Assignment 9

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1 Introduction

- We explore digital fourier transform (DFT) with windowing. This is used to make the signal square integrable, and more specifically, that the function goes sufficiently rapidly towards 0 , also to make infinitely long signal to a finite signal, since to take DFT we need finite aperiodic signal.

- Windowing a simple waveform like $cos(\omega t)$, causes its fourier transform to develop non-zero value at frequencies other than ω. This is called *Spectral Leakage*. This can cause in some applications the stronger peak to smear the weaker contounter parts. So choosing proper windowing functions is essential. The windowing function we use is called **Hamming window** which is generally used in $narrow-$ *band applications*.

2 Python Code

2.1 Question 1:

- Consider the function $\cos^3(\omega_0 t)$. Obtain its spectrum for $\omega_0 = 0.86$ with and without a Hamming window.

Code:

```
def f(t,n,w=None,d=None):
    if(n == 1):
        return sin(sqrt(2)*t)
    elif(n==2):
        if(w is None):
            return pow(cos(0.86*t),3)
        elif(w!=None):
            return pow(cos(w*t),3)
    elif(n==3):
        return cos(16*(1.5+t/(2*pi))*t)
    elif(n==4):
        return t
    elif(n==5):
        if(w is None):
            return cos(0.86*t)
        elif(w!=None and d!=None):
            return cos(w*t+d)
    else:
        return sin(sqrt(2)*t)

def window_fn(n,N):
    return (0.54+0.46*cos(2*pi*n/N))

def findFFT(low_lim,up_lim,no_points,n,window=True,wo=None,d=None,eps=0):
    t = linspace(low_lim,up_lim,no_points+1)[:-1]
    dt=t[1]-t[0]
    fmax=1/dt
    N = no_points
    y = f(t,n,wo,d)+eps*randn(len(t))

    if(window):
        n1=arange(N)
        wnd=fftshift(window_fn(n1,N))
        y=y*wnd

    y[0]=0
```

```
    y=fftshift(y)
    Y=fftshift(fft(y))/N

    w = linspace(-pi*fmax,pi*fmax,N+1)[:-1]
    return t,Y,w

def plot_FFT(t,Y,w,Xlims,plot_title,fig_no,dotted=False,Ylims=None):

    figure()
    subplot(2,1,1)

    if(dotted):
        plot(w,abs(Y),'b',w,abs(Y),'bo',lw=2)
    else:
        plot(w,abs(Y),'b',lw=2)

    xlim(Xlims)

    ylabel(r"$|Y(\omega)| \to$")
    title(plot_title)
    grid(True)

    ax = subplot(2,1,2)
    ii=where(abs(Y)>0.005)
    plot(w[ii],angle(Y[ii]),'go',lw=2)

    if(Ylims!=None):
        ylim(Ylims)

    xlim(Xlims)
    ylabel(r"$\angle Y(j\omega) \to$")
    xlabel(r"$\omega \to$")
    grid(True)
    savefig("fig10-"+fig_no+".png")
    show()

t,Y,w = findFFT(-4*pi,4*pi,256,2,True)
Xlims = [-8,8]
plot_title = r"Figure 8 : Spectrum of Windowed $\cos^3(\omega_o t)$"
plot_FFT(t,Y,w,Xlims,plot_title,"8")
t,Y,w = findFFT(-4*pi,4*pi,256,2,False)
Xlims = [-8,8]
plot_title = r"""Figure 9 : Spectrum of $\cos^3(\omega_o t)$
                 without windowing"""
```

```
plot_FFT(t,Y,w,Xlims,plot_title,"9")
```

