Oxford Mathematics Team Challenge 2026

Maps Round Solutions

		C5 11				**A8** 38		
	C6 61	**C3** 5	**C1** 12		**A3** 64	**A5** 12	**A7** 12	
C7 3	**C4** 3		**R7** 80	**R4** 1332	**A1** 72	**A2** 18	**A4** 63	**A6** 39
	C2 6	**R8** 60	**R5** 21	**R1** 22		**Y3** 121		
		R6 3	**R2** 24	free	**Y2** 12	**Y6** 125		
		R3 8		**Y1** 28	**Y5** 20	**Y8** 24	**B2** 48	
G6 24	**G4** 50	**G2** 16	**G1** 6	**Y4** 13	**Y7** 72		**B4** 20	**B7** 4
	G7 66	**G5** 4852	**G3** 5		**B1** 512	**B3** 16	**B6** 15	
		G8 3201				**B5** 3		

Red Zone (R)

R1. A positive integer is called *portly* if the sum of its digits is equal to the product of its digits. What is the sum of all two-digit portly numbers? *[Hint: expand* $(A-1)(B-1)$*.]*

SOLUTION

Let $n = 10A + B$ be a portly number where A and B are digits. Then we want $A + B = AB$, so $(A-1)(B-1) = 1$. As A, B are whole numbers, either $A - 1 = 1$ and $B - 1 = 1$ or $A - 1 = -1$ and $B - 1 = -1$, so either $(A, B) = (2, 2)$ or $(A, B) = (0, 0)$; we can't use the latter solution as it's not really a two-digit number. So there is only one two-digit portly number, 22, hence the sum is $\boxed{22}$.

R2. When the quadratic equation

$$x^2 + 2Bx + C = 0$$

has exactly one root at $x = \alpha$, what is the value of $-24C/\alpha B$?

SOLUTION

When there is exactly one root, the discriminant $(2B)^2 - 4 \cdot 1 \cdot C = 0$, so $4B^2 - 4C = 0$, so $C/B = B$ (we want this form for the question at hand). By completing the square or using the quadratic formula, we can see that $\alpha = -B$, so $-24C/\alpha B = -24 \cdot B \cdot -1/B = \boxed{24}$.

R3. In the equation

$$(x^2 + x + 1)(y^2 - y + 1) = 3,$$

how many integer solutions are there?

SOLUTION

3 is prime, so one of the brackets equals ± 1 and the other ± 3. First suppose that

$$x^2 + x + 1 = 1 \quad \text{and} \quad y^2 - y + 1 = 3$$

Then $x = 0, -1$ and $y = -1, 2$; we can take any of these combinations so we have four solutions

$$(x, y) = (0, 1),\ (0, -2),\ (-1, 1),\ (-2, -1)$$

Alternatively, if $x^2 + x + 1 = 3$ and $y^2 - y + 1 = 1$ then $x = 1, -2$ and $y = 0, 1$ so we get the four solutions

$$(x, y) = (1, 0),\ (1, 1),\ (-2, 0),\ (-2, 1)$$

For the other two cases, either $x^2 + x + 1 = -3$ or $y^2 - y + 1 = -3$, but neither of these equations have solutions (e.g. because both of their discriminants equal -3) so there are no solutions to be found here. Thus in total there are $\boxed{8}$ solutions.

INVESTIGATION

Sketch the graph of $(x^2 + x + 1)(y^2 - y + 1) = 3$.

R4. What is the sum of all three-digit portly numbers? *[For a definition of portly number, see R1.]*

SOLUTION

Let the digits of the number be A, B and C. Note that if any of them are 0, the product $A \times B \times C$ is 0, meaning all of them are 0 (but then it's not a three-digit number!).

Suppose $A \leq B \leq C$. Then $ABC = A + B + C \leq 3C$, so $AB \leq 3$. As they are digits, we can check cases:

(I) $(A, B) = (1, 1)$ gives $C = 2 + C$, leading to no solution.

(II) $(A, B) = (1, 2)$ gives $2C = 3 + C$, giving $z = 3$.

(III) $(A, B) = (1, 3)$ gives $3C = 4 + C$, which would give $C = 2$, but we want $B \leq C$, so no solutions here.

So the digits must be 1, 2 and 3 in some order; so there are six portly numbers – 123, 132, 213, 231, 312, 321 – whose sum is $\boxed{1332}$.

INVESTIGATION

Find all of the portly numbers.

R5. The integer n satisfies

$$1+\sqrt{n} \leq 1+\sqrt{10+\sqrt{100+\sqrt{1000+\sqrt{10000}}}} \leq 1+\sqrt{n+1}$$

What does n equal?

SOLUTION

Note that $\sqrt{10000} = 100$, so in the innermost surd we really want to estimate $\sqrt{1100}$. It's bounded by

$$30 = \sqrt{900} \leq \sqrt{1100} \leq \sqrt{1600} = 40$$

so the next surd is bounded by $\sqrt{130} \leq \sqrt{100+\sqrt{1100}} \leq \sqrt{140}$. But $\sqrt{140} \leq \sqrt{144}$ and $\sqrt{121} \leq \sqrt{130}$ so our next surd is bounded by $\sqrt{21}$ and $\sqrt{22}$. So $n = \boxed{21}$.

