PROOFS IN MATHEMATICS A1

A1.1 Introduction

The ability to reason and think clearly is extremely useful in our daily life. For example, suppose a politician tells you, 'If you are interested in a clean government, then you should vote for me.' What he actually wants you to believe is that if you do not vote for him, then you may not get a clean government. Similarly, if an advertisement tells you, 'The intelligent wear *XYZ* shoes', what the company wants you to conclude is that if you do not wear *XYZ* shoes, then you are not intelligent enough. You can yourself observe that both the above statements may mislead the general public. So, if we understand the process of reasoning correctly, we do not fall into such traps unknowingly.

The correct use of reasoning is at the core of mathematics, especially in constructing proofs. In Class IX, you were introduced to the idea of proofs, and you actually proved many statements, especially in geometry. Recall that a proof is made up of several mathematical statements, each of which is logically deduced from a previous statement in the proof, or from a theorem proved earlier, or an axiom, or the hypotheses. The main tool, we use in constructing a proof, is the process of deductive reasoning.

We start the study of this chapter with a review of what a mathematical statement is. Then, we proceed to sharpen our skills in deductive reasoning using several examples. We shall also deal with the concept of negation and finding the negation of a given statement. Then, we discuss what it means to find the converse of a given statement. Finally, we review the ingredients of a proof learnt in Class IX by analysing the proofs of several theorems. Here, we also discuss the idea of proof by contradiction, which you have come across in Class IX and many other chapters of this book.

A1.2 Mathematical Statements Revisited

Recall, that a 'statement' is a meaningful sentence which is not an order, or an exclamation or a question. For example, 'Which two teams are playing in the

Cricket World Cup Final?' is a question, not a statement. 'Go and finish your homework' is an order, not a statement. 'What a fantastic goal!' is an exclamation, not a statement.

Remember, in general, statements can be one of the following:

- *always true*
- *always false*
- *ambiguous*

In Class IX, you have also studied that in mathematics, **a statement is acceptable only if it is either always true or always false**. So, ambiguous sentences are not considered as mathematical statements.

Let us review our understanding with a few examples.

Example 1 : State whether the following statements are always true, always false or ambiguous. Justify your answers.

(i) The Sun orbits the Earth.

(ii) Vehicles have four wheels.

(iii) The speed of light is approximately 3×10^5 km/s.

(iv) A road to Kolkata will be closed from November to March.

(v) All humans are mortal.

Solution :

(i) This statement is always false, since astronomers have established that the Earth orbits the Sun.

(ii) This statement is ambiguous, because we cannot decide if it is always true or always false. This depends on what the vehicle is — vehicles can have 2, 3, 4, 6, 10, etc., wheels.

(iii) This statement is always true, as verified by physicists.

(iv) This statement is ambiguous, because it is not clear which road is being referred to.

(v) This statement is always true, since every human being has to die some time.

Example 2 : State whether the following statements are true or false, and justify your answers.

(i) All equilateral triangles are isosceles.

(ii) Some isosceles triangles are equilateral.

(iii) All isosceles triangles are equilateral.

(iv) Some rational numbers are integers.

(v) Some rational numbers are not integers.

(vi) Not all integers are rational.

(vii) Between any two rational numbers there is no rational number.

Solution :

(i) This statement is true, because equilateral triangles have equal sides, and therefore are isosceles.

(ii) This statement is true, because those isosceles triangles whose base angles are $60°$ are equilateral.

(iii) This statement is false. Give a counter-example for it.

(iv) This statement is true, since rational numbers of the form $\frac{p}{q}$, where p is an integer and $q = 1$, are integers (for example, $3 = \frac{3}{1}$).

(v) This statement is true, because rational numbers of the form $\frac{p}{q}$, p, q are integers and q does not divide p, are not integers (for example, $\frac{3}{2}$).

(vi) This statement is the same as saying 'there is an integer which is not a rational number'. This is false, because all integers are rational numbers.

(vii) This statement is false. As you know, between any two rational numbers r and s lies $\frac{r+s}{2}$, which is a rational number.

Example 3 : If $x < 4$, which of the following statements are true? Justify your answers.

(i) $2x > 8$ (ii) $2x < 6$ (iii) $2x < 8$

Solution :

(i) This statement is false, because, for example, $x = 3 < 4$ does not satisfy $2x > 8$.

(ii) This statement is false, because, for example, $x = 3.5 < 4$ does not satisfy $2x < 6$.

(iii) This statement is true, because it is the same as $x < 4$.

Example 4 : Restate the following statements with appropriate conditions, so that they become true statements:

(i) If the diagonals of a quadrilateral are equal, then it is a rectangle.

(ii) A line joining two points on two sides of a triangle is parallel to the third side.

(iii) $\sqrt{p}$ is irrational for all positive integers p.

(iv) All quadratic equations have two real roots.

Solution :

(i) If the diagonals of a parallelogram are equal, then it is a rectangle.

(ii) A line joining the mid-points of two sides of a triangle is parallel to the third side.

(iii) $\sqrt{p}$ is irrational for all primes p.

(iv) All quadratic equations have at most two real roots.

Remark : There can be other ways of restating the statements above. For instance, (iii) can also be restated as '$\sqrt{p}$ is irrational for all positive integers p which are not a perfect square'.

EXERCISE A1.1

1. State whether the following statements are always true, always false or ambiguous. Justify your answers.

(i) All mathematics textbooks are interesting.

(ii) The distance from the Earth to the Sun is approximately 1.5×10^8 km.

(iii) All human beings grow old.

(iv) The journey from Uttarkashi to Harsil is tiring.

(v) The woman saw an elephant through a pair of binoculars.

2. State whether the following statements are true or false. Justify your answers.

(i) All hexagons are polygons.

(ii) Some polygons are pentagons.

(iii) Not all even numbers are divisible by 2.

(iv) Some real numbers are irrational.

(v) Not all real numbers are rational.

3. Let a and b be real numbers such that $ab \neq 0$. Then which of the following statements are true? Justify your answers.

(i) Both a and b must be zero.

(ii) Both a and b must be non-zero.

(iii) Either a or b must be non-zero.

4. Restate the following statements with appropriate conditions, so that they become true.

(i) If $a^2 > b^2$, then $a > b$.

(ii) If $x^2 = y^2$, then $x = y$.

(iii) If $(x + y)^2 = x^2 + y^2$, then $x = 0$.

(iv) The diagonals of a quadrilateral bisect each other.

A1.3 Deductive Reasoning

In Class IX, you were introduced to the idea of deductive reasoning. Here, we will work with many more examples which will illustrate how **deductive reasoning** is used to **deduce** conclusions from given statements that we assume to be true. The given statements are called 'premises' or 'hypotheses'. We begin with some examples.

Example 5 : Given that Bijapur is in the state of Karnataka, and suppose Shabana lives in Bijapur. In which state does Shabana live?

Solution : Here we have two premises:

(i) Bijapur is in the state of Karnataka (ii) Shabana lives in Bijapur

From these premises, we deduce that Shabana lives in the state of Karnataka.

Example 6 : Given that all mathematics textbooks are interesting, and suppose you are reading a mathematics textbook. What can we conclude about the textbook you are reading?

Solution : Using the two premises (or hypotheses), we can deduce that you are reading an interesting textbook.

Example 7 : Given that $y = -6x + 5$, and suppose $x = 3$. What is y?

Solution : Given the two hypotheses, we get $y = -6(3) + 5 = -13$.

Example 8 : Given that ABCD is a parallelogram, and suppose AD = 5 cm, AB = 7 cm (see Fig. A1.1). What can you conclude about the lengths of DC and BC?

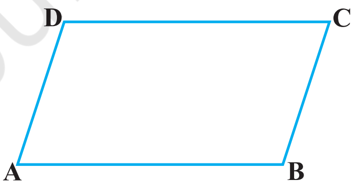


Fig. A1.1

Solution : We are given that ABCD is a parallelogram. So, we deduce that all the properties that hold for a parallelogram hold for ABCD. Therefore, in particular, the property that 'the opposite sides of a parallelogram are equal to each other', holds. Since we know AD = 5 cm, we can deduce that BC = 5 cm. Similarly, we deduce that DC = 7 cm.

Remark : In this example, we have seen how we will often need to find out and use properties hidden in a given premise.

Example 9 : Given that $\sqrt{p}$ is irrational for all primes p, and suppose that 19423 is a prime. What can you conclude about $\sqrt{19423}$?

Solution : We can conclude that $\sqrt{19423}$ is irrational.

In the examples above, you might have noticed that we do not know whether the hypotheses are true or not. We are **assuming** that they are true, and then applying deductive reasoning. For instance, in Example 9, we haven't checked whether 19423 is a prime or not; we assume it to be a prime for the sake of our argument.What we are trying to emphasise in this section is that given a particular statement, how we use deductive reasoning to arrive at a conclusion. What really matters here is that we use the correct process of reasoning, and this process of reasoning does not depend on the trueness or falsity of the hypotheses. However, it must also be noted that if we start with an incorrect premise (or hypothesis), we may arrive at a wrong conclusion.

EXERCISE A1.2

1. Given that all women are mortal, and suppose that A is a woman, what can we conclude about A?

2. Given that the product of two rational numbers is rational, and suppose a and b are rationals, what can you conclude about ab?

3. Given that the decimal expansion of irrational numbers is non-terminating, non-recurring, and $\sqrt{17}$ is irrational, what can we conclude about the decimal expansion of $\sqrt{17}$?

4. Given that $y = x^2 + 6$ and $x = -1$, what can we conclude about the value of y?

5. Given that ABCD is a parallelogram and $\angle B = 80°$. What can you conclude about the other angles of the parallelogram?

6. Given that PQRS is a cyclic quadrilateral and also its diagonals bisect each other. What can you conclude about the quadrilateral?

7. Given that $\sqrt{p}$ is irrational for all primes p and also suppose that 3721 is a prime. Can you conclude that $\sqrt{3721}$ is an irrational number? Is your conclusion correct? Why or why not?

A1.4 Conjectures, Theorems, Proofs and Mathematical Reasoning

Consider the Fig. A1.2. The first circle has one point on it, the second two points, the third three, and so on. All possible lines connecting the points are drawn in each case.

The lines divide the circle into mutually exclusive regions (having no common portion). We can count these and tabulate our results as shown :

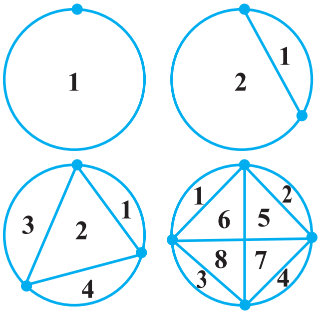


Fig. A1.2

Number of points	Number of regions
1	1
2	2
3	4
4	8
5	
6	
7	

Some of you might have come up with a formula predicting the number of regions given the number of points. From Class IX, you may remember that this intelligent guess is called a '*conjecture*'.

Suppose your conjecture is that given 'n' points on a circle, there are 2^{n-1} mutually exclusive regions, created by joining the points with all possible lines. This seems an extremely sensible guess, and one can check that if $n = 5$, we do get 16 regions. So, having verified this formula for 5 points, are you satisfied that for any n points there are 2^{n-1} regions? If so, how would you respond, if someone asked you, how you can be sure about this for $n = 25$, say? To deal with such questions, you would need a proof which shows beyond doubt that this result is true, or a counter-example to show that this result fails for some 'n'. Actually, if you are patient and try it out for $n = 6$, you will find that there are 31 regions, and for $n = 7$ there are 57 regions. So, $n = 6$, is a counter-example to the conjecture above. This demonstrates the power of a counter-example. You may recall that in the Class IX we discussed that to **disprove a statement, it is enough to come up with a single counter-example**.

You may have noticed that we insisted on a proof regarding the number of regions in spite of verifying the result for $n = 1, 2, 3, 4$ and 5. Let us consider a few more examples. You are familiar with the following result (given in Chapter 5):

$1 + 2 + 3 + ... + n = \dfrac{n(n+1)}{2}$. To establish its validity, it is not enough to verify the result for $n = 1, 2, 3$, and so on, because there may be some 'n' for which this result is not true (just as in the example above, the result failed for $n = 6$). What we need is a proof which establishes its truth beyond doubt. You shall learn a proof for the same in higher classes.

Now, consider Fig. A1.3, where PQ and PR are tangents to the circle drawn from P.

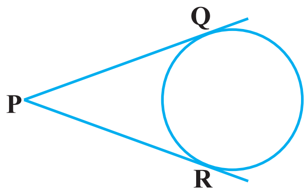


Fig. A1.3

You have proved that PQ = PR (Theorem 10.2). You were not satisfied by only drawing several such figures, measuring the lengths of the respective tangents, and verifying for yourselves that the result was true in each case.

Do you remember what did the proof consist of ? It consisted of a sequence of statements (called **valid arguments**), each following from the earlier statements in the proof, or from previously proved (and known) results independent from the result to be proved, or from axioms, or from definitions, or from the assumptions you had made. And you concluded your proof with the statement PQ = PR, i.e., the statement you wanted to prove. This is the way any proof is constructed.

We shall now look at some examples and theorems and analyse their proofs to help us in getting a better understanding of how they are constructed.

We begin by using the so-called 'direct' or 'deductive' method of proof. In this method, we make several statements. Each is **based on previous statements**. If each statement is logically correct (i.e., a valid argument), it leads to a logically correct conclusion.

Example 10 : The sum of two rational numbers is a rational number.

Solution :

S.No.	Statements	Analysis/Comments
1.	Let x and y be rational numbers.	Since the result is about rationals, we start with x and y which are rational.
2.	Let $x = \frac{m}{n}$, $n \neq 0$ and $y = \frac{p}{q}$, $q \neq 0$ where m, n, p and q are integers.	Apply the definition of rationals.
3.	So, $x + y = \frac{m}{n} + \frac{p}{q} = \frac{mq + np}{nq}$	The result talks about the sum of rationals, so we look at $x + y$.

4.	Using the properties of integers, we see that $mq + np$ and nq are integers.	Using known properties of integers.
5.	Since $n \neq 0$ and $q \neq 0$, it follows that $nq \neq 0$.	Using known properties of integers.
6.	Therefore, $x + y = \frac{mq + np}{nq}$ is a rational number	Using the definition of a rational number.

Remark : Note that, each statement in the proof above is based on a previously established fact, or definition.

Example 11 : Every prime number greater than 3 is of the form $6k + 1$ or $6k + 5$, where k is some integer.

Solution :

S.No.	Statements	Analysis/Comments
1.	Let p be a prime number greater than 3.	Since the result has to do with a prime number greater than 3, we start with such a number.
2.	Dividing p by 6, we find that p can be of the form $6k$, $6k + 1$, $6k + 2$, $6k + 3$, $6k + 4$, or $6k + 5$, where k is an integer.	Using Euclid's division lemma.
3.	But $6k = 2(3k)$, $6k + 2 = 2(3k + 1)$, $6k + 4 = 2(3k + 2)$, and $6k + 3 = 3(2k + 1)$. So, they are not primes.	We now analyse the remainders given that p is prime.
4.	So, p is forced to be of the form $6k + 1$ or $6k + 5$, for some integer k.	We arrive at this conclusion having eliminated the other options.

Remark : In the above example, we have arrived at the conclusion by eliminating different options. This method is sometimes referred to as the **Proof by Exhaustion**.

Theorem A1.1 ***(Converse of the Pythagoras Theorem) :*** *If in a triangle the square of the length of one side is equal to the sum of the squares of the other two sides, then the angle opposite the first side is a right angle.*

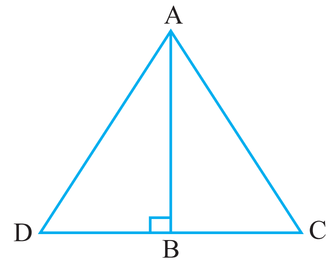


Fig. A1.4

Proof :

S.No.	Statements	Analysis
1.	Let ΔABC satisfy the hypothesis $AC^2 = AB^2 + BC^2$.	Since we are proving a statement about such a triangle, we begin by taking this.
2.	Construct line BD perpendicular to AB, such that BD = BC, and join A to D.	This is the intuitive step we have talked about that we often need to take for proving theorems.
3.	By construction, ΔABD is a right triangle, and from the Pythagoras Theorem, we have $AD^2 = AB^2 + BD^2$.	We use the Pythagoras theorem, which is already proved.
4.	By construction, BD = BC. Therefore, we have $AD^2 = AB^2 + BC^2$.	Logical deduction.
5.	Therefore, $AC^2 = AB^2 + BC^2 = AD^2$.	Using assumption, and previous statement.
6.	Since AC and AD are positive, we have AC = AD.	Using known property of numbers.
7.	We have just shown AC = AD. Also BC = BD by construction, and AB is common. Therefore, by SSS, $\Delta ABC \cong \Delta ABD$.	Using known theorem.
8.	Since $\Delta ABC \cong \Delta ABD$, we get $\angle ABC = \angle ABD$, which is a right angle.	Logical deduction, based on previously established fact.

Remark : Each of the results above has been proved by a sequence of steps, all linked together. Their order is important. Each step in the proof follows from previous steps and earlier known results. (Also see Theorem 6.9.)

EXERCISE A1.3

In each of the following questions, we ask you to prove a statement. List all the steps in each proof, and give the reason for each step.

1. Prove that the sum of two consecutive odd numbers is divisible by 4.
2. Take two consecutive odd numbers. Find the sum of their squares, and then add 6 to the result. Prove that the new number is always divisible by 8.
3. If $p \geq 5$ is a prime number, show that $p^2 + 2$ is divisible by 3.

   [**Hint:** Use Example 11].
4. Let x and y be rational numbers. Show that xy is a rational number.
5. If a and b are positive integers, then you know that $a = bq + r$, $0 \leq r < b$, where q is a whole number. Prove that HCF (a, b) = HCF (b, r).

   [**Hint :** Let HCF $(b, r) = h$. So, $b = k_1 h$ and $r = k_2 h$, where k_1 and k_2 are coprime.]
6. A line parallel to side BC of a triangle ABC, intersects AB and AC at D and E respectively.

   Prove that $\dfrac{\text{AD}}{\text{DB}} = \dfrac{\text{AE}}{\text{EC}}$.

A1.5 Negation of a Statement

In this section, we discuss what it means to 'negate' a statement. Before we start, we would like to introduce some notation, which will make it easy for us to understand these concepts. To start with, let us look at a statement as a single unit, and give it a name. For example, we can denote the statement 'It rained in Delhi on 1 September 2005' by p. We can also write this by

p: It rained in Delhi on 1 September 2005.

Similarly, let us write

q: All teachers are female.

r: Mike's dog has a black tail.

s: $2 + 2 = 4$.

t: Triangle ABC is equilateral.

This notation now helps us to discuss properties of statements, and also to see how we can combine them. In the beginning we will be working with what we call 'simple' statements, and will then move onto 'compound' statements.

Now consider the following table in which we make a new statement from each of the given statements.

Original statement	New statement
p: It rained in Delhi on 1 September 2005	$\sim p$: It is false that it rained in Delhi on 1 September 2005.
q: All teachers are female.	$\sim q$: It is false that all teachers are female.
r: Mike's dog has a black tail.	$\sim r$: It is false that Mike's dog has a black tail.
s: $2 + 2 = 4$.	$\sim s$: It is false that $2 + 2 = 4$.
t: Triangle ABC is equilateral.	$\sim t$: It is false that triangle ABC is equilateral.

Each new statement in the table is a *negation* of the corresponding old statement. That is, $\sim p$, $\sim q$, $\sim r$, $\sim s$ and $\sim t$ are negations of the statements p, q, r, s and t, respectively. Here, $\sim p$ is read as 'not p'. The statement $\sim p$ negates the assertion that the statement p makes. Notice that in our usual talk we would simply mean $\sim p$ as 'It did not rain in Delhi on 1 September 2005.' However, we need to be careful while doing so. You might think that one can obtain the negation of a statement by simply inserting the word 'not' in the given statement at a suitable place. While this works in the case of p, the difficulty comes when we have a statement that begins with 'all'. Consider, for example, the statement q: All teachers are female. We said the negation of this statement is $\sim q$: It is false that all teachers are female. This is the same as the statement 'There are some teachers who are males.' Now let us see what happens if we simply insert 'not' in q. We obtain the statement: 'All teachers are not female', or we can obtain the statement: 'Not all teachers are female.' The first statement can confuse people. It could imply (if we lay emphasis on the word 'All') that all teachers are male! This is certainly not the negation of q. However, the second statement gives the meaning of $\sim q$, i.e., that there is at least one teacher who is not a female. So, be careful when writing the negation of a statement!

So, how do we decide that we have the correct negation? We use the following criterion.

Let p be a statement and $\sim p$ its negation. Then $\sim p$ is false whenever p is true, and $\sim p$ is true whenever p is false.

For example, if it is true that Mike's dog has a black tail, then it is false that Mike's dog does not have a black tail. If it is false that 'Mike's dog has a black tail', then it is true that 'Mike's dog does not have a black tail'.

Similarly, the negations for the statements s and t are:

s: $2 + 2 = 4$; negation, $\sim s$: $2 + 2 \neq 4$.

t: Triangle ABC is equilateral; negation, $\sim t$: Triangle ABC is not equilateral.

Now, what is $\sim(\sim s)$? It would be $2 + 2 = 4$, which is s. And what is $\sim(\sim t)$? This would be 'the triangle ABC is equilateral', i.e., t. In fact, **for any statement *p*, $\sim(\sim p)$ is *p***.

Example 12 : State the negations for the following statements:

(i) Mike's dog does not have a black tail.

(ii) All irrational numbers are real numbers.

(iii) $\sqrt{2}$ is irrational.

(iv) Some rational numbers are integers.

(v) Not all teachers are males.

(vi) Some horses are not brown.

(vii) There is no real number x, such that $x^2 = -1$.

Solution :

(i) It is false that Mike's dog does not have a black tail, i.e., Mike's dog has a black tail.

(ii) It is false that all irrational numbers are real numbers, i.e., some (at least one) irrational numbers are not real numbers. One can also write this as, 'Not all irrational numbers are real numbers.'

(iii) It is false that $\sqrt{2}$ is irrational, i.e., $\sqrt{2}$ is not irrational.

(iv) It is false that some rational numbers are integers, i.e., no rational number is an integer.

(v) It is false that not all teachers are males, i.e., all teachers are males.

(vi) It is false that some horses are not brown, i.e., all horses are brown.

(vii) It is false that there is no real number x, such that $x^2 = -1$, i.e., there is at least one real number x, such that $x^2 = -1$.

Remark : From the above discussion, you may arrive at the following **Working Rule** for obtaining the negation of a statement :

(i) First write the statement with a 'not'.

(ii) If there is any confusion, make suitable modification , specially in the statements involving 'All' or 'Some'.

EXERCISE A1.4

1. State the negations for the following statements :

(i) Man is mortal.

(ii) Line l is parallel to line m.

(iii) This chapter has many exercises.

(iv) All integers are rational numbers.

(v) Some prime numbers are odd.

(vi) No student is lazy.

(vii) Some cats are not black.

(viii) There is no real number x, such that $\sqrt{x} = -1$.

(ix) 2 divides the positive integer a.

(x) Integers a and b are coprime.

2. In each of the following questions, there are two statements. State if the second is the negation of the first or not.

(i) Mumtaz is hungry.
Mumtaz is not hungry.

(ii) Some cats are black.
Some cats are brown.

(iii) All elephants are huge.
One elephant is not huge.

(iv) All fire engines are red.
All fire engines are not red.

(v) No man is a cow.
Some men are cows.

A1.6 Converse of a Statement

We now investigate the notion of the converse of a statement. For this, we need the notion of a 'compound' statement, that is, a statement which is a combination of one or more 'simple' statements. There are many ways of creating compound statements, but we will focus on those that are created by connecting two simple statements with the use of the words 'if' and 'then'. For example, the statement 'If it is raining, then it is difficult to go on a bicycle', is made up of two statements:

p: It is raining

q: It is difficult to go on a bicycle.

Using our previous notation we can say: If p, then q. We can also say 'p implies q', and denote it by $p \Rightarrow q$.

Now, supose you have the statement 'If the water tank is black, then it contains potable water.' This is of the form $p \Rightarrow q$, where the hypothesis is p (the water tank is black) and the conclusion is q (the tank contains potable water). Suppose we interchange the hypothesis and the conclusion, what do we get? We get $q \Rightarrow p$, i.e., if the water in the tank is potable, then the tank must be black. This statement is called the **converse** of the statement $p \Rightarrow q$.

In general, the **converse** of the statement $p \Rightarrow q$ is $q \Rightarrow p$, where p and q are statements. Note that $p \Rightarrow q$ and $q \Rightarrow p$ are the converses of each other.

Example 13 : Write the converses of the following statements :

(i) If Jamila is riding a bicycle, then 17 August falls on a Sunday.

(ii) If 17 August is a Sunday, then Jamila is riding a bicycle.

(iii) If Pauline is angry, then her face turns red.

(iv) If a person has a degree in education, then she is allowed to teach.

(v) If a person has a viral infection, then he runs a high temperature.

(vi) If Ahmad is in Mumbai, then he is in India.

(vii) If triangle ABC is equilateral, then all its interior angles are equal.

(viii) If x is an irrational number, then the decimal expansion of x is non-terminating non-recurring.

(ix) If $x – a$ is a factor of the polynomial $p(x)$, then $p(a) = 0$.

Solution : Each statement above is of the form $p \Rightarrow q$. So, to find the converse, we first identify p and q, and then write $q \Rightarrow p$.

(i) p: Jamila is riding a bicycle, and q: 17 August falls on a Sunday. Therefore, the converse is: If 17 August falls on a Sunday, then Jamila is riding a bicycle.

(ii) This is the converse of (i). Therefore, its converse is the statement given in (i) above.

(iii) If Pauline's face turns red, then she is angry.

(iv) If a person is allowed to teach, then she has a degree in education.

(v) If a person runs a high temperature, then he has a viral infection.

(vi) If Ahmad is in India, then he is in Mumbai.

(vii) If all the interior angles of triangle ABC are equal, then it is equilateral.

(viii) If the decimal expansion of x is non-terminating non-recurring, then x is an irrational number.

(ix) If $p(a) = 0$, then $x – a$ is a factor of the polynomial $p(x)$.

Notice that we have simply written the converse of each of the statements above without worrying if they are true or false. For example, consider the following statement: If Ahmad is in Mumbai, then he is in India. This statement is true. Now consider the converse: If Ahmad is in India, then he is in Mumbai. This need not be true always – he could be in any other part of India.

In mathematics, especially in geometry, you will come across many situations where $p \Rightarrow q$ is true, and you will have to decide if the converse, i.e., $q \Rightarrow p$, is also true.

Example 14 : State the converses of the following statements. In each case, also decide whether the converse is true or false.

(i) If n is an even integer, then $2n + 1$ is an odd integer.

(ii) If the decimal expansion of a real number is terminating, then the number is rational.

(iii) If a transversal intersects two parallel lines, then each pair of corresponding angles is equal.

(iv) If each pair of opposite sides of a quadrilateral is equal, then the quadrilateral is a parallelogram.

(v) If two triangles are congruent, then their corresponding angles are equal.

Solution :

(i) The converse is 'If $2n + 1$ is an odd integer, then n is an even integer.' This is a false statement (for example, $15 = 2(7) + 1$, and 7 is odd).

(ii) 'If a real number is rational, then its decimal expansion is terminating', is the converse. This is a false statement, because a rational number can also have a non-terminating recurring decimal expansion.

(iii) The converse is 'If a transversal intersects two lines in such a way that each pair of corresponding angles are equal, then the two lines are parallel.' We have assumed, by Axiom 6.4 of your Class IX textbook, that this statement is true.

(iv) 'If a quadrilateral is a parallelogram, then each pair of its opposite sides is equal', is the converse. This is true (Theorem 8.1, Class IX).

(v) 'If the corresponding angles in two triangles are equal, then they are congruent', is the converse. This statement is false. We leave it to you to find suitable counter-examples.

EXERCISE A1.5

1. Write the converses of the following statements.

(i) If it is hot in Tokyo, then Sharan sweats a lot.

(ii) If Shalini is hungry, then her stomach grumbles.

(iii) If Jaswant has a scholarship, then she can get a degree.

(iv) If a plant has flowers, then it is alive.

(v) If an animal is a cat, then it has a tail.

2. Write the converses of the following statements. Also, decide in each case whether the converse is true or false.

(i) If triangle ABC is isosceles, then its base angles are equal.

(ii) If an integer is odd, then its square is an odd integer.

(iii) If $x^2 = 1$, then $x = 1$.

(iv) If ABCD is a parallelogram, then AC and BD bisect each other.

(v) If a, b and c, are whole numbers, then $a + (b + c) = (a + b) + c$.

(vi) If x and y are two odd numbers, then $x + y$ is an even number.

(vii) If vertices of a parallelogram lie on a circle, then it is a rectangle.

A1.7 Proof by Contradiction

So far, in all our examples, we used direct arguments to establish the truth of the results. We now explore 'indirect' arguments, in particular, a very powerful tool in mathematics known as 'proof by contradiction'. We have already used this method in Chapter 1 to establish the irrationality of several numbers and also in other chapters to prove some theorems. Here, we do several more examples to illustrate the idea.

Before we proceed, let us explain what a *contradiction* is. In mathematics, a contradiction occurs when we get a statement p such that p is true and $\sim p$, its negation, is also true. For example,

p: $x = \dfrac{a}{b}$, where a and b are coprime.

q: 2 divides both 'a' and 'b'.

If we assume that p is true and also manage to show that q is true, then we have arrived at a contradiction, because q implies that the negation of p is true. If you remember, this is exactly what happened when we tried to prove that $\sqrt{2}$ is irrational (see Chapter 1).

How does proof by contradiction work? Let us see this through a specific example.

Suppose we are given the following :

All women are mortal. A is a woman. Prove that A is mortal.

Even though this is a rather easy example, let us see how we can prove this by contradiction.

- Let us assume that we want to establish the truth of a statement p (here we want to show that p : 'A is mortal' is true).

- So, we begin by assuming that the statement is not true, that is, we assume that the negation of p is true (i.e., A is not mortal).
- We then proceed to carry out a series of logical deductions based on the truth of the negation of p. (Since A is not mortal, we have a counter-example to the statement 'All women are mortal.' Hence, it is false that all women are mortal.)
- If this leads to a contradiction, then the contradiction arises because of our faulty assumption that p is not true. (We have a contradiction, since we have shown that the statement 'All women are mortal' and its negation, 'Not all women are mortal' is true at the same time. This contradiction arose, because we assumed that A is not mortal.)
- Therefore, our assumption is wrong, i.e., p has to be true. (So, A is mortal.)

Let us now look at examples from mathematics.

Example 15 : The product of a non-zero rational number and an irrational number is irrational.

Solution :

Statements	Analysis/Comment
We will use proof by contradiction. Let r be a non-zero rational number and x be an irrational number. Let $r = \frac{m}{n}$, where m, n are integers and $m \neq 0$, $n \neq 0$. We need to prove that rx is irrational.	
Assume rx is rational.	Here, we are assuming the negation of the statement that we need to prove.
Then $rx = \frac{p}{q}$, $q \neq 0$, where p and q are integers.	This follow from the previous statement and the definition of a rational number.
Rearranging the equation $rx = \frac{p}{q}$, $q \neq 0$, and using the fact that $r = \frac{m}{n}$, we get $x = \frac{p}{rq} = \frac{np}{mq}$.	

Since np and mq are integers and $mq \neq 0$, x is a rational number.	Using properties of integers, and definition of a rational number.
This is a contradiction, because we have shown x to be rational, but by our hypothesis, we have x is irrational.	This is what we were looking for — a contradiction.
The contradiction has arisen because of the faulty assumption that rx is rational. Therefore, rx is irrational.	Logical deduction.

We now prove Example 11, but this time using proof by contradiction. The proof is given below:

Statements	**Analysis/Comment**
Let us assume that the statement is note true.	As we saw earlier, this is the starting point for an argument using 'proof by contradiction'.
So we suppose that there exists a prime number $p > 3$, which is not of the form $6n + 1$ or $6n + 5$, where n is a whole number.	This is the negation of the statement in the result.
Using Euclid's division lemma on division by 6, and using the fact that p is not of the form $6n + 1$ or $6n + 5$, we get $p = 6n$ or $6n + 2$ or $6n + 3$ or $6n + 4$.	Using earlier proved results.
Therefore, p is divisible by either 2 or 3.	Logical deduction.
So, p is not a prime.	Logical deduction.
This is a contradiction, because by our hypothesis p is prime.	Precisely what we want!
The contradiction has arisen, because we assumed that there exists a prime number $p > 3$ which is not of the form $6n + 1$ or $6n + 5$.	
Hence, every prime number greater than 3 is of the form $6n + 1$ or $6n + 5$.	We reach the conclusion.

Remark : The example of the proof above shows you, yet again, that there can be several ways of proving a result.

Theorem A1.2 : *Out of all the line segments, drawn from a point to points of a line not passing through the point, the smallest is the perpendicular to the line.*

Proof :

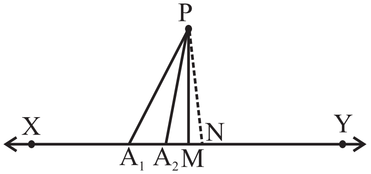


Fig. A1.5

Statements	Analysis/Comment
Let XY be the given line, P a point not lying on XY and PM, PA_1, PA_2, . . . etc., be the line segments drawn from P to the points of the line XY, out of which PM is the smallest (see Fig. A1.5).	Since we have to prove that out of all PM, PA_1, PA_2, . . . etc., the smallest is perpendicular to XY, we start by taking these line segments.
Let PM be not perpendicular to XY	This is the negation of the statement to be proved by contradiction.
Draw a perpendicular PN on the line XY, shown by dotted lines in Fig. A1.5.	We often need constructions to prove our results.
PN is the smallest of all the line segments PM, PA_1, PA_2, . . . etc., which means PN < PM.	Side of right triangle is less than the hypotenuse and known property of numbers.
This contradicts our hypothesis that PM is the smallest of all such line segments.	Precisely what we want!
Therefore, the line segment PM is perpendicular to XY.	We reach the conclusion.

EXERCISE A1.6

1. Suppose $a + b = c + d$, and $a < c$. Use proof by contradiction to show $b > d$.
2. Let r be a rational number and x be an irrational number. Use proof by contradiction to show that $r + x$ is an irrational number.
3. Use proof by contradiction to prove that if for an integer a, a^2 is even, then so is a. [**Hint :** Assume a is not even, that is, it is of the form $2n + 1$, for some integer n, and then proceed.]
4. Use proof by contradiction to prove that if for an integer a, a^2 is divisible by 3, then a is divisible by 3.
5. Use proof by contradiction to show that there is no value of n for which 6^n ends with the digit zero.
6. Prove by contradiction that two distinct lines in a plane cannot intersect in more than one point.

A1.8 Summary

In this Appendix, you have studied the following points :

1. Different ingredients of a proof and other related concepts learnt in Class IX.
2. The negation of a statement.
3. The converse of a statement.
4. Proof by contradiction.

MATHEMATICAL MODELLING A2

A2.1 Introduction

- An adult human body contains approximately 1,50,000 km of arteries and veins that carry blood.
- The human heart pumps 5 to 6 litres of blood in the body every 60 seconds.
- The temperature at the surface of the Sun is about 6,000° C.

Have you ever wondered how our scientists and mathematicians could possibly have estimated these results? Did they pull out the veins and arteries from some adult dead bodies and measure them? Did they drain out the blood to arrive at these results? Did they travel to the Sun with a thermometer to get the temperature of the Sun? Surely not. Then how did they get these figures?

Well, the answer lies in **mathematical modelling,** which we introduced to you in Class IX. Recall that a mathematical model is a mathematical description of some real-life situation. Also, recall that mathematical modelling is the process of creating a mathematical model of a problem, and using it to analyse and solve the problem.

So, in mathematical modelling, we take a real-world problem and convert it to an equivalent mathematical problem. We then solve the mathematical problem, and interpret its solution in the situation of the real-world problem. And then, it is important to see that the solution, we have obtained, 'makes sense', which is the stage of validating the model. Some examples, where mathematical modelling is of great importance, are:

(i) Finding the width and depth of a river at an unreachable place.

(ii) Estimating the mass of the Earth and other planets.

(iii) Estimating the distance between Earth and any other planet.

(iv) Predicting the arrrival of the monsoon in a country.

(v) Predicting the trend of the stock market.

(vi) Estimating the volume of blood inside the body of a person.

(vii) Predicting the population of a city after 10 years.

(viii) Estimating the number of leaves in a tree.

(ix) Estimating the ppm of different pollutants in the atmosphere of a city.

(x) Estimating the effect of pollutants on the environment.

(xi) Estimating the temperature on the Sun's surface.

In this chapter we shall revisit the process of mathematical modelling, and take examples from the world around us to illustrate this. In Section A2.2 we take you through all the stages of building a model. In Section A2.3, we discuss a variety of examples. In Section A2.4, we look at reasons for the importance of mathematical modelling.

A point to remember is that here we aim to make you aware of an important way in which mathematics helps to solve real-world problems. However, you need to know some more mathematics to really appreciate the power of mathematical modelling. In higher classes some examples giving this flavour will be found.

A2.2 Stages in Mathematical Modelling

In Class IX, we considered some examples of the use of modelling. Did they give you an insight into the process and the steps involved in it? Let us quickly revisit the main steps in mathematical modelling.

Step 1 (Understanding the problem) : Define the real problem, and if working in a team, discuss the issues that you wish to understand. Simplify by making assumptions and ignoring certain factors so that the problem is manageable.

For example, suppose our problem is to estimate the number of fishes in a lake. It is not possible to capture each of these fishes and count them. We could possibly capture a sample and from it try and estimate the total number of fishes in the lake.

Step 2 (Mathematical description and formulation) : Describe, in mathematical terms, the different aspects of the problem. Some ways to describe the features mathematically, include:

- define variables
- write equations or inequalities
- gather data and organise into tables
- make graphs
- calculate probabilities

For example, having taken a sample, as stated in Step 1, how do we estimate the entire population? We would have to then mark the sampled fishes, allow them to mix with the remaining ones in the lake, again draw a sample from the lake, and see how many of the previously marked ones are present in the new sample. Then, using ratio and proportion, we can come up with an estimate of the total population. For instance, let us take a sample of 20 fishes from the lake and mark them, and then release them in the same lake, so as to mix with the remaining fishes. We then take another sample (say 50), from the mixed population and see how many are marked. So, we gather our data and analyse it.

One major assumption we are making is that the marked fishes mix uniformly with the remaining fishes, and the sample we take is a good representative of the entire population.

Step 3 (Solving the mathematical problem) : The simplified mathematical problem developed in Step 2 is then solved using various mathematical techniques.

For instance, suppose in the second sample in the example in Step 2, 5 fishes are marked. So, $\frac{5}{50}$, i.e., $\frac{1}{10}$, of the population is marked. If this is typical of the whole population, then $\frac{1}{10}$th of the population = 20.

So, the whole population = $20 \times 10 = 200$.

Step 4 (Interpreting the solution) : The solution obtained in the previous step is now looked at, in the context of the real-life situation that we had started with in Step 1.

For instance, our solution in the problem in Step 3 gives us the population of fishes as 200.

Step 5 (Validating the model) : We go back to the original situation and see if the results of the mathematical work make sense. If so, we use the model until new information becomes available or assumptions change.

Sometimes, because of the simplification assumptions we make, we may lose essential aspects of the real problem while giving its mathematical description. In such cases, the solution could very often be off the mark, and not make sense in the real situation. If this happens, we reconsider the assumptions made in Step 1 and revise them to be more realistic, possibly by including some factors which were not considered earlier.

For instance, in Step 3 we had obtained an estimate of the entire population of fishes. It may not be the actual number of fishes in the pond. We next see whether this is a good estimate of the population by repeating Steps 2 and 3 a few times, and taking the mean of the results obtained. This would give a closer estimate of the population.

Another way of visualising **the process of mathematical modelling** is shown in Fig. A2.1.

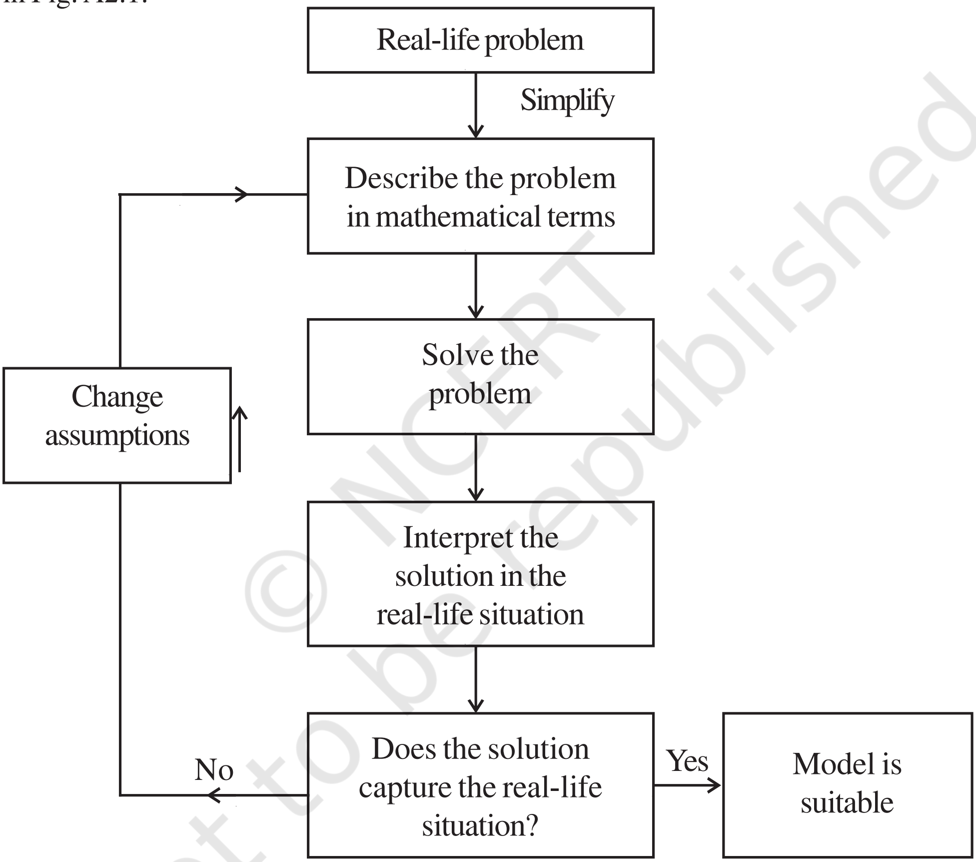


Fig. A2.1

Modellers look for a balance between simplification (for ease of solution) and accuracy. They hope to approximate reality closely enough to make some progress. The best outcome is to be able to predict what will happen, or estimate an outcome, with reasonable accuracy. Remember that different assumptions we use for simplifying the problem can lead to different models. So, there are no perfect models. There are good ones and yet better ones.

EXERCISE A2.1

1. Consider the following situation.

A problem dating back to the early 13th century, posed by Leonardo Fibonacci asks how many rabbits you would have if you started with just two and let them reproduce. Assume that a pair of rabbits produces a pair of offspring each month and that each pair of rabbits produces their first offspring at the age of 2 months. Month by month the number of pairs of rabbits is given by the sum of the rabbits in the two preceding months, except for the 0th and the 1st months.

Month	Pairs of Rabbits
0	1
1	1
2	2
3	3
4	5
5	8
6	13
7	21
8	34
9	55
10	89
11	144
12	233
13	377
14	610
15	987
16	1597

After just 16 months, you have nearly 1600 pairs of rabbits!

Clearly state the problem and the different stages of mathematical modelling in this situation.

A2.3 Some Illustrations

Let us now consider some examples of mathematical modelling.

Example 1 (Rolling of a pair of dice) : Suppose your teacher challenges you to the following guessing game: She would throw a pair of dice. Before that you need to guess the sum of the numbers that show up on the dice. For every correct answer, you get two points and for every wrong guess you lose two points. What numbers would be the best guess?

Solution :

Step 1 (Understanding the problem) : You need to know a few numbers which have higher chances of showing up.

Step 2 (Mathematical description) : In mathematical terms, the problem translates to finding out the probabilities of the various possible sums of numbers that the dice could show.

We can model the situation very simply by representing a roll of the dice as a random choice of one of the following thirty six pairs of numbers.

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

The first number in each pair represents the number showing on the first die, and the second number is the number showing on the second die.

Step 3 (Solving the mathematical problem) : Summing the numbers in each pair above, we find that possible sums are 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12. We have to find the probability for each of them, assuming all 36 pairs are equally likely.

We do this in the following table.

Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Observe that the chance of getting a sum of a seven is 1/6, which is larger than the chances of getting other numbers as sums.

Step 4 (Interpreting the solution) : Since the probability of getting the sum 7 is the highest, you should repeatedly guess the number seven.

Step 5 (Validating the model) : Toss a pair of dice a large number of times and prepare a relative frequency table. Compare the relative frequencies with the corresponding probabilities. If these are not close, then possibly the dice are biased. Then, we could obtain data to evaluate the number towards which the bias is.

Before going to the next example, you may need some background.

