Deep Learning in Finance

Fall 2025

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What this course is ?

- What this course is?
  - In-depth exploration of advanced deep learning techniques for financial time-series & tabular data
  - Seminar format:
    - Theory -> demos -> discussion
  - Pre-requisite:
    - Familiarity with deep neural nets & backpropagation derivations
    - Python/Pytorch
- Logistics & Expectations
  - Seven sessions, each 2.5 hours
  - Come prepared: skim primary text readings; try the starter code each week

Topics at glance

- Theoretical Justifications (approximation, optimization, generalization; double descent; spectral theory)
- Transformers for Financial Time Series Forecasting
- Neural ODEs for Time-Series
- GANs for financial synthetic data
- Diffusion models for financial synthetic data
- Graph Neural Networks for financial data
- Virtue of complexity in finance

Evaluation & Materials

- Evaluation:
  - HomeWorks
  - Final Project
- Materials:
  - Lecture slides
  - Companion GitHub repository (Python/Pytorch)

Theoretical Justification of Deep Learning

Approximation & Scaling Laws

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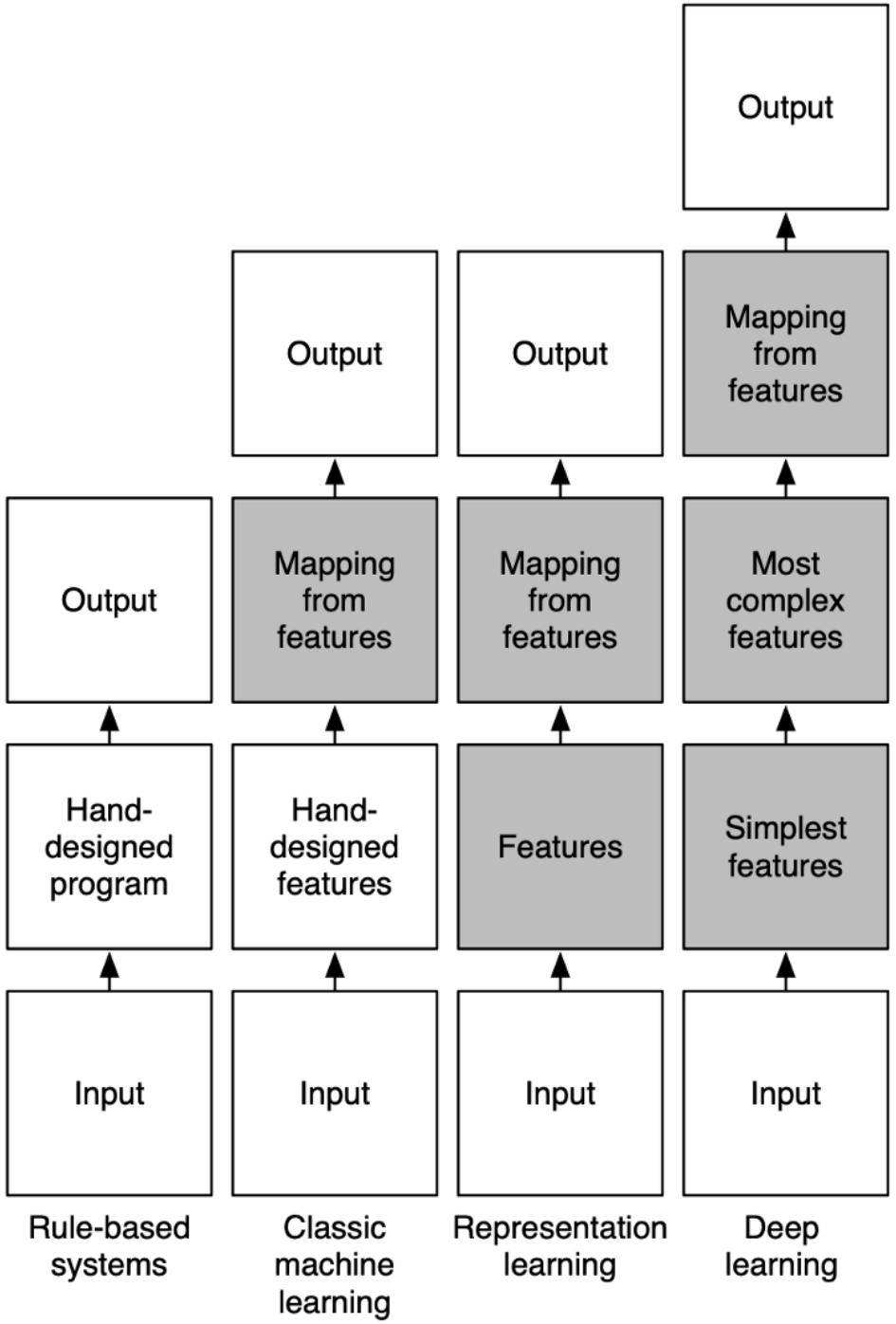
Agenda

- Success of Deep Learning
- Classic statistical learning theory recap
- Approximation
  - Universal Approximation Theorems
  - Beyond Universality: Expressivity and depth
- Neural Scaling Laws
- References

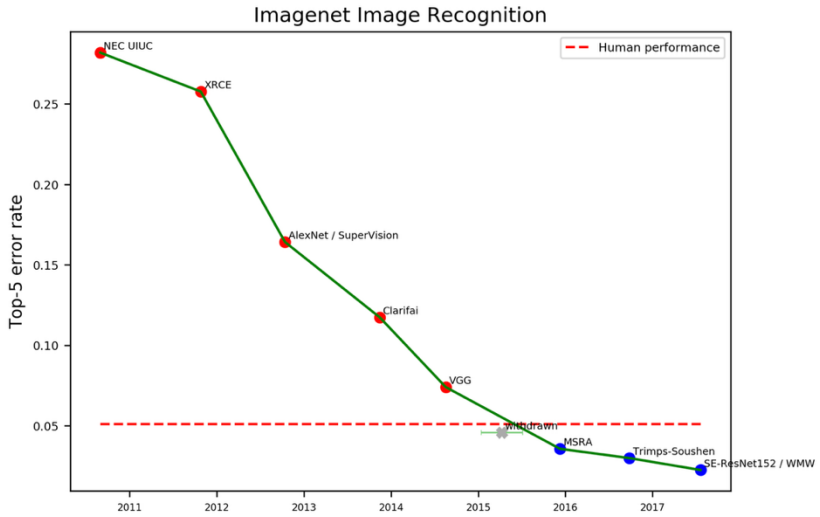
What s deep learning ?

