① (a) $K(x,z) = K_1(x,z) + K_2(x,z)$

$\Rightarrow$ Here K_1 and K_2 are valid kernels

$K_1(x,z) = \phi_1(x)^T \phi_1(z)$ where ϕ_1 is a kernel function

Similarly,

$K_2(x,z) = \phi_2(x)^T \phi_2(z)$

Now $K_1(x,z) + K_2(x,z) = \phi_1(x)^T \phi_1(z) + \phi_2(x)^T \phi_2(z)$

$= [\phi_1(x), \phi_2(x)]^T [\phi_1(z), \phi_2(z)]$

Angalouge to $\phi_\theta^T(x) \phi_\theta(z)$

where $\phi_\theta(x) = [\phi_1(x), \phi_2(x)]$

$\therefore K(x,z) = \phi^T(x)\phi(z)$

where

$\phi(x) = [\phi_1(x), \phi_2(x)]$

$\phi(z) = [\phi_1(z), \phi_2(z)]$

$\therefore$ This is a valid kernel.

(b) $K(x,z) = K_1(x,z)K_2(x,z)$

$$K_1(x,z)K_2(x,z) = \left(\sum_{m=1}^{M} \phi_m^1(x)\phi_m^1(z)\right) \times \left(\sum_{n=1}^{N} \phi_n^2(x)\phi_n^2(z)\right)$$

where $\phi^1(z) = [\phi_1^1(z), \phi_2^1(z) \ldots]$

$$= \sum_{m=1}^{M}\sum_{n=1}^{N} [\phi_m^1(x)\phi_n^2(x)][\phi_m^1(z)\phi_n^2(z)]$$

$$= \sum_{m=1}^{M}\sum_{n=1}^{N} \phi_{mn}(x)\phi(_{mn}(z))$$

$$= \phi^T(x)\phi(z)$$

where $\phi_{mn}(z) = \phi_m^1(z)\phi_m^2(z)$

So, $K(x,z)$ is a valid kernel

(c) $K(x,z) = h(K_1(x,z))$

Since, h is a polynomial function

$$h(x) = a_n x^n + a_{n-1} x^{n-1} \dots \quad a_0$$

As we showed in the above question multiplication of two kernel functions is a kernel function and as shown in part (a) summation of kernel functions is also a kernel function

Thus, $K(x,z)$ is a kernel function

d) $K(x,z) = \exp(K_1(x,z))$

we can write $\exp(x)$ using taylor series as:

$$\exp(x) = \cancel{\frac{x^0}{0!} + \frac{x^1}{1!}} \cdots \sum_{m=1}^{\infty} \frac{x^m}{m!}$$

Now this can be expressed as

$h(x) = a_m x^m$ where $a_m = 1/m!$

So according to part (c) ~~b~~ $K(x,z)$ is a kernel function.

(e) $K(x,z) = \exp\left(-\frac{\|x-z\|_2^2}{\sigma^2}\right)$

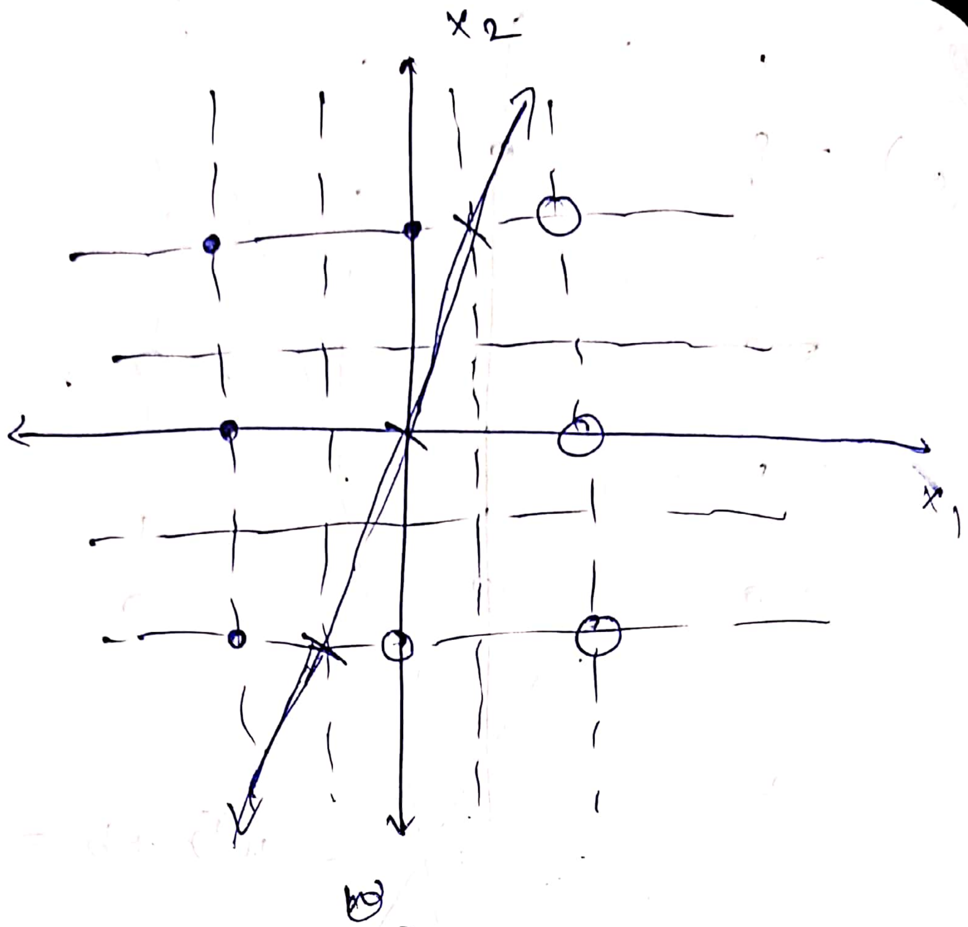
$\Rightarrow K(x,z) = \exp\left(-\left(\frac{\|x\|^2 + \|z\|^2 - 2x^Tz}{\sigma^2}\right)\right)$

