Oxford Mathematics Team Challenge
Scenes Round Solutions

SAMPLE SET

Act	Scene	A	B	C
I	1	6	37	807
	2	898	7	7
	3	1256	204	24
	4	1	35	6
II	1	9	13	10
	2	24	4	20
	3	146	3	8
	4	2877	282	167501

Act I, Scene 1

1A. How many integers a are there such that $x^2 - ax + 50 = 0$ has two distinct integer roots?

SOLUTION

We note that a is the sum of roots, and 50 is the product of the roots. $50 = 2 \cdot 5^2$. The possible integer factorisations of $x^2 - ax + 50$ are when a is the sum of two numbers that multiply to 50, so there are 3 values of a: 51 from $50 \cdot 1$, 27 from $2 \cdot 25$, and 15 from $10 \cdot 5$. But negative values also work! So a can also be -51, -27 and -15; thus there are $\boxed{6}$ values of a giving integer roots.

1B. A *lattice point* in the plane is a point (x, y) where x, y are both integers. How many lattice points lie in the region $x^2 + y^2 \leq 12$?

SOLUTION

Let us first take integers $0 < a \leq b$, with $a^2+b^2 \leq 12$. Then $a < 3$, as $2^2+3^2 = 13$. We can find the solutions $(1, 1), (1, 2), (1, 3), (2, 2)$.

How many solutions does this extend to for x, y? The region is a circle, so we have several symmetries to work with. On the first quadrant we have six points (the (a, b) solutions plus $(3, 1)$ and $(2, 1)$), so across the quadrants we have 24 points. But on the positive x-axis we have three points – $(1, 0), (2, 0), (3, 0)$ – so on the axes we have 12 points, and we also have the origin $(0, 0)$. In total, we have $24 + 12 + 1 = \boxed{37}$ lattice points.

1C. A 3-digit number is called *geometric* if it has 3 distinct digits which, when read from left to right, form a geometric sequence. What is the difference between the largest and smallest geometric numbers?

SOLUTION

One quickly notes that 931 is the largest geometric number, and 124 is the smallest (999, 111 and 100 don't count as they don't have distinct digits). Their difference is $\boxed{807}$.

Act I, Scene 2

2A. What is the third smallest positive integer whose digits sum to 25?

SOLUTION

One quickly spots that having a digit less than or equal to 6 will not suffice. The smallest is 799; the second is 889; the third is $\boxed{898}$.

2B. Each morning, Arav has to decide how to get from his home to his maths class, which is 5 miles due south of his home. He can cycle at 10 mph; alternatively, he can jog at 6 mph to the bus stop a mile due north of his home, then take the bus which arrives every 5 minutes and takes a direct route to the maths class at 20 mph.

If Arav leaves his house at a random time, the probability that the bus will get him to class fater than cycling is p/q, where p/q is fully simplified. What is $p + q$?

SOLUTION

Cycling to class takes 30 minutes. Jogging from his home to the bus stop takes 10 minutes. Taking the bus from the bus stop to his maths class takes 18 minutes. So for the bus to get him to class faster than cycling, Arav would need the bus to arrive within 2 minutes of him getting to the bus stop. As the bus arrives every 5 minutes, and Arav leaves his house at a random time, the probability is thus $\frac{2}{5}$, giving $p + q = \boxed{7}$.

2C. Hannah is standing on the vertex of a cube. For each of the other vertices, one of her friends is standing on it. The sum of the distances from her to each of her friends is $a + b\sqrt{2} + c\sqrt{3}$. What is $a + b + c$?

SOLUTION

The distance to the opposite vertex is $\sqrt{3}$, to the 3 vertices on opposite sides of faces shared with Hannah is $\sqrt{2}$ and the 3 that are connected by edges to Hannah is 1. Thus the sum is $3 + 3\sqrt{2} + \sqrt{3}$, so the sum of the coefficients is $\boxed{7}$.

