Efficient Quantifier Elimination in PA

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Introduction

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  - $a_1x_1 + \cdots + a_nx_n \leq b$
  - $a_1x_1 + \cdots + a_nx_n \equiv b \mod m$

  where $x_1, \ldots, x_n$ are variables and $a_1, \ldots, a_n, b, m \in \mathbb{Z}$ are constants.

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- A formula is quantifier-free if it is a Boolean combination of atomic formulas.

An Example

- Given positive integers $m_1, m_2, \ldots m_n$, what is the largest number that cannot be obtained as a non-negative linear combination of those numbers? The answer, if it exists, is the smallest satisfying assignment of the formula:

$$\Phi(x) = \forall y\, (x < y \longrightarrow (\exists z_1, z_2 \ldots z_n(y = z_1 m_1 + \cdots + z_n m_n \\ \wedge z_1 \geq 0 \wedge \cdots \wedge z_n \geq 0)))$$

(For the case $n = 2$, look up Chicken Mcnugget Theorem!)

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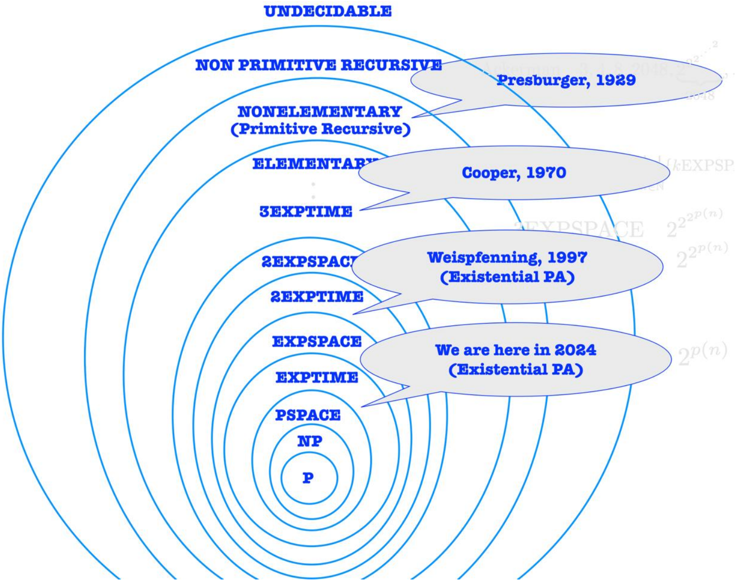
- This result was achieved by using the quantifier elimination method.

Complexity of Quantifier Elimination



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The main ingredient for proving this is the following proposition,

Proposition

Let $A \in \mathbb{Z}^{\ell \times n}$ and $b \in \mathbb{Z}^{\ell}$, and let Δ be an upper bound on all absolute values of the subdeterminants of A. If the system $Ax \leq b$ has an integral solution, then it has one of the form $Db + d$, where $D \in \mathbb{Q}^{n \times \ell}$ and $d \in \mathbb{Q}^n$ with $||D||_{\mathrm{frac}} \leq \Delta$ and $||d||_{\mathrm{frac}} \leq n\Delta^2$.

where $||.||_{\mathrm{frac}}$ denotes the maximal absolute value of all numerators and denominators in the representation.

Example for Proposition

Consider the formula $(x_1 - x_2 \leq 3) \wedge (x_1 + 2x_2 \leq 4)$. This formula can be written as the system,

$$\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Let

$$(D, d) = \left(\begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

Setting $x = Db + d$ gives $(x_1, x_2) = (2, 0)$, which is a satisfying assignment for the formula!

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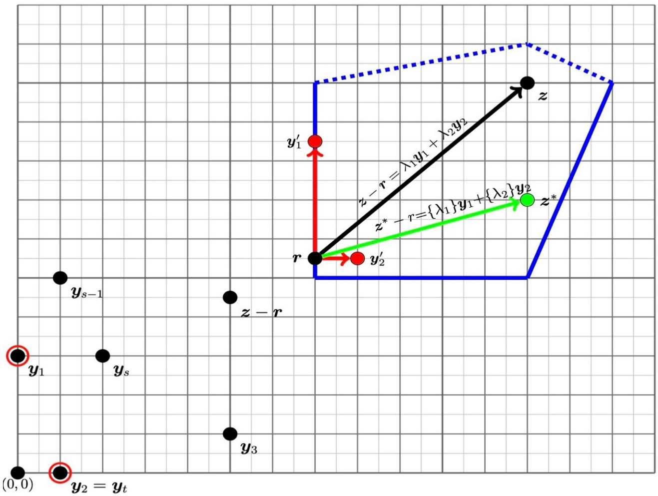
- The proposition claims a small model property for parametric integer programming.
- If the system $Ax \leq b$ has an integral solution, it has a rational solution of a specific form.

Proof of Proposition



For every rational solution r of $Ax \leq b$, we show that there is a close-by integral solution z^*.

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- Formulas Φ_n will require exponential sized quantifier equivalents.

An Application

- A *well-quasi-ordering* (WQO) is a reflexive and transitive ordering $(X, \leq)$ such that for every sequence $x_1, x_2, \ldots \in X$, there are $i < j$ with $x_i \leq x_j$.

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- Given a quantifier-free formula $\varphi(x, y)$, deciding whether the relation $R \subseteq \mathbb{Z}^n \times \mathbb{Z}^n$ defined by φ is a WQO is coNP-complete [BGLZ24].

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- Given a quantifier-free formula $\varphi(x, y)$, deciding whether the relation $R \subseteq \mathbb{Z}^n \times \mathbb{Z}^n$ defined by φ is a WQO is coNP-complete [BGLZ24].
- Our results allow us to settle the complexity for existential formulas:

Corollary

Given an existential PA formula φ, it is coNEXP*-complete to decide whether φ defines a WQO.*

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The Σ_2-fragment of Presburger Arithmetic is in NEXP.

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Monadic decomposability of $\exists$PA formulas is coNEXP*-complete.*

Conclusion

- Main result establishes a quantifier elimination procedure eliminating a block of existentially quantified variables in singly exponential time.
- All known algorithms before required doubly exponential time.
- The technical basis is a small model property for parametric integer programming.
- Implementing optimizations could lead to a more practical use of the algorithm in SMT solvers.

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