Assignment 6

Om Shri Prasath (EE17B113)

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1 Introduction

- We analyse the LTI systems in continuous time using Laplace transform to find the solutions to the equations governing the system with the help of python tools such as Signal toolbox

- *system.impulse* $\rightarrow$ Computes the impulse response of the transfer function
- *sp.lsim* $\rightarrow$ This simulates $y = u(t) * h(t)$ taking $u(t)$ and $\mathcal{H}(s)$ as arguments
- *sp.lti* $\rightarrow$ defines a transfer function from polynomial coefficients of numerator and denominator as inputs.
- $bode() \rightarrow$ It's used to find the magnitude and phase response of transfer function
- We use following method to find the Laplace transform of a time domain signal, here we use these methods to find laplace of system governing differential coefficients
- Some of the equations to follow while finding laplace transform

$$\mathscr{L}\{x(t)\} \rightarrow \mathcal{X}(s) \tag{1}$$

$$\mathscr{L}\{\frac{dx(t)}{dt}\} \rightarrow s\mathcal{X}(s) - x(0^-) \tag{2}$$

$$\mathscr{L}\{\frac{d^2x(t)}{dt^2}\} \rightarrow s^2\mathcal{X}(s) - sx(0^-) - \dot{x}(0^-) \tag{3}$$

- Combining the above equations above, we find the laplace transform of a differential equation and analyse them.

2 Python Code

2.1 Question 1 & 2:

- To solve for the time repsonse of the spring mass system,whose driving force varies as $f(t)$ given as

$$f(t) = \cos(1.5t)e^{-0.5t}u_0(t) \tag{4}$$

- Laplace transform of $f(t)$ using equations (1),(2) & (3) given above

$$\mathcal{F}(s) = \frac{s + 0.5}{(s + 0.5)^2 + 2.25} \tag{5}$$

- Spring satisfies the below equation with $x(0) = 0$ and $\dot{x}(0) = 0$ for $0 \leq t \leq 50s$.

$$\ddot{x} + 2.25x = f(t) \tag{6}$$

- So we take laplace transform of the equation given above with given intial conditions and in generalised form considering $w_0^2 = 2.25$ in the differential equation above, natural frequency of the system is $w_0 = 1.5rads^{-1}$,and decay factor of $f(t)$ as $d = 0.5$ and frequency of the input as $\omega = 1.5rads^{-1}$ in this question.

- In general we get

$$\mathcal{X}(s) = \frac{s + d}{((s + d)^2 + w^2)(s^2 + w_0^2)} \tag{7}$$

- In question 1 we get with given values and $d = 0.5$

$$\mathcal{X}(s) = \frac{s + 0.5}{((s + 0.5)^2 + 2.25)(s^2 + 2.25)} \tag{8}$$

- Solve the above problem with much smaller decay with same initial conditions, now $f(t)$ is as follows

$$f(t) = \cos(1.5t)e^{-0.05t}u_0(t) \tag{9}$$

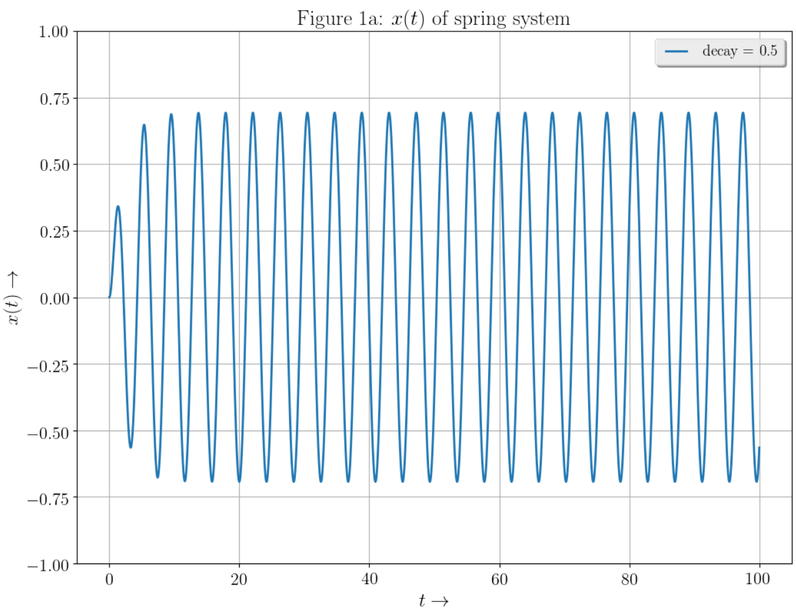
- So in question 2 we get with $d = 0.05$