Act I, Scene 3

3A. Tom is writing a list of letters consisting of O, M, T, C. After writing down 7 letters, he discovers that he has spelled out his name in three consecutive letters precisely once. How many possible sequences of letters could Tom have written?

SOLUTION

We condition on the position of the word TOM in the sequence of 7 letters. If TOM is at the start of the sequence (i.e. the sequence is 'TOM _ _ _ _'), we have $4^4 = 256$ total combinations of other letters. Of these, 8 of these sequences have 'TOM' in the last 4 letters, where the last four letters are either '_ TOM' or 'TOM _' (for these two possibilities, we have four letters we can choose from).

Similarly, the number of possibilities with 'TOM' in the 2nd, 3rd, 4th and 5th position are 252, 256, 252 and 248 respectively. In total, then, there are $\boxed{1256}$ possible sequences Tom could've written.

3B. Alex and Brennan each have positive integers. The highest common factor of their numbers is 12, and their lowest common denominator is 720. What's the smallest possible sum of Alex and Brennan's numbers?

SOLUTION

Write $\text{hcf}(m, n)$ for the highest common factor of m and n, and $\text{lcm}(m, n)$ for the lowest common multiple of m, n.

Let Alex's number be a, and Brennan's be b. For any two numbers, $mn = \text{hcf}(m, n)\,\text{lcm}(m, n)$, so $ab = 12 \cdot 720$. As $\text{hcf}(a, b) = 12$ we can set $a = 12x$ and $b = 12y$ for some x, y with highest common factor 1 to simplify the equation to $xy = 60$. We need to check for solutions where $\text{hcf}(x, y) = 1$:

- $x = 1, y = 60$.
- $x = 3, y = 20$.
- $x = 4, y = 15$.
- $x = 5, y = 12$.

The last gives us the value sum of $x + y$ (and hence $a + b$ as $a + b = 12(x + y)$; so the least value of $a + b$ is $\boxed{204}$.

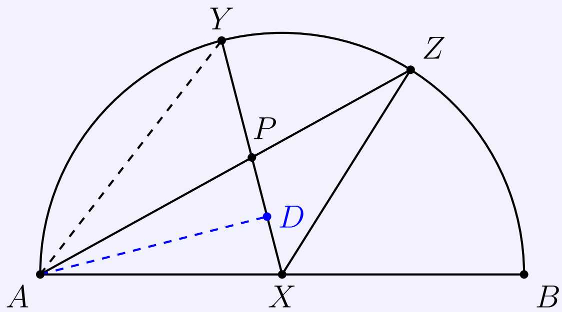
INVESTIGATION

In the solution above, we used the fact that $mn = \text{hcf}(m, n)\,\text{lcm}(m, n)$. Show that $a + b = \min(a, b) + \max(a, b)$, and hence deduce $mn = \text{hcf}(m, n)\,\text{lcm}(m, n)$.

3C. Let AB be the diameter of a semicircle, and let X be the midpoint of A and B. Choose points Y and Z on the arc of the semicircle such that $AY < AZ$. Let P be the intersection of AZ and XY, and suppose that $AP = 4$, $PX = 2$, $PY = 2$, and $PZ = 3$. It follows that $AY = \sqrt{a}$. What does a equal?

SOLUTION

We know that $YX = 4$. But as X is the midpoint of AB which is the diameter of the semicircle, we know that X is indeed the centre of the semicircle. Thus $XA = XY = XB = 4$. Now $XY = XA = 4 = AP$, so XAP is an isosceles triangle. Drawing a perpendicular from A to PX and calling this point D, we know that AP is perpendicular to PX, D bisects PX and thus $DX = 1$. So by Pythagoras $AD = \sqrt{4^2 - 1^2} = \sqrt{15}$ and $AY = \sqrt{AD^2 + YD^2} = \sqrt{15 + 9} = \sqrt{24}$, so $a = \boxed{24}$.



