Assignment 4

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1 Introduction

The report discusses 7 tasks in Python to find Fourier Approximations of two function e^x and $\cos(\cos(x))$ from its integral definition and using Least Squares method.
We will fit two functions, e^x and $\cos(\cos(x))$ over the interval $[0,2\pi)$ using the fourier series

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(nx_i) + b_n \sin(nx_i) \approx f(x_i) \tag{1}$$

The equations used here to find the Fourier coefficients are as follows:

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)dx \tag{2}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x)\cos(nx)dx \tag{3}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x)\sin(nx)dx \tag{4}$$

2 Python code

```
# Importing libraries
from pylab import *
from scipy.integrate import quad
```

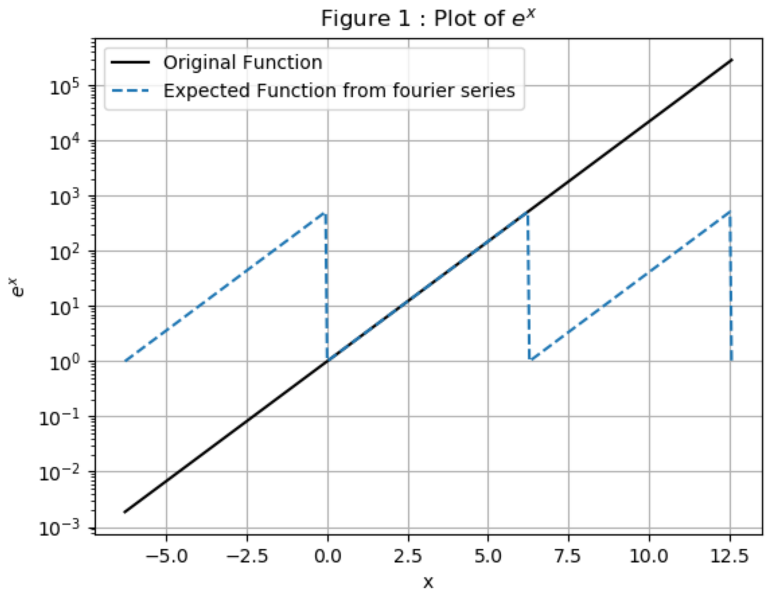
2.1 Question 1

- Define Python functions for the two functions e^x and $\cos(\cos(x))$ which return a vector (or scalar) value.

- Plot the functions over the interval $[2\pi, 4\pi)$.
- Discuss periodicity of both functions
- Plot the expected functions from fourier series

Code:

```
def exp_fn(x):
    return exp(x)

def coscos_fn(x):
    return cos(cos(x))

x = linspace(-2*pi, 4*pi, 400)

period = 2*pi

semilogy(x, exp_fn(x), 'k', label="Original Function")

semilogy(x, exp_fn(x % period), '--',
    label="Expected Function from fourier series")
legend()
title(r"Figure 1 : Plot of $e^{x}$")
xlabel(r"x")
ylabel(r"$e^{x}$")
grid()
show()

plot(x, coscos_fn(x), 'b', linewidth=4, label="Original Function")
plot(x, coscos_fn(x % period), 'y--',
    label="Expected Function from fourier series")
legend(loc='upper right')
title(r"Figure 2 : Plot of $\cos(\cos(x))$")
xlabel(r"x")
ylabel(r"$\cos(\cos(x))$")
grid()
show()
```



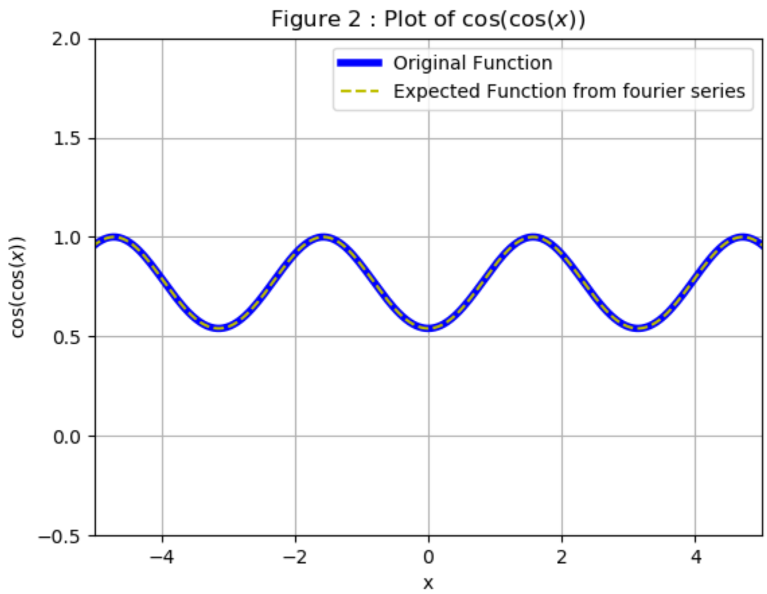


Figure 1: Plot for e^x and $cos(cos(x))$

2.1.1 Results and Discussion :

- We observe that e^x is not periodic, whereas $\cos(\cos(x))$ is periodic as the expected and original function matched for the latter but not for e^x.

