Oxford Mathematics Team Challenge

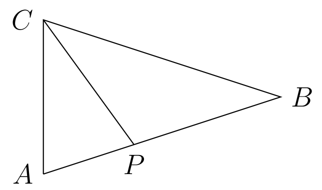
Maps Round Solutions

SAMPLE SET

		C5 26				**A8** 15		
	C6 15	**C3** 4	**C1** 4		**A3** 100	**A5** 7	**A7** 52	
C7 317	**C4** 312		**R7** 39	**R4** 30	**A1** 0	**A2** 4	**A4** 34	**A6** 52
	C2 6	**R8** 13	**R5** 20	**R1** 36		**Y3** 1024		
		R6 28	**R2** 8	free	**Y2** 8	**Y6** 1		
		R3 7		**Y1** 16	**Y5** 91	**Y8** 126	**B2** 49	
G6 24	**G4** 50	**G2** 16	**G1** 6	**Y4** 4	**Y7** 2		**B4** 0251	**B7** 16
	G7 66	**G5** 4852	**G3** 13		**B1** 3	**B3** 10	**B6** 63	
		G8 3101				**B5** 882		

Red Zone (R)

R1. In the diagram, $AB = BC$, $AC = BP$, $BP = CP$ and $\angle BPC \neq \angle BAC$. What is the angle $\angle ABC$ in degrees?



SOLUTION

Let's go angle-chasing! Say $\angle ABC = x°$. Since CPB is isosceles, $\angle PCB = x°$ so $\angle BPC = 180 - 2x°$. Since the angles around P add to $180°$, $\angle APC = 2x°$. Since ACP is isosceles, $\angle PAC = 2x°$, so $\angle ACP = 180 - 4x°$. Now $\angle ACB = \angle ACP + \angle BCP = 180 - 3x$. Since ABC is isosceles, $\angle ACB = \angle CAB$ so $180 - 3x = 2x$, which solves to $x = \boxed{36°}$.

INVESTIGATION

What is the ratio of lengths $AP : PB$? Using the *double angle identity*

$$\cos(2x) = 2\cos^2(x) - 1$$

show that $8\cos^4(36°) - 8\cos^2(36°) + \cos(36°) + 1 = 0$. Hence, find a polynomial with integer coefficients and $\cos(72°)$ as a root.

R2. A circle drawn in the Cartesian plane has diameter with endpoints at $P = (2, 8)$ and $Q = (8, 16)$. What is the shortest distance from the origin to a point on the circle?

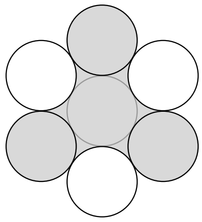
SOLUTION

The circle has centre at $(5, 12)$. So radius of the circle is

$$\sqrt{(8-5)^2 + (16-12)^2} = 5$$

The distance from the centre of the circle to the origin is $\sqrt{12^2 + 5^2} = 13$, thus the shortest distance from the origin to a point on the circle is $13 - 5 = \boxed{8}$.

R3. In the diagram shown, each of the seven circles have radius 1 cm. The area of the shaded region is $a\sqrt{3} + b\pi$. What is $a + b$?



SOLUTION

Let the centres of the outside circles be A, B, C, D, E, F respectively labelled in a clockwise order. Then we know the area of the shaded region is equivalent to the area of $ABCDEF$ + the area of a circle. Now the hexagon $ABCDEF$ is regular with side lengths 2, so it has an area of $6\sqrt{3}$ (add a point at the centre and split the hexagon into six triangles). So the total area of the shaded region is $6\sqrt{3} + \pi$, hence $a + b = \boxed{7}$.

R4. Petar has a straight strip of paper 1 cm wide and lays another straight strip 2 cm wide overlapping it. The resulting overlapping region of the two strips of paper has area 4 cm^2. What is the acute angle between the two strips of paper?

SOLUTION

If we let θ be the angle between the two strips, we note the area of the parallelogram formed by the overlapping region is $2/\sin\theta$ or 4, so $\theta = \boxed{30^\circ}$.

R5. Luke is going on a camping trip with a tent in the shape of a square pyramid that has height 4 m and base length 6 m. He needs a tarp to cover the sides of the tent to protect himself from the rain. In m^2, what is the smallest area of tarp that Luke can buy?

SOLUTION

Consider one of the triangular faces of the square pyramid. These have heights of $\sqrt{3^2 + 4^2} = 5$. So the minimum area needed to cover one of these triangular faces is $\frac{1}{2} \cdot 5 \cdot 6 = 15$. There are 4 such faces, hence the smallest area of tarp that Luke can buy to cover all 4 faces of the pyramid is 60 m^2.

R6. The triangle ABC has lengths $AB = 20$ and $AC = 25$. The midpoint of AB is labelled M, and the midpoint of AC is labelled N. If the circle with diameter BM is tangential to the circle with diameter CN, what is the length of BC?

SOLUTION

Let O_1 be the midpoint of BM and O_2 be the midpoint of CN. Then clearly $4AO_1 = 3AB$ and $4AO_2 = 3AC$ and thus we know $\frac{AO_1}{AO_2} = \frac{AB}{AC}$, and thus O_1O_2 is parallel to BC. But we also know that the point of the tangent of the two circles must lie on the segment O_1O_2. Thus, $O_1O_2 = \frac{1}{4}(AB + AC) = 11.25$. Now as AO_1O_2 is similar to ABC (as O_1O_2 is parallel to BC), we must have $\frac{AO_1}{AB} = \frac{O_1O_2}{BC}$ and hence $BC = \frac{11.25 \times 20}{15} = \boxed{15}$.

R7. How many integer side-length triangles are there where two of the sides are length 20 and 25?