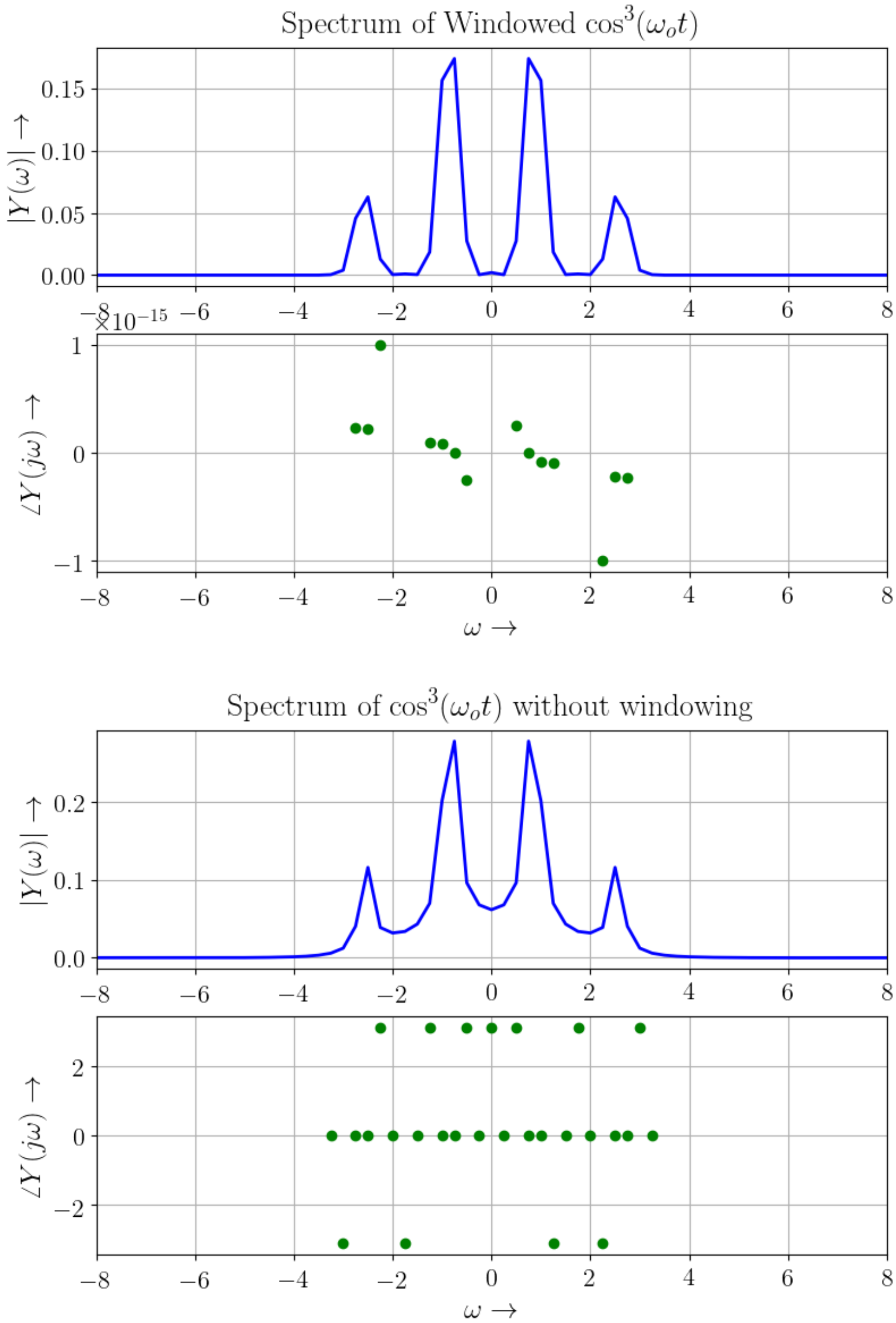


Figure 1: Spectrum of $\cos^3(\omega_o t)$ with and without windowing

2.1.1 Results and Discussion:

- Here we can see clear differences between the windowed fourier transform and fourier transform without application of windows. Peak is being smeared by the windowing function but the stary high frequency components are attenuated by the window function. The *spectral leakage* can also be noticed.

2.2 Question 2:

- Write a program that will take a 128 element vector known to contain $\cos(\omega_0 t + \delta)$ for arbitrary δ and $0.5 < \omega_0 < 1.5$ where $\pi \leq t \leq \pi$.
- You have to extract the digital spectrum of the signal, find the two peaks at $\pm\omega_0$, and estimate ω_0 and δ.