- Period of $\cos(\cos(x))$ is 2π as we observe from graph and e^x monotously increasing hence not periodic.
- We get expected function by:
  - plotting expected function by dividing the x by period and giving remainder as input to the function, so that x values repeat after given period.
  - That is f(x%period) is now the expected periodic function from fourier series.

2.2 Question 2

- Obtain the first 51 coefficients i.e $a_0, a_1, b_1, ...$ for e^x and $\cos(\cos(x))$ using scipy quad function
- And to calculate the function using those coefficients and comparing with original funcitons graphically.

Code:

```
def an_fourier(x, k, f):
    return f(x)*cos(k*x)


def bn_fourier(x, k, f):
    return f(x)*sin(k*x)

def find_coeff(f):

    coeff = []
    coeff.append((quad(f, 0, 2*pi)[0])/(2*pi))
    for i in range(1, 26):
    coeff.append((quad(an_fourier, 0, 2*pi, args=(i, f))[0])/pi)
    coeff.append((quad(bn_fourier, 0, 2*pi, args=(i, f))[0])/pi)
    return coeff

def matrix_create(nrow, ncol, x):
    A = zeros((nrow, ncol))  # allocate space for A
    A[:, 0] = 1  # col 1 is all ones
    for k in range(1, int((ncol+1)/2)):
    A[:, 2*k-1] = cos(k*x)  # cos(kx) column
    A[:, 2*k] = sin(k*x)  # sin(kx) column
    # endfor
    return A

def compute_fn(c):
    A = matrix_create(400, 51, x)
    f_fourier = A.dot(c)
    return f_fourier


exp_coeff = []
coscos_coeff = []
exp_coeff1 = []
```

```
coscos_coeff1 = []

exp_coeff1 = find_coeff(exp_fn)
coscos_coeff1 = find_coeff(coscos_fn)

exp_coeff = np.abs(exp_coeff1)
coscos_coeff = np.abs(coscos_coeff1)

exp_fn_fourier = compute_fn(exp_coeff1)
coscos_fn_fourier = compute_fn(coscos_coeff1)


semilogy(x, exp_fn_fourier, 'ro', label="Function using Fourier Coefficie
ylim([pow(10, -1), pow(10, 4)])
legend()

grid()
show()


plot(x, coscos_fn_fourier, 'ro', label="Function using Fourier Coefficien
legend(loc='upper right')

grid()
show()
```