Not having the money you want when you need it, is a common experience for many people. Whether it is having enough money for buying essentials for daily living, or for buying comforts, we always require money. To enable the customers with limited funds to purchase goods like scooters, refrigerators, televisions, cars, etc., a scheme known as an *instalment scheme* (*or plan*) is introduced by traders.

Sometimes a trader introduces an *instalment* scheme as a marketing strategy to allure customers to purchase these articles. Under the instalment scheme, the customer is not required to make full payment of the article at the time of buying it. She/he is allowed to pay a part of it at the time of purchase, and the rest can be paid in instalments, which could be monthly, quarterly, half-yearly, or even yearly. Of course, the buyer will have to pay more in the instalment plan, because the seller is going to charge some interest on account of the payment made at a later date (called *deferred payment*).

Before we take a few examples to understand the instalment scheme, let us understand the most frequently used terms related to this concept.

The *cash price* of an article is the amount which a customer has to pay as full payment of the article at the time it is purchased. *Cash down payment* is the amount which a customer has to pay as part payment of the price of an article at the time of purchase.

Remark : If the instalment scheme is such that the remaining payment is completely made within one year of the purchase of the article, then simple interest is charged on the deferred payment.

> In the past, charging interest on borrowed money was often considered evil, and, in particular, was long prohibited. One way people got around the law against paying interest was to borrow in one currency and repay in another, the interest being disguised in the exchange rate.

Let us now come to a related mathematical modelling problem.

Example 2 : Juhi wants to buy a bicycle. She goes to the market and finds that the bicycle she likes is available for ₹ 1800. Juhi has ₹ 600 with her. So, she tells the shopkeeper that she would not be able to buy it. The shopkeeper, after a bit of calculation, makes the following offer. He tells Juhi that she could take the bicycle by making a payment of ₹ 600 cash down and the remaining money could be made in two monthly instalments of ₹ 610 each. Juhi has two options one is to go for instalment scheme or to make cash payment by taking loan from a bank which is available at the rate of 10% per annum simple interest. Which option is more economical to her?

Solution :

Step 1 (Understanding the problem) : What Juhi needs to determine is whether she should take the offer made by the shopkeeper or not. For this, she should know the two rates of interest—one charged in the instalment scheme and the other charged by the bank (i.e., 10%).

Step 2 (Mathematical description) : In order to accept or reject the scheme, she needs to determine the interest that the shopkeeper is charging in comparison to the bank. Observe that since the entire money shall be paid in less than a year, simple interest shall be charged.

We know that the cash price of the bicycle = ₹ 1800.

Also, the cashdown payment under the instalment scheme = ₹ 600.

So, the balance price that needs to be paid in the instalment scheme = ₹ (1800 – 600) = ₹ 1200.

Let r % per annum be the rate of interest charged by the shopkeeper.

Amount of each instalment = ₹ 610

Amount paid in instalments = ₹ 610 + ₹ 610 = ₹ 1220

Interest paid in instalment scheme = ₹ 1220 – ₹ 1200 = ₹ 20 (1)

Since, Juhi kept a sum of ₹ 1200 for one month, therefore,

Principal for the first month = ₹ 1200

Principal for the second month = ₹ (1200 – 610) = ₹ 590

Balance of the second principal ₹ 590 + interest charged (₹ 20) = monthly instalment (₹ 610) = 2nd instalment

So, the total principal for one month = ₹ 1200 + ₹ 590 = ₹ 1790

Now, $$\text{interest} = ₹\ \frac{1790 \times r \times 1}{100 \times 12} \qquad (2)$$

Step 3 (Solving the problem) : From (1) and (2)

$$\frac{1790 \times r \times 1}{100 \times 12} = 20$$

or $$r = \frac{20 \times 1200}{1790} = 13.14 \text{ (approx.)}$$

Step 4 (Interpreting the solution) : The rate of interest charged in the instalment scheme = 13.14 %.

The rate of interest charged by the bank = 10%

So, she should prefer to borrow the money from the bank to buy the bicycle which is more economical.

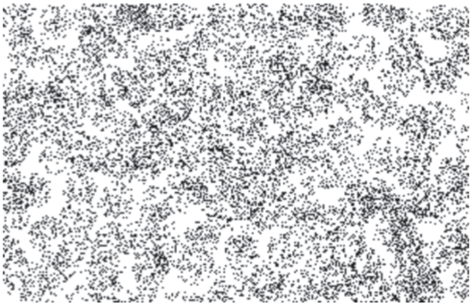
Step 5 (Validating the model) : This stage in this case is not of much importance here as the numbers are fixed. However, if the formalities for taking loan from the bank such as cost of stamp paper, etc., which make the effective interest rate more than what it is the instalment scheme, then she may change her opinion.

Remark : Interest rate modelling is still at its early stages and validation is still a problem of financial markets. In case, different interest rates are incorporated in fixing instalments, validation becomes an important problem.

EXERCISE A2.2

In each of the problems below, show the different stages of mathematical modelling for solving the problems.

1. An ornithologist wants to estimate the number of parrots in a large field. She uses a net to catch some, and catches 32 parrots, which she rings and sets free. The following week she manages to net 40 parrots, of which 8 are ringed.
   (i) What fraction of her second catch is ringed?
   (ii) Find an estimate of the total number of parrots in the field.



2. Suppose the adjoining figure represents an aerial photograph of a forest with each dot representing a tree. Your purpose is to find the number of trees there are on this tract of land as part of an environmental census.

3. A T.V. can be purchased for ₹ 24000 cash or for ₹ 8000 cashdown payment and six monthly instalments of ₹ 2800 each. Ali goes to market to buy a T.V., and he has ₹ 8000 with him. He has now two options. One is to buy TV under instalment scheme or to make cash payment by taking loan from some financial society. The society charges simple interest at the rate of 18% per annum simple interest. Which option is better for Ali?

A2.4 Why is Mathematical Modelling Important?

As we have seen in the examples, mathematical modelling is an interdisciplinary subject. Mathematicians and specialists in other fields share their knowledge and expertise to improve existing products, develop better ones, or predict the behaviour of certain products.

There are, of course, many specific reasons for the importance of modelling, but most are related in some ways to the following :

- *To gain understanding.* If we have a mathematical model which reflects the essential behaviour of a real-world system of interest, we can understand that system better through an analysis of the model. Furthermore, in the process of building the model we find out which factors are most important in the system, and how the different aspects of the system are related.
- *To predict, or forecast, or simulate.* Very often, we wish to know what a real-world system will do in the future, but it is expensive, impractical or impossible to experiment directly with the system. For example, in weather prediction, to study drug efficacy in humans, finding an optimum design of a nuclear reactor, and so on.

  Forecasting is very important in many types of organisations, since predictions of future events have to be incorporated into the decision-making process. For example:

  In marketing departments, reliable forecasts of demand help in planning of the sale strategies.

  A school board needs to able to forecast the increase in the number of school going children in various districts so as to decide where and when to start new schools.

  Most often, forecasters use the past data to predict the future. They first analyse the data in order to identify a pattern that can describe it. Then this data and pattern is extended into the future in order to prepare a forecast. This basic strategy is employed in most forecasting techniques, and is based on the assumption that the pattern that has been identified will continue in the future also.

- *To estimate.* Often, we need to estimate large values. You've seen examples of the trees in a forest, fish in a lake, etc. For another example, before elections, the contesting parties want to predict the probability of their party winning the elections. In particular, they want to estimate how many people in their constituency would vote for their party. Based on their predictions, they may want to decide on the campaign strategy. Exit polls have been used widely to predict the number of seats, a party is expected to get in elections.

EXERCISE A2.3

1. Based upon the data of the past five years, try and forecast the average percentage of marks in Mathematics that your school would obtain in the Class X board examination at the end of the year.

A2.5 Summary

In this Appendix, you have studied the following points :

1. A mathematical model is a mathematical description of a real-life situation. Mathematical modelling is the process of creating a mathematical model, solving it and using it to understand the real-life problem.

2. The various steps involved in modelling are : understanding the problem, formulating the mathematical model, solving it, interpreting it in the real-life situation, and, most importantly, validating the model.

3. Developed some mathematical models.

4. The importance of mathematical modelling.

ANSWERS/HINTS

EXERCISE 1.1

1. (i) $2^2 \times 5 \times 7$ (ii) $2^2 \times 3 \times 13$ (iii) $3^2 \times 5^2 \times 17$

(iv) $5 \times 7 \times 11 \times 13$ (v) $17 \times 19 \times 23$

2. (i) LCM = 182; HCF = 13 (ii) LCM = 23460; HCF = 2 (iii) LCM = 3024; HCF = 6

3. (i) LCM = 420; HCF = 3 (ii) LCM = 11339; HCF = 1 (iii) LCM = 1800; HCF = 1

4. 22338 **7.** 36 minutes

EXERCISE 2.1

1. (i) No zeroes (ii) 1 (iii) 3 (iv) 2 (v) 4 (vi) 3

EXERCISE 2.2

1. (i) $-2, 4$ (ii) $\frac{1}{2}, \frac{1}{2}$ (iii) $-\frac{1}{3}, \frac{3}{2}$

(iv) $-2, 0$ (v) $-\sqrt{15}, \sqrt{15}$ (vi) $-1, \frac{4}{3}$

2. (i) $4x^2 - x - 4$ (ii) $3x^2 - 3\sqrt{2}\,x + 1$ (iii) $x^2 + \sqrt{5}$

(iv) $x^2 - x + 1$ (v) $4x^2 + x + 1$ (vi) $x^2 - 4x + 1$

EXERCISE 3.1

1. (i) Required pair of linear equations is

$x + y = 10$; $x - y = 4$, where x is the number of girls and y is the number of boys.

To solve graphically draw the graphs of these equations on the same axes on graph paper.

Girls = 7, Boys = 3.

(ii) Required pair of linear equations is

$5x + 7y = 50$; $7x + 5y = 46$, where x and y represent the cost (in ₹) of a pencil and of a pen respectively.

To solve graphically, draw the graphs of these equations on the same axes on graph paper.

Cost of one pencil = ₹ 3, Cost of one pen = ₹ 5

2. (i) Intersect at a point (ii) Coincident (iii) Parallel

3. (i) Consistent (ii) Inconsistent (iii) Consistent

(iv) Consistent (v) Consistent

4. (i) Consistent (ii) Inconsistent (iii) Consistent (iv) Inconsistent

The solution of (i) above, is given by $y = 5 – x$, where x can take any value, i.e., there are infinitely many solutions.

The solution of (iii) above is $x = 2$, $y = 2$, i.e., unique solution.

5. Length = 20 m and breadth = 16 m.

6. One possible answer for the three parts:

(i) $3x + 2y – 7 = 0$ (ii) $2x + 3y – 12 = 0$ (iii) $4x + 6y – 16 = 0$

7. Vertices of the triangle are (–1, 0), (4, 0) and (2, 3).

EXERCISE 3.2

1. (i) $x = 9,\ y = 5$ (ii) $s = 9,\ t = 6$ (iii) $y = 3x – 3$,

where x can take any value, i.e., infinitely many solutions.

(iv) $x = 2,\ y = 3$ (v) $x = 0,\ y = 0$ (vi) $x = 2,\ y = 3$

2. $x = –2,\ y = 5;\ m = –1$

3. (i) $x – y = 26,\ x = 3y$, where x and y are two numbers ($x > y$); $x = 39$, $y = 13$.

(ii) $x – y = 18,\ x + y = 180$, where x and y are the measures of the two angles in degrees; $x = 99,\ y = 81$.

(iii) $7x + 6y = 3800$, $3x + 5y = 1750$, where x and y are the costs (in ₹) of one bat and one ball respectively; $x = 500$, $y = 50$.

(iv) $x + 10y = 105,\ x + 15y = 155$, where x is the fixed charge (in ₹) and y is the charge (in ₹ per km); $x = 5,\ y = 10$; ₹ 255.

(v) $11x – 9y + 4 = 0,\ 6x – 5y + 3 = 0$, where x and y are numerator and denominator of the fraction; $\frac{7}{9}$ ($x = 7,\ y = 9$).

(vi) $x – 3y – 10 = 0$, $x – 7y + 30 = 0$, where x and y are the ages in years of Jacob and his son; $x = 40$, $y = 10$.

EXERCISE 3.3

1. (i) $x = \frac{19}{5}, \; y = \frac{6}{5}$ (ii) $x = 2, \; y = 1$ (iii) $x = \frac{9}{13}, y = -\frac{5}{13}$

(iv) $x = 2, \; y = -3$

2. (i) $x - y + 2 = 0$, $2x - y - 1 = 0$, where x and y are the numerator and denominator of the fraction; $\frac{3}{5}$.

(ii) $x - 3y + 10 = 0$, $x - 2y - 10 = 0$, where x and y are the ages (in years) of Nuri and Sonu respectively. Age of Nuri (x) = 50, Age of Sonu (y) = 20.

(iii) $x + y = 9$, $8x - y = 0$, where x and y are respectively the tens and units digits of the number; 18.

(iv) $x + 2y = 40$, $x + y = 25$, where x and y are respectively the number of ₹ 50 and ₹ 100 notes; $x = 10$, $y = 15$.

(v) $x + 4y = 27$, $x + 2y = 21$, where x is the fixed charge (in ₹) and y is the additional charge (in ₹) per day; $x = 15$, $y = 3$.

EXERCISE 4.1

1. (i) Yes (ii) Yes (iii) No (iv) Yes

(v) Yes (vi) No (vii) No (viii) Yes

2. (i) $2x^2 + x - 528 = 0$, where x is breadth (in metres) of the plot.

(ii) $x^2 + x - 306 = 0$, where x is the smaller integer.

(iii) $x^2 + 32x - 273 = 0$, where x (in years) is the present age of Rohan.

(iv) $u^2 - 8u - 1280 = 0$, where u (in km/h) is the speed of the train.

EXERCISE 4.2

1. (i) $-2, 5$ (ii) $-2, \frac{3}{2}$ (iii) $-\frac{5}{\sqrt{2}}, -\sqrt{2}$

(iv) $\frac{1}{4}, \frac{1}{4}$ (v) $\frac{1}{10}, \frac{1}{10}$

2. (i) 9, 36 (ii) 25, 30

3. Numbers are 13 and 14.

4. Positive integers are 13 and 14.

5. 5 cm and 12 cm

6. Number of articles = 6, Cost of each article = ₹ 15

EXERCISE 4.3

1. (i) Real roots do not exist (ii) Equal roots; $\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}$ (iii) Distinct roots; $\frac{3 \pm \sqrt{3}}{2}$

2. (i) $k = \pm\, 2\sqrt{6}$ (ii) $k = 6$

3. Yes. 40 m, 20 m **4.** No **5.** Yes. 20 m, 20 m

EXERCISE 5.1

1. (i) Yes. 15, 23, 31, . . . forms an AP as each succeeding term is obtained by adding 8 in its preceding term.

(ii) No. Volumes are V, $\frac{3V}{4}, \left(\frac{3}{4}\right)^2 V, \cdots$ (iii) Yes. 150, 200, 250, . . . form an AP.

(iv) No. Amounts are $10000\left(1+\frac{8}{100}\right), 10000\left(1+\frac{8}{100}\right)^2, 10000\left(1+\frac{8}{100}\right)^3, \cdots$

2. (i) 10, 20, 30, 40 (ii) – 2, – 2, – 2, – 2 (iii) 4, 1, – 2, – 5

(iv) $-1, -\frac{1}{2}, 0, \frac{1}{2}$ (v) – 1.25, – 1. 50, – 1.75, – 2.0

3. (i) $a = 3,\ d = -2$ (ii) $a = -5,\ d = 4$

(iii) $a = \frac{1}{3},\ d = \frac{4}{3}$ (iv) $a = 0.6,\ d = 1.1$

4. (i) No (ii) Yes. $d = \frac{1}{2}$; 4, $\frac{9}{2}$, 5

(iii) Yes. $d = -2$; – 9.2, –11.2, – 13.2 (iv) Yes. $d = 4$; 6, 10, 14

(v) Yes. $d = \sqrt{2}$; $3 + 4\sqrt{2}, 3 + 5\sqrt{2}, 3 + 6\sqrt{2}$ (vi) No

(vii) Yes. $d = -4$; – 16, – 20, – 24 (viii) Yes. $d = 0$; $-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}$

(ix) No (x) Yes. $d = a$; $5a, 6a, 7a$

(xi) No (xii) Yes. $d = \sqrt{2}$; $\sqrt{50}, \sqrt{72}, \sqrt{98}$

(xiii) No (xiv) No (xv) Yes. $d = 24$; 97, 121, 145

EXERCISE 5.2

1. (i) $a_n = 28$ (ii) $d = 2$ (iii) $a = 46$ (iv) $n = 10$ (v) $a_n = 3.5$

2. (i) C (ii) B

3. (i) $\boxed{14}$ (ii) $\boxed{18}$, $\boxed{8}$ (iii) $\boxed{6\frac{1}{2}}$, $\boxed{8}$

(iv) $\boxed{-2}$, $\boxed{0}$, $\boxed{2}$, $\boxed{4}$ (v) $\boxed{53}$, $\boxed{23}$, $\boxed{8}$, $\boxed{-7}$

4. 16th term
5. (i) 34 (ii) 27
6. No
7. 178
8. 64
9. 5th term
10. 1
11. 65th term
12. 100
13. 128
14. 60
15. 13
16. 4, 10, 16, 22, . . .
17. 20th term from the last term is 158.
18. –13, –8, –3
19. 11th year
20. 10

EXERCISE 5.3

1. (i) 245 (ii) –180 (iii) 5505 (iv) $\frac{33}{20}$

2. (i) $1046\,\frac{1}{2}$ (ii) 286 (iii) –8930

3. (i) $n = 16, S_n = 440$ (ii) $d = \frac{7}{3}, S_{13} = 273$ (iii) $a = 4, S_{12} = 246$

(iv) $d = -1, a_{10} = 8$ (v) $a = -\frac{35}{3}, a_9 = \frac{85}{3}$ (vi) $n = 5, a_n = 34$

(vii) $n = 6, d = \frac{54}{5}$ (viii) $n = 7, a = -8$ (ix) $d = 6$

(x) $a = 4$

4. 12. By putting $a = 9, d = 8, S = 636$ in the formula $S = \frac{n}{2}[2a + (n - 1)d]$, we get a quadratic equation $4n^2 + 5n - 636 = 0$. On solving, we get $n = -\frac{53}{4}, 12$. Out of these two roots only one root 12 is admissible.

5. $n = 16, d = \frac{8}{3}$
6. $n = 38, S = 6973$
7. Sum = 1661
8. $S_{51} = 5610$
9. n^2
10. (i) $S_{15} = 525$ (ii) $S_{15} = -465$

11. $S_1 = 3, S_2 = 4;\ a_2 = S_2 - S_1 = 1;\ S_3 = 3,\ a_3 = S_3 - S_2 = -1,$
$a_{10} = S_{10} - S_9 = -15;\ a_n = S_n - S_{n-1} = 5 - 2n.$

12. 4920 **13.** 960 **14.** 625 **15.** ₹ 27750

16. Values of the prizes (in ₹) are 160, 140, 120, 100, 80, 60, 40.

17. 234 **18.** 143 cm

19. 16 rows, 5 logs are placed in the top row. By putting S = 200, a = 20, d = –1 in the formula $S = \frac{n}{2}[2a + (n-1)d]$, we get, $41n - n^2 = 400$. On solving, n = 16, 25. Therefore, the number of rows is either 16 or 25. $a_{25} = a + 24\,d = -4$

i.e., number of logs in 25th row is – 4 which is not possible. Therefore n = 25 is not possible. For n = 16, a_{16} = 5. Therefore, there are 16 rows and 5 logs placed in the top row.

20. 370 m

EXERCISE 5.4 (Optional)*

1. 32nd term **2.** S_{16} = 20, 76 **3.** 385 cm

4. 35 **5.** 750 m^3

EXERCISE 6.1

1. (i) Similar (ii) Similar (iii) Equilateral
(iv) Equal, Proportional **3.** No

EXERCISE 6.2

1. (i) 2 cm (ii) 2.4 cm

2. (i) No (ii) Yes (iii) Yes

9. Through O, draw a line parallel to DC, intersecting AD and BC at E and F respectively.

EXERCISE 6.3

1. (i) Yes. AAA, Δ ABC ~ Δ PQR (ii) Yes. SSS, Δ ABC ~ Δ QRP
(iii) No (iv) Yes. SAS, Δ MNL ~ Δ QPR
(v) No (vi) Yes. AA, Δ DEF ~ Δ PQR

2. 55°, 55°, 55°

14. Produce AD to a point E such that AD = DE and produce PM to a point N such that PM = MN. Join EC and NR.

15. 42 m

EXERCISE 7.1

1. (i) $2\sqrt{2}$ (ii) $4\sqrt{2}$ (iii) $2\sqrt{a^2+b^2}$

2. 39; 39 km **3.** No **4.** Yes **5.** Champa is correct.

6. (i) Square (ii) No quadrilateral (iii) Parallelogram

7. $(-7, 0)$ **8.** $-9, 3$ **9.** ± 4, QR $= \sqrt{41}$, PR $= \sqrt{82}$, $9\sqrt{2}$

10. $3x + y - 5 = 0$

EXERCISE 7.2

1. $(1, 3)$ **2.** $\left(2, -\frac{5}{3}\right); \left(0, -\frac{7}{3}\right)$

3. $\sqrt{61}$ m; 5th line at a distance of 22.5 m **4.** $2 : 7$

5. $1 : 1; \left(-\frac{3}{2}, 0\right)$ **6.** $x = 6,\ y = 3$ **7.** $(3, -10)$

8. $\left(-\frac{2}{7}, -\frac{20}{7}\right)$ **9.** $\left(-1, \frac{7}{2}\right), (0, 5), \left(1, \frac{13}{2}\right)$ **10.** 24 sq. units

EXERCISE 8.1

1. (i) $\sin A = \frac{7}{25}, \cos A = \frac{24}{25}$ (ii) $\sin C = \frac{24}{25}, \cos C = \frac{7}{25}$

2. 0 **3.** $\cos A = \frac{\sqrt{7}}{4}, \tan A = \frac{3}{\sqrt{7}}$ **4.** $\sin A = \frac{15}{17}, \sec A = \frac{17}{8}$

5. $\sin\theta = \frac{5}{13}, \cos\theta = \frac{12}{13}, \tan\theta = \frac{5}{12}, \cot\theta = \frac{12}{5}, \operatorname{cosec}\theta = \frac{13}{5}$

7. (i) $\frac{49}{64}$ (ii) $\frac{49}{64}$ **8.** Yes

9. (i) 1 (ii) 0 **10.** $\sin P = \frac{12}{13}, \cos P = \frac{5}{13}, \tan P = \frac{12}{5}$

11. (i) False (ii) True (iii) False (iv) False (v) False

EXERCISE 8.2

1. (i) 1 (ii) 2 (iii) $\dfrac{3\sqrt{2}-\sqrt{6}}{8}$ (iv) $\dfrac{43-24\sqrt{3}}{11}$ (v) $\dfrac{67}{12}$

2. (i) A (ii) D (iii) A (iv) C

3. $\angle A = 45°$, $\angle B = 15°$

4. (i) False (ii) True (iii) False (iv) False (v) True

EXERCISE 8.3

1. $\sin A = \dfrac{1}{\sqrt{1+\cot^2 A}}$, $\tan A = \dfrac{1}{\cot A}$, $\sec A = \dfrac{\sqrt{1+\cot^2 A}}{\cot A}$

2. $\sin A = \dfrac{\sqrt{\sec^2 A - 1}}{\sec A}$, $\cos A = \dfrac{1}{\sec A}$, $\tan A = \sqrt{\sec^2 A - 1}$

   $\cot A = \dfrac{1}{\sqrt{\sec^2 A - 1}}$, $\operatorname{cosec} A = \dfrac{\sec A}{\sqrt{\sec^2 A - 1}}$

3. (i) B (ii) C (iii) D (iv) D

EXERCISE 9.1

1. 10 m
2. $8\sqrt{3}$ m
3. 3 m, $2\sqrt{3}$ m
4. $10\sqrt{3}$ m
5. $40\sqrt{3}$ m
6. $19\sqrt{3}$ m
7. $20\left(\sqrt{3}-1\right)$ m
8. $0.8\left(\sqrt{3}+1\right)$ m
9. $16\dfrac{2}{3}$ m
10. $20\sqrt{3}$ m, 20 m, 60 m
11. $10\sqrt{3}$ m, 10 m
12. $7\left(\sqrt{3}+1\right)$ m
13. $75(\sqrt{3}-1)$m
14. $58\sqrt{3}$ m
15. 3 seconds

EXERCISE 10.1

1. Infinitely many
2. (i) One (ii) Secant (iii) Two (iv) Point of contact
3. D

EXERCISE 10.2

1. A
2. B
3. A
6. 3 cm
7. 8 cm
12. AB = 15 cm, AC = 13 cm

EXERCISE 11.1

1. $\frac{132}{7}\text{cm}^2$ **2.** $\frac{77}{8}\text{cm}^2$ **3.** $\frac{154}{3}\text{cm}^2$

4. (i) 28.5 cm^2 (ii) 235.5 cm^2

5. (i) 22 cm (ii) 231 cm^2 (iii) $\left(231 - \frac{441\sqrt{3}}{4}\right)\text{cm}^2$

6. 20.4375 cm^2 ; 686.0625 cm^2 **7.** 88.44 cm^2

8. (i) 19.625 m^2 (ii) 58.875 cm^2 **9.** (i) 285 mm (ii) $\frac{385}{4}\text{ mm}^2$

10. $\frac{22275}{28}\text{ cm}^2$ **11.** $\frac{158125}{126}\text{ cm}^2$ **12.** 189.97 km^2

13. ₹ 162.68 **14.** D

EXERCISE 12.1

1. 160 cm^2 **2.** 572 cm^2 **3.** 214.5 cm^2

4. Greatest diameter = 7 cm,surface area = 332.5 cm^2

5. $\frac{1}{4}l^2\ (\pi + 24)$ **6.** 220 mm^2 **7.** 44 m^2, ₹ 22000

8. 18 cm^2 **9.** 374 cm^2

EXERCISE 12.2

1. $\pi\text{ cm}^3$

2. 66 cm^3. Volume of the air inside the model = Volume of air inside (cone + cylinder + cone)

$= \left(\frac{1}{3}\pi r^2 h_1 + \pi r^2 h_2 + \frac{1}{3}\pi r^2 h_1\right)$, where r is the radius of the cone and the cylinder, h_1 is the height (length) of the cone and h_2 is the height (length) of the cylinder.

Required Volume $= \frac{1}{3}\pi r^2 \left(h_1 + 3h_2 + h_1\right)$.

3. 338 cm^3 **4.** 523.53 cm^3 **5.** 100 **6.** 892.26 kg

7. 1.131 m^3 (approx.) **8.** Not correct. Correct answer is 346.51 cm^3.

EXERCISE 13.1

1. 8.1 plants. We have used direct method because numerical values of x_i and f_i are small.
2. ₹ 545.20
3. $f = 20$
4. 75.9
5. 57.19
6. ₹ 211
7. 0.099 ppm
8. 12.48 days
9. 69.43 %

EXERCISE 13.2

1. Mode = 36.8 years, Mean = 35.37 years. Maximum number of patients admitted in the hospital are of the age 36.8 years (approx.), while on an average the age of a patient admitted to the hospital is 35.37 years.
2. 65.625 hours
3. Modal monthly expenditure = ₹ 1847.83, Mean monthly expenditure = ₹ 2662.5.
4. Mode : 30.6, Mean = 29.2. Most states/U.T. have a student teacher ratio of 30.6 and on an average, this ratio is 29.2.
5. Mode = 4608.7 runs
6. Mode = 44.7 cars

EXERCISE 13.3

1. Median = 137 units, Mean = 137.05 units, Mode = 135.76 units.

   The three measures are approximately the same in this case.
2. $x = 8, y = 7$
3. Median age = 35.76 years
4. Median length = 146.75 mm
5. Median life = 3406.98 hours
6. Median = 8.05, Mean = 8.32, Modal size = 7.88
7. Median weight = 56.67 kg

EXERCISE 14.1

1. (i) 1 (ii) 0, impossible event (iii) 1, sure or certain event

   (iv) 1 (v) 0, 1
2. The experiments (iii) and (iv) have equally likely outcomes.
3. When we toss a coin, the outcomes head and tail are equally likely. So, the result of an individual coin toss is completely unpredictable.
4. B
5. 0.95
6. (i) 0 (ii) 1
7. 0.008
8. (i) $\frac{3}{8}$ (ii) $\frac{5}{8}$
9. (i) $\frac{5}{17}$ (ii) $\frac{8}{17}$ (iii) $\frac{13}{17}$
10. (i) $\frac{5}{9}$ (ii) $\frac{17}{18}$

11. $\frac{5}{13}$

12. (i) $\frac{1}{8}$ (ii) $\frac{1}{2}$ (iii) $\frac{3}{4}$ (iv) 1

13. (i) $\frac{1}{2}$ (ii) $\frac{1}{2}$ (iii) $\frac{1}{2}$

14. (i) $\frac{1}{26}$ (ii) $\frac{3}{13}$ (iii) $\frac{3}{26}$ (iv) $\frac{1}{52}$ (v) $\frac{1}{4}$ (vi) $\frac{1}{52}$

15. (i) $\frac{1}{5}$ (ii) (a) $\frac{1}{4}$ (b) 0

16. $\frac{11}{12}$

17. (i) $\frac{1}{5}$ (ii) $\frac{15}{19}$

18. (i) $\frac{9}{10}$ (ii) $\frac{1}{10}$ (iii) $\frac{1}{5}$

19. (i) $\frac{1}{3}$ (ii) $\frac{1}{6}$

20. $\frac{\pi}{24}$

21. (i) $\frac{31}{36}$ (ii) $\frac{5}{36}$

22. (i)

Sum on 2 dice	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

(ii) No. The eleven sums are not equally likely.

23. $\frac{3}{4}$; Possible outcomes are : HHH, TTT, HHT, HTH, HTT, THH, THT, TTH. Here, THH means tail in the first toss, head on the second toss and head on the third toss and so on.

24. (i) $\frac{25}{36}$ (ii) $\frac{11}{36}$

25. (i) Incorrect. We can classify the outcomes like this but they are not then 'equally likely'. Reason is that 'one of each' can result in two ways — from a head on first coin and tail on the second coin or from a tail on the first coin and head on the second coin. This makes it twicely as likely as two heads (or two tails).

(ii) Correct. The two outcomes considered in the question are equally likely.

EXERCISE A1.1

1. (i) Ambiguous (ii) True (iii) True (iv) Ambiguous
   (v) Ambiguous
2. (i) True (ii) True (iii) False (iv) True (v) True
3. Only (ii) is true.
4. (i) If $a > 0$ and $a^2 > b^2$, then $a > b$.
   (ii) If $xy \geq 0$ and $x^2 = y^2$, then $x = y$.
   (iii) If $(x + y)^2 = x^2 + y^2$ and $y \neq 0$, then $x = 0$.
   (iv) The diagonals of a parallelogram bisect each other.

EXERCISE A1.2

1. A is mortal
2. *ab* is rational
3. Decimal expansion of $\sqrt{17}$ is non-terminating non-recurring.
4. $y = 7$
5. $\angle A = 100°, \angle C = 100°, \angle D = 180°$
6. PQRS is a rectangle.
7. Yes, because of the premise. No, because $\sqrt{3721} = 61$ which is not irrational. Since the premise was wrong, the conclusion is false.

EXERCISE A1.3

1. Take two consecutive odd numbers as $2n + 1$ and $2n + 3$ for some integer n.

EXERCISE A1.4

1. (i) Man is not mortal.
   (ii) Line l is not parallel to line m.
   (iii) The chapter does not have many exercises.
   (iv) Not all integers are rational numbers.
   (v) All prime numbers are not odd.
   (vi) Some students are lazy.
   (vii) All cats are black.
   (viii) There is at least one real number x, such that $\sqrt{x} = -1$.

(ix) 2 does not divide the positive integer a.

(x) Integers a and b are not coprime.

2. (i) Yes (ii) No (iii) No (iv) No (v) Yes

EXERCISE A1.5

1. (i) If Sharan sweats a lot, then it is hot in Tokyo.

(ii) If Shalini's stomach grumbles, then she is hungry.

(iii) If Jaswant can get a degree, then she has a scholarship.

(iv) If a plant is alive, then it has flowers.

(v) If an animal has a tail, then it is a cat.

2. (i) If the base angles of triangle ABC are equal, then it is isosceles. True.

(ii) If the square of an integer is odd, then the integer is odd. True.

(iii) If $x = 1$, then $x^2 = 1$. True.

(iv) If AC and BD bisect each other, then ABCD is a parallelogram. True.

(v) If $a + (b + c) = (a + b) + c$, then a, b and c are whole numbers. False.

(vi) If $x + y$ is an even number, then x and y are odd. False.

(vii) If a parallelogram is a rectangle, its vertices lie on a circle. True.

EXERCISE A1.6

1. Suppose to the contrary $b \leq d$.

3. See Example 10 of Chapter 1.

6. See Theorem 5.1 of Class IX Mathematics Textbook.

EXERCISE A2.2

1. (i) $\frac{1}{5}$ (ii) 160

2. Take 1 cm^2 area and count the number of dots in it. Total number of trees will be the product of this number and the area (in cm^2).

3. Rate of interest in instalment scheme is 17.74 %, which is less than 18 %.

EXERCISE A2.3

1. Students find their own answers.

MATHEMATICS

Textbook for Class X

1062

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NATIONAL COUNCIL OF EDUCATIONAL RESEARCH AND TRAINING

1062 – MATHEMATICS
Textbook for Class X

ISBN 81-7450-634-9

First Edition
December 2006 Pausa 1928

Reprinted
October 2007, January 2009, December 2009, November 2010, January 2012, November 2012, November 2013, November 2014, December 2015, December 2016, December 2017, January 2019, August 2019, January 2021 and November 2021

Revised Edition
October 2022, Ashwina 1944
Reprinted
March 2024 Chaitra 1946

PD 700T SU

₹ 170.00

Printed on 80 GSM paper with NCERT watermark

Published at the Publication Division by the Secretary, National Council of Educational Research and Training, Sri Aurobindo Marg, New Delhi 110 016 and printed at Rohan Pragya Printing & Packing Pvt. Ltd., H-76, Site-V, UPSIDC, Kasna, Greater Noida, G.B. Nagar (U.P.)

OFFICES OF THE PUBLICATION DIVISION, NCERT

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Foreword

The National Curriculum Framework 2005, recommends that children's life at school must be linked to their life outside the school. This principle marks a departure from the legacy of bookish learning which continues to shape our system and causes a gap between the school, home and community. The syllabi and textbooks developed on the basis of NCF signify an attempt to implement this basic idea. They also attempt to discourage rote learning and the maintenance of sharp boundaries between different subject areas. We hope these measures will take us significantly further in the direction of a child-centred system of education outlined in the National Policy on Education (1986).

The success of this effort depends on the steps that school principals and teachers will take to encourage children to reflect on their own learning and to pursue imaginative activities and questions. We must recognise that, given space, time and freedom, children generate new knowledge by engaging with the information passed on to them by adults. Treating the prescribed textbook as the sole basis of examination is one of the key reasons why other resources and sites of learning are ignored. Inculcating creativity and initiative is possible if we perceive and treat children as participants in learning, not as receivers of a fixed body of knowledge.

These aims imply considerable change in school routines and mode of functioning. Flexibility in the daily time-table is as necessary as rigour in implementing the annual calendar so that the required number of teaching days are actually devoted to teaching. The methods used for teaching and evaluation will also determine how effective this textbook proves for making children's life at school a happy experience, rather than a source of stress or boredom. Syllabus designers have tried to address the problem of curricular burden by restructuring and reorienting knowledge at different stages with greater consideration for child psychology and the time available for teaching. The textbook attempts to enhance this endeavour by giving higher priority and space to opportunities for contemplation and wondering, discussion in small groups, and activities requiring hands-on experience.

The National Council of Educational Research and Training (NCERT) appreciates the hard work done by the textbook development committee responsible for this book. We wish to thank the Chairperson of the advisory group in Science and Mathematics, Professor J.V. Narlikar and the Chief Advisors for this book, Professor P. Sinclair of IGNOU, New Delhi and Professor G.P. Dikshit (Retd.) of Lucknow University, Lucknow for guiding the work of this committee. Several teachers contributed to the development of this textbook; we are grateful to their principals for making this possible. We are indebted to the institutions and organisations which have generously permitted us to draw upon their resources, material and personnel. We are especially grateful to the members of the National Monitoring Committee, appointed by the Department of Secondary and Higher Education, Ministry of Human Resource Development under the Chairpersonship of Professor Mrinal Miri and Professor G.P. Deshpande, for their valuable time and contribution. As an organisation committed to systemic reform and continuous improvement in the quality of its products, NCERT welcomes comments and suggestions which will enable us to undertake further revision and refinement.

Director
National Council of Educational
Research and Training

New Delhi
15 November 2006

Rationalisation of Content in the Textbooks

In view of the COVID-19 pandemic, it is imperative to reduce content load on students. The National Education Policy 2020, also emphasises reducing the content load and providing opportunities for experiential learning with creative mindset. In this background, the NCERT has undertaken the exercise to rationalise the textbooks across all classes. Learning Outcomes already developed by the NCERT across classes have been taken into consideration in this exercise.

Contents of the textbooks have been rationalised in view of the following:

- Overlapping with similar content included in other subject areas in the same class
- Similar content included in the lower or higher class in the same subject
- Difficulty level
- Content, which is easily accessible to students without much interventions from teachers and can be learned by children through self-learning or peer-learning
- Content, which is irrelevant in the present context

This present edition, is a reformatted version after carrying out the changes given above.

Preface

Through the years, from the time of the Kothari Commission, there have been several committees looking at ways of making the school curriculum meaningful and enjoyable for the learners. Based on the understanding developed over the years, a National Curriculum Framework (NCF) was finalised in 2005. As part of this exercise, a National Focus Group on Teaching of Mathematics was formed. Its report, which came in 2005, highlighted a constructivist approach to the teaching and learning of mathematics.

The essence of this approach is that children already know, and do some mathematics very naturally in their surroundings, before they even join school. The syllabus, teaching approach, textbooks etc., should build on this knowledge in a way that allows children to enjoy mathematics, and to realise that mathematics is more about a way of reasoning than about mechanically applying formulae and algorithms. The students and teachers need to perceive mathematics as something natural and linked to the world around us. While teaching mathematics, the focus should be on helping children to develop the ability to particularise and generalise, to solve and pose meaningful problems, to look for patterns and relationships, and to apply the logical thinking behind mathematical proof. And, all this in an environment that the children relate to, without overloading them.

This is the philosophy with which the mathematics syllabus from Class I to Class XII was developed, and which the textbook development committee has tried to realise in the present textbook. More specifically, while creating the textbook, the following broad guidelines have been kept in mind.

- The matter needs to be linked to what the child has studied before, and to her experiences.
- The language used in the book, including that for 'word problems', must be clear, simple and unambiguous.
- Concepts/processes should be introduced through situations from the children's environment.
- For each concept/process give several examples and exercises, but not of the same kind. This ensures that the children use the concept/process again and again, but in varying contexts. Here 'several' should be within reason, not overloading the child.
- Encourage the children to see, and come out with, diverse solutions to problems.

- As far as possible, give the children motivation for results used.
- All proofs need to be given in a non-didactic manner, allowing the learner to see the flow of reason. The focus should be on proofs where a short and clear argument reinforces mathematical thinking and reasoning.
- Whenever possible, more than one proof is to be given.
- Proofs and solutions need to be used as vehicles for helping the learner develop a clear and logical way of expressing her arguments.
- All geometric constructions should be accompanied by an analysis of the construction and a proof for the steps taken to do the required construction. Accordingly, the children would be trained to do the same while doing constructions.
- Add such small anecdotes, pictures, cartoons and historical remarks at several places which the children would find interesting.
- Include optional exercises for the more interested learners. These would not be tested in the examinations.
- Give answers to all exercises, and solutions/hints for those that the children may require.
- Whenever possible, propagate constitutional values.

As you will see while studying this textbook, these points have been kept in mind by the Textbook Development Committee. The book has particularly been created with the view to giving children space to explore mathematics and develop the abilities to reason mathematically. Further, two special appendices have been given — Proofs in Mathematics, and Mathematical Modelling. These are placed in the book for interested students to study, and are only optional reading at present. These topics may be considered for inclusion in the main syllabi in due course of time.

As in the past, this textbook is also a team effort. However, what is unusual about the team this time is that teachers from different kinds of schools have been an integral part at each stage of the development. We are also assuming that teachers will contribute continuously to the process in the classroom by formulating examples and exercises contextually suited to the children in their particular classrooms. Finally, we hope that teachers and learners would send comments for improving the textbook to the NCERT.

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Acknowledgements

The Council gratefully acknowledges the valuable contributions of the following participants of the Textbook Review Workshop:

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Special thanks are due to Professor Hukum Singh, *Head* (Retd.)*,* DESM, NCERT for his support during the development of this book.

The Council acknowledges the efforts of Deepak Kapoor, *Incharge,* Computer Station; Purnendu Kumar Barik, *Copy Editor;* Naresh Kumar and Nargis Islam, *D.T.P. Operators;* Yogita Sharma, *Proof Reader*.

The Contribution of APC-Office, administration of DESM, Publication Department and Secretariat of NCERT is also duly acknowledged.

Contents

Constitution of India

Part IV A (Article 51 A)

Fundamental Duties

It shall be the duty of every citizen of India —

(a) to abide by the Constitution and respect its ideals and institutions, the National Flag and the National Anthem;

(b) to cherish and follow the noble ideals which inspired our national struggle for freedom;

(c) to uphold and protect the sovereignty, unity and integrity of India;

(d) to defend the country and render national service when called upon to do so;

(e) to promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practices derogatory to the dignity of women;

(f) to value and preserve the rich heritage of our composite culture;

(g) to protect and improve the natural environment including forests, lakes, rivers, wildlife and to have compassion for living creatures;

(h) to develop the scientific temper, humanism and the spirit of inquiry and reform;

(i) to safeguard public property and to abjure violence;

(j) to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievement;

*(k) who is a parent or guardian, to provide opportunities for education to his child or, as the case may be, ward between the age of six and fourteen years.

Note: The Article 51A containing Fundamental Duties was inserted by the Constitution (42nd Amendment) Act, 1976 (with effect from 3 January 1977).

*(k) was inserted by the Constitution (86th Amendment) Act, 2002 (with effect from 1 April 2010).

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REAL NUMBERS 1

1.1 Introduction

In Class IX, you began your exploration of the world of real numbers and encountered irrational numbers. We continue our discussion on real numbers in this chapter. We begin with two very important properties of positive integers in Sections 1.2 and 1.3, namely the Euclid's division algorithm and the Fundamental Theorem of Arithmetic.

Euclid's division algorithm, as the name suggests, has to do with divisibility of integers. Stated simply, it says any positive integer a can be divided by another positive integer b in such a way that it leaves a remainder r that is smaller than b. Many of you probably recognise this as the usual long division process. Although this result is quite easy to state and understand, it has many applications related to the divisibility properties of integers. We touch upon a few of them, and use it mainly to compute the HCF of two positive integers.

The Fundamental Theorem of Arithmetic, on the other hand, has to do something with multiplication of positive integers. You already know that every composite number can be expressed as a product of primes in a unique way — this important fact is the Fundamental Theorem of Arithmetic. Again, while it is a result that is easy to state and understand, it has some very deep and significant applications in the field of mathematics. We use the Fundamental Theorem of Arithmetic for two main applications. First, we use it to prove the irrationality of many of the numbers you studied in Class IX, such as $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$. Second, we apply this theorem to explore when exactly the decimal expansion of a rational number, say $\frac{p}{q} (q \neq 0)$, is terminating and when it is non-terminating repeating. We do so by looking at the prime factorisation of the denominator q of $\frac{p}{q}$. You will see that the prime factorisation of q will completely reveal the nature of the decimal expansion of $\frac{p}{q}$.

So let us begin our exploration.

1.2 The Fundamental Theorem of Arithmetic

In your earlier classes, you have seen that any natural number can be written as a product of its prime factors. For instance, $2 = 2$, $4 = 2 \times 2$, $253 = 11 \times 23$, and so on. Now, let us try and look at natural numbers from the other direction. That is, can any natural number be obtained by multiplying prime numbers? Let us see.

