Oxford Mathematics Team Challenge 2026
Lock-in Round Question Booklet

Instructions

1. Do not open the paper until the invigilator tells you to do so.
2. Format. This round contains three questions, but you should submit answers to only **two questions**.

   For your answers, write them on sheets of paper and mark the question-part you are answering on the margin. You must also clearly write your Team ID at the top of *each* sheet of paper. Do *not* write your team name. You will be provided treasury tags to fasten sheets of paper together according to which question they answer.

   The questions are long-answer, so you may be required to give detailed explanations, brief descriptions, or mathematical working. The questions may indicate the level of depth you should offer, but you should always exercise your judgement in giving an appropriate level of depth to your answer.
3. Time limit. 60 minutes. You may not write anything on any paper you will submit after the allotted time has expired.
4. No calculators, squared paper or measuring instruments. Lined paper and blank paper for rough working is allowed. You may use a pen or pencil to preference. Other mediums for working (e.g., digital devices, whiteboards, Etch A Sketches) are strictly forbidden.

   The points for each question are in the bottom right of the cells in the Answer Sheet, as well as at the end of the questions in the Question Booklet, and are marked in [square brackets].
5. Scoring rules. Each question is out of 30 points; as you will only submit answers to **two** questions, the paper is out of 60 points.
6. Don't expect to complete the whole paper in the time! The later parts are worth more marks but are generally harder, and they may build up on previous parts of the question.
7. You are also encouraged to think deeply, rather than to guess.
8. Good luck, and enjoy! ☺

1. Birational correspondences

In this question, we will form two *birational correspondences*. A birational correspondence is a one-one correspondence between two sets such that rational elements of one set pair up with rational elements of the other set. We first consider the unit circle $\mathcal{C}$ given by the equation $x^2 + y^2 = 1$, which is depicted in Figure 1:

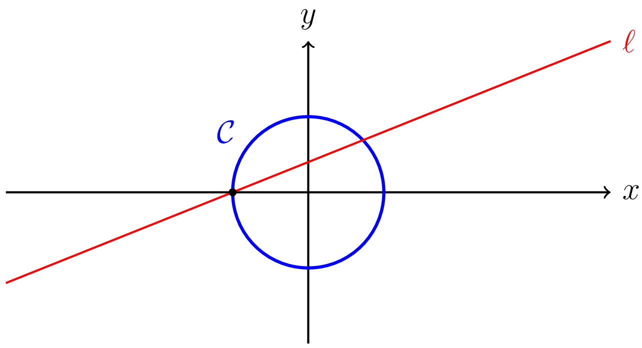


Figure 1

(a) Let ℓ_t be a line with gradient t, where t is some number. We say ℓ_t is a *linking line* if it intercepts $\mathcal{C}$ at $(-1, 0)$.

(i) Any linking line ℓ_t intercepts $\mathcal{C}$ at a second point P_t. Determine the coordinates of P_t in terms of t. [3]

(ii) Verify that the point P_t found in (a)(i) indeed lies on the circle. [1]

(iii) Write down the equation of the line passing $(-1, 0)$ which intercepts with $\mathcal{C}$ at only one point. [1]

(b) (i) Let $\mathcal{C}'$ be the points of $\mathcal{C}$ except for $(-1, 0)$. Let $P = (a, b)$ be any point in $\mathcal{C}'$. Find the equation of the unique linking line ℓ_t which intercepts P. [2]

(ii) Using parts (a)(i) and (b)(i), briefly deduce that the points in $\mathcal{C}'$ pair up naturally with linking lines – that is, they *correspond* to one another. [2]

We now show that this correspondence is in fact a *birational correspondence*, in the sense that the line has a rational gradient if, and only if, the point in $\mathcal{C}'$ is "rational" in the following sense:

(c) (i) Let the linking line ℓ_t correspond to the point $P_t = (a, b)$. Show that t is rational if, and only if, a and b are both rational. [3]