$$\mathcal{X}(s) = \frac{s + 0.05}{((s + 0.05)^2 + 2.25)(s^2 + 2.25)} \tag{10}$$

- To solve for $x(t)$ displacement for each of the cases using Laplace transform with python tools such as *system.impulse* and plot them.
- To analyse the plots obtained and discuss the effect of decay on $x(t)$.

Code:

```
def laplaceSolver(decay, w):
    Xn = poly1d([1, decay])
    Xd = polymul([1, 0, pow(w, 2)], [
                1, 2*decay, (pow(w, 2)+pow(decay, 2))])
    Xs = sp.lti(Xn, Xd)
    t, x = sp.impulse(Xs, None, linspace(0, 100, 10000))
    return Xs, t, x

X, t1, x1 = laplaceSolver(0.5, 1.5)

X, t2, x2 = laplaceSolver(0.05, 1.5)

plot(t1, x1, label="decay = 0.5")
legend()
title(r"Figure 1a: $x(t)$ of spring system")
ylim((-1, 1))
xlabel(r"$t \to $")
ylabel(r"$x(t) \to $")
grid()
show()

plot(t2, x2, label="decay = 0.05")
legend()
ylim((-8, 8))
title(r"Figure 1b: $x(t)$ of spring system")
xlabel(r"$t \to $")
ylabel(r"$x(t) \to $")
grid()
show()
```



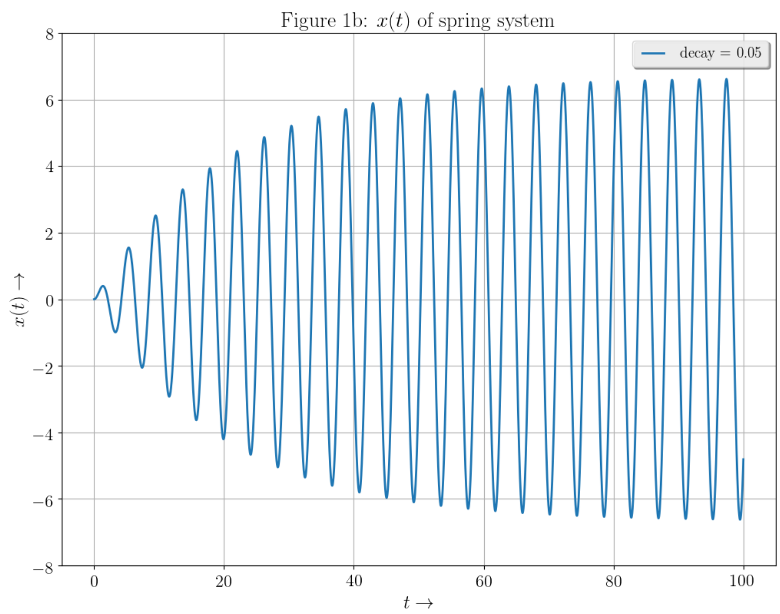


Figure 1: Plots of displacement of block for two different decays

2.1.1 Results and Discussion :

- As we observe the plot that for smaller decay of $e^{-0.05t}$, $x(t)$ has large amplitude and its growing as time increases and oscillates.

- Whereas the for higher decay value the amplitude of $x(t)$ is very small but since its second order system it oscillates.

- And If we observe the Figure 1a, amplitude growth stopped and settles quicker than for smaller decay value whereas for small decay value the $x(t)$ amplitude settling time is larger because as given below:

- Our input $f(t)$ to the system has natural frequency that is $w = w_0$, so its a resonant case,so the solution of differentail equation for sinusoidal inputs from observing the plot can be of the form $te^{-dt}\cos(w_0 t)$ so for smaller decay value the graph takes more time neutralise the growing effect of t in the solution.

