MA-862		2024
	Local Lemma and Linear Arboricity	
Lecturer: Niranjan Balachandran		Scribe: Om Swostik

1 Lecture 1

We start with the statement of the Lovaśz Local Lemma.

Theorem 1 (Lovaśz Local Lemma). *Suppose ξ_i $(i = 1, 2, \ldots, N)$ are events in a probability space $(\Omega, \mathbb{P})$ and suppose D is the* <u>*dependence graph*</u> *of $\{\xi_i\}$ which is constructed such that ξ_i is jointly independent of $\{\xi_j \mid (i, j) \notin E(D)\}$. Suppose $\exists\, 0 \leq x < 1$ $i = 1, 2, \ldots, N$ such that $\mathbb{P}(\xi_i) \leq x_i. \prod_{ij \in E(D)}(1 - x_j)$. Then,*

$$\mathbb{P}\Big(\bigcap_{i=1}^{N} \overline{\xi_i}\Big) \geq \prod_{i=1}^{N}(1 - x_i) > 0.$$

So with positive probability none of the events occur.

Proof. We have,

$$\begin{aligned}&\mathbb{P}(\overline{\xi_1} \cap \overline{\xi_2} \cap \cdots \cap \overline{\xi_N})\\ &= \mathbb{P}(\overline{\xi_1}).\mathbb{P}(\overline{\xi_2} \mid \overline{\xi_1}) \cdots \mathbb{P}(\overline{\xi_N} \mid \overline{\xi_1} \cap \cdots \cap \overline{\xi_{N-1}})\end{aligned}$$

Therefore, it suffices to show, if $S \subseteq \mathbb{N}$ and $i \notin S$,

$$\mathbb{P}(\xi_i \mid \bigcap_{j \in S} \overline{\xi_j}) \leq x_i$$

We will show this by induction on $|S|$. Clearly the statement holds for $|S| = 0$. Let $S_1 = \{j \in S : ij \in E(D)\}$ and $S_2 = S \setminus S_1$. We have,

$$\begin{aligned}&\mathbb{P}\big(\xi_i \mid \bigcap_{j \in S_1} \overline{\xi_j} \cap \bigcap_{l \in S_2} \overline{\xi_l}\big)\\ &= \frac{\mathbb{P}\big(\xi_i \cap \bigcap_{j \in S_1} \overline{\xi_l} \mid \bigcap_{l \in S_2} \overline{\xi_l}\big)}{\mathbb{P}\big(\bigcap_{j \in S_1} \overline{\xi_l} \mid \bigcap_{l \in S_2} \overline{\xi_l}\big)}\\ &\leq \frac{x_i. \prod_{ij \in E(G)}(1 - x_j)}{\mathbb{P}\big(\bigcap_{j \in S_1} \overline{\xi_l} \mid \bigcap_{l \in S_2} \overline{\xi_l}\big)}\end{aligned}$$

It suffices to show that,

$$\mathbb{P}\big(\bigcap_{j \in S_1} \overline{\xi_j} \mid \bigcap_{l \in S_2} \overline{\xi_l}\big) \geq \prod_{ij \in E(G)} (1 - x_j)$$

Suppose $S_1 = \{j_1, j_2, \ldots, j_k\}$. We have,

$$
\begin{aligned}
&\mathbb{P}\Big(\bigcap_{i=1}^{k} \overline{\xi_{j_i}} \mid \bigcap_{l\in S_2} \overline{\xi_l}\Big) \\
&= \prod_{i=1}^{k} \Big(1 - \mathbb{P}\big(\xi_{j_i} \mid \bigcap_{l=1}^{i-1} \overline{\xi_{j_l}} \cap \bigcap_{l\in S_2} \overline{\xi_l}\big)\Big) \\
&\geq \prod_{i=1}^{k} (1 - x_{j_i}) \geq \prod_{ij\in E(D)} (1 - x_j).
\end{aligned}
$$