- Big idea:
  Deep learning learns a *hierarchy of representations* directly from raw data and optimizes the whole pipeline end-to-end.
- Why it matters:
  - Replaces manual feature engineering with learned features at multiple levels of abstraction.
  - Scales with data + compute; performance keeps improving as both grow.
  - Encodes powerful inductive biases (e.g., parameter sharing in convs, attention in Transformers, compositionality).
  - Enables transfer: pretrain on broad data → fine-tune for a task.



https://www.iro.umontreal.ca/~bengioy/talks/mlss-austin.pdf

Success of deep learning



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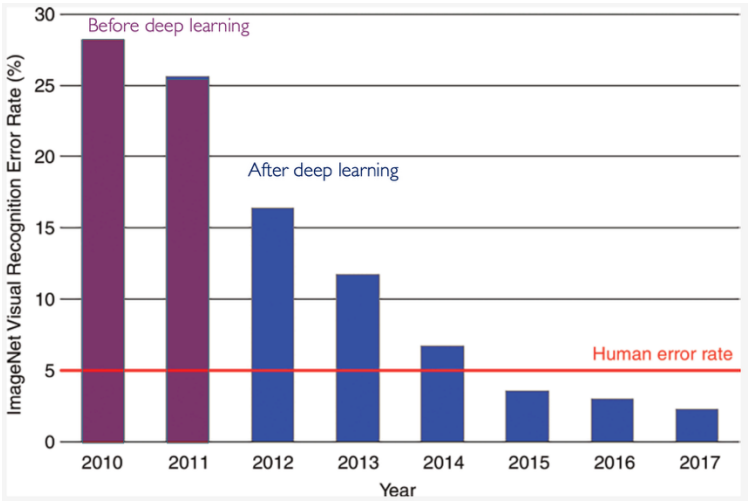

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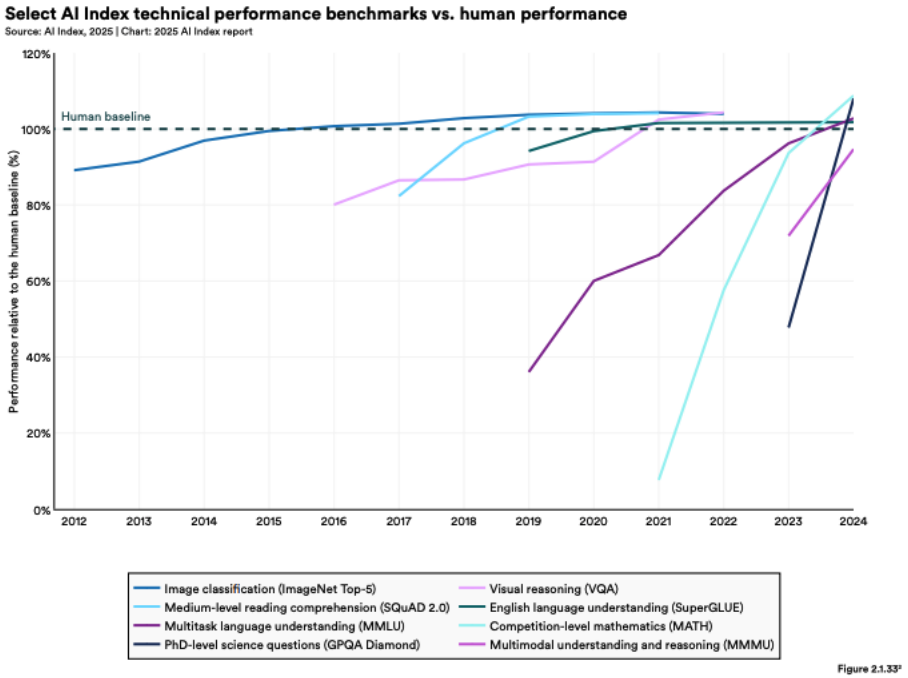

