Oxford Mathematics Team Challenge

Lock-in Round Solutions

SAMPLE SET

1. Circle packing

In this question, we begin with the circle packing problem. In the circle packing problem, we are given a shape and have to place the circles without overlap in the interior of the shape (that is, we have to *pack* them). The goal of the problem is to maximise the radius of the circles.

(a) Figure 1 shows a packing of six circles in a larger circle of radius 3.

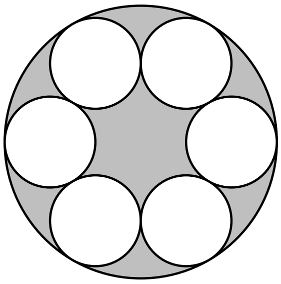
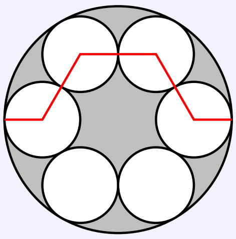


Figure 1

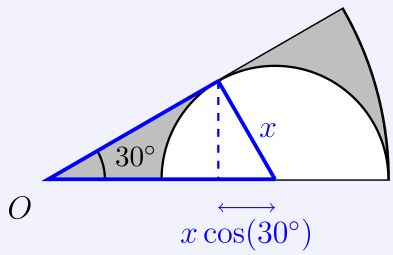
Carefully find the radius of the smaller circles.

SOLUTION

Let x be the radius we wish to find. We can solve for the diameter of the larger circle by considering the red lines:



The horizontal segments add up to $4x$, so we need to calculate the horizontal component of the diagonal segments. Zooming in to a subsection where O is the centre of the large circle:



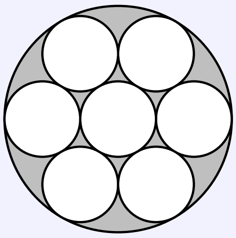
So the horizontal components of the diagonals sum to $4x\cos(30°) = 2x$. Since we know the diameter of the larger circle is 6, and we've just shown that this diameter is also $4x + 2x = 6x$, it follows that $x = 1$.

The packing of the six circles in Figure 1 is *optimal*, meaning that there is no way to fit six congruent circles of a larger radius into a circle of radius 3.

(b) Construct an optimal packing of seven circles in a circle of radius 3. Briefly explain why it is optimal.

SOLUTION

Because the radius is 1, we can fit a seventh circle in the 6-packing we were given:



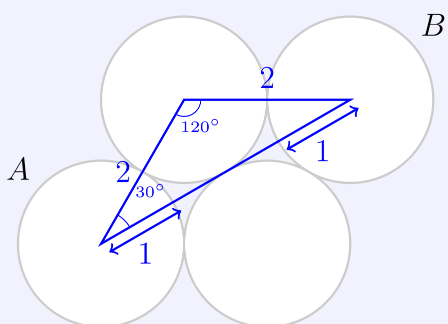
Given two packings of n circles inside a shape, we say that these packings are *equivalent* if we can transform one packing into the other by a series of transformations which slide the circles or rotate the whole packing.

(c) Are all optimal packings of six circles in a circle equivalent? Explain your answer.

SOLUTION

There are inequivalent optimal packings of six circles. From (b), we can consider the following two packings:

These two packings are equivalent if we can slide C from its place in the left figure to its place in the right figure. We can do so only if the diameter of C (which is 2) is less than the shortest distance between A and B.



We can find the shortest distance between the circles A and B (see the figure above) by the sine rule to be

$$\frac{2\sin(120°)}{\sin(30°)} - 2 = 2\sqrt{3} - 2$$

Since $2\sqrt{3} - 2 < 2$, we can't slide C in between them. Therefore, we have two inequivalent optimal packings.

(d) An inconspicuous aside: Figure 2 depicts a triangle with side lengths 13, 13 and 10.