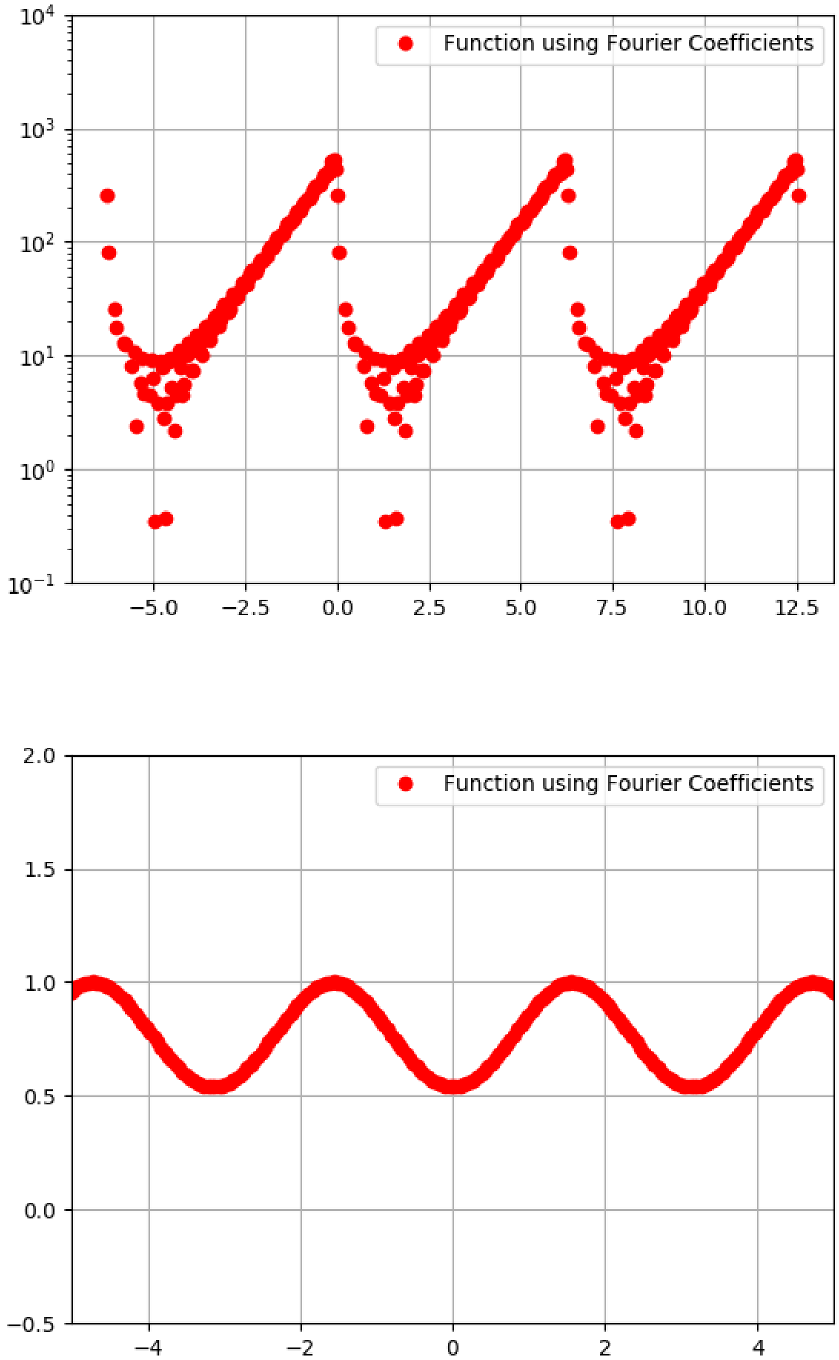


Figure 2: Plot for Fourier Apprixmation of e^x and $cos(cos(x))$

2.3 Question3

- Two different plots for each function using "semilogy" and "loglog" and plot the magnitude of the coefficients vs n
- And to analyse them and to discuss the observations. ## Plots:
- For each function magnitude of a_n and b_n coefficients which are computed using integration are plotted in same figure in semilog as well as loglog plot for simpler comparisons.

Code:

```
semilogy((exp_coeff[1::2]), 'ro', label=r"$a_{n}$ using Integration")
semilogy((exp_coeff[2::2]), 'ko', label=r"$b_{n}$ using Integration")
legend()
title("Figure 3 : Fourier coefficients of $e^{x}$ (semi-log)")
xlabel("n")
ylabel("Magnitude of coeffients")

grid()
show()

loglog((exp_coeff[1::2]), 'ro', label=r"$a_{n}$ using Integration")
loglog((exp_coeff[2::2]), 'ko', label=r"$b_{n}$ using Integration")
legend(loc='upper right')
title("Figure 4 : Fourier coefficients of $e^{x}$ (Log-Log)")
xlabel("n")

grid()
ylabel("Magnitude of coeffients")
show()

semilogy((coscos_coeff[1::2]), 'ro', label=r"$a_{n}$ using Integration")
semilogy((coscos_coeff[2::2]), 'ko', label=r"$b_{n}$ using Integration")
legend(loc='upper right')
title("Figure 5 : Fourier coefficients of $\cos(\cos(x))$ (semi-log)")
xlabel("n")

grid()
ylabel("Magnitude of coeffients")
show()

loglog((coscos_coeff[1::2]), 'ro', label=r"$a_{n}$ using Integration")
loglog((coscos_coeff[2::2]), 'ko', label=r"$b_{n}$ using Integration")
legend(loc='upper right')
```

```
title("Figure 6 : Fourier coefficients of $\cos(\cos(x))$
(Log-Log)")
xlabel("n")

grid()
ylabel("Magnitude of coeffients")
show()
```