Take any collection of prime numbers, say 2, 3, 7, 11 and 23. If we multiply some or all of these numbers, allowing them to repeat as many times as we wish, we can produce a large collection of positive integers (In fact, infinitely many). Let us list a few :

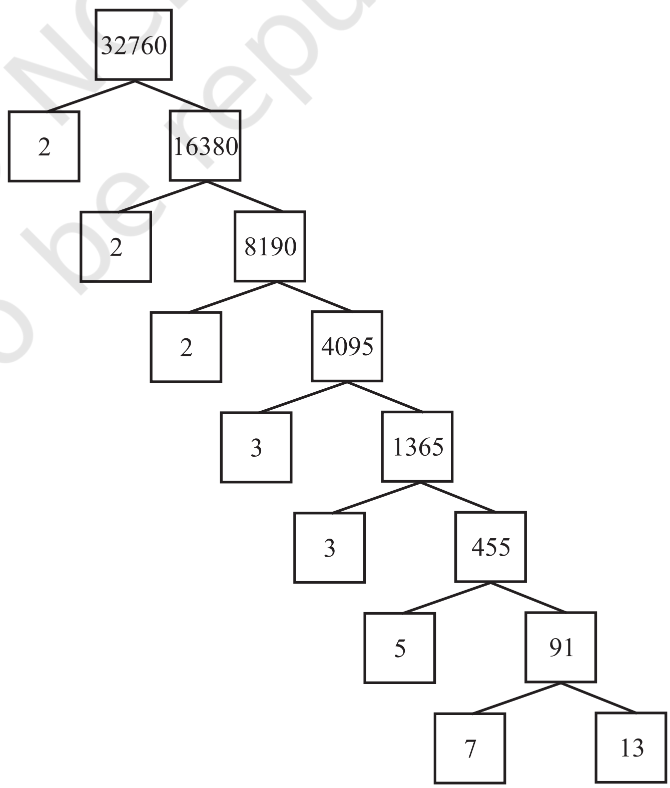
$7 \times 11 \times 23 = 1771$ $\qquad$ $3 \times 7 \times 11 \times 23 = 5313$

$2 \times 3 \times 7 \times 11 \times 23 = 10626$ $\qquad$ $2^3 \times 3 \times 7^3 = 8232$

$2^2 \times 3 \times 7 \times 11 \times 23 = 21252$

and so on.

Now, let us suppose your collection of primes includes all the possible primes. What is your guess about the size of this collection? Does it contain only a finite number of integers, or infinitely many? Infact, there are infinitely many primes. So, if we combine all these primes in all possible ways, we will get an infinite collection of numbers, all the primes and all possible products of primes. The question is – can we produce all the composite numbers this way? What do you think? Do you think that there may be a composite number which is not the product of powers of primes? Before we answer this, let us factorise positive integers, that is, do the opposite of what we have done so far.



We are going to use the factor tree with which you are all familiar. Let us take some large number, say, 32760, and factorise it as shown.

So we have factorised 32760 as $2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7 \times 13$ as a product of primes, i.e., $32760 = 2^3 \times 3^2 \times 5 \times 7 \times 13$ as a product of powers of primes. Let us try another number, say, 123456789. This can be written as $3^2 \times 3803 \times 3607$. Of course, you have to check that 3803 and 3607 are primes! (Try it out for several other natural numbers yourself.) This leads us to a conjecture that every composite number can be written as the product of powers of primes. In fact, this statement is true, and is called the **Fundamental Theorem of Arithmetic** because of its basic crucial importance to the study of integers. Let us now formally state this theorem.

Theorem 1.1 (Fundamental Theorem of Arithmetic) : *Every composite number can be expressed (factorised) as a product of primes, and this factorisation is unique, apart from the order in which the prime factors occur.*

An equivalent version of Theorem 1.2 was probably first recorded as Proposition 14 of Book IX in Euclid's Elements, before it came to be known as the Fundamental Theorem of Arithmetic. However, the first correct proof was given by Carl Friedrich Gauss in his *Disquisitiones Arithmeticae*.

Carl Friedrich Gauss is often referred to as the 'Prince of Mathematicians' and is considered one of the three greatest mathematicians of all time, along with Archimedes and Newton. He has made fundamental contributions to both mathematics and science.

Carl Friedrich Gauss (1777 – 1855)

The Fundamental Theorem of Arithmetic says that every composite number can be factorised as a product of primes. Actually it says more. It says that given any composite number it can be factorised as a product of prime numbers in a **'unique'** way, except for the order in which the primes occur. That is, given any composite number there is one and only one way to write it as a product of primes, as long as we are not particular about the order in which the primes occur. So, for example, we regard $2 \times 3 \times 5 \times 7$ as the same as $3 \times 5 \times 7 \times 2$, or any other possible order in which these primes are written. This fact is also stated in the following form:

The prime factorisation of a natural number is unique, except for the order of its factors.

In general, given a composite number x, we factorise it as $x = p_1 p_2 \ldots p_n$, where $p_1, p_2, \ldots, p_n$ are primes and written in ascending order, i.e., $p_1 \leq p_2 \leq \ldots \leq p_n$. If we combine the same primes, we will get powers of primes. For example,

$$32760 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7 \times 13 = 2^3 \times 3^2 \times 5 \times 7 \times 13$$

Once we have decided that the order will be ascending, then the way the number is factorised, is unique.

The Fundamental Theorem of Arithmetic has many applications, both within mathematics and in other fields. Let us look at some examples.

Example 1 : Consider the numbers 4^n, where n is a natural number. Check whether there is any value of n for which 4^n ends with the digit zero.

Solution : If the number 4^n, for any n, were to end with the digit zero, then it would be divisible by 5. That is, the prime factorisation of 4^n would contain the prime 5. This is not possible because $4^n = (2)^{2n}$; so the only prime in the factorisation of 4^n is 2. So, the uniqueness of the Fundamental Theorem of Arithmetic guarantees that there are no other primes in the factorisation of 4^n. So, there is no natural number n for which 4^n ends with the digit zero.

You have already learnt how to find the HCF and LCM of two positive integers using the Fundamental Theorem of Arithmetic in earlier classes, without realising it! This method is also called the *prime factorisation method*. Let us recall this method through an example.

Example 2 : Find the LCM and HCF of 6 and 20 by the prime factorisation method.

Solution : We have : $6 = 2^1 \times 3^1$ and $20 = 2 \times 2 \times 5 = 2^2 \times 5^1$.

You can find HCF(6, 20) = 2 and LCM(6, 20) = $2 \times 2 \times 3 \times 5 = 60$, as done in your earlier classes.

Note that HCF(6, 20) = 2^1 = **Product of the smallest power of each common prime factor in the numbers.**

LCM (6, 20) = $2^2 \times 3^1 \times 5^1$ = **Product of the greatest power of each prime factor**, **involved in the numbers**.

From the example above, you might have noticed that HCF(6, 20) × LCM(6, 20) = 6 × 20. In fact, we can verify that **for any two positive integers a and b, HCF (a, b) × LCM (a, b) = $a \times b$.** We can use this result to find the LCM of two positive integers, if we have already found the HCF of the two positive integers.

Example 3: Find the HCF of 96 and 404 by the prime factorisation method. Hence, find their LCM.

Solution : The prime factorisation of 96 and 404 gives :

$$96 = 2^5 \times 3, \; 404 = 2^2 \times 101$$

Therefore, the HCF of these two integers is $2^2 = 4$.

Also, $$\text{LCM}(96, 404) = \frac{96 \times 404}{\text{HCF}(96, 404)} = \frac{96 \times 404}{4} = 9696$$

Example 4 : Find the HCF and LCM of 6, 72 and 120, using the prime factorisation method.

Solution : We have :

$$6 = 2 \times 3, \; 72 = 2^3 \times 3^2, \; 120 = 2^3 \times 3 \times 5$$

Here, 2^1 and 3^1 are the smallest powers of the common factors 2 and 3, respectively.

So, $$\text{HCF}(6, 72, 120) = 2^1 \times 3^1 = 2 \times 3 = 6$$

2^3, 3^2 and 5^1 are the greatest powers of the prime factors 2, 3 and 5 respectively involved in the three numbers.

So, $$\text{LCM}(6, 72, 120) = 2^3 \times 3^2 \times 5^1 = 360$$

Remark : Notice, $6 \times 72 \times 120 \neq \text{HCF}(6, 72, 120) \times \text{LCM}(6, 72, 120)$. So, the product of three numbers is not equal to the product of their HCF and LCM.

EXERCISE 1.1

1. Express each number as a product of its prime factors:
   (i) 140 (ii) 156 (iii) 3825 (iv) 5005 (v) 7429
2. Find the LCM and HCF of the following pairs of integers and verify that LCM × HCF = product of the two numbers.
   (i) 26 and 91 (ii) 510 and 92 (iii) 336 and 54
3. Find the LCM and HCF of the following integers by applying the prime factorisation method.
   (i) 12, 15 and 21 (ii) 17, 23 and 29 (iii) 8, 9 and 25
4. Given that HCF (306, 657) = 9, find LCM (306, 657).
5. Check whether 6^n can end with the digit 0 for any natural number n.
6. Explain why $7 \times 11 \times 13 + 13$ and $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 + 5$ are composite numbers.
7. There is a circular path around a sports field. Sonia takes 18 minutes to drive one round of the field, while Ravi takes 12 minutes for the same. Suppose they both start at the

same point and at the same time, and go in the same direction. After how many minutes will they meet again at the starting point?

1.3 Revisiting Irrational Numbers

In Class IX, you were introduced to irrational numbers and many of their properties. You studied about their existence and how the rationals and the irrationals together made up the real numbers. You even studied how to locate irrationals on the number line. However, we did not prove that they were irrationals. In this section, we will prove that $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and, in general, $\sqrt{p}$ is irrational, where p is a prime. One of the theorems, we use in our proof, is the Fundamental Theorem of Arithmetic.

Recall, a number 's' is called *irrational* if it cannot be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. Some examples of irrational numbers, with which you are already familiar, are :

$$\sqrt{2}, \sqrt{3}, \sqrt{15}, \pi, -\frac{\sqrt{2}}{\sqrt{3}}, 0.10110111011110\ldots, \text{ etc.}$$

Before we prove that $\sqrt{2}$ is irrational, we need the following theorem, whose proof is based on the Fundamental Theorem of Arithmetic.

Theorem 1.2 : *Let p be a prime number. If p divides a^2, then p divides a, where a is a positive integer.*

***Proof :** Let the prime factorisation of a be as follows :

$$a = p_1 p_2 \ldots p_n, \text{ where } p_1, p_2, \ldots, p_n \text{ are primes, not necessarily distinct.}$$

Therefore, $a^2 = (p_1 p_2 \ldots p_n)(p_1 p_2 \ldots p_n) = p_1^2 p_2^2 \ldots p_n^2$.

Now, we are given that p divides a^2. Therefore, from the Fundamental Theorem of Arithmetic, it follows that p is one of the prime factors of a^2. However, using the uniqueness part of the Fundamental Theorem of Arithmetic, we realise that the only prime factors of a^2 are $p_1, p_2, \ldots, p_n$. So p is one of $p_1, p_2, \ldots, p_n$.

Now, since $a = p_1 p_2 \ldots p_n$, p divides a.

We are now ready to give a proof that $\sqrt{2}$ is irrational.

The proof is based on a technique called 'proof by contradiction'. (This technique is discussed in some detail in Appendix 1).

Theorem 1.3 : $\sqrt{2}$ *is irrational.*

Proof : Let us assume, to the contrary, that $\sqrt{2}$ is rational.

* Not from the examination point of view.

So, we can find integers r and s ($\neq 0$) such that $\sqrt{2} = \frac{r}{s}$.

Suppose r and s have a common factor other than 1. Then, we divide by the common factor to get $\sqrt{2} = \frac{a}{b}$, where a and b are coprime.

So, $b\sqrt{2} = a$.

Squaring on both sides and rearranging, we get $2b^2 = a^2$. Therefore, 2 divides a^2.

Now, by Theorem 1.3, it follows that 2 divides a.

So, we can write $a = 2c$ for some integer c.

Substituting for a, we get $2b^2 = 4c^2$, that is, $b^2 = 2c^2$.

This means that 2 divides b^2, and so 2 divides b (again using Theorem 1.3 with $p = 2$).

Therefore, a and b have at least 2 as a common factor.

But this contradicts the fact that a and b have no common factors other than 1.

This contradiction has arisen because of our incorrect assumption that $\sqrt{2}$ is rational.

So, we conclude that $\sqrt{2}$ is irrational.

Example 5 : Prove that $\sqrt{3}$ is irrational.

Solution : Let us assume, to the contrary, that $\sqrt{3}$ is rational.

That is, we can find integers a and b ($\neq 0$) such that $\sqrt{3} = \frac{a}{b}$.

Suppose a and b have a common factor other than 1, then we can divide by the common factor, and assume that a and b are coprime.

So, $b\sqrt{3} = a$.

Squaring on both sides, and rearranging, we get $3b^2 = a^2$.

Therefore, a^2 is divisible by 3, and by Theorem 1.3, it follows that a is also divisible by 3.

So, we can write $a = 3c$ for some integer c.

Substituting for a, we get $3b^2 = 9c^2$, that is, $b^2 = 3c^2$.

This means that b^2 is divisible by 3, and so b is also divisible by 3 (using Theorem 1.3 with $p = 3$).

Therefore, a and b have at least 3 as a common factor.

But this contradicts the fact that a and b are coprime.

This contradiction has arisen because of our incorrect assumption that $\sqrt{3}$ is rational.

So, we conclude that $\sqrt{3}$ is irrational.

In Class IX, we mentioned that :

- the sum or difference of a rational and an irrational number is irrational and
- the product and quotient of a non-zero rational and irrational number is irrational.

We prove some particular cases here.

Example 6 : Show that $5-\sqrt{3}$ is irrational.

Solution : Let us assume, to the contrary, that $5-\sqrt{3}$ is rational.

That is, we can find coprime a and b $(b \neq 0)$ such that $5-\sqrt{3}=\frac{a}{b}$.

Therefore, $5-\frac{a}{b}=\sqrt{3}$.

Rearranging this equation, we get $\sqrt{3}=5-\frac{a}{b}=\frac{5b-a}{b}$.

Since a and b are integers, we get $5-\frac{a}{b}$ is rational, and so $\sqrt{3}$ is rational.

But this contradicts the fact that $\sqrt{3}$ is irrational.

This contradiction has arisen because of our incorrect assumption that $5-\sqrt{3}$ is rational.

So, we conclude that $5-\sqrt{3}$ is irrational.

Example 7 : Show that $3\sqrt{2}$ is irrational.

Solution : Let us assume, to the contrary, that $3\sqrt{2}$ is rational.

That is, we can find coprime a and b $(b \neq 0)$ such that $3\sqrt{2}=\frac{a}{b}$.

Rearranging, we get $\sqrt{2}=\frac{a}{3b}$.

Since 3, a and b are integers, $\frac{a}{3b}$ is rational, and so $\sqrt{2}$ is rational.

But this contradicts the fact that $\sqrt{2}$ is irrational.

So, we conclude that $3\sqrt{2}$ is irrational.

EXERCISE 1.2

1. Prove that $\sqrt{5}$ is irrational.
2. Prove that $3 + 2\sqrt{5}$ is irrational.
3. Prove that the following are irrationals :

   (i) $\frac{1}{\sqrt{2}}$ (ii) $7\sqrt{5}$ (iii) $6 + \sqrt{2}$

1.4 Summary

In this chapter, you have studied the following points:

1. The Fundamental Theorem of Arithmetic :

   Every composite number can be expressed (factorised) as a product of primes, and this factorisation is unique, apart from the order in which the prime factors occur.
2. If p is a prime and p divides a^2, then p divides a, where a is a positive integer.
3. To prove that $\sqrt{2}, \sqrt{3}$ are irrationals.

A NOTE TO THE READER

You have seen that :

HCF $(p, q, r) \times$ LCM $(p, q, r) \neq p \times q \times r$, where p, q, r are positive integers (see Example 8). However, the following results hold good for three numbers p, q and r :

$$\text{LCM}(p, q, r) = \frac{p \cdot q \cdot r \cdot \text{HCF}(p, q, r)}{\text{HCF}(p, q) \cdot \text{HCF}(q, r) \cdot \text{HCF}(p, r)}$$

$$\text{HCF}(p, q, r) = \frac{p \cdot q \cdot r \cdot \text{LCM}(p, q, r)}{\text{LCM}(p, q) \cdot \text{LCM}(q, r) \cdot \text{LCM}(p, r)}$$

1062CH02

POLYNOMIALS 2

2.1 Introduction

In Class IX, you have studied polynomials in one variable and their degrees. Recall that if $p(x)$ is a polynomial in x, the highest power of x in $p(x)$ is called **the degree of the polynomial** $p(x)$. For example, $4x + 2$ is a polynomial in the variable x of degree 1, $2y^2 - 3y + 4$ is a polynomial in the variable y of degree 2, $5x^3 - 4x^2 + x - \sqrt{2}$ is a polynomial in the variable x of degree 3 and $7u^6 - \frac{3}{2}u^4 + 4u^2 + u - 8$ is a polynomial in the variable u of degree 6. Expressions like $\frac{1}{x-1}$, $\sqrt{x} + 2$, $\frac{1}{x^2 + 2x + 3}$ etc., are not polynomials.

A polynomial of degree 1 is called a **linear polynomial**. For example, $2x - 3$, $\sqrt{3}x + 5$, $y + \sqrt{2}$, $x - \frac{2}{11}$, $3z + 4$, $\frac{2}{3}u + 1$, etc., are all linear polynomials. Polynomials such as $2x + 5 - x^2$, $x^3 + 1$, etc., are not linear polynomials.

A polynomial of degree 2 is called a **quadratic polynomial**. The name 'quadratic' has been derived from the word 'quadrate', which means 'square'. $2x^2 + 3x - \frac{2}{5}$, $y^2 - 2$, $2 - x^2 + \sqrt{3}x$, $\frac{u}{3} - 2u^2 + 5$, $\sqrt{5}v^2 - \frac{2}{3}v$, $4z^2 + \frac{1}{7}$ are some examples of quadratic polynomials (whose coefficients are real numbers). More generally, any quadratic polynomial in x is of the form $ax^2 + bx + c$, where a, b, c are real numbers and $a \neq 0$. A polynomial of degree 3 is called a **cubic polynomial**. Some examples of a cubic polynomial are $2 - x^3$, x^3, $\sqrt{2}\,x^3$, $3 - x^2 + x^3$, $3x^3 - 2x^2 + x - 1$. In fact, the most general form of a cubic polynomial is

$$ax^3 + bx^2 + cx + d,$$

where, a, b, c, d are real numbers and $a \neq 0$.

Now consider the polynomial $p(x) = x^2 - 3x - 4$. Then, putting $x = 2$ in the polynomial, we get $p(2) = 2^2 - 3 \times 2 - 4 = - 6$. The value '$- 6$', obtained by replacing x by 2 in $x^2 - 3x - 4$, is the value of $x^2 - 3x - 4$ at $x = 2$. Similarly, $p(0)$ is the value of $p(x)$ at $x = 0$, which is $- 4$.

If $p(x)$ is a polynomial in x, and if k is any real number, then the value obtained by replacing x by k in $p(x)$, is called **the value of $p(x)$ at $x = k$**, and is denoted by $p(k)$.

What is the value of $p(x) = x^2 - 3x - 4$ at $x = -1$? We have :

$$p(-1) = (-1)^2 - \{3 \times (-1)\} - 4 = 0$$

Also, note that $\quad p(4) = 4^2 - (3 \times 4) - 4 = 0.$

As $p(-1) = 0$ and $p(4) = 0$, -1 and 4 are called the zeroes of the quadratic polynomial $x^2 - 3x - 4$. More generally, a real number k is said to be a **zero of a polynomial $p(x)$**, if $p(k) = 0$.

You have already studied in Class IX, how to find the zeroes of a linear polynomial. For example, if k is a zero of $p(x) = 2x + 3$, then $p(k) = 0$ gives us $2k + 3 = 0$, i.e., $k = -\dfrac{3}{2}\cdot$

In general, if k is a zero of $p(x) = ax + b$, then $p(k) = ak + b = 0$, i.e., $k = \dfrac{-b}{a}\cdot$

So, the zero of the linear polynomial $ax + b$ is $\dfrac{-b}{a} = \dfrac{-\text{(Constant term)}}{\text{Coefficient of } x}$.

Thus, the zero of a linear polynomial is related to its coefficients. Does this happen in the case of other polynomials too? For example, are the zeroes of a quadratic polynomial also related to its coefficients?

In this chapter, we will try to answer these questions. We will also study the division algorithm for polynomials.

2.2 Geometrical Meaning of the Zeroes of a Polynomial

You know that a real number k is a zero of the polynomial $p(x)$ if $p(k) = 0$. But why are the zeroes of a polynomial so important? To answer this, first we will see the **geometrical** representations of linear and quadratic polynomials and the geometrical meaning of their zeroes.

Consider first a linear polynomial $ax + b$, $a \neq 0$. You have studied in Class IX that the graph of $y = ax + b$ is a straight line. For example, the graph of $y = 2x + 3$ is a straight line passing through the points $(-2, -1)$ and $(2, 7)$.

x	-2	2
$y = 2x + 3$	-1	7

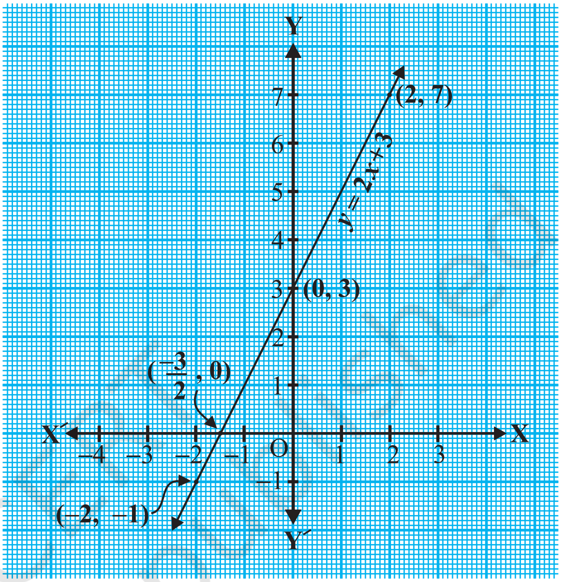


Fig. 2.1

From Fig. 2.1, you can see that the graph of $y = 2x + 3$ intersects the x-axis mid-way between $x = -1$ and $x = -2$, that is, at the point $\left(-\frac{3}{2}, 0\right)$. You also know that the zero of $2x + 3$ is $-\frac{3}{2}$. Thus, the zero of the polynomial $2x + 3$ is the x-coordinate of the point where the graph of $y = 2x + 3$ intersects the x-axis.

In general, for a linear polynomial $ax + b$, $a \neq 0$, the graph of $y = ax + b$ is a straight line which intersects the x-axis at exactly one point, namely, $\left(\frac{-b}{a}, 0\right)$. Therefore, the linear polynomial $ax + b$, $a \neq 0$, has exactly one zero, namely, the x-coordinate of the point where the graph of $y = ax + b$ intersects the x-axis.

Now, let us look for the geometrical meaning of a zero of a quadratic polynomial. Consider the quadratic polynomial $x^2 - 3x - 4$. Let us see what the graph* of $y = x^2 - 3x - 4$ looks like. Let us list a few values of $y = x^2 - 3x - 4$ corresponding to a few values for x as given in Table 2.1.

* Plotting of graphs of quadratic or cubic polynomials is not meant to be done by the students, nor is to be evaluated.

Table 2.1

x	-2	-1	0	1	2	3	4	5
$y = x^2 - 3x - 4$	6	0	-4	-6	-6	-4	0	6

If we locate the points listed above on a graph paper and draw the graph, it will actually look like the one given in Fig. 2.2.

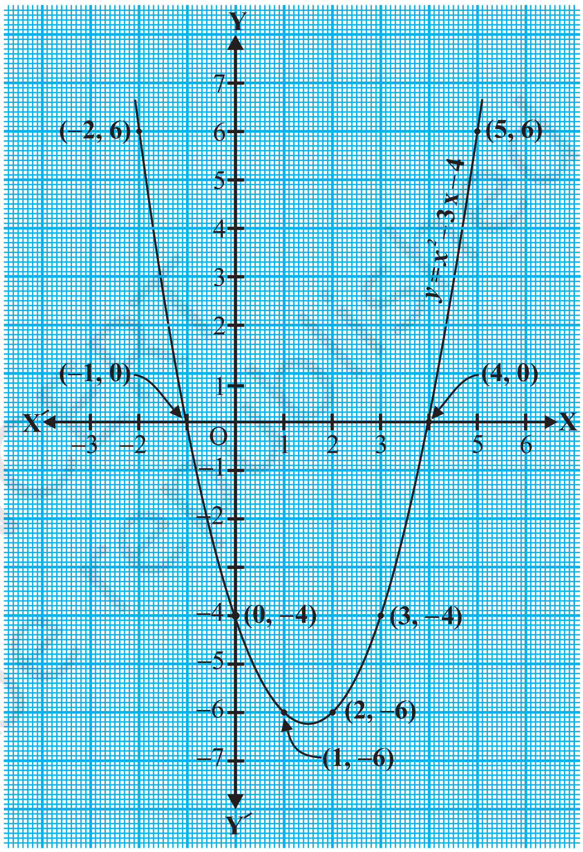


Fig. 2.2

In fact, for any quadratic polynomial $ax^2 + bx + c$, $a \neq 0$, the graph of the corresponding equation $y = ax^2 + bx + c$ has one of the two shapes either open upwards like $\cup$ or open downwards like $\cap$ depending on whether $a > 0$ or $a < 0$. (These curves are called **parabolas**.)

You can see from Table 2.1 that -1 and 4 are zeroes of the quadratic polynomial. Also note from Fig. 2.2 that -1 and 4 are the x-coordinates of the points where the graph of $y = x^2 - 3x - 4$ intersects the x-axis. Thus, the zeroes of the quadratic polynomial $x^2 - 3x - 4$ are x-coordinates of the points where the graph of $y = x^2 - 3x - 4$ intersects the x-axis.

This fact is true for any quadratic polynomial, i.e., the zeroes of a quadratic polynomial $ax^2 + bx + c$, $a \neq 0$, are precisely the x-coordinates of the points where the parabola representing $y = ax^2 + bx + c$ intersects the x-axis.

From our observation earlier about the shape of the graph of $y = ax^2 + bx + c$, the following three cases can happen:

Case (i) : Here, the graph cuts x-axis at two distinct points A and A′.

The x-coordinates of A and A′ are the **two zeroes** of the quadratic polynomial $ax^2 + bx + c$ in this case (see Fig. 2.3).

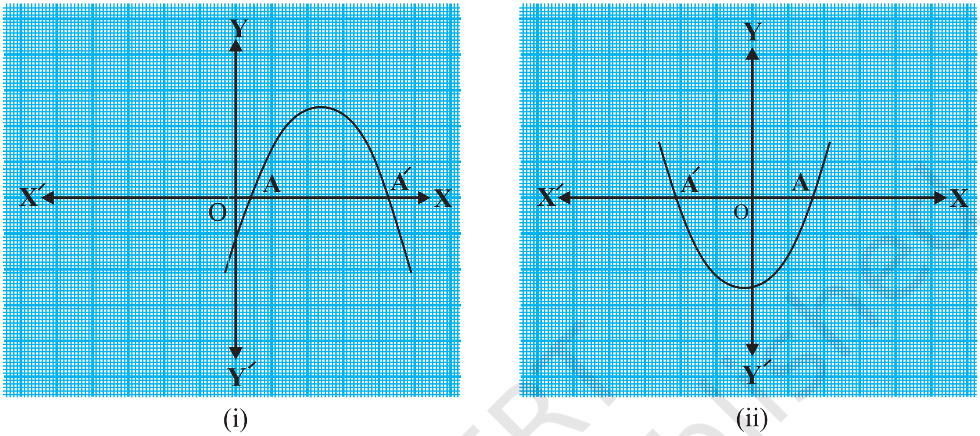


Fig. 2.3

Case (ii) : Here, the graph cuts the x-axis at exactly one point, i.e., at two coincident points. So, the two points A and A′ of Case (i) coincide here to become one point A (see Fig. 2.4).

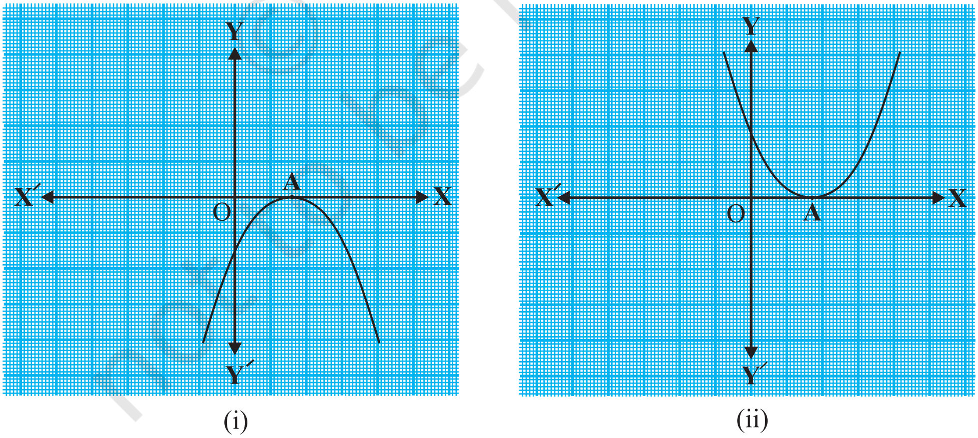


Fig. 2.4

The x-coordinate of A is the **only zero** for the quadratic polynomial $ax^2 + bx + c$ in this case.

Case (iii) : Here, the graph is either completely above the x-axis or completely below the x-axis. So, it does not cut the x-axis at any point (see Fig. 2.5).

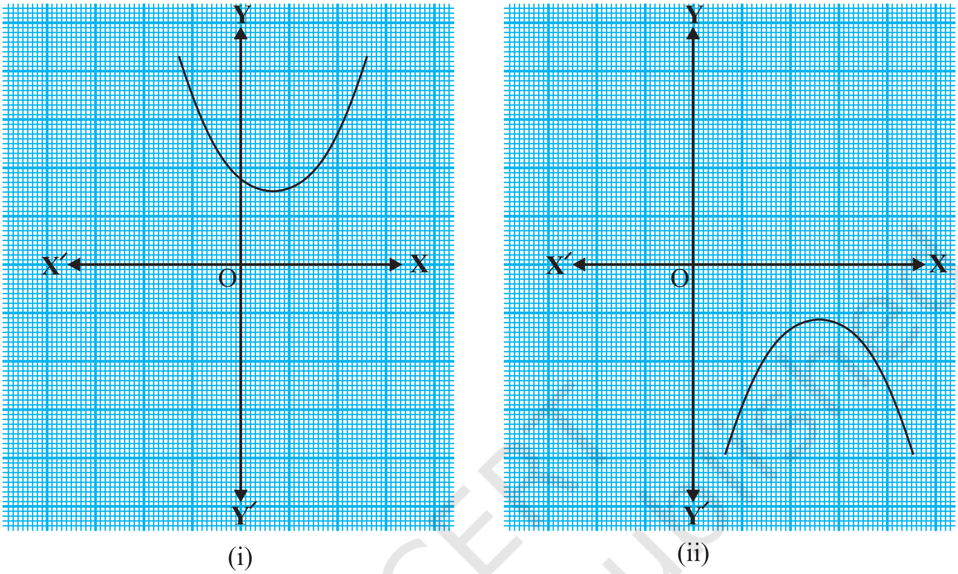


Fig. 2.5

So, the quadratic polynomial $ax^2 + bx + c$ has **no zero** in this case.

So, you can see geometrically that a quadratic polynomial can have either two distinct zeroes or two equal zeroes (i.e., one zero), or no zero. This also means that a polynomial of degree 2 has atmost two zeroes.

Now, what do you expect the geometrical meaning of the zeroes of a cubic polynomial to be? Let us find out. Consider the cubic polynomial $x^3 - 4x$. To see what the graph of $y = x^3 - 4x$ looks like, let us list a few values of y corresponding to a few values for x as shown in Table 2.2.

Table 2.2

x	-2	-1	0	1	2
$y = x^3 - 4x$	0	3	0	-3	0

Locating the points of the table on a graph paper and drawing the graph, we see that the graph of $y = x^3 - 4x$ actually looks like the one given in Fig. 2.6.

We see from the table above that -2, 0 and 2 are zeroes of the cubic polynomial $x^3 - 4x$. Observe that -2, 0 and 2 are, in fact, the x-coordinates of the only points where the graph of $y = x^3 - 4x$ intersects the x-axis. Since the curve meets the x-axis in only these 3 points, their x-coordinates are the only zeroes of the polynomial.

Let us take a few more examples. Consider the cubic polynomials x^3 and $x^3 - x^2$. We draw the graphs of $y = x^3$ and $y = x^3 - x^2$ in Fig. 2.7 and Fig. 2.8 respectively.

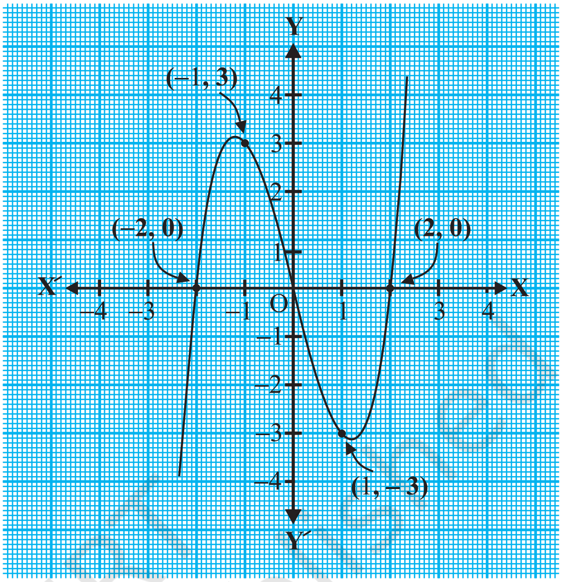


Fig. 2.6

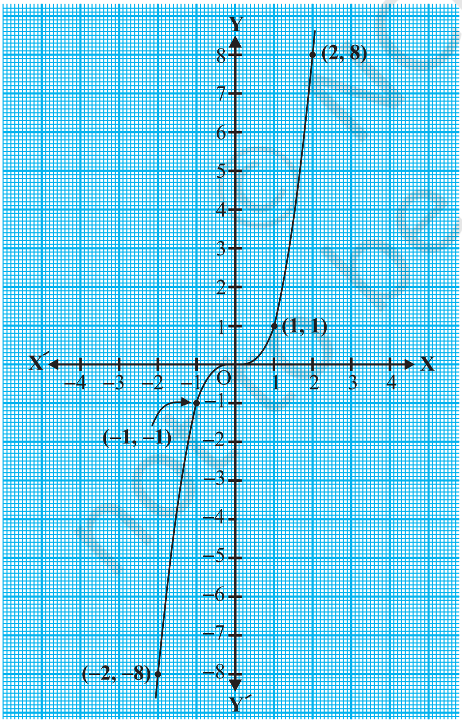


Fig. 2.7

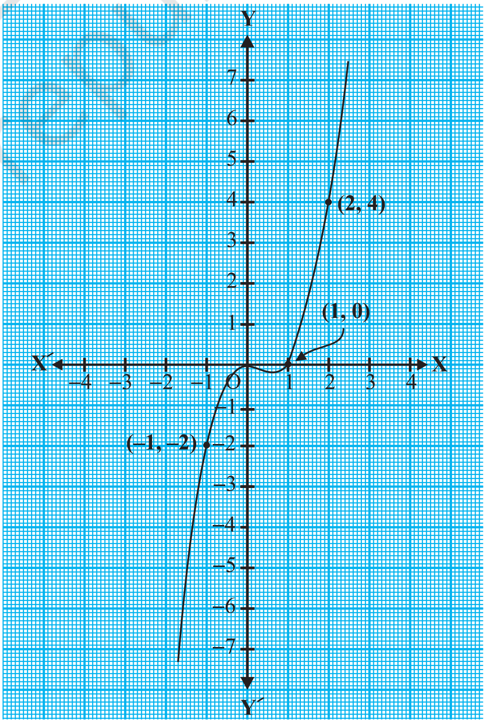


Fig. 2.8

Note that 0 is the only zero of the polynomial x^3. Also, from Fig. 2.7, you can see that 0 is the x-coordinate of the only point where the graph of $y = x^3$ intersects the x-axis. Similarly, since $x^3 - x^2 = x^2(x-1)$, 0 and 1 are the only zeroes of the polynomial $x^3 - x^2$. Also, from Fig. 2.8, these values are the x-coordinates of the only points where the graph of $y = x^3 - x^2$ intersects the x-axis.

From the examples above, we see that there are at most 3 zeroes for any cubic polynomial. In other words, any polynomial of degree 3 can have at most three zeroes.

Remark : In general, given a polynomial $p(x)$ of degree n, the graph of $y = p(x)$ intersects the x-axis at atmost n points. Therefore, a polynomial $p(x)$ of degree n has at most n zeroes.

Example 1 : Look at the graphs in Fig. 2.9 given below. Each is the graph of $y = p(x)$, where $p(x)$ is a polynomial. For each of the graphs, find the number of zeroes of $p(x)$.

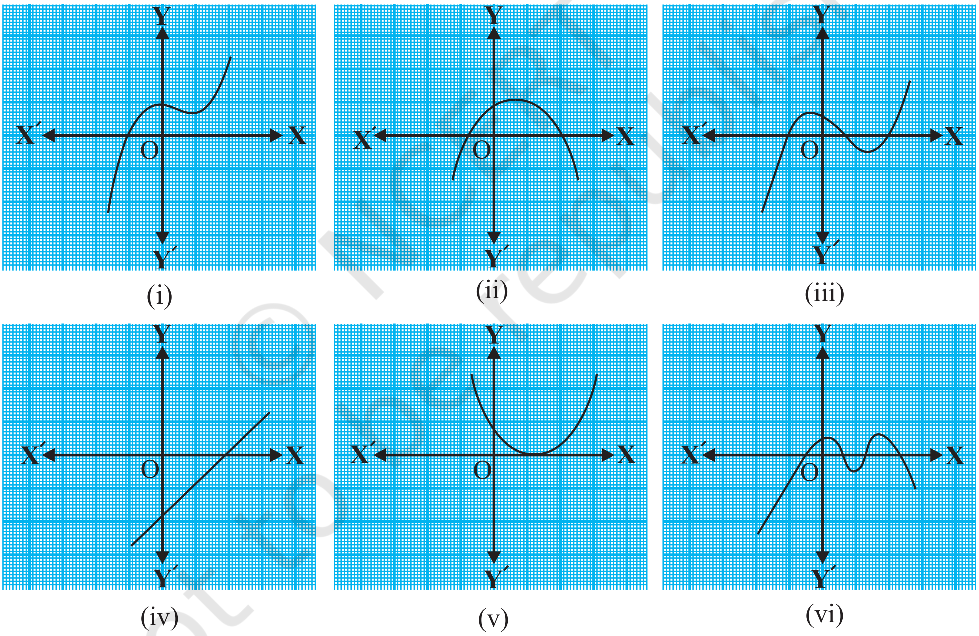


Fig. 2.9

Solution :

(i) The number of zeroes is 1 as the graph intersects the x-axis at one point only.

(ii) The number of zeroes is 2 as the graph intersects the x-axis at two points.

(iii) The number of zeroes is 3. (Why?)

(iv) The number of zeroes is 1. (Why?)

(v) The number of zeroes is 1. (Why?)

(vi) The number of zeroes is 4. (Why?)

EXERCISE 2.1

1. The graphs of $y = p(x)$ are given in Fig. 2.10 below, for some polynomials $p(x)$. Find the number of zeroes of $p(x)$, in each case.

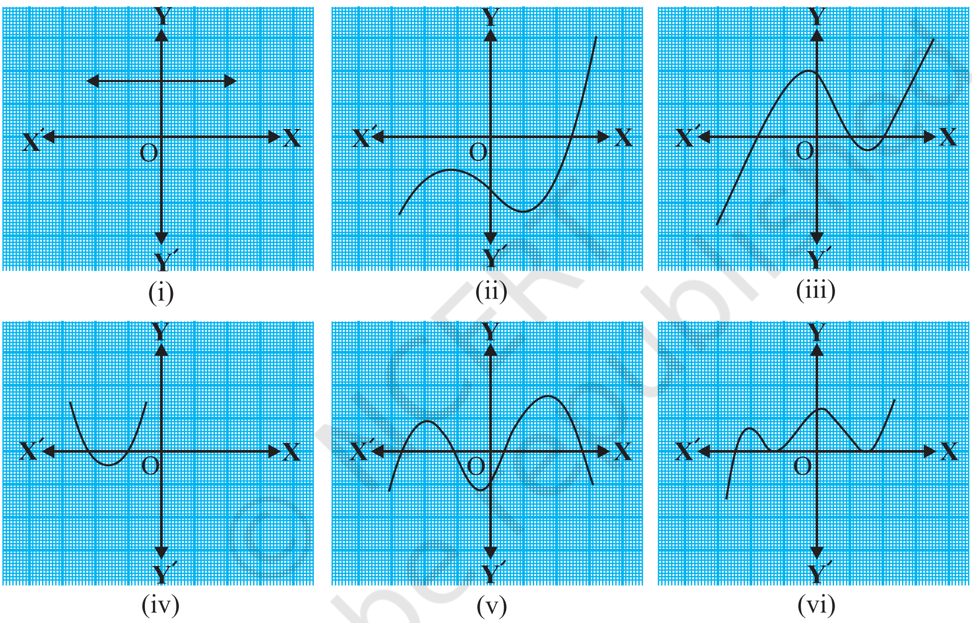


Fig. 2.10

2.3 Relationship between Zeroes and Coefficients of a Polynomial

You have already seen that zero of a linear polynomial $ax + b$ is $-\dfrac{b}{a}$. We will now try to answer the question raised in Section 2.1 regarding the relationship between zeroes and coefficients of a quadratic polynomial. For this, let us take a quadratic polynomial, say $p(x) = 2x^2 - 8x + 6$. In Class IX, you have learnt how to factorise quadratic polynomials by splitting the middle term. So, here we need to split the middle term '$- 8x$' as a sum of two terms, whose product is $6 \times 2x^2 = 12x^2$. So, we write

$$2x^2 - 8x + 6 = 2x^2 - 6x - 2x + 6 = 2x(x - 3) - 2(x - 3)$$

$$= (2x - 2)(x - 3) = 2(x - 1)(x - 3)$$

So, the value of $p(x) = 2x^2 - 8x + 6$ is zero when $x - 1 = 0$ or $x - 3 = 0$, i.e., when $x = 1$ or $x = 3$. So, the zeroes of $2x^2 - 8x + 6$ are 1 and 3. Observe that :

$$\text{Sum of its zeroes} = 1 + 3 = 4 = \frac{-(-8)}{2} = \frac{-(\text{Coefficient of } x)}{\text{Coefficient of } x^2}$$

$$\text{Product of its zeroes} = 1 \times 3 = 3 = \frac{6}{2} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

Let us take one more quadratic polynomial, say, $p(x) = 3x^2 + 5x - 2$. By the method of splitting the middle term,

$$3x^2 + 5x - 2 = 3x^2 + 6x - x - 2 = 3x(x + 2) - 1(x + 2)$$
$$= (3x - 1)(x + 2)$$

Hence, the value of $3x^2 + 5x - 2$ is zero when either $3x - 1 = 0$ or $x + 2 = 0$, i.e., when $x = \frac{1}{3}$ or $x = -2$. So, the zeroes of $3x^2 + 5x - 2$ are $\frac{1}{3}$ and -2. Observe that :

$$\text{Sum of its zeroes} = \frac{1}{3} + (-2) = \frac{-5}{3} = \frac{-(\text{Coefficient of } x)}{\text{Coefficient of } x^2}$$

$$\text{Product of its zeroes} = \frac{1}{3} \times (-2) = \frac{-2}{3} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

In general, if α* and β* are the zeroes of the quadratic polynomial $p(x) = ax^2 + bx + c$, $a \neq 0$, then you know that $x - \alpha$ and $x - \beta$ are the factors of $p(x)$. Therefore,

$$ax^2 + bx + c = k(x - \alpha)(x - \beta), \text{ where } k \text{ is a constant}$$
$$= k[x^2 - (\alpha + \beta)x + \alpha\beta]$$
$$= kx^2 - k(\alpha + \beta)x + k\alpha\beta$$

Comparing the coefficients of x^2, x and constant terms on both the sides, we get

$$a = k,\ b = -k(\alpha + \beta) \text{ and } c = k\alpha\beta.$$

This gives
$$\boldsymbol{\alpha + \beta = \frac{-b}{a}},$$

$$\boldsymbol{\alpha\beta = \frac{c}{a}}$$

* α, β are Greek letters pronounced as 'alpha' and 'beta' respectively. We will use later one more letter 'γ' pronounced as 'gamma'.

i.e., sum of zeroes $= \alpha + \beta = -\dfrac{b}{a} = \dfrac{-(\text{Coefficient of } x)}{\text{Coefficient of } x^2}$,

$$\text{product of zeroes} = \alpha\beta = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}.$$

Let us consider some examples.

Example 2 : Find the zeroes of the quadratic polynomial $x^2 + 7x + 10$, and verify the relationship between the zeroes and the coefficients.

Solution : We have

$$x^2 + 7x + 10 = (x + 2)(x + 5)$$

So, the value of $x^2 + 7x + 10$ is zero when $x + 2 = 0$ or $x + 5 = 0$, i.e., when $x = -2$ or $x = -5$. Therefore, the zeroes of $x^2 + 7x + 10$ are -2 and -5. Now,

$$\text{sum of zeroes} = -2 + (-5) = -(7) = \frac{-(7)}{1} = \frac{-(\text{Coefficient of } x)}{\text{Coefficient of } x^2},$$

$$\text{product of zeroes} = (-2) \times (-5) = 10 = \frac{10}{1} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}.$$

Example 3 : Find the zeroes of the polynomial $x^2 - 3$ and verify the relationship between the zeroes and the coefficients.