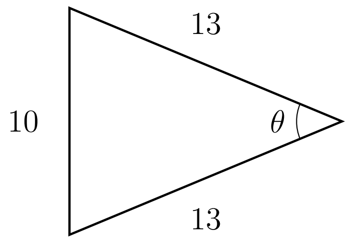
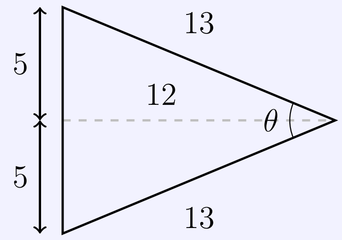


Figure 2

Show that θ, the angle between the sides of length 13, is approximately 45°. Justify your working clearly.

SOLUTION

You can use the cosine rule, but using the sine area rule here is quite nice. Splitting the triangle in two:



We get the 5-12-13 Pythagorean triple, so the area is $2 \times (\frac{1}{2} \times 5 \times 12) = 60$. With the sine area rule, the area is $\frac{1}{2} \times 13^2 \sin\theta$, so equating the two:

$$\begin{aligned} \tfrac{1}{2} \times 13^2 \sin\theta &= 60 \\ 169 \sin\theta &= 120 \\ \therefore \quad \sin\theta &= 120/169 \end{aligned}$$

By calculation, $120/169 = 0.710...$ which we can compare with $\sqrt{2}/2$: $\sqrt{2} = 1.414...$ so

$$\sqrt{2}/2 = 0.707... \approx 0.710... = 120/169$$

thus $\theta \approx 45°$.

(e) Figure 3 shows an optimal packing of nine circles in a circle of radius 3.

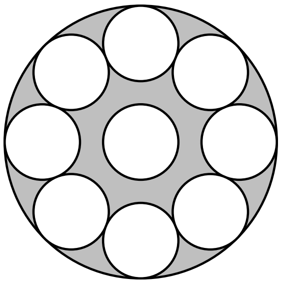
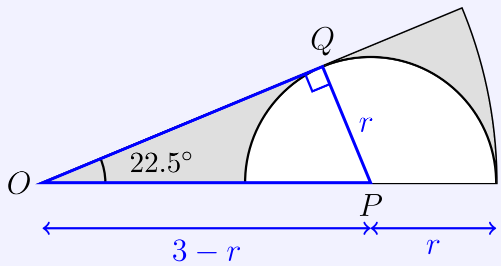


Figure 3

Find the radius r of the smaller circles in terms of $\sin(22.5°)$.

SOLUTION

Take a sixteenth of the diagram and zoom in. For clarity in the diagram we remove the centre circle. Let r be the radii of the smaller circles.



Note that $\angle OQP = 90°$ because the line OQ is tangent to the radius PQ, so we can do some trigonometry on the triangle OPQ to make an equation relating r and $\sin(22.5°)$:

$$\begin{aligned}
\sin(22.5°) &= \frac{r}{3-r} \\
(3-r)\sin(22.5°) &= r \\
3\sin(22.5°) &= r(1+\sin(22.5°)) \\
\therefore \quad r &= \frac{3\sin(22.5°)}{1+\sin(22.5)}
\end{aligned}$$

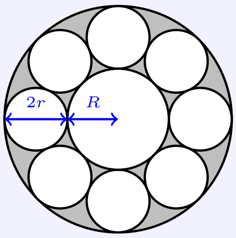
(f) Now suppose the central circle in Figure 3 was enlarged so it is just touching the surrounding circles. Let the central circle have radius R. Estimate the values of r and R. *[Hint: use (d) and (e).]*

SOLUTION

Using the figure from (d), we have an estimate for $\sin(22.5°)$. The dashed line bisects θ, which we estimated to be $45°$; so $\sin(22.5°) \approx 5/13$. We substitute this into r:

$$r \approx \frac{3 \times \frac{5}{13}}{1 + \frac{5}{13}} = \frac{3 \times 5}{13 + 5} = \frac{15}{18} = \frac{5}{6}$$

Lastly, R equals $3 - 2r$ (see figure below), so $R \approx 4/3$.



(g) Hence, or otherwise, verify that the ratio of the areas between the central circle and the smaller circles equals 2.6 when rounded to the nearest tenth.