INVESTIGATION

Define the sequence u_n by $u_1 = 1$, and $u_{n+1} = 1+\sqrt{10u_n}$ for $n \geq 1$. What is the behaviour of u_n as n gets very large?

R6. When expanded, what is the sum of the coefficients of $(x^2+x+1)(x^2-x+1)$?

SOLUTION

$(x^2+x+1)(x^2-x+1) = x^4+x^2+1$ when expanded, so the sum of the coefficients is $\boxed{3}$.

R7. What is the value of

$$\frac{1}{\frac{1}{2001}+\frac{1}{2002}+\cdots+\frac{1}{2025}}$$

when rounded down to the nearest integer?

SOLUTION

We have

$$\frac{1}{\frac{1}{2000}+\cdots+\frac{1}{2000}} < \frac{1}{\frac{1}{2001}+\cdots+\frac{1}{2025}} < \frac{1}{\frac{1}{2025}+\cdots+\frac{1}{2025}}$$

The left-hand expression equals 80 and the right-hand expression equals 81, so rounded down the expression is $\boxed{80}$.

R8. 100010001 has four prime factors less than 100. What is their sum?

SOLUTION

Following the factorisation in R6,

$$\begin{aligned} 100010001 &= 100^4 + 100^2 + 1 = (10^4 + 10^2 + 1)(10^4 - 10^2 + 1) \\ &= (10^2 + 10 + 1)(10^2 - 10 + 1)(9901) = 111 \cdot 91 \cdot 9901 \\ &= 3 \cdot 37 \cdot 7 \cdot 13 \cdot 9901 \end{aligned}$$

Since we are told that there are only four prime factors less than 100, we have found all of them, and their sum is $3+37+7+13 = \boxed{60}$.

INVESTIGATION

Factorise 100020001 and 123454321. Use one of these factorisations to find whole numbers a and b such that $a^2 + 10b^2 = 135797531$.

Yellow Zone (Y)

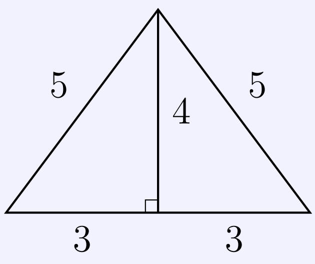
Y1. We say n is a flavourful number if n is an integer such that the function $f(n) = n/(n-7)$ is an integer. What is the sum of all flavourful numbers?

SOLUTION

$\frac{n}{n-7} = 1 - \frac{7}{n-7}$, so to be an integer we need $n-7$ to be a factor of 7. As 7 is a prime, its only factors are ± 7 and ± 1, so our possible values of n are 14, 8, 6 and 0. So the sum of all flavourful numbers is $\boxed{28}$.

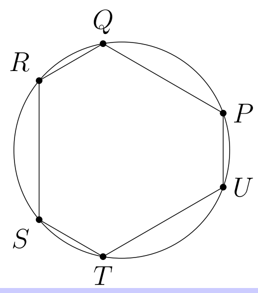
Y2. A triangle has side-lengths 5, 5 and 6. What is its area?

SOLUTION



We can split the triangle into two 3-4-5 triangles. These have area 6. So the area of the original triangle is $6 + 6 = \boxed{12}$.

Y3. Consider the hexagon $PQRSTU$ where P, Q, R, S, T, and U all lie on a circle.



$QR = ST = UP = 3$, and $PQ = RS = TU = 6$. When fully simplified, the area of $PQRSTU$ is $c\sqrt{3}/d$. What is $c + d$?

SOLUTION

Note that every angle in the hexagon is identical and must be 120°. Extend SR, PQ, and TU on both sides. Let the points of intersection points of SR and PQ be A, SR and TU be B, and PQ and TU be C. $\angle$ARQ = $\angle$AQR = $\angle$RAQ = 60° which means that $\triangle$ARQ is an equilateral triangle. Symmetrically, so is $\triangle$BST and $\triangle$CPU. All angles in the $\triangle$ABC are 60°, so $\triangle$ABC is an equilateral triangle. [Area of PQRSTU] = [Area of ABC] - [Area of ARQ] - [Area of BST] - [Area of CPU] $= \frac{\sqrt{3}}{4}12^2 - 3\frac{\sqrt{3}}{4}3^2$, which simplifies to $\frac{117\sqrt{3}}{4}$. So $c + d = \boxed{121}$.

Y4. Shaun tiles the plane with regular n-sided polygons which each have a perimeter of 1 cm. What is the sum of the possible values of n that Shaun could have chosen?

SOLUTION

The interior angle of a regular n-sided polygon is $\frac{180(n-2)^\circ}{n}$. As we want to tile the plane with this n-sided polygon, we need to have polygons fit vertex to vertex, thus we must have $\frac{180(n-2)}{n}^\circ$ be a factor of 360°. This is now the same idea as in Y1, and reduces to $n - 2$ being a factor of 4, so $n - 2 = 1, 2, 4$. This gives us $n = 3, 4, 6$, so the total sum of possible n is $\boxed{13}$.

INVESTIGATION

Draw out tilings with these values of n.