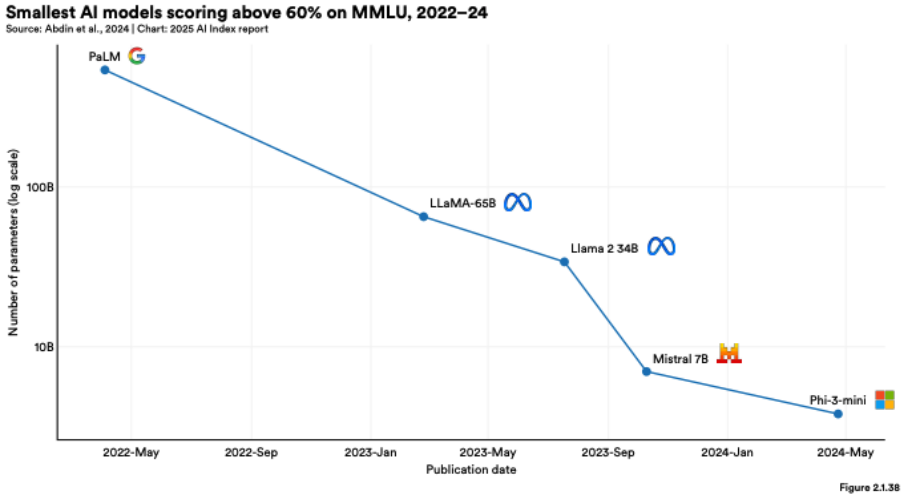

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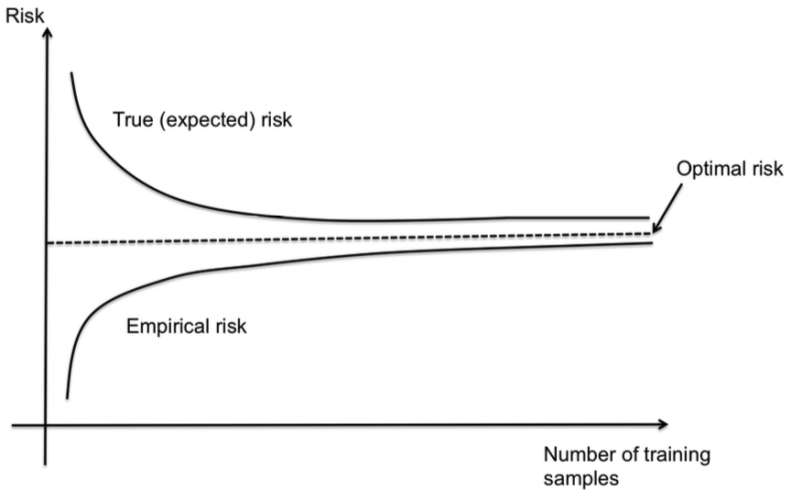
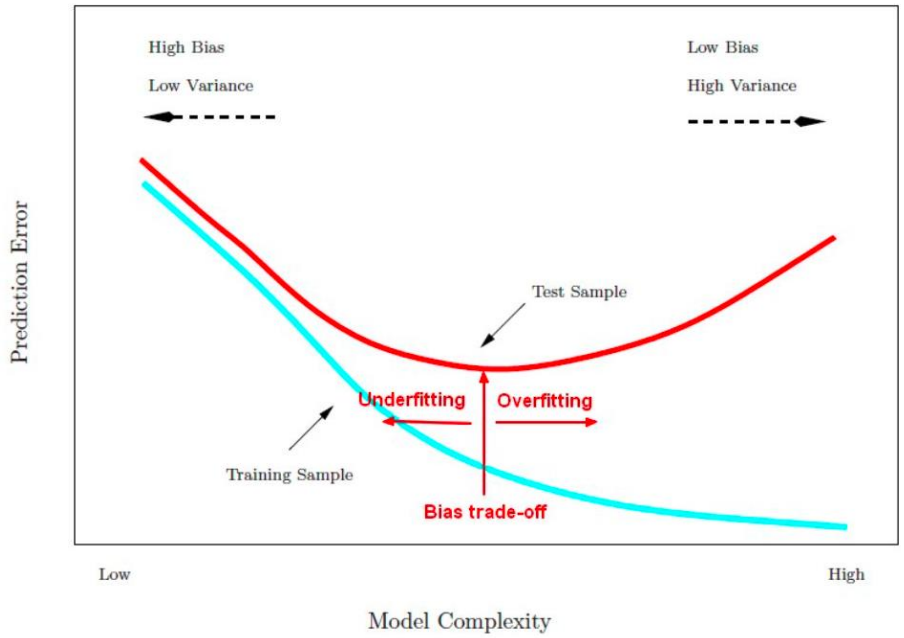
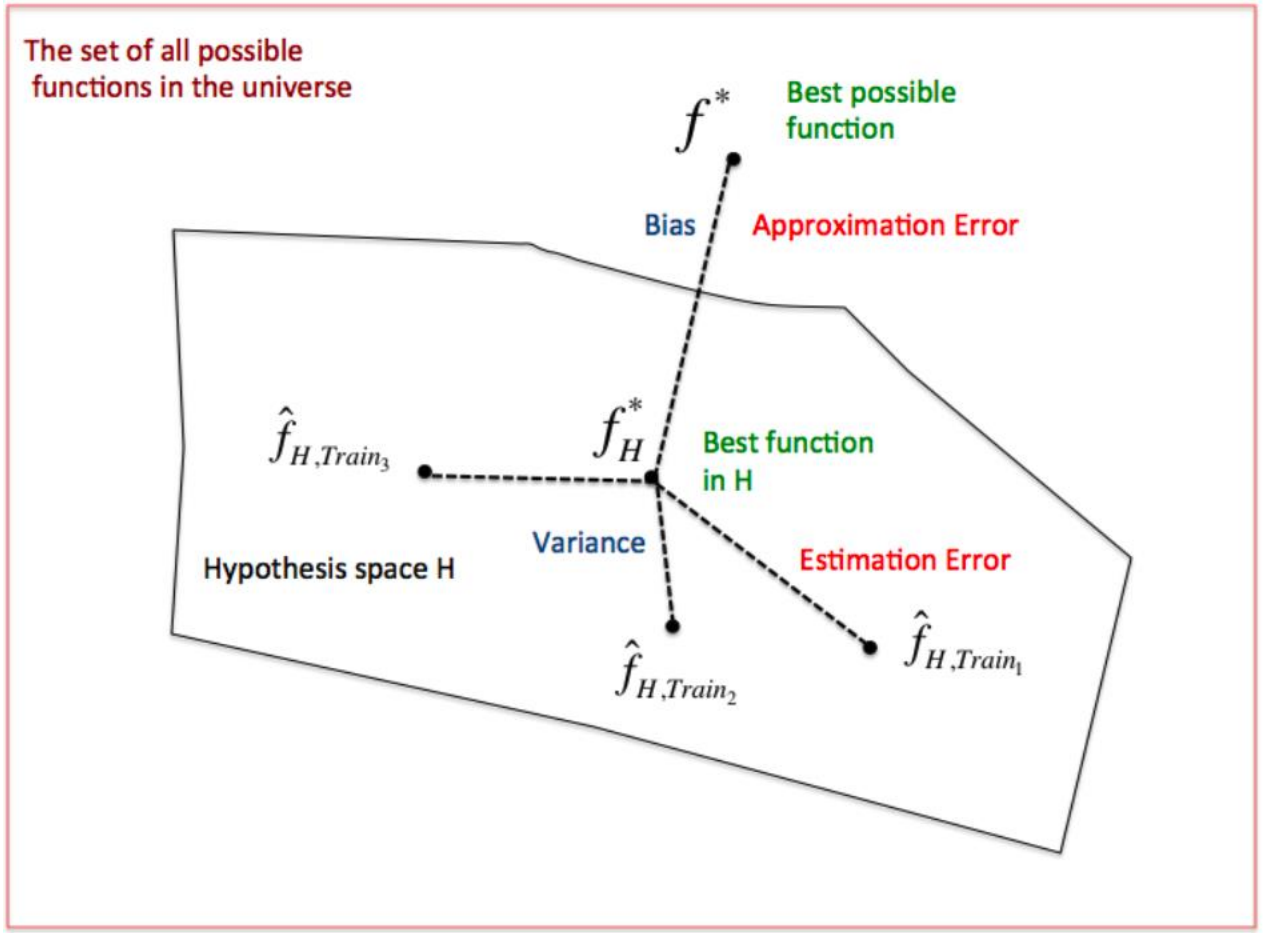
From Breakthrough to Deployment (2024–25)