Act I, Scene 4

4A. Bernie writes down the expression

$$1! + 2! + \cdots + 100!$$

What is the tens digit of Bernie's expression?

SOLUTION

For $n \geq 10$, the tens and ones digit of $n!$ will be 0. Hence we only need calculate the tens digit till 9!. This process is simplified if we discard all but the ones and tens digits. We get the sequence $1, 2, 6, 24, 20, 20, 40, 20, 80$. The last 6 together give us no tens digit, while $1 + 2 + 6 + 4$ gives us 1, hence we get the tens digit as $\boxed{1}$.

INVESTIGATION

How many terms of Bernie's expression can we ignore if we want to find the millions digit of Bernie's expression?

4B. Jimbo wants to draw a decagon, but chooses to be a bit chaotic. Each minute, he rolls a standard 6-sided die and draws that many edges until he has drawn all ten edges in which case he stops.

The probability that Jimbo finishes drawing the decagon after exactly three rolls is p/q, when fully simplified. What does $p+q$ equal?

SOLUTION

We can begin to write down Jimbo's first two rolls and count how many ways his first rolls can sum to at least 10.

First two rolls	Next roll could be...	First two rolls	Next roll could be...
$(1,1)$	-	$(2,1)$	-
$(1,2)$	-	$(2,2)$	6
$(1,3)$	6	$(2,3)$	$5,6$
$(1,4)$	$5,6$	$(2,4)$	$4,5,6$
$(1,5)$	$4,5,6$	$(2,5)$	$3,4,5,6$
$(1,6)$	$3,4,5,6$	$(2,6)$	$2,3,4,5,6$

From writing down the cases where Jimbo's first rolls are 1 or 2, we spot a pattern; each time we increase the first roll by 1, we admit an extra possible third roll to each row (up to six possibilities!). So we count that there are

$$(10) + (15) + (21) + (21 - 1 + 6) + (21 - 3 + 2 \times 6) + (21 - 6 + 3 \times 6) = 135$$

total ways. We need to exclude the number of ways where Jimbo is done in the first two rolls, though; these rolls are $(4,6)$, $(5,5)$, $(6,4)$, $(6,5)$, $(6,6)$ and $(5,6)$, which each account for six possible rolls, so we should minus 36 from our count. There are $6^3 = 216$ possible triples of rolls, so overall the probability is $99/216 = 11/24$; so $p+q = \boxed{35}$.

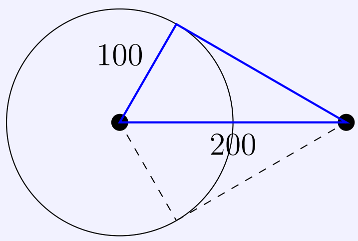
INVESTIGATION

Jimbo is now drawing some regular n-sided polygon. Given he finishes his drawing after 5 rolls, which n-gon(s) was he most likely to have made?

4C. Sam and Owen are stood 200 m apart in a large foggy field. If they ever get within 100 m of each other, they will see each other. Sam and Owen both turn to a random direction and simultaneously begin walking in a straight path. Sam walks at a constant rate of 1 ms^{-1}, and Owen walks at a constant rate of 2 ms^{-1}. The probability that Sam and Owen find each other is $1/p$; what does p equal?

SOLUTION

Because Sam and Owen walk at constant rates, then from Sam's frame of reference, Owen is moving at a constant rate in a random direction. The rate at which Owen moves relative to Sam is unimportant, as we only need to know in what direction Owen needs to move at in order to eventually get within a distance of 100 m of Sam. By drawing a circle of radius 100 m around Sam, we find that the lines intersecting Owen's position that are tangent to the circle form a 60° angle, since the blue right triangle shown below has the side length ratio of a half equilateral triangle.



This means that in Sam's reference frame, as long as Owen moves in a direction within a 60° window, then he and Sam will see each other. This means the probability is $60/360 = 1/6$, so $p = \boxed{6}$.