Y5. Nikky has a regular polygon lying on a table. When she rotates it 54° clockwise about its centre, she notices it covers the precisely same spots on the table as before. What is the smallest number of sides Nikky's polygon could have?

SOLUTION

We need the angle on the centre to be a factor of 54. The angle at the centre of a n-sided polygon is $360/n^\circ$. Now the factors of 54 are 54, 27, 18, 9, 6, 3, 2, 1. Solving for $360/n^\circ$ equal to these values, we want the smallest value of n (where it is an integer!) which gives $n = \boxed{20}$.

Y6. Six points A, B, C, D, E and F lie on (the circumference of) a circle; Stewart measures that $AB = BC = CD = 5$ cm and $DE = EF = FA = 4$ cm. When fully simplified, the area of the hexagon $ABCDEF$ equals $a\sqrt{3}/b$ cm^2. What is $a + b$?

SOLUTION

Rearranging the sides (by slicing into triangles with last vertex at the centre) we can turn the hexagon into the hexagon as in Y3 (but with different side-lengths) This does not change the total area as the area of the slices did not change. Extending it to an equilateral triangle with side-length 13 by extending each of BC, DE and FA we see that the area of this large triangle is $\frac{\sqrt{3}}{4}13^2$; removing the 3 smaller triangles removes $3\frac{\sqrt{3}}{4}4^2$, which simplifies to $121\frac{\sqrt{3}}{4}$. So $a + b = \boxed{125}$.

INVESTIGATION

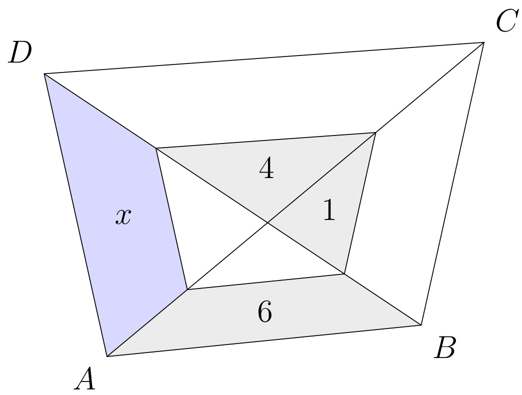
Is the area still $121\sqrt{3}/4$ even if the points A, B, C, D, E, F don't lie on a circle?

Y7. Let ABC be a triangle. The points D and E lie on AC and AB respectively. Let BD and CE intersect at P. Given the areas $PBE = 10$, $PCD = 15$, and $PEAD = 17$, what is the area of ABC?

SOLUTION

Let x be the area of PDE and y the area of PBC. The ratio $x : 15$ is the same as the ratio $10 : y$ as 10 and y have the same height, x and 15 have the same height, and the ratios of the bases are the same; so $y = 150/x$. Now considering the triangles with base AD and AC, we see that $\frac{32}{10+(150/x)} = \frac{17-x}{15+x}$. Solving gives $x = 5$, so the area is $17 + 10 + 15 + 30 = \boxed{72}$.

Y8. A convex quadrilateral $ABCD$ has its diagonals joined, and the quadrilateral within is similar to $ABCD$. Given the shaded-in regions have areas as labelled, what is the area of the shaded region labelled 'x'?



SOLUTION

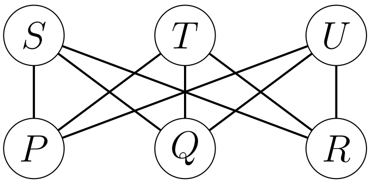
Call the intersection point of AC and BD point P. The ratio of the area of triangle PAD to the area of triangle PAB is the same as 4 to 1 for general length and area reasons (compare to Y7). Similarly, the smaller triangles inside PAD and PAB have the same area ratio of 4 to 1, so we must have $x : 6 = 4 : 1$. Hence $x = \boxed{24}$.

INVESTIGATION

Let the smaller quadrilateral have points A', B', C' and D'. Suppose the scaling factor of $A'B'C'D'$ to $ABCD$ is 2. What is the area of $ABCD$ minus the area of $A'B'C'D'$?

Amber Zone (A)

A1. The following diagram depicts six villages in East Oxfordshire, P, Q, R, S, T, U, connected by nine roads directly between villages. Raph starts at one of the villages, then travels along four roads. He ends up in the village he started in and altogether visited four villages. How many distinct ways can Raph do this?



SOLUTION

There are 6 vertices to choose to start at, 3 places that he can go to in the first trip, 2 in the second, 2 in the third. In the final chance he must return to the first vertex. These are all independent and there is no double counting, so we get $6 \cdot 3 \cdot 2 \cdot 2 = \boxed{72}$.

INVESTIGATION

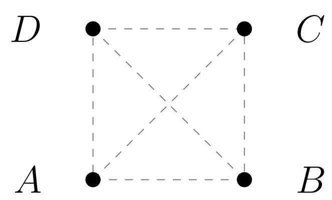
Generalise the result in A1: if we now have 2 rows of n villages connected with roads similarly to A1. Raph now travels along $2m$ roads. He ends up in the village he started in and altogether visited $2m$ villages. How many distinct ways can Raph do this?

A2. Two triangles are placed on top of each other; their overlapping region forms a polygon with n sides. What is the sum of possible values of n?

