Introduction to Quantum Computing

How I Learned to Stop Worrying and Love the Bomb

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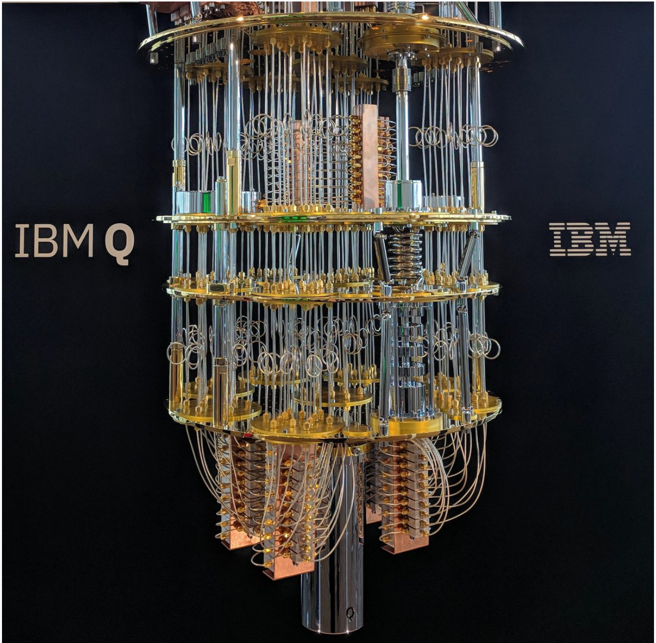
Schmiede.ONE GmbH & Co. KG

Introduction

Ce n'est pas un lustre (This is not a Chandelier)



Big big picture

- New computing paradigm

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- ¯_(ツ)_/¯

Disclaimer

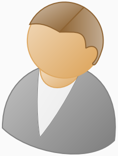
- Quantum Computing is technical
- Quantum Algorithms are hard
- Quantum Mechanics is weird
- Quantum Mechanics is technical

Examples?

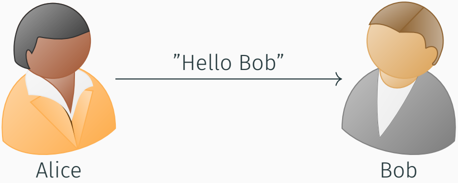
Example: the RSA algorithm

Alice


Alice


Bob

Example: the RSA algorithm



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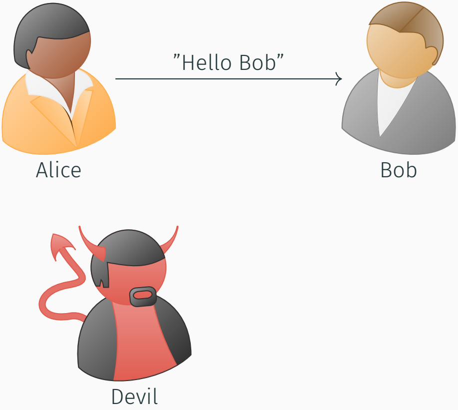

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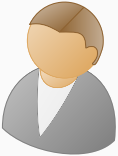

Bob

Generate Public/Private keys

$$e, d, N = pq$$

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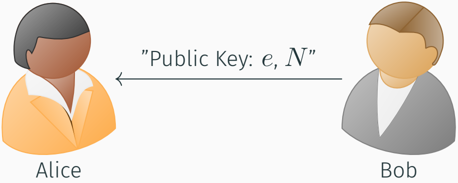

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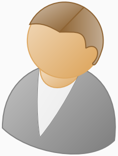

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$$m = \text{encode}(\text{"Hello World"})$$
$$c = m^e \mod N$$

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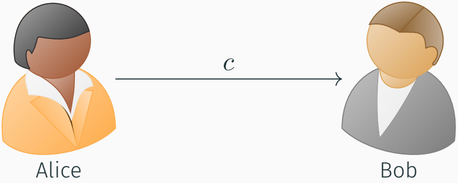

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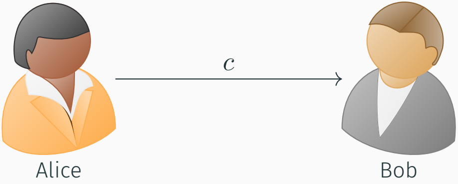

$$m = \text{encode}(\text{"Hello World"})$$
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$$\text{decode}(m)$$

Example: the RSA algorithm - outline

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- $N = pq$ with p, q large prime numbers.

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But it's hard to find p, q, such that $pq = N$.

Factorizing on a Classical Computer

Bits	Time	Notes
128	less than 2 seconds	
192	16 seconds	
256	35 minutes	
260	1 hour	
512	73 days	in 2009

Example: the RSA algorithm - Factorizing is Hard

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Most RSA implementations use between 1024 and 4096 bits.

Example: the RSA algorithm - Shor's Algorithm

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Would require ~ 4000 qubits to break 2048-bit RSA

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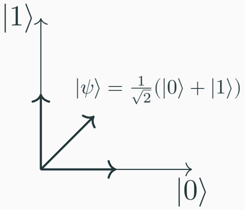

Basics of Quantum Mechanics (simplified)

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- It is impossible to clone the state of the system (no-cloning theorem).