2.2.1 Estimating ω and δ from fourier spectrum

- According to the question the if the spectra is obtained, the resolution is not enough to obtain the ω_0 directly. The peak will not be visible clearly because of the fact that resolution of the frequecny axis is not enough. So a statistic is necessary to estimate value of ω_0
- Let,

$$\mu = Mean(|Y(\omega)|) \tag{1}$$

$$\sigma = Standard\ Deviation(|Y(\omega)|) \tag{2}$$

$$\omega_0 = \frac{\sum \omega_i |Y(\omega_i)|}{\sum |Y(\omega_i)|} \forall \omega_i\ where\ |Y(\omega_i)| > \mu + 0.1\sigma \tag{3}$$

- Which is essentially the weighted average of ω where weights are $|Y(\omega)|$ subject to constraint that $|Y(j\omega)|$ must be greater than a value as specified in the formula.
- Now, δ can be found by two ways :
- Least square fitting of

$$y(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) \tag{4}$$

- Minimizing L2-norm to find the coefficients A, B, we can compute δ by,

$$\delta = -\tan^{-1}(\frac{B}{A}) \tag{5}$$

- Another method which can be used is to find the phase of the discrete fourier transform at ω_o nearest to estimated ω using the above statistic.
- This works because the phase of $\cos(\omega_o t + \delta)$ when $\delta = 0$ is 0, so when its not its δ, so we can estimate it by this approach.

- The latter approach is used in this assignment.

Code:

```
def estimate_omega(low_lim, up_lim, eps):
    w_actual = np.random.uniform(low_lim, up_lim)
    delta_actual = (randn())
    t,Y,w = findFFT(-1*pi,1*pi,128,5,True,w_actual,delta_actual,eps=eps)

    Y_half = Y[int(len(Y)/2):]
    w_half = w[int(len(w)/2):]
    k = 0.1
    idx =
    np.where(abs(Y_half) >= np.mean(abs(Y_half))+
                                   k*sqrt(np.var(abs(Y_half))))

    w0 = np.matmul(w_half[idx],
                 np.transpose(abs(Y_half[idx])))/
                                        (np.sum(abs(Y_half[idx])))

    w_peak_idx = (np.abs(w_half-w0)).argmin()
    delta      = angle(Y_half[w_peak_idx])

    print("Actual w0 : %g , Actual delta : %g"%(w_actual,delta_actual))



    return t,w0,delta,w_actual,delta_actual


def createAmatrix(nrow,t,model,wo):
    A = zeros((nrow,2))
    A[:,0],A[:,1] = model(t,wo)
    return A

def modelA(t,wo):
    return (cos(wo*t),sin(wo*t))
def estimate_delta(t,wo,y):
    M = createAmatrix(len(y),t,modelA,wo)
    c = (lstsq(M,y)[0])

    delta = arccos(c[0]/sqrt(pow(c[0],2)+pow(c[1],2)))
    return delta
def estimator(N,eps):
    est_err_w = []
```

```
    est_err_delta = []
    for i in range(N):
        t,wo_est,delta_est,w_actual,delta_actual =
                            estimate_omega(0.5,1.5,eps)


        est_err_w.append(abs(amax(w_actual-wo_est)))
        est_err_delta.append(abs(amax(delta_actual-delta_est)))

        print("Estimated w0 : %g and delta : %g \n"%((wo_est),delta_est))
    return np.mean(est_err_w),np.mean(est_err_delta)

eps = 0
N = 5

estimated_error_w,estimated_error_delta = estimator(N,eps)

print("\nEstimated Error for %g sample signals without Noise addition
in w0 and delta : %g , %g"%(N,estimated_error_w,estimated_error_delta))
```

2.2.2 Results and Discussion:

- Estimated Error for 5 sample signals without Noise addition in w0 and delta : 0.117576 , 0.0214041
- As we observe that actual ω & δ are generated using uniformly distributed random functions, and using the above center of mass statistic we get estimated ones close with error in order of 1%

2.3 Question 3:

- Now we add **white gaussian noise** to data in Q_3. This can be generated by $randn()$ in python. The extent of this noise is 0.1 in amplitude (i.e., $0.1 * randn(N)$, where N is the number of samples).
- Repeat the problem and find the ω_0 and δ