Solution : Recall the identity $a^2 - b^2 = (a - b)(a + b)$. Using it, we can write:

$$x^2 - 3 = \left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right)$$

So, the value of $x^2 - 3$ is zero when $x = \sqrt{3}$ or $x = -\sqrt{3}$.

Therefore, the zeroes of $x^2 - 3$ are $\sqrt{3}$ and $-\sqrt{3}$.

Now,

$$\text{sum of zeroes} = \sqrt{3} - \sqrt{3} = 0 = \frac{-(\text{Coefficient of } x)}{\text{Coefficient of } x^2},$$

$$\text{product of zeroes} = \left(\sqrt{3}\right)\left(-\sqrt{3}\right) = -3 = \frac{-3}{1} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}.$$

Example 4 : Find a quadratic polynomial, the sum and product of whose zeroes are – 3 and 2, respectively.

Solution : Let the quadratic polynomial be $ax^2 + bx + c$, and its zeroes be α and β. We have

$$\alpha + \beta = -3 = \frac{-b}{a},$$

and $$\alpha\beta = 2 = \frac{c}{a}.$$

If $a = 1$, then $b = 3$ and $c = 2$.

So, one quadratic polynomial which fits the given conditions is $x^2 + 3x + 2$.

You can check that any other quadratic polynomial that fits these conditions will be of the form $k(x^2 + 3x + 2)$, where k is real.

Let us now look at cubic polynomials. Do you think a similar relation holds between the zeroes of a cubic polynomial and its coefficients?

Let us consider $p(x) = 2x^3 - 5x^2 - 14x + 8$.

You can check that $p(x) = 0$ for $x = 4, -2, \frac{1}{2}\cdot$ Since $p(x)$ can have atmost three zeroes, these are the zeores of $2x^3 - 5x^2 - 14x + 8$. Now,

$$\text{sum of the zeroes} = 4 + (-2) + \frac{1}{2} = \frac{5}{2} = \frac{-(-5)}{2} = \frac{-(\text{Coefficient of } x^2)}{\text{Coefficient of } x^3},$$

$$\text{product of the zeroes} = 4 \times (-2) \times \frac{1}{2} = -4 = \frac{-8}{2} = \frac{-\text{Constant term}}{\text{Coefficient of } x^3}.$$

However, there is one more relationship here. Consider the sum of the products of the zeroes taken two at a time. We have

$$\{4 \times (-2)\} + \left\{(-2) \times \frac{1}{2}\right\} + \left\{\frac{1}{2} \times 4\right\}$$

$$= -8 - 1 + 2 = -7 = \frac{-14}{2} = \frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}.$$

In general, it can be proved that if α, β, γ are the zeroes of the cubic polynomial $ax^3 + bx^2 + cx + d$, then

$$\alpha + \beta + \gamma = \frac{-b}{a},$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a},$$

$$\alpha\,\beta\,\gamma = \frac{-d}{a}.$$

Let us consider an example.

Example 5* : Verify that 3, –1, $-\frac{1}{3}$ are the zeroes of the cubic polynomial $p(x) = 3x^3 - 5x^2 - 11x - 3$, and then verify the relationship between the zeroes and the coefficients.

Solution : Comparing the given polynomial with $ax^3 + bx^2 + cx + d$, we get

$a = 3$, $b = -5$, $c = -11$, $d = -3$. Further

$p(3) = 3 \times 3^3 - (5 \times 3^2) - (11 \times 3) - 3 = 81 - 45 - 33 - 3 = 0,$

$p(-1) = 3 \times (-1)^3 - 5 \times (-1)^2 - 11 \times (-1) - 3 = -3 - 5 + 11 - 3 = 0,$

$$p\left(-\frac{1}{3}\right) = 3 \times \left(-\frac{1}{3}\right)^3 - 5 \times \left(-\frac{1}{3}\right)^2 - 11 \times \left(-\frac{1}{3}\right) - 3,$$

$$= -\frac{1}{9} - \frac{5}{9} + \frac{11}{3} - 3 = -\frac{2}{3} + \frac{2}{3} = 0$$

Therefore, 3, –1 and $-\frac{1}{3}$ are the zeroes of $3x^3 - 5x^2 - 11x - 3$.

So, we take $\alpha = 3$, $\beta = -1$ and $\gamma = -\frac{1}{3}$.

Now,

$$\alpha + \beta + \gamma = 3 + (-1) + \left(-\frac{1}{3}\right) = 2 - \frac{1}{3} = \frac{5}{3} = \frac{-(-5)}{3} = \frac{-b}{a},$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = 3 \times (-1) + (-1) \times \left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right) \times 3 = -3 + \frac{1}{3} - 1 = \frac{-11}{3} = \frac{c}{a},$$

$$\alpha\beta\gamma = 3 \times (-1) \times \left(-\frac{1}{3}\right) = 1 = \frac{-(-3)}{3} = \frac{-d}{a}.$$

* Not from the examination point of view.

EXERCISE 2.2

1. Find the zeroes of the following quadratic polynomials and verify the relationship between the zeroes and the coefficients.

(i) $x^2 - 2x - 8$ (ii) $4s^2 - 4s + 1$ (iii) $6x^2 - 3 - 7x$

(iv) $4u^2 + 8u$ (v) $t^2 - 15$ (vi) $3x^2 - x - 4$

2. Find a quadratic polynomial each with the given numbers as the sum and product of its zeroes respectively.

(i) $\frac{1}{4}, -1$ (ii) $\sqrt{2}, \frac{1}{3}$ (iii) $0, \sqrt{5}$

(iv) $1, 1$ (v) $-\frac{1}{4}, \frac{1}{4}$ (vi) $4, 1$

2.4 Summary

In this chapter, you have studied the following points:

1. Polynomials of degrees 1, 2 and 3 are called linear, quadratic and cubic polynomials respectively.
2. A quadratic polynomial in x with real coefficients is of the form $ax^2 + bx + c$, where a, b, c are real numbers with $a \neq 0$.
3. The zeroes of a polynomial $p(x)$ are precisely the x-coordinates of the points, where the graph of $y = p(x)$ intersects the x-axis.
4. A quadratic polynomial can have at most 2 zeroes and a cubic polynomial can have at most 3 zeroes.
5. If α and β are the zeroes of the quadratic polynomial $ax^2 + bx + c$, then

$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}.$$

6. If α, β, γ are the zeroes of the cubic polynomial $ax^3 + bx^2 + cx + d$, then

$$\alpha + \beta + \gamma = \frac{-b}{a},$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a},$$

and $$\alpha\beta\gamma = \frac{-d}{a}.$$

1062CH03

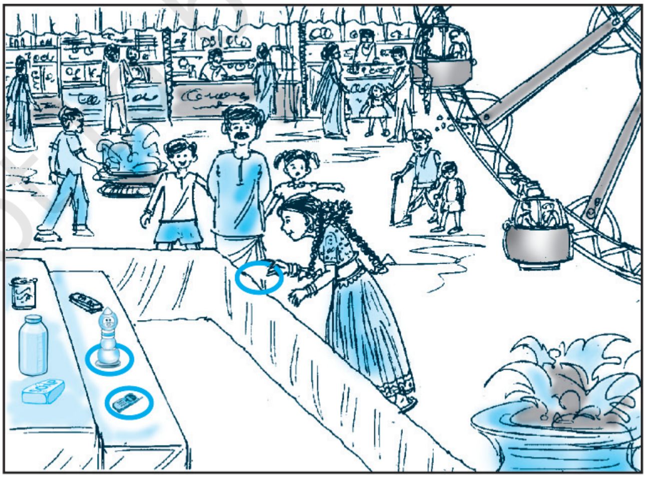
PAIR OF LINEAR EQUATIONS IN TWO VARIABLES 3

3.1 Introduction

You must have come across situations like the one given below :

Akhila went to a fair in her village. She wanted to enjoy rides on the Giant Wheel and play Hoopla (a game in which you throw a ring on the items kept in a stall, and if the ring covers any object completely, you get it). The number of times she played Hoopla is half the number of rides she had on the Giant Wheel. If each ride costs ₹ 3, and a game of Hoopla costs ₹ 4, how would you find out the number of rides she had and how many times she played Hoopla, provided she spent ₹ 20.

May be you will try it by considering different cases. If she has one ride, is it possible? Is it possible to have two rides? And so on. Or you may use the knowledge of Class IX, to represent such situations as linear equations in two variables.



Let us try this approach.

Denote the number of rides that Akhila had by x, and the number of times she played Hoopla by y. Now the situation can be represented by the two equations:

$$y = \frac{1}{2}x \tag{1}$$

$$3x + 4y = 20 \tag{2}$$

Can we find the solutions of this pair of equations? There are several ways of finding these, which we will study in this chapter.

3.2 Graphical Method of Solution of a Pair of Linear Equations

A pair of linear equations which has no solution, is called an *inconsistent* pair of linear equations. A pair of linear equations in two variables, which has a solution, is called a *consistent* pair of linear equations. A pair of linear equations which are equivalent has infinitely many distinct common solutions. Such a pair is called a *dependent* pair of linear equations in two variables. Note that a dependent pair of linear equations is always consistent.

We can now summarise the behaviour of lines representing a pair of linear equations in two variables and the existence of solutions as follows:

(i) the lines may intersect in a single point. In this case, the pair of equations has a unique solution (consistent pair of equations).

(ii) the lines may be parallel. In this case, the equations have no solution (inconsistent pair of equations).

(iii) the lines may be coincident. In this case, the equations have infinitely many solutions [dependent (consistent) pair of equations].

Consider the following three pairs of equations.

(i) $x - 2y = 0$ and $3x + 4y - 20 = 0$ (The lines intersect)

(ii) $2x + 3y - 9 = 0$ and $4x + 6y - 18 = 0$ (The lines coincide)

(iii) $x + 2y - 4 = 0$ and $2x + 4y - 12 = 0$ (The lines are parallel)

Let us now write down, and compare, the values of $\frac{a_1}{a_2}$, $\frac{b_1}{b_2}$ and $\frac{c_1}{c_2}$ in all the three examples. Here, a_1, b_1, c_1 and a_2, b_2, c_2 denote the coefficents of equations given in the general form in Section 3.2.

Table 3.1

Sl No.	Pair of lines	$\frac{a_1}{a_2}$	$\frac{b_1}{b_2}$	$\frac{c_1}{c_2}$	Compare the ratios	Graphical representation	Algebraic interpretation
1.	$x - 2y = 0$ $3x + 4y - 20 = 0$	$\frac{1}{3}$	$\frac{-2}{4}$	$\frac{0}{-20}$	$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$	Intersecting lines	Exactly one solution (unique)
2.	$2x + 3y - 9 = 0$ $4x + 6y - 18 = 0$	$\frac{2}{4}$	$\frac{3}{6}$	$\frac{-9}{-18}$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$	Coincident lines	Infinitely many solutions
3.	$x + 2y - 4 = 0$ $2x + 4y - 12 = 0$	$\frac{1}{2}$	$\frac{2}{4}$	$\frac{-4}{-12}$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$	Parallel lines	No solution

From the table above, you can observe that if the lines represented by the equation

$$a_1x + b_1y + c_1 = 0$$

and $$a_2x + b_2y + c_2 = 0$$

are (i) intersecting, then $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$.

(ii) coincident, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

(iii) parallel, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$.

In fact, the converse is also true for any pair of lines. You can verify them by considering some more examples by yourself.

Let us now consider some more examples to illustrate it.

Example 1 : Check graphically whether the pair of equations

$$x + 3y = 6 \qquad (1)$$

and $$2x - 3y = 12 \qquad (2)$$

is consistent. If so, solve them graphically.

Solution : Let us draw the graphs of the Equations (1) and (2). For this, we find two solutions of each of the equations, which are given in Table 3.2

Table 3.2

x	0	6
$y = \dfrac{6-x}{3}$	2	0

x	0	3
$y = \dfrac{2x-12}{3}$	-4	-2

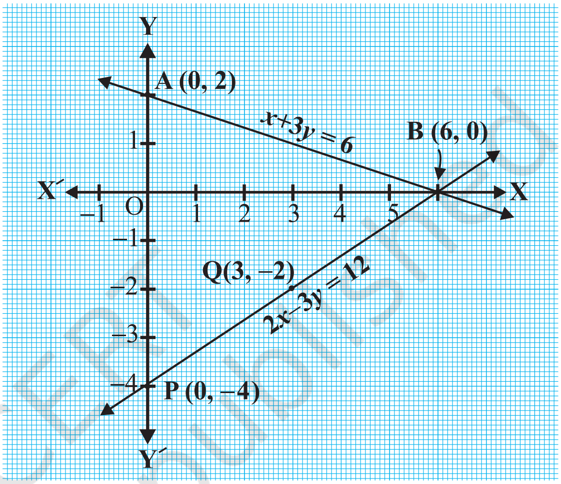


Fig. 3.1

Plot the points A(0, 2), B(6, 0), P(0, – 4) and Q(3, – 2) on graph paper, and join the points to form the lines AB and PQ as shown in Fig. 3.1.

We observe that there is a point B (6, 0) common to both the lines AB and PQ. So, the solution of the pair of linear equations is $x = 6$ and $y = 0$, i.e., the given pair of equations is consistent.

Example 2 : Graphically, find whether the following pair of equations has no solution, unique solution or infinitely many solutions:

$$5x - 8y + 1 = 0 \qquad (1)$$

$$3x - \frac{24}{5}y + \frac{3}{5} = 0 \qquad (2)$$

Solution : Multiplying Equation (2) by $\frac{5}{3}$, we get

$$5x - 8y + 1 = 0$$

But, this is the same as Equation (1). Hence the lines represented by Equations (1) and (2) are coincident. Therefore, Equations (1) and (2) have infinitely many solutions.

Plot few points on the graph and verify it yourself.

Example 3 : Champa went to a 'Sale' to purchase some pants and skirts. When her friends asked her how many of each she had bought, she answered, "The number of skirts is two less than twice the number of pants purchased. Also, the number of skirts is four less than four times the number of pants purchased". Help her friends to find how many pants and skirts Champa bought.

Solution : Let us denote the number of pants by x and the number of skirts by y. Then the equations formed are :

$$y = 2x - 2 \quad (1)$$

and

$$y = 4x - 4 \quad (2)$$

Let us draw the graphs of Equations (1) and (2) by finding two solutions for each of the equations. They are given in Table 3.3.

Table 3.3

x	2	0
$y = 2x - 2$	2	– 2

x	0	1
$y = 4x - 4$	– 4	0

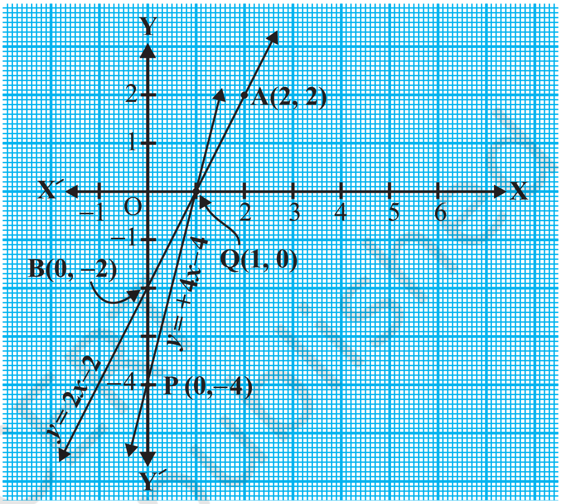


Fig. 3.2

Plot the points and draw the lines passing through them to represent the equations, as shown in Fig. 3.2.

The two lines intersect at the point (1, 0). So, $x = 1$, $y = 0$ is the required solution of the pair of linear equations, i.e., the number of pants she purchased is 1 and she did not buy any skirt.

Verify the answer by checking whether it satisfies the conditions of the given problem.

EXERCISE 3.1

1. Form the pair of linear equations in the following problems, and find their solutions graphically.

(i) 10 students of Class X took part in a Mathematics quiz. If the number of girls is 4 more than the number of boys, find the number of boys and girls who took part in the quiz.

(ii) 5 pencils and 7 pens together cost ₹ 50, whereas 7 pencils and 5 pens together cost ₹ 46. Find the cost of one pencil and that of one pen.

2. On comparing the ratios $\frac{a_1}{a_2}$, $\frac{b_1}{b_2}$ and $\frac{c_1}{c_2}$, find out whether the lines representing the following pairs of linear equations intersect at a point, are parallel or coincident:

(i) $5x - 4y + 8 = 0$
$7x + 6y - 9 = 0$

(ii) $9x + 3y + 12 = 0$
$18x + 6y + 24 = 0$

(iii) $6x - 3y + 10 = 0$
$2x - y + 9 = 0$

3. On comparing the ratios $\frac{a_1}{a_2}$, $\frac{b_1}{b_2}$ and $\frac{c_1}{c_2}$, find out whether the following pair of linear equations are consistent, or inconsistent.

(i) $3x + 2y = 5$; $2x - 3y = 7$

(ii) $2x - 3y = 8$; $4x - 6y = 9$

(iii) $\frac{3}{2}x + \frac{5}{3}y = 7$; $9x - 10y = 14$

(iv) $5x - 3y = 11$; $-10x + 6y = -22$

(v) $\frac{4}{3}x + 2y = 8$; $2x + 3y = 12$

4. Which of the following pairs of linear equations are consistent/inconsistent? If consistent, obtain the solution graphically:

(i) $x + y = 5$, $2x + 2y = 10$

(ii) $x - y = 8$, $3x - 3y = 16$

(iii) $2x + y - 6 = 0$, $4x - 2y - 4 = 0$

(iv) $2x - 2y - 2 = 0$, $4x - 4y - 5 = 0$

5. Half the perimeter of a rectangular garden, whose length is 4 m more than its width, is 36 m. Find the dimensions of the garden.

6. Given the linear equation $2x + 3y - 8 = 0$, write another linear equation in two variables such that the geometrical representation of the pair so formed is:

(i) intersecting lines

(ii) parallel lines

(iii) coincident lines

7. Draw the graphs of the equations $x - y + 1 = 0$ and $3x + 2y - 12 = 0$. Determine the coordinates of the vertices of the triangle formed by these lines and the x-axis, and shade the triangular region.

3.3 Algebraic Methods of Solving a Pair of Linear Equations

In the previous section, we discussed how to solve a pair of linear equations graphically. The graphical method is not convenient in cases when the point representing the solution of the linear equations has non-integral coordinates like $\left(\sqrt{3}, 2\sqrt{7}\right)$, $(-1.75, 3.3)$, $\left(\frac{4}{13}, \frac{1}{19}\right)$, etc. There is every possibility of making mistakes while reading such coordinates. Is there any alternative method of finding the solution? There are several algebraic methods, which we shall now discuss.

3.3.1 Substitution Method : We shall explain the method of substitution by taking some examples.

Example 4 : Solve the following pair of equations by substitution method:

$$7x - 15y = 2 \tag{1}$$

$$x + 2y = 3 \tag{2}$$

Solution :

Step 1 : We pick either of the equations and write one variable in terms of the other. Let us consider the Equation (2) :

$$x + 2y = 3$$

and write it as $$x = 3 - 2y \tag{3}$$

Step 2 : Substitute the value of x in Equation (1). We get

$$7(3 - 2y) - 15y = 2$$

i.e., $$21 - 14y - 15y = 2$$

i.e., $$-29y = -19$$

Therefore, $$y = \frac{19}{29}$$

Step 3 : Substituting this value of y in Equation (3), we get

$$x = 3 - 2\left(\frac{19}{29}\right) = \frac{49}{29}$$

Therefore, the solution is $x = \frac{49}{29}$, $y = \frac{19}{29}$.

Verification : Substituting $x = \frac{49}{29}$ and $y = \frac{19}{29}$, you can verify that both the Equations (1) and (2) are satisfied.

To understand the substitution method more clearly, let us consider it stepwise:

Step 1 : Find the value of one variable, say y in terms of the other variable, i.e., x from either equation, whichever is convenient.

Step 2 : Substitute this value of y in the other equation, and reduce it to an equation in one variable, i.e., in terms of x, which can be solved. Sometimes, as in Examples 9 and 10 below, you can get statements with no variable. If this statement is true, you can conclude that the pair of linear equations has infinitely many solutions. If the statement is false, then the pair of linear equations is inconsistent.

Step 3 : Substitute the value of x (or y) obtained in Step 2 in the equation used in Step 1 to obtain the value of the other variable.

Remark : We have **substituted** the value of one variable by expressing it in terms of the other variable to solve the pair of linear equations. That is why the method is known as the *substitution method*.

Example 5 : Solve the following question—Aftab tells his daughter, "Seven years ago, I was seven times as old as you were then. Also, three years from now, I shall be three times as old as you will be." (Isn't this interesting?) Represent this situation algebraically and graphically by the method of substitution.

Solution : Let s and t be the ages (in years) of Aftab and his daughter, respectively. Then, the pair of linear equations that represent the situation is

$$s - 7 = 7\,(t - 7), \text{ i.e., } s - 7t + 42 = 0 \qquad (1)$$

and

$$s + 3 = 3\,(t + 3), \text{ i.e., } s - 3t = 6 \qquad (2)$$

Using Equation (2), we get $s = 3t + 6$.

Putting this value of s in Equation (1), we get

$$(3t + 6) - 7t + 42 = 0,$$

i.e.,

$$4t = 48, \text{ which gives } t = 12.$$

Putting this value of t in Equation (2), we get

$$s = 3\,(12) + 6 = 42$$

So, Aftab and his daughter are 42 and 12 years old, respectively.

Verify this answer by checking if it satisfies the conditions of the given problems.

Example 6 : In a shop the cost of 2 pencils and 3 erasers is ₹9 and the cost of 4 pencils and 6 erasers is ₹18. Find the cost of each pencil and each eraser.

Solution : The pair of linear equations formed were:

$$2x + 3y = 9 \quad (1)$$

$$4x + 6y = 18 \quad (2)$$

We first express the value of x in terms of y from the equation $2x + 3y = 9$, to get

$$x = \frac{9 - 3y}{2} \quad (3)$$

Now we substitute this value of x in Equation (2), to get

$$\frac{4(9 - 3y)}{2} + 6y = 18$$

i.e., $$18 - 6y + 6y = 18$$

i.e., $$18 = 18$$

This statement is true for all values of y. However, we do not get a specific value of y as a solution. Therefore, we cannot obtain a specific value of x. This situation has arisen because both the given equations are the same. Therefore, Equations (1) and (2) have *infinitely many solutions*. We cannot find a unique cost of a pencil and an eraser, because there are many common solutions, to the given situation.

Example 7 : Two rails are represented by the equations $x + 2y - 4 = 0$ and $2x + 4y - 12 = 0$. Will the rails cross each other?

Solution : The pair of linear equations formed were:

$$x + 2y - 4 = 0 \quad (1)$$

$$2x + 4y - 12 = 0 \quad (2)$$

We express x in terms of y from Equation (1) to get

$$x = 4 - 2y$$

Now, we substitute this value of x in Equation (2) to get

$$2(4 - 2y) + 4y - 12 = 0$$

i.e., $8 - 12 = 0$

i.e., $-4 = 0$

which is a false statement.

Therefore, the equations do not have a common solution. So, the two rails will not cross each other.

EXERCISE 3.2

1. Solve the following pair of linear equations by the substitution method.

(i) $x + y = 14$
$x - y = 4$

(ii) $s - t = 3$
$\frac{s}{3} + \frac{t}{2} = 6$

(iii) $3x - y = 3$
$9x - 3y = 9$

(iv) $0.2x + 0.3y = 1.3$
$0.4x + 0.5y = 2.3$

(v) $\sqrt{2}\,x + \sqrt{3}\,y = 0$
$\sqrt{3}\,x - \sqrt{8}\,y = 0$

(vi) $\frac{3x}{2} - \frac{5y}{3} = -2$
$\frac{x}{3} + \frac{y}{2} = \frac{13}{6}$

2. Solve $2x + 3y = 11$ and $2x - 4y = -24$ and hence find the value of 'm' for which $y = mx + 3$.

3. Form the pair of linear equations for the following problems and find their solution by substitution method.

(i) The difference between two numbers is 26 and one number is three times the other. Find them.

(ii) The larger of two supplementary angles exceeds the smaller by 18 degrees. Find them.

(iii) The coach of a cricket team buys 7 bats and 6 balls for ₹ 3800. Later, she buys 3 bats and 5 balls for ₹ 1750. Find the cost of each bat and each ball.

(iv) The taxi charges in a city consist of a fixed charge together with the charge for the distance covered. For a distance of 10 km, the charge paid is ₹ 105 and for a journey of 15 km, the charge paid is ₹ 155. What are the fixed charges and the charge per km? How much does a person have to pay for travelling a distance of 25 km?

(v) A fraction becomes $\frac{9}{11}$, if 2 is added to both the numerator and the denominator. If, 3 is added to both the numerator and the denominator it becomes $\frac{5}{6}$. Find the fraction.

(vi) Five years hence, the age of Jacob will be three times that of his son. Five years ago, Jacob's age was seven times that of his son. What are their present ages?

3.3.2 Elimination Method

Now let us consider another method of eliminating (i.e., removing) one variable. This is sometimes more convenient than the substitution method. Let us see how this method works.

Example 8 : The ratio of incomes of two persons is 9 : 7 and the ratio of their expenditures is 4 : 3. If each of them manages to save ₹ 2000 per month, find their monthly incomes.

Solution : Let us denote the incomes of the two person by ₹ $9x$ and ₹ $7x$ and their expenditures by ₹ $4y$ and ₹ $3y$ respectively. Then the equations formed in the situation is given by :

$$9x - 4y = 2000 \quad (1)$$

and

$$7x - 3y = 2000 \quad (2)$$

Step 1 : Multiply Equation (1) by 3 and Equation (2) by 4 to make the coefficients of y equal. Then we get the equations:

$$27x - 12y = 6000 \quad (3)$$

$$28x - 12y = 8000 \quad (4)$$

Step 2 : Subtract Equation (3) from Equation (4) to *eliminate* y, because the coefficients of y are the same. So, we get

$$(28x - 27x) - (12y - 12y) = 8000 - 6000$$

i.e.,

$$x = 2000$$

Step 3 : Substituting this value of x in (1), we get

$$9(2000) - 4y = 2000$$

i.e.,

$$y = 4000$$

So, the solution of the equations is $x = 2000$, $y = 4000$. Therefore, the monthly incomes of the persons are ₹ 18,000 and ₹ 14,000, respectively.

Verification : 18000 : 14000 = 9 : 7. Also, the ratio of their expenditures = 18000 – 2000 : 14000 – 2000 = 16000 : 12000 = 4 : 3

Remarks :

1. The method used in solving the example above is called the *elimination* method, because we eliminate one variable first, to get a linear equation in one variable.

In the example above, we eliminated y. We could also have eliminated x. Try doing it that way.

2. You could also have used the substitution, or graphical method, to solve this problem. Try doing so, and see which method is more convenient.

Let us now note down these steps in the elimination method :

Step 1 : First multiply both the equations by some suitable non-zero constants to make the coefficients of one variable (either x or y) numerically equal.

Step 2 : Then add or subtract one equation from the other so that one variable gets eliminated. If you get an equation in one variable, go to Step 3.

If in Step 2, we obtain a true statement involving no variable, then the original pair of equations has infinitely many solutions.

If in Step 2, we obtain a false statement involving no variable, then the original pair of equations has no solution, i.e., it is inconsistent.

Step 3 : Solve the equation in one variable (x or y) so obtained to get its value.

Step 4 : Substitute this value of x (or y) in either of the original equations to get the value of the other variable.

Now to illustrate it, we shall solve few more examples.

Example 9 : Use elimination method to find all possible solutions of the following pair of linear equations :

$$2x + 3y = 8 \tag{1}$$

$$4x + 6y = 7 \tag{2}$$

Solution :

Step 1 : Multiply Equation (1) by 2 and Equation (2) by 1 to make the coefficients of x equal. Then we get the equations as :

$$4x + 6y = 16 \tag{3}$$

$$4x + 6y = 7 \tag{4}$$

Step 2 : Subtracting Equation (4) from Equation (3),

$$(4x - 4x) + (6y - 6y) = 16 - 7$$

i.e., $0 = 9$, which is a false statement.

Therefore, the pair of equations has no solution.

Example 10 : The sum of a two-digit number and the number obtained by reversing the digits is 66. If the digits of the number differ by 2, find the number. How many such numbers are there?

Solution : Let the ten's and the unit's digits in the first number be x and y, respectively. So, the first number may be written as $10x + y$ in the expanded form (for example, $56 = 10(5) + 6$).

When the digits are reversed, x becomes the unit's digit and y becomes the ten's digit. This number, in the expanded notation is $10y + x$ (for example, when 56 is reversed, we get $65 = 10(6) + 5$).

According to the given condition.

$$(10x + y) + (10y + x) = 66$$

i.e., $$11(x + y) = 66$$

i.e., $$x + y = 6 \qquad (1)$$

We are also given that the digits differ by 2, therefore,

either $$x - y = 2 \qquad (2)$$

or $$y - x = 2 \qquad (3)$$

If $x - y = 2$, then solving (1) and (2) by elimination, we get $x = 4$ and $y = 2$. In this case, we get the number 42.

If $y - x = 2$, then solving (1) and (3) by elimination, we get $x = 2$ and $y = 4$. In this case, we get the number 24.

Thus, there are two such numbers 42 and 24.

Verification : Here 42 + 24 = 66 and 4 – 2 = 2. Also 24 + 42 = 66 and 4 – 2 = 2.

EXERCISE 3.3

1. Solve the following pair of linear equations by the elimination method and the substitution method :

(i) $x + y = 5$ and $2x - 3y = 4$ (ii) $3x + 4y = 10$ and $2x - 2y = 2$

(iii) $3x - 5y - 4 = 0$ and $9x = 2y + 7$ (iv) $\frac{x}{2} + \frac{2y}{3} = -1$ and $x - \frac{y}{3} = 3$

2. Form the pair of linear equations in the following problems, and find their solutions (if they exist) by the elimination method :

(i) If we add 1 to the numerator and subtract 1 from the denominator, a fraction reduces to 1. It becomes $\frac{1}{2}$ if we only add 1 to the denominator. What is the fraction?

(ii) Five years ago, Nuri was thrice as old as Sonu. Ten years later, Nuri will be twice as old as Sonu. How old are Nuri and Sonu?

(iii) The sum of the digits of a two-digit number is 9. Also, nine times this number is twice the number obtained by reversing the order of the digits. Find the number.

(iv) Meena went to a bank to withdraw ₹ 2000. She asked the cashier to give her ₹ 50 and ₹ 100 notes only. Meena got 25 notes in all. Find how many notes of ₹ 50 and ₹ 100 she received.

(v) A lending library has a fixed charge for the first three days and an additional charge for each day thereafter. Saritha paid ₹ 27 for a book kept for seven days, while Susy paid ₹ 21 for the book she kept for five days. Find the fixed charge and the charge for each extra day.

3.4 Summary

In this chapter, you have studied the following points:

1. A pair of linear equations in two variables can be represented, and solved, by the:

(i) graphical method

(ii) algebraic method

2. Graphical Method :

The graph of a pair of linear equations in two variables is represented by two lines.

(i) If the lines intersect at a point, then that point gives the unique solution of the two equations. In this case, the pair of equations is **consistent**.

(ii) If the lines coincide, then there are infinitely many solutions — each point on the line being a solution. In this case, the pair of equations is **dependent (consistent)**.

(iii) If the lines are parallel, then the pair of equations has no solution. In this case, the pair of equations is **inconsistent**.

3. Algebraic Methods : We have discussed the following methods for finding the solution(s) of a pair of linear equations :

(i) Substitution Method

(ii) Elimination Method

4. If a pair of linear equations is given by $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then the following situations can arise :

(i) $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_1}$: In this case, the pair of linear equations is consistent.

(ii) $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$: In this case, the pair of linear equations is inconsistent.

(iii) $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$: In this case, the pair of linear equations is dependent and consistent.

5. There are several situations which can be mathematically represented by two equations that are not linear to start with. But we alter them so that they are reduced to a pair of linear equations.

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QUADRATIC EQUATIONS 4

4.1 Introduction

In Chapter 2, you have studied different types of polynomials. One type was the quadratic polynomial of the form $ax^2 + bx + c$, $a \neq 0$. When we equate this polynomial to zero, we get a quadratic equation. Quadratic equations come up when we deal with many real-life situations. For instance, suppose a charity trust decides to build a prayer hall having a carpet area of 300 square metres with its length one metre more than twice its breadth. What should be the length and breadth of the hall? Suppose the breadth of the hall is x metres. Then, its length should be $(2x + 1)$ metres. We can depict this information pictorially as shown in Fig. 4.1.

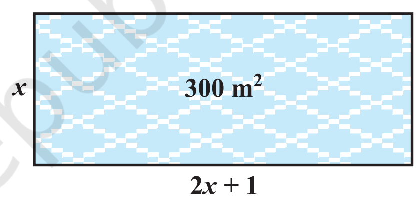

Fig. 4.1

Now, area of the hall $= (2x + 1).\, x$ m^2 $= (2x^2 + x)$ m^2

So, $2x^2 + x = 300$ (Given)

Therefore, $2x^2 + x - 300 = 0$

So, the breadth of the hall should satisfy the equation $2x^2 + x - 300 = 0$ which is a quadratic equation.

Many people believe that Babylonians were the first to solve quadratic equations. For instance, they knew how to find two positive numbers with a given positive sum and a given positive product, and this problem is equivalent to solving a quadratic equation of the form $x^2 - px + q = 0$. Greek mathematician Euclid developed a geometrical approach for finding out lengths which, in our present day terminology, are solutions of quadratic equations. Solving of quadratic equations, in general form, is often credited to ancient Indian mathematicians. In fact, Brahmagupta (C.E.598–665) gave an explicit formula to solve a quadratic equation of the form $ax^2 + bx = c$. Later, Sridharacharya (C.E. 1025) derived a formula, now known as the quadratic formula, (as quoted by Bhaskara II) for solving a quadratic equation by the method of completing the square. An Arab mathematician Al-Khwarizmi (about C.E. 800) also studied quadratic equations of different types. Abraham bar Hiyya Ha-Nasi, in his book 'Liber embadorum' published in Europe in C.E. 1145 gave complete solutions of different quadratic equations.

In this chapter, you will study quadratic equations, and various ways of finding their roots. You will also see some applications of quadratic equations in daily life situations.

4.2 Quadratic Equations

A quadratic equation in the variable x is an equation of the form $ax^2 + bx + c = 0$, where a, b, c are real numbers, $a \neq 0$. For example, $2x^2 + x - 300 = 0$ is a quadratic equation. Similarly, $2x^2 - 3x + 1 = 0$, $4x - 3x^2 + 2 = 0$ and $1 - x^2 + 300 = 0$ are also quadratic equations.

In fact, any equation of the form $p(x) = 0$, where $p(x)$ is a polynomial of degree 2, is a quadratic equation. But when we write the terms of $p(x)$ in descending order of their degrees, then we get the standard form of the equation. That is, $ax^2 + bx + c = 0$, $a \neq 0$ is called the **standard form of a quadratic equation**.

Quadratic equations arise in several situations in the world around us and in different fields of mathematics. Let us consider a few examples.

Example 1 : Represent the following situations mathematically:

(i) John and Jivanti together have 45 marbles. Both of them lost 5 marbles each, and the product of the number of marbles they now have is 124. We would like to find out how many marbles they had to start with.

(ii) A cottage industry produces a certain number of toys in a day. The cost of production of each toy (in rupees) was found to be 55 minus the number of toys produced in a day. On a particular day, the total cost of production was ₹ 750. We would like to find out the number of toys produced on that day.

Solution :

(i) Let the number of marbles John had be x.

Then the number of marbles Jivanti had = $45 - x$ (Why?).

The number of marbles left with John, when he lost 5 marbles = $x - 5$

The number of marbles left with Jivanti, when she lost 5 marbles = $45 - x - 5$

$$= 40 - x$$

Therefore, their product $= (x - 5)(40 - x)$

$= 40x - x^2 - 200 + 5x$

$= - x^2 + 45x - 200$

So, $\quad - x^2 + 45x - 200 = 124 \qquad$ (Given that product = 124)

i.e., $\quad - x^2 + 45x - 324 = 0$

i.e., $\quad x^2 - 45x + 324 = 0$

Therefore, the number of marbles John had, satisfies the quadratic equation

$$x^2 - 45x + 324 = 0$$

which is the required representation of the problem mathematically.

(ii) Let the number of toys produced on that day be x.

Therefore, the cost of production (in rupees) of each toy that day = $55 - x$

So, the total cost of production (in rupees) that day = $x(55 - x)$

Therefore, $\quad x(55 - x) = 750$

i.e., $\quad 55x - x^2 = 750$

i.e., $\quad - x^2 + 55x - 750 = 0$

i.e., $\quad x^2 - 55x + 750 = 0$

Therefore, the number of toys produced that day satisfies the quadratic equation

$$x^2 - 55x + 750 = 0$$

which is the required representation of the problem mathematically.

Example 2 : Check whether the following are quadratic equations:

(i) $(x - 2)^2 + 1 = 2x - 3 \qquad$ (ii) $x(x + 1) + 8 = (x + 2)(x - 2)$

(iii) $x(2x + 3) = x^2 + 1 \qquad$ (iv) $(x + 2)^3 = x^3 - 4$

Solution :

(i) LHS $= (x - 2)^2 + 1 = x^2 - 4x + 4 + 1 = x^2 - 4x + 5$

Therefore, $(x - 2)^2 + 1 = 2x - 3$ can be rewritten as

$$x^2 - 4x + 5 = 2x - 3$$

i.e., $\quad x^2 - 6x + 8 = 0$

It is of the form $ax^2 + bx + c = 0$.

Therefore, the given equation is a quadratic equation.

(ii) Since $x(x + 1) + 8 = x^2 + x + 8$ and $(x + 2)(x - 2) = x^2 - 4$

Therefore, $x^2 + x + 8 = x^2 - 4$

i.e., $x + 12 = 0$

It is not of the form $ax^2 + bx + c = 0$.

Therefore, the given equation is not a quadratic equation.

(iii) Here, LHS $= x(2x + 3) = 2x^2 + 3x$

So, $x(2x + 3) = x^2 + 1$ can be rewritten as

$2x^2 + 3x = x^2 + 1$

Therefore, we get $x^2 + 3x - 1 = 0$

It is of the form $ax^2 + bx + c = 0$.

So, the given equation is a quadratic equation.

(iv) Here, LHS $= (x + 2)^3 = x^3 + 6x^2 + 12x + 8$

Therefore, $(x + 2)^3 = x^3 - 4$ can be rewritten as

$x^3 + 6x^2 + 12x + 8 = x^3 - 4$

i.e., $6x^2 + 12x + 12 = 0$ or, $x^2 + 2x + 2 = 0$

It is of the form $ax^2 + bx + c = 0$.

So, the given equation is a quadratic equation.

Remark : Be careful! In (ii) above, the given equation appears to be a quadratic equation, but it is not a quadratic equation.

In (iv) above, the given equation appears to be a cubic equation (an equation of degree 3) and not a quadratic equation. But it turns out to be a quadratic equation. As you can see, often we need to simplify the given equation before deciding whether it is quadratic or not.

EXERCISE 4.1

1. Check whether the following are quadratic equations :

(i) $(x + 1)^2 = 2(x - 3)$
(ii) $x^2 - 2x = (-2)(3 - x)$
(iii) $(x - 2)(x + 1) = (x - 1)(x + 3)$
(iv) $(x - 3)(2x + 1) = x(x + 5)$
(v) $(2x - 1)(x - 3) = (x + 5)(x - 1)$
(vi) $x^2 + 3x + 1 = (x - 2)^2$
(vii) $(x + 2)^3 = 2x(x^2 - 1)$
(viii) $x^3 - 4x^2 - x + 1 = (x - 2)^3$

2. Represent the following situations in the form of quadratic equations :

(i) The area of a rectangular plot is 528 m^2. The length of the plot (in metres) is one more than twice its breadth. We need to find the length and breadth of the plot.

(ii) The product of two consecutive positive integers is 306. We need to find the integers.

(iii) Rohan's mother is 26 years older than him. The product of their ages (in years) 3 years from now will be 360. We would like to find Rohan's present age.

(iv) A train travels a distance of 480 km at a uniform speed. If the speed had been 8 km/h less, then it would have taken 3 hours more to cover the same distance. We need to find the speed of the train.

4.3 Solution of a Quadratic Equation by Factorisation

Consider the quadratic equation $2x^2 - 3x + 1 = 0$. If we replace x by 1 on the LHS of this equation, we get $(2 \times 1^2) - (3 \times 1) + 1 = 0 =$ RHS of the equation. We say that 1 is a root of the quadratic equation $2x^2 - 3x + 1 = 0$. This also means that 1 is a zero of the quadratic polynomial $2x^2 - 3x + 1$.

In general, a real number α is called a root of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ if $a\alpha^2 + b\alpha + c = 0$. We also say that **$x = \alpha$ is a solution of the quadratic equation**, or that **α satisfies the quadratic equation**. Note that **the zeroes of the quadratic polynomial $ax^2 + bx + c$ and the roots of the quadratic equation $ax^2 + bx + c = 0$ are the same**.

You have observed, in Chapter 2, that a quadratic polynomial can have at most two zeroes. So, any quadratic equation can have atmost two roots.

You have learnt in Class IX, how to factorise quadratic polynomials by splitting their middle terms. We shall use this knowledge for finding the roots of a quadratic equation. Let us see how.

Example 3 : Find the roots of the equation $2x^2 - 5x + 3 = 0$, by factorisation.

Solution : Let us first split the middle term $-5x$ as $-2x - 3x$ [because $(-2x) \times (-3x) = 6x^2 = (2x^2) \times 3$].

So, $2x^2 - 5x + 3 = 2x^2 - 2x - 3x + 3 = 2x(x - 1) - 3(x - 1) = (2x - 3)(x - 1)$

Now, $2x^2 - 5x + 3 = 0$ can be rewritten as $(2x - 3)(x - 1) = 0$.

So, the values of x for which $2x^2 - 5x + 3 = 0$ are the same for which $(2x - 3)(x - 1) = 0$, i.e., either $2x - 3 = 0$ or $x - 1 = 0$.

Now, $2x - 3 = 0$ gives $x = \frac{3}{2}$ and $x - 1 = 0$ gives $x = 1$.

So, $x = \frac{3}{2}$ and $x = 1$ are the solutions of the equation.

In other words, 1 and $\frac{3}{2}$ are the roots of the equation $2x^2 - 5x + 3 = 0$.

Verify that these are the roots of the given equation.

Note that we have found the roots of $2x^2 - 5x + 3 = 0$ by factorising $2x^2 - 5x + 3$ into two linear factors and equating each factor to zero.

Example 4 : Find the roots of the quadratic equation $6x^2 - x - 2 = 0$.

Solution : We have

$$\begin{aligned} 6x^2 - x - 2 &= 6x^2 + 3x - 4x - 2 \\ &= 3x\,(2x + 1) - 2\,(2x + 1) \\ &= (3x - 2)(2x + 1) \end{aligned}$$

The roots of $6x^2 - x - 2 = 0$ are the values of x for which $(3x - 2)(2x + 1) = 0$

Therefore, $3x - 2 = 0$ or $2x + 1 = 0$,

i.e., $x = \dfrac{2}{3}$ or $x = -\dfrac{1}{2}$

Therefore, the roots of $6x^2 - x - 2 = 0$ are $\dfrac{2}{3}$ and $-\dfrac{1}{2}$.

We verify the roots, by checking that $\dfrac{2}{3}$ and $-\dfrac{1}{2}$ satisfy $6x^2 - x - 2 = 0$.

Example 5 : Find the roots of the quadratic equation $3x^2 - 2\sqrt{6}x + 2 = 0$.

Solution : $$\begin{aligned} 3x^2 - 2\sqrt{6}x + 2 &= 3x^2 - \sqrt{6}x - \sqrt{6}x + 2 \\ &= \sqrt{3}x\left(\sqrt{3}x - \sqrt{2}\right) - \sqrt{2}\left(\sqrt{3}x - \sqrt{2}\right) \\ &= \left(\sqrt{3}x - \sqrt{2}\right)\left(\sqrt{3}x - \sqrt{2}\right) \end{aligned}$$

So, the roots of the equation are the values of x for which

$$\left(\sqrt{3}x - \sqrt{2}\right)\left(\sqrt{3}x - \sqrt{2}\right) = 0$$

Now, $\sqrt{3}x - \sqrt{2} = 0$ for $x = \sqrt{\dfrac{2}{3}}$.

So, this root is repeated twice, one for each repeated factor $\sqrt{3}x - \sqrt{2}$.

Therefore, the roots of $3x^2 - 2\sqrt{6}x + 2 = 0$ are $\sqrt{\dfrac{2}{3}}, \sqrt{\dfrac{2}{3}}$.

Example 6 : Find the dimensions of the prayer hall discussed in Section 4.1.

Solution : In Section 4.1, we found that if the breadth of the hall is x m, then x satisfies the equation $2x^2 + x - 300 = 0$. Applying the factorisation method, we write this equation as

$$2x^2 - 24x + 25x - 300 = 0$$

$$2x(x - 12) + 25(x - 12) = 0$$

i.e., $$(x - 12)(2x + 25) = 0$$

So, the roots of the given equation are $x = 12$ or $x = -12.5$. Since x is the breadth of the hall, it cannot be negative.

Thus, the breadth of the hall is 12 m. Its length $= 2x + 1 = 25$ m.

