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**CLTs and Laws of Iterated Logarithm**

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# 1 Kolmogorov's Zero-One Law

Let $\mathbb{R}^\infty \equiv \{(x_1, x_2, \dots) \,:\, x_i \in \mathbb{R}\}$. Define $\mathcal{B}^{(n)}$ as the $\sigma$-algebra generated by $\{x_j : j \geq n\}$, and define $\mathcal{B}_{(n)}$ as the $\sigma$-algebra generated by $\{x_j : 1 \leq j \leq n\}$. As usual, let $\mathcal{B}$ denote the $\sigma$-algebra on $\mathbb{R}^\infty$ generated by finite-dimensional cylinders.

**Definition 1** (Tail $\sigma$-algebra)**.** *Define*

$$\mathcal{T} \coloneqq \bigcap_{n\geq 1} \mathcal{B}^{(n)},$$

*where $\mathcal{T}$ is a $\sigma$-algebra since it is the intersection of $\sigma$-algebras. We will refer to $\mathcal{T}$ as the Tail $\sigma$-algebra.*

**Definition 2** (Tail Event)**.** *Any set $A \in \mathcal{T}$ is called a 'Tail Event'.*

A general proposition about 'tail events' is the following beautiful theorem of Kolmogorov:

**Theorem 3.** *If $A \in \mathcal{T}$, then $\mathbb{P}(A) \in \{0, 1\}$ under any product probability on $\mathbb{R}^\infty$.*

*Proof.* We will show that $A$ is independent of itself, so that

$$\mathbb{P}(A) = \mathbb{P}(A \cap A) = \mathbb{P}(A)^2,$$

which will prove the theorem. Note that since $A \in \mathcal{T}$, we must have $A \in \mathcal{B}^{(n+1)}$ for all $n$. Since we are working with a product measure, we can conclude that $A$ must be independent of $\mathcal{B}_{(n)}$ for all $n$. Therefore, $A$ is independent of all $B \in \mathcal{F} \coloneqq \bigcup_{n\geq 1} \mathcal{B}_{(n)}$. Let $\mathcal{A} \coloneqq \{B \in \mathcal{B} \,:\, B \text{ is independent of } A\}$.

We have the following claim,

**Claim 4.** *$\mathcal{A}$ is a monotone class, and $\mathcal{F} \subseteq \mathcal{A}$.*

Note that $\mathcal{F}$ is a field. By the monotone class theorem, the $\sigma$-algebra generated by $\mathcal{F}$ must be the same as the monotone class generated by $\mathcal{F}$. But $\sigma(\mathcal{F}) = \mathcal{B}$, which implies that $A \in \mathcal{A}$. Proving the claim will complete the proof of theorem.

*Proof of Claim.* Let $A_n \uparrow$ and $\{A_n\}_{n\geq 1} \subseteq \mathcal{A}$, then we need to show that $\bigcup_{n\geq 1} A_n \in \mathcal{A}$. Observe,

$$\mathbb{P}\Big(A \cap \big(\bigcup_{i=1}^{N} A_i\big)\Big) = \mathbb{P}\Big(\bigcup_{i=1}^{N} (A \cap A_N)\Big) = \mathbb{P}(A \cap A_N) = \mathbb{P}(A)\mathbb{P}(A_N).$$

Taking limit $N \to \infty$ concludes proof of Claim. ☐

The theorem follows from the claim. ☐

## 2 Central Limit Theorems

Let $\{X_i\}_{i\geq 1}$ be iid random variables with $\mathbb{E}\left[X_i\right] = 0$ and $\mathbb{E}\left[X_i^2\right] = 1$, $\forall\, i$. Then,

$$\frac{S_n}{\sqrt{n}} \underset{W}{\Longrightarrow} N(0,1),$$

where $S_n = \sum_{i=1}^{n} X_i$.

*Proof.* In light of Levy's continuity theorem, it suffices to show that

$$\phi_{S_n/\sqrt{n}}(t) \underset{n\to\infty}{\longrightarrow} e^{-t^2/2},\ \forall t$$

where $\phi_{S_n/\sqrt{n}}$ is characteristic function of $S_n/\sqrt{n}$. Let characteristic function of $X$ be $\phi$. Then the characteristic function of $(X_1/\sqrt{n} + \cdots + X_n/\sqrt{n})$ is $(\phi(t/\sqrt{n}))^n$. Since $\mathbb{E}[X] = 0$ and $\mathbb{E}\left[X^2\right] = 1$, we have

$$\phi(t/\sqrt{n}) = 1 - \frac{t^2}{2n} + o(1/n),$$

for each fixed $t$ by Taylor's theorem. We can conclude,

$$\Big(\phi(t/\sqrt{n})\Big)^n = \Big(1 - \frac{t^2}{2n} + o(1/n)\Big)^n.$$

Recall that $(1+a_n)^n \to e^z$ if $na_n \to z$ as $n \to \infty$. This tells us that,

$$\Big(\phi(t/\sqrt{n})\Big)^n \underset{n\to\infty}{\longrightarrow} e^{-t^2/2}.$$

$\square$

We have the following sufficient condition for CLT,

**Theorem 5** (Lindeberg's condition)**.** *Suppose $\mu_i$ is the distribution of $X_i$, where the random variables $\{X_i\}$ are independent. Let $\mathbb{E}[X_i] = 0$, $\mathbb{E}\left[X_i^2\right] = \sigma_i^2$ and $s_n^2 = \sum_{i=1}^{n} \sigma_i^2$. Then,*

$$\frac{S_n}{s_n} \underset{W}{\Longrightarrow} N(0,1),$$

*if for any $\epsilon > 0$*

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{i=1}^{n} \Big( \int_{|X|\geq \epsilon s_n} X^2 d\mu_i \Big) = 0.$$

*Note that each $\mu_i$ is a probability distribution on the space $(\Omega_i, \mathcal{B}_i)$.*

Verifying Lindeberg's condition is tough in general. The following gives a sufficient condition for the Lindeberg's condition to hold,

**Corollary 6** (Lyapunov's condition)**.** *Suppose for some $\delta > 0$,*

$$\lim_{n\to\infty} \frac{1}{s_n^{2+\delta}} \sum_{j=1}^{n} \Big( \int |X|^{2+\delta} d\mu_j \Big) = 0,$$

*then Lindeberg's condition holds. So a CLT follows.*

*Proof of Lindeberg's theorem.* Write

$$\frac{S_n}{s_n} \;=\; X_{n,1} + \cdots + X_{n,n}$$

with $X_{n,j} = X_j/s_n$. Let $\phi_{n,j}$ denote the characteristic function of $X_{n,j}$. We have,

$$\phi_{n,j}(t) = \phi_j\Big(\frac{t}{s_n}\Big),$$

where $\phi_j$ is the characteristic function of $X_j$. Let $\widehat{\mu}_n$ denote the characteristic function of $S_n/s_n$,

$$\widehat{\mu}_n(t) = \prod_{j=1}^{n} \phi_j\Big(\frac{t}{s_n}\Big).$$

Write $\psi_{n,j}(t) \coloneqq \phi_{n,j}(t) - 1$, and let

$$\psi_n(t) \coloneqq \prod_{j=1}^{n} \psi_{n,j}(t).$$

Note that to prove the theorem, it suffices to show

$$\widehat{\mu}_n(t) = \prod_{j=1}^{n} \phi_j\Big(\frac{t}{s_n}\Big) \underset{n\to\infty}{\longrightarrow} e^{-t^2/2}, \;\; \forall t.$$

