Oxford Mathematics Team Challenge
Lock-in Round Question Booklet

SAMPLE SET

Instructions

1. Do not open the paper until the invigilator tells you to do so.

2. Format. This round contains three questions, but you should submit answers to only **two questions**.

   For your answers, write them on sheets of paper and mark the question-part you are answering on the margin. You must also clearly write your Team ID at the top of *each* sheet of paper. Do *not* write your team name. You will be provided treasury tags to fasten sheets of paper together according to which question they answer.

   The questions are long-answer, so you may be required to give detailed explanations, brief descriptions, or mathematical working. The questions may indicate the level of depth you should offer, but you should always exercise your judgement in giving an appropriate level of depth to your answer.

3. Time limit. 60 minutes. You may not write anything on any paper you will submit after the allotted time has expired.

4. No calculators, squared paper or measuring instruments. Lined paper and blank paper for rough working is allowed. You may use a pen or pencil to preference. Other mediums for working (e.g., digital devices, whiteboards, thingamabobs) are strictly forbidden.

   The points for each question are in the bottom right of the cells in the Answer Sheet, as well as at the end of the questions in the Question Booklet, and are marked in [square brackets].

5. Scoring rules. Each question is out of 30 points, so the paper is out of 60 points. If you submit answers to all three questions, we will take your two lowest-scoring answers as your official submission.

6. Don't expect to complete the whole paper in the time! The later parts are worth more marks but are generally harder, and they may build up on previous parts of the question.

7. You are also encouraged to think deeply, rather than to guess.

8. Good luck, and enjoy! ☺

1. Circle packing

In this question, we begin with the circle packing problem. In the circle packing problem, we are given a shape and have to place the circles without overlap in the interior of the shape (that is, we have to *pack* them). The goal of the problem is to maximise the radius of the circles.

(a) Figure 1 shows a packing of six circles in a larger circle of radius 3.

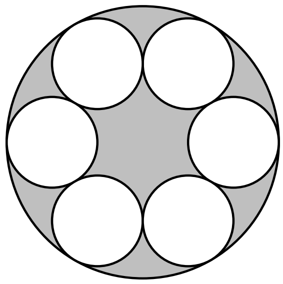

Figure 1

Carefully find the radius of the smaller circles. [3]

The packing of the six circles in Figure 1 is *optimal*, meaning that there is no way to fit six congruent circles of a larger radius into a circle of radius 3.

(b) Construct an optimal packing of seven circles in a circle of radius 3. Briefly explain why it is optimal. [2]

Given two packings of n circles inside a shape, we say that these packings are *equivalent* if we can transform one packing into the other by a series of transformations which slide the circles or rotate the whole packing.

(c) Are all optimal packings of six circles in a circle equivalent? Explain your answer. [3]

(d) An inconspicuous aside: Figure 2 depicts a triangle with side lengths 13, 13 and 10.

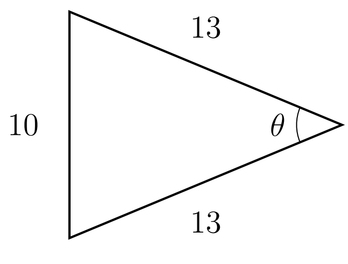

Figure 2

Show that θ, the angle between the sides of length 13, is approximately 45°. Justify your working clearly. [3]

(e) Figure 3 shows an optimal packing of nine circles in a circle of radius 3.

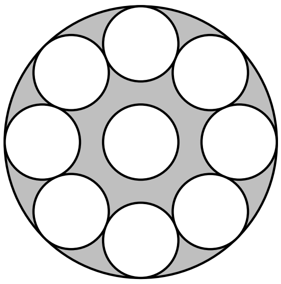

Figure 3

Find the radius r of the smaller circles in terms of $\sin(22.5°)$. [5]

(f) Now suppose the central circle in Figure 3 was enlarged so it is just touching the surrounding circles. Let the central circle have radius R. Estimate the values of r and R. *[Hint: use (d) and (e).]* [5]

(g) Hence, or otherwise, verify that the ratio of the areas between the central circle and the smaller circles equals 2.6 when rounded to the nearest tenth. [2]

We lastly explore ways to pack circles into the (infinite) plane. Sally suggests a *square packing* of the plane, where the centres of the circles form a square grid, whereas Helena suggests a *hexagonal packing* of the plane. Their strategies are depicted in Figure 4:

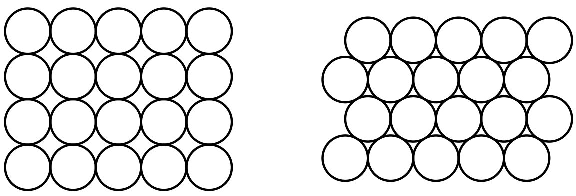

Figure 4

The *density* of a packing on the plane is the percentage of the plane covered by circles.

(h) Determine the density of Sally's packing. [1]

(i) Determine the density of Helena's packing. Hence show that Helena's packing is more dense than Sally's. [2]

(j) It turns out that Helena's packing has the highest possible density for an infinite circle packing on the plane. What are the possible densities of an infinite circle packing on the plane? Justify your answer carefully. [5]

Question 1 is out of 30 points

2. Integer partitions

In this question, we will explore *partition theory*. A *partition* of a positive integer n is a way of writing n as the sum of positive integers (called the *parts*), irrespective of the order of the sum. For example, $5+2+1$ is a partition of 8; this is the same partition as $2+1+5$, however a different partition of 8 is e.g. $3+3+2$. A sum with only one part counts as a partition, e.g. 5 is a partition of 5.