SOLUTION

The ratio of the areas is R^2/r^2, so

$$\frac{R^2}{r^2} = \frac{\frac{16}{9}}{\frac{25}{36}} = \frac{64}{25} = 2.56 \approx 2.6.$$

We lastly explore ways to pack circles into the (infinite) plane. Sally suggests a *square packing* of the plane, where the centres of the circles form a square grid, whereas Helena suggests a *hexagonal packing* of the plane. Their strategies are depicted in Figure 4:

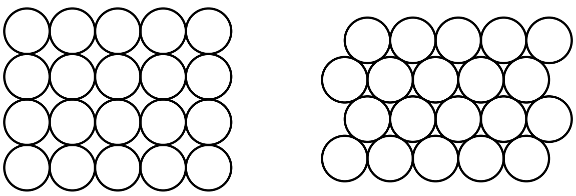


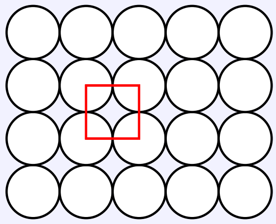
Figure 4

The *density* of a packing on the plane is the percentage of the plane covered by circles.

(h) Determine the density of Sally's packing.

SOLUTION

The trick is to find a tile with circle markings that tessellates the plane. For example, the square tile



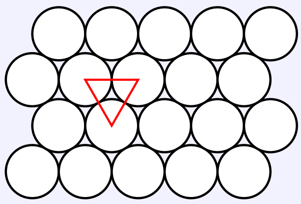
tessellates the plane such that the quarter-circle markings in it can be matched by other circles. Since this same pattern covers the plane, it suffices to determine the ratio of area covered by circles in the tile against the area of the tile.

Letting the circles have radii 1, we have four quarter-circles in the tile so their total area is π. The area of the tile is 4, so the density of Sally's packing is $\pi/4$.

(i) Determine the density of Helena's packing. Hence show that Helena's packing is more dense than Sally's.

SOLUTION

Similarly, we can find another tessellating tile. We have a bit more freedom with what tile we choose; it's nicer to pick a triangle!



This time, we have three sixths of a circle, so their total area is $\pi/2$. The area of the tile is $\frac{1}{2} \cdot 2 \cdot 2 \cdot \sin(60°) = \sqrt{3}$, so the density of Helena's packing is $\frac{\pi}{2\sqrt{3}}$.

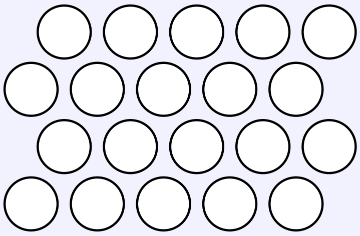
Indeed, Helena's packing is more dense than Sally's:

$$\begin{aligned} \frac{\pi}{2\sqrt{3}} > \frac{\pi}{4} &\iff 4 > 2\sqrt{3} \\ &\iff 2 > \sqrt{3} \\ &\iff \sqrt{4} > \sqrt{3} \end{aligned}$$

(j) It turns out that Helena's packing has the highest possible density for an infinite circle packing on the plane. What are the possible densities of an infinite circle packing on the plane? Justify your answer carefully.

SOLUTION

For any positive number $\leq \frac{\pi}{2\sqrt{3}}$, we can make a packing with that density by taking Helena's packing and evenly spreading the circles out:

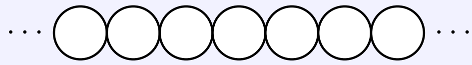


In particular, if we pick a density η, then by setting the distance between the centres of adjacent circles to be

$$r = \frac{2\pi}{\eta\sqrt{3}}$$

we achieve the density η by considering the tile used in (i): the circle area is the same, and the area of the tile becomes $\frac{1}{4}r^2\sqrt{3}$.

Lastly, we can also have a density of 0. One way to achieve this is to have our infinitely many circles just form a line in the plane:



As we take larger subsections of the plane (in such a way that we will eventually contain each point in the plane), the ratio of circle-area to the area of the subsection will become negligible. It cannot have some density $\eta > 0$; as we continue taking larger subsections of the plane, the density will inevitably become less than η. So its density must be 0.

