Warmup Solutions

1. Factorise 399, 396, 391, 384. What do you notice? Hence, what is the sum of the prime factors of 6557?

   A. 129; B. 138; C. 146; D. 154; E. 162.

SOLUTION

$399 = 20^2 - 1^2 = (20+1)(20-1) = 3 \cdot 7 \cdot 19$, by the difference of two squares. Similarly, we can factorise the other numbers via $396 = 20^2 - 2^2$, $391 = 20^2 - 3^2$ and $384 = 20^2 - 4^2$.

We should look for a square number slightly larger than 6557; $80^2 = 6400$, and $81^2 = 6561$. So $6557 = 81^2 - 2^2 = (81+2)(81-2) = 79 \cdot 83$; both of these are prime, so the sum of prime factors is 162. Thus $\boxed{\text{E}}$ is the final answer.

INVESTIGATION

When finding a prime factor of n (or to prove it's prime), we only need to try dividing n by prime numbers less than or equal to $\sqrt{n}$. Why does this work?

The use of difference of two squares like in this question relates to *Fermat's method*. Given an odd number n, let m be the least integer $\geq \sqrt{n}$, then test $m^2 - n$, $(m+1)^2 - n$, $(m+2)^2 - n$ and so on until we find k such that $(m+k)^2 - n$ is a square x^2. Then $n = (m+k-x)(m+j+x)$.

Fermat's method is particularly good for factorising 4-digit numbers by hand; also, if $n = ab$ with a, b odd, then this procedure is quite efficient when $a \approx b$ but not if one is much larger than the other.

Use Fermat's method to factorise **(a)** 8927, **(b)** 6077, **(c)** 8927.

2. Which of the following values is the largest?

A. $2^{(3^4)}$; B. $2^{(4^3)}$; C. $3^{(2^4)}$; D. $4^{(2^3)}$; E. $4^{(3^2)}$.

SOLUTION

Simplifying the exponents, the numbers are (left-to-right) 2^{81}, 2^{64}, 3^{16}, 4^8 and 4^9, so we can eliminate B and D. We can compare A and E as $4^k = 2^{2k}$, so E equals $2^{18} < 2^{81}$, so only A and C remain. However, $3^{16} < 4^{16} = 2^{32} < 2^{81}$, so $\boxed{\text{A}}$ is the largest.

INVESTIGATION

Order the following numbers by size: 2^{24}; 3^{15}; 4^{11}; 5^{10}.

3. 2026 is the product of two primes, 2 and 1013. The Winter Olympics are taking place in 2026. Assuming the Winter Olympics continue to occur every 4 years, in which century will another Olympic games occur on a year which is also the product of two primes?

A. 2000s; B. 2100s; C. 2200s; D. 2300s; E. 2400s.

SOLUTION

2038 is the next such year, so the final answer is $\boxed{\text{A}}$! Since 2026 is divisible by 2, so will every future year that holds the Winter Olympics. So one of the two primes must be 2.

The rest amounts to determining what is the next prime after 1013. 1015 is divisible by 5, 1017 by 3, and one can check that 1019 is prime. Although, for our purposes, it is enough to find a prime number between 1013 and 1049.

INVESTIGATION

Why is it enough to find a prime between 1013 and 1049? What numbers should we check for the other centuries?

4. Let f and g be functions where $f(1) = 1$, $g(1) = 1$ and for $n \geq 1$,

$$f(n+1) = 2f(n) \qquad \text{and} \qquad g(n+1) = g(n) + 50.$$

For how many positive numbers n is it the case that $g(n) > f(n)$?

A. 0; B. 4; C. 8;

D. 128; E. Infinitely many!

SOLUTION

f and g are geometric and arithmetic sequences (respectively) in disguise! Indeed, $f(n) = 2^{n-1}$ and $g(n) = 1 + 50(n-1)$. In any case, we should expect f to eventually become much larger than g, so the final answer shouldn't be E.

Now $f(9) = 256$, $g(9) = 401$, but also $f(10) = 512$ and $g(10) = 451$, so for $n \geq 10$ we will have $f(n) > g(n)$. On the other hand, $f(1) = g(1) = 1$ and also $f(2) = 2$ and $g(2) = 51$.

So for $2 \leq n \leq 9$, we will have $g(n) > f(n)$, so there are 8 such n. The final answer is $\boxed{\text{C}}$.