□

Corollary 2 (Symmetric form of LLL)**.** *Let ξ_i be events in $(\Omega, \mathbb{P})$. Suppose $\mathbb{P}(\xi_i) \leq p$, and if the maximum degree in the dependence graph is d, and suppose $ep(d+1) \leq 1$, then with positive probability none of the ξ_i's occur.*

Proof. Follows from Theorem 1, take $x_i = 1/(d+1)$. □

1.1 A Theorem of Erdós and Lovaśz on a problem of Straus

The following question was proposed by Straus[1]: Given $S \subseteq \mathbb{R}$ such that $|S| < \infty$, is there a k-coloring of $\mathbb{R}$ such that EVERY translate of S is MULTICOLORED?

Definition 3 (MULTICOLORED)**.** *A set $S \subseteq \mathbb{R}$ is multicolored if all k colors appear in the set.*

Definition 4 (k-coloring of $\mathbb{R}$)**.** *A k-coloring of $\mathbb{R}$ is a function $c : \mathbb{R} \to k$.*

Theorem 5 (Erdós-Lovaśz)**.** *If $|S| \geq (3 + o_k(1))k\log(k)$, then the answer is YES.*

Remark. *This bound is optimal upto a constant. There exists a set S of size $k\log(k)$, for which it isn't possible to obtain a k-coloring of $\mathbb{R}$ such that every translate of S is multicolored.*

Proof. First let us fix a finite set X corresponding to the translations (i.e. we will consider the translates $x + S$ for $x \in X$). Fix a large finite set Ω such that $x + S \subseteq \Omega$, $\forall x \in X$. Write $|S| = m$ for simplicity. Color each $w \in \Omega$ independently + randomly in $[k]$. Define the events,

$$\xi_x \coloneqq x + S \text{ isn't multicolored, } \forall x \in X.$$

We have,

$$\mathbb{P}(\xi_x) \leq k(1 - 1/k)^m$$

which follows as the probability of some color being missing from the set $(x + S)$ is $(1 - 1/k)^m$. The dependence graph D looks like,

$$\xi_x \leftrightarrow \xi_y \text{ if and only if } (x + S) \cap (y + S) \neq \emptyset.$$

For fixed x, we want to compute,

$$\#\{y : \xi_x \leftrightarrow \xi_y\}$$

[1] collaborated with both Erdós and Einstein!

where $y \neq x$. If $\exists s_1, s_2 \in S$ such that $x + s_1 = y + s_2$, then $y = x + s_1 - s_2$. Therefore, maximum dependence degree $\leq m(m-1)$. So if

$$ek(1 - 1/k)^m(m(m-1) + 1) \leq 1, \tag{1}$$

holds, then from Corollary 2 it follows that with positive probability none of the events occur i.e. $\mathbb{P}\big(\bigcap_{x \in X} \overline{\xi_x}\big) > 0$. Notice that Equation (1) holds for $m = (3 + o(1)).k \log(k)$. We have thus shown that for every finite set of translates X, the theorem holds. To prove the theorem fully (i.e. for the case of the set of translates being infinite), we'll use Tychonoff's theorem! The space of colorings is $[k]^{\mathbb{R}} \coloneqq \chi$. Endow each component $[k]$ of χ with the discrete topology. Since each component $[k]$ is a finite set, it is compact under discrete toplogy. Since arbitrary product of compact spaces is compact, it follows that $\chi = [k]^{\mathbb{R}}$ is compact under the product topology. Let

$$\mathcal{C}_x = \{c \in [k]^{\mathbb{R}} \,:\, x + S \text{ is multicolored wrt } c\}.$$

Note that $\mathcal{C}_x$ is closed in χ with respect to the product topology. We have shown that $\bigcap_{x \in X} \mathcal{C}_x \neq \emptyset$, for every finite subset X of $\mathbb{R}$. Since $\mathcal{C}_x$ are closed subspaces of the compact space χ and satisy the finite intersection property, it follows that $\bigcap_{x \in \mathbb{R}} \mathcal{C}_x \neq \emptyset$. □

1.2 Linear Arboricity Conjecture of Harary

Definition 6 (Arboricity of a graph)**.** *The arboricity of a graph G is the minimum number of edge-disjoint forests needed to partition $E(G)$.*

As an example, arboricity of the 5-cycle is 2. Note that arboricity of a graph G is 1 if and only if it is a forest.