Act II, Scene 1

1A. What's the sum of the solutions of

$$\log(x-2) + 2\log(x-3) = \log(10x-38)?$$

SOLUTION

Using log properties, one gets the cubic equation $(x-2)(x-3)^2 = 10x - 38$. Simplifying, one gets $x^3 - 8x^2 + 11x + 20 = 0$. We notice that $x = -1$ is a solution, and divide to get $(x+1)(x^2 - 9x + 20)$. The second quadratic factors into $(x-4)(x-5)$.

We note that for $x = -1$, the above logs have negative argument, so we must exclude this solution. The other two have no such problems, hence we get sum of solutions $\boxed{9}$.

1B. The rectangle $ABCD$ has side lengths $AB = 9$, $BC = 6$. Point P is chosen inside the rectangle such that the areas of $ABPD$, BCP and CDP are all equal. The length AP equals $\sqrt{k}$; what does k equal?

SOLUTION

As the areas of BCP and CDP are equal, we know P must lie on the diagonal AC. Now we also know the ratio of areas $ABPD : BCP + CDP = AP : PC = 1 : 2$. And thus, the length of $AP = \frac{1}{3}\sqrt{6^2 + 9^2} = \sqrt{13}$, so $k = \boxed{13}$.

1C. An n-sided regular polygon has an internal angle just less than 145°. What is the largest value of n?

SOLUTION

The interior angle of an n-sided regular polygon is $\frac{n-2}{n} \cdot 180°$. We solve:

$$\frac{x-2}{x} \cdot 180 = 45 \implies 35x = 360 \implies x = \frac{360}{35}$$

Noting that the size of interior angle increases with an increase in n, and $10 < \frac{360}{35} < 11$, we get that the largest value of n is $\boxed{10}$.

Act II, Scene 2

2A. Abigail the Albatross starts on the top-left square of a 50×25 grid. Every move, she can either walk to the square directly below, walk to the square directly to the right, or fly to any square on the board. What is the fewest number of times Abigail needs to fly to visit all the squares on the board?

SOLUTION

Label the squares by (x, y), with $(1, 1)$ denoting top left. Consider the squares on the diagonal $(25, 1), (24, 2), \ldots, (1, 25)$. If at any point we are at one of these, none of the others can be reached by just moving down or right. Thus we need at least 24 flights.

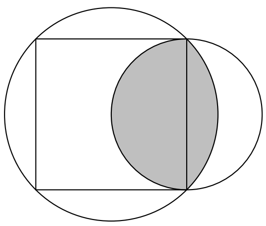
We can check that Abigail can visit each square with 24 flights as well – by moving right on each row, and then flying to the leftmost point of the next row.

Thus the minimum number of required flights is $\boxed{24}$.

INVESTIGATION

Now imagine Abigail is in a $100 \times 50 \times 25$ cuboid grid with $(0, 0, 0)$ the corner Abigail starts on, and can move 1 unit along the positive x-, y- or z-axis or take flight as before. How many times does Abigail need to fly now?

2B. Jimbo is drawing a coffee bean. He first draws a square of side length 2, draws a circle whose diameter is a side of the square, and then draws another circle circumscribing the square. The intersecting region of the two circles is his coffee bean. The area of his coffee bean is $a\pi + b$. What does $5a + b$ equal?



SOLUTION

First, we note that the radius of the bigger circle has a radius of $\sqrt{2}$. To calculate the area of the coffee bean, we split the shaded area into two regions. The first region, given by the area of half the small circle, has an area of $\frac{1}{2}\pi$. The second region has $\frac{1}{4}$ the area of the bigger circle minus the area of the square giving it an area of $\frac{1}{2}\pi - 1$. Thus, the total area of his coffee bean is $\pi - 1$. Thus $5a + b = \boxed{4}$.