2. Integer partitions

In this question, we will explore *partition theory*. A *partition* of a positive integer n is a way of writing n as the sum of positive integers (called the *parts*), irrespective of the order of the sum. For example, $5+2+1$ is a partition of 8; this is the same partition as $2+1+5$, however a different partition of 8 is e.g. $3+3+2$. A sum with only one part counts as a partition, e.g. 5 is a partition of 5.

(a) Write down all the partitions of 4.

SOLUTION

The partitions of 4 are:

$$4, \qquad 3+1, \qquad 2+2, \qquad 2+1+1, \qquad 1+1+1+1.$$

(b) (i) A partition is called *distinct* if no part is repeated in the sum. For example, $5+2+1$ is a distinct partition of 8, whereas $3+3+2$ is not.

Write down all distinct partitions of 7.

SOLUTION

The distinct partitions of 7 are:

$$7, \qquad 6+1, \qquad 5+2, \qquad 4+3, \qquad 4+2+1.$$

(ii) A partition is called *odd* if it only contains odd parts. For example, $5+3$ is an odd partition of 8, whereas $5+2+1$ is not.

Write down all odd partitions of 7.

SOLUTION

The odd partitions of 7 are:

$$7, \quad 5+1+1, \quad 3+3+1, \quad 3+1+1+1+1, \quad 1+1+1+1+1+1+1.$$

(c) For a positive integer n, let $p(n)$ be the number of partitions of n. By convention we also say $p(0) = 1$.

(i) Calculate $p(5)$.

SOLUTION

$p(5) = 7$ – the best way to see this is just to exhaust all partitions of 5:

$$5, \qquad 4+1, \qquad 3+2, \qquad 3+1+1,$$

$$2+2+1, \qquad 2+1+1+1, \qquad 1+1+1+1+1.$$

(ii) Explain why $p(n+1) > p(n)$ for all $n \geq 1$.

SOLUTION

For each partition of n, adding one (as a separate part) results in a unique partition of $n+1$, so at least $p(n+1) \geq p(n)$. These partitions don't include the partition of one part, $n+1$, so $p(n+1) > p(n)$.

(iii) We also write $p_d(n)$ as the number of distinct partitions of n, and $p_o(n)$ as the number of odd partitions of n. Similarly, $p_d(0) = p_o(0) = 1$.

Calculate $p_d(5)$ and $p_o(5)$.

SOLUTION

We have $p_o(5) = p_d(5) = 3$. In fact, you only needed to check one of these, because looking ahead to (e)(ii) we later show that $p_o(n) = p_d(n)$ for all n!

In any case, the odd partitions of 5 are

$$5, \qquad 3+1+1, \qquad 1+1+1+1+1.$$

and the distinct partitions of 5 are

$$5, \qquad 4+1, \qquad 3+2.$$

We can draw out a partition using a Ferrer diagram, which represents the partition as a collection of dots, descending in size of the rows. Figure 5 shows two examples:

For any partition, we can create its *conjugate* by reflecting the Ferrer diagram over the diagonal. For example, the partitions in Figure 5 are conjugates of each other. We also say that a partition is *self-conjugate* if its conjugate is itself.

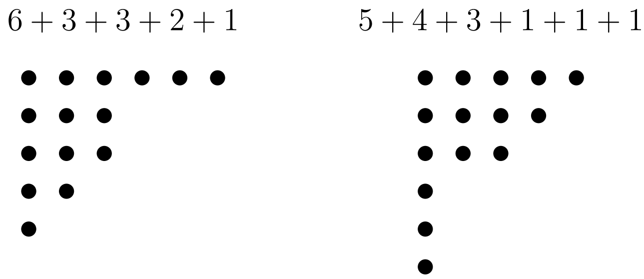
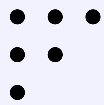


Figure 5

(d) (i) Give an example of a self-conjugate partition of 6.