EXERCISE 4.2

1. Find the roots of the following quadratic equations by factorisation:

   (i) $x^2 - 3x - 10 = 0$ (ii) $2x^2 + x - 6 = 0$

   (iii) $\sqrt{2}\,x^2 + 7x + 5\sqrt{2} = 0$ (iv) $2x^2 - x + \frac{1}{8} = 0$

   (v) $100x^2 - 20x + 1 = 0$

2. Solve the problems given in Example 1.
3. Find two numbers whose sum is 27 and product is 182.
4. Find two consecutive positive integers, sum of whose squares is 365.
5. The altitude of a right triangle is 7 cm less than its base. If the hypotenuse is 13 cm, find the other two sides.
6. A cottage industry produces a certain number of pottery articles in a day. It was observed on a particular day that the cost of production of each article (in rupees) was 3 more than twice the number of articles produced on that day. If the total cost of production on that day was ₹ 90, find the number of articles produced and the cost of each article.

4.4 Nature of Roots

The equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 - 4ac > 0$, we get two distinct real roots $-\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$ and $-\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$.

If $b^2 - 4ac = 0$, then $x = -\frac{b}{2a} \pm 0$, i.e., $x = -\frac{b}{2a}$ or $-\frac{b}{2a}$.

So, the roots of the equation $ax^2 + bx + c = 0$ are both $\frac{-b}{2a}$.

Therefore, we say that the quadratic equation $ax^2 + bx + c = 0$ has two equal real roots in this case.

If $b^2 - 4ac < 0$, then there is no real number whose square is $b^2 - 4ac$. Therefore, there are no real roots for the given quadratic equation in this case.

Since $b^2 - 4ac$ determines whether the quadratic equation $ax^2 + bx + c = 0$ has real roots or not, $b^2 - 4ac$ is called the **discriminant** of this quadratic equation.

So, a quadratic equation $ax^2 + bx + c = 0$ has

(i) two distinct real roots, if $b^2 - 4ac > 0$,

(ii) two equal real roots, if $b^2 - 4ac = 0$,

(iii) no real roots, if $b^2 - 4ac < 0$.

Let us consider some examples.

Example 7: Find the discriminant of the quadratic equation $2x^2 - 4x + 3 = 0$, and hence find the nature of its roots.

Solution : The given equation is of the form $ax^2 + bx + c = 0$, where $a = 2$, $b = -4$ and $c = 3$. Therefore, the discriminant

$$b^2 - 4ac = (-4)^2 - (4 \times 2 \times 3) = 16 - 24 = -8 < 0$$

So, the given equation has no real roots.

Example 8 : A pole has to be erected at a point on the boundary of a circular park of diameter 13 metres in such a way that the differences of its distances from two diametrically opposite fixed gates A and B on the boundary is 7 metres. Is it possible to do so? If yes, at what distances from the two gates should the pole be erected?

Solution : Let us first draw the diagram (see Fig. 4.2).

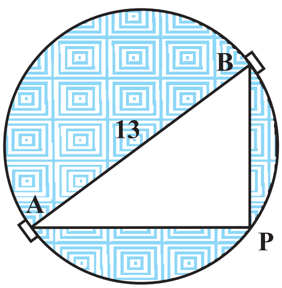

Fig. 4.2

Let P be the required location of the pole. Let the distance of the pole from the gate B be x m, i.e., BP = x m. Now the difference of the distances of the pole from the two gates = AP – BP (or, BP – AP) = 7 m. Therefore, AP = $(x + 7)$ m.

Now, AB = 13m, and since AB is a diameter,

$$\angle APB = 90^\circ \qquad \text{(Why?)}$$

Therefore, $$AP^2 + PB^2 = AB^2 \qquad \text{(By Pythagoras theorem)}$$

i.e., $$(x + 7)^2 + x^2 = 13^2$$

i.e., $$x^2 + 14x + 49 + x^2 = 169$$

i.e., $$2x^2 + 14x - 120 = 0$$

So, the distance 'x' of the pole from gate B satisfies the equation

$$x^2 + 7x - 60 = 0$$

So, it would be possible to place the pole if this equation has real roots. To see if this is so or not, let us consider its discriminant. The discriminant is

$$b^2 - 4ac = 7^2 - 4 \times 1 \times (-60) = 289 > 0.$$

So, the given quadratic equation has two real roots, and it is possible to erect the pole on the boundary of the park.

Solving the quadratic equation $x^2 + 7x - 60 = 0$, by the quadratic formula, we get

$$x = \frac{-7 \pm \sqrt{289}}{2} = \frac{-7 \pm 17}{2}$$

Therefore, $x = 5$ or -12.

Since x is the distance between the pole and the gate B, it must be positive. Therefore, $x = -12$ will have to be ignored. So, $x = 5$.

Thus, the pole has to be erected on the boundary of the park at a distance of 5m from the gate B and 12m from the gate A.

Example 9 : Find the discriminant of the equation $3x^2 - 2x + \frac{1}{3} = 0$ and hence find the nature of its roots. Find them, if they are real.

Solution : Here $a = 3$, $b = -2$ and $c = \frac{1}{3}$.

Therefore, discriminant $b^2 - 4ac = (-2)^2 - 4 \times 3 \times \frac{1}{3} = 4 - 4 = 0.$

Hence, the given quadratic equation has two equal real roots.

The roots are $\frac{-b}{2a}, \frac{-b}{2a}$, i.e., $\frac{2}{6}, \frac{2}{6}$, i.e., $\frac{1}{3}, \frac{1}{3}$.

EXERCISE 4.3

1. Find the nature of the roots of the following quadratic equations. If the real roots exist, find them:

   (i) $2x^2 - 3x + 5 = 0$ (ii) $3x^2 - 4\sqrt{3}\,x + 4 = 0$

   (iii) $2x^2 - 6x + 3 = 0$

2. Find the values of k for each of the following quadratic equations, so that they have two equal roots.

   (i) $2x^2 + kx + 3 = 0$ (ii) $kx\,(x - 2) + 6 = 0$

3. Is it possible to design a rectangular mango grove whose length is twice its breadth, and the area is 800 m^2? If so, find its length and breadth.

4. Is the following situation possible? If so, determine their present ages.
   The sum of the ages of two friends is 20 years. Four years ago, the product of their ages in years was 48.

5. Is it possible to design a rectangular park of perimeter 80 m and area 400 m^2? If so, find its length and breadth.

4.5 Summary

In this chapter, you have studied the following points:

1. A quadratic equation in the variable x is of the form $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$.

2. A real number α is said to be a root of the quadratic equation $ax^2 + bx + c = 0$, if $a\alpha^2 + b\alpha + c = 0$. The zeroes of the quadratic polynomial $ax^2 + bx + c$ and the roots of the quadratic equation $ax^2 + bx + c = 0$ are the same.

3. If we can factorise $ax^2 + bx + c$, $a \neq 0$, into a product of two linear factors, then the roots of the quadratic equation $ax^2 + bx + c = 0$ can be found by equating each factor to zero.

4. Quadratic formula: The roots of a quadratic equation $ax^2 + bx + c = 0$ are given by

   $$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$ provided $b^2 - 4ac \geq 0$.

5. A quadratic equation $ax^2 + bx + c = 0$ has

   (i) two distinct real roots, if $b^2 - 4ac > 0$,

   (ii) two equal roots (i.e., coincident roots), if $b^2 - 4ac = 0$, and

   (iii) no real roots, if $b^2 - 4ac < 0$.

Note

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ARITHMETIC PROGRESSIONS 5

5.1 Introduction

You must have observed that in nature, many things follow a certain pattern, such as the petals of a sunflower, the holes of a honeycomb, the grains on a maize cob, the spirals on a pineapple and on a pine cone, etc.

We now look for some patterns which occur in our day-to-day life. Some such examples are :

(i) Reena applied for a job and got selected. She has been offered a job with a starting monthly salary of ₹ 8000, with an annual increment of ₹ 500 in her salary. Her salary (in ₹) for the 1st, 2nd, 3rd, . . . years will be, respectively

8000, 8500, 9000, ...

(ii) The lengths of the rungs of a ladder decrease uniformly by 2 cm from bottom to top (see Fig. 5.1). The bottom rung is 45 cm in length. The lengths (in cm) of the 1st, 2nd, 3rd, . . ., 8th rung from the bottom to the top are, respectively

45, 43, 41, 39, 37, 35, 33, 31

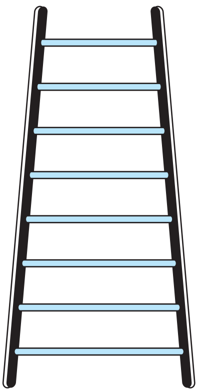

Fig. 5.1

(iii) In a savings scheme, the amount becomes $\frac{5}{4}$ times of itself after every 3 years. The maturity amount (in ₹) of an investment of ₹ 8000 after 3, 6, 9 and 12 years will be, respectively :

10000, 12500, 15625, 19531.25

(iv) The number of unit squares in squares with side 1, 2, 3, . . . units (see Fig. 5.2) are, respectively

$1^2, 2^2, 3^2, \ldots.$

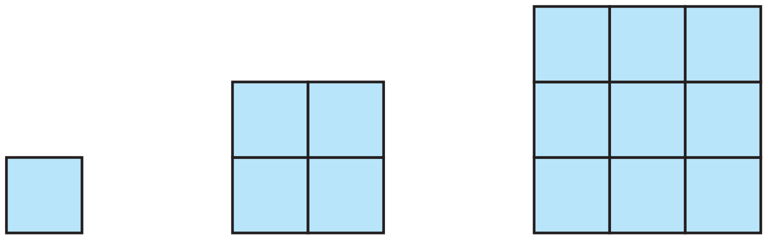

Fig. 5.2

(v) Shakila puts ₹ 100 into her daughter's money box when she was one year old and increased the amount by ₹ 50 every year. The amounts of money (in ₹) in the box on the 1st, 2nd, 3rd, 4th, . . . birthday were

100, 150, 200, 250, . . ., respectively.

(vi) A pair of rabbits are too young to produce in their first month. In the second, and every subsequent month, they produce a new pair. Each new pair of rabbits produce a new pair in their second month and in every subsequent month (see Fig. 5.3). Assuming no rabbit dies, the number of pairs of rabbits at the start of the 1st, 2nd, 3rd, . . ., 6th month, respectively are :

1, 1, 2, 3, 5, 8

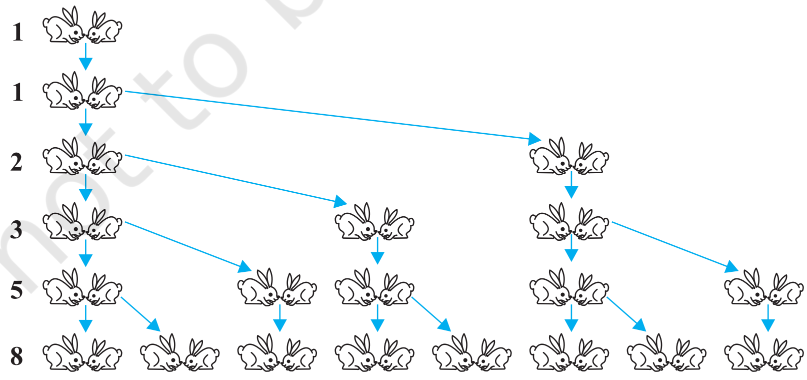

Fig. 5.3

In the examples above, we observe some patterns. In some, we find that the succeeding terms are obtained by adding a fixed number, in other by multiplying with a fixed number, in another we find that they are squares of consecutive numbers, and so on.

In this chapter, we shall discuss one of these patterns in which succeeding terms are obtained by adding a fixed number to the preceding terms. We shall also see how to find their *n*th terms and the sum of *n* consecutive terms, and use this knowledge in solving some daily life problems.

5.2 Arithmetic Progressions

Consider the following lists of numbers :

(i) 1, 2, 3, 4, . . .

(ii) 100, 70, 40, 10, . . .

(iii) –3, –2, –1, 0, . . .

(iv) 3, 3, 3, 3, . . .

(v) –1.0, –1.5, –2.0, –2.5, . . .

Each of the numbers in the list is called a **term**.

Given a term, can you write the next term in each of the lists above? If so, how will you write it? Perhaps by following a pattern or rule. Let us observe and write the rule.

In (i), each term is 1 more than the term preceding it.

In (ii), each term is 30 less than the term preceding it.

In (iii), each term is obtained by adding 1 to the term preceding it.

In (iv), all the terms in the list are 3 , i.e., each term is obtained by adding (or subtracting) 0 to the term preceding it.

In (v), each term is obtained by adding – 0.5 to (i.e., subtracting 0.5 from) the term preceding it.

In all the lists above, we see that successive terms are obtained by adding a fixed number to the preceding terms. Such list of numbers is said to form an **Arithmetic Progression (AP)**.

So, **an arithmetic progression is a list of numbers in which each term is obtained by adding a fixed number to the preceding term except the first term.**

This fixed number is called the **common difference** of the AP. Remember that it **can be positive, negative or zero**.

Let us denote the first term of an AP by a_1, second term by a_2, . . ., nth term by a_n and the common difference by d. Then the AP becomes $a_1, a_2, a_3, \ldots, a_n$.

So, $a_2 - a_1 = a_3 - a_2 = \ldots = a_n - a_{n-1} = d$.

Some more examples of AP are:

(a) The heights (in cm) of some students of a school standing in a queue in the morning assembly are 147 , 148, 149, . . ., 157.

(b) The minimum temperatures (in degree celsius) recorded for a week in the month of January in a city, arranged in ascending order are

$$-3.1, -3.0, -2.9, -2.8, -2.7, -2.6, -2.5$$

(c) The balance money (in ₹) after paying 5 % of the total loan of ₹ 1000 every month is 950, 900, 850, 800, . . ., 50.

(d) The cash prizes (in ₹) given by a school to the toppers of Classes I to XII are, respectively, 200, 250, 300, 350, . . ., 750.

(e) The total savings (in ₹) after every month for 10 months when ₹ 50 are saved each month are 50, 100, 150, 200, 250, 300, 350, 400, 450, 500.

It is left as an exercise for you to explain why each of the lists above is an AP.

You can see that

$$a, a + d, a + 2d, a + 3d, \ldots$$

represents an arithmetic progression where a is the first term and d the common difference. This is called the **general form of an AP.**

Note that in examples (a) to (e) above, there are only a finite number of terms. Such an AP is called a **finite AP.** Also note that each of these Arithmetic Progressions (APs) has a last term. The APs in examples (i) to (v) in this section, are not finite APs and so they are called **infinite Arithmetic Progressions**. Such APs do not have a last term.

Now, to know about an AP, what is the minimum information that you need? Is it enough to know the first term? Or, is it enough to know only the common difference? You will find that you will need to know both – the first term a and the common difference d.

For instance if the first term a is 6 and the common difference d is 3, then the AP is

$$6, 9, 12, 15, \ldots$$

and if a is 6 and d is – 3, then the AP is

$$6, 3, 0, -3, \ldots$$

Similarly, when

$a = -7$, $d = -2$, the AP is $-7, -9, -11, -13, \ldots$

$a = 1.0$, $d = 0.1$, the AP is $1.0, 1.1, 1.2, 1.3, \ldots$

$a = 0$, $d = 1\frac{1}{2}$, the AP is $0, 1\frac{1}{2}, 3, 4\frac{1}{2}, 6, \ldots$

$a = 2$, $d = 0$, the AP is $2, 2, 2, 2, \ldots$

So, if you know what a and d are, you can list the AP. What about the other way round? That is, if you are given a list of numbers can you say that it is an AP and then find a and d? Since a is the first term, it can easily be written. We know that in an AP, every succeeding term is obtained by adding d to the preceding term. So, d found by subtracting any term from its succeeding term, i.e., the term which immediately follows it should be same for an AP.

For example, for the list of numbers :

$$6, 9, 12, 15, \ldots,$$

We have

$$a_2 - a_1 = 9 - 6 = 3,$$
$$a_3 - a_2 = 12 - 9 = 3,$$
$$a_4 - a_3 = 15 - 12 = 3$$

Here the difference of any two consecutive terms in each case is 3. So, the given list is an AP whose first term a is 6 and common difference d is 3.

For the list of numbers : $6, 3, 0, -3, \ldots$,

$$a_2 - a_1 = 3 - 6 = -3$$
$$a_3 - a_2 = 0 - 3 = -3$$
$$a_4 - a_3 = -3 - 0 = -3$$

Similarly this is also an AP whose first term is 6 and the common difference is -3.

In general, for an AP $a_1, a_2, \ldots, a_n$, we have

$$d = a_{k+1} - a_k$$

where a_{k+1} and a_k are the $(k + 1)$th and the kth terms respectively.

To obtain d in a given AP, we need not find all of $a_2 - a_1$, $a_3 - a_2$, $a_4 - a_3$, ... It is enough to find only one of them.

Consider the list of numbers 1, 1, 2, 3, 5, ... By looking at it, you can tell that the difference between any two consecutive terms is not the same. So, this is not an AP.

Note that to find d in the AP : 6, 3, 0, – 3, . . ., we have subtracted 6 from 3 and not 3 from 6, i.e., we should subtract the kth term from the $(k + 1)$ th term even if the $(k + 1)$ th term is smaller.

Let us make the concept more clear through some examples.

Example 1 : For the AP : $\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$, . . ., write the first term a and the common difference d.

Solution : Here, $a = \frac{3}{2}, d = \frac{1}{2} - \frac{3}{2} = -1.$

Remember that we can find d using any two consecutive terms, once we know that the numbers are in AP.

Example 2 : Which of the following list of numbers form an AP? If they form an AP, write the next two terms :

(i) 4, 10, 16, 22, . . . (ii) 1, – 1, – 3, – 5, . . .

(iii) – 2, 2, – 2, 2, – 2, . . . (iv) 1, 1, 1, 2, 2, 2, 3, 3, 3, . . .

Solution : (i) We have
$$a_2 - a_1 = 10 - 4 = 6$$
$$a_3 - a_2 = 16 - 10 = 6$$
$$a_4 - a_3 = 22 - 16 = 6$$

i.e., $a_{k+1} - a_k$ is the same every time.

So, the given list of numbers forms an AP with the common difference $d = 6$.

The next two terms are: 22 + 6 = 28 and 28 + 6 = 34.

(ii)
$$a_2 - a_1 = -1 - 1 = -2$$
$$a_3 - a_2 = -3 - (-1) = -3 + 1 = -2$$
$$a_4 - a_3 = -5 - (-3) = -5 + 3 = -2$$

i.e., $a_{k+1} - a_k$ is the same every time.

So, the given list of numbers forms an AP with the common difference $d = -2$.

The next two terms are:
$$-5 + (-2) = -7 \quad \text{and} \quad -7 + (-2) = -9$$

(iii) $a_2 - a_1 = 2 - (-2) = 2 + 2 = 4$

$a_3 - a_2 = -2 - 2 = -4$

As $a_2 - a_1 \neq a_3 - a_2$, the given list of numbers does not form an AP.

(iv) $a_2 - a_1 = 1 - 1 = 0$
$a_3 - a_2 = 1 - 1 = 0$
$a_4 - a_3 = 2 - 1 = 1$

Here, $a_2 - a_1 = a_3 - a_2 \neq a_4 - a_3$.

So, the given list of numbers does not form an AP.

EXERCISE 5.1

1. In which of the following situations, does the list of numbers involved make an arithmetic progression, and why?

(i) The taxi fare after each km when the fare is ₹ 15 for the first km and ₹ 8 for each additional km.

(ii) The amount of air present in a cylinder when a vacuum pump removes $\frac{1}{4}$ of the air remaining in the cylinder at a time.

(iii) The cost of digging a well after every metre of digging, when it costs ₹ 150 for the first metre and rises by ₹ 50 for each subsequent metre.

(iv) The amount of money in the account every year, when ₹ 10000 is deposited at compound interest at 8 % per annum.

2. Write first four terms of the AP, when the first term a and the common difference d are given as follows:

(i) $a = 10, \quad d = 10$
(ii) $a = -2, \quad d = 0$
(iii) $a = 4, \quad d = -3$
(iv) $a = -1, \quad d = \frac{1}{2}$
(v) $a = -1.25, \quad d = -0.25$

3. For the following APs, write the first term and the common difference:

(i) $3, 1, -1, -3, \ldots$
(ii) $-5, -1, 3, 7, \ldots$
(iii) $\frac{1}{3}, \frac{5}{3}, \frac{9}{3}, \frac{13}{3}, \ldots$
(iv) $0.6, 1.7, 2.8, 3.9, \ldots$

4. Which of the following are APs ? If they form an AP, find the common difference d and write three more terms.

(i) $2, 4, 8, 16, \ldots$
(ii) $2, \frac{5}{2}, 3, \frac{7}{2}, \ldots$
(iii) $-1.2, -3.2, -5.2, -7.2, \ldots$
(iv) $-10, -6, -2, 2, \ldots$
(v) $3, 3+\sqrt{2}, 3+2\sqrt{2}, 3+3\sqrt{2}, \ldots$
(vi) $0.2, 0.22, 0.222, 0.2222, \ldots$
(vii) $0, -4, -8, -12, \ldots$
(viii) $-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \ldots$

(ix) $1, 3, 9, 27, \ldots$

(x) $a, 2a, 3a, 4a, \ldots$

(xi) $a, a^2, a^3, a^4, \ldots$

(xii) $\sqrt{2}, \sqrt{8}, \sqrt{18}, \sqrt{32}, \ldots$

(xiii) $\sqrt{3}, \sqrt{6}, \sqrt{9}, \sqrt{12}, \ldots$

(xiv) $1^2, 3^2, 5^2, 7^2, \ldots$

(xv) $1^2, 5^2, 7^2, 73, \ldots$

5.3 *n*th Term of an AP

Let us consider the situation again, given in Section 5.1 in which Reena applied for a job and got selected. She has been offered the job with a starting monthly salary of ₹ 8000, with an annual increment of ₹ 500. What would be her monthly salary for the fifth year?

To answer this, let us first see what her monthly salary for the second year would be.

It would be ₹ (8000 + 500) = ₹ 8500. In the same way, we can find the monthly salary for the 3rd, 4th and 5th year by adding ₹ 500 to the salary of the previous year.

So, the salary for the 3rd year = ₹ (8500 + 500)

= ₹ (8000 + 500 + 500)

= ₹ (8000 + 2 × 500)

= ₹ [8000 + **(3 – 1)** × 500] (for the **3rd year**)

= ₹ 9000

Salary for the 4th year = ₹ (9000 + 500)

= ₹ (8000 + 500 + 500 + 500)

= ₹ (8000 + 3 × 500)

= ₹ [8000 + **(4 – 1)** × 500] (for the **4th year**)

= ₹ 9500

Salary for the 5th year = ₹ (9500 + 500)

= ₹ (8000+500+500+500 + 500)

= ₹ (8000 + 4 × 500)

= ₹ [8000 + **(5 – 1)** × 500] (for the **5th year**)

= ₹ 10000

Observe that we are getting a list of numbers

8000, 8500, 9000, 9500, 10000, . . .

These numbers are in AP. (Why?)

Now, looking at the pattern formed above, can you find her monthly salary for the 6th year? The 15th year? And, assuming that she will still be working in the job, what about the monthly salary for the 25th year? You would calculate this by adding ₹ 500 each time to the salary of the previous year to give the answer. Can we make this process shorter? Let us see. You may have already got some idea from the way we have obtained the salaries above.

Salary for the 15th year

$$= \text{Salary for the 14th year} + ₹\ 500$$

$$= ₹\left[8000 + \underbrace{500 + 500 + 500 + \ldots + 500}_{13 \text{ times}}\right] + ₹\ 500$$

$$= ₹\ [8000 + 14 \times 500]$$

$$= ₹\ [8000 + \mathbf{(15 - 1)} \times 500] = ₹\ 15000$$

i.e., First salary + (15 – 1) × Annual increment.

In the same way, her monthly salary for the 25th year would be

$$₹\ [8000 + \mathbf{(25 - 1)} \times 500] = ₹\ 20000$$

$$= \text{First salary} + \mathbf{(25 - 1)} \times \text{Annual increment}$$

This example would have given you some idea about how to write the 15th term, or the 25th term, and more generally, the nth term of the AP.

Let $a_1, a_2, a_3, \ldots$ be an AP whose first term a_1 is a and the common difference is d.

Then,

the **second term** $a_2 = a + d = a + \mathbf{(2 - 1)}\, d$

the **third** term $a_3 = a_2 + d = (a + d) + d = a + 2d = a + \mathbf{(3 - 1)}\, d$

the **fourth** term $a_4 = a_3 + d = (a + 2d) + d = a + 3d = a + \mathbf{(4 - 1)}\, d$

. . . .

. . . .

Looking at the pattern, we can say that the ***n*th** term $a_n = a + (n - 1)\, d$.

So, **the *n*th term a_n of the AP with first term a and common difference d is given by $a_n = a + (n - 1)\, d$.**

$\boldsymbol{a_n}$ is also called the **general term of the AP**. If there are m terms in the AP, then $\boldsymbol{a_m}$ represents the **last term which is sometimes also denoted by** $\boldsymbol{l}$.

Let us consider some examples.

Example 3 : Find the 10th term of the AP : 2, 7, 12, . . .

Solution : Here, $a = 2$, $d = 7 - 2 = 5$ and $n = 10$.

We have $\quad a_n = a + (n - 1)\,d$

So, $\quad a_{10} = 2 + (10 - 1) \times 5 = 2 + 45 = 47$

Therefore, the 10th term of the given AP is 47.

Example 4 : Which term of the AP : 21, 18, 15, . . . is -81? Also, is any term 0? Give reason for your answer.

Solution : Here, $a = 21$, $d = 18 - 21 = -3$ and $a_n = -81$, and we have to find n.

As $\quad a_n = a + (n - 1)\,d,$

we have $\quad -81 = 21 + (n - 1)(-3)$

$$-81 = 24 - 3n$$

$$-105 = -3n$$

So, $\quad n = 35$

Therefore, the 35th term of the given AP is -81.

Next, we want to know if there is any n for which $a_n = 0$. If such an n is there, then

$$21 + (n - 1)(-3) = 0,$$

i.e., $\quad 3(n - 1) = 21$

i.e., $\quad n = 8$

So, the eighth term is 0.

Example 5 : Determine the AP whose 3rd term is 5 and the 7th term is 9.

Solution : We have

$$a_3 = a + (3 - 1)\,d = a + 2d = 5 \qquad (1)$$

and $$a_7 = a + (7 - 1)\,d = a + 6d = 9 \qquad (2)$$

Solving the pair of linear equations (1) and (2), we get

$$a = 3, \quad d = 1$$

Hence, the required AP is 3, 4, 5, 6, 7, . . .

Example 6 : Check whether 301 is a term of the list of numbers 5, 11, 17, 23, . . .

Solution : We have :

$$a_2 - a_1 = 11 - 5 = 6, \quad a_3 - a_2 = 17 - 11 = 6, \quad a_4 - a_3 = 23 - 17 = 6$$

As $a_{k+1} - a_k$ is the same for $k = 1, 2, 3$, etc., the given list of numbers is an AP.

Now, $a = 5$ and $d = 6$.

Let 301 be a term, say, the nth term of this AP.

We know that

$$a_n = a + (n - 1)\,d$$

So, $$301 = 5 + (n - 1) \times 6$$

i.e., $$301 = 6n - 1$$

So, $$n = \frac{302}{6} = \frac{151}{3}$$

But n should be a positive integer (Why?). So, 301 is not a term of the given list of numbers.

Example 7 : How many two-digit numbers are divisible by 3?

Solution : The list of two-digit numbers divisible by 3 is :

$$12, 15, 18, \ldots, 99$$

Is this an AP? Yes it is. Here, $a = 12$, $d = 3$, $a_n = 99$.

As $$a_n = a + (n - 1)\,d,$$

we have $$99 = 12 + (n - 1) \times 3$$

i.e., $$87 = (n - 1) \times 3$$

i.e., $$n - 1 = \frac{87}{3} = 29$$

i.e., $$n = 29 + 1 = 30$$

So, there are 30 two-digit numbers divisible by 3.

Example 8 : Find the 11th term from the last term (towards the first term) of the AP : 10, 7, 4, . . ., – 62.

Solution : Here, $a = 10$, $d = 7 - 10 = -3$, $l = -62$,

where $$l = a + (n - 1)\,d$$

To find the 11th term from the last term, we will find the total number of terms in the AP.

So, $$-62 = 10 + (n-1)(-3)$$

i.e., $$-72 = (n-1)(-3)$$

i.e., $$n - 1 = 24$$

or $$n = 25$$

So, there are 25 terms in the given AP.

The 11th term from the last term will be the 15th term. (Note that it will not be the 14th term. Why?)

So, $$a_{15} = 10 + (15-1)(-3) = 10 - 42 = -32$$

i.e., the 11th term from the last term is -32.

Alternative Solution :

If we write the given AP in the reverse order, then $a = -62$ and $d = 3$ (Why?)

So, the question now becomes finding the 11th term with these a and d.

So, $$a_{11} = -62 + (11-1) \times 3 = -62 + 30 = -32$$

So, the 11th term, which is now the required term, is -32.

Example 9 : A sum of ₹ 1000 is invested at 8% simple interest per year. Calculate the interest at the end of each year. Do these interests form an AP? If so, find the interest at the end of 30 years making use of this fact.

Solution : We know that the formula to calculate simple interest is given by

$$\text{Simple Interest} = \frac{P \times R \times T}{100}$$

So, the interest at the end of the 1st year = ₹ $\frac{1000 \times 8 \times 1}{100}$ = ₹ 80

The interest at the end of the 2nd year = ₹ $\frac{1000 \times 8 \times 2}{100}$ = ₹ 160

The interest at the end of the 3rd year = ₹ $\frac{1000 \times 8 \times 3}{100}$ = ₹ 240

Similarly, we can obtain the interest at the end of the 4th year, 5th year, and so on.

So, the interest (in ₹) at the end of the 1st, 2nd, 3rd, . . . years, respectively are

$$80, 160, 240, \ldots$$

It is an AP as the difference between the consecutive terms in the list is 80, i.e., $d = 80$. Also, $a = 80$.

So, to find the interest at the end of 30 years, we shall find a_{30}.

Now, $a_{30} = a + (30 - 1)\, d = 80 + 29 \times 80 = 2400$

So, the interest at the end of 30 years will be ₹ 2400.

Example 10 : In a flower bed, there are 23 rose plants in the first row, 21 in the second, 19 in the third, and so on. There are 5 rose plants in the last row. How many rows are there in the flower bed?

Solution : The number of rose plants in the 1st, 2nd, 3rd, . . ., rows are :

$$23, 21, 19, \ldots, 5$$

It forms an AP (Why?). Let the number of rows in the flower bed be n.

Then $a = 23, \quad d = 21 - 23 = -2, \quad a_n = 5$

As, $a_n = a + (n - 1)\, d$

We have, $5 = 23 + (n - 1)(-2)$

i.e., $-18 = (n - 1)(-2)$

i.e., $n = 10$

So, there are 10 rows in the flower bed.

EXERCISE 5.2

1. Fill in the blanks in the following table, given that a is the first term, d the common difference and a_n the nth term of the AP:

	a	d	n	a_n
(i)	7	3	8	. . .
(ii)	– 18	. . .	10	0
(iii)	. . .	– 3	18	– 5
(iv)	– 18.9	2.5	. . .	3.6
(v)	3.5	0	105	. . .

2. Choose the correct choice in the following and justify :

(i) 30th term of the AP: 10, 7, 4, . . . , is

(A) 97 (B) 77 (C) –77 (D) – 87

(ii) 11th term of the AP: – 3, $-\frac{1}{2}$, 2, . . ., is

(A) 28 (B) 22 (C) –38 (D) $-48\frac{1}{2}$

3. In the following APs, find the missing terms in the boxes :

(i) 2, $\boxed{}$, 26

(ii) $\boxed{}$, 13, $\boxed{}$, 3

(iii) 5, $\boxed{}$, $\boxed{}$, $9\frac{1}{2}$

(iv) – 4, $\boxed{}$, $\boxed{}$, $\boxed{}$, $\boxed{}$, 6

(v) $\boxed{}$, 38, $\boxed{}$, $\boxed{}$, $\boxed{}$, – 22

4. Which term of the AP : 3, 8, 13, 18, . . . ,is 78?

5. Find the number of terms in each of the following APs :

(i) 7, 13, 19, . . . , 205 (ii) 18, $15\frac{1}{2}$, 13, . . . , – 47

6. Check whether – 150 is a term of the AP : 11, 8, 5, 2 . . .

7. Find the 31st term of an AP whose 11th term is 38 and the 16th term is 73.

8. An AP consists of 50 terms of which 3rd term is 12 and the last term is 106. Find the 29th term.

9. If the 3rd and the 9th terms of an AP are 4 and – 8 respectively, which term of this AP is zero?

10. The 17th term of an AP exceeds its 10th term by 7. Find the common difference.

11. Which term of the AP : 3, 15, 27, 39, . . . will be 132 more than its 54th term?

12. Two APs have the same common difference. The difference between their 100th terms is 100, what is the difference between their 1000th terms?

13. How many three-digit numbers are divisible by 7?

14. How many multiples of 4 lie between 10 and 250?

15. For what value of n, are the nth terms of two APs: 63, 65, 67, . . . and 3, 10, 17, . . . equal?

16. Determine the AP whose third term is 16 and the 7th term exceeds the 5th term by 12.

17. Find the 20th term from the last term of the AP : 3, 8, 13, . . ., 253.

18. The sum of the 4th and 8th terms of an AP is 24 and the sum of the 6th and 10th terms is 44. Find the first three terms of the AP.

19. Subba Rao started work in 1995 at an annual salary of ₹ 5000 and received an increment of ₹ 200 each year. In which year did his income reach ₹ 7000?

20. Ramkali saved ₹ 5 in the first week of a year and then increased her weekly savings by ₹ 1.75. If in the nth week, her weekly savings become ₹ 20.75, find n.

5.4 Sum of First *n* Terms of an AP

Let us consider the situation again given in Section 5.1 in which Shakila put ₹ 100 into her daughter's money box when she was one year old, ₹ 150 on her second birthday, ₹ 200 on her third birthday and will continue in the same way. How much money will be collected in the money box by the time her daughter is 21 years old?

Here, the amount of money (in ₹) put in the money box on her first, second, third, fourth . . . birthday were respectively 100, 150, 200, 250, . . . till her 21st birthday. To find the total amount in the money box on her 21st birthday, we will have to write each of the 21 numbers in the list above and then add them up. Don't you think it would be a tedious and time consuming process? Can we make the process shorter? This would be possible if we can find a method for getting this sum. Let us see.

We consider the problem given to Gauss (about whom you read in Chapter 1), to solve when he was just 10 years old. He was asked to find the sum of the positive integers from 1 to 100. He immediately replied that the sum is 5050. Can you guess how did he do? He wrote :

$$S = 1 + 2 + 3 + . . . + 99 + 100$$

And then, reversed the numbers to write

$$S = 100 + 99 + . . . + 3 + 2 + 1$$

Adding these two, he got

$$2S = (100 + 1) + (99 + 2) + . . . + (3 + 98) + (2 + 99) + (1 + 100)$$

$$= 101 + 101 + . . . + 101 + 101 \quad (100 \text{ times})$$

So, $$S = \frac{100 \times 101}{2} = 5050, \text{ i.e., the sum} = 5050.$$

We will now use the same technique to find the sum of the first n terms of an AP :

$$a, a + d, a + 2d, \ldots$$

The nth term of this AP is $a + (n - 1)\, d$. Let S denote the sum of the first n terms of the AP. We have

$$S = a + (a + d) + (a + 2d) + \ldots + [a + (n - 1)\, d] \quad (1)$$

Rewriting the terms in reverse order, we have

$$S = [a + (n - 1)\, d] + [a + (n - 2)\, d] + \ldots + (a + d) + a \quad (2)$$

On adding (1) and (2), term-wise. we get

$$2S = \underbrace{[2a + (n-1)d] + [2a + (n-1)d] + \ldots + [2a + (n-1)d] + [2a + (n-1)d]}_{n \text{ times}}$$

or, $\quad 2S = n\, [2a + (n - 1)\, d] \qquad$ (Since, there are n terms)

or, $\quad S = \dfrac{n}{2}\, [2a + (n - 1)\, d]$

So, **the sum of the first n terms of an AP is given by**

$$\mathbf{S = \frac{n}{2}\, [2a + (n - 1)\, d]}$$

We can also write this as $\quad S = \dfrac{n}{2}\, [a + a + (n - 1)\, d]$

i.e., $$\mathbf{S = \frac{n}{2}\, (a + a_n)} \quad (3)$$

Now, if there are only n terms in an AP, then $a_n = l$, the last term. From (3), we see that

$$\mathbf{S = \frac{n}{2}\, (a + l)} \quad (4)$$

This form of the result is useful when the first and the last terms of an AP are given and the common difference is not given.

Now we return to the question that was posed to us in the beginning. The amount of money (in Rs) in the money box of Shakila's daughter on 1st, 2nd, 3rd, 4th birthday, . . ., were 100, 150, 200, 250, . . ., respectively.

This is an AP. We have to find the total money collected on her 21st birthday, i.e., the sum of the first 21 terms of this AP.

Here, $a = 100$, $d = 50$ and $n = 21$. Using the formula :

$$S = \frac{n}{2}[2a + (n-1)\,d],$$

we have
$$S = \frac{21}{2}[2 \times 100 + (21-1) \times 50] = \frac{21}{2}[200 + 1000]$$

$$= \frac{21}{2} \times 1200 = 12600$$

So, the amount of money collected on her 21st birthday is ₹ 12600.

Hasn't the use of the formula made it much easier to solve the problem?

We also use S_n in place of S to denote the sum of first n terms of the AP. We write S_{20} to denote the sum of the first 20 terms of an AP. The formula for the sum of the first n terms involves four quantities S, a, d and n. If we know any three of them, we can find the fourth.

Remark : The nth term of an AP is the difference of the sum to first n terms and the sum to first $(n - 1)$ terms of it, i.e., $a_n = S_n - S_{n-1}$.

Let us consider some examples.

Example 11 : Find the sum of the first 22 terms of the AP : 8, 3, –2, . . .

Solution : Here, $a = 8$, $d = 3 - 8 = -5$, $n = 22$.

We know that

$$S = \frac{n}{2}[2a + (n-1)\,d]$$

Therefore,
$$S = \frac{22}{2}[16 + 21\,(-5)] = 11(16 - 105) = 11(-89) = -979$$

So, the sum of the first 22 terms of the AP is – 979.

Example 12 : If the sum of the first 14 terms of an AP is 1050 and its first term is 10, find the 20th term.

Solution : Here, $S_{14} = 1050$, $n = 14$, $a = 10$.

As
$$S_n = \frac{n}{2}[2a + (n-1)d],$$

so,
$$1050 = \frac{14}{2}[20 + 13d] = 140 + 91d$$

i.e., $910 = 91d$

or, $d = 10$

Therefore, $a_{20} = 10 + (20 - 1) \times 10 = 200$, i.e. 20th term is 200.

Example 13 : How many terms of the AP : 24, 21, 18, . . . must be taken so that their sum is 78?

Solution : Here, $a = 24$, $d = 21 - 24 = -3$, $S_n = 78$. We need to find n.

We know that $$S_n = \frac{n}{2}[2a + (n-1)d]$$

So, $$78 = \frac{n}{2}[48 + (n-1)(-3)] = \frac{n}{2}[51 - 3n]$$

or $3n^2 - 51n + 156 = 0$

or $n^2 - 17n + 52 = 0$

or $(n - 4)(n - 13) = 0$

or $n = 4$ or 13

Both values of n are admissible. So, the number of terms is either 4 or 13.

Remarks:

1. In this case, the sum of the first 4 terms = the sum of the first 13 terms = 78.
2. Two answers are possible because the sum of the terms from 5th to 13th will be zero. This is because a is positive and d is negative, so that some terms will be positive and some others negative, and will cancel out each other.

Example 14 : Find the sum of :

(i) the first 1000 positive integers (ii) the first n positive integers

Solution :

(i) Let $S = 1 + 2 + 3 + . . . + 1000$

Using the formula $S_n = \frac{n}{2}(a + l)$ for the sum of the first n terms of an AP, we have

$$S_{1000} = \frac{1000}{2}(1 + 1000) = 500 \times 1001 = 500500$$

So, the sum of the first 1000 positive integers is 500500.

(ii) Let $S_n = 1 + 2 + 3 + . . . + n$

Here $a = 1$ and the last term l is n.

Therefore, $$S_n = \frac{n(1+n)}{2} \quad \text{or} \quad S_n = \frac{n(n+1)}{2}$$

So, **the sum of first n positive integers is given by**

$$\mathbf{S_n = \frac{n(n+1)}{2}}$$

Example 15 : Find the sum of first 24 terms of the list of numbers whose nth term is given by

$$a_n = 3 + 2n$$

Solution :

As $a_n = 3 + 2n$,

so, $a_1 = 3 + 2 = 5$

$a_2 = 3 + 2 \times 2 = 7$

$a_3 = 3 + 2 \times 3 = 9$

$\vdots$

List of numbers becomes 5, 7, 9, 11, . . .

Here, $7 - 5 = 9 - 7 = 11 - 9 = 2$ and so on.

So, it forms an AP with common difference $d = 2$.

To find S_{24}, we have $n = 24$, $a = 5$, $d = 2$.

Therefore, $$S_{24} = \frac{24}{2}\left[2\times5+(24-1)\times2\right] = 12\left[10+46\right] = 672$$

So, sum of first 24 terms of the list of numbers is 672.

Example 16 : A manufacturer of TV sets produced 600 sets in the third year and 700 sets in the seventh year. Assuming that the production increases uniformly by a fixed number every year, find :

(i) the production in the 1st year

(ii) the production in the 10th year

(iii) the total production in first 7 years

Solution : (i) Since the production increases uniformly by a fixed number every year, the number of TV sets manufactured in 1st, 2nd, 3rd, . . ., years will form an AP.

Let us denote the number of TV sets manufactured in the nth year by a_n.

Then, $a_3 = 600$ and $a_7 = 700$

or, $a + 2d = 600$

and $a + 6d = 700$

Solving these equations, we get $d = 25$ and $a = 550$.

Therefore, production of TV sets in the first year is 550.

(ii) Now $a_{10} = a + 9d = 550 + 9 \times 25 = 775$

So, production of TV sets in the 10th year is 775.

(iii) Also, $S_7 = \frac{7}{2}[2 \times 550 + (7-1) \times 25]$

$$= \frac{7}{2}[1100 + 150] = 4375$$

Thus, the total production of TV sets in first 7 years is 4375.

EXERCISE 5.3

1. Find the sum of the following APs:

(i) 2, 7, 12, . . ., to 10 terms.

(ii) –37, –33, –29, . . ., to 12 terms.

(iii) 0.6, 1.7, 2.8, . . ., to 100 terms.

(iv) $\frac{1}{15}, \frac{1}{12}, \frac{1}{10}$, . . ., to 11 terms.

2. Find the sums given below :

(i) $7 + 10\frac{1}{2} + 14 + \ldots + 84$

(ii) $34 + 32 + 30 + \ldots + 10$

(iii) $-5 + (-8) + (-11) + \ldots + (-230)$

3. In an AP:

(i) given $a = 5$, $d = 3$, $a_n = 50$, find n and S_n.

(ii) given $a = 7$, $a_{13} = 35$, find d and S_{13}.

(iii) given $a_{12} = 37$, $d = 3$, find a and S_{12}.

(iv) given $a_3 = 15$, $S_{10} = 125$, find d and a_{10}.

(v) given $d = 5$, $S_9 = 75$, find a and a_9.

(vi) given $a = 2$, $d = 8$, $S_n = 90$, find n and a_n.

(vii) given $a = 8$, $a_n = 62$, $S_n = 210$, find n and d.

(viii) given $a_n = 4$, $d = 2$, $S_n = -14$, find n and a.

(ix) given $a = 3$, $n = 8$, $S = 192$, find d.

(x) given $l = 28$, $S = 144$, and there are total 9 terms. Find a.

4. How many terms of the AP : 9, 17, 25, . . . must be taken to give a sum of 636?

5. The first term of an AP is 5, the last term is 45 and the sum is 400. Find the number of terms and the common difference.

6. The first and the last terms of an AP are 17 and 350 respectively. If the common difference is 9, how many terms are there and what is their sum?

7. Find the sum of first 22 terms of an AP in which $d = 7$ and 22nd term is 149.

8. Find the sum of first 51 terms of an AP whose second and third terms are 14 and 18 respectively.

9. If the sum of first 7 terms of an AP is 49 and that of 17 terms is 289, find the sum of first n terms.

10. Show that $a_1, a_2, \ldots, a_n, \ldots$ form an AP where a_n is defined as below :

(i) $a_n = 3 + 4n$ (ii) $a_n = 9 - 5n$

Also find the sum of the first 15 terms in each case.

11. If the sum of the first n terms of an AP is $4n - n^2$, what is the first term (that is S_1)? What is the sum of first two terms? What is the second term? Similarly, find the 3rd, the 10th and the nth terms.

