MA-5103		2024
	The Szemeredi-Trotter Theorem	
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Template Problem

Let $\mathcal{L}$ be a set of L lines and $\mathcal{P}$ be a set of P points in $\mathbb{R}^2$. Define the number of incidences,

$$I(\mathcal{P},\mathcal{L}) = \#\{(p,l) \in \mathcal{P} \times \mathcal{L};\ p \in \mathcal{L}\}.$$

Apriori, we have $I(\mathcal{P},\mathcal{L}) \le PL$.

Theorem 1 (Szemeredi-Trotter, 84)**.** *Let $\mathcal{L}$ be a set of L lines and $\mathcal{P}$ be a set of P points in $\mathbb{R}^2$. We have,*

$$I(\mathcal{P},\mathcal{L}) \lesssim P^{2/3}L^{2/3} + P + L. \tag{1}$$

Further, the above bound is tight over $\mathbb{R}^2$.

Can we obtain the above upper bound over any field $\mathbb{F}$? Let $\mathbb{F} = \mathbb{F}_q$, we claim that many more incidences are possible! Consider the projective plane of order q, this has q^2+q+1 points and equal number of lines. Every point is incident on $q+1$ lines, therefore there are $O(q^2.q) = O(q^3)$ incidences. We have,

$$P^{2/3}L^{2/3} \approx q^{4/3}.q^{4/3} = O(q^{8/3}) < O(q^3).$$

So the previous bound doesn't work!

Observation. *If $P^{2/3} \le L^{1/3}$ ($L \ge P^2$), then $L^{2/3}P^{2/3} = O(L)$. Similarly if $L^{2/3} \le P^{1/3}$, then $L^{2/3}P^{2/3} \le P$. So the dominant term in Equation (1) is $P^{2/3}L^{2/3}$ only when $\sqrt{P} \le L \le P^2$ and $\sqrt{L} \le P \le L^2$.*

A Tight Construction

We will show that the bound in Theorem 1 is tight. Consider the 'grid' $\{(i,j) \in \mathbb{N}^2 : 1 \le i \le r, 1 \le j \le 2rs\}$ and define,

$$l_{a,b} \coloneqq \{Y = aX + b\}$$

$\forall a,b$ such that $1 \le a \le s$ and $1 \le b \le rs$. Observe that $1 \le X \le r \Rightarrow 1 \le Y = aX + b \le 2rs$. Let $|\mathcal{P}| = P$ be the number of points and $|\mathcal{L}| = L$ be the number of lines in the grid, we have $P = 2r^2s$ and $L = rs^2$. Note that each $l \in \mathcal{L}$ touches each line $(X = i)$ at exactly one point in $\mathcal{P}$. Therefore,

$$I(\mathcal{P},\mathcal{L}) = Lr = r^2s^2 \approx P^{2/3}L^{2/3}.$$

Pick r,s such that $\sqrt{L} \le P \le L^2$ and $\sqrt{P} \le L \le P^2$, so as to make the term $P^{2/3}L^{2/3}$ dominate over both linear P, L terms. This construction shows that the bound of Theorem 1 is asymptotically tight.

An Easier Upper Bound

Proposition 2. *$I(\mathcal{P},\mathcal{L}) \lesssim P\sqrt{L}+L$, and $I(\mathcal{P},\mathcal{L}) \lesssim L\sqrt{P}+P$.*

Proof. Let $J = \{(p,q,l) \,:\, p,q \in \mathcal{P},\ l \in \mathcal{L} \text{ and } p,q \in l\}$, we have

$$|J| = \#\{(p,q,l) \,:\, p \neq q\} + \#\{(p,p,l)\} \leq 2\binom{P}{2} + I(\mathcal{P},\mathcal{L}).$$

This follows from the fact that two distinct points in the plane have a unique line passing through both of them. On the other hand, for each line $l \in \mathcal{L}$, define $m_l = \#$ of points on l and observe,

$$|J| = \sum_{l \in \mathcal{L}} m_l^2.$$

Note that,

$$\sum_{l \in \mathcal{L}} m_l = I(\mathcal{P},\mathcal{L}).$$

Applying Cauchy-Schwarz gives the inequality,

$$\begin{aligned} I(\mathcal{P},\mathcal{L})^2 = (\sum_{l \in \mathcal{L}} m_l)^2 &\leq (\sum_{l \in \mathcal{L}} m_l^2)L \\ &\leq LP(P-1) + L\, I(\mathcal{P},\mathcal{L}), \end{aligned}$$

which is quadratic in $I(\mathcal{P},\mathcal{L})$. The first bound follows by analyzing the discriminant of the above quadratic. For the second bound, the argument is the same except points are replaced by lines, and we use the observation that two lines can intersect in atmost one point in the plane. $\square$

Remark. *The bound is infact tight, the equality case is only when equality holds in the application of Cauchy-Schwarz, and this is precisely when each line has same number of points. More concretely, consider the projective plane of order q over $\mathbb{F}_q$. This has q^2+q+1 points and equal number of lines, and $I(\mathcal{P},\mathcal{L}) = (q^2+q+1)(q+1) \approx P\sqrt{L}$, as the number of lines passing through each point is $(q+1) \approx \sqrt{L}$. The argument for the above proof doesn't take into account that we are working on $\mathbb{R}^2$ (which Theorem 1 does), infact it works for any general characteristic.*

We will use the tool of polynomial partitioning to prove Theorem 1, which we state below. Note that $Poly_D(\mathbb{R},n)$ is the set of all n-variable polynomials over $\mathbb{R}$ with deg $\leq D$.

Theorem 3 (Guth-Katz [1])**.** *Given a set $X \subseteq \mathbb{R}^n$, $|X| < \infty$ and an integer D, there exists $f(X_1,\ldots,X_n) \in Poly_D(\mathbb{R},n)$ such that,*

- $\mathbb{R}^n \setminus Z(f) = \bigcup_{i=1}^N O_i$ *with* $N \lesssim D^n$,
- $|X \cap O_i| \lesssim |X|/D^n$,

where O_i are open and connected 'cells' of the decomposition.

Remark. *It is very much possible that a lot of points of X might lie in $Z(f)$.*

The following provides an example which illustrates the above remark, i.e. we have $X \subseteq Z(f)$.

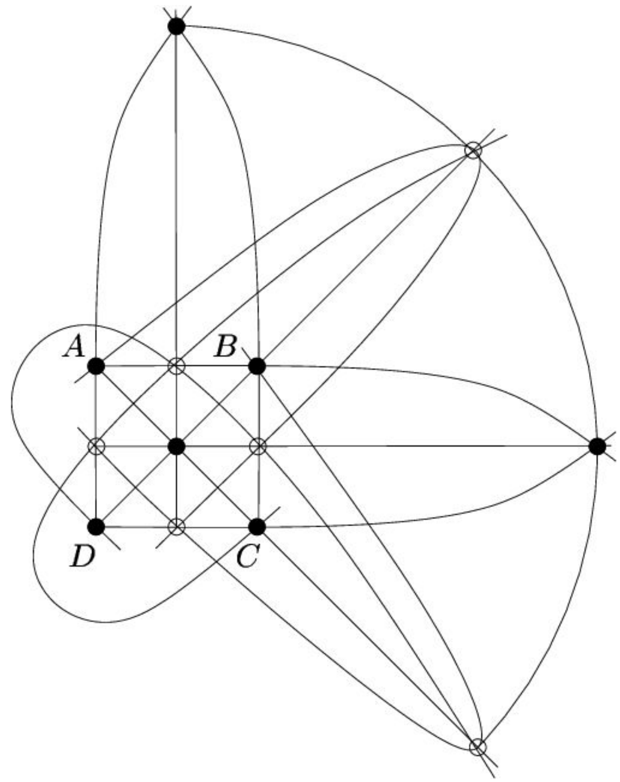

Figure 1: Projective plane of order 3, such a construction isn't possible in the plane!