SOLUTION

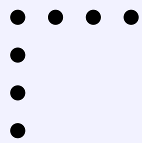
A partition being self-conjugate is equivalent to a partition's Ferrer diagram having reflective symmetry along the diagonal. The only self-conjugate partition of 6 is $3 + 2 + 1$:



(ii) Give an example of a self-conjugate partition of 7.

SOLUTION

The only self-conjugate partition of 7 is $4 + 1 + 1 + 1$:



(iii) Explain why the number of partitions with six parts is the same as the number of partitions with the largest part equal to 6.

SOLUTION

The conjugate operation creates a one-to-one correspondence between partitions (potentially mapping a partition to itself, if the partition is self-conjugate). We can also think of the conjugate as reading the partition column-wise, as opposed to row-wise (the normal way).

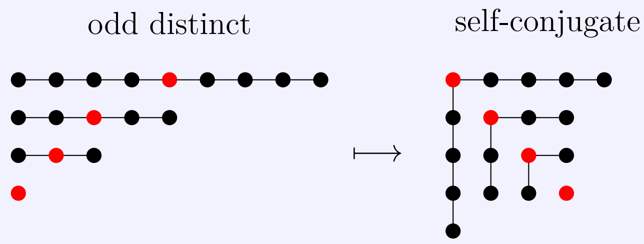
If the largest part of a partition is 6, the Ferrer diagram has six columns, so its conjugate will have six parts. Conversely, if a partition has six parts, the Ferrer diagram has six rows; as the Ferrer diagram descends in the size of rows, its tallest column must be 6, i.e. the conjugate has largest part equal to 6.

This establishes a one-to-one correspondence between partitions with largest part equal to 6, and partitions with six parts, hence they are equal in number for a partition of any n.

(iv) Explain why the number of partitions of n into distinct odd parts is the same number of partitions of n into self-conjugate parts.

SOLUTION

Start with an odd distinct partition. Take the middle point in each row and use it as a "hinge" for a new Ferrer diagram:



The odd distinct partition forms a self-conjugate diagram as each hinge reflects along the diagonal. This is also a *valid Ferrer diagram* as each part in the original partition is distinct (each hinge needs to be at least 2 greater than the last, which is guaranteed by the partition being distinct)!

We can go backwards, too, as each self-conjugate partition gives us the hinges along the diagonal. It's odd because there's an equal number of points to the right of and below each hinge; it's Ferrer, and moreover distinct, because the groupings form a strictly decreasing sequence. This establishes a one-to-one correspondence between odd distinct partitions and self-conjugate partitions, hence they are equal in number for a partition of any n.

(e) Let's return to $p_o(n)$ and $p_d(n)$.

(i) Consider the infinite products

$$A = (1+x)(1+x^2)(1+x^3)\cdots$$
$$B = (1+x+x^2+\cdots)(1+x^3+x^6+\cdots)(1+x^5+x^{10}+\cdots)\cdots$$

Explain and justify how the coefficients of x^n the infinite products relate to $p_d(n)$ and $p_o(n)$.

SOLUTION

Recall that we defined

$$A = (1+x)(1+x^2)(1+x^3)\cdots$$
$$B = (1+x+x^2+\cdots)(1+x^3+x^6+\cdots)(1+x^5+x^{10}+\cdots)\cdots$$

We can associate the coefficient of x^n in A to $p_d(n)$, and the coefficient of x^n in B to $p_o(n)$. Why? Consider $p_d(3)$, for example. The only distinct partitions for 3 are 3 and $2+1$. Now consider the coefficient of x^n in A. We can find it by thinking what we need to multiply together to make a term of x^3; e.g., we can choose

$$(\mathbf{1}+x)(\mathbf{1}+x^2)(1+\boldsymbol{x^3})(\mathbf{1}+x^4)\cdots$$

or

$$(1+\boldsymbol{x})(1+\boldsymbol{x^2})(\mathbf{1}+x^3)(\mathbf{1}+x^4)\cdots$$

These correspond to the distinct partitions 3 and $1+2$, respectively. For example, choosing to multiply by the x^2 is the same as including 2 as a part in your partition; choosing to multiply by 1 instead of x^2 is the same as not including 2 in your partition.