In fact, we will show

$$\lim_{n\to\infty} \Big( \log\big(\widehat{\mu}_n(t)\big) + \frac{t^2}{2}\Big) = 0.$$

This makes sense since we can always define a branch of logarithm unambiguously on

$$\mathbb{C} \setminus \{iy \,:\, y \in \mathbb{R}, y \leq 0\}.$$

Fix $T > 0$, observe

$$\begin{aligned}
\lim_{n\to\infty} \sup_{|t|\leq T} \Big| \log\big(\widehat{\mu}_n(t)\big) - \log\big(\psi_n(t)\big)\Big| &\leq \lim_{n\to\infty} \sup_{|t|\leq T} \sum_{j=1}^{n} \Big| \log\big(\phi_{n,j}(t)\big) - \big(\phi_{n,j}(t) - 1\big)\Big| \\
&\leq \lim_{n\to\infty} \sup_{|t|\leq T} C \left| \sum_{j=1}^{n} \big|\phi_{n,j}(t) - 1\big|^2 \right|,
\end{aligned}$$

for some absolute constant $C > 0$, since $\log(1+x) = x + O(x^2)$. Now,

$$\sup_{|t|\le T}\sum_{j=1}^{n}\left|\phi_{n,j}(t)-1\right|^2 \le \sup_{|t|\le T}\left(\sup_{1\le j\le n}\left|\phi_{n,j}(t)-1\right|\right).\sup_{|t|\le T}\left(\sum_{j=1}^{n}\left|\phi_{n,j}(t)-1\right|\right)$$

since the inequality $\sum_{i=1}^{n} a_i^2 \le \max_i |a_i| \sum_{i=1}^{n} |a_i|$ always holds. We now claim the following,

$$\lim_{n\to\infty}\sup_{|t|\le T}\left(\sup_{1\le j\le n}\left|\phi_{n,j}(t)-1\right|\right) \tag{1}$$

$$\sup_{n}\sup_{|t|\le T}\left(\sum_{j=1}^{n}\left|\phi_{n,j}(t)-1\right|\right) < \infty \tag{2}$$

This will allows us to conclude,

$$\lim_{n\to\infty}\left[\sup_{|t|\le T}\left(\sup_{1\le j\le n}\left|\phi_{n,j}(t)-1\right|\right).\sup_{|t|\le T}\left(\sum_{j=1}^{n}\left|\phi_{n,j}(t)-1\right|\right)\right] = 0.$$

So if we had proven Statement 1 and Statement 2, then we have

$$\lim_{n\to\infty}\sup_{|t|\le T}\left|\log\left(\widehat{\mu}_n(t)\right) - \log\left(\psi_n(t)\right)\right| = 0,$$

for $T > 0$. To complete the proof, it will then suffice to show the following

$$\lim_{n\to\infty}\sup_{|t|\le T}\left|\log\left(\psi_n(t)\right) + \frac{t^2}{2}\right| = 0 \tag{3}$$

Thus, in view of Statement 1, Statement 2 and Statement 3, the proof is complete except for the proofs of these three statements.

**Remark.** *Note the inequality* $\sum_{i=1}^{n} a_i^2 \le \sup_i |a_i| \sum_{i=1}^{n} |a_i|$, *which holds for all reals* $a_1, \dots, a_n$.

*Proof of Statement 1.* Observe,

$$\begin{aligned}\sup_{|t|\le T}\left|\phi_{n,j}(t)-1\right| &= \sup_{|t|\le T}\left|\int\left(e^{itx/s_n}-1\right)d\mu_j\right|\\ &= \sup_{|t|\le T}\left|\int\left(e^{itx/s_n}-1-\frac{itx}{s_n}\right)d\mu_j\right|\end{aligned}$$

as $\phi_{n,j}$ is the characteristic function of $X_{n,j} = X_j/s_n$ and all $X_{n,j}$ have mean 0. Since $\left|e^{iy}-1-iy\right| = O(y^2)$, we have

$$\sup_{|t|\le T}\left|\phi_{n,j}(t)-1\right| \le C_T\int\left(X^2/s_n^2\right)d\mu_j$$

for some absolute constant $C_T > 0$ depending only on $T$. Note that,

$$\begin{aligned} C_T \int (X^2/s_n^2) d\mu_j &= C_T \int_{|X|<\epsilon s_n} (X^2/s_n^2) d\mu_j + C_T \int_{|X|\geq \epsilon s_n} (X^2/s_n^2) d\mu_j \\ &\leq C_T \epsilon^2 + C_T \int_{|X|\geq \epsilon s_n} (X^2/s_n^2) d\mu_j \end{aligned}$$

Hence,

$$\sup_{1\leq j\leq n} \sup_{|t|\leq T} \Big|\phi_{n,j}(t) - 1\Big| \leq C_T \epsilon^2 + C_T \sum_{j=1}^{n} \left( \int_{|X|\geq \epsilon s_n} (X^2/s_n^2) d\mu_j \right)$$

By Lindeberg's condition,

$$C_T \sum_{j=1}^{n} \left( \int_{|X|\geq \epsilon s_n} (X^2/s_n^2) d\mu_j \right) \underset{n\to\infty}{\longrightarrow} 0$$

which gives,

$$\limsup_{n\to\infty} \sup_{1\leq j\leq n} \sup_{|t|\leq T} \Big|\phi_{n,j}(t) - 1\Big| \leq C_T \epsilon^2, \ \forall \epsilon > 0.$$

Statement 1 follows by taking $\epsilon \to 0$. □

*Proof of Statement 2.* Recall that we need to show,

$$\sup_n \sup_{|t|\leq T} \left( \sum_{j=1}^{n} \big|\phi_{n,j}(t) - 1\big| \right) < \infty.$$

As in the proof of Statement 1, we have

$$\begin{aligned} \sup_{|t|\leq T} \left( \sum_{j=1}^{n} \big|\phi_{n,j}(t) - 1\big| \right) &\leq C_T \sum_{j=1}^{n} \int (X^2/s_n^2) d\mu_j \\ &\leq \frac{C_T}{s_n^2} \sum_{j=1}^{n} \int X^2 d\mu_j \\ &= \frac{C_T}{s_n^2} \sum_{j=1}^{n} \sigma_j^2 = C_T, \end{aligned}$$

uniformly over $n$ for some absolute constant $C_T > 0$. Thus,

$$\sup_n \sup_{|t|\leq T} \left( \sum_{j=1}^{n} \big|\phi_{n,j}(t) - 1\big| \right) \leq C_T < \infty.$$

