Graph Theory		2023
	Perfect Graphs	
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If G is a line graph, then $w(G) \leq \chi(G) \leq w(G) + 1$ ($w(G)$ = clique number of G).
This follows as a corollary of Vizing's theorem.
Natural question: When is $w(G) = \chi(G)$? In other words, can we characterize graphs for which $w(G) = \chi(G)$?
Consider vertex-disjoint K_k and Erdos graph with $\chi = k$ and large girth.
The above illustrates that the question is poorly posed.
Better Question: Given a graph G, is $\chi(H) = w(H)$, $\forall$ induced subgraphs H of G?

Definition 1 (C.Berge, late 60s). *A graph G satisfying $\chi(H) = w(H)$, $\forall\ H \subseteq G$, where H must be an induced subgraph, is called perfect.*

Here are some examples of perfect graphs:

- K_n
- Bipartite graphs
- Line graphs of bipartite graphs

Theorem 2. *If G is bipartite, then $\chi'(G) = \Delta(G)$.*

Proof. WLOG assume G is Δ-regular (else embed G into a bipartite graph that is Δ-regular). If $G(A, B, E)$ is Δ-regular, then note that $|A| = |B|$. It suffices to show: G has a perfect matching i.e. there is a set of edges $\mathcal{M} = \{(a_i, b_i)\}$ that are pair-wise disjoint and cover $V(G)$ (every vertex is adjacent to exactly one edge). To establish this, we need a classical theorem:

Theorem 3 (Hall). *$G(A, B, E)$ admits a matching that saturates A i.e. a matching in which every $a \in A$ is incident with a matching edge iff for every $S \subseteq A$, $|N(S)| \geq |S|$ where $N(S) = \cup_{a\in S} N(s)$.*

Fix $S \subseteq A$. Since $G(A, B, E)$ is Δ-regular, we have, $e(S, N(S)) = \Delta|S|$.

$$\Delta|N(S)| \geq e(N(S), N(N(S))) \geq e(S, N(S)) \geq \Delta|S| \Rightarrow |N(S)| \geq |S|$$

From Hall's theorem, the result follows. □

From Theorem 2, it follows that line graphs of bipartite graphs are perfect.

Definition 4 (Vertex Cover). *A set of vertices that (together) touch every edge.*

Theorem 5 (Konig). *If G is bipartite, then size of a minimum vertex cover is the same as the size of a maximum matching.*

Using Theorem 5, we can establish that complement of bipartite graphs are also perfect. From the next theorem, it will follow that, complement of $L(G)$, where G is bipartite is also perfect.

Conjecture 6 (Weak Perfect Graph Conjecture). *G is perfect $\Leftrightarrow$ $\overline{G}$ is perfect.*

Theorem 7 (Lovasz, mid 70's). *G is perfect $\Leftrightarrow$ $\overline{G}$ is perfect.*

We'll see a proof this due to Gasparian (1996).

Definition 8 (Minimally imperfect graphs)**.** *Call a graph G minimally imperfect if G is not perfect, but every proper induced subgraph of G is perfect.*

For example, all odd cycles are minimally imperfect. We need the following observations:

Observation 9. *If G is minimally imperfect, then $\chi(G) = w(G) + 1$. Note that for any vertex x of G, $w(G \setminus \{x\}) = w(G)$.*

Observation 10. *Call a clique $\mathcal{C}$ large if $|\mathcal{C}| = w(G)$. Then for any non-empty, independent set $I \subseteq G$, there exists some large clique $\mathcal{C}$ of G such that $I \cap \mathcal{C} = \Phi$.*

Suppose I meets every large clique. Note that, $|I \cap \mathcal{C}| \in \{0, 1\}$, $\forall\, I \neq \Phi$. We have, $\chi(G \setminus I) = w(G \setminus I) \leq w(G) - 1$. So $G \setminus I$ can be $w(G) - 1$ colored $\Rightarrow G$ can be $w(G)$ colored. Contradiction. Write $\alpha = \alpha(G)$, $w = w(G)$. Let $I_0 = \{v_1, v_2 \ldots v_\alpha\}$ be a maximum independent set.

$$V \setminus \{v_i\} \;=\; I_1^{(i)} \uplus I_2^{(i)} \uplus \cdots \uplus I_w^{(i)}$$

since $\chi(V \setminus \{i\}) = w(G)$.

$$\mathcal{I} = \{I_0, I_j^{(i)} \;:\; 1 \leq i \leq \alpha,\; 1 \leq j \leq w\}$$

where $\mathcal{I}$ is the set of all independent sets. Let $N = |\mathcal{I}| = 1 + \alpha w$.