Definition 7 (Linear)**.** *Each tree in the forest decomposition of G must be a path.*

For a given graph G, we denote its linear arboricity by $la(G)$. Every graph G can be embedded in a d-regular subgraph by adding more vertices and edges. Let G be a graph on n vertices such that $\Delta(G) \leq d$. Let $F_1, F_2, \ldots, F_r$ be a (linear) forest decomposition of G. Since each F_i is a forest, $e(F_i) \leq (n-1)$. We have,

$$(n-1)r \leq \sum_{i=1}^{r} e(F_i) = \sum_{i=1}^{r} d_i/2 \leq dn/2$$

This gives (taking $r = la(G)$),

$$d/2 < dn/2(n-1) \leq la(G).$$

We have the following conjecture by Harary [1] which essentially says that this bound is tight!

Conjecture 8 (Harary,1980)**.** $la(G) \leq \lceil (d+1)/2 \rceil$*, where $d =$ maximum degree of the graph G.*

Following is a directed version of the conjecture, which if true, will imply the undirected version.

Conjecture 9 (Directed version)**.** *Suppose D is a directed d-regular digraph. For each v, if $N^+(v) = \{u \,:\, (v,u) \in E(D)\}$, then $d^+(v) = |N^+(v)| = d$ and similarly, $d^-(v) = d$. If D is directed and d-regular, then $dla(D) = d + 1$.*

Theorem 10 (Alon)**.** *If G is directed and d-regular, then $dla(G) \leq d + O(d^{3/4}\log^{1/2}(d)) = d(1 + o_d(1))$.*

Given D directed and d-regular, create two copies V and V' of the vertex set of D such that $(u, v') \in E(T_D)$ if and only if $(u, v) \in E(D)$, where $u \in V$ and $v' \in V'$. By construction, T_D is a d-regular, bipartite graph. From Hall's theorem, it follows that

$$E(T_D) = M_1 \uplus M_2 \uplus \cdots \uplus M_d$$

where M_i's are perfect matchings in T_D. Each perfect matching in the bipartite graph T_D corresponds to a union of disjoint cycles in the graph D. Therefore,

$$E(D) = F_1 \uplus F_2 \uplus \cdots \uplus F_d$$

where each F_i is a union of disjoint cycles in the graph D. So clearly, $\text{dla}(D) \leq 2d$.
Idea: If it possible to choose one edge from each cycle such that the resulting edges form a matching, then we have $\text{dla}(D) \leq d + 1$. Look at the line graph, we want an independent set there! More generally, suppose we have

$$V(G) = V_1 \uplus \cdots \uplus V_r$$

Can one pick a <u>TRANSVERSAL INDEPENDENT</u> set with respect to this partition? Pick one vertex $v_i \in V_i$ such that the resulting graph is independent!

Theorem 11 (Alon)**.** *Suppose*

$$V(G) = V_1 \uplus \cdots \uplus V_r$$

If $\Delta(G) \leq d$ and $|V_i| \geq 2ed$, then G admits an independent transversal for this partition.

Proof. Pick $u \in V_i$ uniformally at random. For $1 \leq i < j \leq r$, $\xi_{ij} \equiv v_i v_j \in E(G)$. Notice, $\mathbb{P}(\xi_{ij}) = e(V_i, V_j)/4e^2d^2 = 1/2e$. This doesn't work out! □

Remark. *In the above attempt, things didn't work out cause we had too few bad events!*

2 Lecture 2

2.1 Directed Linear Arboricity Conjecture

Suppose D is a d-regular directed graph ($d^+(v) = d^-(v) = d$, $\forall\, v$). Then $\text{dla}(D) \leq (d + 1)$. Recall that we defined $\text{dla}(D)$ as the minimum number of colors needed to color $E(D)$ such that each color class induces a <u>LINEAR FOREST</u> i.e. each connected component is a directed path. We have already seen that $\text{dla}(D) > d$. To avoid the issue faced before in finding the transversal independent set, we will sparsify the bad events!