□

*Proof of Statement 3.* Recall that we need to show,

$$\lim_{n\to\infty} \sup_{|t|\leq T} \left| \log\big(\psi_n(t)\big) + \frac{t^2}{2} \right| = 0,$$

where $\psi_n(t) = \prod_{j=1}^n \big(\phi_{n,j}(t) - 1\big)$, and $\phi_{n,j}$ is the characteristic function corresponding to $X_{n,j} = X_j/s_n$. Observe,

$$\begin{aligned}
\lim_{n\to\infty} \sup_{|t|\le T} \left| \log\big(\psi_n(t)\big) + \frac{t^2}{2} \right| &= \lim_{n\to\infty} \sup_{|t|\le T} \left| \sum_{j=1}^n \big(\phi_{n,j}(t) - 1\big) + \frac{t^2}{2} \right| \\
&= \lim_{n\to\infty} \sup_{|t|\le T} \left| \sum_{j=1}^n \Big(\phi_{n,j}(t) - 1 + \frac{\sigma_j^2 t^2}{2s_n^2}\Big) \right| \\
&= \lim_{n\to\infty} \sup_{|t|\le T} \left| \sum_{j=1}^n \int \Big(e^{itx/s_n} - 1 - \frac{itx}{s_n} + \frac{t^2x^2}{2s_n^2}\Big) d\mu_j \right|
\end{aligned}$$

since $s_n^2 = \displaystyle\sum_{j=1}^n \sigma_j^2$, and each $X_{n,j}$ has mean 0. Recall Lindeberg's condition,

$$\lim_{n\to\infty} \sum_{j=1}^n \Big( \int_{|X|\ge \epsilon s_n} \frac{X^2}{s_n^2} d\mu_j \Big) = 0.$$

Also by Taylor's theorem,

$$\left| e^{itx} - 1 - itx + \frac{t^2x^2}{2} \right| \le C_T |x|^3$$

in a small interval around 0 if $|t| \le T$. Note that $C_T > 0$ is an absolute constant depending only on $T$. So,

$$\begin{aligned}
\sum_{j=1}^n \int_{|X|<\epsilon s_n} \left| e^{itx/s_n} - 1 - \frac{itx}{s_n} + \frac{t^2x^2}{2s_n^2} \right| d\mu_j &\le \sum_{j=1}^n C_T \int_{|X|<\epsilon s_n} \frac{|x|^3}{s_n^3} d\mu_j \\
&\le \sum_{j=1}^n C_T \epsilon \int_{|X|<\epsilon s_n} \frac{|x|^2}{s_n^2} d\mu_j \\
&\le \epsilon C_T.
\end{aligned}$$

We can pick a large enough constant $C_T > 0$ (depending only on $T$) such that,

$$\left| e^{itx} - 1 - itx + \frac{t^2x^2}{2} \right| \le C_T |x|^2$$

holds for all $x \in \mathbb{R}$ if $|t| \le T$. We use this to bound,

$$\sum_{j=1}^n \int_{|X|\ge\epsilon s_n} \left| e^{itx/s_n} - 1 - \frac{itx}{s_n} + \frac{t^2x^2}{2s_n^2} \right| d\mu_j \le \sum_{j=1}^n C_T \int_{|X|\ge\epsilon s_n} \frac{|x|^2}{s_n^2} d\mu_j$$

Observe that the RHS in the above equation goes to 0 as $n$ goes to $\infty$ by Lindeberg's condition. We can conclude,

$$\sup_{|t|\le T} \left| \sum_{j=1}^n \int \Big(e^{itx/s_n} - 1 - \frac{itx}{s_n} + \frac{t^2x^2}{2s_n^2}\Big) d\mu_j \right| \xrightarrow[n\to\infty]{} 0.$$

$\square$

Since all three statements are proved, proof of Theorem 5 follows. □

Recall the following stated before,

**Corollary 7** (Lyapunov)**.** *Suppose for some $\delta > 0$,*

$$\lim_{n\to\infty} \frac{1}{s_n^{2+\delta}} \sum_{j=1}^{n} \left( \int |X|^{2+\delta} d\mu_j \right) = 0,$$

*then Lindeberg's condition holds.*

*Proof.* Observe,

$$\frac{1}{s_n^{2+\delta}} \sum_{j=1}^{n} \left( \int_{|X| \geq \epsilon s_n} |X|^{2+\delta} d\mu_j \right) \geq \frac{\epsilon^\delta}{s_n^2} \sum_{j=1}^{n} \left( \int_{|X| \geq \epsilon s_n} |X|^2 d\mu_j \right)$$

for any fixed $\epsilon > 0$. By the given hypothesis,

$$\frac{1}{s_n^{2+\delta}} \sum_{j=1}^{n} \left( \int_{|X| \geq \epsilon s_n} |X|^{2+\delta} d\mu_j \right) \leq \frac{1}{s_n^{2+\delta}} \sum_{j=1}^{n} \left( \int |X|^{2+\delta} d\mu_j \right) \underset{n\to\infty}{\longrightarrow} 0.$$

Therefore, for any fixed $\epsilon > 0$,

$$\frac{1}{s_n^2} \sum_{j=1}^{n} \left( \int_{|X| \geq \epsilon s_n} |X|^2 d\mu_j \right) \underset{n\to\infty}{\longrightarrow} 0.$$

□

Is Lyapunov's condition easy to verify? Consider the following example,

**Example 8.** *Let $\{X_i\}_{i\geq 1}$ be iid random variables, with $|X_i| \leq C$ for some constant $C > 0$. Then the Lyapunov condition works if we take $\delta = 1$. In more details,*

$$\frac{1}{s_n^3} \sum_{j=1}^{n} \left( \int |X|^3 d\mu_j \right) \leq O\left( \frac{n}{n^{3/2}} \right) \underset{n\to\infty}{\longrightarrow} 0.$$

## 3 Kolmogorov's Three Series Theorem

We will show the necessary part of the following theorem,

**Theorem 9** (Kolmogorov's Three Series Theorem)**.** *Let $\{X_i\}_{i\geq 1}$ be an independent sequence of random variables. Then $\sum_i X_i < \infty$ if and only if,*