$= \exp\left(-\frac{x^Tx}{\sigma^2}\right) \cdot \exp\left(-\frac{z^Tz}{\sigma^2}\right) \exp\left(\frac{2x^Tz}{\sigma^2}\right)$

Now exp() part is proved in part d so all the three terms are valid kernels so this is a valid kernel.

Also x^Tz is a linear for some x, z.

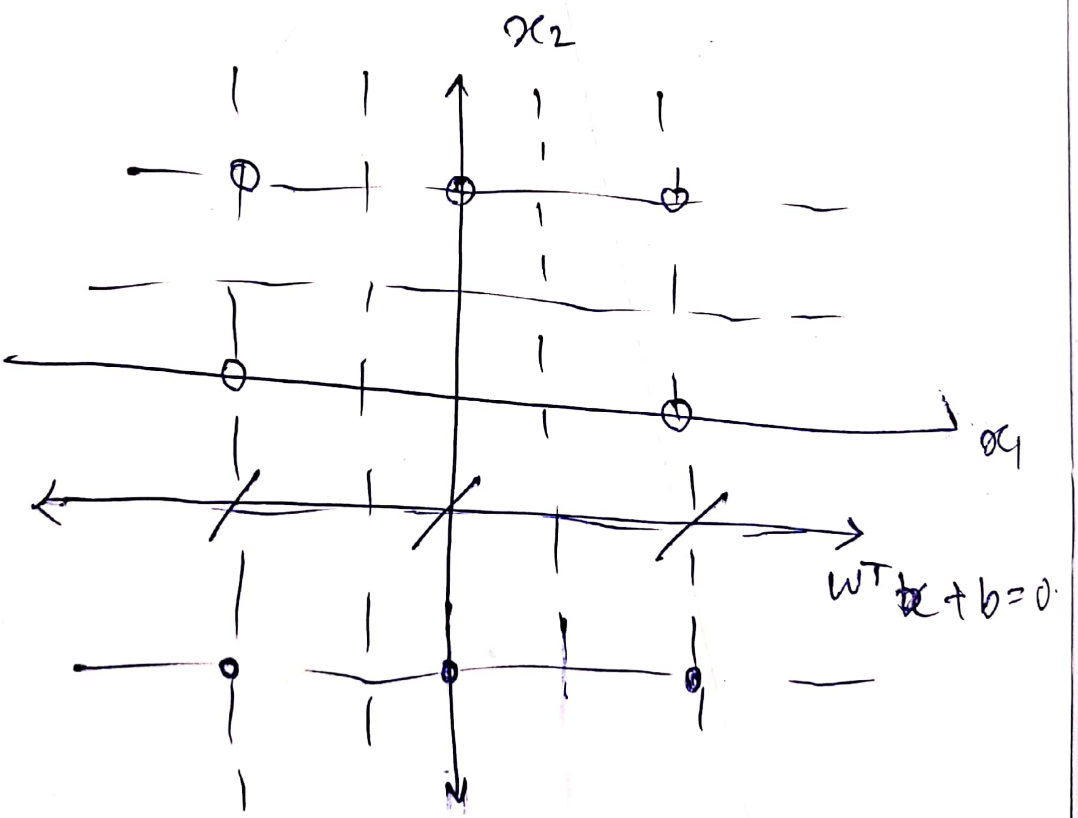
(2) a



Now the equation of line be $w^T x + b = 0$

$2x_1 - x_2 + 0 = 0$ ∴ $w^T = [2, -1]$, $b = 0$

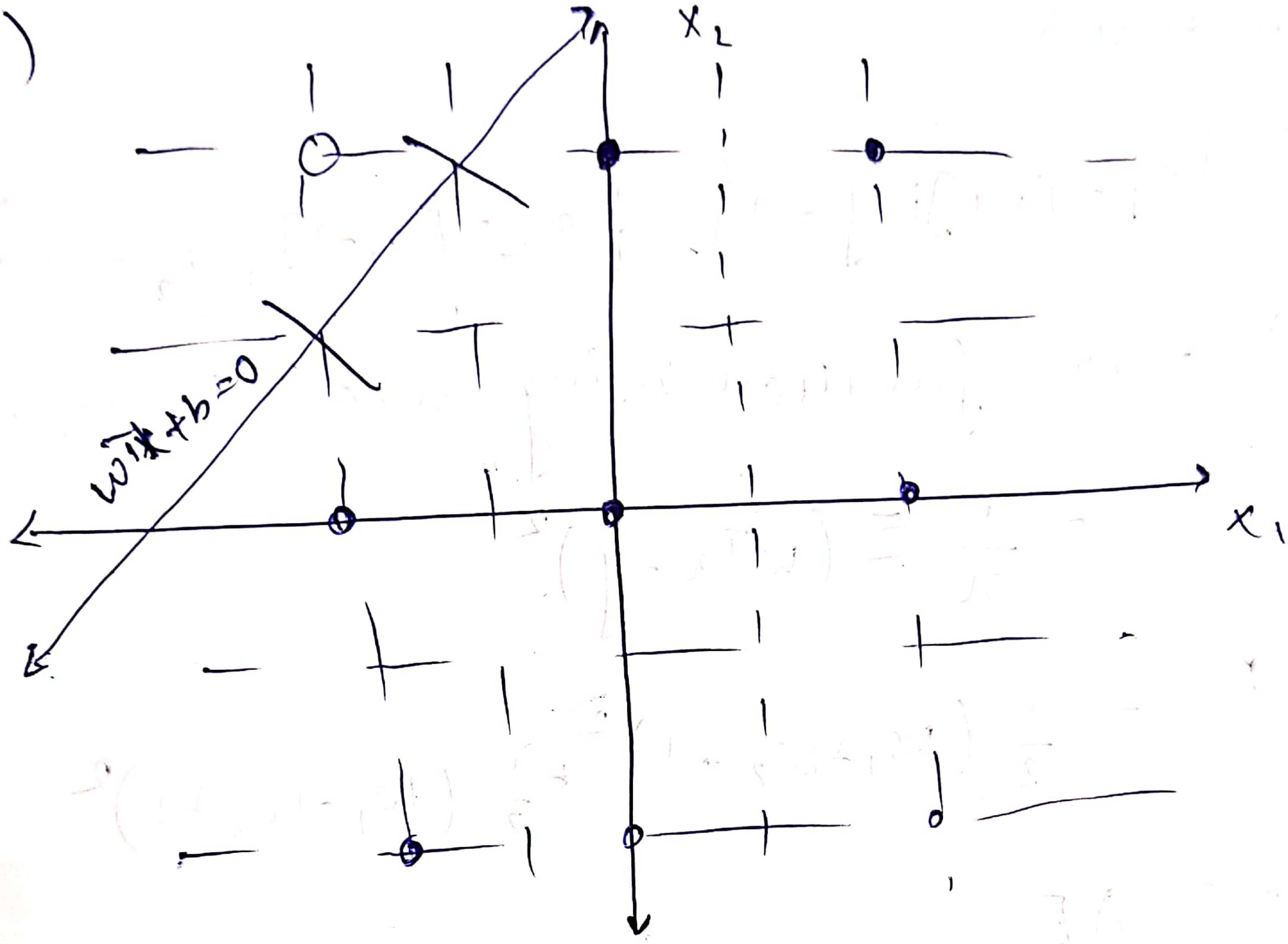
(b)



∴ $0x_1 + x_2 + 1 = 0$

$w^T = [0, 1]$ $b = 1$

(c)



$$x_1 - x_2 + 3 = 0$$

$$w^T = [1, -1] \quad b = 3$$

Question 8

(a) $\{x_1 = [1\ 1];\ y_1 = 1\}$, $\{x_2 = [1, -1];\ y_2 = -1\}$

a) Error function using mean Error,

$$= \frac{1}{n} \Sigma (w^T x - y)^2$$

$$E = \frac{1}{2} (w_1 + w_2 - 1)^2 + \frac{1}{2} (w_1 - w_2 + 1)^2$$

Now $$\frac{\partial E}{\partial w_1} = (w_1 + w_2 - 1) + (w_1 - w_2 + 1)$$

$$= 2w_1$$

Now for minima $\frac{\partial E}{\partial w_1} = 0$

$\therefore w_1 = 0$.

$$\frac{\partial E}{\partial w_2} = w_1 + w_2 - 1 - (w_1 - w_2 + 1)$$

$$= 2w_2 - 2$$

for minima $\frac{\partial E}{\partial w_2} = 0 \Rightarrow w_2 = 1$.

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Now, $\frac{\partial^2 E}{\partial w_1^2} = 2 > 0$ & $\frac{\partial^2 E}{\partial w_2^2} = 2 > 0$.

& $\frac{\partial^2 E}{\partial w_1 w_2} = 0$ & $\frac{\partial^2 E}{\partial w_2 w_1} = 0$.

$\therefore$ Hessian $= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$\Rightarrow$ $|\text{Hessian}| = 4 > 0$

So $(0,1)$ is the minimum and the center of the error surface

(b) Hessian $= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ as above

Since it is an upper triangular matrix eigenvalues $= 2, 2$

so the eigenvalues are repeated and positive.