INVESTIGATION

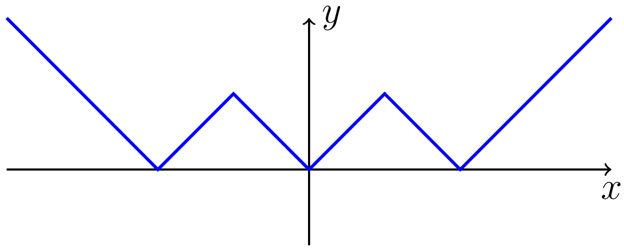
Find closed formulae for the following functions:

(a) $h(1) = 1$, $h(n+1) = h(n) + 2n + 1$;

(b) $j(1) = 1$, $j(n+1) = j(n) + 2^{-n}$;

(c) (Hard!) $F(1) = 1$, $F(2) = 1$, $F(n+2) = F(n+1) + F(n)$.

5. Which of the following equations is drawn below?



A. $y = |x|$;

B. $y = \big||x| - 1\big|$;

C. $y = \Big|\big||x| - 1\big| - 1\Big|$;

D. $y = \bigg|\Big|\big||x| - 1\big| - 1\Big| - 1\bigg|$;

E. $y = \Bigg|\bigg|\Big|\big||x| - 1\big| - 1\Big| - 1\bigg| - 1\Bigg|$.

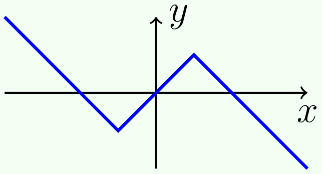
SOLUTION

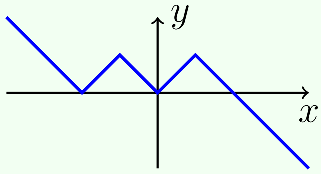
We need our graph to have three distinct roots (i.e. three distinct x-values where $y = 0$). A only has one root ($x = 0$) and B has two ($x = \pm 1$), so they won't do. C has three: when $x = \pm 2$, $y = ||2 - 1| - 1| = |1 - 1| = 0$, and when $x = 0$, $y = || - 1| - 1| = |1 - 1| = 0$.

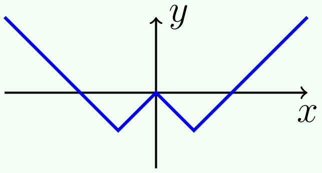
We might suspect that D and E have more roots, and we would be right: for D, we have roots at ± 1 and ± 3; for E we have ± 4, ± 2 and 0. So it has to be $\boxed{\text{C}}$.

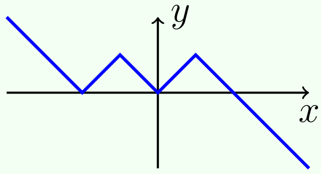
INVESTIGATION

Find equations for the following graphs:

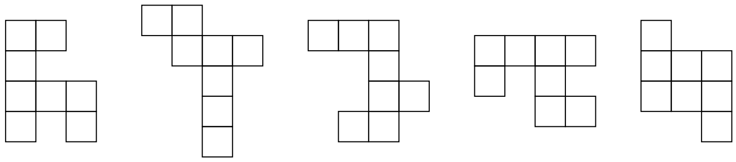








6. A polyomino is constructed by connecting 1×1 square tiles to each other. Below are examples of polyominoes each with eight tiles.

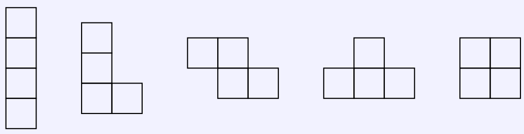


Two polyominoes are considered the same if one can be transformed into the other with rotations and reflections (so, for example, the first and fourth polyominoes above are the same). How many distinct polyominoes can be constructed with exactly 4 tiles?

A. 3; B. 4; C. 5; D. 6; E. 7.

SOLUTION

We have five tetrominoes:



Since we only have four tiles, we must always be able to fit a tetromino in a 4×4 grid. However, if it has a length of 4 then it must be the 1×4 tile, so we can certainly reduce it to a 3×4 grid; by rotating tiles, we can further reduce the grid to 2×4.

With a 2×4 grid, it is much easier to find all five tetrominoes by trial and error. The final answer is C.