Theorem 12 (Alon, [2,3])**.** *Suppose $V(G) = V_1 \uplus \cdots \uplus V_r$ with $\Delta(G) \leq d$, and $|V_i| \geq \lceil 2ed \rceil$. Then the collection $\{V_i\}$ admits an independent transversal, i.e. $\exists v_i \in V_i$ such that $I = \{v_1, \ldots, v_r\}$ is independent in G.*

Proof. WLOG $|V_i| = \lceil 2ed \rceil$ (throw vertices out!). Pick $v_i \in V_i$ independently + uniformly, i.e. one v_i is picked randomly from each V_i. For each edge $e = \{u, v\}$, let $\xi_e =$ both u, v are picked, where $e = (u, v)$. We have, 1+ dependence degree $\leq 2.2ed.d = 4ed^2$. This follows as $|V_i| = |V_j| = 2ed, \forall\, i \neq j$ and degree of each vertex is d. If $u \in V_i$ and $v \in V_j$ $(i \neq j)$,

$$\mathbb{P}(\xi_e) \leq 1/4e^2d^2 \Rightarrow e(1/4e^2d^2)(4ed^2) = 1.$$

The local lemma applies. □

Remark. *The best constant c such that if $|V_i| \geq cd$, then there is an independent transversal is ≤ 2 (best constant has to be > 1).*

Given D, construct H bipartite as $H = (V, V', E)$, where $V' \simeq V = V(D)$ and $(u, v') \in E(H)$ if and only if $(u, v) \in E(D)$. Since D is d-regular, H is also d-regular. We have,

$$E(H) = M_1 \uplus \cdots \uplus M_d$$

where each M_i is a matching in H. We obtain this by applying Hall's theorem repeatedly and reducing the graph by removing perfect matchings at each step. Note that such a reduction preserves the regularity of graph. Each matching M_i induces a partition of $V(D)$ into vertex-disjoint cycles.

Observation 13. *Consider Figure 1, if one can pick one edge from each of these cycles (in these cycle decompositions) such that the chosen edges form a matching, then we can color each $\mathcal{M}_i$ with color i and color the matching formed by the 'chosen' edges with one extra color. Total $(d + 1)$ colors will be used in this process.*

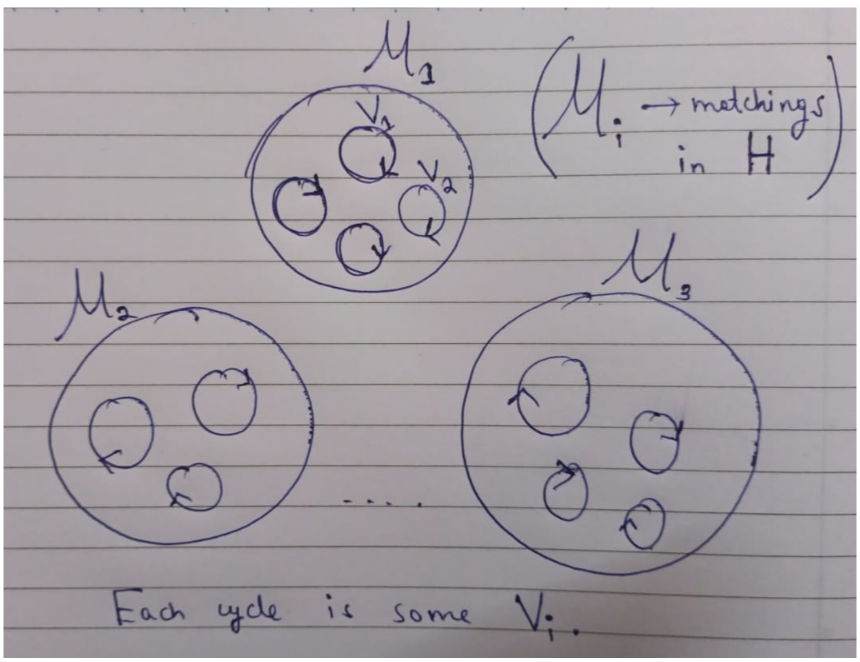

Figure 1: We work with the line graph of D to check if the chosen edges from each cycle form a matching