SOLUTION

Let the remaining side-length be x. We can make two "triangles" of area 0, where $x = 45$ or $x = 5$, obtained by having the sides of lengths 20 and 25 parallel. We can make an actual triangle with x taking any value between 5 and 45, so there are $\boxed{39}$ integer values.

R8. Three circles X, Y, Z having centres A, B, C respectively are externally tangential to each other. Let D on AB, E on BC, F on CA be the intersection points of each pair of circles. Let T_D, T_E, T_F be the circles' tangent lines at points D, E, F respectively.

Suppose the lines T_D, T_E form an angle $90°$, the lines T_E, T_F form an angle $120°$, the lines T_F, T_D form an angle $150°$, and $AB = 5$. Then $BC = \frac{a\sqrt{3}}{b}$ for some positive integers a, b. What is $10a + b$?

SOLUTION

First we solve for each angle of $\triangle ABC$: Draw the lines AB, BC, CA. Notice that the lines AB, T_D, T_F, AC bound a quadrilateral Q_A.

Since T_D is the tangent line to circles X and Y, it must be perpendicular to the radius (vector) of each circle, i.e. $T_D \perp AD$ and $T_D \perp DB$. Therefore the angles of Q_A at D and F are right angles.

Recall that angles in a quadrilateral sum to $360°$, which gives us $\angle BAC = 360° - 90° - 90° - \angle(T_D, T_F) = 30°$.

Similarly we obtain $\angle ABC = 90°$ and $\angle BCA = 60°$. (TIP: check that these three angles really add to $180°$!)

Now that we have all angles, use the sine formula:

$$\begin{aligned} \frac{AB}{\sin \angle BCA} &= \frac{BC}{\sin \angle BAC} \\ \frac{5}{\sqrt{3}/2} &= \frac{BC}{1/2} \\ BC &= \boxed{\frac{5}{\sqrt{3}}}. \end{aligned}$$

Hence $a = 5$, $b = 3$, Hence $10a + b = \boxed{53}$

Yellow Zone (Y)

Y1. The median of the dataset $\{4, 6, 7, 7, 9, x\}$ equals the mean of the dataset $\{4, 6, 7, 7, 9, x, y\}$ where x and y are both positive integers. What is $x + y$?

SOLUTION

The mean of the dataset $\{4, 6, 7, 7, 9, x, y\}$ is equal to $\frac{33+x+y}{7}$. As we need both x and y to be positive integers, we must have that the median of the data set $\{4, 6, 7, 7, 9, x\}$ is also an integer. Thus we need that $x \geq 7$ (else the median would be 6.5). When $x \geq 7$, the median of $\{4, 6, 7, 7, 9, x\} = 7$ and solving $\frac{33+x+y}{7} = 7$, we obtain $x + y = \boxed{16}$.

Y2. What is the sum of the coefficients of

$$(1 + x)(1 + x^2)(1 + x^4)$$

including the constant term?

SOLUTION

Expanding this out, we get

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7.$$

The sum of the coefficients is $\boxed{8}$. Note that we can get the sum of the coefficients by setting $x = 1$, so we didn't need to expand; rather, we could've calculated $(1 + 1)(1 + 1^2)(1 + 1^4)$.

INVESTIGATION

One way to see the expansion is by considering each power of x and finding all of the choices of terms in each bracket which multiply to get the right power of x. For example, there is only one way to get x^3: by choosing x, x^2 and 1 respectively. Determine the coefficients of the specialised *Jacobi triple product*

$$\prod_{n=1}^{\infty}(1 - x^{2n})(1 + x^{2n-1})^2$$

That is,

$$(1 - x^2)(1 + x)^2(1 - x^4)(1 + x^3)^2(1 - x^6)(1 + x^5)^2 \cdots$$

Y3. What is the sum of the coefficients of

$$(1+x)(5-x^2)(7+x^4)(17-x^8)$$

including the constant term?

SOLUTION

Using the idea in Y2, set $x = 1$: we get $2 \times 4 \times 8 \times 16 = 2^{10} = \boxed{1024}$.

Y4. What is the sum of all distinct solutions to $(x^2 - 6x + 9)^{(x^2+2x)} = 1$?

SOLUTION

The left-hand side expression equals 1 in two cases: either (i) $x^2-6x+9 = 1$; or (ii) $x^2 + 2x = 0$ and $x^2 - 6x + 9 \neq 0$. If both the base and the exponent are 0, the expression is undefined!

In the first case, we get $x = 2, 4$; in the second case, we get $x = 0, -2$. None of these solutions overlap (so the 0^0 worry doesn't materialise), so the sum of all solutions is $\boxed{4}$.

Y5. 2025 is a square number. How many years is it (from the year 2025) until the next year which is a square number?

SOLUTION

$2025 = 45^2$, so the next year which is a square number is 46^2. This happens in $46^2 - 45^2$ years; by the difference of two squares, $46^2 - 45^2 = (46-45)(46+45) = \boxed{91}$.

Y6. How many distinct values of x satisfy $(1+x)(1+x^2)(1+x^4) = 8$?

SOLUTION

From Y2, we may spot instantly that $x = 1$ satisfies the equation. We argue by cases that there are no other solutions:

- If $x > 1$, then $(1+x)(1+x^2)(1+x^4) > 8$; no solutions here.
- If $-1 < x < 1$, then $(1+x)(1+x^2)(1+x^4) < 8$; none here either.
- If $x \leq -1$, then $(1+x)$ is non-positive so the left-hand expression certainly isn't equal to 8.

So there is precisely $\boxed{1}$ solution.

Y7. Let $x_0 = 1$. We define the sequence x_n iteratively by $x_{n+1} = \dfrac{x_n + 6}{3x_n + 1}$. As n gets very large, x_n approaches $\sqrt{k}$. What is k?

SOLUTION

As x_n gets very large, $x_n \approx x_{n+1}$; set both to x. Then $x(3x+1) = x+6$, so $x^2 = 2$; each x_n is positive so $x_n \to \sqrt{2}$ as $n \to \infty$; $k = \boxed{2}$.

