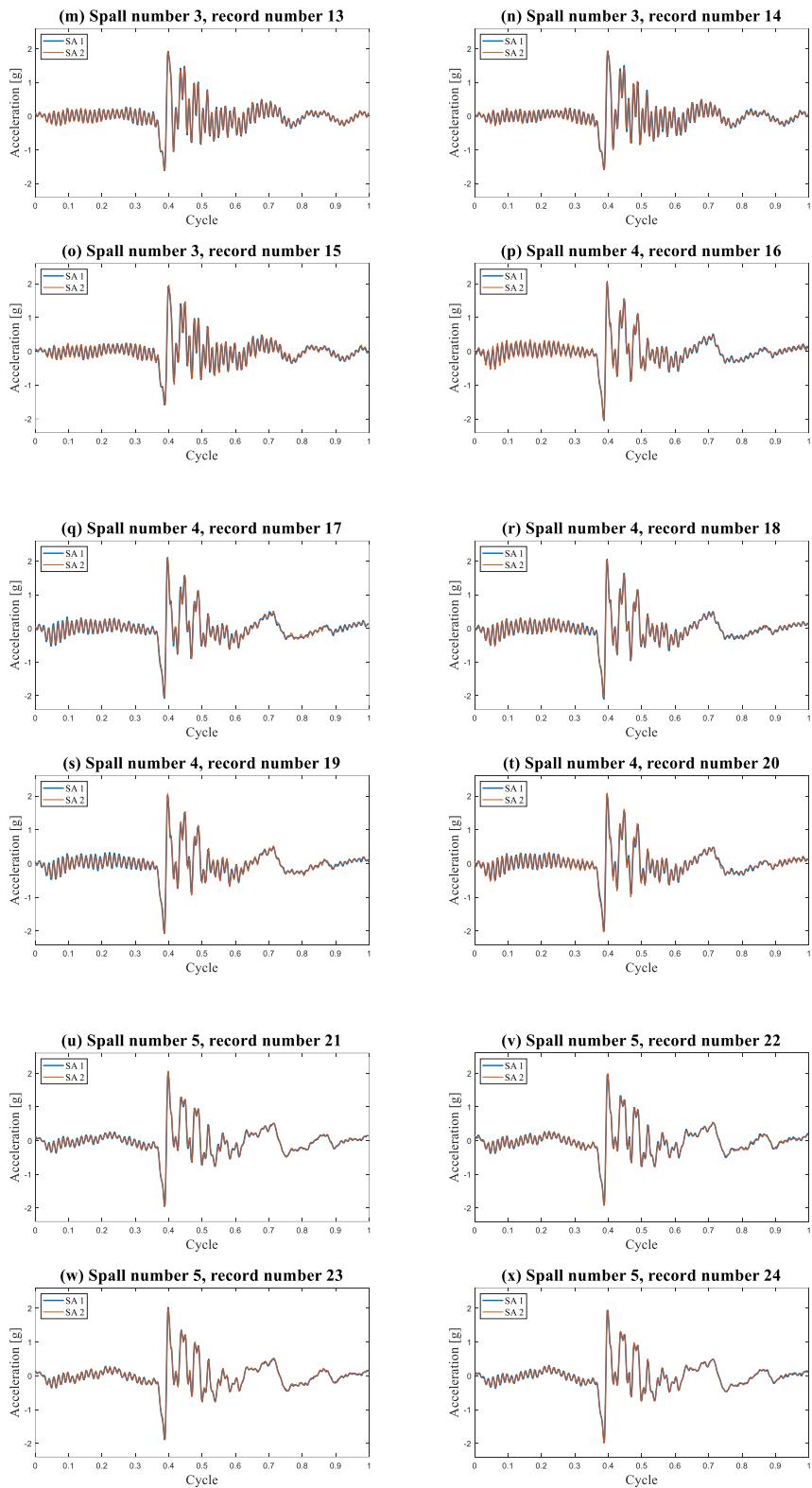


Fig. 4 – Results of Angular Synchronization SA on the endurance bearing test, Y-axis.