12. Find the sum of the first 40 positive integers divisible by 6.

13. Find the sum of the first 15 multiples of 8.

14. Find the sum of the odd numbers between 0 and 50.

15. A contract on construction job specifies a penalty for delay of completion beyond a certain date as follows: ₹ 200 for the first day, ₹ 250 for the second day, ₹ 300 for the third day, etc., the penalty for each succeeding day being ₹ 50 more than for the preceding day. How much money the contractor has to pay as penalty, if he has delayed the work by 30 days?

16. A sum of ₹ 700 is to be used to give seven cash prizes to students of a school for their overall academic performance. If each prize is ₹ 20 less than its preceding prize, find the value of each of the prizes.

17. In a school, students thought of planting trees in and around the school to reduce air pollution. It was decided that the number of trees, that each section of each class will plant, will be the same as the class, in which they are studying, e.g., a section of Class I will plant 1 tree, a section of Class II will plant 2 trees and so on till Class XII. There are three sections of each class. How many trees will be planted by the students?

18. A spiral is made up of successive semicircles, with centres alternately at A and B, starting with centre at A, of radii 0.5 cm, 1.0 cm, 1.5 cm, 2.0 cm, . . . as shown in Fig. 5.4. What is the total length of such a spiral made up of thirteen consecutive semicircles? (Take $\pi = \dfrac{22}{7}$)

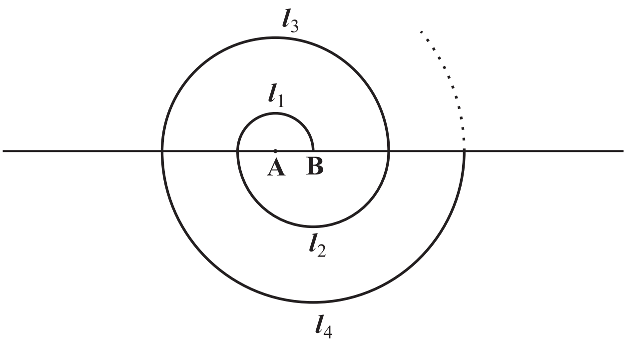

Fig. 5.4

[**Hint :** Length of successive semicircles is $l_1, l_2, l_3, l_4, \ldots$ with centres at A, B, A, B, . . ., respectively.]

19. 200 logs are stacked in the following manner: 20 logs in the bottom row, 19 in the next row, 18 in the row next to it and so on (see Fig. 5.5). In how many rows are the 200 logs placed and how many logs are in the top row?

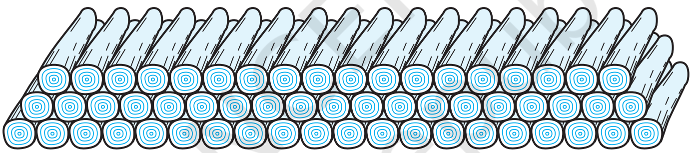

Fig. 5.5

20. In a potato race, a bucket is placed at the starting point, which is 5 m from the first potato, and the other potatoes are placed 3 m apart in a straight line. There are ten potatoes in the line (see Fig. 5.6).

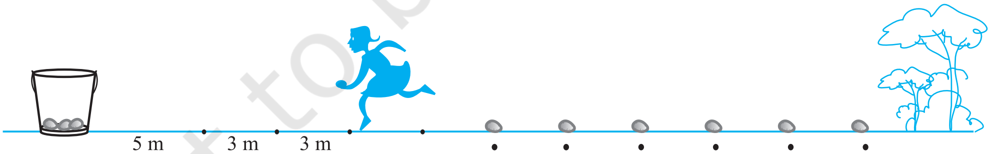

Fig. 5.6

A competitor starts from the bucket, picks up the nearest potato, runs back with it, drops it in the bucket, runs back to pick up the next potato, runs to the bucket to drop it in, and she continues in the same way until all the potatoes are in the bucket. What is the total distance the competitor has to run?

[**Hint :** To pick up the first potato and the second potato, the total distance (in metres) run by a competitor is $2 \times 5 + 2 \times (5 + 3)$]

EXERCISE 5.4 (Optional)*

1. Which term of the AP : 121, 117, 113, . . ., is its first negative term?

[**Hint :** Find n for $a_n < 0$]

2. The sum of the third and the seventh terms of an AP is 6 and their product is 8. Find the sum of first sixteen terms of the AP.

3. A ladder has rungs 25 cm apart. (see Fig. 5.7). The rungs decrease uniformly in length from 45 cm at the bottom to 25 cm at the top. If the top and the bottom rungs are $2\frac{1}{2}$ m apart, what is the length of the wood required for the rungs?

[**Hint :** Number of rungs $= \frac{250}{25} + 1$]

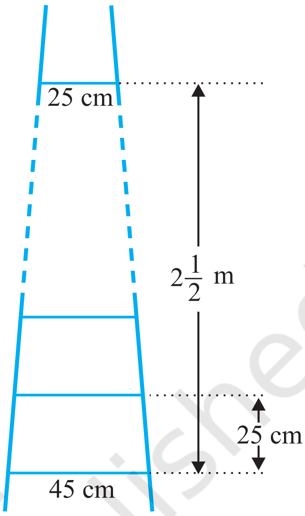

Fig. 5.7

4. The houses of a row are numbered consecutively from 1 to 49. Show that there is a value of x such that the sum of the numbers of the houses preceding the house numbered x is equal to the sum of the numbers of the houses following it. Find this value of x.

[**Hint :** $S_{x-1} = S_{49} - S_x$]

5. A small terrace at a football ground comprises of 15 steps each of which is 50 m long and built of solid concrete.

Each step has a rise of $\frac{1}{4}$ m and a tread of $\frac{1}{2}$ m. (see Fig. 5.8). Calculate the total volume of concrete required to build the terrace.

[**Hint :** Volume of concrete required to build the first step $= \frac{1}{4} \times \frac{1}{2} \times 50 \text{ m}^3$]

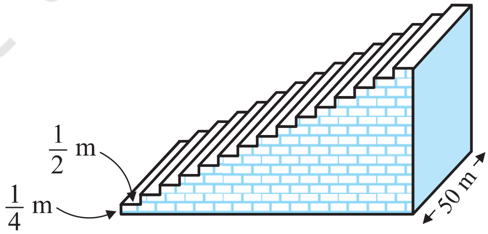

Fig. 5.8

* These exercises are not from the examination point of view.

5.5 Summary

In this chapter, you have studied the following points :

1. An **arithmetic progression** (AP) is a list of numbers in which each term is obtained by adding a fixed number d to the preceding term, except the first term. The fixed number d is called the **common difference**.

   The general form of an AP is $a, \ a + d, \ a + 2d, \ a + 3d, \ldots$

2. A given list of numbers $a_1, a_2, a_3, \ldots$ is an AP, if the differences $a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots$, give the same value, i.e., if $a_{k+1} - a_k$ is the same for different values of k.

3. In an AP with first term a and common difference d, the nth term (or the general term) is given by $a_n = a + (n - 1)\, d.$

4. The sum of the first n terms of an AP is given by :

$$S = \frac{n}{2}\left[2a + (n-1)d\right]$$

5. If l is the last term of the finite AP, say the nth term, then the sum of all terms of the AP is given by :

$$S = \frac{n}{2}(a + l)$$

A NOTE TO THE READER

If a, b, c are in AP, then $b = \dfrac{a+c}{2}$ and b is called the arithmetic mean of a and c.

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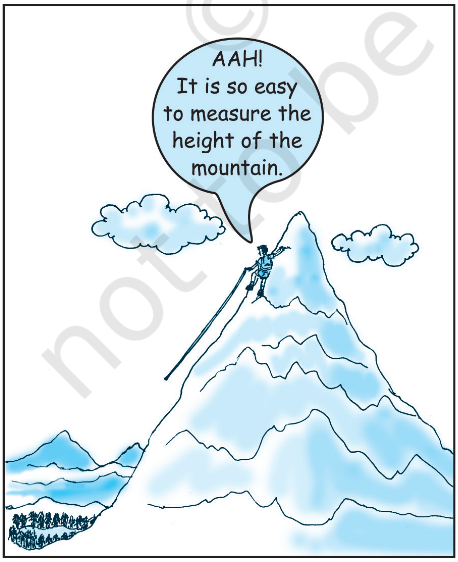
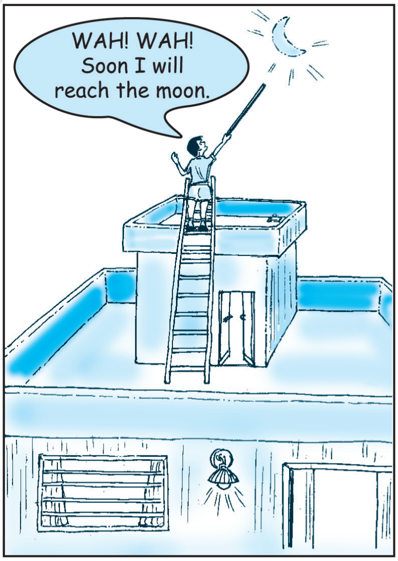
TRIANGLES 6

6.1 Introduction

You are familiar with triangles and many of their properties from your earlier classes. In Class IX, you have studied congruence of triangles in detail. Recall that two figures are said to be *congruent*, if they have the same shape and the same size. In this chapter, we shall study about those figures which have the same shape but not necessarily the same size. Two figures having the same shape (and not necessarily the same size) are called *similar figures.* In particular, we shall discuss the similarity of triangles and apply this knowledge in giving a simple proof of Pythagoras Theorem learnt earlier.

Can you guess how heights of mountains (say Mount Everest) or distances of some long distant objects (say moon) have been found out? Do you think these have





been measured directly with the help of a measuring tape? In fact, all these heights and distances have been found out using the idea of indirect measurements, which is based on the principle of similarity of figures (see Example 7, Q.15 of Exercise 6.3 and also Chapters 8 and 9 of this book).

6.2 Similar Figures

In Class IX, you have seen that all circles with the same radii are congruent, all squares with the same side lengths are congruent and all equilateral triangles with the same side lengths are congruent.

Now consider any two (or more) circles [see Fig. 6.1 (i)]. Are they congruent? Since all of them do not have the same radius, they are not congruent to each other. Note that some are congruent and some are not, but all of them have the same shape. So they all are, what we call, *similar*. Two similar figures have the same shape but not necessarily the same size. Therefore, all circles are similar. What about two (or more) squares or two (or more) equilateral triangles [see Fig. 6.1 (ii) and (iii)]? As observed in the case of circles, here also all squares are similar and all equilateral triangles are similar.

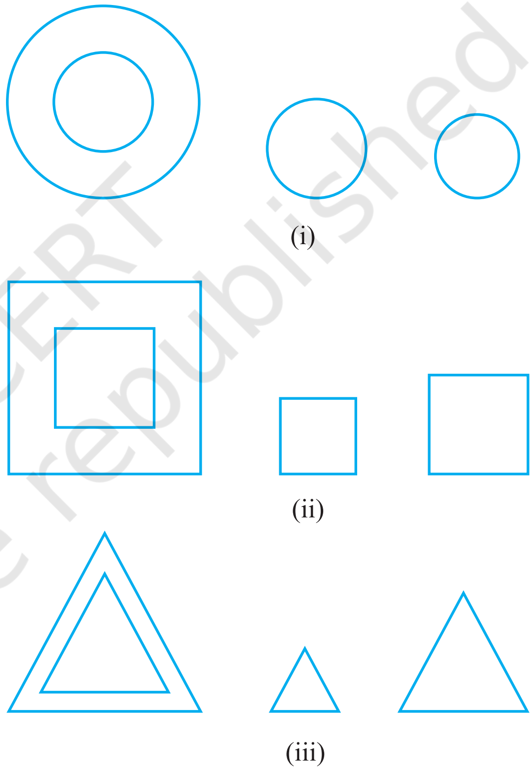


Fig. 6.1

From the above, we can say *that all congruent figures are similar but the similar figures need not be congruent.*

Can a circle and a square be similar? Can a triangle and a square be similar? These questions can be answered by just looking at the figures (see Fig. 6.1). Evidently these figures are not similar. (Why?)

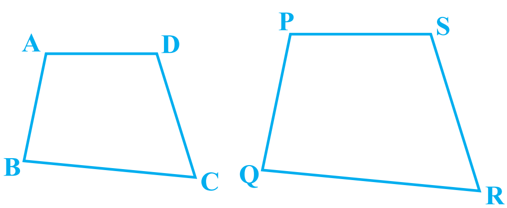


Fig. 6.2

What can you say about the two quadrilaterals ABCD and PQRS (see Fig 6.2)?Are they similar? These figures appear to be similar but we cannot be certain about it.Therefore, we must have some definition of similarity of figures and based on this definition some rules to decide whether the two given figures are similar or not. For this, let us look at the photographs given in Fig. 6.3:

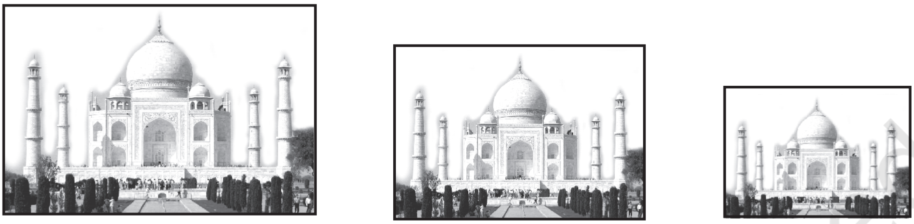


Fig. 6.3

You will at once say that they are the photographs of the same monument (Taj Mahal) but are in different sizes. Would you say that the three photographs are similar? Yes,they are.

What can you say about the two photographs of the same size of the same person one at the age of 10 years and the other at the age of 40 years? Are these photographs similar? These photographs are of the same size but certainly they are not of the same shape. So, they are not similar.

What does the photographer do when she prints photographs of different sizes from the same negative? You must have heard about the stamp size, passport size and postcard size photographs. She generally takes a photograph on a small size film, say of 35mm size and then enlarges it into a bigger size, say 45mm (or 55mm). Thus, if we consider any line segment in the smaller photograph (figure), its corresponding line segment in the bigger photograph (figure) will be $\frac{45}{35}\left(\text{or } \frac{55}{35}\right)$ of that of the line segment. This really means that every line segment of the smaller photograph is enlarged (increased) *in the ratio* 35:45 (or 35:55). It can also be said that every line segment of the bigger photograph is reduced (decreased) in the ratio 45:35 (or 55:35). Further, if you consider inclinations (or angles) between any pair of corresponding line segments in the two photographs of different sizes, you shall see that these inclinations(or angles) *are always equal.* This is the essence of the similarity of two figures and in particular of two polygons. We say that:

Two polygons of the same number of sides are similar, if (i) their corresponding angles are equal and (ii) their corresponding sides are in the same ratio (or proportion).

Note that the same ratio of the corresponding sides is referred to as *the scale factor* (or the *Representative Fraction*) for the polygons. You must have heard that world maps (i.e., global maps) and blue prints for the construction of a building are prepared using a suitable scale factor and observing certain conventions.

In order to understand similarity of figures more clearly, let us perform the following activity:

Activity 1 : Place a lighted bulb at a point O on the ceiling and directly below it a table in your classroom. Let us cut a polygon, say a quadrilateral ABCD, from a plane cardboard and place this cardboard parallel to the ground between the lighted bulb and the table. Then a shadow of ABCD is cast on the table. Mark the outline of this shadow as A′B′C′D′ (see Fig.6.4).

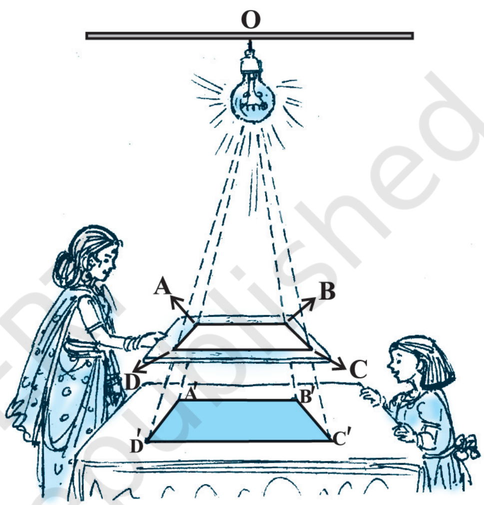


Fig. 6.4

Note that the quadrilateral A′B′C′D′ is an enlargement (or magnification) of the quadrilateral ABCD. This is because of the property of light that light propogates in a straight line. You may also note that A′ lies on ray OA, B′ lies on ray OB, C′ lies on OC and D′ lies on OD. Thus, quadrilaterals A′B′C′D′ and ABCD are of the same shape but of different sizes.

So, quadrilateral A′B′C′D′ is similiar to quadrilateral ABCD. We can also say that quadrilateral ABCD is similar to the quadrilateral A′B′C′D′.

Here, you can also note that vertex A′ corresponds to vertex A, vertex B′ corresponds to vertex B, vertex C′ corresponds to vertex C and vertex D′ corresponds to vertex D. Symbolically, these correspondences are represented as $A' \leftrightarrow A$, $B' \leftrightarrow B$, $C' \leftrightarrow C$ and $D' \leftrightarrow D$. By actually measuring the angles and the sides of the two quadrilaterals, you may verify that

(i) $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, $\angle D = \angle D'$ and

(ii) $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CD}{C'D'} = \dfrac{DA}{D'A'}$.

This again emphasises that *two polygons of the same number of sides are similar, if* (*i*) *all the corresponding angles are equal and* (*ii*) *all the corresponding sides are in the same ratio* (*or proportion*).

From the above, you can easily say that quadrilaterals ABCD and PQRS of Fig. 6.5 are similar.

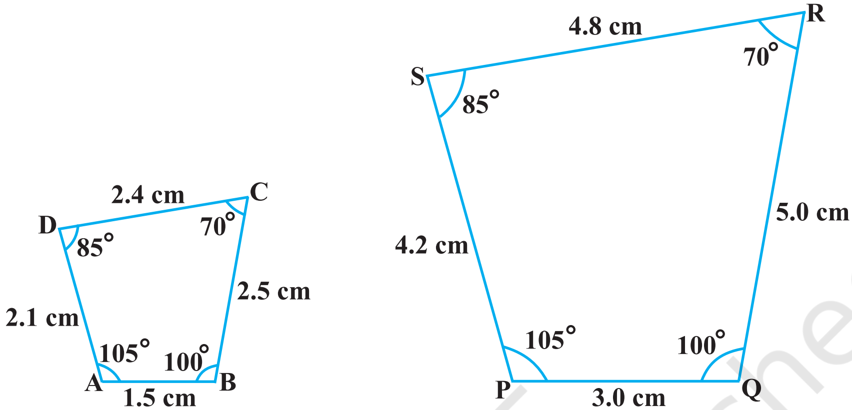


Fig. 6.5

Remark : You can verify that if one polygon is similar to another polygon and this second polygon is similar to a third polygon, then the first polygon is similar to the third polygon.

You may note that in the two quadrilaterals (a square and a rectangle) of Fig. 6.6, corresponding angles are equal, but their corresponding sides are not in the same ratio.

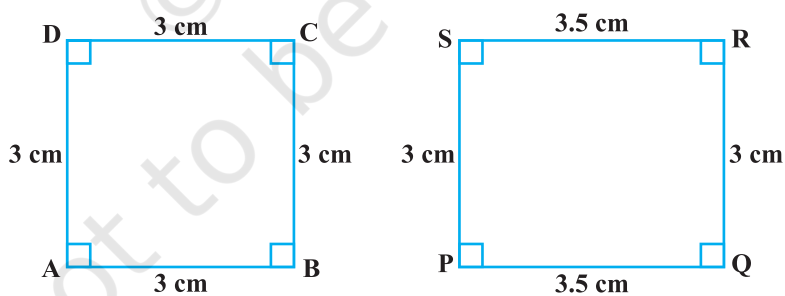


Fig. 6.6

So, the two quadrilaterals are not similar. Similarly, you may note that in the two quadrilaterals (a square and a rhombus) of Fig. 6.7, corresponding sides are in the same ratio, but their corresponding angles are not equal. Again, the two polygons (quadrilaterals) are not similar.

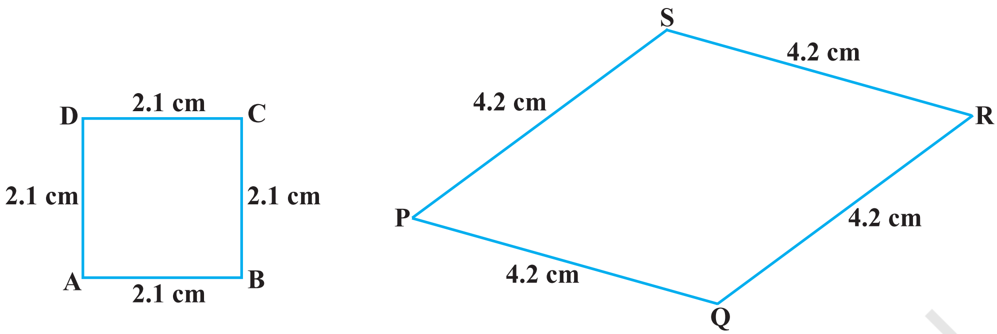


Fig. 6.7

Thus, *either of the above two conditions (i) and (ii) of similarity of two polygons is not sufficient for them to be similar.*

EXERCISE 6.1

1. Fill in the blanks using the correct word given in brackets :

(i) All circles are ________. (congruent, similar)

(ii) All squares are ________. (similar, congruent)

(iii) All ________ triangles are similar. (isosceles, equilateral)

(iv) Two polygons of the same number of sides are similar, if (a) their corresponding angles are ________ and (b) their corresponding sides are ________. (equal, proportional)

2. Give two different examples of pair of

(i) similar figures. (ii) non-similar figures.

3. State whether the following quadrilaterals are similar or not:

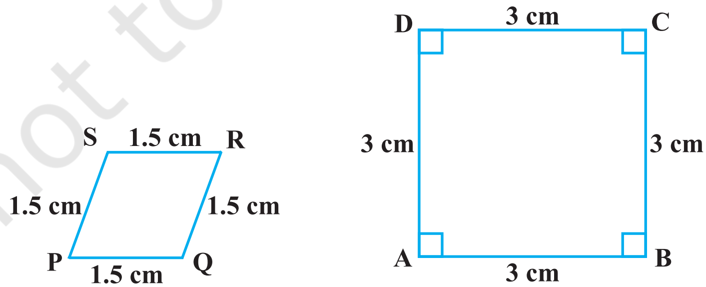


Fig. 6.8

6.3 Similarity of Triangles

What can you say about the similarity of two triangles?

You may recall that triangle is also a polygon. So, we can state the same conditions for the similarity of two triangles. That is:

Two triangles are similiar, if

(*i*) *their corresponding angles are equal and*

(*ii*) *their corresponding sides are in the same ratio* (*or proportion*).

Thales (640 – 546 B.C.)

Note that if corresponding angles of two triangles are equal, then they are known as *equiangular triangles*. A famous Greek mathematician Thales gave an important truth relating to two equiangular triangles which is as follows:

The ratio of any two corresponding sides in two equiangular triangles is always the same.

It is believed that he had used a result called the *Basic Proportionality Theorem* (now known as the *Thales Theorem*) for the same.

To understand the Basic Proportionality Theorem, let us perform the following activity:

Activity 2 : Draw any angle XAY and on its one arm AX, mark points (say five points) P, Q, D, R and B such that AP = PQ = QD = DR = RB.

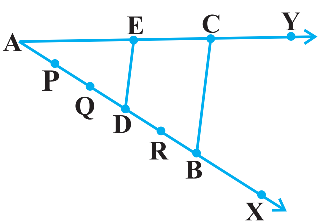


Fig. 6.9

Now, through B, draw any line intersecting arm AY at C (see Fig. 6.9).

Also, through the point D, draw a line parallel to BC to intersect AC at E. Do you observe from your constructions that $\frac{\text{AD}}{\text{DB}} = \frac{3}{2}$? Measure AE and EC. What about $\frac{\text{AE}}{\text{EC}}$? Observe that $\frac{\text{AE}}{\text{EC}}$ is also equal to $\frac{3}{2}$. Thus, you can see that in Δ ABC, DE || BC and $\frac{\text{AD}}{\text{DB}} = \frac{\text{AE}}{\text{EC}}$. Is it a coincidence? No, it is due to the following theorem (known as the Basic Proportionality Theorem):

Theorem 6.1 : *If a line is drawn parallel to one side of a triangle to intersect the other two sides in distinct points, the other two sides are divided in the same ratio.*

Proof : We are given a triangle ABC in which a line parallel to side BC intersects other two sides AB and AC at D and E respectively (see Fig. 6.10).

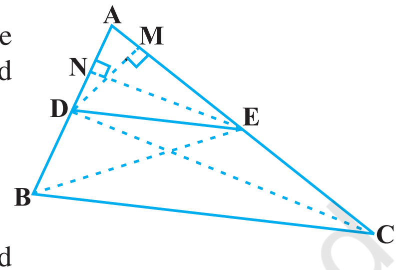


Fig. 6.10

We need to prove that $\frac{AD}{DB} = \frac{AE}{EC}$.

Let us join BE and CD and then draw DM $\perp$ AC and EN $\perp$ AB.

Now, area of Δ ADE (= $\frac{1}{2}$ base $\times$ height) = $\frac{1}{2}$ AD $\times$ EN.

Recall from Class IX, that area of Δ ADE is denoted as ar(ADE).

So, $\quad \text{ar(ADE)} = \frac{1}{2} \text{AD} \times \text{EN}$

Similarly, $\quad \text{ar(BDE)} = \frac{1}{2} \text{DB} \times \text{EN},$

$$\text{ar(ADE)} = \frac{1}{2} \text{AE} \times \text{DM} \text{ and } \text{ar(DEC)} = \frac{1}{2} \text{EC} \times \text{DM}.$$

Therefore, $$\frac{\text{ar(ADE)}}{\text{ar(BDE)}} = \frac{\frac{1}{2}\text{AD} \times \text{EN}}{\frac{1}{2}\text{DB} \times \text{EN}} = \frac{\text{AD}}{\text{DB}} \qquad (1)$$

and $$\frac{\text{ar(ADE)}}{\text{ar(DEC)}} = \frac{\frac{1}{2}\text{AE} \times \text{DM}}{\frac{1}{2}\text{EC} \times \text{DM}} = \frac{\text{AE}}{\text{EC}} \qquad (2)$$

Note that Δ BDE and DEC are on the same base DE and between the same parallels BC and DE.

So, $\quad \text{ar(BDE)} = \text{ar(DEC)} \qquad (3)$

Therefore, from (1), (2) and (3), we have :

$$\frac{AD}{DB} = \frac{AE}{EC}$$

Is the converse of this theorem also true (For the meaning of converse, see Appendix 1)? To examine this, let us perform the following activity:

Activity 3 : Draw an angle XAY on your notebook and on ray AX, mark points B_1, B_2, B_3, B_4 and B such that $AB_1 = B_1B_2 = B_2B_3 = B_3B_4 = B_4B$.

Similarly, on ray AY, mark points C_1, C_2, C_3, C_4 and C such that $AC_1 = C_1C_2 = C_2C_3 = C_3C_4 = C_4C$. Then join B_1C_1 and BC (see Fig. 6.11).

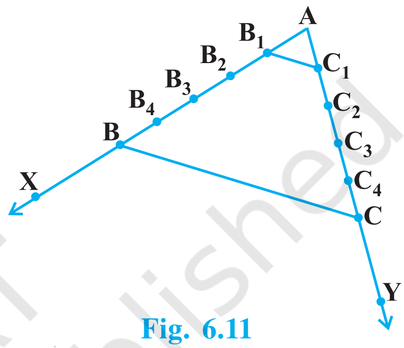


Fig. 6.11

Note that $\quad \dfrac{AB_1}{B_1B} = \dfrac{AC_1}{C_1C}$ (Each equal to $\dfrac{1}{4}$)

You can also see that lines B_1C_1 and BC are parallel to each other, i.e.,

$$B_1C_1 \parallel BC \qquad (1)$$

Similarly, by joining B_2C_2, B_3C_3 and B_4C_4, you can see that:

$$\frac{AB_2}{B_2B} = \frac{AC_2}{C_2C}\left(=\frac{2}{3}\right) \text{ and } B_2C_2 \parallel BC \qquad (2)$$

$$\frac{AB_3}{B_3B} = \frac{AC_3}{C_3C}\left(=\frac{3}{2}\right) \text{ and } B_3C_3 \parallel BC \qquad (3)$$

$$\frac{AB_4}{B_4B} = \frac{AC_4}{C_4C}\left(=\frac{4}{1}\right) \text{ and } B_4C_4 \parallel BC \qquad (4)$$

From (1), (2), (3) and (4), it can be observed that if a line divides two sides of a triangle in the same ratio, then the line is parallel to the third side.

You can repeat this activity by drawing any angle XAY of different measure and taking any number of equal parts on arms AX and AY . Each time, you will arrive at the same result. Thus, we obtain the following theorem, which is the converse of Theorem 6.1:

Theorem 6.2 : *If a line divides any two sides of a triangle in the same ratio, then the line is parallel to the third side.*

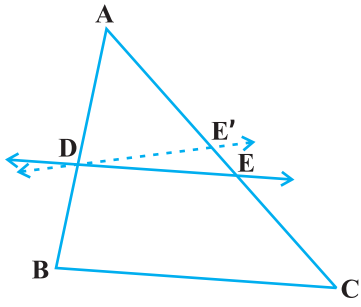


Fig. 6.12

This theorem can be proved by taking a line DE such that $\frac{AD}{DB} = \frac{AE}{EC}$ and assuming that DE is not parallel to BC (see Fig. 6.12).

If DE is not parallel to BC, draw a line DE′ parallel to BC.

So, $$\frac{AD}{DB} = \frac{AE'}{E'C} \quad \text{(Why ?)}$$

Therefore, $$\frac{AE}{EC} = \frac{AE'}{E'C} \quad \text{(Why ?)}$$

Adding 1 to both sides of above, you can see that E and E′ must coincide. (Why ?)

Let us take some examples to illustrate the use of the above theorems.

Example 1 : If a line intersects sides AB and AC of a Δ ABC at D and E respectively and is parallel to BC, prove that $\frac{AD}{AB} = \frac{AE}{AC}$ (see Fig. 6.13).

Solution : DE || BC (Given)

So, $$\frac{AD}{DB} = \frac{AE}{EC} \quad \text{(Theorem 6.1)}$$

or, $$\frac{DB}{AD} = \frac{EC}{AE}$$

or, $$\frac{DB}{AD} + 1 = \frac{EC}{AE} + 1$$

or, $$\frac{AB}{AD} = \frac{AC}{AE}$$

So, $$\frac{AD}{AB} = \frac{AE}{AC}$$

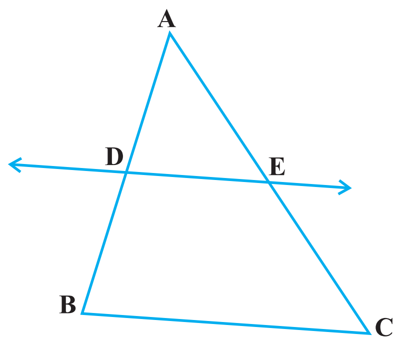


Fig. 6.13

Example 2 : ABCD is a trapezium with AB || DC. E and F are points on non-parallel sides AD and BC respectively such that EF is parallel to AB (see Fig. 6.14). Show that $\frac{AE}{ED} = \frac{BF}{FC}$.

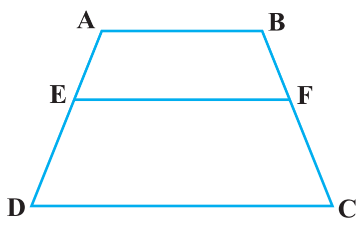


Fig. 6.14

Solution : Let us join AC to intersect EF at G (see Fig. 6.15).

AB || DC and EF || AB (Given)

So, EF || DC (Lines parallel to the same line are parallel to each other)

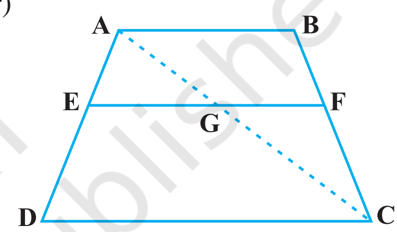


Fig. 6.15

Now, in Δ ADC,

EG || DC (As EF || DC)

So, $\frac{AE}{ED} = \frac{AG}{GC}$ (Theorem 6.1) (1)

Similarly, from Δ CAB,

$$\frac{CG}{AG} = \frac{CF}{BF}$$

i.e., $$\frac{AG}{GC} = \frac{BF}{FC} \quad (2)$$

Therefore, from (1) and (2),

$$\frac{AE}{ED} = \frac{BF}{FC}$$

Example 3 : In Fig. 6.16, $\frac{PS}{SQ} = \frac{PT}{TR}$ and ∠ PST = ∠ PRQ. Prove that PQR is an isosceles triangle.

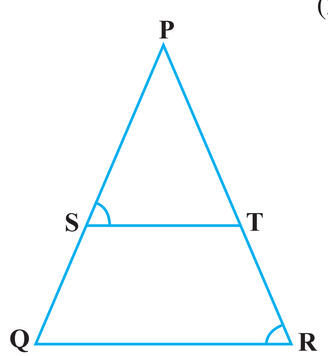


Fig. 6.16

Solution : It is given that $\frac{PS}{SQ} = \frac{PT}{TR}$.

So, ST || QR (Theorem 6.2)

Therefore, ∠ PST = ∠ PQR (Corresponding angles) (1)

Also, it is given that

$$\angle PST = \angle PRQ \qquad (2)$$

So, $\angle PRQ = \angle PQR$ [From (1) and (2)]

Therefore, PQ = PR (Sides opposite the equal angles)

i.e., PQR is an isosceles triangle.

EXERCISE 6.2

1. In Fig. 6.17, (i) and (ii), DE || BC. Find EC in (i) and AD in (ii).

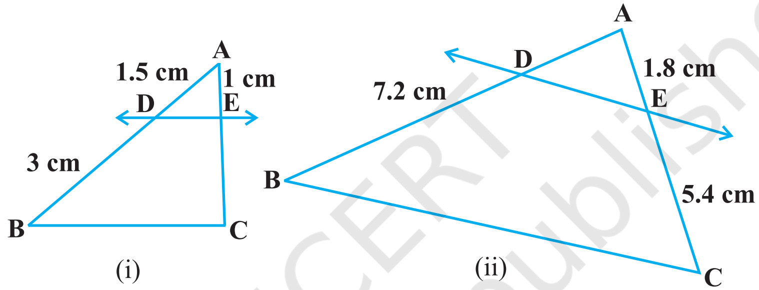


Fig. 6.17

2. E and F are points on the sides PQ and PR respectively of a Δ PQR. For each of the following cases, state whether EF || QR :

(i) PE = 3.9 cm, EQ = 3 cm, PF = 3.6 cm and FR = 2.4 cm

(ii) PE = 4 cm, QE = 4.5 cm, PF = 8 cm and RF = 9 cm

(iii) PQ = 1.28 cm, PR = 2.56 cm, PE = 0.18 cm and PF = 0.36 cm

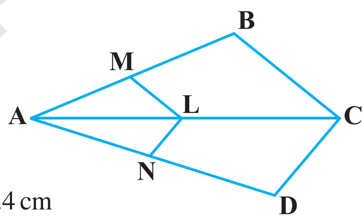


Fig. 6.18

3. In Fig. 6.18, if LM || CB and LN || CD, prove that

$$\frac{AM}{AB} = \frac{AN}{AD}.$$

4. In Fig. 6.19, DE || AC and DF || AE. Prove that

$$\frac{BF}{FE} = \frac{BE}{EC}.$$

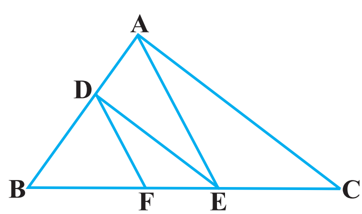


Fig. 6.19

5. In Fig. 6.20, DE || OQ and DF || OR. Show that EF || QR.

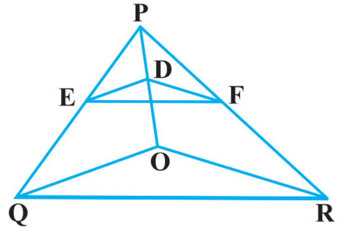


Fig. 6.20

6. In Fig. 6.21, A, B and C are points on OP, OQ and OR respectively such that AB || PQ and AC || PR. Show that BC || QR.

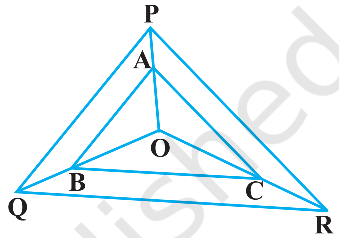


Fig. 6.21

7. Using Theorem 6.1, prove that a line drawn through the mid-point of one side of a triangle parallel to another side bisects the third side. (Recall that you have proved it in Class IX).

8. Using Theorem 6.2, prove that the line joining the mid-points of any two sides of a triangle is parallel to the third side. (Recall that you have done it in Class IX).

9. ABCD is a trapezium in which AB || DC and its diagonals intersect each other at the point O. Show that $\frac{AO}{BO} = \frac{CO}{DO}$.

10. The diagonals of a quadrilateral ABCD intersect each other at the point O such that $\frac{AO}{BO} = \frac{CO}{DO}$. Show that ABCD is a trapezium.

6.4 Criteria for Similarity of Triangles

In the previous section, we stated that two triangles are similar, if (i) their corresponding angles are equal and (ii) their corresponding sides are in the same ratio (or proportion).

That is, in Δ ABC and Δ DEF, if

(i) $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$ and

(ii) $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$, then the two triangles are similar (see Fig. 6.22).

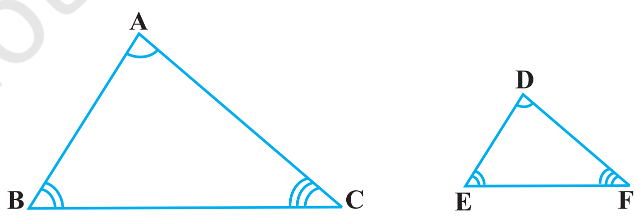


Fig. 6.22

Here, you can see that A corresponds to D, B corresponds to E and C corresponds to F. Symbolically, we write the similarity of these two triangles as 'Δ ABC ~ Δ DEF' and read it as 'triangle ABC is similar to triangle DEF'. The symbol '~' stands for 'is similar to'. Recall that you have used the symbol '≅' for 'is congruent to' in Class IX.

It must be noted that as done in the case of congruency of two triangles, the similarity of two triangles should also be expressed symbolically, using correct correspondence of their vertices. For example, for the triangles ABC and DEF of Fig. 6.22, we cannot write Δ ABC ~ Δ EDF or Δ ABC ~ Δ FED. However, we can write Δ BAC ~ Δ EDF.

Now a natural question arises : For checking the similarity of two triangles, say ABC and DEF, should we always look for all the equality relations of their corresponding angles ($\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$) and all the equality relations of the ratios of their corresponding sides $\left(\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}\right)$? Let us examine. You may recall that in Class IX, you have obtained some criteria for congruency of two triangles involving only three pairs of corresponding parts (or elements) of the two triangles. Here also, let us make an attempt to arrive at certain criteria for similarity of two triangles involving relationship between less number of pairs of corresponding parts of the two triangles, instead of all the six pairs of corresponding parts. For this, let us perform the following activity:

Activity 4 : Draw two line segments BC and EF of two different lengths, say 3 cm and 5 cm respectively. Then, at the points B and C respectively, construct angles PBC and QCB of some measures, say, 60° and 40°. Also, at the points E and F, construct angles REF and SFE of 60° and 40° respectively (see Fig. 6.23).

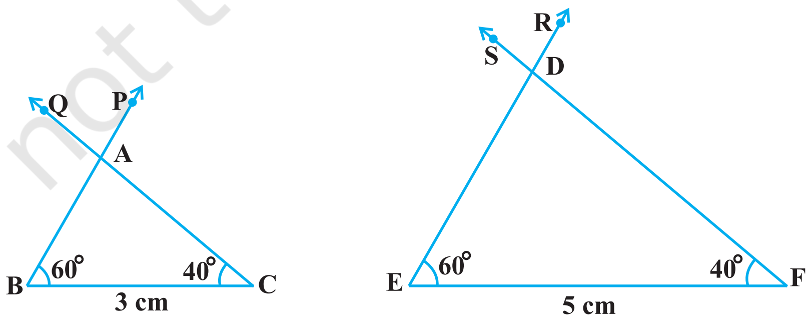


Fig. 6.23

Let rays BP and CQ intersect each other at A and rays ER and FS intersect each other at D. In the two triangles ABC and DEF, you can see that $\angle B = \angle E$, $\angle C = \angle F$ and $\angle A = \angle D$. That is, corresponding angles of these two triangles are equal. What can you say about their corresponding sides ? Note that $\frac{BC}{EF} = \frac{3}{5} = 0.6.$ What about $\frac{AB}{DE}$ and $\frac{CA}{FD}$? On measuring AB, DE, CA and FD, you will find that $\frac{AB}{DE}$ and $\frac{CA}{FD}$ are also equal to 0.6 (or nearly equal to 0.6, if there is some error in the measurement). Thus, $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}.$ You can repeat this activity by constructing several pairs of triangles having their corresponding angles equal. Every time, you will find that their corresponding sides are in the same ratio (or proportion). This activity leads us to the following criterion for similarity of two triangles.

Theorem 6.3 : *If in two triangles, corresponding angles are equal, then their corresponding sides are in the same ratio (or proportion) and hence the two triangles are similar.*

This criterion is referred to as the AAA (Angle–Angle–Angle) criterion of similarity of two triangles.

This theorem can be proved by taking two triangles ABC and DEF such that $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$ (see Fig. 6.24)

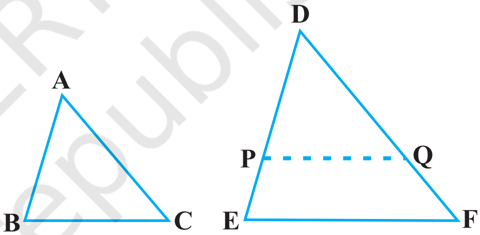


Fig. 6.24

Cut DP = AB and DQ = AC and join PQ.

So, $\Delta\, ABC \cong \Delta\, DPQ$ (Why ?)

This gives $\angle B = \angle P = \angle E$ and $PQ \parallel EF$ (How?)

Therefore, $$\frac{DP}{PE} = \frac{DQ}{QF} \quad \text{(Why?)}$$

i.e., $$\frac{AB}{DE} = \frac{AC}{DF} \quad \text{(Why?)}$$

Similarly, $\frac{AB}{DE} = \frac{BC}{EF}$ and so $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$.

Remark : If two angles of a triangle are respectively equal to two angles of another triangle, then by the angle sum property of a triangle their third angles will also be equal. Therefore, AAA similarity criterion can also be stated as follows:

If two angles of one triangle are respectively equal to two angles of another triangle, then the two triangles are similar.

This may be referred to as the AA *similarity criterion* for two triangles.

You have seen above that if the three angles of one triangle are respectively equal to the three angles of another triangle, then their corresponding sides are proportional (i.e., in the same ratio). What about the converse of this statement? Is the converse true? In other words, if the sides of a triangle are respectively proportional to the sides of another triangle, is it true that their corresponding angles are equal? Let us examine it through an activity :

Activity 5 : Draw two triangles ABC and DEF such that AB = 3 cm, BC = 6 cm, CA = 8 cm, DE = 4.5 cm, EF = 9 cm and FD = 12 cm (see Fig. 6.25).

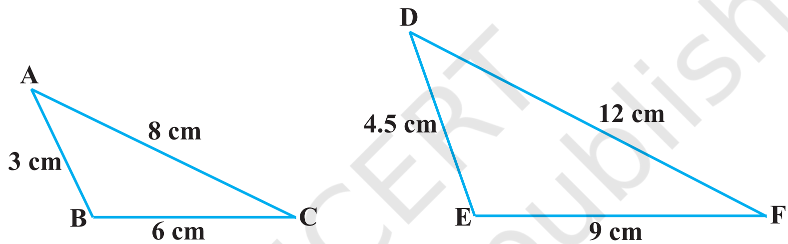


Fig. 6.25

So, you have : $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$ (each equal to $\frac{2}{3}$)

Now measure ∠ A, ∠ B, ∠ C, ∠ D, ∠ E and ∠ F. You will observe that ∠ A = ∠ D, ∠ B = ∠ E and ∠ C = ∠ F, i.e., the corresponding angles of the two triangles are equal.

You can repeat this activity by drawing several such triangles (having their sides in the same ratio). Everytime you shall see that their corresponding angles are equal. It is due to the following criterion of similarity of two triangles:

Theorem 6.4 : *If in two triangles, sides of one triangle are proportional to (i.e., in the same ratio of) the sides of the other triangle, then their corresponding angles are equal and hence the two triangles are similiar.*

This criterion is referred to as the SSS (Side–Side–Side) *similarity criterion for two triangles*.