INVESTIGATION

Let $x_0 = 1$, and define the sequence x_n iteratively by

$$x_{n+1} = \frac{ax_n + b}{cx_n + d}$$

What are the conditions for which x_n approaches $\sqrt{k}$ as n gets large?

Y8. Rebekah chooses 5 random distinct non-zero digits A, B, C, D, E and computes the product $(A^A - 1)(B^B - 1)(C^C - 1)(D^D - 1)(E^E - 1)$. Let p be the probability that the last digit of the product is 0. What is $1/p$?

SOLUTION

One of A, B, C, D, E must be odd (there aren't enough non-zero even digits), so without loss of generality suppose A is odd. Then $A^A - 1$ is even. So for the last digit of the product to be 0, we need one of $X \in \{A, B, C, D, E\}$ to be such that $(X^X - 1)$ has a factor of 5. The only X that satisfy this are $X = 1, 4, 6, 8$ thus the only way to *not* have the last digit of the product be 0 is if (A, B, C, D, E) is some permutation of $(2, 3, 5, 7, 9)$. Thus $p = \frac{1}{126}$ so the answer is $\boxed{126}$.

Amber Zone (A)

A1. Let $g(x) = \cos(x - 270^\circ) + \cos(90^\circ - x)$. Let K be the maximum value of g, and k the minimum value. What is $K - k$?

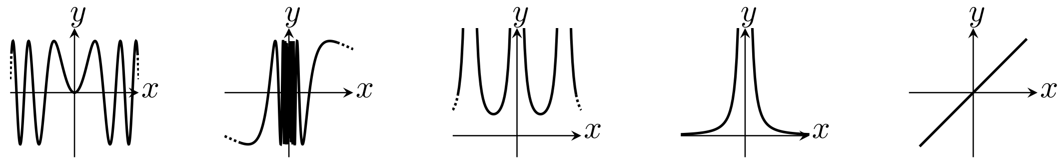
SOLUTION

We have

$$\begin{aligned}\cos(x - 270^\circ) &= \cos(x + 90^\circ) \\ &= -\cos(180^\circ - (x + 90^\circ)) \\ &= -\cos(90^\circ - x)\end{aligned}$$

so g is identically 0. Thus $k = K = 0$, so $K - k = \boxed{0}$.

A2. Let $p(x) = \sin x$, $q(x) = 1/x$ and $r(x) = x^2$. Cora chooses one of p, q, r, then Derek applies one of p, q, r (possibly the same) to Cora's function, then sketches the function. How many functions depicted below could be Derek's sketch?



SOLUTION

The graphs, from left to right, are

$$y = \sin(x^2), \quad y = \sin\left(\frac{1}{x}\right), \quad y = \frac{1}{\sin^2 x}, \quad y = \frac{1}{x^2}, \quad y = x$$

The only one which can't be made with two of p, q, r is the third one – note the last one is obtained by $q(q(x))$. So there are $\boxed{4}$ graphs which could be Derek's sketch.

INVESTIGATION

What are the properties of the graphs above which let us identify what functions Cora and Derek could've used?

A3. Let $f(x) = \sin x - \cos^2 x + \sin^3 x - \cos^4 x + \cdots + \sin^{99} x - \cos^{100} x$. Let M be the maximum value of f, and m the minimum value. What is $M - m$?

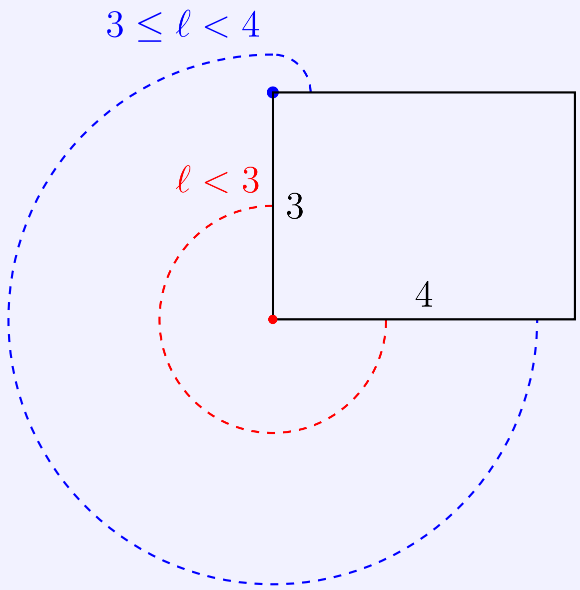
SOLUTION

Note $f(0^\circ) = -50$ and $f(90^\circ) = 50$. For any other values, either $|\sin x| < 1$, in which case the subsequent sine terms become negligible, or $|\cos x| < 1$, in which case the subsequent cosine terms become negligible. There is therefore no way for $|f(x)| > 50$; so $M - m = \boxed{100}$.

A4. A goat is tied at the corner of a 3×4 shed by a rope of length ℓ, where ℓ is a rational number. How long does the rope need to be for the goat to be able to graze in an area 28π? Submit your answer as $a + b$, where $\ell = a/b$ is the simplest fraction for ℓ.

SOLUTION

It's useful to split the goat's roaming area into cases. If $\ell \leq 3$, the roaming area is just $\frac{3}{4}\pi\ell^2 \leq \frac{27}{4}\pi$ – far too small. Something interesting happens for $\ell > 3$: first, if $3 < \ell \leq 4$, the goat can wrap around the shorter side of the shed to get some extra area. In the diagram below, the dotted red line represents the boundary for $\ell < 3$, and the blue represents for some $3 \leq \ell < 4$.