Example. *Suppose $S \subseteq \mathbb{R} \subseteq \mathbb{R}^2$, where $\mathbb{R}$ is interpreted to be the x-axis in the plane. Let f be as in the polynomial partitioning theorem. For a suitable D (not too large!) we have the following,*

Claim. $Y \mid f(X, Y)$ *i.e. the x-axis completely lies inside $Z(f)$.*

To prove the claim, we will use the following,

Theorem 4 (Bezout)**.** *Over any field $\mathbb{F}$, we have $|Z(f,g)| \leq \deg(f)\deg(g)$, where f, g are relatively prime polynomials over $\mathbb{F}$ and $Z(f,g)$ denotes the common zero set of f, g.*

Proof of Claim. Assume for contradiction $Y \nmid f(X, Y)$. Therefore, we must have Y and f are relatively prime polynomials. By Bezout's theorem, $|Z(f) \cap \{Y = 0\}| \leq D$. The number of cells is $O(D)$, as f can have atmost D zeroes on the x- axis and the region between any two consecutive zeroes on the x-axis can be captured by a single cell. Each cell has $\lesssim (|S|/D^2)$ points which gives that the total number of points is bounded by $(|S|/D^2)\, O(D) < |S|$. Since this is not possible, we have a contradiction and the claim follows. $\square$

Proof of Theorem 1 using Polynomial Partitioning

Let D be a parameter to be determined, and let $f(X, Y)$ be a polynomial of deg $\leq D$ satisfying the conclusion of the polynomial partitioning theorem for $S = \mathcal{P}$. Write

$$\mathcal{P} = \mathcal{P}_{cell} \cup \mathcal{P}_{alg},$$
$$\mathcal{L} = \mathcal{L}_{cell} \cup \mathcal{L}_{alg},$$

where for each cell O_i, $\mathcal{P}_i = \mathcal{P} \cap O_i$, $\mathcal{L}_i$ is the set of lines of $\mathcal{L}$ passing through O_i, $\mathcal{P}_{cell} = \bigcup_i \mathcal{P}_i$, $\mathcal{P}_{alg} = \mathcal{P} \cap Z(f)$, $\mathcal{L}_{cell} = \bigcup_i \mathcal{L}_i$, and $\mathcal{L}_{alg}$ is the set of lines of $\mathcal{L}$ contained in $Z(f)$. Following this notation, we have

$$I(\mathcal{P}, \mathcal{L}) = \sum_i I(\mathcal{P}_i, \mathcal{L}_i) + I(\mathcal{P}_{alg}, \mathcal{L}_{cell}) + I(\mathcal{P}_{alg}, \mathcal{L}_{alg}).$$

Let $P_i = |\mathcal{P}_i|$ and $L_i = |\mathcal{L}_i|$. We have the following claim,

Claim. $I(\mathcal{P}_i, \mathcal{L}_i) \leq L_i + P_i^2$. *More generally,* $I(\mathcal{P}, \mathcal{L}) \leq L + P^2$ *and* $I(\mathcal{P}, \mathcal{L}) \leq P + L^2$.

Proof of Claim. For each $p \in \mathcal{P}$, let L_p be the number of lines through p and no other point of $\mathcal{P}$. Then if $I(p, \mathcal{L}) := \#\{(p, l) : l \in \mathcal{L},\ p \in l\}$, we have

$$I(p, \mathcal{L}) \leq L_p + P\,, \tag{2}$$

since there exactly L_p lines only containing the point p, and the number of lines containing p and some other point of $\mathcal{P}$ is atmost P (atmost one line can pass between any two points in the plane). Further, observe that $\sum_{p \in \mathcal{P}} L_p \leq L$. Summing Equation (2) over all points of $\mathcal{P}$ gives the first inequality. To show the second inequality $I(\mathcal{P}, \mathcal{L}) \leq P + L^2$, we can replace points by lines and repeat the same argument. □

Now we will bound $I(\mathcal{P}_i, \mathcal{L}_i)$. Fix $l \in \mathcal{L} \setminus \mathcal{L}_{alg}$, then l meets $\leq D + 1$ regions O_i. When we alternate between regions, we must see a zero of f and $l \notin \mathcal{L}_{alg}$ can have atmost D zeroes of f. Therefore,

$$\sum_i L_i \leq (D + 1)L,$$

where $L_i = |\mathcal{L}_i|$. By counting the number of pairs (l, O) such that line l crosses region O, we have

$$\sum_i L_i = \sum_l \#\{O_i : l \text{ meets } O_i\} \leq (D + 1)L.$$

Hence,

$$\sum_i I(\mathcal{P}_i, \mathcal{L}_i) \leq \sum_i L_i + \sum_i P_i^2 \lesssim (D + 1)L + P^2/D^2,$$

since each $P_i \lesssim P/D^2$, whence we get $\sum_i P_i^2 \lesssim (P/D^2) \sum_i P_i \leq (P^2/D^2)$. Recall that,

$$I(\mathcal{P}, \mathcal{L}) = \sum_i I(\mathcal{P}_i, \mathcal{L}_i) + I(\mathcal{P}_{alg}, \mathcal{L}_{alg}) + I(\mathcal{P}_{alg}, \mathcal{L}_{cell}). \tag{3}$$

By the analysis above, we already have $\sum_i I(\mathcal{P}_i, \mathcal{L}_i) \lesssim DL + P/D^2$. To bound the second term in Equation (3), observe that each $l \in \mathcal{L}_{alg}$ is a linear factor of f. This gives $|\mathcal{L}_{alg}| \leq D$, and hence

$$I(\mathcal{P}_{alg}, \mathcal{L}_{alg}) \leq P + D^2,$$

where we have used the claim above. For the third term, note that each $l \in \mathcal{L}_{cell}$ touches $Z(f)$ in $\leq D$ points. This gives,

$$I(\mathcal{P}_{alg}, \mathcal{L}_{cell}) \leq DL.$$

Putting everything together, we have

$$I(\mathcal{P}, \mathcal{L}) \lesssim DL + P^2/D^2 + D^2 + P.$$

Optimizing over the parameter D (i.e. set $D = (2P^2/L)^{1/3}$), we get

$$I(\mathcal{P}, \mathcal{L}) \lesssim P^{2/3}L^{2/3} + P + L,$$

and we conclude the proof of Theorem 1.

Proof of Polynomial Partitioning Theorem

We have seen the proof of Theorem 1 via Polynomial Partitioning, which we shall prove now. Recall,

Theorem 5 (Guth-Katz [1])**.** *Given a set $S \subseteq \mathbb{R}^n$ ($|S| < \infty$) and $D \in \mathbb{N}$, there exists a polynomial $f \in \mathbb{R}[X_1, \ldots, X_n]$ such that,*

- $\deg(f) \leq D$,
- $\mathbb{R}^n \setminus Z(f)$ *has* $\lesssim D^n$ *(open) connected components (each called cell),*
- *In each cell* O_i, $|O_i \cap S| \lesssim |S|/D^n$.

In this sense, the partitioning of $\mathbb{R}^n$ via the polynomial f is 'equitable'. For the proof of Theorem 5, we will need a topological tool called the generalized ham-sandwich theorem.

Theorem 6 (Generalized Ham-Sandwich)**.** *Suppose V is a vector space of real-valued continuous functions, and suppose $U_1, \ldots, U_N$ are disjoint open sets in $\mathbb{R}^n$. If $\dim_{\mathbb{R}} V > N$ and $Z(f)$ has lebesgue measure 0 in $\mathbb{R}^n$ for each $f \in V \setminus \{0\}$, then there exists $f \in V \setminus \{0\}$ such that f bisects U_i $\forall\, i$, i.e. for each U_i,*

$$m(U_i^+) = m(U_i^-),$$

where $U_i^+ = \{x \in U_i : f(x) > 0\}$ and $U_i^- = \{x \in U_i : f(x) < 0\}$.