This theorem can be proved by taking two triangles ABC and DEF such that $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$ (< 1) (see Fig. 6.26):

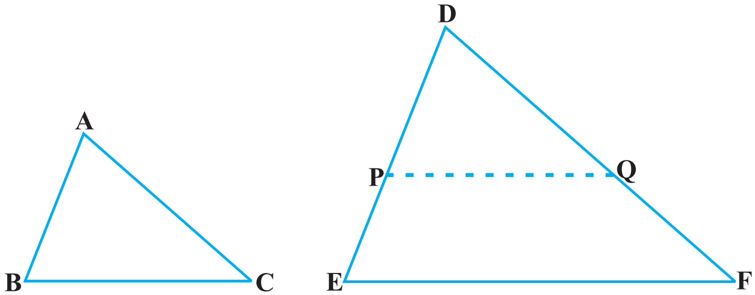


Fig. 6.26

Cut DP = AB and DQ = AC and join PQ.

It can be seen that $\frac{DP}{PE} = \frac{DQ}{QF}$ and PQ || EF (How?)

So, $\angle P = \angle E$ and $\angle Q = \angle F$.

Therefore, $\frac{DP}{DE} = \frac{DQ}{DF} = \frac{PQ}{EF}$

So, $\frac{DP}{DE} = \frac{DQ}{DF} = \frac{BC}{EF}$ (Why?)

So, BC = PQ (Why?)

Thus, Δ ABC $\cong$ Δ DPQ (Why ?)

So, $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$ (How ?)

Remark : You may recall that either of the two conditions namely, (i) corresponding angles are equal and (ii) corresponding sides are in the same ratio is not sufficient for two polygons to be similar. However, on the basis of Theorems 6.3 and 6.4, you can now say that in case of similarity of the two triangles, it is not necessary to check both the conditions as one condition implies the other.

Let us now recall the various criteria for congruency of two triangles learnt in Class IX. You may observe that SSS similarity criterion can be compared with the SSS congruency criterion.This suggests us to look for a similarity criterion comparable to SAS congruency criterion of triangles. For this, let us perform an activity.

Activity 6 : Draw two triangles ABC and DEF such that AB = 2 cm, $\angle$ A = 50°, AC = 4 cm, DE = 3 cm, $\angle$ D = 50° and DF = 6 cm (see Fig.6.27).

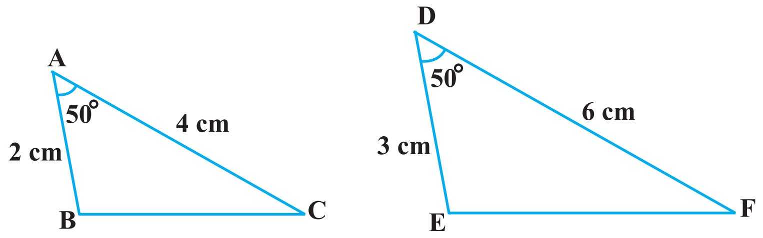


Fig. 6.27

Here, you may observe that $\frac{AB}{DE} = \frac{AC}{DF}$ (each equal to $\frac{2}{3}$) and $\angle$ A (included between the sides AB and AC) = $\angle$ D (included between the sides DE and DF). That is, one angle of a triangle is equal to one angle of another triangle and sides including these angles are in the same ratio (i.e., proportion). Now let us measure $\angle$ B, $\angle$ C, $\angle$ E and $\angle$ F.

You will find that $\angle$ B = $\angle$ E and $\angle$ C = $\angle$ F. That is, $\angle$ A = $\angle$ D, $\angle$ B = $\angle$ E and $\angle$ C = $\angle$ F. So, by AAA similarity criterion, Δ ABC ~ Δ DEF. You may repeat this activity by drawing several pairs of such triangles with one angle of a triangle equal to one angle of another triangle and the sides including these angles are proportional. Everytime, you will find that the triangles are similar. It is due to the following criterion of similarity of triangles:

Theorem 6.5 : *If one angle of a triangle is equal to one angle of the other triangle and the sides including these angles are proportional, then the two triangles are similar.*

This criterion is referred to as the SAS (Side–Angle–Side) similarity criterion for two triangles.

As before, this theorem can be proved by taking two triangles ABC and DEF such that $\frac{AB}{DE} = \frac{AC}{DF}$ (< 1) and $\angle$ A = $\angle$ D (see Fig. 6.28). Cut DP = AB, DQ = AC and join PQ.

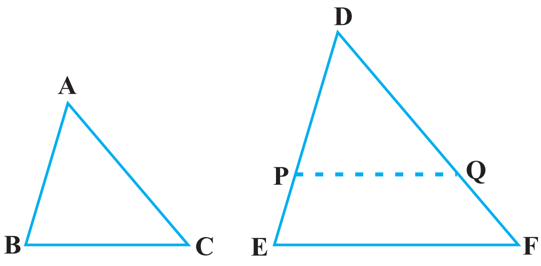


Fig. 6.28

Now, $PQ \parallel EF$ and $\Delta\ ABC \cong \Delta\ DPQ$ (How ?)

So, $\angle A = \angle D,\ \angle B = \angle P$ and $\angle C = \angle Q$

Therefore, $\Delta\ ABC \sim \Delta\ DEF$ (Why?)

We now take some examples to illustrate the use of these criteria.

Example 4 : In Fig. 6.29, if $PQ \parallel RS$, prove that $\Delta\ POQ \sim \Delta\ SOR$.

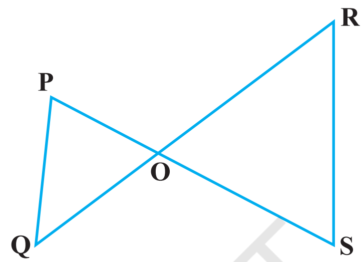


Fig. 6.29

Solution : $PQ \parallel RS$ (Given)

So, $\angle P = \angle S$ (Alternate angles)

and $\angle Q = \angle R$

Also, $\angle POQ = \angle SOR$ (Vertically opposite angles)

Therefore, $\Delta\ POQ \sim \Delta\ SOR$ (AAA similarity criterion)

Example 5 : Observe Fig. 6.30 and then find $\angle P$.

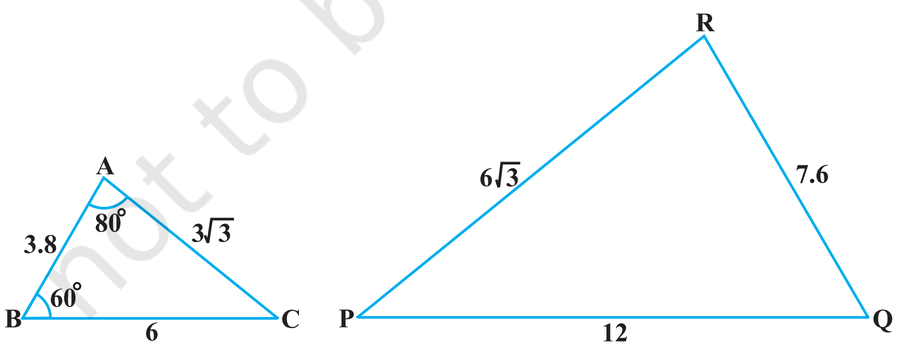


Fig. 6.30

Solution : In $\Delta\ ABC$ and $\Delta\ PQR$,

$$\frac{AB}{RQ} = \frac{3.8}{7.6} = \frac{1}{2},\ \frac{BC}{QP} = \frac{6}{12} = \frac{1}{2} \text{ and } \frac{CA}{PR} = \frac{3\sqrt{3}}{6\sqrt{3}} = \frac{1}{2}$$

That is, $\frac{AB}{RQ} = \frac{BC}{QP} = \frac{CA}{PR}$

So, $\Delta\,ABC \sim \Delta\,RQP$ (SSS similarity)

Therefore, $\angle C = \angle P$ (Corresponding angles of similar triangles)

But $\angle C = 180° - \angle A - \angle B$ (Angle sum property)

$= 180° - 80° - 60° = 40°$

So, $\angle P = 40°$

Example 6 : In Fig. 6.31,

$OA \cdot OB = OC \cdot OD.$

Show that $\angle A = \angle C$ and $\angle B = \angle D$.

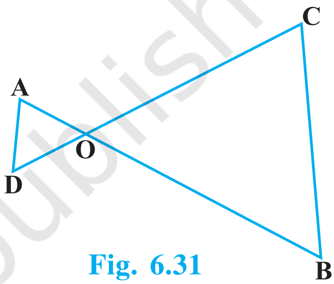


Fig. 6.31

Solution : $OA \cdot OB = OC \cdot OD$ (Given)

So, $\frac{OA}{OC} = \frac{OD}{OB}$ (1)

Also, we have $\angle AOD = \angle COB$ (Vertically opposite angles) (2)

Therefore, from (1) and (2), $\Delta\,AOD \sim \Delta\,COB$ (SAS similarity criterion)

So, $\angle A = \angle C$ and $\angle D = \angle B$

(Corresponding angles of similar triangles)

Example 7 : A girl of height 90 cm is walking away from the base of a lamp-post at a speed of 1.2 m/s. If the lamp is 3.6 m above the ground, find the length of her shadow after 4 seconds.

Solution : Let AB denote the lamp-post and CD the girl after walking for 4 seconds away from the lamp-post (see Fig. 6.32).

From the figure, you can see that DE is the shadow of the girl. Let DE be x metres.

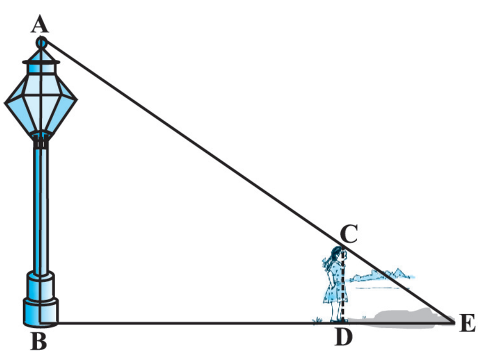


Fig. 6.32

Now, BD = 1.2 m × 4 = 4.8 m.

Note that in Δ ABE and Δ CDE,

$\angle B = \angle D$ (Each is of 90° because lamp-post as well as the girl are standing vertical to the ground)

and $\angle E = \angle E$ (Same angle)

So, $\Delta\, ABE \sim \Delta\, CDE$ (AA similarity criterion)

Therefore, $$\frac{BE}{DE} = \frac{AB}{CD}$$

i.e., $$\frac{4.8 + x}{x} = \frac{3.6}{0.9} \qquad \left(90 \text{ cm} = \frac{90}{100} \text{ m} = 0.9 \text{ m}\right)$$

i.e., $4.8 + x = 4x$

i.e., $3x = 4.8$

i.e., $x = 1.6$

So, the shadow of the girl after walking for 4 seconds is 1.6 m long.

Example 8 : In Fig. 6.33, CM and RN are respectively the medians of Δ ABC and Δ PQR. If Δ ABC ~ Δ PQR, prove that :

(i) Δ AMC ~ Δ PNR

(ii) $\frac{CM}{RN} = \frac{AB}{PQ}$

(iii) Δ CMB ~ Δ RNQ

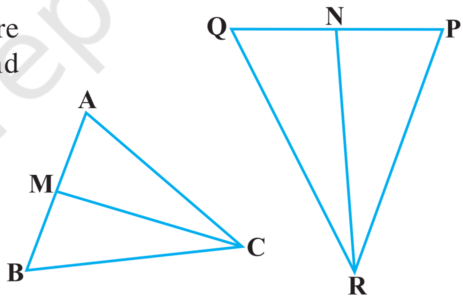


Fig. 6.33

Solution : (i) $\Delta\, ABC \sim \Delta\, PQR$ (Given)

So, $$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP} \qquad (1)$$

and $\angle A = \angle P,\ \angle B = \angle Q$ and $\angle C = \angle R$ (2)

But AB = 2 AM and PQ = 2 PN

(As CM and RN are medians)

So, from (1), $$\frac{2\,AM}{2\,PN} = \frac{CA}{RP}$$

i.e., $$\frac{AM}{PN} = \frac{CA}{RP} \quad (3)$$

Also, $$\angle MAC = \angle NPR \quad \text{[From (2)]} \quad (4)$$

So, from (3) and (4),

$$\Delta\ AMC \sim \Delta\ PNR \quad \text{(SAS similarity)} \quad (5)$$

(ii) From (5), $$\frac{CM}{RN} = \frac{CA}{RP} \quad (6)$$

But $$\frac{CA}{RP} = \frac{AB}{PQ} \quad \text{[From (1)]} \quad (7)$$

Therefore, $$\frac{CM}{RN} = \frac{AB}{PQ} \quad \text{[From (6) and (7)]} \quad (8)$$

(iii) Again, $$\frac{AB}{PQ} = \frac{BC}{QR} \quad \text{[From (1)]}$$

Therefore, $$\frac{CM}{RN} = \frac{BC}{QR} \quad \text{[From (8)]} \quad (9)$$

Also, $$\frac{CM}{RN} = \frac{AB}{PQ} = \frac{2\ BM}{2\ QN}$$

i.e., $$\frac{CM}{RN} = \frac{BM}{QN} \quad (10)$$

i.e., $$\frac{CM}{RN} = \frac{BC}{QR} = \frac{BM}{QN} \quad \text{[From (9) and (10)]}$$

Therefore, $$\Delta\ CMB \sim \Delta\ RNQ \quad \text{(SSS similarity)}$$

[**Note :** You can also prove part (iii) by following the same method as used for proving part (i).]

EXERCISE 6.3

1. State which pairs of triangles in Fig. 6.34 are similar. Write the similarity criterion used by you for answering the question and also write the pairs of similar triangles in the symbolic form :

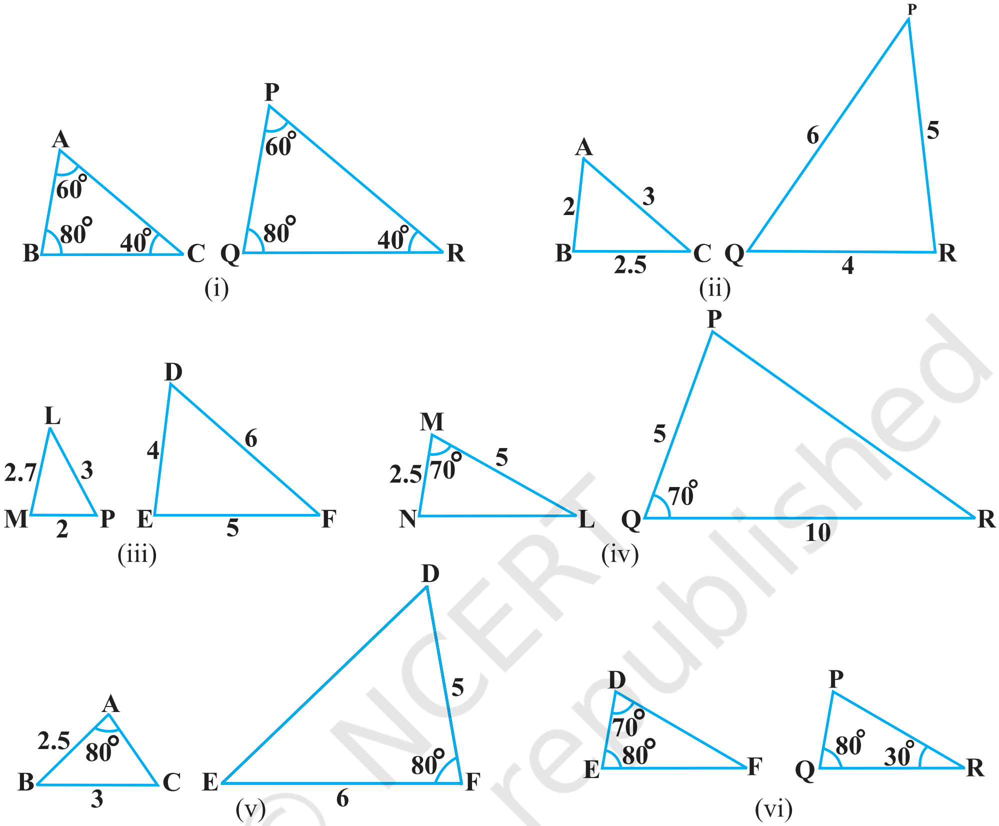


Fig. 6.34

2. In Fig. 6.35, $\Delta\, ODC \sim \Delta\, OBA$, $\angle\, BOC = 125°$ and $\angle\, CDO = 70°$. Find $\angle\, DOC$, $\angle\, DCO$ and $\angle\, OAB$.

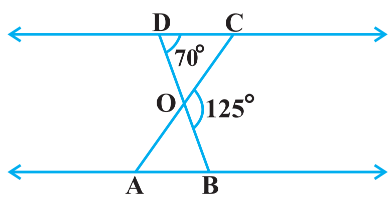


Fig. 6.35

3. Diagonals AC and BD of a trapezium ABCD with AB || DC intersect each other at the point O. Using a similarity criterion for two triangles, show that $\dfrac{OA}{OC} = \dfrac{OB}{OD}$.

4. In Fig. 6.36, $\frac{QR}{QS} = \frac{QT}{PR}$ and $\angle 1 = \angle 2$. Show that Δ PQS ~ Δ TQR.

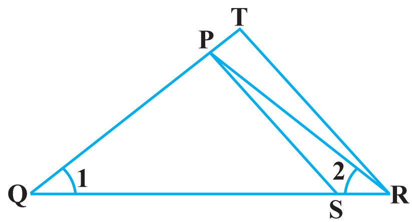


Fig. 6.36

5. S and T are points on sides PR and QR of Δ PQR such that $\angle$ P = $\angle$ RTS. Show that Δ RPQ ~ Δ RTS.

6. In Fig. 6.37, if Δ ABE $\cong$ Δ ACD, show that Δ ADE ~ Δ ABC.

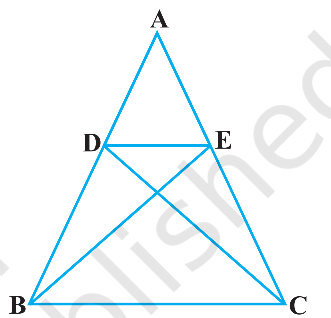


Fig. 6.37

7. In Fig. 6.38, altitudes AD and CE of Δ ABC intersect each other at the point P. Show that:

(i) Δ AEP ~ Δ CDP

(ii) Δ ABD ~ Δ CBE

(iii) Δ AEP ~ Δ ADB

(iv) Δ PDC ~ Δ BEC

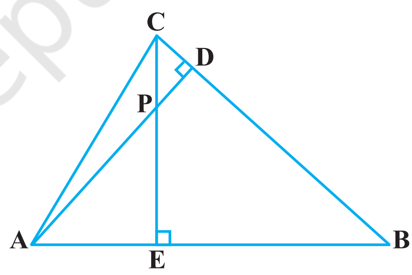


Fig. 6.38

8. E is a point on the side AD produced of a parallelogram ABCD and BE intersects CD at F. Show that Δ ABE ~ Δ CFB.

9. In Fig. 6.39, ABC and AMP are two right triangles, right angled at B and M respectively. Prove that:

(i) Δ ABC ~ Δ AMP

(ii) $\frac{CA}{PA} = \frac{BC}{MP}$

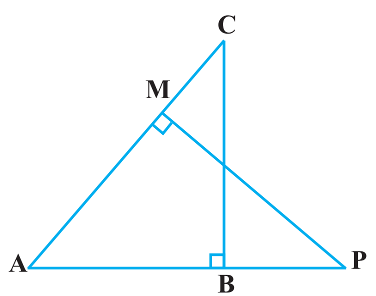


Fig. 6.39

10. CD and GH are respectively the bisectors of $\angle$ACB and $\angle$EGF such that D and H lie on sides AB and FE of Δ ABC and Δ EFG respectively. If Δ ABC ~ Δ FEG, show that:

(i) $\frac{CD}{GH} = \frac{AC}{FG}$

(ii) Δ DCB ~ Δ HGE

(iii) Δ DCA ~ Δ HGF

11. In Fig. 6.40, E is a point on side CB produced of an isosceles triangle ABC with AB = AC. If AD ⊥ BC and EF ⊥ AC, prove that Δ ABD ~ Δ ECF.

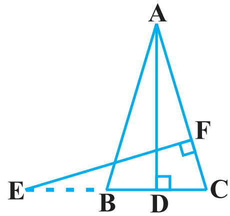


Fig. 6.40

12. Sides AB and BC and median AD of a triangle ABC are respectively proportional to sides PQ and QR and median PM of Δ PQR (see Fig. 6.41). Show that Δ ABC ~ Δ PQR.

13. D is a point on the side BC of a triangle ABC such that ∠ ADC = ∠ BAC. Show that $CA^2 = CB.CD$.

14. Sides AB and AC and median AD of a triangle ABC are respectively proportional to sides PQ and PR and median PM of another triangle PQR. Show that Δ ABC ~ Δ PQR.

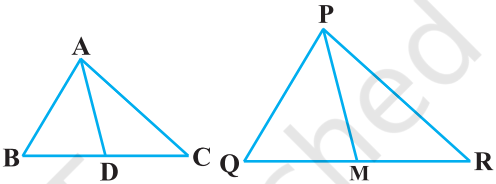


Fig. 6.41

15. A vertical pole of length 6 m casts a shadow 4 m long on the ground and at the same time a tower casts a shadow 28 m long. Find the height of the tower.

16. If AD and PM are medians of triangles ABC and PQR, respectively where

Δ ABC ~ Δ PQR, prove that $\dfrac{AB}{PQ} = \dfrac{AD}{PM}$.

6.5 Summary

In this chapter you have studied the following points :

1. Two figures having the same shape but not necessarily the same size are called similar figures.
2. All the congruent figures are similar but the converse is not true.
3. Two polygons of the same number of sides are similar, if (i) their corresponding angles are equal and (ii) their corresponding sides are in the same ratio (i.e., proportion).
4. If a line is drawn parallel to one side of a triangle to intersect the other two sides in distinct points, then the other two sides are divided in the same ratio.
5. If a line divides any two sides of a triangle in the same ratio, then the line is parallel to the third side.
6. If in two triangles, corresponding angles are equal, then their corresponding sides are in the same ratio and hence the two triangles are similar (AAA similarity criterion).
7. If in two triangles, two angles of one triangle are respectively equal to the two angles of the other triangle, then the two triangles are similar (AA similarity criterion).

8. If in two triangles, corresponding sides are in the same ratio, then their corresponding angles are equal and hence the triangles are similar (SSS similarity criterion).
9. If one angle of a triangle is equal to one angle of another triangle and the sides including these angles are in the same ratio (proportional), then the triangles are similar (SAS similarity criterion).

A NOTE TO THE READER

If in two right triangles, hypotenuse and one side of one triangle are proportional to the hypotenuse and one side of the other triangle, then the two triangles are similar. This may be referred to as the RHS Similarity Criterion.

If you use this criterion in Example 2, Chapter 8, the proof will become simpler.

1062CH07

COORDINATE GEOMETRY 7

7.1 Introduction

In Class IX, you have studied that to locate the position of a point on a plane, we require a pair of coordinate axes. The distance of a point from the y-axis is called its ***x*-coordinate**, or **abscissa**. The distance of a point from the x-axis is called its ***y*-coordinate**, or **ordinate**. The coordinates of a point on the x-axis are of the form $(x, 0)$, and of a point on the y-axis are of the form $(0, y)$.

Here is a play for you. Draw a set of a pair of perpendicular axes on a graph paper. Now plot the following points and join them as directed: Join the point A(4, 8) to B(3, 9) to C(3, 8) to D(1, 6) to E(1, 5) to F(3, 3) to G(6, 3) to H(8, 5) to I(8, 6) to J(6, 8) to K(6, 9) to L(5, 8) to A. Then join the points P(3.5, 7), Q (3, 6) and R(4, 6) to form a triangle. Also join the points X(5.5, 7), Y(5, 6) and Z(6, 6) to form a triangle. Now join S(4, 5), T(4.5, 4) and U(5, 5) to form a triangle. Lastly join S to the points (0, 5) and (0, 6) and join U to the points (9, 5) and (9, 6). What picture have you got?

Also, you have seen that a linear equation in two variables of the form $ax + by + c = 0$, (a, b are not simultaneously zero), when represented graphically, gives a straight line. Further, in Chapter 2, you have seen the graph of $y = ax^2 + bx + c$ ($a \neq 0$), is a parabola. In fact, coordinate geometry has been developed as an algebraic tool for studying geometry of figures. It helps us to study geometry using algebra, and understand algebra with the help of geometry. Because of this, coordinate geometry is widely applied in various fields such as physics, engineering, navigation, seismology and art!

In this chapter, you will learn how to find the distance between the two points whose coordinates are given. You will also study how to find the coordinates of the point which divides a line segment joining two given points in a given ratio.

7.2 Distance Formula

Let us consider the following situation:

A town B is located 36 km east and 15 km north of the town A. How would you find the distance from town A to town B without actually measuring it. Let us see. This situation can be represented graphically as shown in Fig. 7.1. You may use the Pythagoras Theorem to calculate this distance.

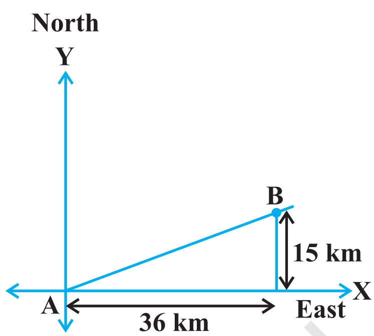


Fig. 7.1

Now, suppose two points lie on the x-axis. Can we find the distance between them? For instance, consider two points A(4, 0) and B(6, 0) in Fig. 7.2. The points A and B lie on the x-axis.

From the figure you can see that OA = 4 units and OB = 6 units.

Therefore, the distance of B from A, i.e., AB = OB – OA = 6 – 4 = 2 units.

So, if two points lie on the x-axis, we can easily find the distance between them.

Now, suppose we take two points lying on the y-axis. Can you find the distance between them. If the points C(0, 3) and D(0, 8) lie on the y-axis, similarly we find that CD = 8 – 3 = 5 units (see Fig. 7.2).

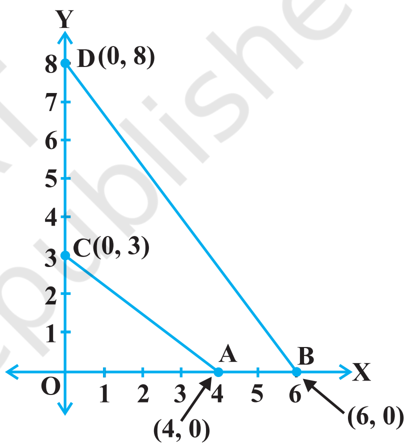


Fig. 7.2

Next, can you find the distance of A from C (in Fig. 7.2)? Since OA = 4 units and OC = 3 units, the distance of A from C, i.e., AC = $\sqrt{3^2 + 4^2}$ = 5 units. Similarly, you can find the distance of B from D = BD = 10 units.

Now, if we consider two points not lying on coordinate axis, can we find the distance between them? Yes! We shall use Pythagoras theorem to do so. Let us see an example.

In Fig. 7.3, the points P(4, 6) and Q(6, 8) lie in the first quadrant. How do we use Pythagoras theorem to find the distance between them? Let us draw PR and QS perpendicular to the x-axis from P and Q respectively. Also, draw a perpendicular from P on QS to meet QS at T. Then the coordinates of R and S are (4, 0) and (6, 0), respectively. So, RS = 2 units. Also, QS = 8 units and TS = PR = 6 units.

Therefore, QT = 2 units and PT = RS = 2 units.

Now, using the Pythagoras theorem, we have

$$PQ^2 = PT^2 + QT^2$$

$$= 2^2 + 2^2 = 8$$

So, $PQ = 2\sqrt{2}$ units

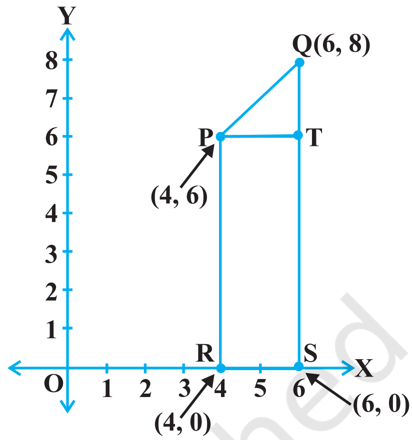


Fig. 7.3

How will we find the distance between two points in two different quadrants?

Consider the points P(6, 4) and Q(–5, –3) (see Fig. 7.4). Draw QS perpendicular to the *x*-axis. Also draw a perpendicular PT from the point P on QS (extended) to meet *y*-axis at the point R.

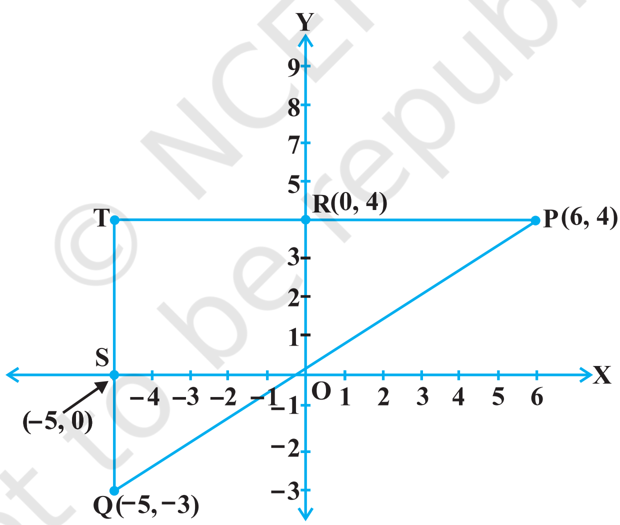


Fig. 7.4

Then PT = 11 units and QT = 7 units. (Why?)

Using the Pythagoras Theorem to the right triangle PTQ, we get $PQ = \sqrt{11^2 + 7^2} = \sqrt{170}$ units.

Let us now find the distance between any two points $P(x_1, y_1)$ and $Q(x_2, y_2)$. Draw PR and QS perpendicular to the x-axis. A perpendicular from the point P on QS is drawn to meet it at the point T (see Fig. 7.5).

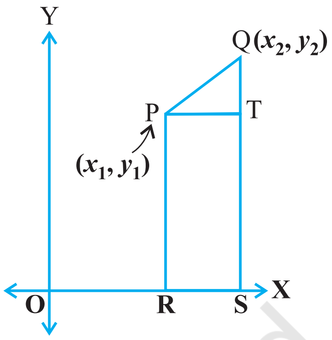


Fig. 7.5

Then, $OR = x_1$, $OS = x_2$. So, $RS = x_2 - x_1 = PT$.

Also, $SQ = y_2$, $ST = PR = y_1$. So, $QT = y_2 - y_1$.

Now, applying the Pythagoras theorem in Δ PTQ, we get

$$PQ^2 = PT^2 + QT^2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Therefore, $$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note that since distance is always non-negative, we take only the positive square root. So, the distance between the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$\mathbf{PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}},$$

which is called the **distance formula**.

Remarks :

1. In particular, the distance of a point $P(x, y)$ from the origin $O(0, 0)$ is given by

$$OP = \sqrt{x^2 + y^2}.$$

2. We can also write, $PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. (Why?)

Example 1 : Do the points (3, 2), (–2, –3) and (2, 3) form a triangle? If so, name the type of triangle formed.

Solution : Let us apply the distance formula to find the distances PQ, QR and PR, where P(3, 2), Q(–2, –3) and R(2, 3) are the given points. We have

$$PQ = \sqrt{(3+2)^2 + (2+3)^2} = \sqrt{5^2 + 5^2} = \sqrt{50} = 7.07 \text{ (approx.)}$$

$$QR = \sqrt{(-2-2)^2 + (-3-3)^2} = \sqrt{(-4)^2 + (-6)^2} = \sqrt{52} = 7.21 \text{ (approx.)}$$

$$PR = \sqrt{(3-2)^2 + (2-3)^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2} = 1.41 \text{ (approx.)}$$

Since the sum of any two of these distances is greater than the third distance, therefore, the points P, Q and R form a triangle.

Also, $PQ^2 + PR^2 = QR^2$, by the converse of Pythagoras theorem, we have $\angle P = 90°$. Therefore, PQR is a right triangle.

Example 2 : Show that the points (1, 7), (4, 2), (–1, –1) and (– 4, 4) are the vertices of a square.

Solution : Let A(1, 7), B(4, 2), C(–1, –1) and D(– 4, 4) be the given points. One way of showing that ABCD is a square is to use the property that all its sides should be equal and both its digonals should also be equal. Now,

$$AB = \sqrt{(1-4)^2 + (7-2)^2} = \sqrt{9+25} = \sqrt{34}$$

$$BC = \sqrt{(4+1)^2 + (2+1)^2} = \sqrt{25+9} = \sqrt{34}$$

$$CD = \sqrt{(-1+4)^2 + (-1-4)^2} = \sqrt{9+25} = \sqrt{34}$$

$$DA = \sqrt{(1+4)^2 + (7-4)^2} = \sqrt{25+9} = \sqrt{34}$$

$$AC = \sqrt{(1+1)^2 + (7+1)^2} = \sqrt{4+64} = \sqrt{68}$$

$$BD = \sqrt{(4+4)^2 + (2-4)^2} = \sqrt{64+4} = \sqrt{68}$$

Since, AB = BC = CD = DA and AC = BD, all the four sides of the quadrilateral ABCD are equal and its diagonals AC and BD are also equal. Thereore, ABCD is a square.

Alternative Solution : We find the four sides and one diagonal, say, AC as above. Here $AD^2 + DC^2 = 34 + 34 = 68 = AC^2$. Therefore, by the converse of Pythagoras theorem, $\angle D = 90°$. A quadrilateral with all four sides equal and one angle 90° is a square. So, ABCD is a square.

Example 3 : Fig. 7.6 shows the arrangement of desks in a classroom. Ashima, Bharti and Camella are seated at A(3, 1), B(6, 4) and C(8, 6) respectively. Do you think they are seated in a line? Give reasons for your answer.

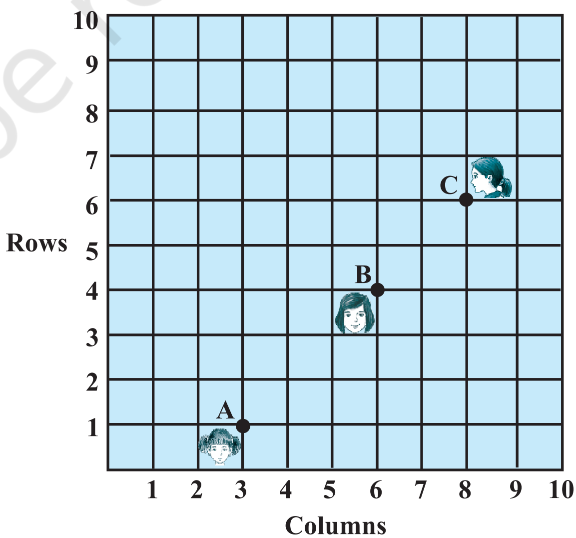


Fig. 7.6

Solution : Using the distance formula, we have

$$AB = \sqrt{(6-3)^2 + (4-1)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$$

$$BC = \sqrt{(8-6)^2 + (6-4)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

$$AC = \sqrt{(8-3)^2 + (6-1)^2} = \sqrt{25+25} = \sqrt{50} = 5\sqrt{2}$$

Since, $AB + BC = 3\sqrt{2} + 2\sqrt{2} = 5\sqrt{2} = AC,$ we can say that the points A, B and C are collinear. Therefore, they are seated in a line.

Example 4 : Find a relation between x and y such that the point (x, y) is equidistant from the points (7, 1) and (3, 5).

Solution : Let P(x, y) be equidistant from the points A(7, 1) and B(3, 5).

We are given that AP = BP. So, $AP^2 = BP^2$

i.e., $$(x-7)^2 + (y-1)^2 = (x-3)^2 + (y-5)^2$$

i.e., $$x^2 - 14x + 49 + y^2 - 2y + 1 = x^2 - 6x + 9 + y^2 - 10y + 25$$

i.e., $$x - y = 2$$

which is the required relation.

Remark : Note that the graph of the equation $x - y = 2$ is a line. From your earlier studies, you know that a point which is equidistant from A and B lies on the perpendicular bisector of AB. Therefore, the graph of $x - y = 2$ is the perpendicular bisector of AB (see Fig. 7.7).

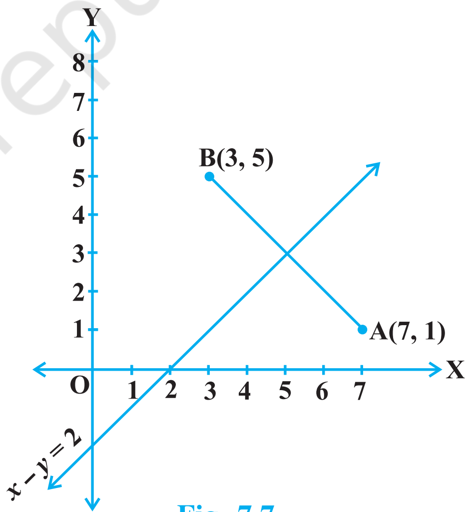


Fig. 7.7

Example 5 : Find a point on the y-axis which is equidistant from the points A(6, 5) and B(– 4, 3).

Solution : We know that a point on the y-axis is of the form $(0, y)$. So, let the point P(0, y) be equidistant from A and B. Then

$$(6-0)^2 + (5-y)^2 = (-4-0)^2 + (3-y)^2$$

i.e., $$36 + 25 + y^2 - 10y = 16 + 9 + y^2 - 6y$$

i.e., $$4y = 36$$

i.e., $$y = 9$$

So, the required point is (0, 9).

Let us check our solution : $\text{AP} = \sqrt{(6-0)^2 + (5-9)^2} = \sqrt{36+16} = \sqrt{52}$

$$\text{BP} = \sqrt{(-4-0)^2 + (3-9)^2} = \sqrt{16+36} = \sqrt{52}$$

Note : Using the remark above, we see that (0, 9) is the intersection of the y-axis and the perpendicular bisector of AB.

EXERCISE 7.1

1. Find the distance between the following pairs of points :

(i) (2, 3), (4, 1) (ii) (– 5, 7), (– 1, 3) (iii) $(a, b), (-a, -b)$

2. Find the distance between the points (0, 0) and (36, 15). Can you now find the distance between the two towns A and B discussed in Section 7.2.

3. Determine if the points (1, 5), (2, 3) and (– 2, – 11) are collinear.

4. Check whether (5, – 2), (6, 4) and (7, – 2) are the vertices of an isosceles triangle.

5. In a classroom, 4 friends are seated at the points A, B, C and D as shown in Fig. 7.8. Champa and Chameli walk into the class and after observing for a few minutes Champa asks Chameli, "Don't you think ABCD is a square?" Chameli disagrees. Using distance formula, find which of them is correct.

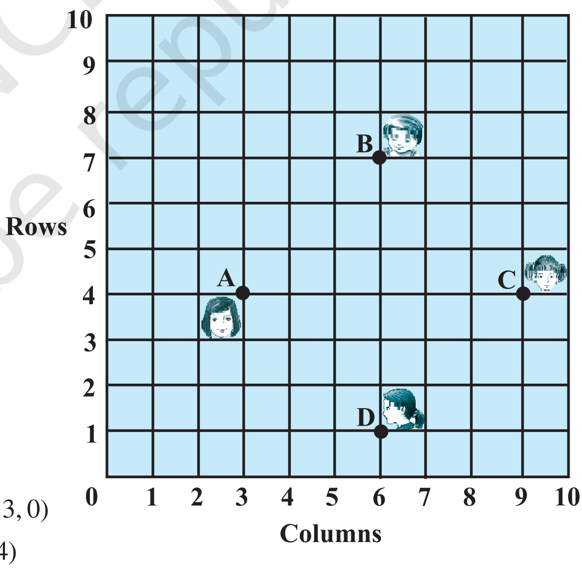


Fig. 7.8

6. Name the type of quadrilateral formed, if any, by the following points, and give reasons for your answer:

(i) (– 1, – 2), (1, 0), (– 1, 2), (– 3, 0)

(ii) (–3, 5), (3, 1), (0, 3), (–1, – 4)

(iii) (4, 5), (7, 6), (4, 3), (1, 2)

7. Find the point on the x-axis which is equidistant from (2, –5) and (–2, 9).

8. Find the values of y for which the distance between the points P(2, – 3) and Q(10, y) is 10 units.

9. If Q(0, 1) is equidistant from P(5, –3) and R(x, 6), find the values of x. Also find the distances QR and PR.

10. Find a relation between x and y such that the point (x, y) is equidistant from the point (3, 6) and (– 3, 4).

7.3 Section Formula

Let us recall the situation in Section 7.2. Suppose a telephone company wants to position a relay tower at P between A and B is such a way that the distance of the tower from B is twice its distance from A. If P lies on AB, it will divide AB in the ratio 1 : 2 (see Fig. 7.9). If we take A as the origin O, and 1 km as one unit on both the axis, the coordinates of B will be (36, 15). In order to know the position of the tower, we must know the coordinates of P. How do we find these coordinates?

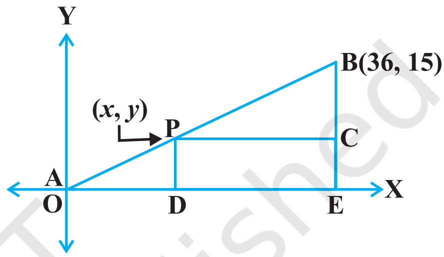


Fig. 7.9

Let the coordinates of P be (x, y). Draw perpendiculars from P and B to the x-axis, meeting it in D and E, respectively. Draw PC perpendicular to BE. Then, by the AA similarity criterion, studied in Chapter 6, Δ POD and Δ BPC are similar.

Therefore , $\dfrac{\text{OD}}{\text{PC}} = \dfrac{\text{OP}}{\text{PB}} = \dfrac{1}{2}$, and $\dfrac{\text{PD}}{\text{BC}} = \dfrac{\text{OP}}{\text{PB}} = \dfrac{1}{2}$

So, $\dfrac{x}{36 - x} = \dfrac{1}{2}$ and $\dfrac{y}{15 - y} = \dfrac{1}{2}$.

These equations give $x = 12$ and $y = 5$.

You can check that P(12, 5) meets the condition that OP : PB = 1 : 2.

Now let us use the understanding that you may have developed through this example to obtain the general formula.

Consider any two points A(x_1, y_1) and B(x_2, y_2) and assume that P (x, y) divides AB internally in the ratio m_1 : m_2, i.e., $\dfrac{\text{PA}}{\text{PB}} = \dfrac{m_1}{m_2}$ (see Fig. 7.10).

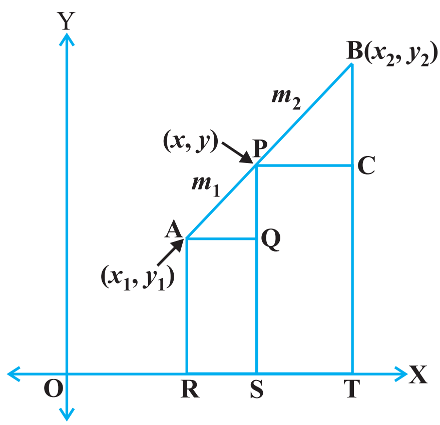


Fig. 7.10

Draw AR, PS and BT perpendicular to the x-axis. Draw AQ and PC parallel to the x-axis. Then, by the AA similarity criterion,

$$\Delta \text{ PAQ} \sim \Delta \text{ BPC}$$

Therefore, $$\frac{\text{PA}}{\text{BP}} = \frac{\text{AQ}}{\text{PC}} = \frac{\text{PQ}}{\text{BC}} \qquad (1)$$

Now, $$\text{AQ} = \text{RS} = \text{OS} - \text{OR} = x - x_1$$
$$\text{PC} = \text{ST} = \text{OT} - \text{OS} = x_2 - x$$
$$\text{PQ} = \text{PS} - \text{QS} = \text{PS} - \text{AR} = y - y_1$$
$$\text{BC} = \text{BT} - \text{CT} = \text{BT} - \text{PS} = y_2 - y$$

Substituting these values in (1), we get

$$\frac{m_1}{m_2} = \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y}$$

Taking $$\frac{m_1}{m_2} = \frac{x - x_1}{x_2 - x}, \text{ we get } x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}$$

Similarly, taking $$\frac{m_1}{m_2} = \frac{y - y_1}{y_2 - y}, \text{ we get } y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}$$

So, the coordinates of the point P(x, y) which divides the line segment joining the points A(x_1, y_1) and B(x_2, y_2), internally, in the ratio $m_1 : m_2$ are

$$\left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}\right) \qquad (2)$$

This is known as the **section formula**.

This can also be derived by drawing perpendiculars from A, P and B on the y-axis and proceeding as above.

If the ratio in which P divides AB is $k : 1$, then the coordinates of the point P will be

$$\left(\frac{kx_2 + x_1}{k+1}, \frac{ky_2 + y_1}{k+1}\right).$$

Special Case : The mid-point of a line segment divides the line segment in the ratio 1 : 1. Therefore, the coordinates of the mid-point P of the join of the points A(x_1, y_1) and B(x_2, y_2) is

$$\left(\frac{1 \cdot x_1 + 1 \cdot x_2}{1+1}, \frac{1 \cdot y_1 + 1 \cdot y_2}{1+1}\right) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Let us solve a few examples based on the section formula.

Example 6 : Find the coordinates of the point which divides the line segment joining the points (4, – 3) and (8, 5) in the ratio 3 : 1 internally.