Code:

```
eps = 0.1
N   = 5

estimated_error_w , estimated_error_delta = estimator(N, eps)

print("\nEstimated Error for %g sample signals with Noise addition in
w0 and delta : %g , %g"% (N, estimated_error_w , estimated_error_delta))
```

2.3.1 Results and Discussion:

- Estimated Error for 5 sample signals with Noise addition in w0 and delta : 0.12154 , 0.0247817
- Now we have added noise to the function and tried to estimate ω & δ from the spectrum.
- We follow same procedure as above case and we can observe that ω & δ are generated using uniformly distributed random functions, and using the above center of mass statistic we get estimated ones close with error in order of 10% in ω and 1% in δ
- This error is slightly higher compared to the case without noise as we expected.

2.4 Question 4 - Analysis of Chirped Signal Spectrum

- Plot the *DFT* of the function $\cos(16\ (1.5 + \frac{t}{2\pi})\ t)$ where $-\pi \leq t \leq \pi$ in 1024 steps. This is known as a *chirped* signal.
- Its frequency continuously changes from 16 to 32 radians per second. This also means that the period is 64 samples near π and is 32 samples near $+\pi$.

Code:

```
def chirp(t):
    return cos(16*(1.5+t/(2*pi))*t)
t = linspace(-pi,pi,1025)[:-1]
dt=t[1]-t[0]
fmax=1/dt
N = 1024
y = chirp(t)
Y=fftshift(fft(y))/N
w = linspace(-pi*fmax,pi*fmax,N+1)[:-1]
Xlims = [-100,100]
plot_title = r"Spectrum of Non-Windowed Chirped Signal"
plot_FFT(t,Y,w,Xlims,plot_title,"10")
t = linspace(-pi,pi,1025)[:-1]
dt=t[1]-t[0]
fmax=1/dt
N = 1024
y = chirp(t)

n1=arange(N)
wnd=fftshift(window_fn(n1,N))
y=y*wnd
y[0] = 0
y = fftshift(y)
Y = fftshift(fft(y))/1024.0
w = linspace(-pi*fmax,pi*fmax,1025)[:-1]

Xlims = [-100,100]
plot_title = r"Spectrum of Windowed Chirped Signal"
plot_FFT(t,Y,w,Xlims,plot_title,"11")
```