(a) Write down all the partitions of 4. [1]

(b) (i) A partition is called *distinct* if no part is repeated in the sum. For example, $5+2+1$ is a distinct partition of 8, whereas $3+3+2$ is not.

Write down all distinct partitions of 7. [2]

(ii) A partition is called *odd* if it only contains odd parts. For example, $5+3$ is an odd partition of 8, whereas $5+2+1$ is not.

Write down all odd partitions of 7. [2]

(c) For a positive integer n, let $p(n)$ be the number of partitions of n. By convention we also say $p(0) = 1$.

(i) Calculate $p(5)$. [1]

(ii) Explain why $p(n+1) > p(n)$ for all $n \geq 1$. [2]

(iii) We also write $p_d(n)$ as the number of distinct partitions of n, and $p_o(n)$ as the number of odd partitions of n. Similarly, $p_d(0) = p_o(0) = 1$.

Calculate $p_d(5)$ and $p_o(5)$. [2]

We can draw out a partition using a Ferrer diagram, which represents the partition as a collection of dots, descending in size of the rows. Figure 5 shows two examples:

$6+3+3+2+1$ $\quad$ $5+4+3+1+1+1$

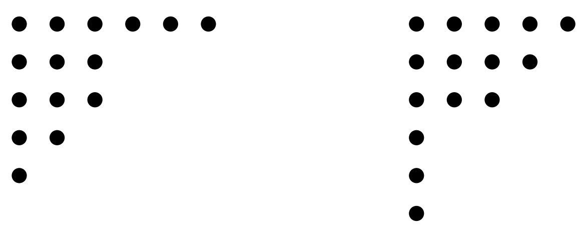

Figure 5

For any partition, we can create its *conjugate* by reflecting the Ferrer diagram over the diagonal. For example, the partitions in Figure 5 are conjugates of each other. We also say that a partition is *self-conjugate* if its conjugate is itself.

(d) (i) Give an example of a self-conjugate partition of 6. [1]

(ii) Give an example of a self-conjugate partition of 7. [1]

(iii) Explain why the number of partitions with six parts is the same as the number of partitions with the largest part equal to 6. [4]

(iv) Explain why the number of partitions of n into distinct odd parts is the same number of partitions of n into self-conjugate parts. [4]

(e) Let's return to $p_o(n)$ and $p_d(n)$.

(i) Consider the infinite products

$$A = (1+x)(1+x^2)(1+x^3)\cdots$$
$$B = (1+x+x^2+\cdots)(1+x^3+x^6+\cdots)(1+x^5+x^{10}+\cdots)\cdots$$

Explain and justify how the coefficients of x^n the infinite products relate to $p_d(n)$ and $p_o(n)$. [5]

(ii) Show that $p_d(n) = p_o(n)$. *[Hint: consider* $\dfrac{1-x^{2k}}{1-x^k}$ *for each positive integer k.]* [5]

Question 2 is out of 30 points

3. Random tic-tac-toe

Two robots, Xeep and Obot, play a game of *tic-tac-toe* (a.k.a. noughts and crosses) on a 3×3 grid. Xeep, who goes first, marks its squares with Xs, and Obot marks with Os. They take turns choosing a random unfilled square to mark. Each robot is equally likely to choose any one of the unfilled squares on its turn. A robot wins by being the first to achieve a 3-in-a-row of their own symbols, either horizontally, vertically, or diagonally. If the grid gets filled completely with no such 3-in-a-row, then the game ends in a draw.

We give the robots additional instructions so that even if a robot has won before the grid is completely filled, the robots will continue to make random moves until the grid is completely filled. We shall call the final completely-filled grid of this game the *end grid*. Note that every possible end grid has 5 Xs and 4 Os.

The first three parts lead to finding the probability of a draw.

(a) Briefly describe how you can determine whether or not a game of tic-tac-toe ended in a draw if you are provided the end grid of the game. [2]

(b) Given an end grid of a game that didn't end in a draw, is it always possible to determine who the winner was? Briefly explain. [3]

(c) There are exactly 126 possible end grids. What is the probability that a game of random tic-tac-toe ends in a draw? [7]

The last four parts lead to finding the probabilities of each of the robots winning.

(d) How many end grids contain a 3-in-a-row of Os, but not a 3-in-a-row of Xs? [4]

(e) How many end grids contain *both* a 3-in-a-row of Os and a 3-in-a-row of Xs? [4]

(f) Consider a variant of tic-tac-toe called *ric-rac-roe* which takes place on the partially-shaded board as shown in Figure 6. In this variant, Xeep can only make moves in the unshaded squares, and Obot can only make moves in the shaded squares. Both robots are still equally likely to choose any one of its legal moves for each turn.

What is the probability that Xeep wins in ric-rac-roe? *[Hint: consider the probability that Obot can achieve a 3-in-a-row in three moves.]* [6]

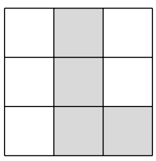

Figure 6

(g) What is the probability that Xeep wins a game of random tic-tac-toe? What is the probability of Obot winning? Express your answers as simplified fractions. [4]

Question 3 is out of 30 points