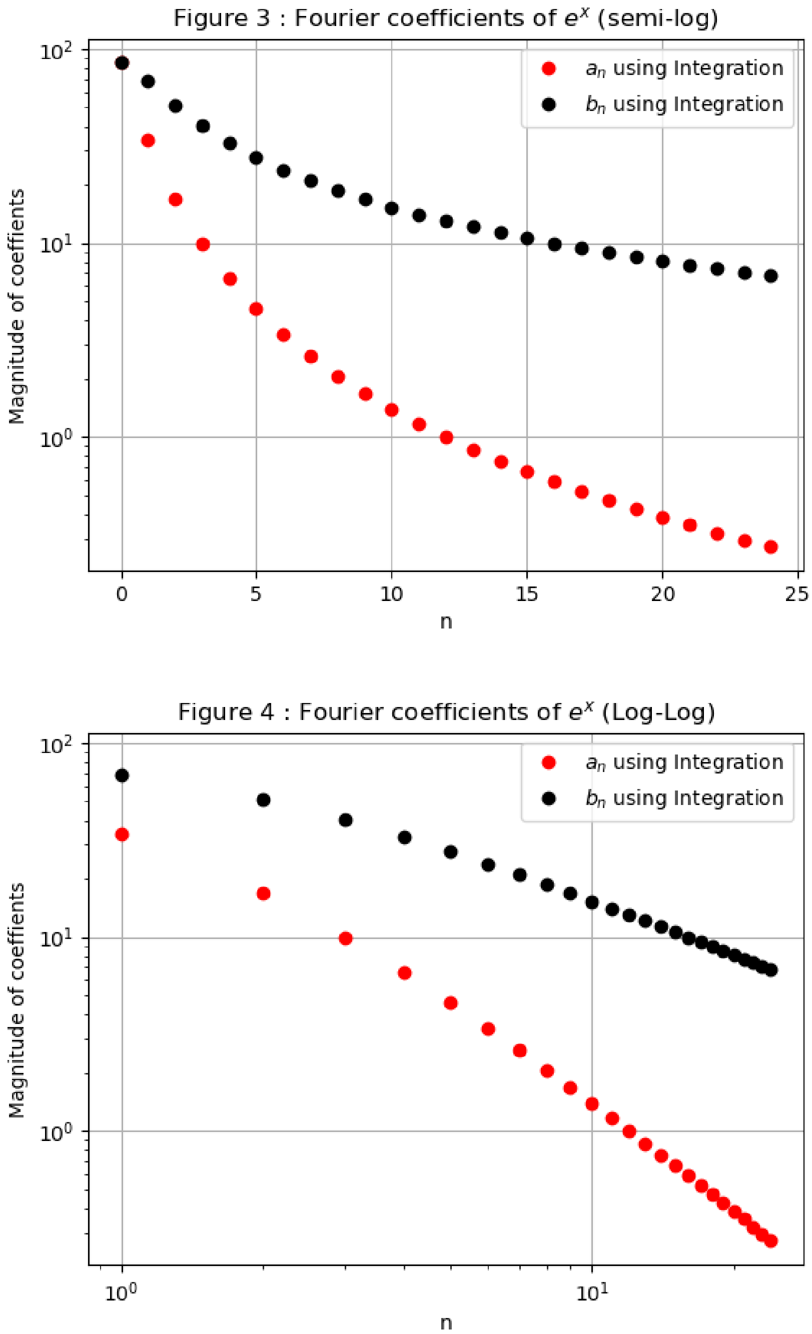


Figure 3: Semi-Log ,Log-Log plots of Fourier coefficients of e^x (Integration)

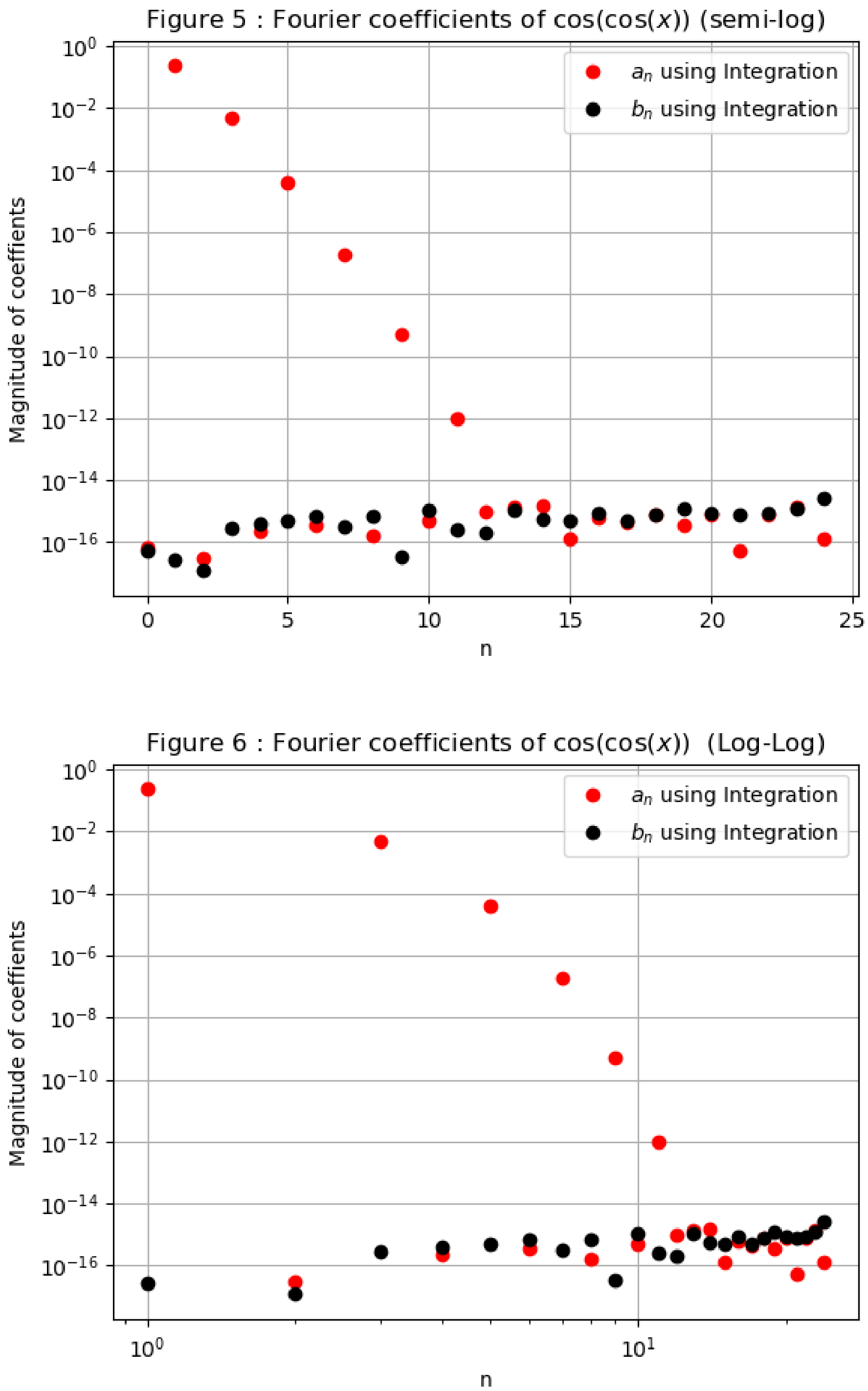


Figure 4: Semi-Log ,Log-Log plots of Fourier coefficients of $cos(cos(x))$ (Integration)

2.3.1 Results and Observations :

- The b_n coefficients in the second case should be nearly zero. Why does this happen?