- So to conclude for small decay , $x(t)$ has large amplitude and the time required for it settle or saturate to a certain maximum amplitude is higher compared to large decay case

2.2 Question 3:

- In an LTI system. $f(t)$ is the input, and $x(t)$ is the output.
- To Obtain the system transfer function $\frac{\mathcal{X}(s)}{\mathcal{F}(s)}$
- we use *signal.lsim* to simulate the problem.
- Here we try to plot the system response for different values of excitation frequecies i.e input frequencies with natural frequency of the system as $w_0 = 1.5rads^{-1}$
- So using a for loop, we sweep the frequency w of the $f(t)$ from 1.4 to 1.6 in steps of 0.05 keeping the exponent as $e^{0.05t}$ that is $d = 0.05$ and plot the resulting responses.
- So with above conditions laplace transform of $x(t)$ is

$$\mathcal{X}(s) = \frac{s + 0.05}{((s + 0.05)^2 + w^2)(s^2 + 2.25)} \tag{11}$$

- So we transfer function of the system is

$$\mathcal{H}(s) = \frac{s + 0.05}{((s + 0.05)^2 + w^2)(s^2 + 2.25)} \tag{12}$$

$$\mathcal{H}(s) = \frac{s + 0.05}{((s + 0.05)^2 + w^2)(s^2 + 2.25)} \tag{13}$$

- Using this we analyse the results.

Code:

```
def cosf(t, w, decay):
    return cos(w*t)*exp(-decay*t)

for w in arange(1.4, 1.6, 0.05):
    decay = 0.05
    H, a, b, = laplaceSolver(decay, 1.5)
    t = linspace(0, 200, 20000)
    t, y, svec = sp.lsim(H, cosf(t, w, decay), t)
    plot(t, y, label="$w$ = %g rad/s" % (w))
    legend()
    plot()
xlabel(r"$t \to $")
ylabel(r"$x(t) \to $")
ylim((-80, 80))
```

```
title(r"Figure 2: $x(t)$ of spring system with varying frequencies")
grid()
show()
```

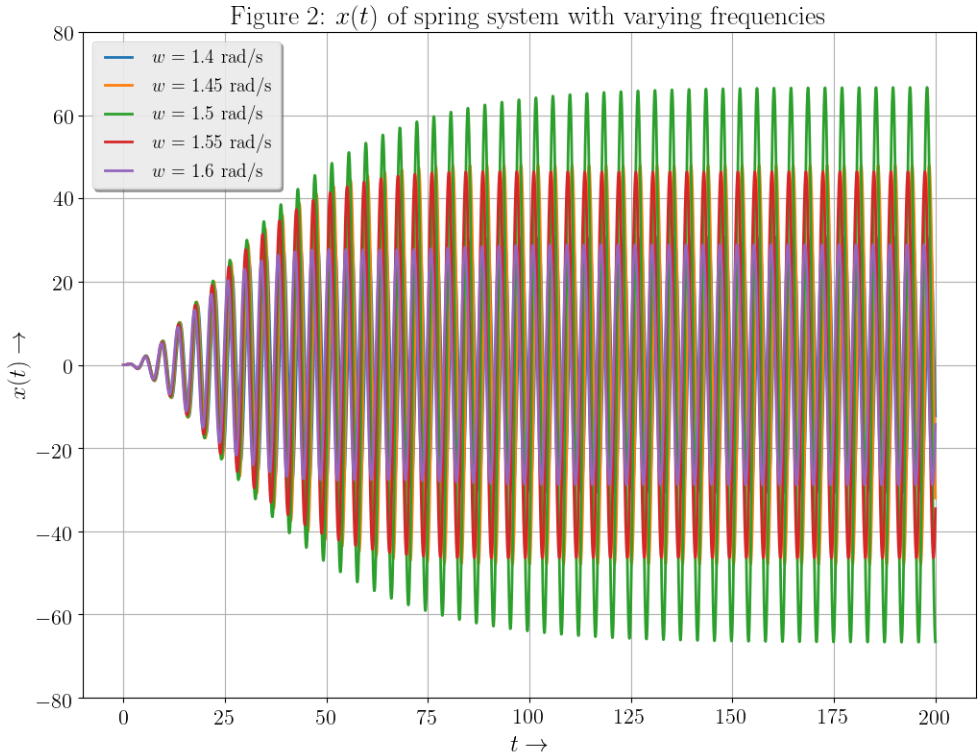


Figure 2: Plots of displacement of block for different frequencies

2.2.1 Results and Discussion:

- We see that for frequency $\omega = 1.5rads^{-1}$, the amplitude is the largest since it is the resonance frequency of the system.