(ii) Hence, write down the general form for points (a, b) in $\mathcal{C}'$ where a and b are both rational. [1]

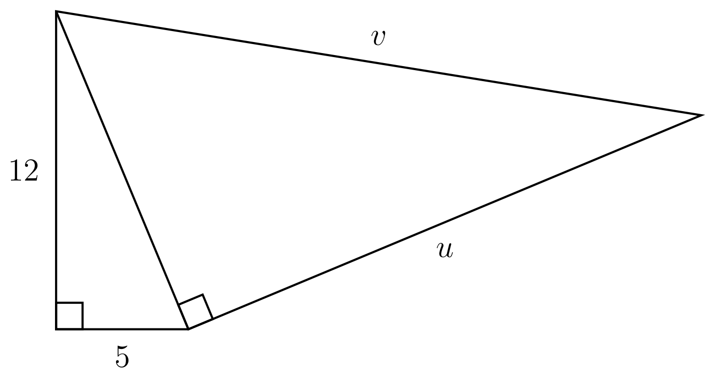
(d) (i) A *Pythagorean triple* is a triple of positive whole numbers (a, b, c) such that $a^2 + b^2 = c^2$.

Using previous parts, or otherwise, find a general form for a Pythagorean triple in terms of two positive whole numbers p, q with $q > p > 0$. *[The general form* $(a, b, \sqrt{a^2 + b^2})$ *is* not *general because* $\sqrt{a^2 + b^2}$ *is not necessarily whole.]* [3]

(ii) Suppose (a, b, c) is a Pythagorean triple where a, b, c have highest common factor 1. Carefully explain why (a, b, c) *or* (b, a, c) is part of the general form you found in (d)(i). [3]

(iii) If (a, b, c) is a Pythagorean triple where a, b, c have highest common factor $h > 1$, will they be part of the general form you found in (d)(i)? *[Hint: consider* $a = 9$*.]* [2]

(iv) Hence, find the unique positive whole numbers u, v which satisfy the following diagram (which is very much *not* to scale):



[3]

Now let $\mathcal{E}$ be the ellipse given by $x^2 + y^2 - xy = 1$, as depicted in Figure 2:

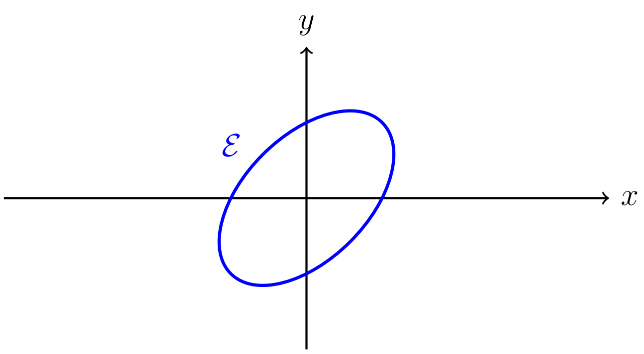


Figure 2

(e) By considering lines that intercept $(1, 0)$, or otherwise, find all pairs of rational numbers (x, y) which satisfy $x^2 + y^2 - xy = 1$. [6]

Question 1 is out of 30 points

2. Look-and-say sequences

In this question, we will discuss *look-and-say sequences*, invented by John Conway. Define a *good number* to mean a positive integer that does not contain ten or more equal digits in a row. A look-and-say sequence is an infinite sequence $L_1, L_2, L_3, \ldots$, where the first term L_1 is a good number, and each term afterwards is defined by this recursive rule:

1. Partition the digits of the current term into non-overlapping groups of consecutive digits, where each group consists of equal digits and neighbouring groups have different digits. We will refer to these groups as *runs*.

2. From left to right, for each run, say the number of digits in that run, followed by the digit of that run. The numbers you say are the digits of the next term.

For example, if $L_1 = 113$, then the runs are 11 and 3, so you read out 113 as "two 1s, one 3", giving you $L_2 = 2113$. Then the runs of 2113 are 2, 11, and 3, so you read out 2113 as "one 2, two 1s, one 3", giving you $L_3 = 122113$, and so on.