- **Conclusions**
  - Error collapsed: perception tasks surpassed human level; gains have plateaued on classic benchmarks.
  - Capability broadened: strong progress across vision, language, multimodal; uneven on math/logic.
  - Efficiency jumped: “good-enough” performance now with small models (<10B), slashing cost/latency.
- **Implications**
  - New bottleneck = reasoning, reliability, and transfer; less about raw recognition.
  - Best ROI: domain-tuned small models + tool use + retrieval/knowledge graphs.
  - Eval must shift to cost-quality-safety: robustness, data provenance, and verifiability.

Classic statistical learning theory recap







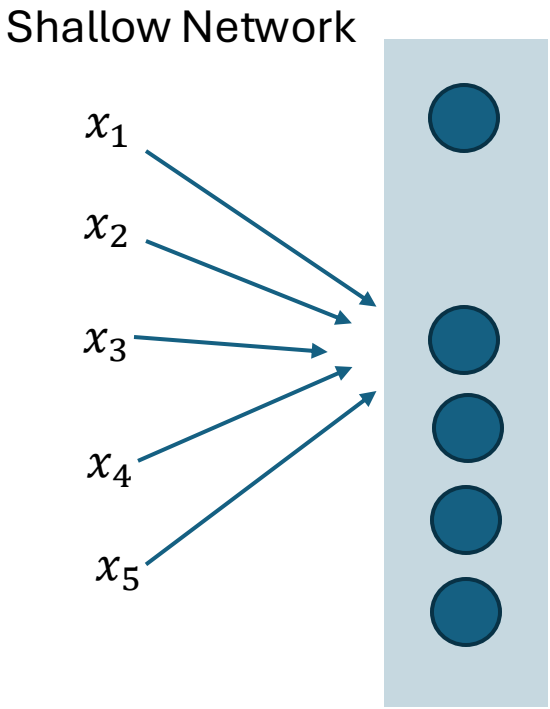
Bias–Variance: what to remember

- Test risk combines irreducible noise, approximation error, and estimation error.
- Increasing model capacity usually lowers approximation error but raises estimation error unless you add data or regularization.
- Underfitting appears when both training and validation errors remain high.
- Overfitting appears when training error is low but validation error is high.
- When both errors are high because of noisy labels or poor targets, focus on improving the data.
- You control three dials: model capacity, regularization, and the quality and quantity of data.
- Practical regularization includes weight decay, dropout, data augmentation, and early stopping.
- In deep learning, optimization provides implicit regularization and can produce double descent, yet the bias–variance trade-off still governs generalization.
- Plot learning and capacity curves first so you know whether to buy data, add regularization, or scale the model.
- Aim for the best outcome per dollar by favoring compact, well-regularized models supported by high-quality data.

The Universal Approximation Theorem

Theorem. *Let $g(x)$ be a continuous function defined in a compact subset $S \subset \mathbb{R}^l$ and any $\epsilon > 0$. Then there is a two layer neural network $\hat{g}(x)$ with $K(\epsilon)$ hidden nodes so that*

$$|g(x) - \hat{g}(x)| < \epsilon \quad \forall x \in S$$



- A single hidden-layer network with a non-linear activation can approximate any continuous function on a bounded input domain.
- *Compact subset* just means we restrict inputs to a normalized, bounded region; exactly what we do in practice.
- The result is about representational capacity only; it does not guarantee that training will find the best weights or need only a little data.

George Cybenko. Approximation by superpositions of a sigmoidal function. Mathematics of control, signals and systems, 2(4):303-314, 1989.

Proof of the Universal Approximation Theorem (1)

Definition. *A function σ is discriminatory if given a measure $\mu \in M(I_n)$ such that*

$$\int_{I_n} \sigma(w^T x + b) d\mu(x) = 0, \forall w \in \mathbb{R}^n, b \in \mathbb{R}$$

implies that $\mu = 0$

Definition. *A function $\sigma : \mathbb{R} \to \mathbb{R}$ is called sigmoidal if it is non-decreasing and satisfies the asymptotic conditions*

$$\lim_{z \to -\infty} \sigma(z) = 0 \quad \textit{and} \quad \lim_{z \to \infty} \sigma(z) = 1.$$

An example of such a function is the logistic function $\sigma(z) = \frac{1}{1+e^{-z}}$.

Lemma. *Any bounded, measurable, sigmoidal function is discriminatory. In particular, any continuous sigmoidal function is discriminatory.*

Proof of the Universal Approximation Theorem (2)

Proof. Let $C(S)$ denote the space of continuous functions on the compact set $S \subset \mathbb{R}^n$, equipped with the uniform norm. Define the set of two-layer neural networks:

$$\mathcal{N} = \left\{ \hat{g}(x) = \sum_{k=1}^{K} \alpha_k \sigma(\beta_k^T x + \gamma_k) \,\middle|\, \alpha_k, \gamma_k \in \mathbb{R},\, \beta_k \in \mathbb{R}^n,\, K \in \mathbb{N} \right\},$$

where $\sigma : \mathbb{R} \to \mathbb{R}$ is a continuous, sigmoidal activation function. We will show that $\mathcal{N}$ is dense in $C(S)$.

Suppose, for contradiction, that $\mathcal{N}$ is **not dense** in $C(S)$. By the Hahn-Banach theorem, there exists a non-zero bounded linear functional $L : C(S) \to \mathbb{R}$ that vanishes on $\mathcal{N}$, i.e., $L(f) = 0$ for all $f \in \mathcal{N}$.