We also have the polynomial version of the ham-sandwich theorem,

Theorem 7 (Polynomial Ham-Sandwich)**.** *Suppose $U_1, \ldots, U_N$ are disjoint open sets in $\mathbb{R}^n$, and suppose $D \in \mathbb{N}$ such that $N < \binom{n+D}{n}$. Then there exists non-zero $f \in Poly_{\leq D}(\mathbb{R}^n)$ such that f bisects all the U_i.*

This is a special case of Theorem 6 when $V \equiv$ Polynomials of deg $\leq D$. We also state the finite set ham-sandwich theorem,

Definition. *Given $S \subseteq \mathbb{R}^n$ ($|S| < \infty$), a function f bisects S if for*

$$S^+ := \{x \in S : f(x) > 0\}$$
$$S^- := \{x \in S : f(x) < 0\}$$

we have, $|S^+| \leq |S|/2$ and $|S^-| \leq |S|/2$.

Theorem 8 (Finite Set Ham-Sandwich). *Given finite sets $S_1, \ldots, S_N$ in $\mathbb{R}^n$. If $N < \binom{n+D}{n}$, there exists non-zero polynomial f of $\deg \leq D$ such that f bisects each S_i.*

With these tools in hand, first we will establish the polynomial partitioning theorem.

Proof of Theorem 5. Given $S \subseteq \mathbb{R}^n$ with $|S| < \infty$. By Theorem 8, there exists polynomial f_1 of $\deg = 1$ that bisects S into S_1 and S_2. Get another polynomial f_2 such that f_2 bisects both S_1 and S_2 into $S_{11}, S_{12}, S_{21}, S_{22}$. At stage j, get f_j of $\deg \leq n2^{j/n}$ that simultaneously bisects 2^{j-1} sets into 2^j sets. Stop after k steps, and let $f = f_1 f_2 \ldots f_k$, where k is a parameter TBD. Observe that the number of components of $\mathbb{R}^n \setminus Z(f) \leq 2^k$, since each f_i $(1 \leq i \leq k)$ has the same sign in each component. Choose k such that,

$$\begin{aligned} & 2n2^{k/n} \leq D \\ \Rightarrow\ & 2^k \leq (D/2n)^n \\ \Rightarrow\ & 2^k \leq C(n)D^n, \end{aligned}$$

where $C(n)$ is a constant depending on n. So $\mathbb{R}^n \setminus Z(f)$ has atmost $C(n)D^n$ connected components. Note that after step j, there are $\leq |S|/2^j$ points of S in each connected component, by construction. Therefore, at end, each cell has $\leq |S|/2^k \lesssim |S|/D^n$ elements of S in each cell of the partition by f (constant depending on n is absorbed, and $2^k = O(D^n)$). $\square$

Now we will look at a proof of the finite set ham-sandwich, assuming the polynomial ham-sandwich theorem.

Proof of Theorem 8 assuming Theorem 7. Given $S_1, \ldots, S_N$ finite disjoint sets. We want a non-zero polynomial f of $\deg \leq D$ such that f bisects each S_i, where D is such that $\binom{n+D}{n} > N$. For $\delta > 0$ (sufficiently small), let $U_i^{(\delta)}$ = union of balls of radius δ centered at the elements of S_i. For

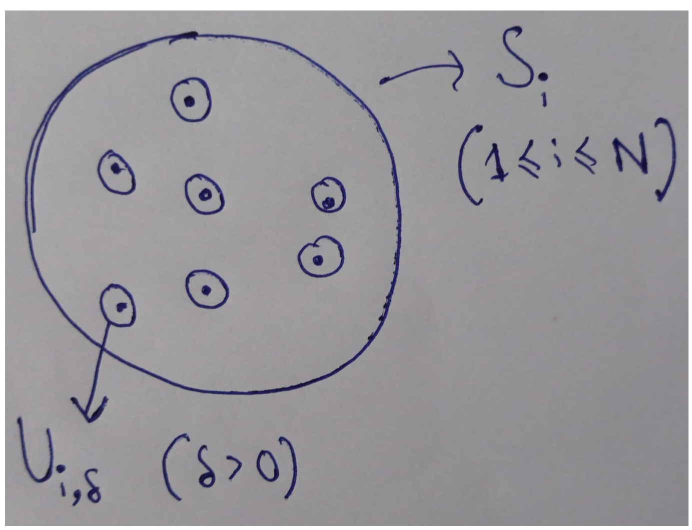

Figure 2: Disjoint balls of radius $\delta > 0$ covering elements of S_i

the collection $\{U_i^{(\delta)}\}_{i=1}^N$, get f_δ by Theorem 7 that bisects each $U_i^{(\delta)}$. For any polynomial p, define

$\|p\|$ to be the maximal absolute value of the coefficients of p. By scaling each f_δ, we may assume WLOG that $\|f_\delta\| = 1$, $\forall\, \delta > 0$. Now we can find a sequence $(\delta_m) \downarrow 0$ such that f_{δ_m} converges in the space of all deg $\leq D$ polynomials. Let f be the limit polynomial, observe that $\deg(f) \leq D$ and $\|f\| = 1$. In particular, f is non-zero. Since the coefficients of f_{δ_m} converge to the coefficients of polynomial f, we have that $f_{\delta_m} \to f$ uniformly on compact sets.

Claim. *The polynomial f bisects S_i, $\forall\, 1 \leq i \leq N$.*

Proof. We will proceed by contradiction, assume instead that $f > 0$ on more than half the points of some S_i (the case $f < 0$ is similar). Let $S_i^+ \subset S_i$ denote the set of points of S_i where $f > 0$. By choosing ϵ sufficiently small, we can assume that $f > \epsilon$ on the ϵ-ball around each point of S_i^+. Also, we can choose ϵ small enough such that the ϵ-balls around the points of S_i are disjoint. Since $f_{\delta_m} \to f$ uniformly on compact sets, we can find m large enough that $f_{\delta_m} > 0$ on the ϵ-ball around each point of S_i^+. By making m large, we can also arrange that $\delta_m < \epsilon$. Therefore, $f_{\delta_m} > 0$ on the δ_m-ball around each point of S_i^+. But then $f_{\delta_m} > 0$ on more than half of $U_i^{(\delta_m)}$. This contradiction proves that f bisects S_i, $\forall\, 1 \leq i \leq N$. ☐

The above claim completes the proof of Theorem 8 assuming Theorem 7. ☐

Now we will prove the generalized ham-sandwich theorem,

Proof of Theorem 6. Given sets $U_1, \ldots, U_N$ open and disjoint in $\mathbb{R}^n$. WLOG, $V \simeq \mathbb{R}^{N+1}$ (span-wise, not topological). We will use the following theorem,

Theorem 9 (Borsuk-Ulam)**.** *Suppose $\phi : S^N \to \mathbb{R}^N$ is continuous and anti-podal, i.e. $\phi(-x) = -\phi(x)$, $\forall\, x \in S^n$. Then there exists $\tilde{x} \in S^n$ such that $\phi(\tilde{x}) = 0$.*

For each U_i, define $\phi_i : V \to \mathbb{R}$ as follows,

$$\phi_i(f) = m(U_i^+) - m(U_i^-),$$

where each ϕ_i is continuous and $\phi_i(-f) = -\phi_i(f)$, $\forall\, 1 \leq i \leq N$. Define $\phi = (\phi_1, \ldots, \phi_N)$, we have $\phi : V \simeq \mathbb{R}^{n+1} \to \mathbb{R}^n$ is continuous and anti-podal. This gives by restriction (scaling) $\phi : S^N \subseteq \mathbb{R}^{n+1} \to \mathbb{R}^N$ and by Borsuk-Ulam, there exists $f \in S^N \subset V$ such that $\phi(f) = \tilde{0}$. Therefore, for this choice of f, $m(U_i^+) = m(U_i^-)$ $(\forall\, i)$, and the theorem follows. ☐

A Brief Sojourn into Algebraic Geometry

The following definitions are over $\mathbb{R}$.