7. Let p be any prime number, and let n be any whole number. Which option can always be uniquely expressed as a difference of two squares?

A. p; B. $4p$; C. $17n$; D. np; E. $(2n+1)^2$.

SOLUTION

The prime factorisation of $4p$ is $2^2 \cdot p$, so if $4p = a^2 - b^2$ then the factor pairs $(a+b)$ and $(a-b)$ are either $(4, p)$ or $(2, 2p)$ in some order. However the factors need to be both odd or both even, so $a - b = 2$ and $a + b = 2p$, which solves to $a = p + 1$ and $b = p - 1$. Indeed, $4p = (p+1)^2 - (p-1)^2$, and with the above argument this must be unique. So the answer is B.

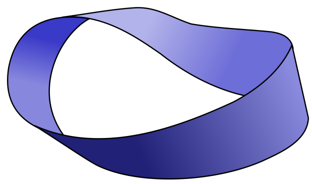
8. For the polynomial $q(x) = x^4 + ux^2 + v$, $q(\sqrt{2} + \sqrt{3}) = 0$, where u and v are whole numbers. What is the sum of u and v?

A. -9; B. -4; C. 0; D. 4; E. 9.

SOLUTION

Let $\alpha = \sqrt{2} + \sqrt{3}$. *[Idea: "square out the roots".]* Squaring both sides, $\alpha^2 = 2 + 2\sqrt{6} + 3$, so $\alpha^2 - 5 = 2\sqrt{6}$. Squaring this, we get $(\alpha^2 - 5)^2 = 24$, which expanded out gives $\alpha^4 - 10\alpha^2 + 1 = 0$, so we take $q(x) = x^4 - 10x^2 + 1$; thus the final answer is $\boxed{\text{A}}$.

9. A Möbius strip is constructed by taking a strip of paper, twisting one of the ends by 180° (a 'twist') and gluing the ends:

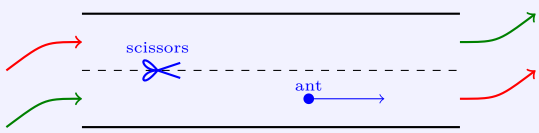


What do you obtain if you cut a Möbius strip along the middle?

A. One loop with no twists; B. A Möbius strip;
C. One loop with two twists; D. Two separate Möbius strips;
E. Two interlocked Möbius strips.

SOLUTION

We can represent the Möbius strip diagramatically:



Imagine we are an ant on the Möbius strip and travel in the direction of the arrow. By the twist (represented by the red and green arrows) we will end up on the other side of the scissors, and continuing to travel around we will end up where we started; thus there is only one strip (so it's not D nor E).

So how many twists? Each arrow will add one twist on the way round, so overall the ant will make two twists before returning to starting position. So the answer is $\boxed{\text{C}}$.

10. Which of the following functions is not equal to the rest?

A. $\sin(x + 720°)$;

B. $\cos(x - 90°)$;

C. $\sin(\arccos(\cos x))$;

D. $\sin^3 x + \sin x \cos^2 x$;

E. $(\tan^2(x) \sin^2(x + 90°) + \cos^2 x) \sin x$.

SOLUTION

We show each of A, B, D and E are equal to $\sin x$:

A. $\sin(x + 720°) = \sin(x + 360°) = \sin x$ by periodicity;

B. $\cos(x - 90°) = \sin x$ by the cos-sin relation;

D. $\sin^3 x + \sin x \cos^2 x = \sin^3 + \sin x(1 - \sin^2 x) = \sin x$ by the fact that $\sin^2 x + \cos^2 x \equiv 1$;

E. First note $\sin(x + 90°) = \cos x$, so using $\tan x = \sin x / \cos x$ we get

$$\tan^2(x) \sin^2(x + 90°) + \cos^2 x = 1$$

so this expression is just $\sin x$.

That leaves C: intuitively, arccos "loses information" about where x is. For example, $\sin(\arccos(\cos 720°)) = \sin(\arccos(1)) = \sin(0°)$. This is sometimes fine, but not always:

$$\begin{aligned} \sin((\cos(270°))) &= \sin(\arccos(-1)) \\ &= \sin(-90°) \\ &= -1 \end{aligned}$$

which is not $\sin(270°) = 1$, so $\boxed{\text{C}}$ is the odd one out.