- As we observe in Figure 2 that when we force (we are basically exciting the spring mass system with $f(t)$) the system with $w \neq w_0$ where $w_0 = 1.5$ and $w \approx w_0$ i.e around 1.5 $rads^{-1}$,since intially the system is at rest,so when f(t) is forced upon it,they are in phase,so constructive superposition occurs so the amplitude of $x(t)$ increases in same fashion for w very close to w_0 at starting , after that since the forced and natural frequency are not same,it is detuned so amplitude after peak amplitude starts falling slightly but at steady state all forced response dies out,so basically $x(t)$ will vary only with natural mode since forced response died out eventually.

- Whereas for $w = w_0$ case resonant occurs,so system and forcing input have same frequency so,the forced response adds up to natural response basically 'tuned',so the amplitude is very high in steady state.

2.3 Question 4:

- To Solve for a coupled spring problem:
- System satisfies the below equation with $x(0) = 1$ and $\dot{x}(0) = y(0) = \dot{y}(0) = 0$.

$$\ddot{x} + (x - y) = 0 \tag{14}$$

$$\ddot{y} + 2(y - x) = 0 \tag{15}$$

- Solve for $x(t)$ and $y(t)$ for $0 \leq t \leq 20s$ by taking laplace transform of both equations given above and solve for $\mathcal{X}(s)$ and $\mathcal{Y}(s)$ using substitution method.
- Now from $\mathcal{X}(s)$ and $\mathcal{Y}(s)$ find $x(t)$ and $y(t)$ using *system.impulse*.
- Plot $x(t)$ and $y(t)$ in the same graph and analyse them

Code:

```
def coupledSysSolver(n_coeff, d_coeff):
    H_n = poly1d(n_coeff)
    H_d = poly1d(d_coeff)

    Hs = sp.lti(H_n, H_d)
    t, h = sp.impulse(Hs, None, linspace(0, 100, 1000))
    return t, h


t1, x = coupledSysSolver([1, 0, 2], [1, 0, 3, 0])
t2, y = coupledSysSolver([2], [1, 0, 3, 0])

plot(t1, x, 'b', label="$x(t)$")
plot(t2, y, 'r', label="$y(t)$")
legend()
title(r"Figure 3: Time evolution of $x(t)$ and $y(t)$ for $0 \leq t
\leq 100$. of Coupled spring system ")
xlabel(r"$t \to $")
ylabel(r"$x(t),y(t) \to $")
ylim((-0.5, 2))
grid()
show()
```