Definition (Variety)**.** *A set $V \subset \mathbb{R}^d$ is a variety if $V = Z(f_1, \ldots, f_m)$ for some $f_1, \ldots, f_m \in \mathbb{R}[X_1, \ldots, X_d]$.*

Remark. *All varieties over $\mathbb{R}$ are given by a single polynomial.*

Definition (Irreducibility)**.** *A variety V is <u>reducible</u> if $V = U \cup W$ with $U, W \neq \emptyset$ and U, W being varieties. A variety is <u>irreducible</u> if it isn't reducible.*

Example. *Let $f = XY \in \mathbb{R}[X, Y]$, note that $V(f) = (x - axis) \cup (y - axis)$. As each of the axes form non-empty varieties, $V(f)$ is reducible.*

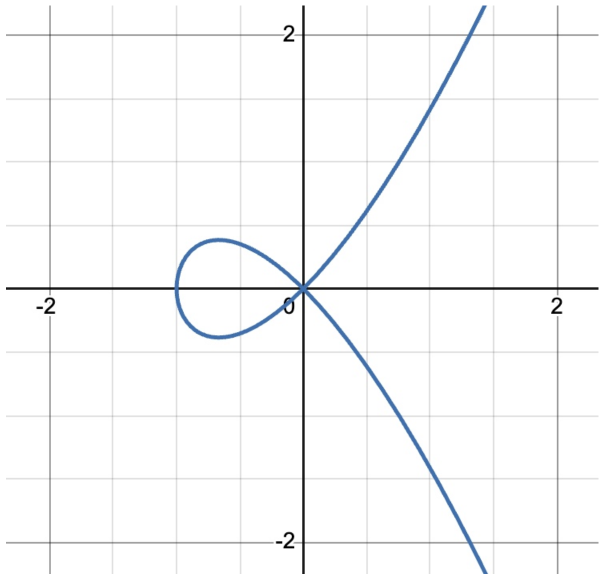

Figure 3: $V(f) = \langle Y^2 - X^3 - X^2 \rangle$ is a curve in $\mathbb{R}^2$

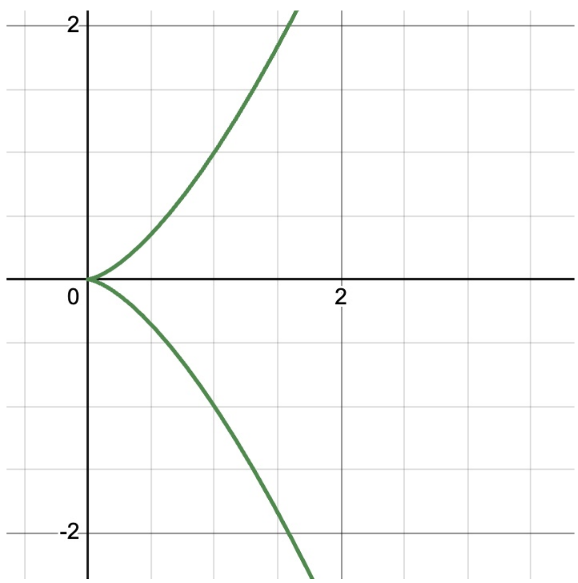

Figure 4: $V(f) = \langle X^3 - Y^2 \rangle$, yet another example of a curve in $\mathbb{R}^2$

Now we will define the dimension of a variety in $\mathbb{R}^d$.

Definition (Dimension)**.** *Let $V \subset \mathbb{R}^d$ be a variety. If V is irreducible, then* $\dim V$ *is the maximum $k \in \mathbb{N}_0$ such that there exists a chain,*

$$W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_k = V$$

with each $W_i \neq \emptyset$ being an irreducible sub-variety of V.

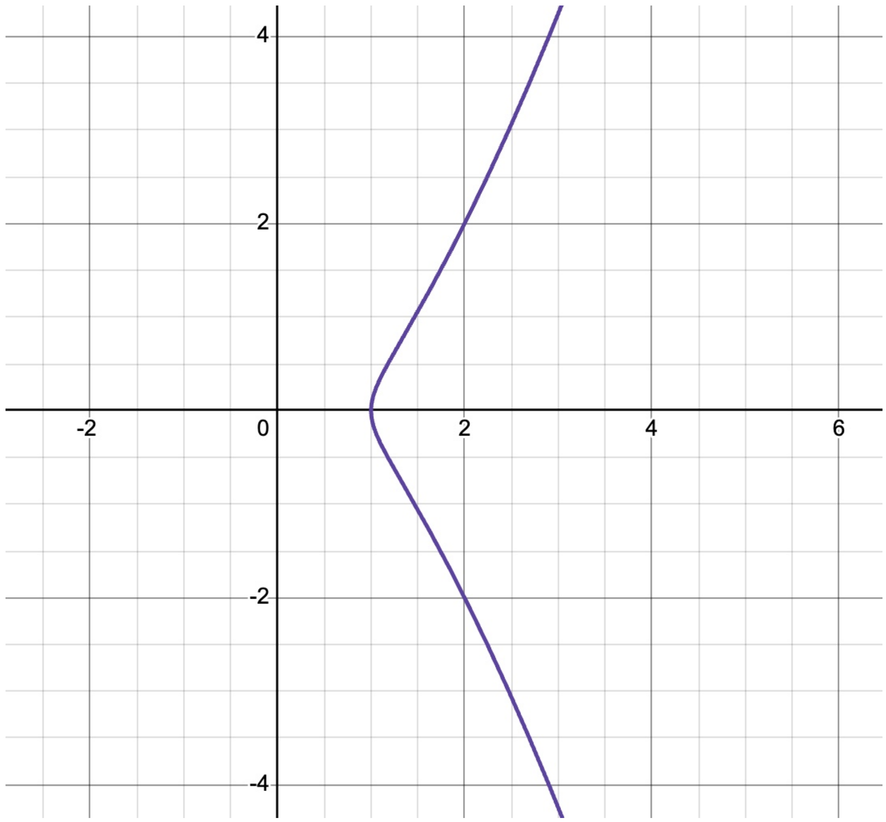

Figure 5: $V(f) = \langle Y^2 - X^3 + X^2 \rangle$, curve in $\mathbb{R}^2$ (the point $(0,0)$ is isolated)

If $V \subseteq \mathbb{R}^d$ is reducible, let $(U_1, \ldots, U_r)$ be its (unique!) irreducible components. Then

$$\dim V = \max_i (\dim U_i).$$

Remark. *For any variety $V \subseteq \mathbb{R}^d$, we always have $\dim V \leq d$.*

Definition (Complexity)**.** *The complexity of a variety V is the minimum $k \in \mathbb{N}_0$ such that V can be expressed as the 'variety generated' by $\leq k$ polynomials of $\deg \leq k$ each.*

Example (Reducibility and Dimension)**.** *Let $V_1 = \langle XY \rangle$ and $V_2 = \langle XY - 1 \rangle$ be two varieties over $\mathbb{R}$. Notice that $\dim(V_1) = \dim(V_2) = 1$, however V_1 is reducible and V_2 is irreducible.*

Remark (Warning!)**.** *Irreducible components aren't necessarily topologically connected.*

Definition (Curve)**.** *A curve is an irreducible variety in $\mathbb{R}^2$ of dimension 1.*

Definition (Degree of a curve)**.** *Let $\gamma \subset \mathbb{R}^2$ be a curve, $\deg(\gamma)$ is the minimum degree d such that $\gamma = V(f)$, where f has degree d.*

Proposition 10. *Suppose $\gamma \subset \mathbb{R}^2$ is an irreducible curve of degree d. Then $\exists f \in \mathbb{R}[X,Y]$ such that $\gamma = V(f)$ and if $g \in \mathbb{R}[X,Y]$ such that $V \subseteq V(g)$ then $f \mid g$ in $\mathbb{R}[X,Y]$.*

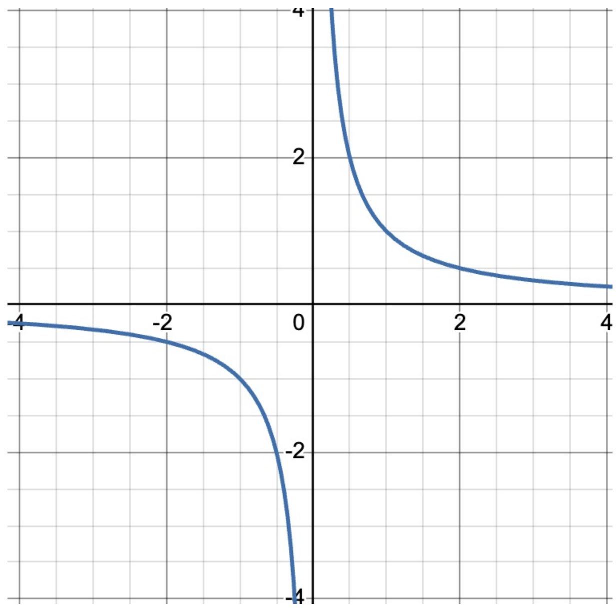

Figure 6: $\langle XY - 1\rangle$ is irreducible, but isn't topologically connected!