  - Because $\cos(\cos(x))$ is an even function and for finding b_n we use Eq.(4) so the whole integral can be integrated in any interval with length of 2π, so for convenience we choose $[-\pi, \pi)$, then the integrand is odd since $\sin(nx)$ is there. so the integral becomes zero analytically. Where as here we compute using quad function which uses numerical methods so b_n is very small but not exactly zero.

- In the first case, the coefficients do not decay as quickly as the coefficients for the second case. Why not?

  - Rate of decay of fourier coefficients is determined by how smooth the function is,if a function is infinitely differentiable then its fourier coefficients decays very faster, where as if k^{th} derivative of function is discontinous the coefficients falls as $\frac{1}{n^{k+1}}$. to atleast converge.So in first case i.e is e^x is not periodic hence discontinous at $2n\pi$ so the function itself is discontinous so coefficients falls as $\frac{1}{n}$ so we need more coefficients for more accuracy,coefficients doesn't decay as quickly as for $\cos(\cos(x))$ as it is infinitely differentiable and smooth so we need less no of coefficients to reconstruct the function so it decays faster.

- Why does loglog plot in Figure 4 look linear, wheras the semilog plot in Figure 5 looks linear?

  - Because the coefficients of e^x varies as n^k where as $\cos(\cos(x))$ varies exponentially with 'n' means α^n , thats why loglog looks linear in first case and semilog in second case.

2.4 Question 4 & 5

- Uses least squares method approach to find the fourier coefficients of e^x and $\cos(\cos(x))$
- Evaluate both the functions at each x values and call it b. Now this is approximated by $a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$
- such that

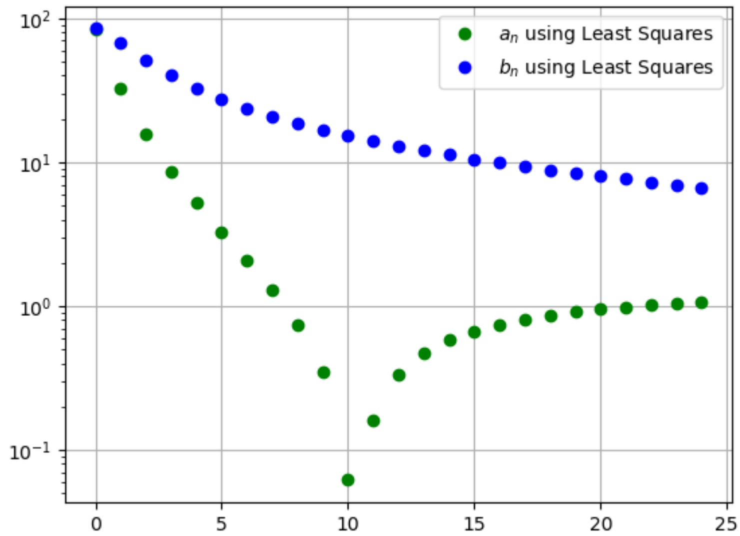
$$a_0 + \sum_{n=1}^{\infty} a_n \cos(nx_i) + b_n \sin(nx_i) \approx f(x_i) \tag{5}$$

- To implement this we use matrices to find the coefficients using Least Squares method using inbuilt python function 'lstsq'

Code:

```
def lstsq_coeff(f, low_lim, upp_lim, no_points):
    x1 = linspace(low_lim, upp_lim, no_points)
    # drop last term to have a proper periodic integral
    x1 = x1[:-1]
    b = []
    b = f(x1)
    A = matrix_create(no_points-1, 51, x1)
    c = []
    c = lstsq(A, b, rcond=None)[0]

    return c

coeff_exp = lstsq_coeff(exp_fn, 0, 2*pi, 401)
coeff_coscos = lstsq_coeff(coscos_fn, 0, 2*pi, 401)

c1 = np.abs(coeff_exp)
c2 = np.abs(coeff_coscos)

semilogy((c1[1::2]), 'go', label=r"$a_{n}$ using Least Squares")
semilogy((c1[2::2]), 'bo', label=r"$b_{n}$ using Least Squares")

grid()
legend(loc='upper right')
show()
```

```
loglog((c1[1::2]), 'go', label=r"$a_{n}$ using Least Squares ")
loglog((c1[2::2]), 'bo', label=r"$b_{n}$ using Least Squares")

grid()
legend(loc='lower left')
show()


semilogy((c2[1::2]), 'go', label=r"$a_{n}$ using Least Squares")
semilogy((c2[2::2]), 'bo', label=r"$b_{n}$ using Least Squares")


grid()
legend(loc='upper right')
show()

loglog((c2[1::2]), 'go', label=r"$a_{n}$ using Least Squares ")
loglog((c2[2::2]), 'bo', label=r"$b_{n}$ using Least Squares")

grid()
legend(loc=0)
show()
```