2C. The sequence a_n has initial values $a_0 = 1$ and $a_1 = 3$. For $n \geq 2$,

$$a_n = 6a_{n-1} - 9a_{n-2}$$

If $a_{20} = 2^\ell 3^m 5^n 7^o$, what does $\ell + m + n + o$ equal?

SOLUTION

We note the first few values of the sequence to be $1, 3, 9, 27$.

Recursively, we argue that $a_n = 3^n$. If we have $a_{n-2} = 3^{n-2}$ and $a_{n-1} = 3^{n-1}$, then

$$a_n = 6 \times 3^{n-1} - 9 \times 3^{n-2} = 2 \times 3^n - 3^n = 3^n$$

As our sequence starts on powers of 3, the pattern continues so $a_n = 3^n$. Thus $a_{20} = 3^{20}$, which sets $m = 20$ and the rest 0. Thus $\ell + m + n + o = \boxed{20}$.

Act II, Scene 3

3A. Let S be the set containing the terms in the arithmetic sequence $1, 8, 15, \ldots,$ that are less than 2000. Mr Healy takes a subset of S with n elements. No matter what subset of size n Mr Healy chose, n is just large enough such that Mr Dobby can always find two elements in Mr Healy's subset which sum to 2025.

What does n equal?

SOLUTION

The nth term of the arithmetic sequence is given by the formula $a_n = 1 + 7(n-1)$.

Suppose we group the members of the arithmetic sequence as follows: $\{1\}$, $\{8\}$, $\{15\}$, $\{22\}$, $\{29, 1996\}$, $\{36, 1989\}$, $\ldots$, $\{1+7n, 2024-7n\}$, $\ldots$, $\{1009, 1016\}$. We have constructed 145 groups in this grouping.

Now if there are two numbers in the same group within the n numbers we picked from S, there will necessarily be a pair of numbers that add up to 2025. If $n = 145$, then Mr Healy could've picked one number from each of the 145 groups, which means the elements Mr Healy and Mr Dobby choose cannot sum to 2025. With $n = 146$ though, Mr Healy's subset must have one of the pairs, which Mr Dobby can then pick.

Thus n equals $\boxed{146}$.

3B. Zach spends £1 to play a game. Each round, a coin is flipped. If it comes up as heads, he doubles his money; if it comes up as tails, he loses all of his money. He can withdraw at any point and claim all of the money he has won.

Zach maximises the probability that he wins at least £1 back; this probability is p/q, where p, q have highest common factor 1. What is $p + q$?

SOLUTION

The probability of getting at-least £1 back is equal to 1 minus the probability of losing all the money, which with every turn played (until loss) is $\frac{1}{2^n}$. Thus the maximum probability of winning at least £1 back is playing the game once giving a probability of $\frac{1}{2}$ and hence an answer of $\boxed{3}$.

3C. The graph $\ln x$ is stretched vertically by a factor b and results in the function $f(x)$. Given that $8^{f(x)} = x^2$ for all x, what is the value of $e^{2/b}$?

SOLUTION

A stretch by b implies that the new function is $f(x) = b \ln x$. Taking natural logarithms on our equation,

$$\begin{aligned} x^2 = 8^{f(x)} &\implies \ln\left(x^2\right) = \ln\left(8^{f(x)}\right) \\ &\implies 2\ln x = f(x)\ln 8 \\ &\implies \frac{2\ln x}{\ln 8} = b\ln x \\ &\implies b = \frac{2}{\ln 8} \end{aligned}$$

Thus $e^{2/b} = e^{\ln 8} = \boxed{8}$.