1. *For some $C > 0$,*

$$\sum_i \mathbb{P}(|X_i| \geq C) < \infty.$$

*2. Let* $Y_i^{(C)} = X_i \mathbb{1}_{|X_i| \leq C}$*, then*

$$\sum_i \mathbb{E}\left[Y_i^{(C)}\right] < \infty.$$

*3. With* $Y_i^{(C)}$ *same as in* 2*,*

$$\sum_i \mathit{Var}\left(Y_i^{(C)}\right) < \infty.$$

*Proof of Necessity.* Assume $\sum_i X_i < \infty$, we will show that each of the three conditions must hold. We claim that condition 1 must hold for any $C > 0$. Assume for contradiction there exist $C > 0$ such that,

$$\sum_i \mathbb{P}(|X_i| \geq C) = \infty.$$

Define the event $A_i \coloneqq \{|X_i| > C\}$. Since the random variables $X_i$ are independent, Borel-Cantelli lemma gives

$$\mathbb{P}\Big(\limsup_{i\to\infty} A_i\Big) = 1.$$

We can conclude $|X_i| > C$ holds infinitely often w.p. 1. On the other hand, since $\sum_i X_i < \infty$,

$$\lim_{i\to\infty} X_i = 0$$

holds w.p. 1. This contradicts our intial assumption, hence condition 1 follows. We will first show condition 2 assuming condition 3. Since $\sum_i X_i < \infty$ holds w.p. 1, we must have

$$\lim_{i\to\infty} X_i = 0,$$

holds w.p. 1 and hence,

$$Y_i^{(C)} = X_i \quad \forall i \geq i_0$$

with $i_0$ large enough. Therefore, we can conclude

$$\sum_i Y_i^{(C)} < \infty.$$

Observe that since $\{X_i\}_{i\geq 1}$ is a sequence of independent random variables, $\{Y_i^{(C)}\}_{i\geq 1}$ is also a sequence of independent random variables. We have the following claim,

**Claim 10.**

$$\sum_i \left(Y_i^{(C)} - \mathbb{E}\left[Y_i^{(C)}\right]\right) < \infty.$$

*Proof of Claim.* Define $Z_i \coloneqq \left(Y_i^{(C)} - \mathbb{E}\left[Y_i^{(C)}\right]\right)$ and note that $\mathbb{E}[Z_i] = 0$, $\mathrm{Var}[Z_i] = \mathrm{Var}\left[Y_i^C\right]$ $\forall\, i$. Since $\left\{Y_i^{(C)}\right\}_{i\geq 1}$ is a sequence of independent random variables, $\{Z_i\}_{i\geq 1}$ is also a sequence of independent random variables. Since we know,

$$\sum_i \mathrm{Var}\left(Z_i\right) = \sum_i \mathrm{Var}\left(Y_i^{(C)}\right) < \infty$$

it must follow that $\sum_i Z_i < \infty$ from the sufficiency part of the three series theorem. □

Since $\sum_i Y_i^{(C)} < \infty$ and $\sum_i \left(Y_i^{(C)} - \mathbb{E}\left[Y_i^{(C)}\right]\right) < \infty$, we can conclude

$$\sum_i \mathbb{E}\left[Y_i^{(C)}\right] < \infty.$$

It remains to prove condition 3. To this end, define

$$\left(s_n^{(C)}\right)^2 := \sum_{i=1}^{n} \mathrm{Var}\left(Y_i^{(C)}\right).$$

Suppose if possible that $s_n^{(C)} \to \infty$ as $n \to \infty$. Observe that the random variables $Y_i^{(C)}$ are uniformly bounded by constant $C$ and independent. So by sufficiency of the Lyapunov condition, we have

$$\frac{\sum_{i=1}^{n}\left(Y_i^{(C)} - \mathbb{E}\left[Y_i^{(C)}\right]\right)}{s_n} \underset{W}{\Longrightarrow} N(0,1).$$

Let $\sum_{i=1}^{n} Y_i^{(C)} = S_n^{(C)}$, and let $\sum_{i=1}^{n} \mathbb{E}\left[Y_i^{(C)}\right] = M_n^{(C)}$. In other words, for any $x < y$,

$$\mathbb{P}\left(x < \frac{S_n^{(C)} - M_n^{(C)}}{s_n^{(C)}} < y\right) \underset{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{2\pi}} \int_x^y e^{-t^2/2} dt. \tag{4}$$

We also have the following claim,

**Claim 11.** *The following,*

$$\frac{S_n^{(C)}}{s_n^{(C)}} \underset{n\to\infty}{\longrightarrow} 0$$

*holds w.p.* 1.

*Proof of Claim.* We know $\sum_i X_i$ converges w.p. 1. This implies,

$$\lim_{i\to\infty} X_i = 0,$$

holds w.p. 1 and hence,

$$Y_i^{(C)} = X_i \quad \forall i \geq i_0$$

with $i_0$ large enough. Therefore, we can conclude

$$\sum_i Y_i^{(C)} < \infty.$$

The claim follows since the sequence $S_n^{(C)}$ is convergent (and hence bounded) w.p. 1, and $s_n^{(C)} \to \infty$ as $n \to \infty$. □

Since almost-sure convergence implies convergence in probability, we have $\forall \epsilon > 0$,

$$\mathbb{P}\left(\frac{\left|S_n^{(C)}\right|}{s_n^{(C)}} \geq \epsilon\right) \underset{n\to\infty}{\longrightarrow} 0. \tag{5}$$

Equations 4 and 5 imply that,

$$\mathbb{P}\left(x + \epsilon < \frac{-M_n^{(C)}}{s_n^{(C)}} < y - \epsilon\right) > 0$$

must hold for all $n$ large enough. In particular, if $(x_1 + \epsilon, y_1 - \epsilon)$ and $(x_2 + \epsilon, y_2 - \epsilon)$ are disjoint intervals, then this is a contradiction. Therefore,

$$\sum_{i\geq 1} \mathrm{Var}(Y_i^{(C)}) < \infty$$

and this establishes the necessity of all conditions of the theorem. □

# 4 Law of Iterated Logarithm

Recall the following (vanilla) version of CLT,

**Theorem 12.** *Let $\{X_i\}_{i\geq 1}$ be iid random variables such that $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[X_i^2] = 1 \; \forall \, i$. Then we have,*

$$\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}} \underset{W}{\Longrightarrow} N(0, 1).$$

We have the following fact,

**Fact.** *For any sequence $(n_k)_{k\geq 1}$ such that $n_k \to \infty$,*

$$\mathbb{P}\left(\limsup_{k\to\infty} \frac{X_1 + X_2 + \cdots + X_{n_k}}{\sqrt{n_k}} = \infty\right) = 1.$$

*Proof of Fact.* Define the random variable

$$Z \coloneqq \limsup_{k\to\infty} \frac{X_1 + X_2 + \cdots + X_{n_k}}{\sqrt{n_k}},$$

which can possibly be $+\infty$. Fix $a \in \mathbb{R}$. Because the normal distribution has an infinitely long tail, i.e., the probability of exceeding any given value is positive, we must have

$$\mathbb{P}(Z \geq a) > 0.$$

Since $\{Z \geq a\}$ is an event in the tail $\sigma$-algebra, by Kolmogorov's 0-1 law

$$\mathbb{P}(Z \geq a) \in \{0, 1\}.$$

Proof of fact follows as the probability cannot be zero by the previous observation. □