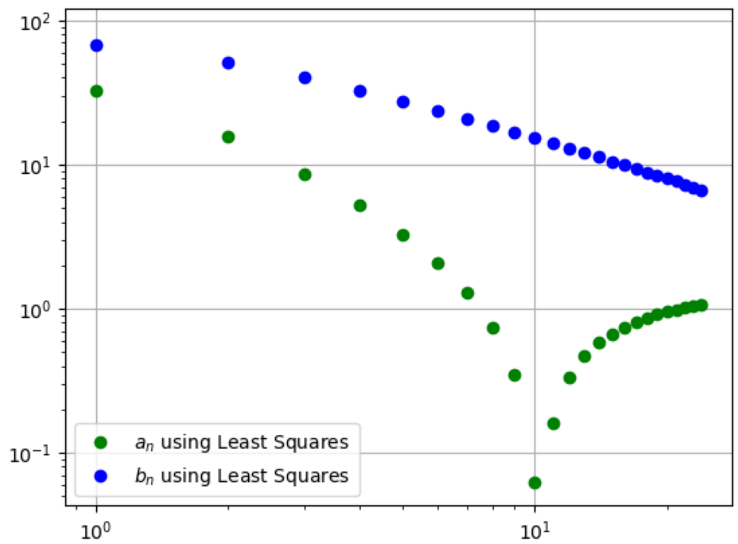
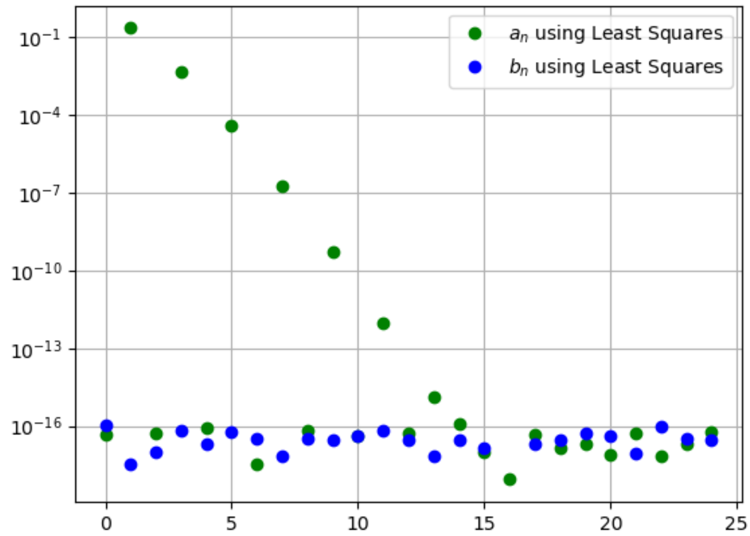


Figure 5: Semi-Log ,Log-Log plots of coefficients of e^x(Least Squares)



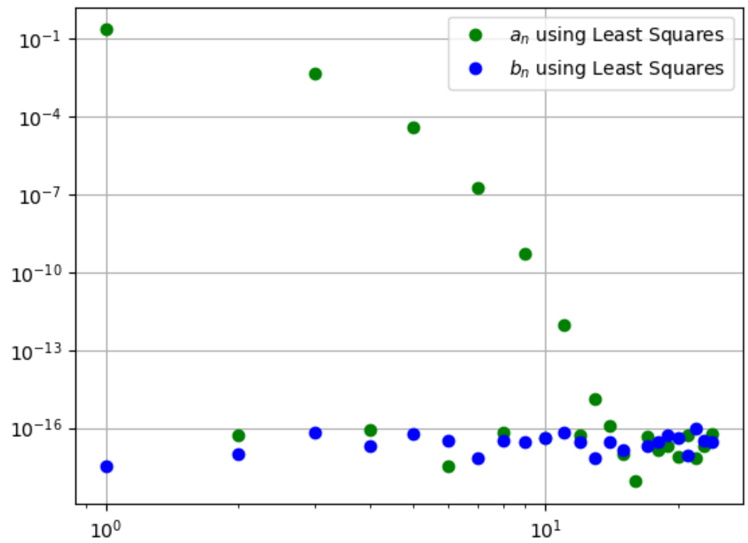


Figure 6: Semi-Log ,Log-Log plots of coefficients of $cos(cos(x))$(Least Squares)

2.5 Question 6

- To compare the answers got by least squares and by the direct integration.
- And finding deviation between them and find the largest deviation using Vectors

Code:

```
def coeff_compare(f):
    deviations = []
    max_dev = 0
    if(f == 1):
        deviations = np.abs(exp_coeff1 - coeff_exp)
    elif(f == 2):
        deviations = np.abs(coscos_coeff1 - coeff_coscos)

    max_dev = np.amax(deviations)
    return deviations, max_dev


dev1, maxdev1 = coeff_compare(1)
dev2, maxdev2 = coeff_compare(2)

print("Maximum deviation in exp coefficients : ", maxdev1)
print("Maximum deviation in cos_cos coefficients : ", maxdev2)
plot(dev1, 'g')
title(r"Figure 7 : Deviation between Coefficients for $e^{x}$")
grid()
xlabel("n")
ylabel("Magnitude of Deviations")
show()

plot(dev2, 'g')
title(r"Figure 8 : Deviation between coefficients for $\cos(\cos(x))$")
grid()
xlabel("n")
ylabel("Magnitude of Deviations")
show()
```