Consider the line graph L of the given graph D. Let $V(L) = E(D) = \biguplus_{i=1}^{N} V_i$, where each V_i is a set of edges forming a cycle in some matching $\mathcal{M}_j$. So in terms of the line graph, we have seen that if each $|V_i| \geq 2ed$, then $\{V_i\}$ admits an independent transversal. Small cycles are a nuisance! If a small cycle pops up, then it might not be possible to choose an independent transversal among the sets $\{V_i\}$. Observe that the degree of each vertex in the line graph L is $\leq (4d-2)$. This follows by looking at all the possible edges incident on one of u or v, given the edge $e = (u, v)$. Since $2e(4d-2) \leq 8ed$, it is an easy consequence of Theorem 11 to see that if the girth of the graph D is $\geq 8ed$, then $\mathrm{dla}(D) = (d+1)$. The conjecture holds for graphs of large girth! How do we proceed from here? Given D, if we can partition

$$E(D) = D_0 \uplus D_2 \uplus \cdots \uplus D_{p-1}$$

for some integer p such that each D_i has large girth and "proportionally small" degree, then could apply this result on each D_i. Let degree of each vertex in some D_i be $\approx d/p$. Then we have,

$$\mathrm{dla}(D) \leq (d/p + 1)p = d + p$$

If $p = o(d)$, then this gives a bound $\mathrm{dla}(D) = d + o(D)$.

The main idea: Pick a p (shall see how to do this!) and if $E(D)$ can be partitioned into digraphs $D_0, D_1, \ldots, D_{p-1}$ such that,

- $\Delta^+(D_i), \Delta^-(D_i) \approx d/p$
- $\mathrm{girth}(D_i) >> (d/p)$
- $p = o(d)$,

then one can repeatedly use the result for digraphs of "large girth" to get $\mathrm{dla}(D) \leq (d/p+1).p = d + o(d)$. Let $p >> \sqrt{d}$. We will show that the minimum cycle length is $> p$ by constructing D_i's and using $p \geq \mathrm{girth}(D_i) \geq d/p$. The inequality $p > d/p$ holds as we chose $p >> \sqrt{d}$.

Remark. *It is possible to attain (1) by locally splitting at each vertex to obtain (d/p) degree for each split and then using Chernoff type bound to obtain a global split.*

Here is the strategy: First we pick a large prime p (of order $\sqrt{d}$, as one back of the envelope calculation). We shall color the edges of D using colors $\{0, 1, \ldots, p-1\}$ such that $\forall v \in V(D)$ and each $i \in \{0, 1, \ldots, p-1\}$, we have

$$N^+(v, i) \coloneqq \#\{u \,:\, (v, u) \in E(D) \text{ and } (v, u) \text{ has color } i\}.$$

Similarly, we define,

$$N^-(v, i) \coloneqq \#\{u \,:\, (u, v) \in E(D) \text{ and } (u, v) \text{ has color } i\}.$$

Suppose

$$N^+(v, i) = N^-(v, i) = d/p \pm O(\sqrt{d/p \log(d)})$$

has been achieved (Chernoff). Define the digraph $D_i = (V, E_i)$ (for $1 \leq i \leq (p-1)$), where $(u, v) \in E_i$ if and only if $\chi(v) = \chi(u) + i \mod (p)$. Figure 2 gives a representation of the D_i's and the coloring idea, exploiting the fact that p is a prime. Choosing p to be prime is integral to ensuring that each split has 'large enough' girth.