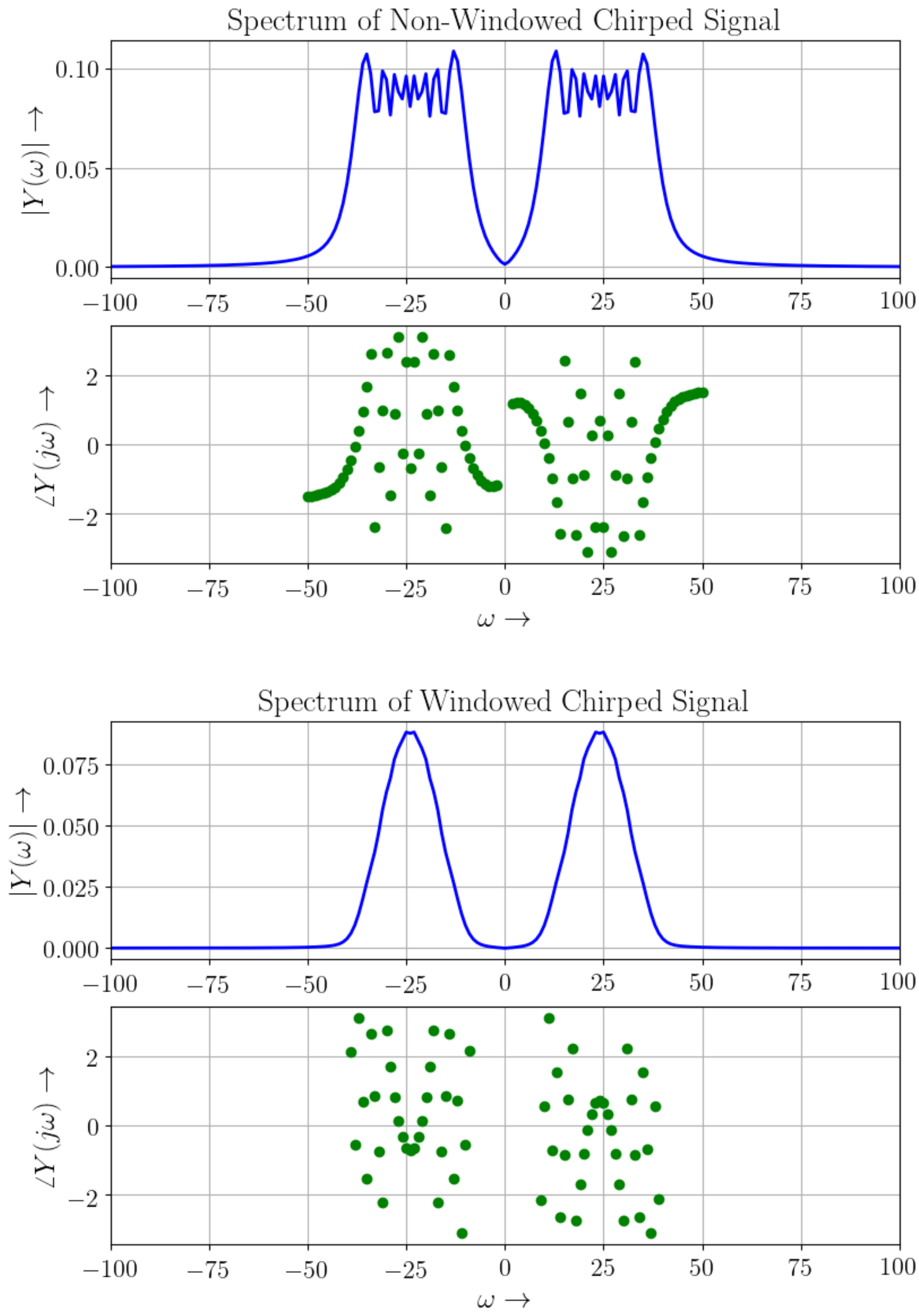


Figure 2: Spectrum of chirped signal with and without windowing

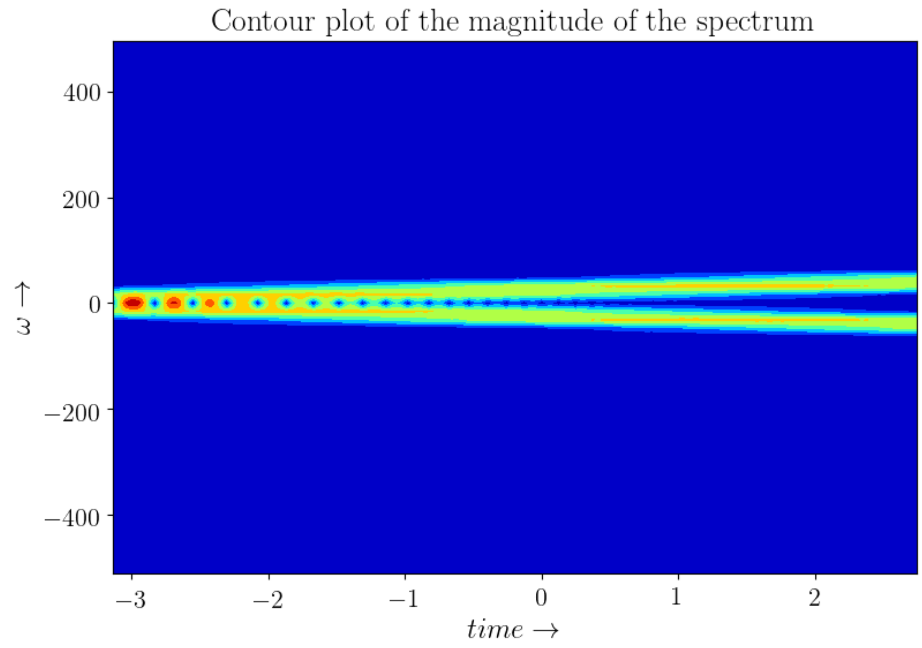
2.5 Question 5:

- For the same chirped signal, break the 1024 vector into pieces that are 64 samples wide. Extract the DFT of each and store as a column in a 2D array.
- Then plot the array as a surface plot to show how the frequency of the signal varies with time.
- Plot and analyse the **time frequency** plot, where we get localized $DFTs$ and show how the spectrum evolves in time.
- We are going to implement this using **Short time Fourier Transform** (STFT)
- From the 1024 dimensional vector, we take 64 dimensional sub-vector, and find the fourier transform and we will see how it evolves in time.This is known as **Short time Fourier Transform** (STFT)

Code:

```
def partition(t,n):
  t_batches = [t[i:n+i] for i in range(len(t)-n)]
  return t_batches
def STFT(t,no_samples,n):
  dt=t[1]-t[0]

  fmax=1/dt
  N = no_samples
  y = f(t,n)

  n1=arange(N)
  wnd=fftshift(window_fn(n1,N))
  y=y*wnd

  Y=fftshift(fft(y))/N

  w = linspace(-pi*fmax,pi*fmax,N+1)[:-1]
  return t,Y,w
n = 64
t_batches = partition(t,n)

batch_dfts = []
batch_ts   = []

for i in range(len(t_batches)):
```

```
    t,Y,w = STFT(t_batches[i],n,3)
    batch_dfts.append(Y)
    batch_ts.append(t)
t = linspace(-pi,pi,1025)[:-1]
T, W = np.meshgrid(t[:960],w)
Z = abs(np.array(batch_dfts))
fig = figure()
ax = fig.add_subplot(111)
ax.contourf(T,W,Z.T,cmap='jet')
title("Contour plot of the magnitude of the spectrum")
xlabel(r"$time \to$")
ylabel(r"$\omega \to$")
plt.savefig("fig10-12.png")
show()
fig = plt.figure()
ax = fig.gca(projection='3d')

surf = ax.plot_surface(T,W,(Z.T), cmap=cm.jet,
                       linewidth=0.1)

title(r"Surface plot of $|Y(\omega)|$")
ax.set_xlabel(r'$t \to$')
ax.set_ylabel(r'$\omega \to$')
ax.set_zlabel(r'$|Y(\omega)| \to$')


fig.colorbar(surf, shrink=0.5, aspect=5)
plt.savefig("fig10-13.png")
plt.show()
```



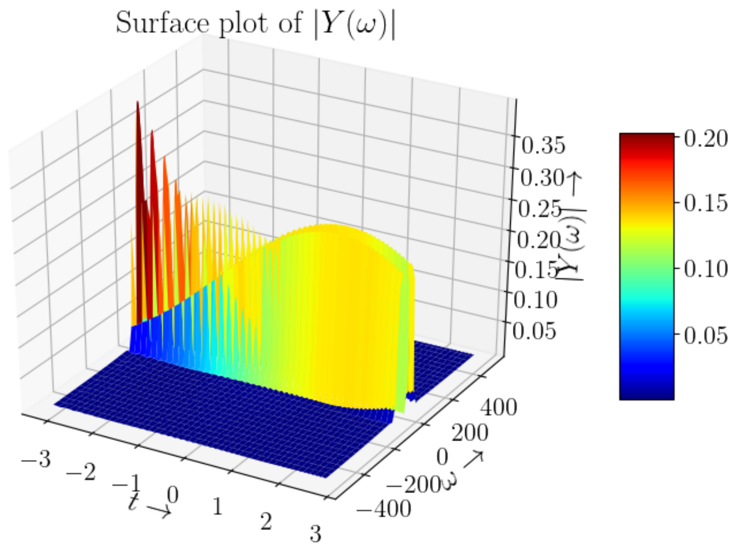


Figure 3: Contour plot and Surface plot of the magnitude of broken chirped signal

2.5.1 Results & Discussion:

- We observe that the magnitude of the fourier transform splits as time progresses as the frequency of the signal increases.
- In the surface plot of magnitude of spectrum vs time vs frequency, we observe strong peaks and as inferred from the contour plot of the magnitude spectrum we see 2 lobes whose separation increases as time increases.

3 Conclusion :

- Here in this assignment we implemented windowed fourier transform and also understood the need for windowing and also effects of windowing.
- Moreover,we implemented **Short time Fourier Transform** and witnessed how fourier transform evolves in time.