This generalises to any x^n in A: its coefficient tells us the number of distinct partitions.

The same goes for B with odd partitions. The first bracket tells us the number of 1's we add to the sum; the second bracket tells us the number of 3's we add to the sum; and so on.

(ii) Show that $p_d(n) = p_o(n)$. *[Hint: consider* $\dfrac{1-x^{2k}}{1-x^k}$ *for each positive integer* k*.]*

SOLUTION

Time to crunch some algebra! The hint tells us to consider $\dfrac{1-x^{2k}}{1-x^k}$. On the one hand,

$$\frac{1-x^{2k}}{1-x^k} = \frac{(1-x^k)(1+x^k)}{1-x^k} = (1+x^k)$$

so we can think of A as the infinite product

$$\begin{aligned} A &= (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\cdots \\ &= \frac{1-x^2}{1-x} \cdot \frac{1-x^4}{1-x^2} \cdot \frac{1-x^6}{1-x^3} \cdot \frac{1-x^8}{1-x^4} \cdot \frac{1-x^{10}}{1-x^5} \cdots \end{aligned}$$

We can see that all of the numerators $(1-x^{2k})$ will eventually cancel with a denominator later on:

$$= \frac{1-x^2}{1-x} \cdot \frac{1-x^4}{1-x^2} \cdot \frac{1-x^6}{1-x^3} \cdot \frac{1-x^8}{1-x^4} \cdot \frac{1-x^{10}}{1-x^5} \cdots$$

This leaves us with A equal to

$$A = \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdots$$

These are each infinite geometric sums, with first term 1 and ratio x, x^3, x^5, and so on respectively. That is,

$$\begin{aligned} A &= (1+x+x^2+\cdots)(1+x^3+x^6+\cdots)(1+x^5+x^{10}+\cdots)\cdots \\ &= B. \end{aligned}$$

INVESTIGATION

This was a lovely proof which Leonhard Euler came up with in 1748 – that the number of distinct partitions of n equals the number of odd partitions of n, and that this can be proved using these types of infinite products which we call *generating functions*. For example, we saw that $(1+x)(1+x^2)(1+x^3)\cdots$ was a generating function for the sequence $p_d(n)$.

If you're interested, you might want to look at Wilf's *generatingfunctionology*: `https://www2.math.upenn.edu/~wilf/gfology2.pdf`

3. Random tic-tac-toe

Two robots, Xeep and Obot, play a game of *tic-tac-toe* (a.k.a. noughts and crosses) on a 3×3 grid. Xeep, who goes first, marks its squares with Xs, and Obot marks with Os. They take turns choosing a random unfilled square to mark. Each robot is equally likely to choose any one of the unfilled squares on its turn. A robot wins by being the first to achieve a 3-in-a-row of their own symbols, either horizontally, vertically, or diagonally. If the grid gets filled completely with no such 3-in-a-row, then the game ends in a draw.

We give the robots additional instructions so that even if a robot has won before the grid is completely filled, the robots will continue to make random moves until the grid is completely filled. We shall call the final completely-filled grid of this game the *end grid*. Note that every possible end grid has 5 Xs and 4 Os.

The first three parts lead to finding the probability of a draw.

(a) Briefly describe how you can determine whether or not a game of tic-tac-toe ended in a draw if you are provided the end grid of the game.

SOLUTION

A game of tic-tac-toe ended in a draw if and only if the end grid does not contain any 3-in-a-rows of all Xs or all Os. If there was a 3-in-a-row, then at some point during the game, the robot would complete that 3-in-a-row, so that game ends with a robot winning.

(b) Given an end grid of a game that didn't end in a draw, is it always possible to determine who the winner was? Briefly explain.

SOLUTION

Given an end grid of a game that didn't end in a draw, we cannot always determine who the winner was. If we get an end grid with a 3-in-a-row from both X and O, such as the grid shown below, then either Xeep or Obot could've won this game, depending on whose 3-in-a-row was formed first in the sequence.