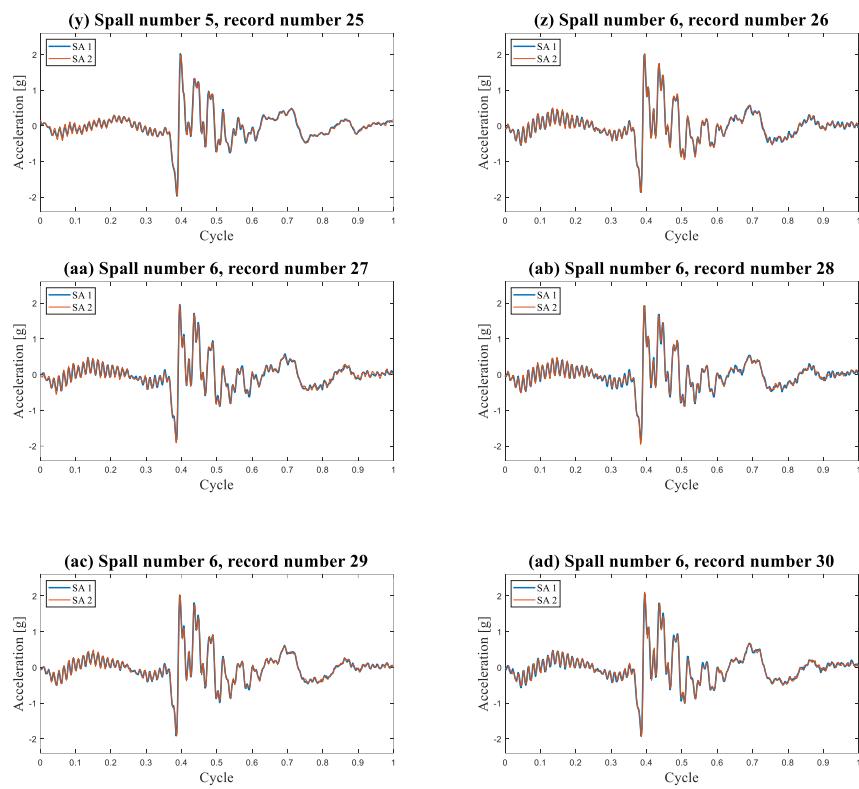
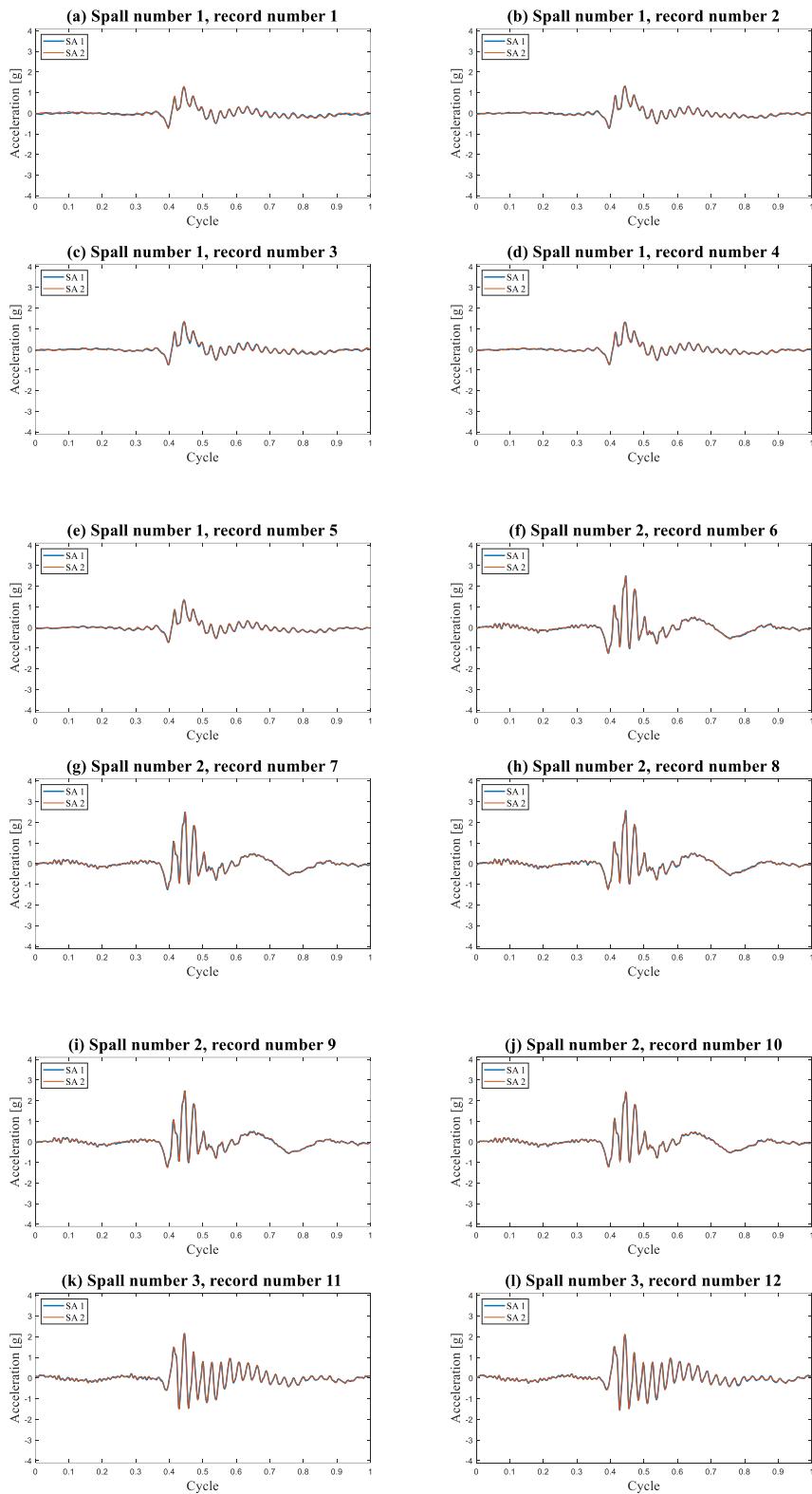