With the blue boundary, the goat can travel on the $\frac{3}{4}\pi\ell^2$ as before, but also gets an extra $\frac{1}{4}\pi(\ell - 3)^2$ (you can think of the goat trying to travel around the corner, and the rope is effectively tied to the blue dot with 3 units of length less). The area encompassed by the blue boundary is

$\frac{3}{4}\pi\ell^2 + \frac{1}{4}\pi(\ell-3)^2$. The maximum area in this case is $\frac{49}{4}\pi$ (taking $\ell = 4$) which isn't enough still.

If $4 \le \ell < 7$, with similar reasoning to the blue boundary the area the goat can graze in becomes $\frac{3}{4}\pi\ell^2 + \frac{1}{4}\pi(\ell-3)^2 + \frac{1}{4}\pi(\ell-4)^2$, with a maximum of 43π (with $\ell = 7$), which is enough! Set this formula equal to 28π. Then:

$$\begin{aligned} \tfrac{3}{4}\pi\ell^2 + \tfrac{1}{4}\pi(\ell-3)^2 + \tfrac{1}{4}\pi(\ell-4)^2 &= 28\pi \\ 3\ell^2 + (\ell-3)^2 + (\ell-4)^2 &= 112 \\ 5\ell^2 - 14\ell - 87 &= 0 \\ (5\ell - 29)(\ell+3) &= 0 \end{aligned}$$

Since $\ell > 0$, $\ell = 29/5$, so our final answer is $\boxed{34}$.

A5. For $0^\circ \le x < 90^\circ$, the minimum value of $\tan^2 x - 4\tan x + 5$ is achieved when the sine of x equals $a/\sqrt{b}$ where a and b are integers that share no common factors. What is the value of $a + b$?

SOLUTION

Completing the square, we get $(\tan x - 2)^2 + 1$, so this expression is minimised when $\tan x = 2$. Since $-90^\circ < x < 90^\circ$, we can think of x in a right-angled triangle with the angle x°, the opposite side having length 2 and the adjacent length 1. By Pythagoras' Theorem, the hypotenuse has length $\sqrt{5}$, so $\sin x = 2/\sqrt{5}$. Thus $a + b = \boxed{7}$.

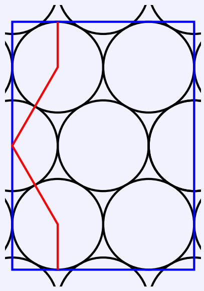
A6. Consider a cylinder with an army of 6 ants living on its curved surface. The ants each have a territory, which is the region of points which are less than 1 cm away when distance is measured along the surface. Territories aren't allowed to cross the edges of the cylinder, and – because they don't get along – no two ants' territories can overlap.

Let P, H be its base perimeter and height of the cylinder respectively. The quantity $C = 2(P + H)$ has a minimum value of $a + b\sqrt{3}$ cm, where a, b are integers. What is the value of $a + 10b$? *[Hint: what does C represent geometrically?]*

SOLUTION

It may be hard to visualise what the ants' territory look like, but there is a sneaky trick: we can flatten the cylinder by cutting it along its height and "unrolling" it onto a plane rectangle. Flattening the cylinder does not change the distance between any pair of points. So the ants' territories correspond to discs in a rectangle. However, we have to keep in mind that circle packings on this rectangle must loop around on one of the pairs of sides. Moreover, notice that C is equal to the perimeter of this rectangle. Intuitively, the closer the rectangle is to a square, the smaller its perimeter. So we should try to pack 6 circles into a rectangle close to a square.

Here we use another trick: recall that for a circle packing of the whole plane, the tightest arrangement is that of a hexagonal grid (where each circle touches 6 other congruent circles). We can find the optimal answer by drawing the rectangle with smallest perimeter on this grid, such that circles it contains 6 circles which can only cross its looping edges. From here, working by trial and error, we see that the following arrangement is the optimal rectangle:



This gives the minimum $C_{\min}$ to be $2(4+(2+2\sqrt{3})) = 12+4\sqrt{3}$, so $a+10b = \boxed{52}$.

INVESTIGATION

Intuitively, the closer the rectangle is to a square, the smaller its perimeter. Prove this is true. *[Hint: complete the square.]*

A7. Consider an equilateral triangle ABC of side length 1. Let P be a point in the interior, and consider the region of points within ABC that is closer to P than to any of A, B, C. For different points, P, this region has different areas. The maximum area is M and the minimum area is m. What is the ratio M/m equal to?

SOLUTION

It suffices to vary P across a sixth of the triangle ABC (e.g. between A, the midpoint of AB — call it X — and the centroid — call it G) by symmetry of ABC. Let Z_p be the area described by the point p. Construct the vectors $\mathbf{a} = AX$ and $\mathbf{b} = AG$. For the points across $\lambda\mathbf{a}$, $\lambda\mathbf{b}$ with $0 \leq \lambda \leq 1$, the area is extreme for $\lambda = 0$ or $\lambda = 1$ (between, clearly maxima and minima aren't achieved). Anywhere between these points are also not extreme. One can also verify that $Z_A < Z_X < Z_G$, so $m = Z_A$ and $M = Z_G$. Moreover m is a third of the area of the triangle, and M is two thirds, so $\frac{M}{m} = \boxed{2}$.

A8. Longname's quadrilateral $ABCD$ satisfies $AB = BC = CD = 5$ and $\angle ABC = \angle ADB = 90°$. What is the area of $ABCD$?

SOLUTION

Let E be the foot of the perpendicular from C to BD. Now as $BC = CD$, BCD is an isosceles triangle. As $\angle ADB = \angle ABC = 90°$,

$$\begin{aligned} \angle DAB &= 90° - \angle ABD \\ &= \angle DBC \\ &= \angle BDC \end{aligned}$$

But we also know that $AB = BC = CD$ and thus by ASA congruency we can deduce that the triangles DAB, CBE, and CDE are congruent and hence have the same area. Now setting DA to have a length of x, we can deduce that $DB = 2x$ and thus the area of $DAB = x^2$. By Pythagoras we know that

$$\begin{aligned} 5 &= AB \\ &= \sqrt{DA^2 + DB^2} \\ &= \sqrt{5x^2} \end{aligned}$$

Hence $x^2 = 5$. And thus the area of $ABCD = \boxed{15}$.