SOLUTION

All $3 \leq n \leq 6$ are possible. $n = 3$ is possible when two identical triangles are placed on each other (or one is placed in the other); $n = 6$ is possible with a hexagram. By carefully transforming between the $n = 3$ and $n = 6$ case, we can reach $n = 4$ and $n = 5$. On the other hand, we can't have $n > 6$ as there will be at most $3 + 3 = 6$ non-parallel sides, and the intersection of the triangles must be convex (and we can't have $n < 3$). So the sum is $3+4+5+6 = \boxed{18}$.

A3. West Oxfordshire contains the four villages Alvescot, Broadwell, Combe and Ducklington. A *plan for West Oxfordshire* consists of connecting each pair of villages by either a road or a river. How many distinct plans for West Oxfordshire are there?



SOLUTION

There are $\binom{4}{2} = 6$ distinct pairs of villages. For each one, we have an independent choice of whether we have a road between them or a river; multiplying these together by the product rule of counting, we have $2^6 = \boxed{64}$ distinct plans for West Oxfordshire.

A4. A convex pentagon and a convex hexagon are placed on top of each other; their overlapping region forms a polygon with n sides. What is the sum of possible values of n? *[Neither are necessarily regular polygons.]*

SOLUTION

The minimum is 3 (overlap a corner and a side) while the maximum is $5 + 6 = 11$, which is achievable by aligning corners and edges.

The intermediates are all achievable by just aligning some corners together. So the answer is $3 + 4 + \cdots + 11 = \boxed{63}$.

INVESTIGATION

Generalise the result in A4: if a convex p-sided polygon and a convex q-sided polygon are placed on top of each other, their overlapping region forms a polygon with n sides. What are the possible n?

Does the result still hold if one of the shapes isn't convex?

A5. In West Oxfordshire, cyclists can only travel along roads, and boatsmen along rivers. How many plans for West Oxfordshire allow both cyclists and boatsmen to reach any of the villages? *[For a definition of a plan for West Oxfordshire, see A3.]*

SOLUTION

The only way this is possible is if there are three roads and three rivers, since you can't connect four cities with just two roads or rivers (and there's six roads and rivers altogether).

Focussing on the roads, the roads must be connected head-to-tail (not looping), otherwise we would have one village with three roads which means no rivers can reach it.

We have 6 ways to choose one path to make a road, and then 4 choices to extend that road along a connected path. The last road is forced in order to avoid a loop, so there are 24 ways of choosing roads this way, however this method of counting roads double counted (we get the same choice of roads by choosing in reverse) so there are actually $\boxed{12}$ such ways of choosing roads.

Quite conveniently, a path of roads forces the rivers to form a path too, so we are done.

A6. Draw a straight line on a 2×1 cm^2 piece of paper that splits the paper into two regions, then fold the paper along that line. The outline of the folded shape forms a polygon with k sides. What is the sum of possible values of k?

SOLUTION

Any number from 4 to 9 is possible, and 9 is maximal because you can treat this scenario as having 2 trapeziums that have 3 "free" sides (since 2 of the sides have to be identified with each other), and folding so that the corners are on top of the edges (which is the best you can do to create a shape with lots of edges) you get 9 edges. Trial and error (maybe you want diagrams) shows that everything in between is possible, so its $4 + 5 + \cdots + 9 = \boxed{39}$

A7. With the construction in A6, the two regions overlap each other to form a polygon with ℓ sides. What is the sum of possible values of ℓ?

SOLUTION

The minimum value 3 can be achieved (make a fold where one region is subsumed by the other). We cannot get more than five sides: one of the sides is the folding line (which is shared side of the two overlapping regions). As the overlapping regions are quadrilaterals, there are six other sides, and in particular four of these sides connect with the folding line, however on either side one of the lines will be in the overlapping regions and the other won't (because its angle to the folding line is more obtuse than the other), so we have at most $1 + 2 \times 3 - 2 = 5$ sides. We can now show that 5 sides is possible (compare with the case in A6 with nine sides), and 4 is also possible. Thus the sum of possible ℓ-values is $3 + 4 + 5 = \boxed{12}$.

A8. How many plans for West Oxfordshire allow cyclists (but not necessarily boatsmen) to reach any of the villages? *[For a definition of a plan for West Oxfordshire, see A3.]*

SOLUTION

There are $2^6 = 64$ possible road-river arrrangements, as we have an independent choice for each edge of whether it's a road or a river. If a road-river arrangement isn't road-connecting, then by flipping what edges are roads and rivers we get a road-connecting arrangement, so there are at least 32 road-connecting arrangements. It suffices to check how many road-river arrangements are road-connecting and have a road-connecting flip; we calculated precisely this in A4! So we should add on $12/2 = 6$ to our total, giving us $\boxed{38}$ road-connecting arrangements overall.

Green Zone (G)

G1. Alice, Bob and Charlie say the following:

Alice (1):	Bob is lying.
Bob (2):	Charlie is telling the truth.
Charlie (4):	We are all lying.

For each person who is lying, add their bracketed number to get the *liars' score* of this problem (for example, if Alice and Charlie are lying, the liars' score is $1 + 4 = 5$). What is the liars' score in this case?