Recall the SLLN,

**Theorem 13.** *Let $\{X_i\}_{i\geq 1}$ be iid random variables such that $\mathbb{E}[X_i]=0$ and $\mathbb{E}[X_i^2]=1\ \forall\, i$. Then we have,*

$$\mathbb{P}\left(\lim_{n\to\infty}\frac{X_1+X_2+\cdots+X_n}{n}=0\right)=1.$$

This motivates the following theorem,

**Theorem 14** (Law of Iterated Logarithm)**.** *Let $\{X_i\}_{i\geq 1}$ be iid random variables with $\mathbb{E}[X_i]=0$ and $\mathbb{E}[X_i^2]=1\ \forall\, i$. Then,*

$$\mathbb{P}\left(\limsup_{n\to\infty}\frac{X_1+X_2+....+X_n}{\phi(n)}=\sqrt{2}\right)=1 \tag{6}$$

*for $\phi(n)=\sqrt{n\log\log n}$.*

*Proof.* We will see the proof of (6) with an additional condition $\mathbb{E}|X_i|^{2+\alpha}<\infty$ for some $\alpha>0$. First we will prove the theorem for the case when $X_i \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$. As usual, let $S_n := \sum_{i=1}^{n} X_i$. It suffices to prove the following two claims,

**Claim 15.** *For all $\lambda>\sqrt{2}$,*

$$\mathbb{P}\left(\limsup_{n\to\infty}\frac{S_n}{\phi(n)}\geq\lambda\right)=0.$$

**Claim 16.** *For all $\lambda<\sqrt{2}$,*

$$\mathbb{P}\left(\limsup_{n\to\infty}\frac{S_n}{\phi(n)}\geq\lambda\right)=1.$$

*Proof of Claim 15.* Given $\lambda>\sqrt{2}$. We are interested in showing,

$$\mathbb{P}(S_n\geq\lambda\phi(n)\ \text{ i.o.})=0$$

It would be sufficient because of the Borel-Cantelli lemma to show that,

$$\sum_n \mathbb{P}\left(\frac{S_n}{\phi(n)}\geq\lambda\right)<\infty.$$

However, this is too strong! Notice that if we have sequence $k_n\uparrow\infty$ such that

$$\sup_{k_{n-1}\leq j\leq k_n} S_j\geq\lambda\phi(k_{n-1})$$

happens infinitely often, this also gives us what we want. We shall pick a sequence $k_n:=\rho^n$ (for some $\rho>1$) and we will compute

$$\sum_n \mathbb{P}\left(\sup_{k_{n-1}\leq j\leq k_n} S_j\geq\lambda\phi(k_{n-1})\right) \tag{7}$$

and if the above sum is $< \infty$, then it implies

$$\limsup_{n\to\infty} \frac{\sup_{k_{n-1}\le j\le k_n} S_n}{\phi(k_{n-1})} \le \lambda$$

holds with probability 1. Since $\phi$ is a monotonically increasing function then in particular, this implies that

$$\limsup_{n\to\infty} \frac{S_n}{\phi(n)} \le \lambda$$

holds with probability 1. It will then follow that

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{S_n}{\phi(n)} > \lambda\right) = 0$$

and we are done! To show that the sum in (7) is $< \infty$ for the chosen sequence $\{k_n\}$, we shall use Levy's inequality in the following (slightly) general form,

**Theorem 17** (Levy)**.** *Let $\{X_i\}_{i=1}^n$ be independent random variables. If*

$$\sup_{1\le j\le n} \mathbb{P}\Big(|S_n - S_j| \ge r\Big) \le \delta,$$

*then we must have,*

$$\mathbb{P}\left(\sup_{1\le j\le n} |S_j| \ge l\right) \le \frac{\mathbb{P}\Big(|S_n| \ge l - r\Big)}{1-\delta}.$$

*Proof.* Same proof as the original version of the inequality with slight tweaks. □

Let $0 < \sigma < 1$ be sufficiently small so that $\lambda' = \lambda - \sigma > \sqrt{2}$. To apply the above inequality, set $l = \lambda\phi(k_{n-1})$ and $r = \sigma\phi(k_{n-1})$. So we want to first get a good $\delta$ in order to apply Levy's inequality in the form we stated earlier. Observe that for any $1 \le i \le k_n$,

$$\begin{aligned}\mathbb{P}\Big(|S_i| \ge \sigma\phi(k_{n-1})\Big) &\le \frac{\mathbb{E}\big[|S_i|^2\big]}{\sigma^2\phi(k_{n-1})^2} \\ &\le \frac{k_n}{\sigma^2 k_{n-1}\log\log k_{n-1}}.\end{aligned}$$

Recall that we chose $k_n = \rho^n$ for some $\rho > 1$. We have,

$$\begin{aligned}\sup_{1\le i\le k_n} \mathbb{P}\Big(|S_i| \ge \sigma\phi(k_{n-1})\Big) &\le \frac{k_n}{\sigma^2 k_{n-1}\log\log k_{n-1}} \\ &\le \frac{C\rho}{\sigma^2\log(n)}\end{aligned}$$

where $C > 0$ is a constant. Note that $C\rho/\sigma^2\log(n) \to 0$ as $n \to \infty$. Choose $\delta = 1/2$ in the hypothesis of Levy's inequality so that we have,

$$\mathbb{P}\left(\sup_{k_{n-1}\le j\le k_n} S_j \ge \lambda\phi(k_{n-1})\right) \le 2\mathbb{P}\Big(|S_{k_n}| \ge \lambda'\phi(k_{n-1})\Big).$$

We want to show,

$$\sum_n \mathbb{P}\Big( \sup_{k_{n-1}\le j\le k_n} S_j \ge \lambda\phi(k_{n-1})\Big) < \infty.$$

To this end, it suffices to prove,

$$\sum_n \mathbb{P}\Big( |S_{k_n}| \ge \lambda'\phi(k_{n-1})\Big) < \infty.$$

Observe that if $X \sim \mathcal{N}(0,1)$, we have for any $a \ge 1$,

$$\begin{aligned}\mathbb{P}(X \ge a) &= \frac{1}{\sqrt{2\pi}}\int_a^\infty e^{-t^2/2}dt \\ &\le \frac{1}{\sqrt{2\pi}}\int_a^\infty te^{-t^2/2}dt \\ &= \frac{e^{-a^2/2}}{\sqrt{2\pi}}.\end{aligned}$$

So, this gives us $\mathbb{P}(X \ge a) \le \frac{1}{\sqrt{2\pi}}e^{-a^2/2}$. Further,

$$\sum_n \mathbb{P}\Big( |S_{k_n}| \ge \lambda'\phi(k_{n-1})\Big) < \infty \iff \sum_n \mathbb{P}\Big( \frac{|S_{k_n}|}{\sqrt{k_n}} \ge \frac{\lambda'\phi(k_{n-1})}{\sqrt{k_n}}\Big) < \infty.$$