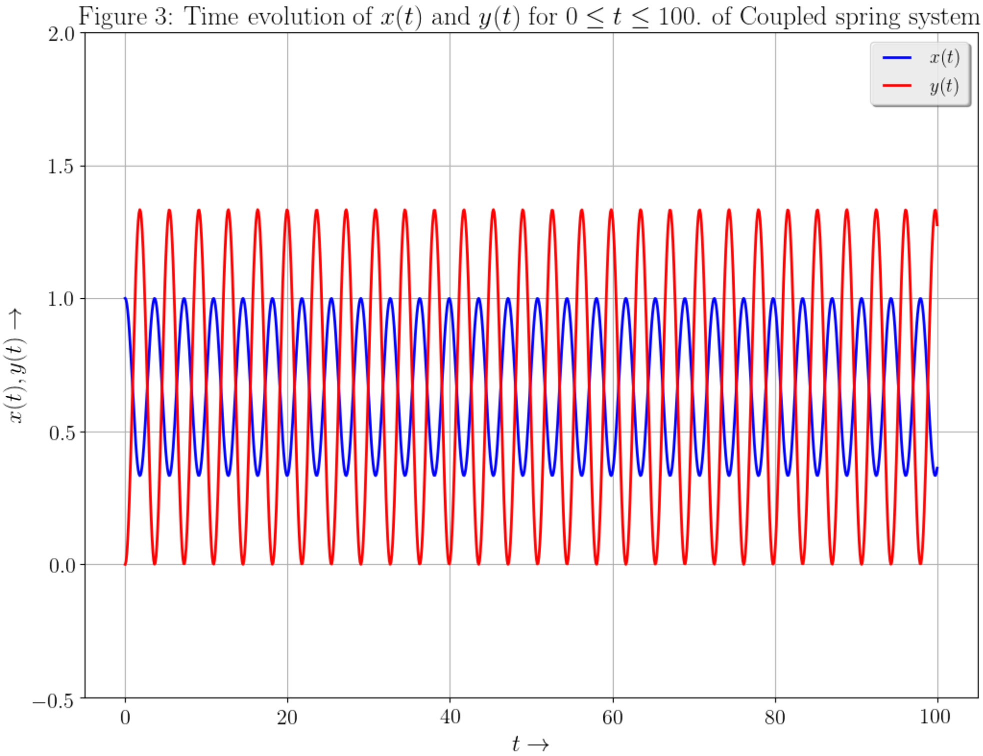


Figure 3: Plots of displacement of two coupled blocks

2.3.1 Results and Discussion:

- As we observe figure 4 that, the $x(t)$ and $y(t)$ obtained satisfies the given initial conditions,and oscillating sinusoidally with 180^o out of phase

2.4 Question 5:

- To Obtain the magnitude and phase response of the Steady State Transfer function of the following two-port network
- Transfer function of the RLC network from input to voltage at capacitor in general for given Network is

$$\frac{\mathcal{V}_0(s)}{\mathcal{V}_i(s)} = \mathcal{H}(s) = \frac{1}{s^2LC + sRC + 1} \tag{16}$$

- For the given values of $R = 100\Omega$, $L = 1\mu H, C = 1\mu F$
- We get

$$\mathcal{H}(s) = \frac{1}{s^2 10^{-12} + s10^{-4} + 1} \tag{17}$$

- can be written as

$$\mathcal{H}(s) = \frac{1}{(1 + \frac{s}{10^8})(1 + \frac{s}{10^4})} \tag{18}$$

- So system has poles on left half s plane, that too real poles with $s = -10^4, -10^8\ rads^{-1}$, we will observe the effect of poles on magnitude and phase response by analysing plots of them
- Magnitude Response is $|H(s)|$ evaluated at any point on imaginary axis so basically $|H(j\omega)|$
- Phase response is $\angle H(j\omega)$
- Using $\mathcal{H}(s)$ we calculate magnitude and phase response of it using $Bode()$ and plot them and analyse.

Code:

```
def RLCnetwork(R, C, L):
  Hnum = poly1d([1])
  Hden = poly1d([L*C, R*C, 1])

  Hs = sp.lti(Hnum, Hden)

  w, mag, phi = Hs.bode()
  return w, mag, phi, Hs
```

```
R = 100
L = 1e-6
C = 1e-6

w, mag, phi, H = RLCnetwork(R, L, C)

semilogx(w, mag, 'b', label="$Magnitude Response$")
legend()
title(r"Figure 4: Magnitude Response of $H(jw)$ of Series
RLC network")
xlabel(r"$ \log w \to $")
ylabel(r"$ 20\log|H(jw)|  \to $")
grid()
show()

semilogx(w, phi, 'r', label="$Phase Response$")
legend()
title(r"Figure 5: Phase response of the $H(jw)$ of Series RLC
network for")
xlabel(r"$ \log w \to $")
ylabel(r"$ \angle H(j\omega)$ $\to $")
grid()
show()
```