Green Zone (G)

G1. For what c does the equation

$$c\,(y-x)^2 - 4x - cy = 0$$

have reflective symmetry about the line $y = x$?

SOLUTION

A graph is symmetric about $y = x$ iff swapping y and x doesn't change the equation of the graph. For them to have the same expression then, we need $c = \boxed{4}$.

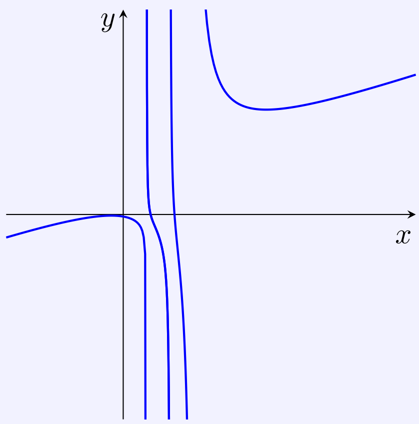
G2. For $a \neq 0$, how many segments does the curve

$$\frac{1}{x-a} + \frac{x}{x-2a} + \frac{x^2}{x-3a}$$

split the plane into?

SOLUTION

The expression goes to ∞ as $x \to \infty$, and to $-\infty$ as $x \to -\infty$. Further, there are vertical asymptotes at $a, 2a, 3a$, which are all distinct since $a \neq 0$. This divides the plane into $\boxed{5}$ regions as can be seen in a graph.



G3. Your friend is thinking of a function f that takes positive integer inputs and outputs positive integers. She tells you that f has the following properties:

- If p is a prime number, then $f(p)$ is also a prime number.
- For all integers $n > 1$, $f(n^2 - 1) = (f(n))^2 + 1$.

Given these properties, there exists a positive integer N such that $f(N)$ must be a 3-digit number. What is N?

SOLUTION

We get a foothold by noting $f(2)$, $f(3)$ are prime and $f(3) = f(2)^2 + 1$; so $f(2) = 2$, $f(3) = 5$, because $f(2)$ and $f(3)$ must have opposite parity, and 2 is the only even prime. Now $f(8) = f(3)^2 + 1 = 26$, and $f(63) = f(8)^2 + 1 = \boxed{677}$.

G4. The curve

$$\frac{1}{x-a} + \frac{x}{x-b} + \frac{x^2}{x-c}$$

is graphed, alongside a parabola $Ax^2 + B$. Let M and m be the maximum and minimum possible number of intersections of the parabola and the curve respectively. What is $M - m$?

SOLUTION

As seen in the graph from above in G2, we can see that there is at most 5 intersections. The minimum is 1, when $a = b = c$. This is the best we can do, as the parabola must grow faster than the curve, as the curve becomes linear as $x \to \infty$. Alternatively, we can set the two equations equal to each other and find roots:

$$\frac{1}{x-a} + \frac{x}{x-b} + \frac{x^2}{x-c} = Ax^2 + B$$

then multiplying out the fractions, we get

$$\begin{aligned}&(x-b)(x-c) + x(x-a)(x-c) + x^2(x-a)(x-b)\\ =\ &(Ax^2+B)(x-a)(x-b)(x-c)\end{aligned}$$

We don't care too much about simplifying this expression; what matters is that the highest power of x is x^5, meaning there can be at most five solutions, and at least 1.

INVESTIGATION

Odd-degree polynomials – that is, a polynomial whose highest power of x is odd – always have a root. Why?

G5. Gunther foolishly claims that he has proven $\log(x+y) = \log(x) + \log(y)$, as he has found positive integers $a \leq b \leq c$ such that $\log(a+b+c) = \log(a) + \log(b) + \log(c)$. How many different (a, b, c) could Gunther have found?

SOLUTION

As $\log(a) + \log(b) + \log(c) = \log(abc)$, we have $abc = a + b + c$. As $a \leq b \leq c$, $abc = a + b + c \leq 3c$, so in particular $ab \leq 3$, we check cases:

(a) $a = b = 1$, gives $c = 2 + c$, leading to no solution.

(b) $a = 1, b = 2$ gives $2c = 3 + c$, giving $c = 3$.

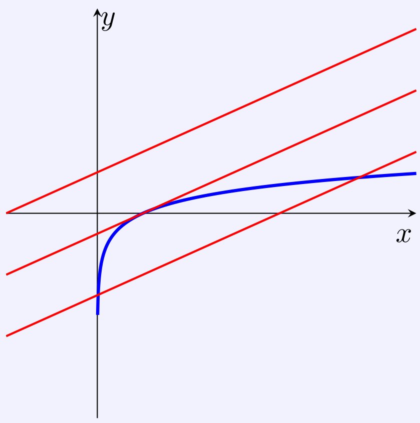
(c) $a = 1, b = 3$ gives $3c = 4 + c$, giving $c = 2$, but we want $b \leq c$

Hence we have a unique solution $a = 1, b = 2, c = 3$, giving us the answer of $\boxed{1}$.

G6. $2x - \ln(x^2 - y^2) = c$ is graphed for positive x and y. What is the minimum value of c for which the graph has an x-intercept? *[Hint: the tangent to* $\ln(x)$ *has gradient equal to* $1/x$ *for each* x*.]*

SOLUTION

For the graph have a x-intercept, we plug $y = 0$ to get $x - \ln(x) = \frac{c}{2}$.



In the graph above we have plotted $\ln(x)$, and $x - \frac{c}{2}$ for 3 different values of c, the higher lines having smaller c value.