By the Riesz representation theorem, any bounded linear functional on $C(S)$ can be represented by a finite signed Borel measure μ on S:

$$L(f) = \int_S f(x)\, d\mu(x) \quad \forall f \in C(S).$$

Since L vanishes on $\mathcal{N}$, we have:

$$\int_S \sigma(\beta^T x + \gamma)\, d\mu(x) = 0 \quad \forall \beta \in \mathbb{R}^n,\, \gamma \in \mathbb{R}.$$

By the lemma established above, any continuous sigmoidal function σ is discriminatory. Therefore, this integral condition implies $\mu = 0$, which contradicts the assumption that L is a non-zero functional.

Hence, $\mathcal{N}$ must be dense in $C(S)$. Density implies that for any $g \in C(S)$ and $\epsilon > 0$, there exists $\hat{g} \in \mathcal{N}$ such that:

$$\sup_{x \in S} |g(x) - \hat{g}(x)| < \epsilon.$$

In particular, $\hat{g}$ can be written as $\hat{g}(x) = \sum_{k=1}^{K(\epsilon)} \alpha_k \sigma(\beta_k^T x + \gamma_k)$, where $K(\epsilon)$ is finite. This completes the proof.

□

The proof does not help up construct such a shallow network !

Complexity of approximation

Theorem. *Let $\sigma : \mathbb{R} \to \mathbb{R}$ be infinitely differentiable, and not a polynomial. For $f \in W_m^n$ the complexity of shallow networks that provide accuracy at least ϵ is*

$$N = \mathcal{O}(\epsilon^{-\frac{n}{m}})$$

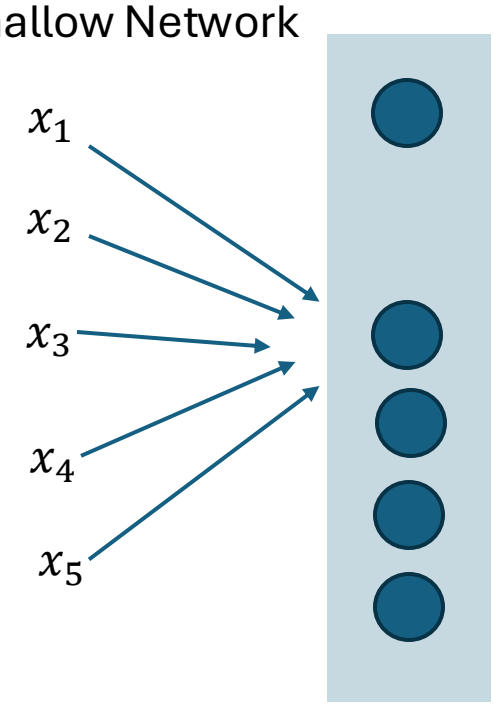
and is the best possible

Key interpretations (using n, m, ε, N)

- **Curse of dimensionality (role of n).** The exponent has n in the numerator: reducing error becomes exponentially harder as dimension grows.
  Rule of thumb: halving the error multiplies the required width by $2^{n/m}$. Example: $n = 10,\ m = 2 \Rightarrow$ need $2^5 = 32\times$ more units to go from ε to $\varepsilon/2$.
- **Smoothness helps (role of m).** Larger m (smoother f) lowers the exponent n/m, so far fewer units are needed for the same ε.
- **Tightness / optimality.** There exist constants $c, C > 0$ (independent of ε) such that

$$\left(\frac{c}{\varepsilon}\right)^{n/m} \le N(\varepsilon) \le \left(\frac{C}{\varepsilon}\right)^{n/m}.$$

  No one–hidden–layer architecture can beat the $\varepsilon^{-n/m}$ scaling uniformly over W_m^n.
- **What "complexity" means here.** "Complexity" $= N$ (the width; parameter count scales linearly with N up to constants). Training/sample requirements typically grow with N.
- **When this bound can be beaten** If f has additional structure (e.g., compositional structure, low-dimensional manifold structure, Barron class) or if depth is allowed, rates can improve and the effective n in the exponent can drop. This theorem applies to general W_m^n targets with shallow nets.



Hrushikesh N. Mhaskar and Tomaso Poggio (2016) , Deep vs. Shallow Networks: an Approximation Theory Perspective

Compositional functions

Formal definition (binary case). Let $k = 2$. For $L \in \mathbb{N}$ and $n = 2^L$, define the class $\mathcal{C}_{2,L}$ of compositional functions $f : \mathbb{R}^n \to \mathbb{R}$ with a binary-tree computational graph:

$$f(x_1, \ldots, x_{2^L}) = h_L\Big(h_{L-1,1}(h_{L-2,1}(x_1, x_2),\ h_{L-2,2}(x_3, x_4)),\ h_{L-1,2}(h_{L-2,3}(x_5, x_6),\ h_{L-2,4}(x_7, x_8))\Big),$$

where each constituent map $h_{\ell,j} : \mathbb{R}^2 \to \mathbb{R}$ belongs to W_m^2 (two-variable Sobolev class of order m). Depth $= L = \log_2 n$.

Why this matches real data.

- Images: local edges → motifs → objects (CNN hierarchy).
- Time series: short-range dynamics → multi-scale patterns (stacked/dilated).
- Tabular: feature pairs/crosses → segment-level aggregations.

Practical benefits: locality & reuse, parameter sharing, better sample efficiency, and a natural inductive bias.

Takeaway: Depth aligns the network with the computation graph of f. When f is compositional, we replace one hard n-D approximation by many easy 2-D ones, shrinking the ε-exponent from n to 2.