Movie

The Technical Part

"Shut up and calculate"

David Mermin

- Representing computation with linear algebra
- Qubits, superposition and quantum logic gates
- Simplest problem where a quantum computer outperforms a classical one
- Bonus: Quantum entanglement and quantum teleportation

One bit with value 0, also written as $|0\rangle$ (Dirac vector notation)

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

One bit with value 1, also written as $|1\rangle$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Review: matrix multiplication

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax + by + cz \\ dx + ey + fz \\ gx + hy + iz \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

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Identity	$f(x) = x$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
Negation	$f(x) = \neg x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
Constant-0	$f(x) = 0$	$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
Constant-1	$f(x) = 1$	$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Reversible computing

- Given the operation and the input, you can always infer the output.
  - For $Ax = b$, given b and A, you can uniquely find x.

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- Permutations are reversible; erasing and overwriting are not
    - Identity and negation are reversible.
    - Constant-0 and Constant-1 are not reversible.
- Quantum computers use only reversible operations.
    - In fact, all quantum operations are their own inverse.

Review: tensor product of vectors

$$\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \otimes \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \\ x_1 \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} x_0 y_0 \\ x_0 y_1 \\ x_1 y_0 \\ x_1 y_1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \otimes \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \otimes \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \otimes \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{pmatrix} x_0 y_0 z_0 \\ x_0 y_0 z_1 \\ x_0 y_1 z_0 \\ x_0 y_1 z_1 \\ x_1 y_0 z_0 \\ x_1 y_0 z_1 \\ x_1 y_1 z_0 \\ x_1 y_1 z_1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Representing multiple cbits

$$|00\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad |01\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$|10\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad |11\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$|100\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{pmatrix}$$

- The tensor representation is called the product state.
- It can be factored back into the individual state representation.
- The product state of n bits is a vector of size 2^n.

Operations on multiple cbits: CNOT

- Takes two bits, one control bit and one target bit.
- If the control bit is set, flip the target bit, otherwise leave it.

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- If most significant bit is control, and least-significant is target, then:

$\lvert 00\rangle$	$\lvert 00\rangle$
$\lvert 01\rangle$	$\lvert 01\rangle$
$\lvert 10\rangle$	$\lvert 10\rangle$
$\lvert 11\rangle$	$\lvert 11\rangle$

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$$
\begin{array}{lcl}
|00\rangle & \longrightarrow & |00\rangle \\
|01\rangle & & |01\rangle \\
|10\rangle & & |10\rangle \\
|11\rangle & & |11\rangle
\end{array}
$$

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$$\begin{tikzcd}
|00\rangle \arrow[rr] & & |00\rangle \\
|01\rangle \arrow[rr] & & |01\rangle \\
|10\rangle \arrow[drr, red] & & |10\rangle \\
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\end{tikzcd}$$

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|00\rangle \arrow[r] & |00\rangle \\
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$$\begin{tikzcd} |00\rangle \arrow[r] & |00\rangle \\ |01\rangle \arrow[r] & |01\rangle \\ |10\rangle \arrow[dr] & |10\rangle \\ |11\rangle \arrow[ur] & |11\rangle \end{tikzcd}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Operations on multiple cbits: CNOT

$$C|10\rangle = C\left(\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} = |11\rangle$$

$$C|11\rangle = C\left(\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} = |10\rangle$$

Operations on multiple cbits: CNOT

$$C|00\rangle = C\left(\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} = |00\rangle$$

$$C|01\rangle = C\left(\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} = |01\rangle$$

Qubits and superposition

- Cbits are a special case of Qubits!
- A qubit is represented by $\binom{a}{b}$ where a and b are complex numbers such that $||a||^2 + ||b||^2 = 1$.
  - The cbit vectors $\binom{1}{0}$ and $\binom{0}{1}$ fit this definition.
- Example qubit values:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \quad \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$$

What does that mean?

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Superposition

The qubit is in a state of both $|0\rangle$ and $|1\rangle$. We can write this as:

$$\begin{pmatrix} a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} = a\,|0\rangle + b\,|1\rangle\,.$$

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a and b are called amplitudes. $||a||^2$ is the probability of the qubit being 0 when measured; $||b||^2$ is the probability of measuring 1.

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Measurement

The measurement of the qubit collapses its state. It will be in the state $|0\rangle$ if we measured 0 and $|1\rangle$ if we measured $1^{\dagger}$.

For example

The qubit $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ has a $\left|\left|\frac{1}{\sqrt{2}}\right|\right|^2 = \frac{1}{2}$ chance of collapsing to $|0\rangle$ or $|1\rangle$.

The qubit $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ has 100% chance of collapsing to $|0\rangle$, and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ has a 100% chance of collapsing to $|1\rangle$.

- Multiple qubits are represented by the tensor product:
  $\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ab \\ ad \\ bc \\ bd \end{pmatrix}$ with $||ac||^2 + ||ad||^2 + ||bc||^2 + ||bd||^2 = 1.$

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- For example:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}; \quad \left\|\frac{1}{2}\right\|^2 = \frac{1}{4}; \quad \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$$

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$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}; \quad \left\|\frac{1}{2}\right\|^2 = \frac{1}{4}; \quad \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$$

$\rightarrow$ There's an equal chance (25%) of measuring $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$.

- We operate on qubits in the same way as on cbits: with matrices.
- All the operations we saw so far (bit flip, CNOT, etc...) work on qubits as well.
- In reality, the matrix operations model some device that manipulates the real qubits without measurement.
- Some gates only make sense in the quantum context...

The Hadamard gate

- The Hadaramd gate puts a $|0\rangle$ or $|1\rangle$ bit into exact superposition: $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

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$$H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

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$$H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

- Note that $H^2 = HH = \mathbf{1}$ so $H^2|0\rangle = |0\rangle$ and $H^2|1\rangle = |1\rangle$.

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- This allows us to get out of superposition without measurement! So we can structure computations deterministically.