So,

$$\begin{aligned}\mathbb{P}\Big( |S_{k_n}| \ge \lambda'\phi(k_{n-1})\Big) &\lesssim \exp\left[\frac{-1}{2}(\lambda')^2\frac{\rho^{n-1}}{\rho^n}\big[\log n + \log\log\rho\big]\right] \\ &= \exp\left[\frac{-1}{2}\frac{(\lambda')^2}{\rho}\big[\log n + \log\log\rho\big]\right].\end{aligned}$$

Note that $\log\log\rho^n = \log n + \log\log\rho$, and the term $\log\log\rho$ is an absolute constant. If

$$\frac{(\lambda')^2}{2\rho} = 1 + c_0$$

for a small constant $c_0 > 0$, then the expression simplifies to

$$\lesssim \frac{1}{n^{1+c_0}}.$$

This can be achieved by picking $\rho$ sufficiently close to 1 such that $\lambda'\sqrt{\rho} > \sqrt{2}$. Since the sum $\sum_{n\ge1} 1/n^{1+c_0} < \infty$, we are done with the proof of Claim 15. $\square$

*Proof of Claim 16.* For the next case, we will again use Borel-Cantelli lemma but in the converse direction. We will restate the lemma here for convenience,

**Lemma 18.** *Let $\{A_n\} \subseteq \Omega$. If $\sum_{n\ge1}\mathbb{P}(A_n) = \infty$, and $\{A_n\}$ is independent, then*

$$\mathbb{P}\Big(\limsup_{n\to\infty} A_n\Big) = 1.$$

Again, we will find a sequence $k_n = \rho^n$ (for a suitable $\rho > 1$) such that if we define

$$Y_n := S_{k_n} - S_{k_{n-1}},$$

then the random variables $Y_n$ are independent. Note that $Y_n \sim \mathcal{N}(0, k_n - k_{n-1})$. Fix $\lambda < \sqrt{2}$. We will try to show,

$$\mathbb{P}\Big(Y_n \geq \lambda\phi(k_n) \text{ i.o.}\Big) = \mathbb{P}\bigg(\frac{Y_n}{\sqrt{k_n - k_{n-1}}} \geq \frac{\lambda\phi(k_n)}{\sqrt{k_n - k_{n-1}}} \text{ i.o.}\bigg) = 1,$$

and to do this, we will get a lower bound for the expression $\mathbb{P}(X \geq a)$ when $X \sim \mathcal{N}(0,1)$. Observe,

$$\begin{aligned}
\mathbb{P}(X \geq a) &= \frac{1}{\sqrt{2\pi}} \int_a^\infty e^{-t^2/2} dt \\
&\geq \frac{1}{\sqrt{2\pi}} \int_a^\infty (1+t) e^{-(t + \frac{t^2}{2})} dt \\
&\geq \frac{1}{\sqrt{2\pi}} e^{\frac{-(a+1)^2}{2}}.
\end{aligned}$$

So, we have

$$\begin{aligned}
\mathbb{P}\Big(Y_n \geq \lambda\phi(k_n)\Big) &\geq \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-1}{2}\left(1 + \frac{\lambda\phi(k_n)}{\sqrt{k_n - k_{n-1}}}\right)^2\right] \\
&\gtrsim \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-\rho\lambda^2 \log n}{2(\rho-1)}\right] \\
&\gtrsim \exp\left[-\log\left(n^{\frac{\lambda^2\rho}{2(\rho-1)}}\right)\right] \\
&\gtrsim \frac{1}{n^{\frac{\lambda^2\rho}{2(\rho-1)}}}.
\end{aligned}$$

So this series diverges if

$$\frac{\lambda^2\rho}{2(\rho-1)} < 1,$$

which happens for $\lambda < \sqrt{2}$ (fixed) and when the constant $\rho$ is large. So, the upshot is the following: If $\lambda^2\rho/2(\rho-1) < 1$, then

$$\sum_{n\geq 1} \mathbb{P}\Big(Y_n \geq \lambda\phi(k_n)\Big) = \infty.$$

By Borel-Cantelli lemma, we have

$$\mathbb{P}\Big(Y_n \geq \lambda\phi(k_n) \text{ i.o.}\Big) = 1.$$

Or equivalently, we have $S_{k_n} - S_{k_{n-1}} \geq \lambda\phi(k_n)$ occurs infinitely often w.p. 1.
If $X \sim \mathcal{N}(0,1)$, then the random variable $-X \sim \mathcal{N}(0,1)$ as well. Consider the following 'trick': Replace $X_i$ by $-X_i$ in the upper bound obtained in previous claim to get,

$$\mathbb{P}\bigg(\limsup_{n\to\infty} \frac{-S_{k_{n-1}}}{\phi(k_n)} \leq \frac{\sqrt{2}}{\sqrt{\rho}}\bigg) = 1. \tag{8}$$

This follows as we have $\frac{\phi(k_{n-1})}{\phi(k_n)} = \frac{1}{\sqrt{\rho}}$. Setting $\lambda = \sqrt{\frac{2(\rho-1)}{\rho}}$ and combining this with the above observation gives,

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{S_{k_n}}{\phi(k_n)} \geq \sqrt{\frac{2(\rho-1)}{\rho}} - \frac{\sqrt{2}}{\sqrt{\rho}}\right) = 1,$$

and therefore,

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{S_n}{\phi(n)} \geq \sqrt{\frac{2(\rho-1)}{\rho}} - \frac{\sqrt{2}}{\sqrt{\rho}}\right) = 1.$$

Taking $\rho$ arbitrarily large concludes the proof of Claim 16. □

This establishes the proof when $X_i \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$. To get to the general case, we note that the proof needed two important inequalities,

- For any $a > 0$, an upper bound of the form:

$$\mathbb{P}\left(\frac{S_n}{\phi(n)} \geq a\right) \leq \frac{C}{(\log n)^{1+\epsilon}}.$$

- For any $a > 0$, a lower bound of the form:

$$\mathbb{P}\left(\frac{S_n}{\phi(n)} \geq a\right) \geq \frac{C'}{(\log n)^{1-\epsilon}}.$$

In particular, if the tail probability bounds for $S_n$ are 'very close' to the bounds we obtained for the normal distribution, then the same proof will work! Now we will use the hypothesis that $\mathbb{E}|X|^{2+\alpha} < \infty$ for some $\alpha > 0$.

**Theorem 19** (Berry-Esseen)**.** *Let $X_i \overset{iid}{\sim}$ with mean 0 and variance 1. Let $S_n = \sum_{i=1}^n X_i$. Suppose $\alpha > 0$, and $\mathbb{E}|X|^{2+\alpha} < \infty$. Then we have the estimate,*

$$\sup_{a\in\mathbb{R}} \left| \mathbb{P}\left(\frac{S_n}{\sqrt{n}} \geq a\right) - \frac{1}{\sqrt{2\pi}} \int_a^\infty e^{-t^2/2} dt \right| \leq \frac{C}{n^\delta},$$

*for some $\delta = \delta(\alpha) > 0$.*

*Proof of Berry-Esseen.* We will prove the following three lemmas before proving the theorem.