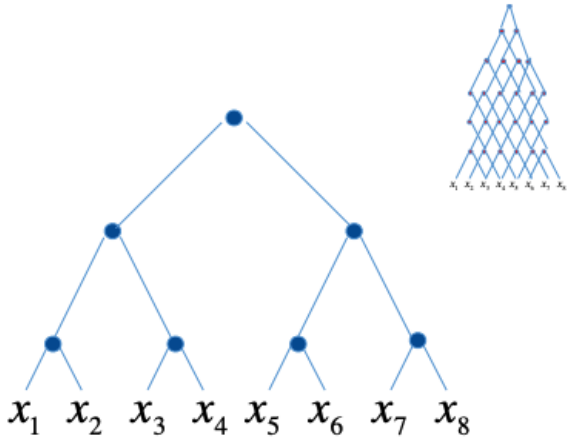


Figure 3: *A binary tree hierarchical network in 8 variables, which approximates well functions of the form (2.2). Each of the nodes consists of n units and computes the ridge function* $\sum_{i=1}^{n} a_i \sigma(\langle \mathbf{v}_i, \mathbf{x} \rangle + t_i)$, *with* $\mathbf{v}_i, \mathbf{x} \in \mathbb{R}^2$, $a_i, t_i \in \mathbb{R}$. *Similar to the shallow network such a hierarchical network can approximate any continuous function; the text proves how it approximates compositional functions better than a shallow network. Shift invariance may additionally hold implying that the weights in each layer are the same. The inset at the top right shows a network similar to ResNets: our results on binary trees apply to this case as well with obvious changes in the constants*

Hrushikesh N. Mhaskar and Tomaso Poggio (2016) , Deep vs. Shallow Networks: an Approximation Theory Perspective

Depth Efficiency

Theorem. *For $f \in W_m^{n,2}$ consider a deep network with the same compositional architecture and with an activation function $\sigma : \mathbb{R} \to \mathbb{R}$ which is infinitely differentiable, and not a polynomial. The complexity of the network to provide approximation with accuracy at least ϵ is:*

$$N = \mathcal{O}((n-1)\epsilon^{-\frac{2}{m}})$$

Clearly, deep networks, when aligned with the compositional structure of the target function, can approximate much more efficiently, with complexity scaling only linearly in dimension and polynomially in accuracy.

Feature	Shallow Networks	Deep Networks
Function class	$f \in W_m^n$ (general Sobolev)	$f \in W_m^{n,2}$ (compositional structure)
Approximation complexity	$\mathcal{O}(\epsilon^{-n/m})$	$\mathcal{O}((n-1)\epsilon^{-2/m})$
Curse of dimensionality	Yes (exponential in n)	Avoided (linear in n)
Optimality	Yes (best possible)	Not necessarily, but much better
Assumes structure in f	No	Yes (hierarchical/compositional)

Telgarsky, M. (2016). *Benefits of depth in neural networks*.
Hrushikesh N. Mhaskar and Tomaso Poggio. (2016) Deep vs. Shallow Networks: an Approximation Theory Perspective
Tomaso Poggioa, Andrzej Banburskia , and Qianli Liao (2019) Theoretical Issues in Deep Networks

Architecture x Function class comparison

Width $N(\varepsilon)$ needed for error $\leq \varepsilon$

Architecture	Function class: General Sobolev W_m^n	Function class: Compositional (binary; nodes $h \in W_m^2$)
Shallow (1 hidden layer)	$\Theta(\varepsilon^{-n/m})$	$\Theta(\varepsilon^{-n/m})$
Deep compositional (depth $\log_2 n$)	**Unknown**	$\mathcal{O}(n\,\varepsilon^{-2/m})$

- "Unknown" = no established general improvement theorem for a compositional **architecture** when f is an arbitrary W_m^n function (worst-case curse $\Theta(\varepsilon^{-n/m})$ persists for general deep classes).
- Rates hide constants/log factors and depend on the domain/norm.
- For a k-ary compositional graph, replace 2 by k: $\mathcal{O}(n\,\varepsilon^{-k/m})$.

Beyond approximation: Empirical Laws of Neural Scaling

- We've just seen how deep networks can overcome the curse of dimensionality for *structured functions*
  - they offer provably better approximation rates than shallow networks when the function has a compositional architecture
- These theorems give us a *theoretical foundation* for why deep networks are more powerful. But in practice, the success of deep learning has gone far beyond what these theorems predict; especially in large-scale models.
- This raises a natural question: *As we make models bigger*, train on *more data*, and use *more compute*, how does performance improve? This is not just a theoretical curiosity; it's become a central principle in modern machine learning.
- This brings us to **neural scaling laws:** empirical laws that describe how error decreases as a power law in model size, dataset size, or compute. While the approximation results give us asymptotic rates for a fixed function class, scaling laws describe behavior across model classes as we scale the system.
- In a sense, scaling laws are the *empirical counterparts* to the approximation bounds we just saw. Instead of asking 'how complex a network do I need to approximate a fixed function?', we ask 'how does performance scale as I increase capacity?

Scaling Laws

Scaling laws are empirical regularities—typically **power laws**—that describe how a model's error-like metric (e.g., pre-training loss) changes **predictably** as we scale core resources under a fixed training *recipe* (same objective, architecture family, optimizer/schedule, and data distribution). In practice, single-factor "slices" (varying one resource while the others are not bottlenecking) look like straight lines on log–log plots.

A commonly used aggregate form for the *reducible* part of the loss is:

$$\boxed{L(N, D, C) \approx L_\infty + A N^{-\alpha} + B D^{-\beta} + E C^{-\gamma}}$$

where:

- L_∞ is the irreducible loss (noise floor),
- N = non-embedding parameters, D = training tokens, C = training compute (FLOPs),
- $A, B, E > 0$ are fit constants and $\alpha, \beta, \gamma > 0$ are **scaling exponents**.
  Single-factor slices then have slopes $-\alpha$, $-\beta$, and $-\gamma$ on log–log axes.

Why this matters. These laws let you **forecast** performance when scaling up and **plan budgets** (e.g., under $C \approx kND$ you can derive compute-optimal $N^\star(C)$ and $D^\star(C)$). They're descriptive (recipe-dependent), and can bend under data duplication/quality changes, optimization-regime shifts, or distribution shift.