SOLUTION

Charlie is lying, so Bob is too, so Alice is telling the truth. The liars' score is $\boxed{6}$.

G2. Felix, Gabby, Harry, Ivy and Jacob say the following:

Felix (1):	My middle name is Sheldon.
Gabby (2):	Between myself, Harry and Ivy, an even number of us are lying.
Harry (4):	There are a square number of liars.
Ivy (8):	$1 + 1 = 2$.
Jacob (16):	Harry is lying.

What is the liars' score? *[See G1. for the definition of liars' score.]*

SOLUTION

Ivy is telling the truth. Look at Gabby: if Gabby is telling the truth then Harry must be too; if Gabby is lying then Harry is telling the truth (else there are 2 liars which makes Gabby's statement true). Either way, then, Harry is telling the truth. So Jacob is lying. Lastly the only way for there to be a square number of liars is for Jacob to be the only liar. So the liars' score is $\boxed{16}$.

G3. Kai, Laurel, Miguel, Nina and Oliver say the following:

Kai (1):	There are exactly three liars.
Laurel (2):	There are at least two liars.
Miguel (4):	There are at least three liars.
Nina (8):	There are at most two liars.
Oliver (16):	There are at most three liars.

What is the liars' score? *[See G1. for the definition of liars' score.]*

SOLUTION

Suppose there are n liars. Which n are possible? If $n \leq 1$, Kai and Miguel are lying, so n can't be at most 1. If $n = 3$, Kai and Nina are the only people who are lying, so there are not enough liars. If $n > 3$, then at least Laurel, Miguel are telling the truth (again, not enough liars). This leaves $n = 2$, in which Kai and Miguel are lying. The liars' score is $\boxed{5}$.

G4. Let $50 \leq k \leq 100$. The following 100 people, Terence and T_2 up to T_{100}, say:

Terence:	2 is a prime number.
T_2:	At least $k\%$ of the people who spoke before me are lying.
$\vdots$	$\vdots$
T_n:	At least $k\%$ of the people who spoke before me are lying.
$\vdots$	$\vdots$
T_{100}:	At least $k\%$ of the people who spoke before me are lying.

If $k = 50$, how many people are lying?

SOLUTION

Terence is telling the truth. Consider the proportion of people who lie before T_n against people who tell the truth before T_n, r_n: if $n = 2m + 1$, then m liars spoke before T_n and m truth-tellers spoke before T_n so $r_n = \frac{1}{2} \geq 50\%$ so T_n is telling the truth. On the other hand, if $n = 2m$, then $m-1$ liars spoke before T_n and m truth-tellers spoke before T_n, so $r_n < 50\%$ hence T_n is lying.

So the even T_n are lying, hence there are $\boxed{50}$ liars.

G5. The following 99 people, P_1 up to P_{99}, say the following:

P_1 (1): There is exactly one liar.
P_2 (2): There are exactly two liars.
$\vdots$ $\vdots$
P_n (n): There are exactly n liars.
$\vdots$ $\vdots$
P_{99} (99): There are exactly 99 liars.

What is the liars' score? *[See G1. for the definition of liars' score.]*

SOLUTION

Exactly 98 people are lying. There can't be 2 truth-tellers as any pair of people can't both be telling the truth; if there are 99 liars then P_{99} is telling the truth (contradiction!), so there must be 98 liars, in which case P_{98} is telling the truth. Hence the score is

$$(1 + 2 + \cdots + 99) - 98 = \boxed{4852}.$$

G6. With the scenario in G4., let k' be the least percentage such that T_{100} is the twelfth person to tell the truth. Given that $k' = AB.\dot{C}\%$ for digits A, B, C, what is $A + B + C$?

SOLUTION

For T_{100} to be the twelfth truth-teller, there must be 11 truth-tellers before T_{100} and 88 liars, so the ratio of liars to truth-tellers must be at least 88/99 which simplifies to 8/9. Furthermore, choosing $k = 8/9$ we see that the truth-tellers are Terence ("T_1"), T_{10}, T_{19}, T_{28}, ... T_{100}; indeed T_{100} is the twelfth truth-teller. So $k' = 8/9$, which in the form $AB.\dot{C}\%$ is $88.\dot{8}\%$, so $A = B = C = 8$, so $A + B + C = \boxed{24}$.

INVESTIGATION

(a) Consider the graph of L against T, given by

$$L = \frac{kT}{1-k}$$

where $0 \leq k < 1$. A robot R_K starts at $(0,0)$ and obeys the following algorithm, starting at $n = 1$:

> Look at the identity of T_n: if they're telling the truth, move by $\binom{0}{1}$; otherwise, move by $\binom{1}{0}$. Then repeat this process with $n+1$.

For $k \geq 1/2$, devise an algorithm with the same outcome as R_k's algorithm with reference to the graph instead of the identities of any of the T's. How would an algorithm for $k < 1/2$ differ?

(b) Suppose x is irrational. Is $x/(1-x)$ necessarily irrational too? If $x/(1-x)$ is irrational, is x necessarily irrational?

The following two investigations are very difficult!

(c) Prove that, if $k = p/q$ with $p, q \in \mathbb{N}$, there are exactly p truth-tellers among every q consecutive people.