**Lemma 20.** *Let $-\infty < a < b < \infty$ and $0 < h < \frac{b-a}{2}$ be a small positive number. Define*

$$f_{a,h}(x) = \begin{cases} 0, & \text{for } -\infty \leq x < a-h \\ \frac{x-a+h}{2h}, & \text{for } a-h \leq x \leq a+h \\ 1, & \text{for } a+h \leq x < \infty \end{cases}$$

$$f_{a,b,h}(x) = \begin{cases} 0, & \text{for } -\infty \leq x < a-h \\ \frac{x-a+h}{2h}, & \text{for } a-h \leq x \leq a+h \\ 1, & \text{for } a+h \leq x \leq b-h \\ 1-\frac{x-b+h}{2h}, & \text{for } b-h \leq x \leq b+h \\ 0, & \text{for } b+h \leq x < \infty \end{cases}$$

*Then,* $f_{a,h}(x) = \lim_{b\to\infty} f_{a,b,h}(x)$ *holds pointwise.*

*Proof of Lemma.* If $x \in (-\infty, a+h]$, then $f_{a,b,x}(x) = f_{a,h}(x)$ for all $b$. For any fixed $b$, $f_{a,b,x}(x) = f_{a,h}(x)$ for $x \in (a+h, b-h]$. If $x \in (a+h, \infty)$, then we pick $b > x+h$. And thus $x < b - h$ will imply $f_{a,b,x}(x) = f_{a,h}(x)$. Hence, $f_{a,h}(x) = \lim_{b\to\infty} f_{a,b,h}(x)$ pointwise. □

**Lemma 21.** *For any probability measure* $\mu$

$$\int_{-\infty}^{\infty} f_{a,b,h}(x)d\mu(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\mu}(t)\left(\frac{e^{-iat}-e^{-ibt}}{it}\right)\frac{\sin ht}{ht}dt$$

*where* $\hat{\mu} \equiv$ *characteristic function of* $\mu$.

*Proof of Lemma.* This is essentially the Fourier inversion formula. Note that

$$f_{a,b,h}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixy}\frac{e^{-iay}-e^{-iby}}{iy}\frac{\sin(hy)}{hy}\,dy.$$

We can start with the double integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ixy}\frac{e^{-iay}-e^{-iby}}{iy}\frac{\sin(hy)}{hy}\,dy\,d\mu(x)$$

We can then apply Fubini's theorem to obtain

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ixy}\frac{e^{-iay}-e^{-iby}}{iy}\frac{\sin(hy)}{hy}\,d\mu(x)\,dy\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-iay}-e^{-iby}}{iy}\frac{\sin(hy)}{hy}\left(\int_{-\infty}^{\infty} e^{ixy}\,d\mu(x)\right)dy\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\mu}(y)\frac{e^{-iay}-e^{-iby}}{iy}\frac{\sin(hy)}{hy}\,dy
\end{aligned}$$

Note that Fubini's theorem is applicable as,

$$\left|e^{ixy}\frac{e^{-iay}-e^{-iby}}{iy}\frac{\sin(hy)}{hy}\right| \le O_h\left(\frac{1}{y^2}\right)$$

which is $L^1$-integrable with respect to the given product measure.

□

**Lemma 22.** *If* $\mu$ *is a probability measure and* $\widehat{\mu}(\cdot)$ *denotes its characteristic function, then*

$$\int_{-\infty}^{\infty} f_{a,h}(x)d\mu(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{\mu}(t)\frac{e^{-iat}}{it}\frac{sin(ht)}{ht}dt.$$

*Proof of Lemma.* We will use the following result,

**Fact** (Riemann Lebesgue Lemma)**.** *Let $f \in L^1(\mathbb{R}^n)$ be an integrable function, i.e., $f\colon \mathbb{R}^n \to \mathbb{C}$ is a measurable function such that*

$$\|f\|_{L^1} = \int_{\mathbb{R}^n} |f(x)|\,\mathrm{d}x < \infty,$$

*and let $\hat{f}$ be the Fourier transform of $f$, i.e.,*

$$\hat{f}\colon \mathbb{R}^n \to \mathbb{C}, \quad \xi \mapsto \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\,\mathrm{d}x.$$

*Then $\hat{f}$ vanishes at infinity, i.e., $|\hat{f}(\xi)| \to 0$ as $|\xi| \to \infty$.*

Now we just let $b \to \infty$ in the previous lemma. This gives us,

$$\begin{aligned}
&\frac{1}{2\pi} \lim_{b\to\infty} \int_{-\infty}^{\infty} \widehat{\mu}(y) \frac{e^{-iay} - e^{-iby}}{iy} \frac{\sin(hy)}{hy}\, dy \\
&= \frac{1}{2\pi} \left( \lim_{b\to\infty} \left( \int_{-\infty}^{\infty} \hat{\mu}(y) \frac{e^{-iay}}{iy} \frac{\sin(hy)}{hy}\, dy \right) - \lim_{b\to\infty} \left( \int_{-\infty}^{\infty} \widehat{\mu}(y) \frac{e^{-iby}}{iy} \frac{\sin(hy)}{hy}\, dy \right) \right) \\
&= \frac{1}{2\pi} \left( \int_{-\infty}^{\infty} \widehat{\mu}(y) \frac{e^{-iay}}{iy} \frac{\sin(hy)}{hy}\, dy \right).
\end{aligned}$$

The term $\lim_{b\to\infty} \int_{-\infty}^{\infty} \hat{\mu}(y) \frac{e^{-iby}}{iy} \frac{\sin(hy)}{hy} dy$ goes to 0 after applying Riemann Lebesgue lemma where we substitute $f(x) = \left( \hat{\mu}(y) \frac{1}{iy} \frac{\sin(hy)}{hy} \right)$. By applying DCT, we obtain

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_{a,b,h}(x) d\mu(x) = \int_{-\infty}^{\infty} f_{a,h}(x) d\mu(x).$$

We can conclude,

$$\int_{-\infty}^{\infty} f_{a,h}(x) d\mu(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\mu}(y) \frac{e^{-iay}}{iy} \frac{\sin(hy)}{hy}\, dy$$

and the lemma follows. □

Since the proofs of the three lemmas are complete, we will now proceed to prove the theorem. Note that for any $a > 0$,

$$\mu\left([a,\infty)\right) \le \int_{\mathbb{R}} f_{a-2h,h}(x) d\mu(x) \le \mu\left([a-2h,\infty)\right)$$

$$\lambda\left([a,\infty)\right) \le \int_{\mathbb{R}} f_{a-2h,h}(x) d\lambda(x) \le \lambda\left([a-2h,\infty)\right)$$

where $\lambda, \mu$ are probability measures.