- L: training/pretraining loss (e.g., cross-entropy)
- L_∞: irreducible loss (asymptote / noise floor)
- N: non-embedding parameters (model capacity)
- D: training tokens (effective after curation/dedup)
- C: training compute in FLOPs actually performed
- k: proportionality constant in $C \approx kND$ (depth, sequence length, efficiency)
- $A, B, E > 0$: fit constants ("amplitudes") for the N, D, and C terms
- α: parameter-scaling exponent (slope of $\log(L - L_\infty)$ vs $\log N$)
- β: data-scaling exponent (slope vs $\log D$)
- γ: compute-scaling exponent (slope vs $\log C$, single-factor slice)
- $\eta = \frac{\alpha\beta}{\alpha+\beta}$: envelope exponent for compute-optimal frontier $L^\star(C)$

Data Scaling Laws

Equation of the Law

$$L(D) \approx L_\infty + B D^{-\beta}, \quad \text{so on log–log axes: slope} = -\beta.$$

As training tokens D grow; when model capacity and steps aren't limiting; the **reducible loss** $L - L_\infty$ follows a power law.

How to fit well

- Use reducible loss $L - L_\infty$ (estimate L_∞ jointly or from a floor).
- Vary D over ≥ 1 order of magnitude; keep N (params) and steps large enough.
- Deduplicate/curate; report CIs for β.

Interpretation (elasticity)

- More tokens reduce loss with predictable elasticity

$$\varepsilon_D = \frac{\partial \log(L - L_\infty)}{\partial \log D} = -\beta.$$

Pitfalls

- Heavy repetition / low quality data flatten β.
- Domain shift moves slope and the asymptote L_∞.

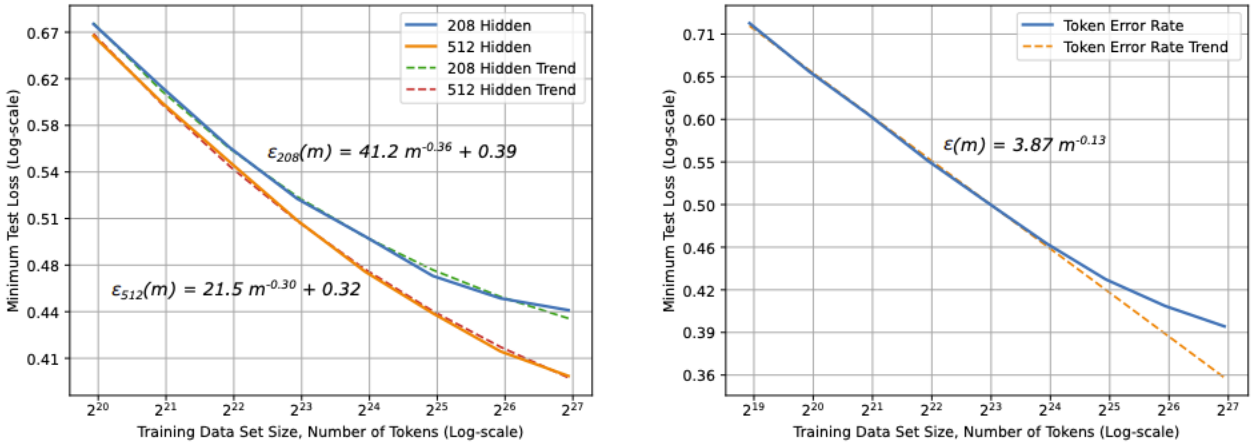


Figure 1: Neural machine translation learning curves. Left: the learning curves for separate models follow $\varepsilon(m) = \alpha m^{\beta_g} + \gamma$. Right: composite learning curve of best-fit model at each data set size.

Joel Hestness et al (2017), Deep Learning Scaling is Predictable, Empirically

Compute Scaling laws

Because each token “touches” essentially all N parameters

Equation (single-factor slice)

$$L(C) \approx L_\infty + EC^{-\gamma}.$$

Compute constraint (decoder LM, fixed recipe)

$$C \approx kND$$

where N = parameters, D = token exposures, and $k > 0$ absorbs depth/seq length/efficiency.

Compute-optimal allocation (derivation sketch)

Minimize

$$L(N, D) = AN^{-\alpha} + BD^{-\beta}$$

s.t. $ND = C/k$. Then

$$N^\star(C) = \kappa_N C^{\frac{\beta}{\alpha+\beta}}, \qquad D^\star(C) = \kappa_D C^{\frac{\alpha}{\alpha+\beta}},$$

and the **optimal envelope** becomes

$$L^\star(C) = L_\infty + KC^{-\eta}, \qquad \eta = \frac{\alpha\beta}{\alpha+\beta}.$$

Inference-aware note

- If serving dominates TCO, training **smaller** N longer (larger D) can beat a huge model at fixed budget.
- Test-time compute (best-of-n, self-consistency) also “scales” quality at fixed N.

Ops checklist

- Keep optimizer/schedule fixed while fitting γ; regime changes bend the curve.
- Track achieved TFLOPs/MFU to avoid miscounting C.

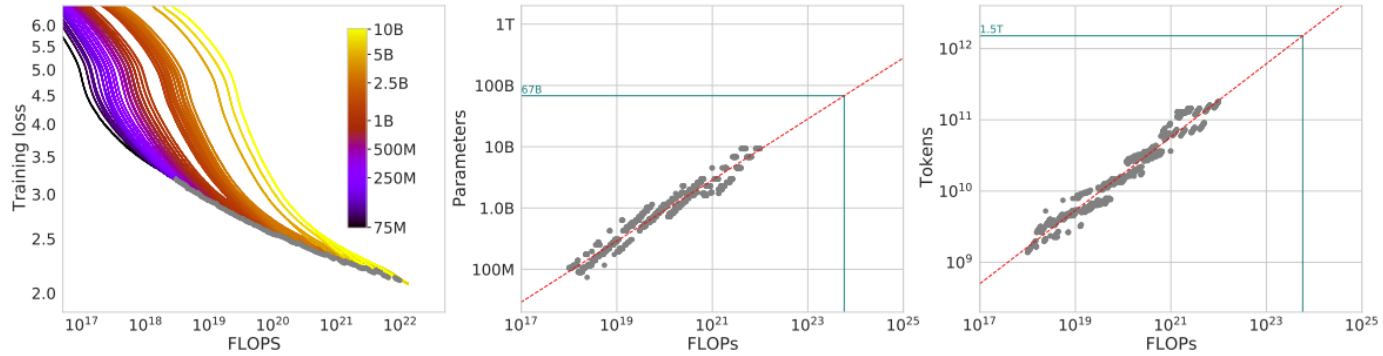


Figure 2 | **Training curve envelope.** On the **left** we show all of our different runs. We launched a range of model sizes going from 70M to 10B, each for four different cosine cycle lengths. From these curves, we extracted the envelope of minimal loss per FLOP, and we used these points to estimate the optimal model size (**center**) for a given compute budget and the optimal number of training tokens (**right**). In green, we show projections of optimal model size and training token count based on the number of FLOPs used to train *Gopher* (5.76×10^{23}).