(d) Prove that k is rational *if, and only if,* the sequence of truth-tellers and liars is periodic. That is, show that there is some N such that the identity of T_n is the same as T_{n+N} for all n.

G7. With the scenario in G4., if $k = 66.\dot{6}\%$, how many people are lying now?

SOLUTION

Every third person (starting from Terence) is telling the truth, by considering a similar iterative argument as in G4. There are 34 truth-tellers, so $\boxed{66}$ liars.

G8. The following 99 people, Fawkes, R_1 up to R_{97}, and Tooth, say the following:

$$\begin{aligned}
\text{Fawkes } (0)\text{:}\quad & \pi = 4. \\
R_1\ (1)\text{:}\quad & \text{Fawkes is lying if and only if } R_2 \text{ is.} \\
R_2\ (2)\text{:}\quad & R_1 \text{ is lying if and only if } R_3 \text{ is.} \\
\vdots\quad & \quad\vdots \\
R_n\ (n)\text{:}\quad & R_{n-1} \text{ is lying if and only if } R_{n+1} \text{ is.} \\
\vdots\quad & \quad\vdots \\
R_{97}\ (97)\text{:}\quad & R_{96} \text{ is lying if and only if Tooth is.} \\
\text{Tooth } (98)\text{:}\quad & \pi < 4.
\end{aligned}$$

What is the liars' score? *[See G1. for the definition of liars' score.]*

SOLUTION

The truthfulness of R_2 is the opposite of R_1: for if R_1 is true, then R_2 is lying; if R_1 is false, then R_2 is telling the truth. By R_2, R_3 is *always* false (if R_2 is telling the truth, then R_1 is lying so R_3 is lying by the truth of R_2; if R_2 is lying, then R_1 is telling the truth and R_3 is lying by the falsity of R_2).

This pattern repeats (we can think of R_3 as the same as Fawkes but shifted down) to get that the truth-tellers are either R_1, R_4, R_7, ... or R_2, R_5, R_8, ...; this pattern needs to hit R_{95} to line up with Tooth, so the truth-tellers must be $R_2, R_5, R_8 \dots$. Now we need to sum the liars which are Fawkes (0), R_1, R_4, R_7, ... (this sums to 1617 from an arithmetic series); and R_3, R_6, R_9, ... (this sums to 1584 from an arithmetic series). In total, the liars' score is $\boxed{3201}$.

INVESTIGATION

What happens if there is no R_{97}? What if there is no R_{96} nor R_{97}?

Cyan Zone (C)

C1. The *Ackermann function* is defined on non-negative whole numbers m, n as follows:

$$\begin{aligned} \mathrm{A}(0,n) &= n+1 \\ \mathrm{A}(m+1,0) &= \mathrm{A}(m,1) \\ \mathrm{A}(m+1,n+1) &= \mathrm{A}(m,\mathrm{A}(m+1,n)) \end{aligned}$$

Find the value of $\mathrm{A}(1,0)+\mathrm{A}(1,1)+\mathrm{A}(1,2)+\mathrm{A}(1,3)$.

SOLUTION

$\mathrm{A}(1,1) = \mathrm{A}(0,\mathrm{A}(1,0)) = \mathrm{A}(0,\mathrm{A}(0,1)) = \mathrm{A}(0,2) = 3$, $\mathrm{A}(1,2) = \mathrm{A}(0,\mathrm{A}(1,1)) = \mathrm{A}(0,3) = 4$, $\mathrm{A}(1,3) = \mathrm{A}(0,\mathrm{A}(1,2)) = \mathrm{A}(0,4) = 5$. Hence the answer is $3+4+5 = \boxed{12}$.

C2. The Fibonacci numbers are given by

$$F_1 = 1, \quad F_2 = 1 \qquad \text{and} \qquad F_{n+2} = F_{n+1} + F_n \quad \text{for } n > 0.$$

Let p/q be the probability that a randomly-picked Fibonacci number is odd, when fully simplified. What does pq equal?

SOLUTION

Every third Fibonacci is even; as F_1 and F_2 are odd, F_3 is even; so F_4 is odd (it's the sum of an odd and even number); F_5 is also odd as (sum of an even and an odd number). As F_4 and F_5 are both odd, the cycle repeats! So the probability a randomly-picked Fibonacci number being odd is $2/3$, so the answer is $\boxed{6}$.

INVESTIGATION

What is the probability that a Fibonacci number is **(a)** a multiple of 3; **(b)** a multiple of 5; **(c)** a multiple of the kth Fibonacci number F_k?

C3. Find the value of $\mathrm{A}(2,1)$. *[For the definition of* A*, see C1.]*

SOLUTION

From the working in C1, we know that $\mathrm{A}(1,1)=3, \mathrm{A}(1,3)=5$. So $\mathrm{A}(2,1)=\mathrm{A}(1,A(2,0))=\mathrm{A}(1,A(1,1))=\mathrm{A}(1,3)=\boxed{5}$

C4. Define the Fibonacci numbers F_n as we did in C2. When the series

$$F_1x+F_2x^2+F_3x^3+\cdots$$

has a limit for a given value of x, it tends to

$$\frac{x}{a-bx-cx^2}$$

where a,b,c are positive integers. What is $a+b+c$? *[Hint: consider the fact that $F_{n+2}x^n=F_{n+1}x^n+F_nx^n$.]*

SOLUTION

$F_{n+2}x^n=F_{n+1}x^n+F_nx^n$. Summing this from $n=1$ to ∞, we get $S+(\frac{S}{x}-1)=\frac{\frac{S}{x}-1}{x}-1$ where $S=\sum_{n=1}^{\infty}F_nx^n$. Rearranging, $S=\frac{x}{1-x-x^2}$, so $a=b=c=1$; hence their sum is $\boxed{3}$.