Let $\mu_n$ denote the distribution of $\frac{S_n}{\sqrt{n}}$ and $\lambda$ denote the normal distribution of $\mathcal{N}(0,1)$. And, let the integral $\int f\, d(\mu - \lambda) := \int f\, d\mu - \int f\, d\lambda$. So we get the inequalities,

$$\mu\left([a,\infty)\right) - \lambda\left([a,\infty)\right) - \left(\lambda\left([a-2h,\infty)\right) - \lambda\left([a,\infty)\right)\right) \le \int f_{a-h,h}(x)\, d(\mu - \lambda),$$

$$\int f_{a-h,h}(x)\, d(\mu - \lambda) \le \mu\left([a-2h,\infty)\right) - \lambda\left([a-2h,\infty)\right) - \left(\lambda\left([a,\infty)\right) - \lambda\left([a-2h,\infty)\right)\right).$$

Since $\lambda$ has a density that is bounded, it follows that $|\lambda([a-2h,\infty)) - \lambda([a,\infty))|$ can be bounded by $Ch$ for some absolute constant $C > 0$. Hence, it follows that,

$$\sup_{a\in\mathbb{R}} |\mu[a,\infty) - \lambda[a,\infty)| \le \left| \int f_{a-h,h}(x)\, d(\mu-\lambda) \right| + Ch$$

Substituting for the expression in the RHS gives,

$$\sup_{a\in\mathbb{R}} |\mu([a,\infty)) - \lambda([a,\infty))| \le \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \widehat{\mu}(t) - e^{\frac{t^2}{2}} \right| \frac{|\sin ht|}{ht} \frac{1}{t}\, dt\, + Ch. \tag{9}$$

If $\phi(t)$ is a characteristic function of $X_i$, then we know that $\widehat{\mu}_n(t) = \left(\phi\left(\frac{t}{\sqrt{n}}\right)\right)^n$. Now the given hypothesis $\mathbb{E}|X|^{2+\alpha} < \infty$ implies that for $|t| \le 1$,

$$\phi(t) = 1 - \frac{t^2}{2} + O\left(|t|^{2+\alpha}\right)$$

for some $\alpha > 0$. So,

$$\widehat{\mu}_n(t) = \left(1 - \frac{t^2}{2n} + O\left(\frac{|t|^{2+\alpha}}{n^{1+\frac{\alpha}{2}}}\right)\right)^n$$
$$\implies \widehat{\mu}_n(t) = \exp\left(n\log\left(1 - \frac{t^2}{2n} + O\left(\frac{|t|^{2+\alpha}}{n^{1+\frac{\alpha}{2}}}\right)\right)\right).$$

Using the Taylor series expansion for $\log(1+x)$, we get

$$\begin{aligned}\widehat{\mu}_n(t) &\simeq \exp\left(n\left(-\frac{t^2}{2n} + O\left(\frac{|t|^{2+\alpha}}{n^{1+\frac{\alpha}{2}}}\right)\right)\right)\\ &\simeq \exp\left(-\frac{t^2}{2}\right)\exp\left(n\,O\left(\frac{|t|^{2+\alpha}}{n^{1+\frac{\alpha}{2}}}\right)\right).\end{aligned}$$

Fix $|t| \le n^{\frac{\alpha}{2(\alpha+2)}}$. Then we have,

$$\exp\left(n\,O\left(\frac{|t|^{2+\alpha}}{n^{1+\frac{\alpha}{2}}}\right)\right) = \exp\left(O\left(\frac{|t|^{2+\alpha}}{n^{\frac{\alpha}{2}}}\right)\right) \le 1 + C\frac{|t|^{2+\alpha}}{n^{\frac{\alpha}{2}}},$$

where $C > 0$ is an absolute constant. Combining this with the previous bound gives,

$$\left| \widehat{\mu}_n(t) - \exp\left(-\frac{t^2}{2}\right) \right| \le O\left(\frac{|t|^{2+\alpha}}{n^{\frac{\alpha}{2}}}\right)$$

provided $|t| \le n^{\frac{\alpha}{2(\alpha+2)}}$. Let $\theta = \frac{\alpha}{2(\alpha+2)}$, we can split the integral

$$\int \left| \widehat{\mu}_n(t) - e^{\frac{-t^2}{2}} \right| \frac{\sin ht}{ht^2} dt = (I_1 + I_2) := \int_{|t|\le n^\theta} \left| \widehat{\mu}_n(t) - e^{\frac{-t^2}{2}} \right| \frac{\sin ht}{ht^2} dt + \int_{|t|>n^\theta} \left| \widehat{\mu}_n(t) - e^{\frac{-t^2}{2}} \right| \frac{\sin ht}{ht^2} dt.$$

We have the following bounds on these integrals,

$$\begin{aligned} I_1 &\le \frac{C_1}{n^{\frac{\alpha}{2}}} \int_{-n^\theta}^{+n^\theta} \frac{|t|^{2+\alpha}}{ht^2} dt \le \frac{C_1 t^{\alpha+1}}{hn^{\frac{\alpha}{2}}} \le \frac{C_1 n^{\theta(\alpha+1)-\frac{\alpha}{2}}}{h} \le \frac{C_1}{h} n^{-\frac{\alpha}{2(\alpha+2)}} = \frac{C_1}{hn^\theta}\\ I_2 &\le \frac{1}{h}\int_{|t|>n^\theta} \frac{C_1}{t^2} dt \le \frac{C_1}{hn^\theta}, \end{aligned}$$

where $C_1 > 0$ is an absolute constant. Hence, it follows from (9),

$$\sup_{a\in\mathbb{R}} |\mu([a,\infty)) - \lambda([a,\infty))| \leq Ch + I_1 + I_2 \leq \frac{2C_1}{hn^\theta} + Ch.$$

Setting $h = n^{-\frac{\theta}{2}}$ gives,

$$\sup_{a\in\mathbb{R}} |\mu([a,\infty)) - \lambda([a,\infty))| \leq O\left(\frac{1}{n^{\frac{\theta}{2}}}\right),$$

and we obtain the desired conclusion for $\delta = \frac{\theta}{2} = \frac{\alpha}{4(\alpha+2)}$. $\square$

Theorem 14 follows with the additional hypothesis that $\mathbb{E}|X|^{2+\alpha} < \infty$ for some $\alpha > 0$. $\square$

**Remark** (Application of Berry-Esseen's theorem). *We can use the Berry-Esseen's theorem to get an approximaton,*

$$\mathbb{P}\left(\frac{S_n}{\sqrt{n}} \geq a\right) \simeq (1 - \Phi(a)) + \epsilon$$

*where $\Phi$ denotes the cumulative distribution function of $\mathcal{N}(0,1)$, and $\epsilon$ denotes the error term decaying at the rate $O\left(\frac{1}{n^\delta}\right)$.*

# References

[1] SR Srinivasa Varadhan. *Probability theory.* Number 7. American Mathematical Soc., 2001.