Proof. Suppose $\deg(\gamma) = d$. Then $\exists f \in \mathbb{R}[X,Y]$ of deg d such that $\gamma = V(f)$. We have the following claim,

Claim. *f is irreducible.*

Proof of Claim. Let $f = f_1 f_2$ with f_1, f_2 being non-constant. Then $\deg(f_i) < d$ for $i = 1,2$ and as f is the least degree polynomial defining V, we have $V(f_1), V(f_2) \neq \emptyset$. In particular, $V(f_i) \subsetneq V(f)$ for $i = 1,2$. But then $\gamma = V(f) = V(f_1) \cup V(f_2)$, contradicting that γ is irreducible as a variety. $\square$

Assume $f \nmid g$. Since f is irreducible, this gives that f and g don't have any common factors. By Bezout, $|V(f) \cap V(g)| \leq \deg(f)\deg(g) < \infty$. As $V(f) \subseteq V(f) \cap V(g)$, we get that the variety generated by the polynomial f only has finitely many points. This isn't possible as γ is a curve in $\mathbb{R}^2$! $\square$

Fact. *The number of irreducible components of a variety $V \subseteq \mathbb{R}^d$ of complexity k is $O_{k,d}(1)$.*

A Combinatorial Perspective

Suppose $\mathcal{P} \subseteq \mathbb{R}^2$, $|\mathcal{P}| = P$, and suppose Γ is a set of N curves in $\mathbb{R}^2$ of deg $\leq k$ (for some k). Define the graph $G(\mathcal{P}, \Gamma)$ to be the bipartite graph with partition parts $\mathcal{P}$ and Γ, and (p, γ) is an edge in G if and only $p \in \gamma$ (incidence in $\mathbb{R}^2$). If Γ is a set of lines, since two lines can intersect in atmost one point in the plane, $K_{2,2}$ is forbidden in G. If Γ only consists of unit circles, $K_{2,3}$ is forbidden in G as three (distinct) unit circles in the plane can't all have two points in common.

Theorem 11 (Kovari-Sos-Turan)**.** *$G(m,n)$ is bipartite with parts of size m, n and suppose that $K_{s,t} \not\subset G$, for $2 \leq s \leq t$. Then,*

$$e(G) \leq O_{s,t}(mn^{1-1/s} + n).$$

Proof. Let G have parts X, Y with $|X| = m$ and $|Y| = n$. Pick $x \in X$, and any set T of t neighbours of x. We have,

$$\#\{(x,T) : T \subseteq N(X),\ |T| = t\} = \sum_{x \in X} \binom{d(x)}{t}.$$

On the other hand, any set $T \subseteq Y$ of size t can arise from $\leq (s-1)$ different $x \in X$. Therefore,

$$\sum_{x \in X} \binom{d(x)}{t} \leq (s-1)\binom{n}{t}.$$

Since $\binom{x}{t}$ is convex, we can apply Jensen and this gives,

$$m\binom{\frac{\sum_{x\in X} d(x)}{m}}{t} \leq (s-1)\binom{n}{t}$$
$$\Rightarrow\ m\binom{e/m}{t} \leq (s-1)\binom{n}{t}.$$

Simplifying gives the required bound for $e = e(G)$. $\square$

Theorem 12. *Suppose Γ is a collection of irreducible curves (with $|\Gamma| = N$) of $\deg \leq k$ in $\mathbb{R}^2$ and let $\mathcal{P} \subseteq \mathbb{R}^2$, $|\mathcal{P}| = P$. If the graph $G(\mathcal{P}, \Gamma)$ has no $K_{s,t}$ then*

$$I(\mathcal{P}, \Gamma) \leq O_{s,t,k}(P^{s/2s-1} N^{2s-2/2s-1} + P + N).$$

Note that just forbidding $K_{s,t}$ gives a bound (purely combinatorial!) of $O_{s,t}(PN^{1-1/s} + N)$.

Proof. We will use polynomial partition again, more precisely, suppose $f \in \mathbb{R}[X, Y]$ of $\deg \leq r$ (r TBD) such that,

- $\mathbb{R}^2 \setminus Z(f)$ has $\lesssim r^2$ connected components,
- Each cell O_i has $|O_i \cap \mathcal{P}| \lesssim P/r^2$.

Again, define for a cell O_i, $\mathcal{P}_i = \mathcal{P} \cap O_i$, $\Gamma_i = \{\gamma \in \Gamma\ :\ \gamma \cap O_i \neq \emptyset\}$, $\mathcal{P}_{cell} = \cup_i \mathcal{P}_i$, $\Gamma_{cell} = \cup_i \Gamma_i$, $\mathcal{P}_{alg} = \mathcal{P} \cap Z(f)$, and $\Gamma_{alg} = \{\gamma \in \Gamma : \gamma \subseteq Z(f)\}$. Following this notation, $\mathcal{P} = \mathcal{P}_{alg} \cup \mathcal{P}_{cell}$ and $\Gamma = \Gamma_{alg} \cup \Gamma_{cell}$. Let $|\mathcal{P}_i| = P_i$ and $|\Gamma_i| = N_i$. We have,

$$I(\mathcal{P}, \Gamma) = I(\mathcal{P}_{cell}, \Gamma_{cell}) + I(\mathcal{P}_{alg}, \Gamma_{cell}) + I(\mathcal{P}_{alg}, \Gamma_{alg}). \tag{4}$$

For the first term in Equation (4), we have the split

$$I(\mathcal{P}_{cell}, \Gamma_{cell}) = \sum_i I(\mathcal{P}_i, \Gamma_i).$$

By the combinatorial bound on each cell, we have

$$\sum_i I(\mathcal{P}_i, \Gamma_i) \leq C_{s,t,k}\Big(\sum_i P_i N_i^{1-1/s} + \sum_i N_i\Big)$$
$$\leq O_{s,t,k}(1).\Big(P/r^2 \sum_i N_i^{1-1/s} + \sum_i N_i\Big).$$

If we have an upper bound for $\sum_i N_i$, then we can upper bound $\sum_i N_i^{1-1/s}$ by using Holder. Observe that,

$$\sum_i N_i = \sum_i \sum_{\gamma \in \Gamma} \mathbb{1}_{\gamma \cap O_i \neq \emptyset} = \sum_{\gamma \in \Gamma} \#\{i : \gamma \cap O_i \neq \emptyset\}.$$

Fix $\gamma \in \Gamma$, we will bound $\#\{i : \gamma \cap O_i \neq \emptyset\}$. Imagine a small circle $C_\epsilon(p)$ centered at an intersection p of γ and $Z(f)$. Then the # of regions O_i that γ enters around $p \leq$ # of intersection points of $C_\epsilon(p)$ and γ. Observe that $C_\epsilon(p)$ and γ don't have common factors because $\gamma \in \Gamma$ is irreducible. By Bezout's theorem, the number of such intersection points between $C_\epsilon(p)$ and $\gamma \leq \deg(\gamma)\deg(C_\epsilon(p)) \leq 2k$ (as $deg(C_\epsilon(p)) = 2$). Therefore,

$$\#\{i : \gamma \cap O_i \neq \emptyset\} \leq (2k)(kr),$$

as $|\gamma \cap Z(f)| \leq \deg(\gamma)\deg(f) \leq kr$. Summing the above over all $\gamma \in \Gamma$ gives,

$$\sum_i N_i = \sum_\gamma \#\{i : \gamma \cap O_i \neq \emptyset\} \leq O_k(rN).$$