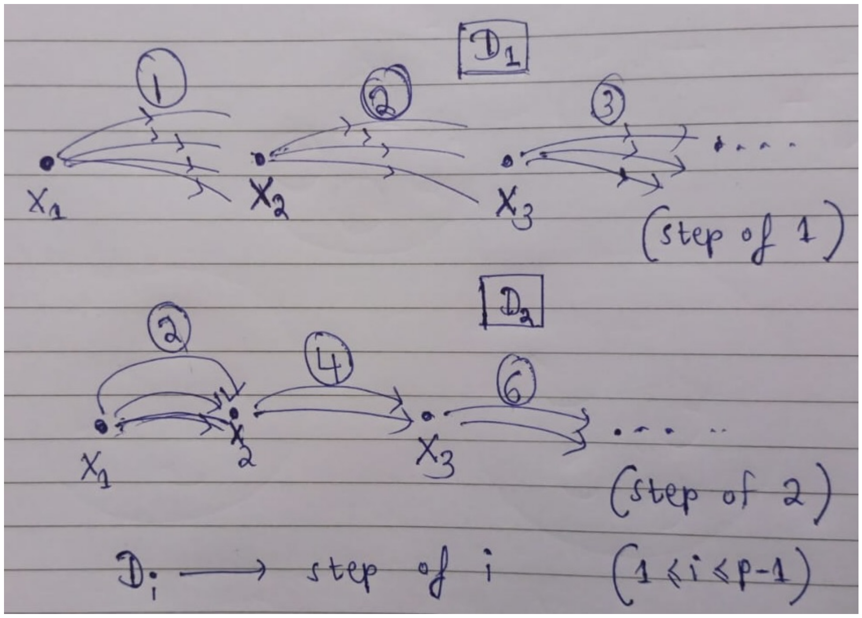

Figure 2: We define each equitable split D_i in such a way as to ensure $\mathrm{girth}(D_i) > p$

3 Lecture 3

Recap: If $\mathrm{girth}(D) \geq 8ed$, then $\mathrm{dla}(D) = d + 1$. Given D, we want to partition D into $D_0, D_1, \ldots, D_{p-1}$ such that

- $\mathrm{girth}(D_i)$ is large ($> 9e(d/p)$, where p is a parameter to be determined)
- $\Delta^+(D_i), \Delta^-(D_i) = (1 \pm o(1))d/p$.

In particular, each D_i can be partitioned into $(1 \pm o(1))d/p$ linear forests. So, D can be partitioned into $(1 \pm o(1))d/p.p = (d + o(d))$ linear forests.

Remark. *Note that any d-regular digraph has $dla(D) \leq 2d$.*

As before, construct T_D bipartite such that $(u, v') \in E(T_D)$ if and only if $(u, v) \in E(D)$. By Hall's theorem, the graph T_D can be partitioned into d matchings, where each matching of T_D gives rise to a disjoint union of cycles among the edges of D. We want to obtain D_i as an (almost) equitable split of every vertex. To get this, let p be a prime (sufficiently large?!) and let $\chi : V(D) \to \{0, 1, \ldots, p-1\}$ be a uniformly random map, i.e. $\chi(v) = i$ with probability $1/p$ for each i and independently for $v \in V(D)$.

Claim 14. *With + probability for each v, and each $i \in \{0, 1, \ldots, p-1\}$, if*

$$N^+(v, i) = \#\{u : (v, u) \in E(D) \text{ and } \chi(u) = i\}$$
$$\text{and } N^-(v, i) = \#\{u : (u, v) \in E(D) \text{ and } \chi(u) = i\},$$

then $N^+(v, i), N^-(v, i) = (1 \pm o(1))d/p$.