Throughout this problem, you should disregard cases where a term has a run of ten or more digits, such as 1111111111. You may assume without proof that if L_1 is a good number, then all terms L_n are good numbers.

(a) Consider the look-and-say sequence with $L_1 = 1$.

(i) Compute L_7 and L_8. No justification needed. [2]

(ii) Find the last two digits of L_{2026}. No justification needed. [2]

(b) (i) Call a digit in a number *switched* if it is a non-leading digit that is not equal to the digit immediately before it. Given that L_n has d switched digits, write an expression in terms of d for the number of digits in L_{n+1}. No justification needed. [2]

(ii) Briefly explain why all the terms in a look-and-say sequence share the same last digit. [2]

(c) A good number x is called a *fixed point* if setting $L_1 = x$ results in a sequence where all L_n are equal. In this part, you will work towards proving that there only is one fixed point, namely 22.

(i) Using the definition, show that 22 is a fixed point. Explain carefully. [2]

(ii) Suppose x is a fixed point with leading digit d. Explain briefly why the first run of x contains d digits. [1]

(iii) By referencing the statement in (c)(ii), or otherwise, show that any fixed point must have a leading digit of 2. [5]

(iv) By referencing the statements in the previous subparts, or otherwise, show that 22 is the only fixed point. [3]

(d) Returning to the look-and-say sequence with $L_1 = 1$, let D_n be the number of digits in L_n. It turns out that the sequence $D_1, D_2, D_3, \ldots$ grows approximately exponentially, with the ratio D_{n+1}/D_n approaching λ as n grows; λ is known as *Conway's constant*. We will approximate Conway's constant using probability.

(i) Explain why the only digits that appear in the look-and-say sequence with $L_1 = 1$ are 1, 2, and 3. [2]

(ii) Let n be a large positive integer. In order to make your approximation, you should make the following simplifying assumption. Since L_n only consists of 1, 2, and 3, assume that the digits in L_n are sampled independently at random with a 1/3 probability of each of 1, 2, and 3.

Let r be an approximation of the growth rate D_{n+1}/D_n. Give a value for r, and show how you arrived at this value. Your approximation need not match the official answer if you justify your answer quantitatively. [4]

(iii) The actual value of Conway's constant is $\lambda \approx 1.3036$. Compare your r from the previous subpart to λ. If $r > \lambda$, explain one way in which the simplifying assumption may have led you to overestimate λ. If $r < \lambda$, explain one way in which the simplifying assumption may have led you to underestimate λ. [2]

(iv) It turns out for any good number $L_1 \neq 22$, the sequence $D_1, D_2, D_3, \ldots$ grows exponentially at the same rate λ. If a look-and-say sequence with a different L_1 consists only of the digits 1, 2, and 3, then the simplifying assumption still applies, and r will be the same as before.

Explain why even if a look-and-say sequence contains digits other than 1, 2, and 3, you can still justifiably use the simplifying assumption to produce the same r. [3]

Question 2 is out of 30 points

3. Confused coats

Carol has a collection of six coats labelled A, B, C, D, E, F, and six coat hooks labelled 1, 2, 3, 4, 5, 6. Currently the coats are in *starting order*: coat A is on hook 1, coat B is on hook 2, and so on. The starting order *reads* from left to right as $ABCDEF$.

A *swap* is a pair of distinct numbers written $(m\ n)$ indicating that we should swap the coats on hooks m and n. For example, if we perform (2 5) from starting order, then the coats read $AECDBF$ from left to right.

(a) Steve performs the swap $(m\ n)$ on starting order. Write down the swap that Carol should perform to return the coats to starting order. [1]

We can perform multiple swaps in a row by performing the left-most swap first. For example, (1 2)(2 3) denotes performing the swap (1 2) first, followed by the swap (2 3).