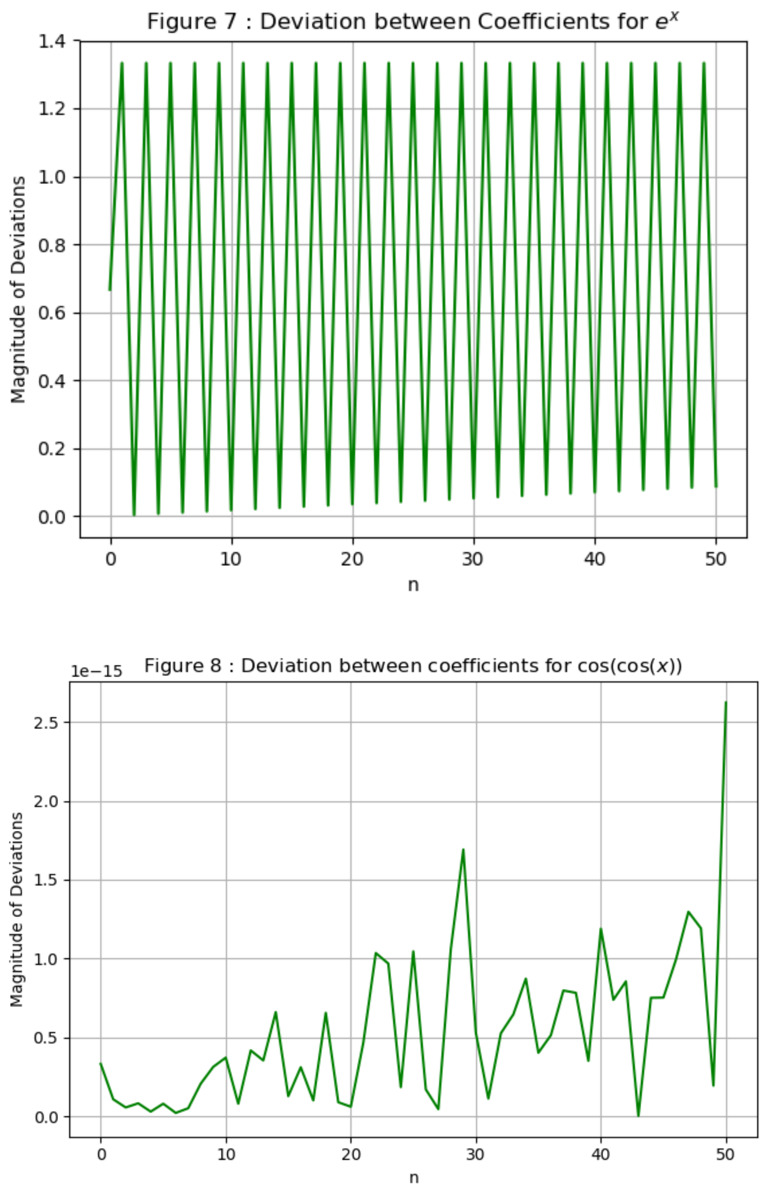


Figure 7: Deviation plots of Fourier coefficients of both the functions found via integration and least-squares

2.5.1 Results and Discussion :

- The maximum deviation for :
  - exp(x) = 1.3327308703353111
  - cos(cos(x)) = 2.622704669603838e-15

2.6 Question 7

- Computing Ac i.e multiplying Matrix A and Vector C from the estimated values of Coeffient Vector C by Least Squares Method.
- To Plot them (with green circles) in Figures 1 and 2 respectively for the two functions.

Code:

```
x1 = linspace(0, 2*pi, 400)

def fn_create_lstsq(c):
     f_lstsq = []
     A = matrix_create(400, 51, x1)
     f_lstsq = A.dot(c)
     return f_lstsq


exp_fn_lstsq = fn_create_lstsq(coeff_exp)
coscos_fn_lstsq = fn_create_lstsq(coeff_coscos)

semilogy(x1, exp_fn_lstsq, 'go',
         label="Inverse Fourier Transform From Least Squares")
legend()

grid()
ylim([pow(10, -2), pow(10, 3)])
xlim([0, 2*pi])
show()

plot(x1, coscos_fn_lstsq, 'go', markersize=4,
     label="Inverse Fourier Transform From Least Squares")
ylim([0.5, 1.3])
xlim([0, 2*pi])
grid()
legend()
show()
```