C5. By considering $\mathrm{A}(2,i)$ for $i=0,1,2,3$, or otherwise, find the value of $\mathrm{A}(2,4)$. *[For the definition of* A*, see C1.]*

SOLUTION

$\mathrm{A}(1,n)=\mathrm{A}(0,\mathrm{A}(1,n-1))=\mathrm{A}(1,n-1)+1$, so we get that $\mathrm{A}(1,n)=\mathrm{A}(1,1)+n-1=n+2$. Now $\mathrm{A}(2,n)=\mathrm{A}(1,\mathrm{A}(2,n-1))=\mathrm{A}(2,n-1)+2$, so we get $\mathrm{A}(2,n)=\mathrm{A}(2,1)+2(n-1)=2n+3$. So $\mathrm{A}(2,4)=\boxed{11}$

C6. By considering $\mathrm{A}(3,i)$ for $i=0,1,2$, or otherwise, find the value of $\mathrm{A}(3,3)$. *[For the definition of* A*, see C1.]*

SOLUTION

Since $\mathrm{A}(2,n) = 2n+3$, we have

$$\begin{aligned}
\mathrm{A}(3,1) &= \mathrm{A}(2,\mathrm{A}(3,0) = \mathrm{A}(2,\mathrm{A}(2,1)) = \mathrm{A}(2,5) = 13, \\
\mathrm{A}(3,2) &= \mathrm{A}(2,\mathrm{A}(2,3)) = \mathrm{A}(2,13) = 29, \\
\mathrm{A}(3,3) &= \mathrm{A}(2,\mathrm{A}(3,2)) = \mathrm{A}(2,29) = \boxed{61}.
\end{aligned}$$

C7. The series in C4 has a limit for a given value of x if, and only if,

$$|x| < \frac{(d\sqrt{5}-1)}{e}$$

where d and e are positive integers. What is $d+e$? *[Hint: factorise the quadratic $(a - bx - cx^2)$ where a, b, c are the answers in C4.]*

SOLUTION

$1 - x - x^2 = -(x - \frac{1+\sqrt{5}}{2})(x - \frac{1-\sqrt{5}}{2})$. This implies that the series converges when $|x| < \frac{1}{(1+\sqrt{5})/2}, \frac{1}{(\sqrt{5}-1)/2}$, i.e. $|x| < \frac{1}{(1+\sqrt{5})/2} = 2\frac{\sqrt{5}-1}{4} = \frac{\sqrt{5}-1}{2}$. Hence $d = 1, e = 2$, so $d + e = 1 + 2 = \boxed{3}$

Blue Zone (B)

B1. The graph

$$|x| + |y| = 16$$

describes a polygon with an area A. What does A equal?

SOLUTION

The polygon is a square with vertices $(0,16),(16,0),(0,-16)$ and $(-16,0)$. So the side-length is $16\sqrt{2}$, so the area is $(16\sqrt{2})^2 = \boxed{512}$.

B2. Sophie starts with a sphere of radius 1. Cai puts the sphere inside the smallest possible cube that can contain it. Sophie then puts the cube is into the smallest possible sphere that can contain it. The volume of the larger sphere is $k\pi$; what does k^2 equal?

SOLUTION

The cube has side-length 2, and the distance from the the centre of the cube to one of its vertices is $\sqrt{1^2+1^2+1^2} = \sqrt{3}$. Hence the sphere has volume $\frac{4}{3}\pi\sqrt{3}^3 = 4\sqrt{3}\pi$. So $k^2 = \boxed{48}$.

B3. The graph

$$|x+y| + |x-y| = 4$$

describes a polygon with area A. What does A equal?

SOLUTION

The shape is a square with corners $(2,2),(2,-2),(-2,-2),(-2,2)$, so has side-length 4, hence has area $\boxed{16}$.

B4. Starting with a sphere of radius $\frac{\sqrt{3}}{2}$, Cai and Sophie take turns putting the previous object in the smallest cube and sphere that contains it, respectively, in the fashion of B2. Eventually, Cai puts a cube of side-length 243 around a sphere. At this point, how many spheres and cubes are there altogether?

SOLUTION

The first cube has side-length $\sqrt{3}$, so the second sphere has radius $\sqrt{\frac{3}{4}+\frac{3}{4}+\frac{3}{4}} = \frac{3}{2}$, and the second cube then has side-length 3. Continuing this way, we see that the nth cube has side-length $(\sqrt{3})^n$. So the tenth cube has side-length 243, hence there are $10+10 = \boxed{20}$ spheres and cubes all together.