By an application of Holder, we have

$$\sum_i N_i^{1-1/s} \leq \Big(\sum_i N_i\Big)^{(s-1)/s} \Big(r^2\Big)^{1/s},$$

since there are $\lesssim r^2$ cells in the polynomial partition. Finally we obtain the bound on the first term,

$$\sum_i I(\mathcal{P}_i, \Gamma_i) = O_{s,t,k}(P(N/r)^{1-1/s} + rN).$$

For the second term, note that

$$I(\mathcal{P}_{alg}, \Gamma_{cell}) = \#\{(p, \gamma) : p \in \gamma \cap Z(f),\ \gamma \not\subset Z(f)\}.$$

Since $\gamma \in \Gamma_{cell}$, we have that γ cannot lie inside $Z(f)$. Therefore, since γ is irreducible, we can apply Bezout and this gives,

$$I(\mathcal{P}_{alg}, \Gamma_{cell}) \leq \sum_\gamma \deg(f)\deg(\gamma) \leq rkN \leq O_k(rN).$$

Finally, for $I(\mathcal{P}_{alg}, \Gamma_{alg})$, note that

$$\Gamma_{alg} = \{\gamma \in \Gamma : \gamma \subseteq Z(f)\}.$$

To bound $I(\mathcal{P}_{alg}, \Gamma_{alg})$, we need to consider all the components of $Z(f)$. Isolated points of $Z(f)$ won't contribute to these incidences (as curves can't be singletons!). Consider the dim 1 components of $Z(f)$. We have the following definitions,

Definition (Singularity)**.** *A point $p \in Z(f)$ is called singular if $\nabla f(p) = \tilde{0}$. In particular, if f is a min-degree polynomial that defines a variety V, a point $p \in V$ is singular if $\nabla f(p) = \tilde{0}$.*

Definition (Regularity)**.** *A point $p \in V$ is called regular if it isn't singular.*

We have the following (easy) fact,

Fact. *If the polynomial f has no repeated factors, then f and f_x have no common factors (where f 'depends' on x).*

Let $f = \prod_{i=1}^{o} g_i$, where each g_i is irreducible. Since $\deg(f) \le r$, we must have $o \le r$. Fix $\gamma \in \Gamma_{alg}$. Incidences $(p, \gamma) \in \mathcal{P}_{alg} \times \Gamma_{alg}$ are of two kinds:

- p is regular in $Z(f)$,
- p is singular in $Z(f)$.

If $p \in \mathcal{P}_{alg}$ is regular with respect to $Z(f)$, then it lies on ≤ 1 of the factors g_i of f, since a regular point can't come in more than one component of $Z(f)$. So,

$$I(\mathcal{P}_{reg} \cap \Gamma_{alg}) \le P_{reg} \le P.$$

Finally, if p is singular, WLOG we can assume that no line component of f is parallel to the x-axis (else apply a rotation). If p is singular, then $f_x(p) = 0$. Then $f, f_x(\not\equiv 0)$ have no common factor (if the polynomial f that comes from polynomial partitioning is chosen to be one of least degree, then it has no repeated factor). We conclude that f, f_x have no common factors, therefore such a point p must lie on $Z(f_x) \cap Z(\gamma)$, and Bezout applies to give that for each fixed $\gamma \in \Gamma_{alg}$, the number of singular point incidences is $\le deg(f_x)deg(f) \le (k-1)r$. We obtain,

$$I(\mathcal{P}_{sing}, \Gamma_{alg}) \le O_k(rN).$$

Therefore, the bound for the third term becomes,

$$I(\mathcal{P}_{alg}, \Gamma_{alg}) = I(\mathcal{P}_{reg}, \Gamma_{alg}) + I(\mathcal{P}_{sing}, \Gamma_{alg}) \le P + O_k(rN).$$

Hence,

$$I(\mathcal{P}, \Gamma) \le O_{s,t,k}(P(N/r)^{1-1/s} + rN + P).$$

We need to optimize over the parameter r to obtain the best bound in the above inequality. Setting $r = P^{s/2s-1}/N^{1/2s-1}$ does the job! □

Next we will see a version of Theorem 12 in $\mathbb{R}^d$. Formally we will show the following,

Theorem 13. *Suppose $\mathcal{P} \subseteq \mathbb{R}^d$, with $|\mathcal{P}| = P$. Let Γ be a set of N varieties of* dim ≤ 1 *and complexity $\le k$. If $G(\mathcal{P}, \Gamma)$ has no $K_{s,t}$, then*

$$I(\mathcal{P}, \Gamma) \le O_{s,t,k,d}(P^{s/2s-1}N^{2s-2/2s-1} + P + N).$$

Before proving the above, we will look at the version of Szemeredi-Trotter over $\mathbb{R}^3$ as a warm-up!

Theorem 14 (Szemeredi-Trotter over $\mathbb{R}^3$)**.** *Let $\mathcal{P}$ be a set of points and $\mathcal{L}$ be a set of lines in $\mathbb{R}^3$. We have,*

$$I(\mathcal{P}, \mathcal{L}) \le O(P^{2/3}L^{2/3} + P + L).$$

Proof. Project to a 'generic' plane of $\mathbb{R}^3$. Let $\mathcal{P}'$ be the set of points, and let $\mathcal{L}'$ be the set of lines obtained after projection. We have,

$$I(\mathcal{P}, \mathcal{L}) = I(\mathcal{P}', \mathcal{L}') \le O(P^{2/3}L^{2/3} + P + L),$$

from Theorem 1, since incidences are preserved after projection onto a generic plane. □

We shall resort to the same idea for the proof of Theorem 13, i.e. projecting onto a generic point in $\mathbb{R}^d$. We will be using the following facts,

Definition (Zariski Closure)**.** *The Zariski closure of a set $U \subseteq \mathbb{R}^d$ denoted by $\overline{U}$ is the smallest variety in $\mathbb{R}^d$ containing U.*

Fact. *Suppose $U \subseteq \mathbb{R}^d$ is a variety of $\dim d_U$ and complexity k. Then for any projection $\pi : \mathbb{R}^d \to \mathbb{R}^l$ (with $l < d$), $\overline{\pi(U)}$ (Zariski closure) is a variety of $\dim \leq d$ and complexity $O_{k,d}(1)$.*

Fact. *The number of irreducible components of $U \subseteq \mathbb{R}^d$ and complexity k is $O_{k,d}(1)$.*

Recall that we need to prove $I(\mathcal{P}, \Gamma) \leq O_{s,t,k,d}(P^{s/2s-1}N^{2s-2/2s-1} + P + N)$.

Proof of Theorem 13. Let Γ be the collection of N varieties of $\dim \leq 1$, not necessarily irreducible. Break each $\gamma \in \Gamma$ into its irreducible components. Each γ splits into $\leq O_{k,d}(1)$ components by the second fact above. Let the maximum number of components of any $\gamma \in \Gamma$ be $c = O_{k,d}(1)$. Recall that a dim 0 variety in $\mathbb{R}^d$ is a finite set of points, futher if the variety is irreducible, it must be a single point. Collect all the dim 0 components, i.e. the points in Γ (call this Γ_0) counting with respect to multiplicity of points occuring in components. We have,

$$I(\mathcal{P}, \Gamma_0) \leq |\Gamma_0| \leq O_{k,d}(N).$$

So we will now work with only dim 1 irreducible components. Our first (natural) idea is to take Γ' to be the multiset of all dim 1 components arising from all $\gamma \in \Gamma$ and consider working with $I(\mathcal{P}, \Gamma')$ instead, repeat irreducible components if it occurs in both $\gamma_1, \gamma_2 \in \Gamma$. However, this may be problematic. We know $K_{s,t} \not\subset G(\mathcal{P}, \Gamma)$, but can we conclude $K_{s,t} \not\subset G(\mathcal{P}, \Gamma')$?
More concretely, let γ be the union of four circles in the plane, all of which pass through two common points p and q. Let $\Gamma = \{\gamma\}$, if we split γ into four irreducible components, the points p, q occur in each circle, which implies that $K_{2,4}$ must be present in $G(\mathcal{P}, \Gamma')$!