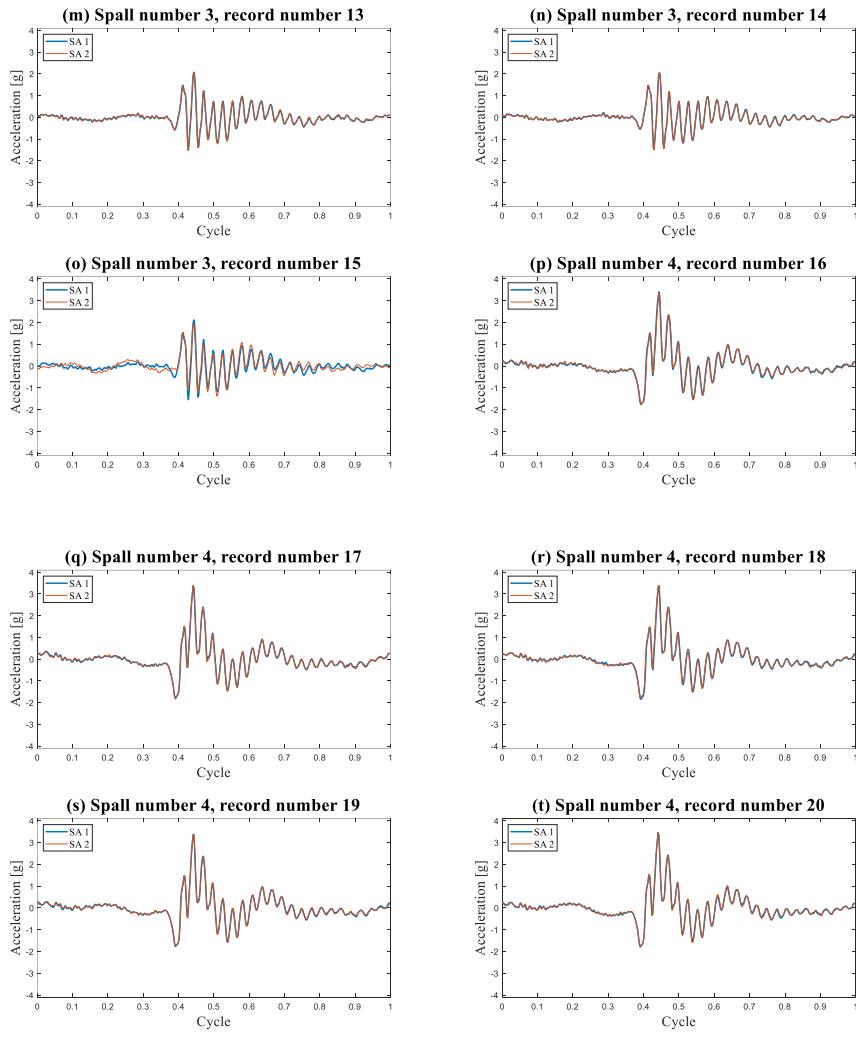