X	O	X
X	O	X
X	O	O

(c) There are exactly 126 possible end grids. What is the probability that a game of random tic-tac-toe ends in a draw?

SOLUTION

Since all legal moves are made with equal probability, the resulting end grid of a game of random tic-tac-toe is equally likely to be any one of the 126 possible end grids. From (a), the end grid indicates a drawn game if there is no 3-in-a-row. So if we count the number of drawn end grids, then divide that by 126, that result is the proportion of drawn end grids, which is the probability that a game of random tic-tac-toe ended in a draw.

The big task is counting the number of drawn end grids, i.e. those without a 3-in-a-row. There are various systems you may use to keep track of your grids, to make sure you don't miss any cases or count the same grid twice. The system we show here is just one way to approach this.

We approach this by divide and conquer. We'll separate the end grids into smaller buckets, and then calculate the number of drawn end grids within each smaller bucket. In any end grid, there are either 0, 1, 2, 3, or 4 Os at the corners, so one way you may divide and conquer this problem is to consider cases of the number of Os at the corners.

Four Os at the corners. Recall that all end grids have 4 Os and 5 Xs. There is only 1 end grid with 4 Os at the corners, and it has a 3-in-a-row of Xs, as shown below. So there are **0** drawn end grids in this case.

O	X	O
X	X	X
O	X	O

Three Os at the corners. We draw in the corners first, shown below on the left. To avoid making a 3-in-a-row, we're forced to draw 3 more Xs between the Os as shown on the right.

O		O
O		X

O	X	O
X	X	
O		X

But now we're stuck, because we need to place one more X, which will form a 3-in-a-row. So there are in fact **0** drawn end grids in this case.

Two Os at the corners. We can't have the Os at opposite corners because then no matter our choice for the centre square, we will form a 3-in-a-row. Therefore, we'll consider arrangements with the Os at neighbouring corners.

O		X
X		O

O		O
X		X

In the grid on the right, we force the top-middle to be X and the bottom-middle to be O. We have one more O to place in the grid, and it turns out that any of the placements results in a drawn end grid!

O	X	O
X	O	X

→

O	X	O
O	X	X
X	O	X

O	X	O
X	O	X
X	O	X

O	X	O
X	X	O
X	O	X

Don't forget that we each rotate each of these grids to produce 4 distinct end grids. So in total, these are **12** drawn end grids in this case.

One O at the corners. Having 1 O and 3 Xs at the corners forces 3 more Os as shown below, then that forces the remaining squares to be Xs. This resulting grid has no 3-in-a-row, so it counts toward our total. Since this also can be rotated, there are **4** drawn end grids in this case.

O		X
X		X

O		X
	O	O
X	O	X

O	X	X
X	O	O
X	O	X

No Os at the corners. In this case, we place 4 Xs at the corners, but we need to place one more X, which cannot be done without forming a 3-in-a-row. So there are **0** drawn end grids in this case.

Any end grid falls into exactly one of the above cases, so we have accounted for all possible drawn end grids. The figure below shows the drawn grids we've found (note that any rotation of these grids is also a drawn grid).

O	X	O
O	X	X
X	O	X

O	X	O
X	O	X
X	O	X

O	X	O
X	X	O
X	O	X

O	X	X
X	O	O
X	O	X

Adding up the subtotals from each case, there are in total 16 drawn end grids. Therefore, the probability of random tic-tac-toe ending in a draw is 16/126, or 8/63.

The last four parts lead to finding the probabilities of each of the robots winning.

(d) How many end grids contain a 3-in-a-row of Os, but not a 3-in-a-row of Xs?

SOLUTION

A 3-in-a-row of Os can run either vertically, horizontally, or diagonally. But notice how if we have our 3-in-a-row of Os run vertically, that leaves two more vertical rows and only one more O left to block them, so here we will also have a vertical 3-in-a-row of Xs. The same applies to a horizontal 3-in-a-row of Xs. In order to get an end grid with a 3-in-a-row of Os, but not a 3-in-a-row of Xs, we must have the Os run diagonally. There are 2 diagonals to choose from for our 3-in-a-row of Os, and then there are 6 spots remaining to place the final O. In total, there are $2 \times 6 = 12$ such end grids.