We note that the graphs $x - \frac{c}{2}$ and $\ln(x)$ have no intersections for large negative c, have 2 intersections for large positive c, and have 1 intersection at exactly one point. It it at this one point that the graph will first have an x-intercept. At this point, the line $x - \frac{c}{2}$ is tangent to $\ln(x)$. As suggested in the hint, the slope of the tangent of $\ln x$ is $1/x$, and the slope of $x - \frac{c}{2}$ is 1, hence $\frac{1}{x} = 1$, we see that the graphs meet at $(1, 0)$. Thus we get $c = 2$, hence the minimum value is $\boxed{2}$.

G7. Let $f(x)$ be a function with the property $f(f(x)) = x$. If $f(x) = 0$ whenever x is an integer, what is the greatest integer that is in the range of f?

SOLUTION

Let $f(y) = n$, where n is an integer. Then $y = f(f(y)) = f(n) = 0$. But then $f(f(n)) = f(0) = 0$ as 0 is an integer, and $f(f(n)) = n$, so $n = 0$. Thus the only integer in the range of f is 0, hence the largest integer value in the range of f is just $\boxed{0}$.

G8. Brad has a function that satisfies $f(xy) = f(x) + f(y)$, and also knows the values of f at n distinct points. What is the least value of n such that Brad can know the values of $f(1), f(2), \ldots, f(50)$?

SOLUTION

We plug in $x = y = 1$ to the function property and find $f(1) = 0$. It is then sufficient to know the values of $f(x)$ on the primes less than or equal to 50, of which there are 15.

This is also necessary, as one could arbitrarily change values that f takes on the primes, which would not contradict any other values.

Noting that less than 15 values can only determine less than 15 primes (linear equations), we conclude that the least value of n is $\boxed{15}$.

Cyan Zone (C)

C1. We say a function f is *shrike* if it satisfies the following properties: (i) the domain and range of f is the set $\{1, 2, 3, 4, 5\}$; (ii) for all x in the domain, $f(f(x)) = x$.

How many shrike functions are there such that $f(4) = 4$ and $f(5) = 5$?

SOLUTION

As $f(4) = 4$ and $f(5) = 5$, our only “choices” are where we map 1, 2 and 3. One option is that f is the identity map, i.e. $f(1) = 1$, $f(2) = 2$ and $f(3) = 3$. Otherwise, without loss of generality suppose $f(1) = 2$. In order for $f(f(x)) = x$, we need $f(2) = 1$, which forces $f(3)$ to equal 3. So there are three non-identity functions with $f(4) = 4$ and $f(5) = 5$, which correspond to choosing two of 1, 2 and 3 to pair together.

The final answer is $\boxed{4}$.

C2. Rosie, Angie and Ella are standing in a line, with Rosie being at the front and Ella at the back. Each of them are wearing a jersey with a distinct natural number picked from the set $\{1, 2, 5, 10, 19\}$ on their back, which they cannot see themselves. Each person can see the numbers of people ahead of them.

First, Ella says that she doesn't know if her number is even or odd. Then Angie says she doesn't know if her number is even or odd. Then Rosie then says she knows whether her number is even or odd.

What are the sum of the numbers that Rosie could be wearing?

SOLUTION

If Ella *did* know whether her number was odd or even, then it would be that she can see 2 and 10 ahead of her, leaving her with only odd numbers. The fact that she doesn't know means that there can't both be 2 and 10 on Angie and Rosie's backs.

Now if Angie then *did* know whether her number was odd or even, then she must see either 2 or 10, forcing her number to be odd. The fact that she doesn't know means that she sees an odd number.

So Rosie's number must be odd. Thus the sum of numbers she could be wearing are $1 + 5 + 19 = \boxed{25}$.

INVESTIGATION

What are the possible yes-no responses from Rosie, Angie and Ella?

C3. How many shrike functions are there such that $f(1) = 5$? *[See C1. for the definition of a shrike function.]*

SOLUTION

This is exactly the same as in C1; we have three free choices of mappings (2, 3 and 4) so there are again $\boxed{4}$ such shrike functions.

C4. Yoshi starts with a capital of £150. A game consists of a sequence of up to fifty **A**'s and **B**'s, where **A** and **B** are the following actions:

A: Yoshi loses £1.

B: If Yoshi's capital is even, he wins £3. Otherwise, he loses £5.

Yoshi breaks even after a sequence of games if his capital is £150. What is the sum of all game-lengths where Yoshi can break even?

SOLUTION

Considering the capital's remainder when divided by 4, after either action of **A** or **B** it always decreases by one. Thus one can only break even on games of lengths which are multiples of 4. But which such games? It is not too hard to check that one can break even on a game of length 4 (e.g., **BAAA**) so we can break even on any game length which is a multiple of 4 less than 50 (by repeating **BAAA**, for example) so we should add $4 + 8 + 12 + \cdots + 48$. This gives $\boxed{312}$.

C5. How many shrike functions are there in total? *[See C1. for the definition of a shrike function.]*

SOLUTION

We can split the shrike functions into how many swapping-pairs there are (for example, in C3 we would say that 1 and 5 are swapping-pairs).

- If there are no swapping pairs, there is 1 shrike function, the identity function.
- If there is one swapping-pair, then we have ${}^5C_2 = 10$ swapping-pairs; the remaining part of the map is determined so there are 10 such shrike functions.

- If there are two swapping-pairs, then we have $({}^5C_2 \times {}^3C_2)/2$ mappings (this is the same problem as C6) so there are 15 such maps.

In total, there are $\boxed{26}$ shrike functions.

C6. Bobby has five distinguishable socks. How many ways can he make two pairs of socks?

SOLUTION

Suppose the socks are labelled A, B, C, D, E.