Jordan Hoffmann et al (2022) , Training Compute-Optimal Large Language Models

Parameters scaling laws

Equation (single-factor slice)

$$L(N) \approx L_\infty + A N^{-\alpha}, \quad \text{so on log–log axes: slope} = -\alpha.$$

Meaning

In the **capacity-limited** regime (enough tokens & steps), increasing N steadily reduces loss.

How to fit well

- Ensure enough tokens/steps (avoid undertraining); vary N over ≥ 1–2 orders.
- Fit $L - L_\infty$ vs $\log N$; bootstrap CIs for α.

Interpretation (elasticity)

$$\varepsilon_N = \frac{\partial \log(L - L_\infty)}{\partial \log N} = -\alpha.$$

Pitfalls

- Too few tokens/steps flatten the apparent $-\alpha$ slope ("big models don't help" = bottleneck elsewhere).
- Architecture/parametrization changes (e.g., depth/width, μP) shift constants and can nudge α.

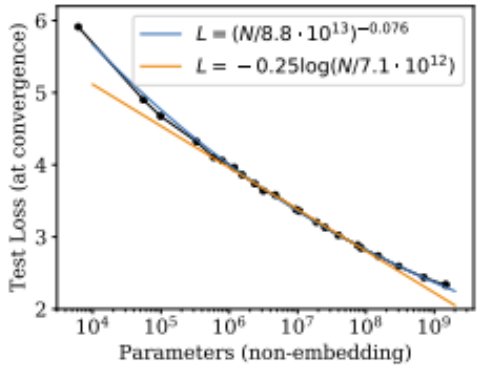


Figure 23 The trend for performance as a function of parameter count, $L(N)$, is fit better by a power law than by other functions such as a logarithm at a qualitative level.

Jared Kaplan et al (2020), Scaling Laws for Neural Language Models

Optimal Allocation of Resources: Scaling laws

- **Compute-Optimal Training:** For a fixed compute budget C, model size N and dataset size D should scale approximately equally: $N \propto C^{0.50}$ and $D \propto C^{0.50}$. Doubling compute means ~1.4× larger models trained on ~1.4× more data.
- **Sample Efficiency:** Larger models achieve better performance per training token. A 10× larger model can reach the same loss with ~5-10× fewer training examples, making larger models more data-efficient during training.
- **Loss Scaling:** Performance improves predictably with compute as a power law: $L \propto C^{-\alpha}$ where $\alpha \approx 0.05 - 0.07$. Each 10× increase in compute reduces loss by roughly 10-15%.
- **Inference-Optimal Strategy:** If inference costs dominate deployment, favor smaller models trained longer on more data ($D \propto C^{0.55}$, $N \propto C^{0.45}$). Training compute is one-time; inference costs scale with usage.
- **Over-training vs. Under-training:** Training beyond the compute-optimal point shows diminishing returns. Under-trained large models underperform smaller, well-trained ones with the same training compute.
- **Emergent Scaling:** Some capabilities appear suddenly at specific scales rather than smoothly, though this remains debated. Critical thresholds may exist for certain reasoning tasks.

What Universal Approximation Theorems Really Tell Us ?

Advantages:

They guarantee representational capacity.

Limitations:

- They do **not** explain why or how deep nets generalize.
- They do **not** address training dynamics or sample efficiency.

- So if representational power isn't enough, what explains generalization in overparameterized deep networks ?
- **That is what we will explore in** Optimization & Generalization in the second part of the lecture.

What Scaling Laws Really Tell Us ?

- **Advantages:**
  - They provide **empirical regularities**: loss typically falls as a **power law** with model size and data.
  - They guide **budget allocation**: show the **compute-optimal tradeoff** between parameters, data, and training time.
  - They enable **predictability**: small pilot runs can forecast returns at larger scale.
  - They highlight **diminishing returns** and when to spend on **data quality** instead of more parameters.

- **Limitations:**
  - They are **descriptive, not causal** explanations of why models work.
  - Exponents **depend on task, data, and architecture**; they can shift with changes or distribution shift.
  - They say **nothing about optimization stability** (learning rates, curvature, mode collapse).
  - They do **not guarantee generalization** or sample efficiency; they average over many confounders.

- **So if scaling alone isn't enough, what explains generalization and efficient training? That is what we will explore** next: optimization & generalization—implicit bias of SGD, flat vs sharp minima, margins, regularization & augmentation, data curriculum, and how architecture + data shape out-of-sample performance.

Illustration notebooks

- Approximation
  - https://github.com/omroot/DLinFinance_BaruchMFE2025/blob/master/src/notebooks/theory/approximation.ipynb
- Scaling Laws
  - https://github.com/omroot/DLinFinance_BaruchMFE2025/blob/master/src/notebooks/theory/scaling_laws.ipynb

References

- George Cybenko. Approximation by superpositions of a sigmoidal function. Mathematics of control, signals and systems, 2(4):303-314, 1989.
- Jared Kaplan et al (2020), Scaling Laws for Neural Language Models
- Joel Hestness et al (2017), Deep Learning Scaling is Predictable, Empirically
- Telgarsky, M. (2016). *Benefits of depth in neural networks*.
- Hrushikesh N. Mhaskar and Tomaso Poggio. (2016) Deep vs. Shallow Networks: an Approximation Theory Perspective
- Tomaso Poggioa, Andrzej Banburskia , and Qianli Liao (2019) Theoretical Issues in Deep Networks
- Hrushikesh N. Mhaskar and Tomaso Poggio (2016) , Deep vs. Shallow Networks: an Approximation Theory Perspective