Figure 4: Magnitude Response of $H(jw)$ of Series RLC network

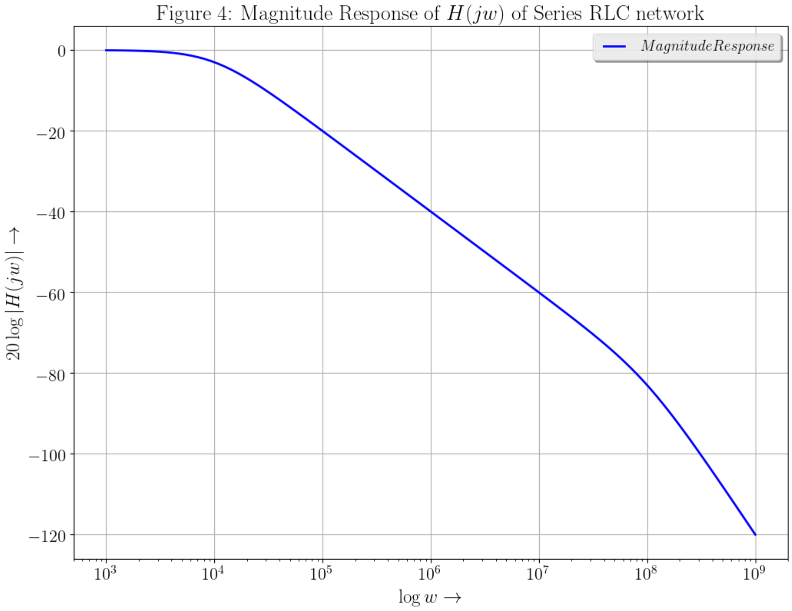


Figure 5: Phase response of the $H(jw)$ of Series RLC network

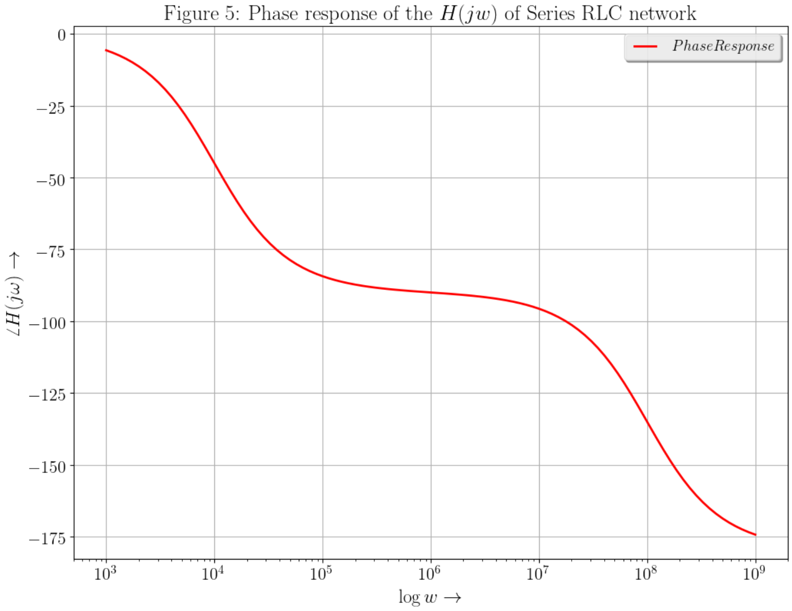


Figure 4: Bode Plots of the LCR system

2.4.1 Results and Discussion:

- As we observe figure 6 that,$0 \leq \angle H(j\omega) < 180^o$.
- So the system is unconditionally stable since phase does not go to 180^o.
- Since each pole adds 90^o to the phase its clear that the system is second order because it has two poles hence phase go till 180^o.
- And since all the poles are in left half s plane RLC Network given is unconditionally stable for given values.

2.5 Question 6:

- Consider the problem in $Q5$.If the input signal $v_i(t)$ is given by

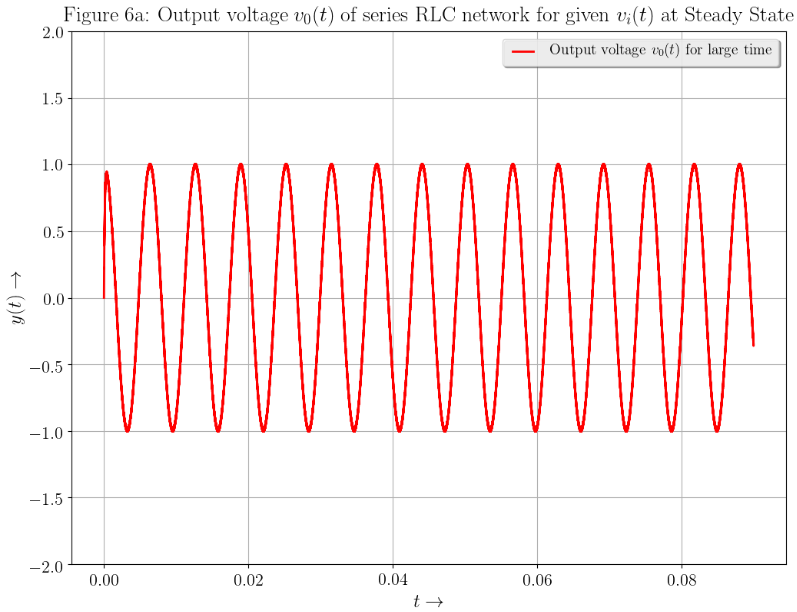
$$v_i(t) = \cos(10^3 t)u(t)\cos(10^6 t)u(t) \tag{19}$$