To pick our first pair, we have $(5 \times 4)/2$ choices: 5 for the first pick, 4 for the second; we divide by 2 because the order we picked the pair doesn't matter (for example, we want to count picking A then B as the same as B then A). Then for our second pair, we have $(3 \times 2)/2$ choices. Lastly, the order by which we picked the pairs don't matter (picking the pair AB then CD should be counted the same as picking CD then AB). Multiplying these together, we get $\boxed{15}$.

C7. A friendly enemy writes the numbers $1, 2, 3, 4, 5$ onto a dark blue wall. Repeatedly, you erase two numbers – call them a and b, with $a > b$ – and write either $2a + b$ or $a + 7b$ onto the wall. This process repeats until there is only one number left. What is the largest number you can create?

SOLUTION

There's a naïve approach, where we take the biggest two numbers at each available opportunity. This gives

$$(5,4) \mapsto 33; \quad (33,3) \mapsto 69; \quad (69,2) \mapsto 140; \quad (140,1) \mapsto 281.$$

But we can do better by trying to take advantage of the $a + 7b$ option. If we get two medium-sized numbers then $a + 7b$ can be quite big! Indeed, the approach

$$(3,4) \mapsto 25; \quad (2,5) \mapsto 19; \quad (19,25) \mapsto 158; \quad (158,1) \mapsto 317$$

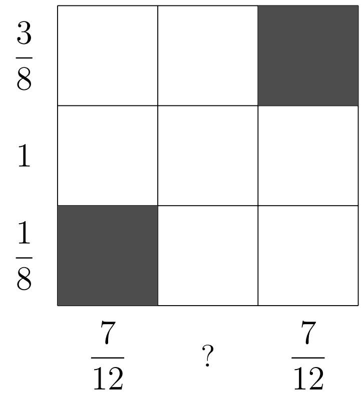
is optimal, so the largest number you can create is $\boxed{317}$.

Blue Zone (B)

B1. With the unit fractions

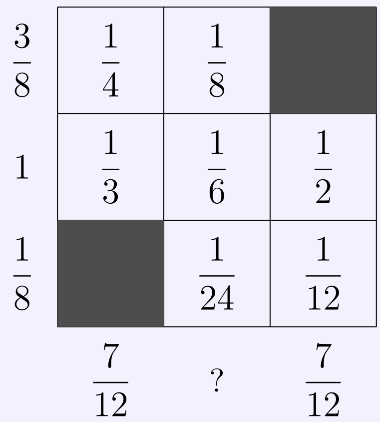
$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{12}, \frac{1}{24},$$

Angus arranges them in the white squares in the grid below, and then writes the sums of the rows and columns as shown in the diagram. What's the reciprocal of the sum of the middle column?



SOLUTION

Filled in, the grid looks like



So $? = \frac{1}{3}$, so its reciprocal is $\boxed{3}$. To get started on filling in the grid, it's helpful to note that $1/8$ must be the sum of $1/12$ and $1/24$ (all the other fractions are too big); this can then help with the right-hand $7/12$ column, which in turn helps narrows down the possibilities for the left-hand $7/12$ column.

B2. For a positive integer N, let d be the sum of its digits. We say N is well-fed if $2d < N < 4d$. What is the largest well-fed number?

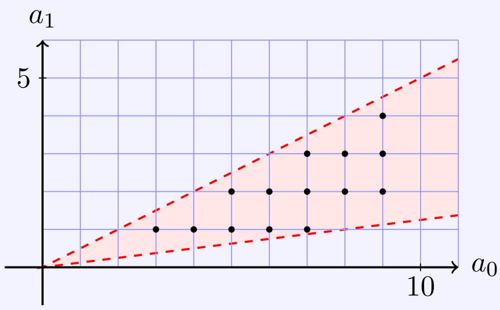
SOLUTION

We'll start with 1-digit N, then move onto 2- and 3-digit N. If $N = a_0$ for a_0 a digit, then $d = a_0$; we need $2d < N$ so $2a_0 < a_0$, but this is impossible.

If $N = 10a_1 + a_0$ then $d = a_1 + a_0$. We need $2d < N < 4d$, so we have

$$\begin{aligned} 2a_1 + 2a_0 < 10a_1 + a_0 &\implies a_0 < 8a_1 \\ 10a_1 + a_0 < 4a_1 + 4a_0 &\implies 2a_1 < a_0 \end{aligned}$$

We can plot this graphically:



So the largest 2-digit well-fed number is 49.

Can we have a well-fed number with more digits? No – if $N = 100a_2 + 10a_1 + a_0$ then $d = a_2 + a_1 + a_0$. We need $N < 4d$, so $100a_2 + 10a_1 + a_0 < 4a_2 + 4a_1 + 4a_0$, which is equivalent to $a_0 > 2a_1 + 32a_2$; this is impossible if $a_2 \neq 0$. So the largest well-fed number is $\boxed{49}$.

B3. The unit fractions $1/a$, $1/b$ and $1/c$ sum to 1 where a, b, c are positive integers such that $a \leq b \leq c$. How many distinct solutions of (a, b, c) are there?

SOLUTION

Without loss of generality, suppose $a \leq b \leq c$. We can spot the solutions $(a, b, c) = (2, 3, 6)$ and $(2, 4, 4)$. Suppose for a contradiction that $a = 2$ and $b > 4$; then $\frac{1}{c} = 1 - \frac{1}{a} - \frac{1}{b} > \frac{1}{2} - \frac{1}{4}$. So $\frac{1}{c} > 1/4$ so $c < 4$: a contradiction (by ordering of b, c). If $a = 3$, we may also spot $(a, b, c) = (3, 3, 3)$. One can make a similar argument that, if $a = 3$ and $b > 3$, we get no well-ordered solutions. Lastly, if $a > 3$ then since $a \leq b \leq c$ the sum can only be as big as $\frac{3}{4}$, which is not big enough. Now, removing the well-ordering restriction we get $\boxed{10}$ solutions.