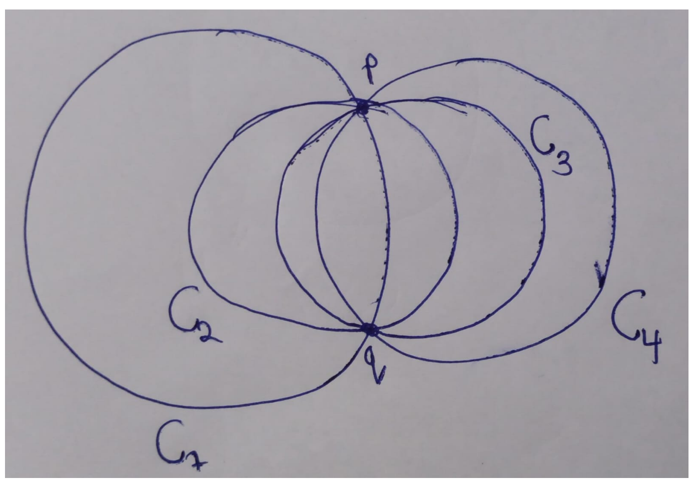

Figure 7: 4 circles passing through two common points p, q in the plane, a $K_{2,4}$ is present in $G(\mathcal{P}, \Gamma')$

We shall project onto a generic plane $\pi : \mathbb{R}^d \to \mathbb{R}^2$ so that $|\pi(\mathcal{P})| = P$, $|\{\overline{\pi(\gamma)}\}_{\gamma \in \Gamma}| = N$, and there are no additional incidences generated by the projection. So we will be able to use the previous theorem provided we show that $K_{s,t}$ is forbidden in the bipartite graph between the projected points and the irreducible curve components of $\gamma \in \Gamma$. Note that each $\gamma \in \Gamma$ has $\leq c = O_{k,d}(1)$ components. If $\Gamma' =$ multiset of all dim 1 irreducibles coming from all $\gamma \in \Gamma$, we can write

$$\Gamma' = \Gamma'_1 \cup \cdots \cup \Gamma'_c,$$

where each Γ'_j contains ≤ 1 component of each $\gamma \in \Gamma$. We will work with $I(\mathcal{P}, \Gamma'_i)$, instead of $I(\mathcal{P}, \Gamma')$. Now project Γ'_i onto $\mathbb{R}^2$ for each i, Theorem 12 is now applicable! Observe that $K_{s,t} \not\subset G(\mathcal{P}, \Gamma'_i)$ after projection, therefore we get

$$I(\mathcal{P}, \Gamma'_i) \leq O_{s,t,k,d}(P^{2/3}N^{2/3} + P + N).$$

Summing the above inequality over i gives,

$$I(\mathcal{P}, \Gamma) \leq O_{s,t,k,d}(P^{2/3}N^{2/3} + P + N).$$

$\square$

Szemeredi-Trotter over $\mathbb{C}^2$

In this section, we will see a version of Szemeredi-Trotter over $\mathbb{C}^2$.

Theorem 15 (Solymosi-Tao [2])**.** *Let $\epsilon > 0$. Then there exists $\alpha_1 = \alpha_1(\epsilon)$, $\alpha_2 = \alpha_2(\epsilon)$ such that if $\mathcal{P} \subseteq \mathbb{C}^2$ and $\mathcal{L}$ is a set of lines in $\mathbb{C}^2$ with $|\mathcal{P}| = P$, $|\mathcal{L}| = L$, then*

$$I(\mathcal{P}, \mathcal{L}) \leq \alpha_1(P^{2/3+\epsilon}N^{2/3}) + \alpha_2(P + N).$$

Remark. *The best known result doesn't have the extra ϵ in the exponent [3]!*

Consider the following map $\phi : \mathbb{C}^2 \to \mathbb{R}^4$, given by $(z_1, z_2) \mapsto (x_1, y_1, x_2, y_2)$, where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

Definition (Lines in $\mathbb{C}^2$)**.** *Given $\alpha, \beta, \gamma \in \mathbb{C}$ such that $(\alpha, \beta) \neq (0, 0)$, we have*

$$\{(z_1, z_2) \in \mathbb{C}^2 : \alpha z_1 + \beta z_2 + \gamma = 0\}$$

is a line in $\mathbb{C}^2$.

Proposition 16. *Let $l_{\alpha,\beta,\gamma} = \{(z_1, z_2) : \alpha z_1 + \beta z_2 + \gamma = 0\}$ be a line in $\mathbb{C}^2$ where $\alpha = a + ia'$, $\beta = b + ib'$, and $\gamma = c + ic'$ $(a, a', b, b', c, c' \in \mathbb{R})$. The point $(x_1 + iy_1, x_2 + iy_2)$ is on line $l_{\alpha,\beta,\gamma}$ if and only if,*

$$ax_1 - a'y_1 + bx_2 - b'y_2 + c = 0,$$
$$a'x_1 + ay_1 + b'x_2 + by_2 + c' = 0.$$

This motivates the following definition,

Definition (Special 2-Flat)**.** *A special 2-flat in $\mathbb{R}^4$ is an affine 2-plane of the form,*

$$ax_1 - a'y_1 + bx_2 - b'y_2 + c = 0$$
$$a'x_1 + ay_1 + b'x_2 + by_2 + c' = 0$$

for reals a, a', b, b', c, c' with atleast one of $(a, a'), (b, b') \neq (0, 0)$.

To prove Theorem 15, it suffices to prove the following,

Theorem 17. *Given $\epsilon > 0$, there exists α_1, α_2 (depending on ϵ) such that if $\mathcal{P} \subseteq \mathbb{R}^4$, $|\mathcal{P}| = P$, $\Gamma \subseteq \mathbb{R}^4$ is a set of N special 2-flats, then*

$$I(\mathcal{P}, \Gamma) \leq \alpha_1 P^{2/3+\epsilon} N^{2/3} + \alpha_2(P + N).$$

Proof. We will proceed by induction on $P + N$. For 'small' values of $P + N$, the statement follows by blowing up α_1, α_2 (precisely, make α_2 large enough such that $\alpha_2(P + N) \geq PN$). We will also assume $N \leq CP^2$ for some absolute constant $C > 0$. This is allowed, since $I(\mathcal{P}, \Gamma) \leq O(P\sqrt{N} + N)$, which comes from the combinatorial bound arising from the fact that $K_{2,2}$ is forbidden in $G(\mathcal{P}, \Gamma)$. From Theorem 5, we get a polynomial $f \in \mathbb{R}[X_1, X_2, X_3, X_4]$ of deg $\leq Cr$ (where $C > 0$ is absolute constant) such that

- number of 'cells' (connected components) of $\mathbb{R}^4 \setminus Z(f) \leq r^4$,
- $|\mathcal{P} \cap O_i| \leq P/r^4, \ \forall\, i$.