Fig. 5 – Results of Angular Synchronization SA on the monitored spall bearing test, Z-axis.





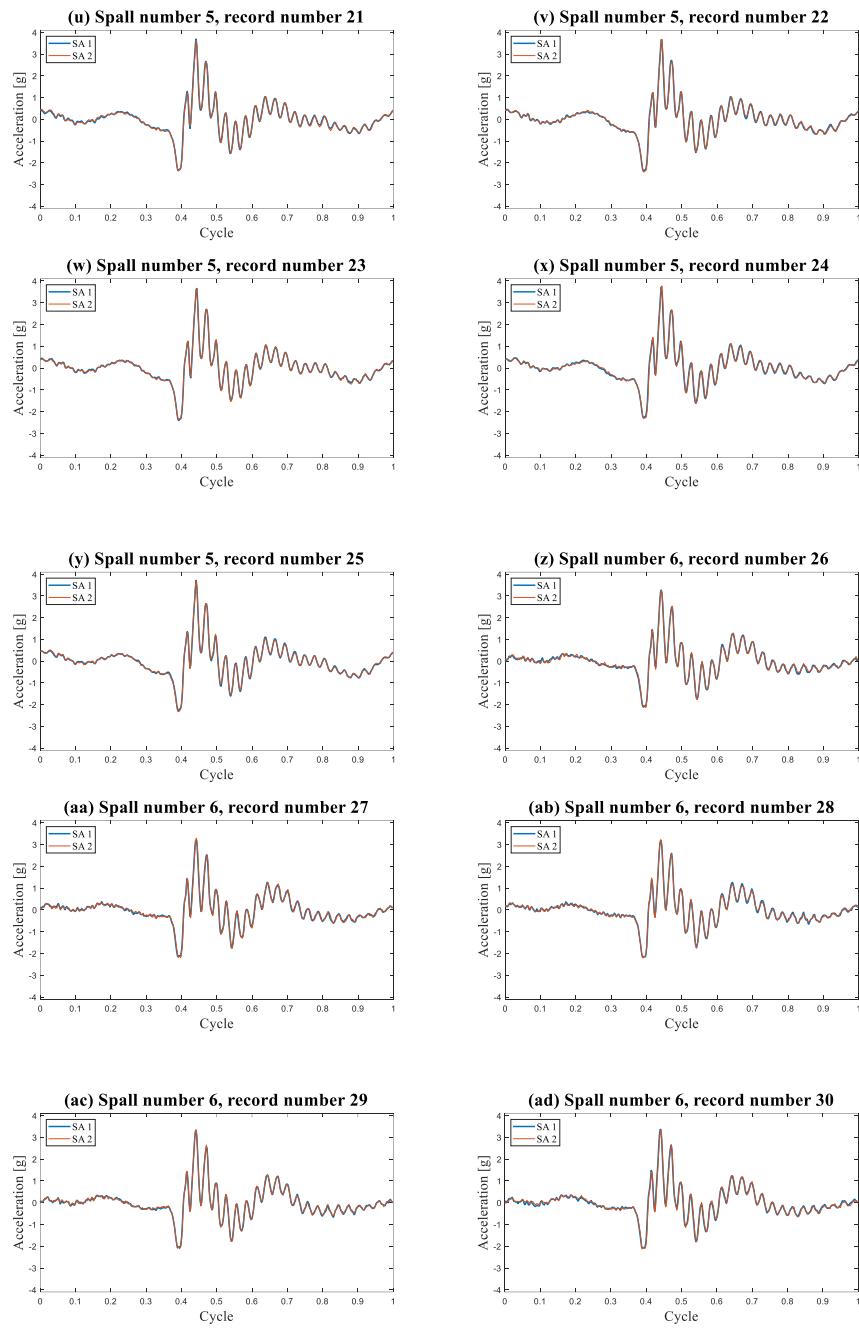


Fig. 6 – Results of Angular Synchronization SA on the monitored spall bearing test, Y-axis.