B4. What, from left to right, are the last 4 digits of 11^{2025}?

SOLUTION

Consider the Binomial expansion of $(10+1)^{2025}$. All of the terms with 10^4 or higher are essentially irrelevant in how they affect the last four digits, so we only need to consider the last four digits of

$$^{2025}C_3 10^3 1^{2022} + {}^{2025}C_2 10^2 1^{2023} + {}^{2025}C_1 10^1 1^{2024} + 1$$

We know $^{2025}C_1 = 2025$, we can calculate

$$^{2025}C_2 = \frac{2025!}{2!2023!} = \frac{2025 \times 2024}{2} = 2025 \times 1012 = 2049300$$

In fact, it suffices to calculate 25×12 because

$$2025 \times 1012 = (2000 + 25)(1000 + 12)$$

and the terms with factors of 1000 don't affect the last four digits (since we're multiplying this term by 10^2). For $^{2025}C_3$, as long as we know there's a factor of 100, the 10^3 means this has no effect. Indeed,

$$^{2025}C_3 = \frac{2025!}{3!2022!} = \frac{2025 \times 2024 \times 2023}{6} = 675 \times 1012 \times 2023$$

675 has a factor of 25, and 1012 has a factor of 4, so we have a factor of 100.

Therefore, the only terms affecting the last four digits is 1 and 20250, so the last four digits are $\boxed{0251}$.

B5. A cuboid has side-lengths that are all integers in cm; its surface area is N cm^2, and its volume is N cm^3. What is the maximum possible value of N?

SOLUTION

Let (positive) integers x, y, z be the three side lengths of the cuboid in cm. The surface area is equal to the sum of six rectangles, and hence equals $2(xy + yz + zx)$. The volume is equal to xyz.

We are given that the surface area (measured in cm^2) and volume (measured in cm^3) are both equal to some number N; so we set up the equation $2(xy + yz + zx) = xyz$ and try to solve for x, y, z. Dividing both sides by xyz we obtain the equation

$$\frac{1}{2} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

Now without loss of generality, suppose $x \leq y \leq z$. If $x > 6$ then we have

$\frac{1}{2} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} < \frac{3}{6}$, a contradiction. So we must have $x \leq 6$. But note we also must have $x > 2$ as both y and z are positive integers, so we have 4 cases for x to solve for.

Now $2(xy + yz + zx) = xyz$ gives $2y + \frac{(2-x)yz}{x} + 2z = 0$. Hence

$$y\left(2 + \frac{(2-x)z}{x}\right) + 2z + \frac{4x}{(2-x)} = \frac{4x}{(2-x)}$$

$$\left(y + \frac{2x}{(2-x)}\right)\left(2 + \frac{(2-x)z}{x}\right) = \frac{4x}{(2-x)}$$

$$\left(y + \frac{2x}{(2-x)}\right)\left(z + \frac{2x}{(2-x)}\right) = \frac{4x^2}{(2-x)^2}$$

If $x = 3$, we have

$$(y-6)(z-6) = 36$$

giving us the five solutions $(x, y, z) = (3, 7, 42), (3, 8, 24), (3, 9, 18), (3, 10, 15), (3, 12, 12)$.

If $x = 4$, we have

$$(y-4)(z-4) = 16$$

giving us the solutions $(x, y, z) = (4, 5, 20), (4, 6, 12), (4, 8, 8)$.

If $x = 5$, we have

$$\left(y - \frac{10}{3}\right)\left(z - \frac{10}{3}\right) = \frac{100}{9}$$

giving us the solutions $(x, y, z) = (5, 5, 10)$.

And finally if $x = 6$, we have

$$(y-3)(z-3) = 9$$

giving us the solutions $(x, y, z) = (6, 6, 6)$.

Solving for N, we find the maximum value N is $\boxed{882}$.

INVESTIGATION

Show that there is a maximum value of N.

B6. There is only one integer n between 1 and 100 such that the sum of the digits of n is half of the sum of the digits of $3n$. What is n?

SOLUTION

For brevity, denote $S(n)$ denote the sum of the digits of n. We can write $2S(n) = S(3n)$. Note $S(3n)$ is a multiple of 3, so $S(n)$ is a multiple of 3, so n is a multiple of 3, so $S(3n)$ is a multiple of 9, so n is a multiple of 9. With this insight, our search of choices is reduced greatly, so only the multiples of 9 needing to be checked. We can then see that 63 is the only such number with this property, so the answer is $\boxed{63}$.

B7. When $25!$ is multiplied by 5^n, it has the highest possible number of zero digits at the end of the number. What is the least value of n?

SOLUTION

To solve this problem, we count the highest power of 2 that divides $25!$. This is given by

$$\left\lfloor \frac{25}{2} \right\rfloor + \left\lfloor \frac{25}{4} \right\rfloor + \left\lfloor \frac{25}{8} \right\rfloor + \left\lfloor \frac{25}{16} \right\rfloor = 22$$

where $\lfloor x \rfloor$ is x rounded down to the nearest integer (so $\lfloor 25/k \rfloor$ is the number of whole k's that can fit in 25). We also count the highest power of 5 that divides $25!$ which is given by

$$\left\lfloor \frac{25}{5} \right\rfloor + \left\lfloor \frac{25}{25} \right\rfloor = 6$$

To get a zero at the end of a number, we need a factor of $10 = 2 \cdot 5$; so the highest possible number of zero digits at the end of $25! \cdot 5^n$ is 22. As we already have six 5's in $25!$, the least value of n is $\boxed{16}$.

INVESTIGATION

Let f be the function defined on positive integers with the following rules:

$$f(2n+1) = 0 \text{ for all even } n, \qquad f(2n) = f(n) + 1 \text{ for all } n$$

What is the value of the sum $f(1) + f(2) + \cdots + f(100)$?