(e) How many end grids contain *both* a 3-in-a-row of Os and a 3-in-a-row of Xs?

SOLUTION

Using the logic from (d), we now want our 3-in-a-row of Os to run either horizontally or vertically. There are 3 horizontal rows and 3 vertical rows to choose from, giving 6 options for where to place our 3-in-a-row of Os. Then there are 6 spots remaining from which to place our final O, and any choice also creates a 3-in-a-row of Xs. In total, there are $6 \times 6 = 36$ such end grids.

(f) Consider a variant of tic-tac-toe called *ric-rac-roe* which takes place on the partially-shaded board as shown in Figure 6. In this variant, Xeep can only make moves in the unshaded squares, and Obot can only make moves in the shaded squares. Both robots are still equally likely to choose any one of its legal moves for each turn.

What is the probability that Xeep wins in ric-rac-roe? *[Hint: consider the probability that Obot can achieve a 3-in-a-row in three moves.]*

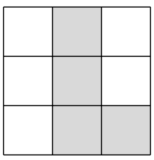


Figure 6

SOLUTION

There is a 1/4 probability that Obot could achieve a 3-in-a-row in 3 moves, since after 3 moves, 1 grey square will be empty, and we need that to be the bottom-right square. There is a 1/10 probability that Xeep gets a 3-in-a-row in 3 moves, since there are 10 total pairs of white squares, and we need the pair of white squares on the right to be empty after three moves. There is a 2/5 probability that Xeep gets a 3-in-a-row in 4 moves or fewer, since there would be 1 white square empty, and we need it to be one of the 2 white squares on the right.

If we let the game play out until the grid is filled, in the 1/4-probability case that Obot gets a 3-in-a-row in 3 moves, Xeep must hit its own 3-in-a-row in 3 moves in order to win. This case is a $\frac{1}{4} \times \frac{1}{10} = \frac{1}{40}$ probability win for Xeep. The other case is the 3/4-probability case is that Obot only gets a 3-in-a-row in exactly 4 moves. For Xeep to win it must hit a 3-in-a-row in 4 moves or fewer, so the probability for this case is $\frac{3}{4} \times \frac{2}{5} = \frac{3}{10}$. If Xeep doesn't hit a 3-in-a-row in 4 moves or fewer then it must lose.

In total, the probability that Xeep wins ric-rac-roe is $\dfrac{1}{40} + \dfrac{3}{10} = \dfrac{13}{40}$.

(g) What is the probability that Xeep wins a game of random tic-tac-toe? What is the probability of Obot winning? Express your answers as simplified fractions.

SOLUTION

To find the probability Obot wins random tic-tac-toe, consider the end grids that could result in a win for Obot. In (d), we showed there are 12 grids with a 3-in-a-row for O and not for X. If a game has this end grid, we can be certain that Obot won this game. In (e), we showed there are 36 grids with a 3-in-a-row for both O and X, and in this case we cannot be certain of who won. In (f), which shows one such ambiguous case, there is a $1 - \frac{13}{40} = \frac{27}{40}$ probability that Obot wins if we assume that the final end grid has a 3-in-a-row for both O and X. So if we are given one of the 36 end grids with a 3-in-a-row for both players, we can say with $\frac{27}{40}$ certainty that Obot won that game.

Since no other end grids have a 3-in-a-row of Os, these are the only cases which Obot could win. The probability Obot wins in random tic-tac-toe is therefore

$$\frac{12}{126} + \frac{36}{126} \cdot \frac{27}{40} = \frac{121}{420}$$

The probabilities of Obot winning, of Xeep winning, and of a draw must add up to 1, since these are the only possible outcomes. Since from (c) the probability of a draw is $\frac{8}{63}$, the probability that Xeep wins is

$$1 - \frac{8}{63} - \frac{121}{420} = \frac{737}{1260}$$