(b) (i) Perform (1 3)(2 4)(3 5) on starting order. What do the coats read from left to right? [1]

(ii) Steve performs the swaps in (b)(i) on starting order. Write down a series of swaps that Carol can perform to return the coats to starting order. [1]

(c) Imagine, for this part only, that Carol in fact has one hundred coats labelled C_1, C_2, $\ldots$, C_{100}, and one hundred coat hooks labelled 1, 2, 3, $\ldots$, 100. For each of the following series of swaps, write down what the coats read from left to right:

(i) $(1\ 2)(1\ 3)(1\ 4)\ldots(1\ 100)$; [1]

(ii) $(1\ 2)(2\ 3)(3\ 4)\ldots(99\ 100)$. [1]

A *cycle* is a series of distinct numbers $(n_1\ n_2\ \ldots\ n_k)$, indicating that move each coat to the next position in the cycle, and move the last coat to the first position. For example, performing (2 5 4) on starting order reads $ADCEBF$ from left to right.

Similarly with swaps, we may perform a series of cycles starting with the left-most cycle first. A series of cycles is called a *permutation*; we say two permutations are *the same* if they produce the same left-to-right reading of coats on the hooks.

(d) (i) Show that any cycle is the same as a series of swaps. [2]

(ii) Hence, deduce that for any permutation σ that Steve applies to the coats, there is some permutation that returns the coats to starting order. [2]

We call this permutation its *inverse* and denote it by σ^{-1}. You may use, without proof, the fact that it is unique (that is, if σ_1^{-1} and σ_2^{-1} both return the coats to starting order after Steve applies σ, then σ_1^{-1} is the same as σ_2^{-1}).

(iii) Using (d)(i), or otherwise, find a series of *exactly four* swaps which is the same as the permutation $(2\ 6\ 4\ 5)(2\ 4\ 5)(4\ 1\ 3\ 5)$. *[Hint: it is useful to note that, for example, the cycle $(m\ n\ p\ q)$ is the same as the cycle $(n\ p\ q\ m)$.]* [2]

(iv) How many different permutations are there? Briefly explain. [2]

Let σ be a permutation and write σ^k to denote applying σ on the coats k times in a row (if $k = 0$, we do nothing). The *order* of σ is the least $k > 0$ such that σ^k is the same as doing nothing, or put differently, applying σ^k on starting order reads $ABCDEF$.

(e) Find the orders of the following permutations:

(i) $\sigma_1 = (2\ 3\ 5)(1\ 4\ 6)$;

(ii) $\sigma_2 = (2\ 3\ 5)(1\ 4)$;

(iii) $\sigma_3 = (2\ 3\ 5)(5\ 6)$. [2]

We also say two cycles $(a_1\ a_2\ \dots\ a_k)$ and $(b_1\ b_2\ \dots\ b_\ell)$ are *disjoint* if $a_i \neq b_j$ for all i, j.

(f) (i) By following one of the coats, show that

$$(a_1\ a_2\ \dots\ a_k)(b_1\ b_2\ \dots\ b_\ell) \qquad \text{and} \qquad (b_1\ b_2\ \dots\ b_\ell)(a_1\ a_2\ \dots\ a_k)$$

are the same permutation if the two cycles are disjoint. [2]

(ii) Does the result in (g)(i) hold even if the two permutations are not disjoint? Briefly justify your answer. [1]

(iii) Let σ be a permutation. By considering the permutations

$$\sigma^0,\ \sigma^1,\ \sigma^2,\ \sigma^3,\ \dots$$

deduce that there is some $k > 0$ such that σ^k keeps coat A on hook 1. [3]

It turns out that, using (f)(iii), one can show that *any* permutation is the same as a series of disjoint cycles. You may use this result in part (g) without proof.

(g) (i) How many different permutations are there with order 6? [2]

(ii) How many different permutations are there with order 3? [3]

(iii) How many different permutations are there with order 2? [4]

Question 3 is out of 30 points