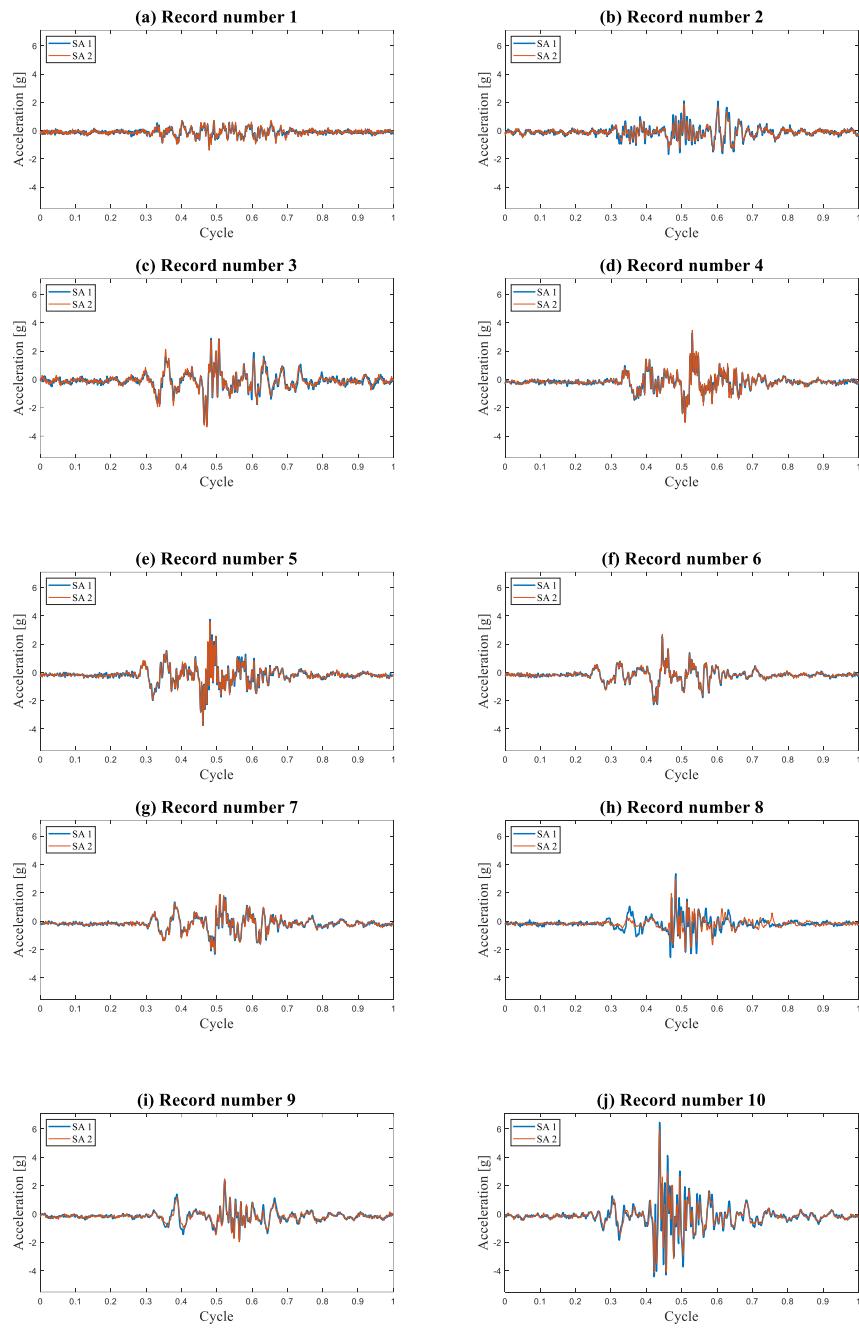


Fig. 7 – Results of Angular Synchronization SA on the MFPT dataset.

References:

- [1] S. Braun, The synchronous (time domain) average revisited, *Mech Syst Signal Process.* 25 (2011) 1087–1102. <https://doi.org/10.1016/J.YMSSP.2010.07.016>.