- Obtain the output voltage $v_0(t)$ using the transfer function of the system obtained.
- To explain the output signal for $0<t<30\mu s$
- And explain the long term response on the *msec* timescale

Code:

```
t = linspace(0, 90*pow(10, -3), pow(10, 6))
vi = cos(t*pow(10, 3))-cos(t*pow(10, 6))

t, vo, svec = sp.lsim(H, vi, t)
vo_ideal = cos(1e3*t)

plot(t, vo, 'r', label="Output voltage $v_0(t)$ for large time")
legend()
title(r"Figure 6a: Output voltage $v_0(t)$  of series RLC network
 for given $v_i(t)$ at Steady State")
xlabel(r"$ t \to $")
ylabel(r"$ y(t) \to $")
grid()
show()

plot(t, vo, 'r', label="Output voltage $v_0(t)$ - zoomed in ")
plot(t, vo_ideal, 'g', label="Ideal Low Pass filter Output with
 cutoff at $10^4$")
xlim(0.0505, 0.051)
ylim(0.75, 1.1)
legend()
title(r"Figure 6b: Output voltage $v_0(t)$  Vs Ideal Low pass
 filter Output")
xlabel(r"$ t \to $")
ylabel(r"$ y(t) \to $")
grid()
show()

plot(t, vo, 'r', label="Output voltage $v_0(t)$ : $0<t<30\mu sec$")
```

```
legend()
title(r"Figure 7: Output voltage $v_0(t)$ for $0<t<30\mu sec$")
xlim(0, 3e-5)
ylim(-1e-5, 0.3)
xlabel(r"$ t \to $")
ylabel(r"$ v_0(t) \to $")
grid()
show()
```