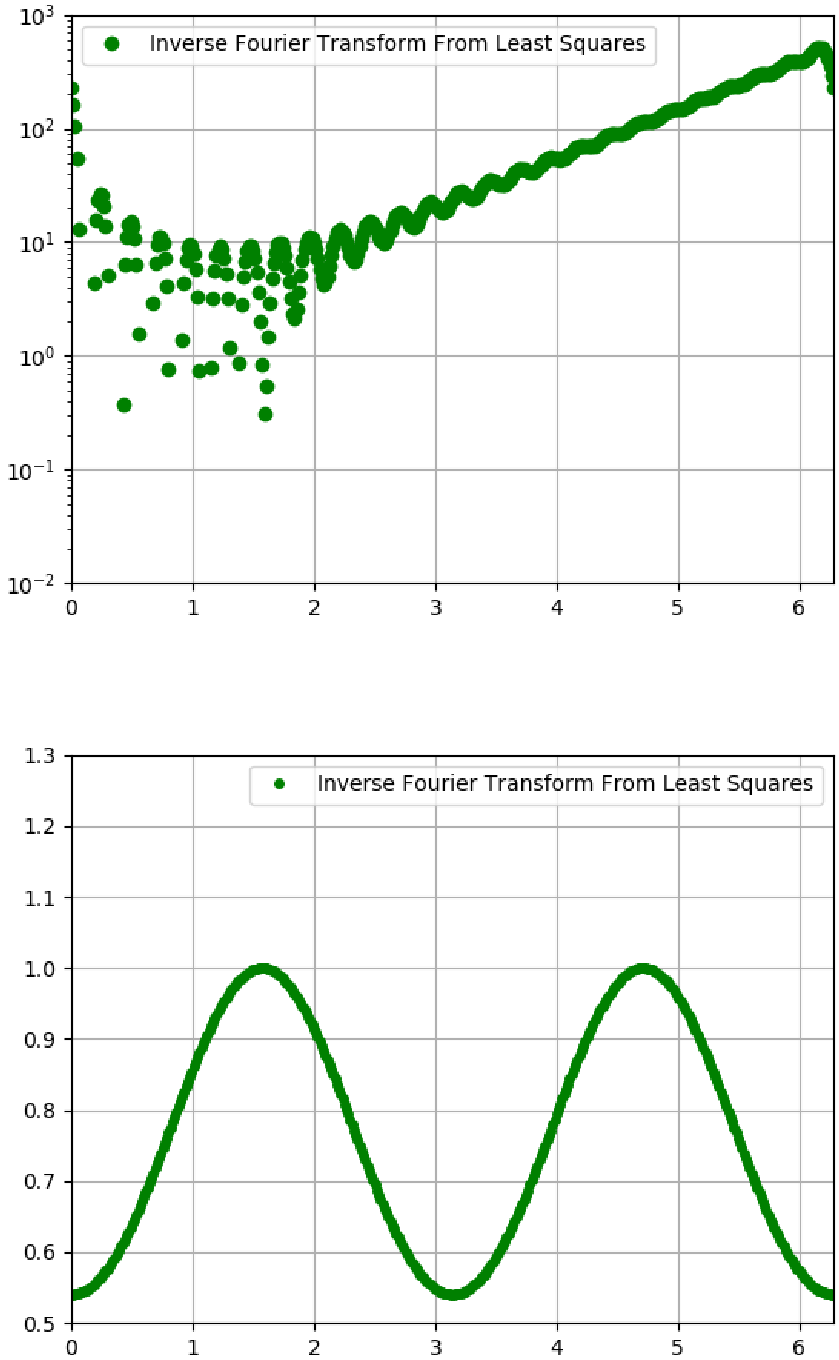


Figure 8: Plots of the functions got by Inverse Fourier Transform

2.6.1 Results and Discussion :

- As we observe that there is a significant deviation for e^x as it has discontinuites at $2n\pi$ which can be observed in Figure 1 and so there will be **Gibbs phenomenon** i.e there will be oscillations around the discontinuity points and their ripple amplitude will decrease as we go close to discontinuity. In this case it is at 2π for e^x.

- As we observe that rimples are high in starting and reduces and oscillate with more frequency as we go towards 2π. This phenomenon is called **Gibbs Phenomenon**

- Due to this. the orginal function and one which is reconstructed using least squares will not fit exactly.

- And as we know that Fourier series is used to define periodic signals in frequency domain and e^x is a aperiodic signal so you can't define an aperiodic signal on an interval of finite length (if you try, you'll lose information about the signal), so one must use the Fourier transform for such a signal.

- Thats why there are significant deviations for e^x from original function.

- Whereas for $\cos(\cos(x))$ the curves fit almost perfectly because the function itself is a periodic function and it is a continous function in entire x range so we get very negligible deviation and able to reconstruct the signal with just the fourier coefficients.

3 Conclusion

We see that the fourier estimation of e^x does not match significantly with the function close to 0, but matches near perfectly in the case of $\cos(\cos(x))$. This is due to the presence of a discontiuity at $x = 0$ for the periodic extension of e^x. This discontiuity leads to non-uniform convergence of the fourier series, with different rates for both the functions.

The difference in the rates of convergence leads to the **Gibb's phenomenon**, which is the ringing observed at discontiuities in the fourier estimation of a discontiuous function. This explains the mismatch in the fourier approximation for e^x.

Thus we can conclude that the Fourier Series Approximation Method works extremely well for smooth periodic functions, but gives bad results for discontinuos periodic functions