For $0 \leq i \leq (p-1)$, let D_i be those edges (u, v) such that $\chi(v) = \chi(u) + i$. Then note that for $1 \leq i \leq p-1$, $\text{girth}(D_i) \geq p$. So we want, $p^2 \geq 9ed$, which reduces to $p \geq \Omega(\sqrt{d})$. So, armed with these observations, presciently(!) pick a prime p with,

$$10\sqrt{d} < p < 20\sqrt{d}.$$

Assume the claim holds with error term $t = 10d^{1/4}(log(d))^{1/2}$. This gets us that for $1 \leq i \leq p-1$,

$$\text{dla}(D_i) \leq 1 + d/p + 10d^{1/4}log(d)^{1/2} \tag{2}$$

Summing Equation (2) over $1 \leq i \leq p-1$,

$$\text{dla}\Big(\bigcup_{i=1}^{n} D_i\Big) \leq 20\sqrt{d} + d + O(d^{3/4}log(d)^{1/2}) \qquad (p < 20\sqrt{d})$$

Further, we have

$$\text{dla}(D_0) \leq 2(d/p + O(d^{1/4}log(d)^{1/2})) = O(\sqrt{d}).$$

This gives that, $\text{dla}(D) \leq d + O(d^{3/4}log(d)^{1/2})$. Note that $N^+(v, i) \sim \text{Bin}(d, 1/p)$ i.e. it is a sum of independent Bernoulli indicator variables. Fix v and i. Following is a useful formulation of the Chernoff bound,

Theorem 15 (Chernoff)**.** *Let $X \sim Bin(n, p)$. For any $0 \leq t \leq np$,*

$$\mathbb{P}(|X - np| \geq t) \leq 2e^{-t^2/3np}$$
$$\Rightarrow \mathbb{P}(|N^+(v,i) - d/p| \geq t) \leq O\big(e^{-O(t^2/\sqrt{d})}\big)$$

So we may take $t = 10d^{1/4}(log(d))^{1/2}$ as the error term.

Remark. *Note that Chernoff works only for a single vertex equitable split. If we apply Chernoff for each vertex and then union over all, we bring n into the picture (not good!). Since we only want +ve probability, local lemma serves the purpose.*

So it suffices to prove Claim 14. We will do this by the local lemma!

Proof of Claim 14. Let $A^+(v, i)$ and $B^-(v, i)$ be the BAD events where,

$$A^+(v,i): \ |N^+(v,i) - d/p| > 10d^{1/4}(log(d))^{1/2}$$
$$B^-(v,j): \ |N^-(v,j) - d/p| > 10d^{1/4}(log(d))^{1/2}$$

By Chernoff, $\mathbb{P}(A^+(v,i))$, $\mathbb{P}(B^-(v,j)) \leq O(1/d^{10})$. To ensure indpendence of the bad events corresponding to vertices u and v, we want $\text{dist}(u, v) \geq 3$. It follows that the maximum dependence degree is $\leq O(d^{5/2}) = O(d^2 p)$. Local lemma applies with room to spare! □

Alon and others [4] brought down the error term in the conjecture to $O(d^{2/3})$. Following is the best known result [5],

Theorem 16 (Ferber, Fox, Jain)**.** *We can get a sharper bound $dla(D) \leq d + O(d^{2/3-\alpha})$, for some $\alpha > 0$.*

References

[1] Jin Akiyama, Geoffrey Exoo, and Frank Harary. Covering and packing in graphs. iii: Cyclic and acyclic invariants. *Mathematica Slovaca*, 30(4):405–417, 1980.

[2] Noga Alon. The linear arboricity of graphs. *Israel Journal of Mathematics*, 62(3):311–325, 1988.

[3] Noga Alon and Joel H Spencer. *The probabilistic method.* John Wiley & Sons, 2016.

[4] Noga Alon, Vanessa J Teague, and Nicholas C Wormald. Linear arboricity and linear k-arboricity of regular graphs. *Graphs and Combinatorics*, 17:11–16, 2001.

[5] Asaf Ferber, Jacob Fox, and Vishesh Jain. Towards the linear arboricity conjecture. *Journal of Combinatorial Theory, Series B*, 142:56–79, 2020.