B5. For $\alpha > 1$, the graph

$$|x+\alpha y| + |x-\alpha y| + |\alpha x + y| + |\alpha x - y| = 1$$

describes an octagon. For the particular value $\alpha = \sqrt{a}+b$ (where a, b are integers), this octagon is regular. What is $a+b$?

SOLUTION

Suppose all expressions are positive, so $x + ay \geq 0$, $x - ay \geq 0$, $x + \frac{1}{a}y \geq 0$ and $x + \frac{1}{a}y \geq 0$. Then the equation simplifies to the line $2(\alpha+1)x = 1$, so in this region $x = 1/2(\alpha+1)$. But how big is this region actually? Does it even exist?

The first two inequalities are equivalent to $x \geq \alpha|y|$ and the second two to $x \geq \frac{1}{\alpha}|y|$. As $\alpha > 1$, $\alpha > \frac{1}{\alpha}$ so the "stricter" region is $x \geq \alpha|y|$; this region definitely has area, so the line $2(\alpha+1)x = 1$ is actually a side of the octagon. By looking at the boundary of this region (i.e. when $x = \alpha y$), we get that the points

$$A = \left(\frac{1}{2(\alpha+1)}, \frac{1}{2\alpha(\alpha+1)}\right), \qquad B = \left(\frac{1}{2(\alpha+1)}, \frac{1}{2\alpha(\alpha+1)}\right)$$

are vertices of the octagon. We also note there is symmetry about

$y = x$, so in particular we also have the vertex

$$C = \left(\frac{1}{2\alpha(\alpha+1)}, \frac{1}{2(\alpha+1)} \right)$$

The vertices B and C are the ones next to A (we know this because, e.g. the *only* vertices in the region $x \geq \alpha y$ are A and B), so we can now compute the distance AC and equate it to the distance AB.

Clearly $AB = 1/\alpha(\alpha+1)$. By Pythagoras' Theorem,

$$\begin{aligned} AC &= \sqrt{\left(\frac{1}{2(\alpha+1)} - \frac{1}{2\alpha(\alpha+1)} \right)^2 + \left(\frac{1}{2\alpha(\alpha+1)} - \frac{1}{2(\alpha+1)} \right)^2} \\ &= \sqrt{2 \left(\frac{1}{2(\alpha+1)} - \frac{1}{2\alpha(\alpha+1)} \right)^2} \\ &= \sqrt{2} \left(\frac{1}{2(\alpha+1)} - \frac{1}{2\alpha(\alpha+1)} \right) \end{aligned}$$

Simplifying this and equating it to AB,

$$\frac{(\alpha-1)\sqrt{2}}{2\alpha(\alpha+1)} = \frac{1}{\alpha(\alpha+1)}$$

Multiply up by $\alpha(\alpha+1)$: we get $\alpha = \sqrt{2} + 1$, so $a + b = \boxed{3}$.

INVESTIGATION

Find a graph which describes a regular hexagon.

B6. Dianna starts with the shape described by the graph $|x| + |y| = 16$ as in B1. Scott finds the value r_1 such that the shape described by $|x + y| + |x - y| = r_1$ fits exactly into Dianna's first shape. Dianna then finds the value r_2 such that the shape described by $|x| + |y| = r_2$ fits exactly into Scott's shape.

If they keep repeating this process, the areas of Dianna's shapes minus the areas of Scott's shapes approaches p^a/q where p, q are prime numbers. What is $p + a + q$?

SOLUTION

Dianna's first shape is a square with vertices on the x- and y-axes, while Scott's first shape is a square with sides parallel to the x- and y-axes, with vertices on Dianna's squares. It is easy to see that the area of Scott's first shape is half that of Dianna's first shape, and that the area of Dianna's second shape is half of the area of Scott's first shape, hence a quarter of Dianna's first shape. Similarly, Scott's second shape has area one quarter of Scott's first shape.

Dianna's first shape has area 512, Scott's first shape has area $512/2 = 256$, so the areas of Dianna's shapes minus the areas of Scott's shapes is

$$\begin{aligned} &512\left(1 + \frac{1}{4} + \frac{1}{16} + \cdots\right) - 256\left(1 + \frac{1}{4} + \frac{1}{16} + \cdots\right) \\ &= 256\left(1 + \frac{1}{4} + \frac{1}{16} + \cdots\right) \\ &= 256 \times 4/3 \\ &= 2^{10}/3 \end{aligned}$$

So $p + a + q = 2 + 10 + 3 = \boxed{15}$.

B7. Starting with a cube of side-length 2, Sophie puts the cube into the smallest possible sphere that can contain it and positions this cube to be parallel to the x-, y- and z-axes. The cube cannot slide, so instead Sophie spins it the cube in the sphere about the y-axis. The volume of the region that the cube occupies whilst being spun is $a\pi$; what does a equal?

SOLUTION

While spinning, the cube turns into a cylinder. Consider the face parallel to the x-z-plane. Then after being rotated, it turns into the base of the cylinder. This base now has radius being half the length of the diagonal of the square, which is $\sqrt{2}$. The height of the cylinder is clearly equal to the side-length, which is 2. Hence the volume of the cylinder is $\pi(\sqrt{2})^2(2) = 4\pi$, so $k = \boxed{4}$.