Define (as before) $\mathcal{P}_{cell} = \cup_i \mathcal{P}_i$, $\Gamma_{cell} = \cup_i \Gamma_i$, $\mathcal{P}_{alg} = \mathcal{P} \cap Z(f)$, $\Gamma_{alg} = \{\Pi : \Pi \subseteq Z(f)\}$, $\mathcal{P}_i = \mathcal{P} \cap O_i$, $\Gamma_i = \{\Pi \in \Gamma : \Pi \cap O_i \neq \emptyset\}$. Further, let $P' = |\mathcal{P}_{cell}|$ and $P_0 = |\mathcal{P}_{alg}|$. We have,

$$I(\mathcal{P}, \Gamma) = \sum_i I(\mathcal{P}_i, \Gamma_i) + I(\mathcal{P}_{alg}, \Gamma_{cell}) + I(\mathcal{P}_{alg}, \Gamma_{alg}).$$

By induction hypothesis,

$$\begin{aligned}\sum_{i=1}^{c} I(\mathcal{P}_i, \Gamma_i) &\leq \sum_{i=1}^{c} \left[\alpha_1 P_i^{2/3+\epsilon} N_i^{2/3} + \alpha_2(P_i + N_i)\right] \\ &\leq \alpha_1 (P/r^4)^{2/3+\epsilon} \sum_i N_i^{2/3} + \alpha_2 P' + \alpha_2 \sum_i N_i.\end{aligned}$$

First we will establish an upper bound for $\sum_i N_i$, then we will use Holder to bound $\sum_i N_i^{2/3}$. Observe that,

$$\sum_{i=1}^{c} N_i = \#\{(O_i, \Pi) : \Pi \cap O_i \neq \emptyset\}.$$

Fix Π, a special 2-flat in Γ_{cell}. We will bound $\#\{i : \Pi \cap O_i \neq \emptyset\}$. This comes from the following,

Fact. *Let $U \subseteq \mathbb{R}^d$ be a variety of* dim d_U*, complexity k and let $f \in \mathbb{R}[X_1, \ldots, X_d]$ of* deg $\leq r$*. The number of components of $U \setminus Z(f)$ is $O_{k,d}(r^{d_U})$.*

In our case, complexity $k \leq 2$ (two linear equations), $d = 4$ (constant), $\dim U = 2$ (as U is 2-flat). Therefore, $\Pi \setminus Z(f)$ has $\leq O(r^2)$ components. Since each of these components determine a cell (it must lie in some cell O_i as $\Pi \setminus Z(f)$ can't cross $Z(f)$),

$$\#\{i : \Pi \cap O_i \neq \emptyset\} = O(r^2).$$

Summing the above gives,

$$\sum_i N_i = O(r^2 N).$$

By Holder,

$$\Big(\sum_i N_i^{2/3}\Big) \lesssim \Big(\sum_i N_i\Big)^{2/3}\Big(r^4\Big)^{1/3} = O(N^{2/3}r^{8/3}),$$

since the number of cells of $\mathbb{R}^4 \setminus Z(f)$ is $\leq r^4$. Continuing,

$$\sum_i I(\mathcal{P}_i, \Gamma_i) \leq \alpha_1 (P/r^4)^{2/3+\epsilon} + \alpha_2 \mathcal{P}' + \Big(\sum_i N_i^{2/3}\Big) + \alpha_2 \sum_i N_i.$$

Choose r large enough, and $r \ll \alpha_1 \ll \alpha_2$ such that this bound is $\leq (\alpha_1/3)P^{2/3+\epsilon}N^{2/3} + \alpha_2 P'$. For $I(\mathcal{P}_{alg}, \Gamma_{cell})$, let $\Lambda = \{\Pi \cap Z(f) : \Pi \in \Gamma_{cell}\}$. We have,

$$I(\mathcal{P}_{alg}, \Gamma_{cell}) = I(\mathcal{P}_{alg}, \Lambda).$$

Therefore it suffices to bound $I(\mathcal{P}_{alg}, \Lambda)$. For any fixed $\Pi \in \Gamma_{cell}$, note that $\Pi \cap Z(f)$ is variety of dim $<$ dim Π. This is because Π is defined by two linear equations, and $\Pi \not\subset Z(f)$. The variety $\Pi \cap Z(f)$ is defined by two linear equations, the polynomial f and thus is a variety of dim ≤ 1. Each of the varieties (in Λ) has complexity $\leq O(r)$. Let the set of points $\mathcal{P} \subseteq \mathbb{R}^4$, $\Gamma \equiv$ set of varieties of dim ≤ 1 and complexity $\leq O(r)$. If $G(\mathcal{P}, \Gamma)$ has no $K_{s,t}$ ($K_{2,2}$ here!), applying Theorem 13 gives

$$I(\mathcal{P}, \Gamma) \leq O_r(P_0^{2/3}N^{2/3} + P_0 + N).$$

By picking α_1, α_2 large enough (depending on ϵ),

$$O_r(P_0^{2/3}N^{2/3} + P_0 + N) \leq \Big[\frac{\alpha_1}{3}P_0^{2/3+\epsilon}N^{2/3} + P_0\Big].$$

At this point, we fix r to be absolute constant > 0. Finally, we will bound $I(\mathcal{P}_{alg}, \Gamma_{alg})$. We make an observation,

Observation. *If $p \in V(f)$ is a regular point, then p lies on ≤ 1 of the special 2-flats contained in $V(f)$.*

Recall that $p \in V(f)$ is regular if $\nabla f(p) \neq 0$. Suppose not, i.e. suppose Π_1, Π_2 are two different special 2-flats that contain p (assume $p = \tilde{0}$ WLOG). Both Π_1, Π_2 are two-dimensional subspaces that together span $\mathbb{R}^4$. Also, since $T_p(\Pi_i) = \Pi_i$ (tangent space), it follows that $\Pi_i \subseteq T_p(V(f))$ but $T_p(V(f))$ is a three-dimensional hypersurface in $\mathbb{R}^4$. As before, we have,

$$I(\mathcal{P}_{alg}, \Gamma_{alg}) = I(\mathcal{P}_{reg}, \Gamma_{alg}) + I(\mathcal{P}_{sing}, \Gamma_{alg})$$

where $\mathcal{P}_{reg}$ and $\mathcal{P}_{sing}$ refer to the regular and singular points in $\mathcal{P}$. Note that,

$$I(\mathcal{P}_{reg}, \Gamma_{alg}) \leq |\mathcal{P}_{reg}| \leq P_0,$$

by the observation above. Therefore, it suffices to find an upper bound for $I(\mathcal{P}_{sing}, \Gamma_{alg})$. We will need the following,

Fact. *$U \subseteq \mathbb{R}^d$ is a variety of* dim d_U *and complexity $\leq k$, then $U_{sing} = \{x \in U : x$ is singular$\}$ is a variety of* dim $< d_U$ *and complexity $\leq k^d d^{d_U}$.*

In our case $U = V(f)$, so U has dim ≤ 3 and complexity $\leq O(r)$. Therefore, by fact, it follows that U_{sing} is a variety of dim < 3 and complexity $O_r(1)$. Suppose $\Pi \in \Gamma_{alg}$ is a component of U_{sing}. Then since the number of irreducible components of U_{sing} is $O_r(1)$, the number of such Π that lie in U_{sing} is $\leq O_r(1)$. Hence, the number of incidences between singular points of $V(f)$ that are also in $\mathcal{P}$, and special 2-flats that are contained in U_{sing} is $O_r(P_0)$.
(Really!) Finally, we need to bound $I(\mathcal{P}_{sing}, \Gamma_{alg} \setminus U_{sing})$, i.e. incidents between $\mathcal{P}_{sing}$ and the special 2-flats not contained in U_{sing}. Define,

$$\Lambda' = \{\Pi \cap U_{sing} : \Pi \in \Gamma_{alg}, \Pi \not\subset U_{sing}\}.$$

We have,

$$I(\mathcal{P}_{sing}, \Gamma_{alg} \setminus U_{sing}) = I(\mathcal{P}_{sing}, \Lambda').$$

Bounding this gives,

$$I(\mathcal{P}_{sing}, \Lambda') \leq O_r(P_0^{2/3} N^{2/3} + P_0 + N) \leq (\alpha_1/3) P^{2/3} N^{2/3} + \alpha_2 P_0.$$

by choosing α_1, α_2 large enough. The theorem follows. □

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