Claim 11. *If $\mathcal{C}$ is a large clique in G, then $\mathcal{C}$ is disjoint from ≤ 1 of the sets of $\mathcal{I}$.*

Proof. Suppose $\mathcal{C} \cap I_0 = \Phi$, then $\mathcal{C} \cap I_j^{(i)} \neq \Phi$, $\forall\, i, j$. To show this, notice that

$$\mathcal{C} \subseteq V \setminus \{v_i\} \;=\; I_1^{(i)} \uplus I_2^{(i)} \uplus \cdots \uplus I_w^{(i)}$$

Since $|\mathcal{C}| = w$, it must touch each $I_j^{(i)}$, $\forall\, i, j$. Now, suppose $\mathcal{C} \cap I_j^{(i)} = \Phi$, for some i, j. This means, following a similar line of argument as before, $\mathcal{C} \cap I_0 = \{v_i\}$. We have, $\mathcal{C} \cap I_k^{(l)} \neq \Phi$, $\forall k, l$ with $l \neq i$. Further, since $|\mathcal{C}| = w$, we have $\mathcal{C} \cap I_k^{(i)} \neq \Phi$, $\forall k$ with $k \neq j$. The claim follows. $\square$

Let $C_1, C_2 \ldots, C_N$ be large cliques such that each C_i is disjoint with exactly one of the independent sets of $\mathcal{I}$. Let $\mathcal{C}$ be the collection of sets $\{C_i\}$. Reorder the sets in $\mathcal{C}$ such that C_i is disjoint with I_i in $\mathcal{I}$. Let A be an $N \times n$ adjacency matrix such that $A(i, j) = 1$ iff I_i contains vertex j, otherwise 0. Similarly, define B as an $N \times n$ adjacency matrix with the rows labelled by cliques (in order) and columns labelled by vertices of G. Since I_i misses C_i, $I_i \cap C_i = \Phi$, but $|I_j \cap C_i| = 1$ for all $i \neq j$ $\Rightarrow (AB^T)_{n \times n} = J - I$. Since $J - I$ is invertible, we have (by rank arguments),

$$N = 1 + \alpha w \leq n.$$

Note that $n \leq \alpha(G)\chi(G)$, for any graph G. In particular,

$$\chi(G \setminus v) = w(G) \;\text{ and }\; \alpha(G \setminus v) \leq \alpha(G).$$

So,

$$
\begin{aligned}
&n-1 \leq \alpha(G\setminus v)\chi(G\setminus v) \leq \alpha(G)w(G) = \alpha w\\
\Rightarrow\ & n \leq 1+\alpha w = N\\
\Rightarrow\ & N = n\\
\Rightarrow\ & n = N = 1+\alpha(G)w(G) = 1+\alpha(\overline{G})w(\overline{G}).
\end{aligned}
$$

In particular, if G is minimally imperfect, $\overline{G}$ cannot be perfect either. This is because, if $\overline{G}$ is perfect, then

$$
\begin{aligned}
&w(\overline{G}) = \chi(\overline{G})\\
\Rightarrow\ & n \leq \alpha(\overline{G})\chi(\overline{G}) = \alpha(\overline{G})w(\overline{G}) < n.
\end{aligned}
$$

which is a contradiction.
We have proved so far that G being minimally imperfect $\Rightarrow \overline{G}$ is not perfect. Say G is perfect and $\overline{G}$ is not perfect. There exists a minimally imperfect induced subgraph $\overline{H}$ of $\overline{G}$. But $H = \overline{\overline{H}}$ is not perfect, which is a contradiction! Theorem 7 follows.
For an account of Lovasz's proof, see the book Modern Graph Theory by Bollobas.

Definition 12 (Odd Hole)**.** *An odd hole is an induced odd cycle of size ≥ 5.*

Definition 13 (Anti-Hole)**.** *An anti-hole is a complement of a hole.*

Conjecture 14 (Strong Perfect Graph Conjecture, Berge, 70s)**.** *G is perfect iff G has no odd holes/antiholes.*

This became a theorem in 2006.

Theorem 15 (Chudnovsky, Robertson, Seymour, Thomas, 2006)**.** *The conjecture is true.*

See [**?**] for the (150-page) proof!