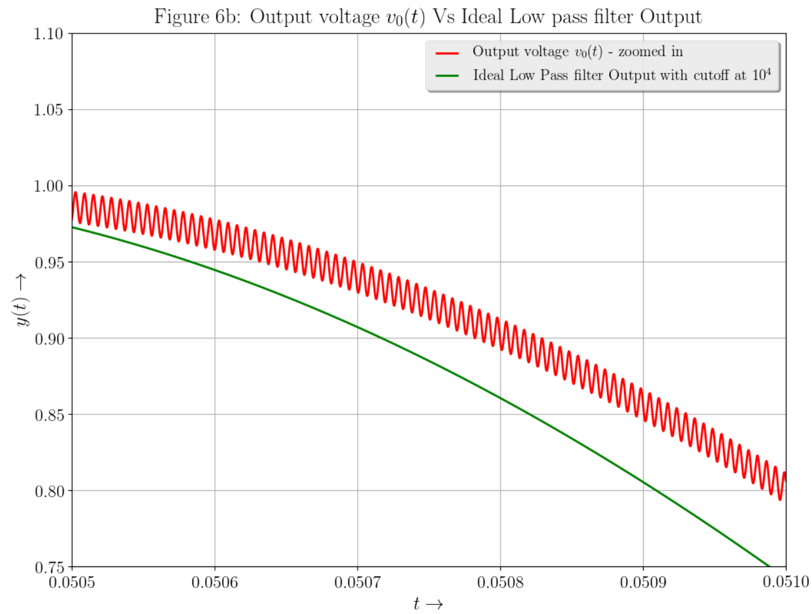


Figure 5: Output of the LCR system(LPF) and comparision with Ideal LPF

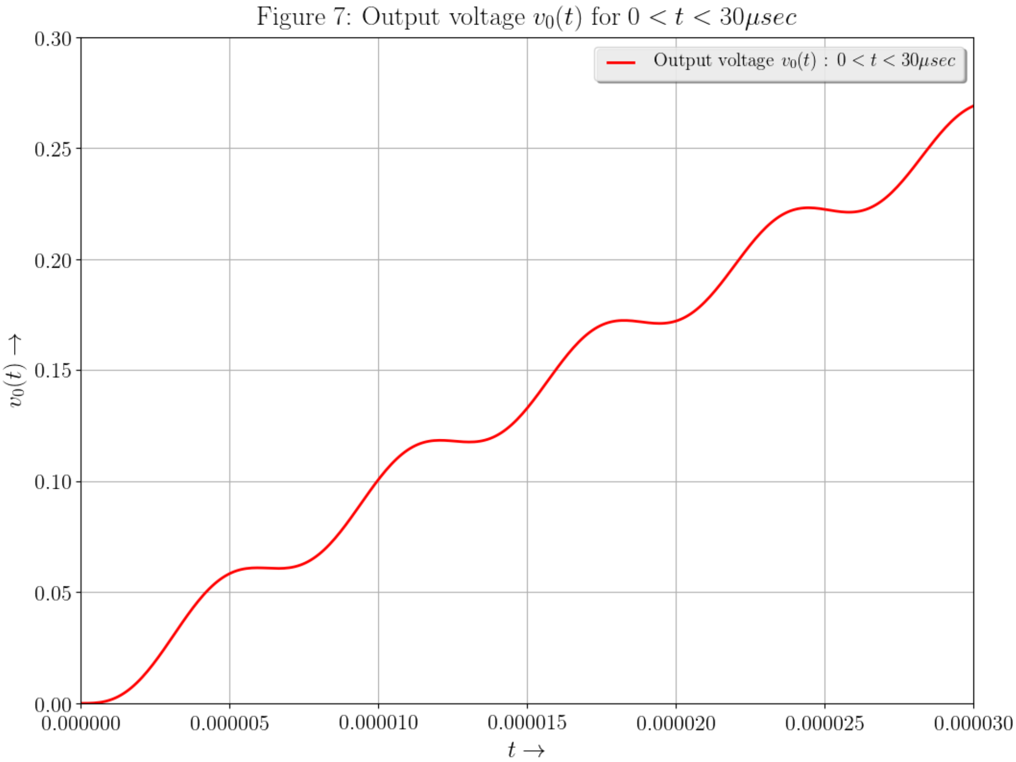


Figure 6: Initial plot for time between 0 to 30 μs

2.5.1 Results and Discussion:

- As we observe the plot and the circuit that we know it is a Low pass filter with bandwidth $0 < \omega < 10^4$.
- So when the circuit will only pass input with frequencies which are in range of bandwidth only. But since its not a ideal low pass filter as its gain doesn't drop abruptly at 10^4 rather gradual decrease which is observed from magnitude response plot.
- So the output $V_o(t)$ will be mainly of $\cos(10^3t)$ with higher frequencies riding over it in long term response i.e Steady state solution.
- This behaviour is observed in the plot that,the $v_o(t) \approx \cos(10^3t)$ with higher frequencies riding over it for large time.
- The curve is very flickery or not smooth because of the high frequency components only.
- In the next plot we've zoomed large enough to observe those components.
- The oscillatory behaviour in the graph is because of high frequency components riding over the main signal which is $\cos(10^3t)$ since gain of the system ≈ 1 for lower frequencies than $\omega < 10^4$ and gradually decreases as 20dB/dec which is observed from magnitude response plot and for very high frequencies 40dB/dec.
- So there will be some higher frequencies but the gain will be very less since its low pass filter ,that's why we can see there are very small sinusoidal oscillations over the main output signal compared to ideal low pass filter output which is $\cos(10^3t)$
- As we observe the plot of Figure 6 , for $v_0(t)$ from $0 < t < 30\mu sec$ increases very fast from 0 to 0.3 in just $30\mu s$ because of transients that is we apply sinusoidal step input to the system i.e the input is zero for $t < 0$, so when abruptly the input becomes non zero for $t \geq 0$, the system output jumps or raises suddenly from 0 to non-zero values in less time.
- That's why we observe a sharp rise in output at the start.
- And the oscillatory behaviour in the graph is because of high frequency components riding over the main signal which is $\cos(10^3t)$ since gain of the system ≈ 1 for lower frequencies than $\omega < 10^4$.

3 Conclusion:

- So to conclude we analysed a way to find the solution of continuous time LTI systems using laplace transform with help of Python signals toolbox and got familiarised with solving of differential equations by taking laplace transform instead of doing arduous time domain analysis.