Act II, Scene 4: Estimathon

4A. How many prime factors does 1000! have? *[For example, $12 = 2^2 \times 3$ has three prime factors.]*

SOLUTION

Factorising 1000! as

$$1 \times 2 \times 3 \times 4 \times \cdots \times 999 \times 1000$$

Of these 1000 factors, 500 factors which are divisible by 2; of these, half of them are divisible by 4; and of those, half of them are divisible by 8... In total, then, there are

$$\left\lfloor \frac{1000}{2} \right\rfloor + \left\lfloor \frac{1000}{4} \right\rfloor + \left\lfloor \frac{1000}{8} \right\rfloor + \cdots$$

factors of 2 in 1000!, where $\lfloor x \rfloor$ rounds x down to the nearest integer. Eventually, the 2^k in the denominator will be large enough that the terms reach 0 – namely, $2^{10} = 1024$ so $1000/1024 < 1$ hence $\lfloor 1000/1024 \rfloor = 0$, so we can calculate that there are

$$500 + 250 + 125 + 62 + 31 + 15 + 7 + 3 + 1 = 994$$

factors of 2. We should repeat this procedure on all prime factors of 1000! for an exact answer. For an approximate answer, as the prime factors get larger, they make a smaller contribution to the number of prime factors (for example, the prime number 997 only appears once).

When all is said and done, we find that 1000! has $\boxed{2877}$ prime factors.

4B. *Recamán's alternative sequence* is defined as follows. Initially, $u_0 = 1$. For all subsequent terms,

$$u_{n+1} = \begin{cases} u_n/n & \text{if } n \text{ is a factor of } u_n \\ u_n \times n & \text{otherwise} \end{cases}$$

Which is the first term to exceed 100 digits in length?

ANSWER

This question is quite hard to find the exact value of, so we would expect contestants to use rougher estimations to try and determine the answer.

Upon computation, one finds that the $\boxed{\text{282th}}$ term is the first to exceed 100 digits.

4C. Ivy has a perfectly circular pizza and an atomically precise pizza cutter. Every second, she chooses two random points on the circumference of the pizza and then makes a perfectly straight cut connecting those two points. After she makes 1000 cuts, what is the expected number of pieces of pizza she makes? That is to say, if Ivy were to repeat this procedure sufficiently many times and take the mean of all her results, what number would this mean approach?

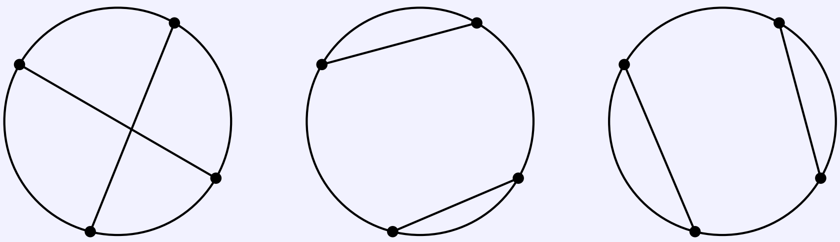
SOLUTION

The kth cut adds $1 + (\#\text{ of intersections with previous } k-1 \text{ cuts})$ many regions. So for 1000 cuts, the number of pizza regions will be

$$\begin{aligned}(\#\text{ of pizza regions}) &= 1 + \sum_{k=1}^{1000} \big(1 + (\#\text{ of intersections with previous } k-1 \text{ cuts})\big) \\ &= 1 + 1000 + (\#\text{ of intersections within the circle})\end{aligned}$$

We can calculate the number of expected cuts if we can find the probability of two random cuts intersecting; as each pair of cuts is independent, we may multiply the probability of two cuts intersecting with the number of pairs of cuts.

So what is the probability of two random cuts intersecting? Any two random chords drawn in a circle are jointly formed by four endpoints on the circle, so we can count the number of ways to pair up the points and count how many of these have an intersection.



Given an arrangement of 4 points, there are 3 different pairs of chords that can be drawn between them, only one of which intersect. Hence, the probability for two randomly drawn chords to intersect is $\frac{1}{3}$.

There are ${}^{1000}C_2 = \frac{1}{2} \times 1000 \times 999 = 499500$ pairs of cuts, so the expected number of intersections is $\frac{1}{3} \times 499500 = 166500$. In total, then, we expect Ivy to have $1 + 1000 + 166500 = \boxed{167501}$ cuts.