Solution : Let P(x, y) be the required point. Using the section formula, we get

$$x = \frac{3(8)+1(4)}{3+1} = 7, \quad y = \frac{3(5)+1(-3)}{3+1} = 3$$

Therefore, (7, 3) is the required point.

Example 7 : In what ratio does the point (– 4, 6) divide the line segment joining the points A(– 6, 10) and B(3, – 8)?

Solution : Let (– 4, 6) divide AB internally in the ratio $m_1 : m_2$. Using the section formula, we get

$$(-4, 6) = \left(\frac{3m_1 - 6m_2}{m_1 + m_2}, \frac{-8m_1 + 10m_2}{m_1 + m_2}\right) \qquad (1)$$

Recall that if $(x, y) = (a, b)$ then $x = a$ and $y = b$.

So, $$-4 = \frac{3m_1 - 6m_2}{m_1 + m_2} \text{ and } 6 = \frac{-8m_1 + 10m_2}{m_1 + m_2}$$

Now, $$-4 = \frac{3m_1 - 6m_2}{m_1 + m_2}$$ gives us

$$-4m_1 - 4m_2 = 3m_1 - 6m_2$$

i.e., $$7m_1 = 2m_2$$

i.e., $$m_1 : m_2 = 2 : 7$$

You should verify that the ratio satisfies the y-coordinate also.

Now, $$\frac{-8m_1 + 10m_2}{m_1 + m_2} = \frac{-8\dfrac{m_1}{m_2} + 10}{\dfrac{m_1}{m_2} + 1}$$ (Dividing throughout by m_2)

$$= \frac{-8 \times \dfrac{2}{7} + 10}{\dfrac{2}{7} + 1} = 6$$

Therefore, the point (– 4, 6) divides the line segment joining the points A(– 6, 10) and B(3, – 8) in the ratio 2 : 7.

Alternatively : The ratio $m_1 : m_2$ can also be written as $\frac{m_1}{m_2} : 1$, or k : 1. Let (– 4, 6) divide AB internally in the ratio k : 1. Using the section formula, we get

$$(-4, 6) = \left(\frac{3k-6}{k+1}, \frac{-8k+10}{k+1}\right) \qquad (2)$$

So, $$-4 = \frac{3k-6}{k+1}$$

i.e., $$-4k - 4 = 3k - 6$$

i.e., $$7k = 2$$

i.e., $$k : 1 = 2 : 7$$

You can check for the y-coordinate also.

So, the point (– 4, 6) divides the line segment joining the points A(– 6, 10) and B(3, – 8) in the ratio 2 : 7.

Note : You can also find this ratio by calculating the distances PA and PB and taking their ratios provided you know that A, P and B are collinear.

Example 8 : Find the coordinates of the points of trisection (i.e., points dividing in three equal parts) of the line segment joining the points A(2, – 2) and B(– 7, 4).

Solution : Let P and Q be the points of trisection of AB i.e., AP = PQ = QB (see Fig. 7.11).

Fig. 7.11

Therefore, P divides AB internally in the ratio 1 : 2. Therefore, the coordinates of P, by applying the section formula, are

$$\left(\frac{1(-7)+2(2)}{1+2}, \frac{1(4)+2(-2)}{1+2}\right), \text{ i.e., } (-1, 0)$$

Now, Q also divides AB internally in the ratio 2 : 1. So, the coordinates of Q are

$$\left(\frac{2(-7)+1(2)}{2+1}, \frac{2(4)+1(-2)}{2+1}\right), \text{ i.e., } (-4, 2)$$

Therefore, the coordinates of the points of trisection of the line segment joining A and B are (–1, 0) and (– 4, 2).

Note : We could also have obtained Q by noting that it is the mid-point of PB. So, we could have obtained its coordinates using the mid-point formula.

Example 9 : Find the ratio in which the y-axis divides the line segment joining the points (5, – 6) and (–1, – 4). Also find the point of intersection.

Solution : Let the ratio be k : 1. Then by the section formula, the coordinates of the point which divides AB in the ratio k : 1 are $\left(\dfrac{-k+5}{k+1}, \dfrac{-4k-6}{k+1}\right)$.

This point lies on the y-axis, and we know that on the y-axis the abscissa is 0.

Therefore, $$\frac{-k+5}{k+1} = 0$$

So, $$k = 5$$

That is, the ratio is 5 : 1. Putting the value of k = 5, we get the point of intersection as $\left(0, \dfrac{-13}{3}\right)$.

Example 10 : If the points A(6, 1), B(8, 2), C(9, 4) and D(p, 3) are the vertices of a parallelogram, taken in order, find the value of p.

Solution : We know that diagonals of a parallelogram bisect each other.

So, the coordinates of the mid-point of AC = coordinates of the mid-point of BD

i.e., $$\left(\frac{6+9}{2}, \frac{1+4}{2}\right) = \left(\frac{8+p}{2}, \frac{2+3}{2}\right)$$

i.e., $$\left(\frac{15}{2}, \frac{5}{2}\right) = \left(\frac{8+p}{2}, \frac{5}{2}\right)$$

so, $$\frac{15}{2} = \frac{8+p}{2}$$

i.e., $$p = 7$$

EXERCISE 7.2

1. Find the coordinates of the point which divides the join of (–1, 7) and (4, –3) in the ratio 2 : 3.
2. Find the coordinates of the points of trisection of the line segment joining (4, –1) and (–2, –3).
3. To conduct Sports Day activities, in your rectangular shaped school ground ABCD, lines have been drawn with chalk powder at a distance of 1m each. 100 flower pots have been placed at a distance of 1m from each other along AD, as shown in Fig. 7.12. Niharika runs $\frac{1}{4}$ th the distance AD on the 2nd line and posts a green flag. Preet runs $\frac{1}{5}$ th the distance AD on the eighth line and posts a red flag. What is the distance between both the flags? If Rashmi has to post a blue flag exactly halfway between the line segment joining the two flags, where should she post her flag?

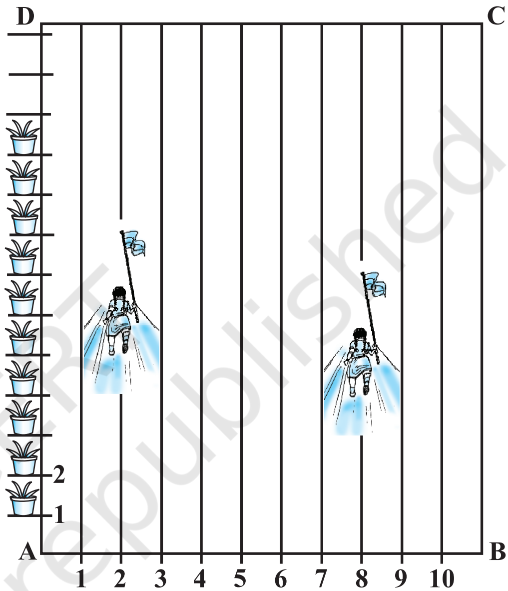


Fig. 7.12

4. Find the ratio in which the line segment joining the points (– 3, 10) and (6, – 8) is divided by (– 1, 6).
5. Find the ratio in which the line segment joining A(1, – 5) and B(– 4, 5) is divided by the x-axis. Also find the coordinates of the point of division.
6. If (1, 2), (4, y), (x, 6) and (3, 5) are the vertices of a parallelogram taken in order, find x and y.
7. Find the coordinates of a point A, where AB is the diameter of a circle whose centre is (2, – 3) and B is (1, 4).
8. If A and B are (– 2, – 2) and (2, – 4), respectively, find the coordinates of P such that AP = $\frac{3}{7}$ AB and P lies on the line segment AB.
9. Find the coordinates of the points which divide the line segment joining A(– 2, 2) and B(2, 8) into four equal parts.
10. Find the area of a rhombus if its vertices are (3, 0), (4, 5), (– 1, 4) and (– 2, – 1) taken in order. [**Hint :** Area of a rhombus = $\frac{1}{2}$ (product of its diagonals)]

7.4 Summary

In this chapter, you have studied the following points :

1. The distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
2. The distance of a point $P(x, y)$ from the origin is $\sqrt{x^2 + y^2}$.
3. The coordinates of the point $P(x, y)$ which divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $m_1 : m_2$ are $\left(\dfrac{m_1x_2 + m_2x_1}{m_1 + m_2}, \dfrac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$.
4. The mid-point of the line segment joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

A NOTE TO THE READER

Section 7.3 discusses the Section Formula for the coordinates (x, y) of a point P which divides internally the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m_1 : m_2$ as follows :

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \quad y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$$

Note that, here, PA : PB = $m_1 : m_2$.

However, if P does not lie between A and B but lies on the line AB, outside the line segment AB, and PA : PB = $m_1 : m_2$, we say that P divides externally the line segment joining the points A and B. You will study Section Formula for such case in higher classes.

1062CH08

INTRODUCTION TO TRIGONOMETRY 8

There is perhaps nothing which so occupies the middle position of mathematics as trigonometry.

– J.F. Herbart (1890)

8.1 Introduction

You have already studied about triangles, and in particular, right triangles, in your earlier classes. Let us take some examples from our surroundings where right triangles can be imagined to be formed. For instance :

1. Suppose the students of a school are visiting Qutub Minar. Now, if a student is looking at the top of the Minar, a right triangle can be imagined to be made, as shown in Fig 8.1. Can the student find out the height of the Minar, without actually measuring it?

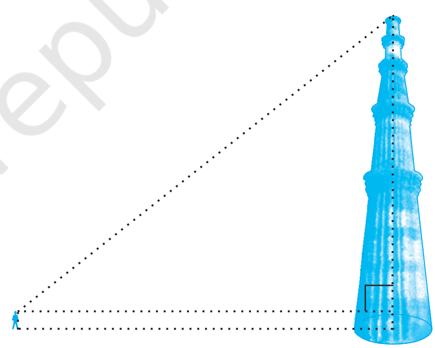


Fig. 8.1

2. Suppose a girl is sitting on the balcony of her house located on the bank of a river. She is looking down at a flower pot placed on a stair of a temple situated nearby on the other bank of the river. A right triangle is imagined to be made in this situation as shown in Fig.8.2. If you know the height at which the person is sitting, can you find the width of the river?

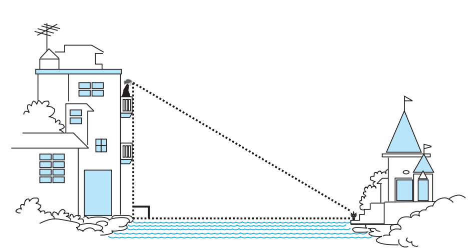


Fig. 8.2

3. Suppose a hot air balloon is flying in the air. A girl happens to spot the balloon in the sky and runs to her mother to tell her about it. Her mother rushes out of the house to look at the balloon.Now when the girl had spotted the balloon intially it was at point A. When both the mother and daughter came out to see it, it had already travelled to another point B. Can you find the altitude of B from the ground?

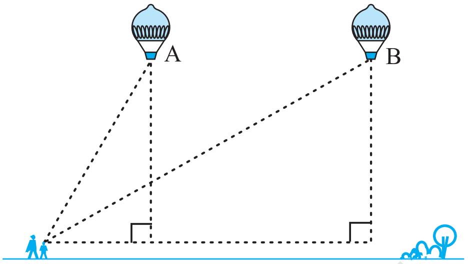


Fig. 8.3

In all the situations given above, the distances or heights can be found by using some mathematical techniques, which come under a branch of mathematics called 'trigonometry'. The word 'trigonometry' is derived from the Greek words 'tri' (meaning three), 'gon' (meaning sides) and 'metron' (meaning measure). In fact, **trigonometry** is the study of relationships between the sides and angles of a triangle. The earliest known work on trigonometry was recorded in Egypt and Babylon. Early astronomers used it to find out the distances of the stars and planets from the Earth. Even today, most of the technologically advanced methods used in Engineering and Physical Sciences are based on trigonometrical concepts.

In this chapter, we will study some ratios of the sides of a right triangle with respect to its acute angles, called **trigonometric ratios of the angle.** We will restrict our discussion to acute angles only. However, these ratios can be extended to other angles also. We will also define the trigonometric ratios for angles of measure 0° and 90°. We will calculate trigonometric ratios for some specific angles and establish some identities involving these ratios, called **trigonometric identities.**

8.2 Trigonometric Ratios

In Section 8.1, you have seen some right triangles imagined to be formed in different situations.

Let us take a right triangle ABC as shown in Fig. 8.4.

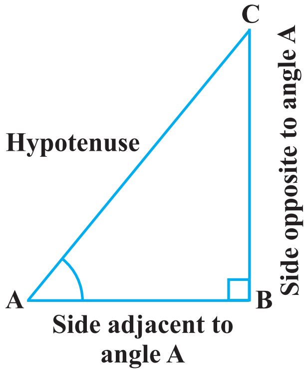


Fig. 8.4

Here, $\angle$ CAB (or, in brief, angle A) is an acute angle. Note the position of the side BC with respect to angle A. It faces $\angle$ A. We call it the *side opposite* to angle A. AC is the *hypotenuse* of the right triangle and the side AB is a part of $\angle$ A. So, we call it the *side adjacent* to angle A.

Note that the position of sides change when you consider angle C in place of A (see Fig. 8.5).

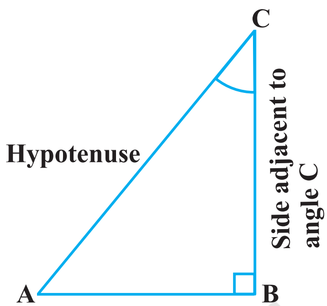


Fig. 8.5

You have studied the concept of 'ratio' in your earlier classes. We now define certain ratios involving the sides of a right triangle, and call them trigonometric ratios.

The trigonometric ratios of the angle A in right triangle ABC (see Fig. 8.4) are defined as follows :

$$\text{sine of } \angle A = \frac{\text{side opposite to angle A}}{\text{hypotenuse}} = \frac{BC}{AC}$$

$$\text{cosine of } \angle A = \frac{\text{side adjacent to angle A}}{\text{hypotenuse}} = \frac{AB}{AC}$$

$$\text{tangent of } \angle A = \frac{\text{side opposite to angle A}}{\text{side adjacent to angle A}} = \frac{BC}{AB}$$

$$\text{cosecant of } \angle A = \frac{1}{\text{sine of } \angle A} = \frac{\text{hypotenuse}}{\text{side opposite to angle A}} = \frac{AC}{BC}$$

$$\text{secant of } \angle A = \frac{1}{\text{cosine of } \angle A} = \frac{\text{hypotenuse}}{\text{side adjacent to angle A}} = \frac{AC}{AB}$$

$$\text{cotangent of } \angle A = \frac{1}{\text{tangent of } \angle A} = \frac{\text{side adjacent to angle A}}{\text{side opposite to angle A}} = \frac{AB}{BC}$$

The ratios defined above are abbreviated as sin A, cos A, tan A, cosec A, sec A and cot A respectively. Note that the ratios **cosec A, sec A and cot A** are respectively, the reciprocals of the ratios sin A, cos A and tan A.

Also, observe that $\tan A = \frac{BC}{AB} = \frac{\frac{BC}{AC}}{\frac{AB}{AC}} = \frac{\sin A}{\cos A}$ and $\cot A = \frac{\cos A}{\sin A}$.

So, the **trigonometric ratios** of an acute angle in a right triangle express the relationship between the angle and the length of its sides.

Why don't you try to define the trigonometric ratios for angle C in the right triangle? (See Fig. 8.5)

The first use of the idea of **'sine'** in the way we use it today was in the work *Aryabhatiyam* by Aryabhata, in A.D. 500. Aryabhata used the word ***ardha-jya*** for the half-chord, which was shortened to ***jya*** or ***jiva*** in due course. When the *Aryabhatiyam* was translated into Arabic, the word ***jiva*** was retained as it is. The word ***jiva*** was translated into ***sinus,*** which means curve, when the Arabic version was translated into Latin. Soon the word ***sinus***, also used as ***sine***, became common in mathematical texts throughout Europe. An English Professor of astronomy Edmund Gunter (1581–1626), first used the abbreviated notation ***'sin'.***

Aryabhata
C.E. 476 – 550

The origin of the terms **'cosine'** and **'tangent'** was much later. The cosine function arose from the need to compute the sine of the complementary angle. Aryabhatta called it **kotijya**. The name ***cosinus*** originated with Edmund Gunter. In 1674, the English Mathematician Sir Jonas Moore first used the abbreviated notation ***'cos'***.

Remark : Note that the symbol sin A is used as an abbreviation for 'the sine of the angle A'. sin A is *not* the product of 'sin' and A. 'sin' separated from A has no meaning. Similarly, cos A is *not* the product of 'cos' and A. Similar interpretations follow for other trigonometric ratios also.

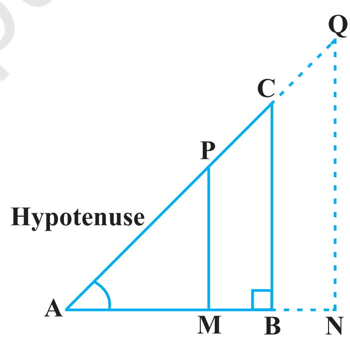


Fig. 8.6

Now, if we take a point P on the hypotenuse AC or a point Q on AC extended, of the right triangle ABC and draw PM perpendicular to AB and QN perpendicular to AB extended (see Fig. 8.6), how will the trigonometric ratios of $\angle$ A in Δ PAM differ from those of $\angle$ A in Δ CAB or from those of $\angle$ A in Δ QAN?

To answer this, first look at these triangles. Is Δ PAM similar to Δ CAB? From Chapter 6, recall the AA similarity criterion. Using the criterion, you will see that the triangles PAM and CAB are similar. Therefore, by the property of similar triangles, the corresponding sides of the triangles are proportional.

So, we have $$\frac{AM}{AB} = \frac{AP}{AC} = \frac{MP}{BC}.$$

From this, we find $\dfrac{MP}{AP} = \dfrac{BC}{AC} = \sin A$.

Similarly, $\dfrac{AM}{AP} = \dfrac{AB}{AC} = \cos A$, $\dfrac{MP}{AM} = \dfrac{BC}{AB} = \tan A$ and so on.

This shows that the trigonometric ratios of angle A in Δ PAM not differ from those of angle A in Δ CAB.

In the same way, you should check that the value of sin A (and also of other trigonometric ratios) remains the same in Δ QAN also.

From our observations, it is now clear that **the values of the trigonometric ratios of an angle do not vary with the lengths of the sides of the triangle, if the angle remains the same.**

Note : For the sake of convenience, we may write $\sin^2 A$, $\cos^2 A$, etc., in place of $(\sin A)^2$, $(\cos A)^2$, etc., respectively. But $\operatorname{cosec} A = (\sin A)^{-1} \neq \sin^{-1} A$ (it is called sine inverse A). $\sin^{-1} A$ has a different meaning, which will be discussed in higher classes. Similar conventions hold for the other trigonometric ratios as well. Sometimes, the Greek letter θ (theta) is also used to denote an angle.

We have defined six trigonometric ratios of an acute angle. If we know any one of the ratios, can we obtain the other ratios? Let us see.

If in a right triangle ABC, $\sin A = \dfrac{1}{3}$, then this means that $\dfrac{BC}{AC} = \dfrac{1}{3}$, i.e., the lengths of the sides BC and AC of the triangle ABC are in the ratio 1 : 3 (see Fig. 8.7). So if BC is equal to k, then AC will be $3k$, where k is any positive number. To determine other trigonometric ratios for the angle A, we need to find the length of the third side AB. Do you remember the Pythagoras theorem? Let us use it to determine the required length AB.

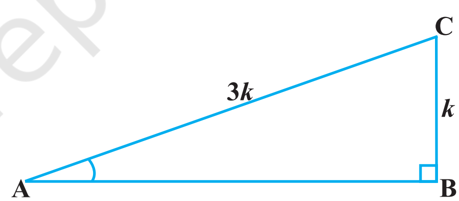


Fig. 8.7

$$AB^2 = AC^2 - BC^2 = (3k)^2 - (k)^2 = 8k^2 = (2\sqrt{2}\,k)^2$$

Therefore, $AB = \pm 2\sqrt{2}\,k$

So, we get $AB = 2\sqrt{2}\,k$ (Why is AB not $-2\sqrt{2}\,k$?)

Now, $\cos A = \dfrac{AB}{AC} = \dfrac{2\sqrt{2}\,k}{3k} = \dfrac{2\sqrt{2}}{3}$

Similarly, you can obtain the other trigonometric ratios of the angle A.

Remark : Since the hypotenuse is the longest side in a right triangle, the value of sin A or cos A is always less than 1 (or, in particular, equal to 1).

Let us consider some examples.

Example 1 : Given $\tan A = \frac{4}{3}$, find the other trigonometric ratios of the angle A.

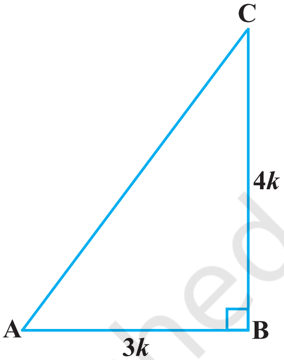


Fig. 8.8

Solution : Let us first draw a right Δ ABC (see Fig 8.8).

Now, we know that $\tan A = \frac{BC}{AB} = \frac{4}{3}$.

Therefore, if BC = $4k$, then AB = $3k$, where k is a positive number.

Now, by using the Pythagoras Theorem, we have

$$AC^2 = AB^2 + BC^2 = (4k)^2 + (3k)^2 = 25k^2$$

So, $\quad AC = 5k$

Now, we can write all the trigonometric ratios using their definitions.

$$\sin A = \frac{BC}{AC} = \frac{4k}{5k} = \frac{4}{5}$$

$$\cos A = \frac{AB}{AC} = \frac{3k}{5k} = \frac{3}{5}$$

Therefore, $\cot A = \frac{1}{\tan A} = \frac{3}{4}$, $\operatorname{cosec} A = \frac{1}{\sin A} = \frac{5}{4}$ and $\sec A = \frac{1}{\cos A} = \frac{5}{3}$.

Example 2 : If $\angle$ B and $\angle$ Q are acute angles such that sin B = sin Q, then prove that $\angle$ B = $\angle$ Q.

Solution : Let us consider two right triangles ABC and PQR where sin B = sin Q (see Fig. 8.9).

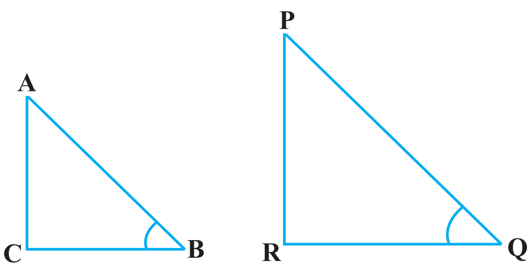


Fig. 8.9

We have $\quad \sin B = \frac{AC}{AB}$

and $\quad \sin Q = \frac{PR}{PQ}$

Then $$\frac{AC}{AB} = \frac{PR}{PQ}$$

Therefore, $$\frac{AC}{PR} = \frac{AB}{PQ} = k,\ \text{say} \qquad (1)$$

Now, using Pythagoras theorem,

$$BC = \sqrt{AB^2 - AC^2}$$

and $$QR = \sqrt{PQ^2 - PR^2}$$

So, $$\frac{BC}{QR} = \frac{\sqrt{AB^2 - AC^2}}{\sqrt{PQ^2 - PR^2}} = \frac{\sqrt{k^2PQ^2 - k^2PR^2}}{\sqrt{PQ^2 - PR^2}} = \frac{k\sqrt{PQ^2 - PR^2}}{\sqrt{PQ^2 - PR^2}} = k \qquad (2)$$

From (1) and (2), we have

$$\frac{AC}{PR} = \frac{AB}{PQ} = \frac{BC}{QR}$$

Then, by using Theorem 6.4, Δ ACB ~ Δ PRQ and therefore, $\angle$ B = $\angle$ Q.

Example 3 : Consider Δ ACB, right-angled at C, in which AB = 29 units, BC = 21 units and $\angle$ ABC = θ (see Fig. 8.10). Determine the values of

(i) $\cos^2\theta + \sin^2\theta$,

(ii) $\cos^2\theta - \sin^2\theta$.

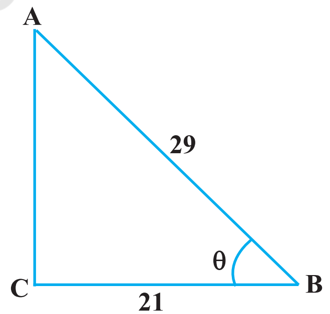


Fig. 8.10

Solution : In Δ ACB, we have

$$AC = \sqrt{AB^2 - BC^2} = \sqrt{(29)^2 - (21)^2}$$

$$= \sqrt{(29-21)(29+21)} = \sqrt{(8)(50)} = \sqrt{400} = 20 \text{ units}$$

So, $$\sin\theta = \frac{AC}{AB} = \frac{20}{29},\ \cos\theta = \frac{BC}{AB} = \frac{21}{29}.$$

Now, (i) $$\cos^2\theta + \sin^2\theta = \left(\frac{20}{29}\right)^2 + \left(\frac{21}{29}\right)^2 = \frac{20^2 + 21^2}{29^2} = \frac{400 + 441}{841} = 1,$$

and (ii) $$\cos^2\theta - \sin^2\theta = \left(\frac{21}{29}\right)^2 - \left(\frac{20}{29}\right)^2 = \frac{(21+20)(21-20)}{29^2} = \frac{41}{841}.$$

Example 4 : In a right triangle ABC, right-angled at B, if tan A = 1, then verify that

2 sin A cos A = 1.

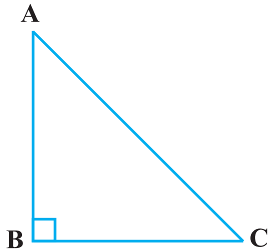


Fig. 8.11

Solution : In Δ ABC, $\tan A = \frac{BC}{AB} = 1$ (see Fig 8.11)

i.e., $BC = AB$

Let AB = BC = k, where k is a positive number.

Now, $AC = \sqrt{AB^2 + BC^2}$

$= \sqrt{(k)^2 + (k)^2} = k\sqrt{2}$

Therefore, $\sin A = \frac{BC}{AC} = \frac{1}{\sqrt{2}}$ and $\cos A = \frac{AB}{AC} = \frac{1}{\sqrt{2}}$

So, $2 \sin A \cos A = 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = 1$, which is the required value.

Example 5 : In Δ OPQ, right-angled at P, OP = 7 cm and OQ – PQ = 1 cm (see Fig. 8.12). Determine the values of sin Q and cos Q.

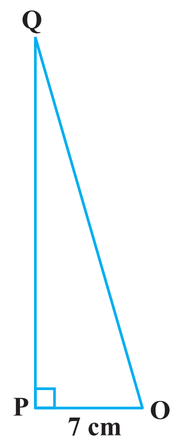


Fig. 8.12

Solution : In Δ OPQ, we have

$OQ^2 = OP^2 + PQ^2$

i.e., $(1 + PQ)^2 = OP^2 + PQ^2$ (Why?)

i.e., $1 + PQ^2 + 2PQ = OP^2 + PQ^2$

i.e., $1 + 2PQ = 7^2$ (Why?)

i.e., PQ = 24 cm and OQ = 1 + PQ = 25 cm

So, $\sin Q = \frac{7}{25}$ and $\cos Q = \frac{24}{25}$.

EXERCISE 8.1

1. In Δ ABC, right-angled at B, AB = 24 cm, BC = 7 cm. Determine :
   (i) sin A, cos A
   (ii) sin C, cos C

2. In Fig. 8.13, find tan P – cot R.

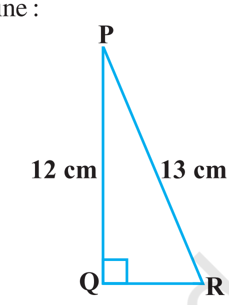


Fig. 8.13

3. If $\sin A = \frac{3}{4}$, calculate cos A and tan A.

4. Given 15 cot A = 8, find sin A and sec A.

5. Given $\sec \theta = \frac{13}{12}$, calculate all other trigonometric ratios.

6. If $\angle$ A and $\angle$ B are acute angles such that cos A = cos B, then show that $\angle$ A = $\angle$ B.

7. If $\cot \theta = \frac{7}{8}$, evaluate : (i) $\frac{(1+\sin\theta)(1-\sin\theta)}{(1+\cos\theta)(1-\cos\theta)}$, (ii) $\cot^2\theta$

8. If 3 cot A = 4, check whether $\frac{1-\tan^2 A}{1+\tan^2 A} = \cos^2 A - \sin^2 A$ or not.

9. In triangle ABC, right-angled at B, if $\tan A = \frac{1}{\sqrt{3}}$, find the value of:
   (i) sin A cos C + cos A sin C
   (ii) cos A cos C – sin A sin C

10. In Δ PQR, right-angled at Q, PR + QR = 25 cm and PQ = 5 cm. Determine the values of sin P, cos P and tan P.

11. State whether the following are true or false. Justify your answer.
    (i) The value of tan A is always less than 1.
    (ii) $\sec A = \frac{12}{5}$ for some value of angle A.
    (iii) cos A is the abbreviation used for the cosecant of angle A.
    (iv) cot A is the product of cot and A.
    (v) $\sin\theta = \frac{4}{3}$ for some angle θ.

8.3 Trigonometric Ratios of Some Specific Angles

From geometry, you are already familiar with the construction of angles of 30°, 45°, 60° and 90°. In this section, we will find the values of the trigonometric ratios for these angles and, of course, for 0°.

Trigonometric Ratios of 45°

In Δ ABC, right-angled at B, if one angle is 45°, then the other angle is also 45°, i.e., ∠ A = ∠ C = 45° (see Fig. 8.14).

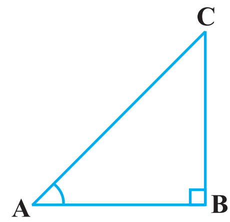


Fig. 8.14

So, BC = AB (Why?)

Now, Suppose BC = AB = a.

Then by Pythagoras Theorem, $AC^2 = AB^2 + BC^2 = a^2 + a^2 = 2a^2$,

and, therefore, $AC = a\sqrt{2}$.

Using the definitions of the trigonometric ratios, we have :

$$\sin 45° = \frac{\text{side opposite to angle } 45°}{\text{hypotenuse}} = \frac{BC}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\cos 45° = \frac{\text{side adjacent to angle } 45°}{\text{hypotenuse}} = \frac{AB}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\tan 45° = \frac{\text{side opposite to angle } 45°}{\text{side adjacent to angle } 45°} = \frac{BC}{AB} = \frac{a}{a} = 1$$

Also, $\text{cosec } 45° = \frac{1}{\sin 45°} = \sqrt{2}$, $\sec 45° = \frac{1}{\cos 45°} = \sqrt{2}$, $\cot 45° = \frac{1}{\tan 45°} = 1$.

Trigonometric Ratios of 30° and 60°

Let us now calculate the trigonometric ratios of 30° and 60°. Consider an equilateral triangle ABC. Since each angle in an equilateral triangle is 60°, therefore, ∠ A = ∠ B = ∠ C = 60°.

Draw the perpendicular AD from A to the side BC (see Fig. 8.15).

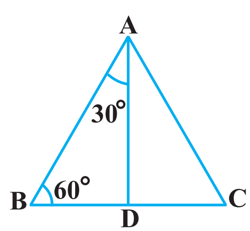


Fig. 8.15

Now Δ ABD ≅ Δ ACD (Why?)

Therefore, BD = DC

and ∠ BAD = ∠ CAD (CPCT)

Now observe that:

Δ ABD is a right triangle, right-angled at D with ∠ BAD = 30° and ∠ ABD = 60° (see Fig. 8.15).

As you know, for finding the trigonometric ratios, we need to know the lengths of the sides of the triangle. So, let us suppose that AB = $2a$.

Then, $$\text{BD} = \frac{1}{2}\text{BC} = a$$

and $$\text{AD}^2 = \text{AB}^2 - \text{BD}^2 = (2a)^2 - (a)^2 = 3a^2,$$

Therefore, $$\text{AD} = a\sqrt{3}$$

Now, we have :

$$\sin 30° = \frac{\text{BD}}{\text{AB}} = \frac{a}{2a} = \frac{1}{2},\ \cos 30° = \frac{\text{AD}}{\text{AB}} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$$

$$\tan 30° = \frac{\text{BD}}{\text{AD}} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

Also, $$\operatorname{cosec} 30° = \frac{1}{\sin 30°} = 2,\quad \sec 30° = \frac{1}{\cos 30°} = \frac{2}{\sqrt{3}}$$

$$\cot 30° = \frac{1}{\tan 30°} = \sqrt{3}.$$

Similarly,

$$\sin 60° = \frac{\text{AD}}{\text{AB}} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2},\ \cos 60° = \frac{1}{2},\ \tan 60° = \sqrt{3},$$

$$\operatorname{cosec} 60° = \frac{2}{\sqrt{3}},\ \sec 60° = 2 \text{ and } \cot 60° = \frac{1}{\sqrt{3}}.$$

Trigonometric Ratios of 0° and 90°

Let us see what happens to the trigonometric ratios of angle A, if it is made smaller and smaller in the right triangle ABC (see Fig. 8.16), till it becomes zero. As ∠ A gets smaller and smaller, the length of the side BC decreases.The point C gets closer to point B, and finally when ∠ A becomes very close to 0°, AC becomes almost the same as AB (see Fig. 8.17).

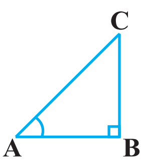


Fig. 8.16

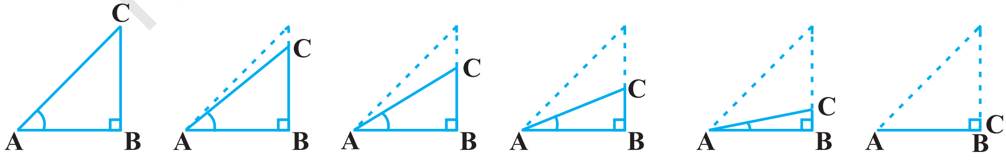


Fig. 8.17

When $\angle$ A is very close to 0°, BC gets very close to 0 and so the value of $\sin A = \frac{BC}{AC}$ is very close to 0. Also, when $\angle$ A is very close to 0°, AC is nearly the same as AB and so the value of $\cos A = \frac{AB}{AC}$ is very close to 1.

This helps us to see how we can define the values of sin A and cos A when A = 0°. We define : **sin 0° = 0 and cos 0° = 1.**

Using these, we have :

$$\tan 0° = \frac{\sin 0°}{\cos 0°} = 0, \cot 0° = \frac{1}{\tan 0°}, \text{ which is not defined. (Why?)}$$

$$\sec 0° = \frac{1}{\cos 0°} = 1 \text{ and } \operatorname{cosec} 0° = \frac{1}{\sin 0°}, \text{ which is again not defined.(Why?)}$$

Now, let us see what happens to the trigonometric ratios of $\angle$ A, when it is made larger and larger in Δ ABC till it becomes 90°. As $\angle$ A gets larger and larger, $\angle$ C gets smaller and smaller. Therefore, as in the case above, the length of the side AB goes on decreasing. The point A gets closer to point B. Finally when $\angle$ A is very close to 90°, $\angle$ C becomes very close to 0° and the side AC almost coincides with side BC (see Fig. 8.18).

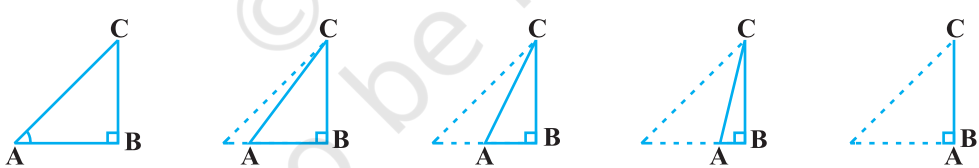


Fig. 8.18

When $\angle$ C is very close to 0°, $\angle$ A is very close to 90°, side AC is nearly the same as side BC, and so sin A is very close to 1. Also when $\angle$ A is very close to 90°, $\angle$ C is very close to 0°, and the side AB is nearly zero, so cos A is very close to 0.

So, we define : **sin 90° = 1 and cos 90° = 0.**

Now, why don't you find the other trigonometric ratios of 90°?

We shall now give the values of all the trigonometric ratios of 0°, 30°, 45°, 60° and 90° in Table 8.1, for ready reference.

Table 8.1

$\angle$ A	0°	30°	45°	60°	90°
sin A	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos A	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan A	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
cosec A	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
sec A	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
cot A	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Remark : From the table above you can observe that as $\angle$ A increases from 0° to 90°, sin A increases from 0 to 1 and cos A decreases from 1 to 0.

Let us illustrate the use of the values in the table above through some examples.

Example 6 : In Δ ABC, right-angled at B, AB = 5 cm and $\angle$ ACB = 30° (see Fig. 8.19). Determine the lengths of the sides BC and AC.

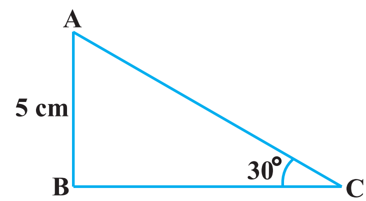


Fig. 8.19

Solution : To find the length of the side BC, we will choose the trigonometric ratio involving BC and the given side AB. Since BC is the side adjacent to angle C and AB is the side opposite to angle C, therefore

$$\frac{AB}{BC} = \tan C$$

i.e.,
$$\frac{5}{BC} = \tan 30° = \frac{1}{\sqrt{3}}$$

which gives
$$BC = 5\sqrt{3} \text{ cm}$$

To find the length of the side AC, we consider

$$\sin 30^\circ = \frac{AB}{AC} \qquad \text{(Why?)}$$

i.e., $$\frac{1}{2} = \frac{5}{AC}$$

i.e., $$AC = 10 \text{ cm}$$

Note that alternatively we could have used Pythagoras theorem to determine the third side in the example above,

i.e., $$AC = \sqrt{AB^2 + BC^2} = \sqrt{5^2 + (5\sqrt{3})^2} \text{ cm} = 10 \text{cm}.$$

Example 7 : In Δ PQR, right-angled at Q (see Fig. 8.20), PQ = 3 cm and PR = 6 cm. Determine $\angle$ QPR and $\angle$ PRQ.

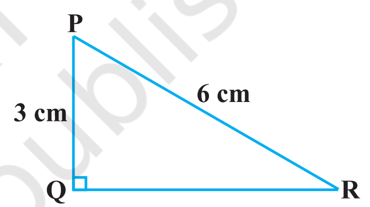


Fig. 8.20

Solution : Given PQ = 3 cm and PR = 6 cm.

Therefore, $$\frac{PQ}{PR} = \sin R$$

or $$\sin R = \frac{3}{6} = \frac{1}{2}$$

So, $$\angle PRQ = 30^\circ$$

and therefore, $$\angle QPR = 60^\circ. \qquad \text{(Why?)}$$

You may note that if one of the sides and any other part (either an acute angle or any side) of a right triangle is known, the remaining sides and angles of the triangle can be determined.

Example 8 : If $\sin (A - B) = \frac{1}{2}$, $\cos (A + B) = \frac{1}{2}$, $0^\circ < A + B \le 90^\circ$, $A > B$, find A and B.

Solution : Since, $\sin (A - B) = \frac{1}{2}$, therefore, $A - B = 30^\circ$ (Why?) (1)

Also, since $\cos (A + B) = \frac{1}{2}$, therefore, $A + B = 60^\circ$ (Why?) (2)

Solving (1) and (2), we get : $A = 45^\circ$ and $B = 15^\circ$.

EXERCISE 8.2

1. Evaluate the following :

(i) $\sin 60° \cos 30° + \sin 30° \cos 60°$

(ii) $2 \tan^2 45° + \cos^2 30° - \sin^2 60°$

(iii) $\dfrac{\cos 45°}{\sec 30° + \operatorname{cosec} 30°}$

(iv) $\dfrac{\sin 30° + \tan 45° - \operatorname{cosec} 60°}{\sec 30° + \cos 60° + \cot 45°}$

(v) $\dfrac{5 \cos^2 60° + 4 \sec^2 30° - \tan^2 45°}{\sin^2 30° + \cos^2 30°}$

2. Choose the correct option and justify your choice :

(i) $\dfrac{2 \tan 30°}{1 + \tan^2 30°} =$

(A) $\sin 60°$ (B) $\cos 60°$ (C) $\tan 60°$ (D) $\sin 30°$

(ii) $\dfrac{1 - \tan^2 45°}{1 + \tan^2 45°} =$

(A) $\tan 90°$ (B) 1 (C) $\sin 45°$ (D) 0

(iii) $\sin 2A = 2 \sin A$ is true when $A =$

(A) 0° (B) 30° (C) 45° (D) 60°

(iv) $\dfrac{2 \tan 30°}{1 - \tan^2 30°} =$

(A) $\cos 60°$ (B) $\sin 60°$ (C) $\tan 60°$ (D) $\sin 30°$

3. If $\tan (A + B) = \sqrt{3}$ and $\tan (A - B) = \dfrac{1}{\sqrt{3}}$; $0° < A + B \leq 90°$; $A > B$, find A and B.

4. State whether the following are true or false. Justify your answer.

(i) $\sin (A + B) = \sin A + \sin B$.

(ii) The value of $\sin \theta$ increases as θ increases.

(iii) The value of $\cos \theta$ increases as θ increases.

(iv) $\sin \theta = \cos \theta$ for all values of θ.

(v) $\cot A$ is not defined for $A = 0°$.

8.4 Trigonometric Identities

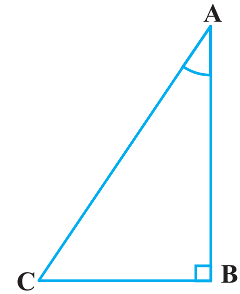


Fig. 8.21

You may recall that an equation is called an identity when it is true for all values of the variables involved. Similarly, an equation involving trigonometric ratios of an angle is called a **trigonometric identity,** if it is true for all values of the angle(s) involved.

In this section, we will prove one trigonometric identity, and use it further to prove other useful trigonometric identities.

In Δ ABC, right-angled at B (see Fig. 8.21), we have:

$$AB^2 + BC^2 = AC^2 \qquad (1)$$

Dividing each term of (1) by AC^2, we get

$$\frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} = \frac{AC^2}{AC^2}$$

i.e., $$\left(\frac{AB}{AC}\right)^2 + \left(\frac{BC}{AC}\right)^2 = \left(\frac{AC}{AC}\right)^2$$

i.e., $$(\cos A)^2 + (\sin A)^2 = 1$$

i.e., $$\mathbf{\cos^2 A + \sin^2 A = 1} \qquad (2)$$

This is true for all A such that $0° \leq A \leq 90°$. So, this is a trigonometric identity.

Let us now divide (1) by AB^2. We get

$$\frac{AB^2}{AB^2} + \frac{BC^2}{AB^2} = \frac{AC^2}{AB^2}$$

or, $$\left(\frac{AB}{AB}\right)^2 + \left(\frac{BC}{AB}\right)^2 = \left(\frac{AC}{AB}\right)^2$$

i.e., $$\mathbf{1 + \tan^2 A = \sec^2 A} \qquad (3)$$

Is this equation true for $A = 0°$? Yes, it is. What about $A = 90°$? Well, tan A and sec A are not defined for $A = 90°$. So, (3) is true for all A such that $0° \leq A < 90°$.

Let us see what we get on dividing (1) by BC^2. We get

$$\frac{AB^2}{BC^2} + \frac{BC^2}{BC^2} = \frac{AC^2}{BC^2}$$

i.e., $$\left(\frac{AB}{BC}\right)^2 + \left(\frac{BC}{BC}\right)^2 = \left(\frac{AC}{BC}\right)^2$$

i.e., $$\mathbf{cot^2\,A + 1 = cosec^2\,A} \qquad (4)$$

Note that cosec A and cot A are not defined for A = 0°. Therefore (4) is true for all A such that 0° < A ≤ 90°.

Using these identities, we can express each trigonometric ratio in terms of other trigonometric ratios, i.e., if any one of the ratios is known, we can also determine the values of other trigonometric ratios.

Let us see how we can do this using these identities. Suppose we know that $\tan A = \frac{1}{\sqrt{3}}$. Then, $\cot A = \sqrt{3}$.

Since, $\sec^2 A = 1 + \tan^2 A = 1 + \frac{1}{3} = \frac{4}{3}$, $\sec A = \frac{2}{\sqrt{3}}$, and $\cos A = \frac{\sqrt{3}}{2}$.

Again, $\sin A = \sqrt{1-\cos^2 A} = \sqrt{1-\frac{3}{4}} = \frac{1}{2}$. Therefore, cosec A = 2.

Example 9 : Express the ratios cos A, tan A and sec A in terms of sin A.

Solution : Since $\cos^2 A + \sin^2 A = 1$, therefore,

$$\cos^2 A = 1 - \sin^2 A, \text{ i.e., } \cos A = \pm\sqrt{1-\sin^2 A}$$

This gives $$\cos A = \sqrt{1-\sin^2 A} \qquad \text{(Why?)}$$

Hence, $$\tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\sqrt{1-\sin^2 A}} \text{ and } \sec A = \frac{1}{\cos A} = \frac{1}{\sqrt{1-\sin^2 A}}$$

Example 10 : Prove that sec A (1 – sin A)(sec A + tan A) = 1.

Solution :

$$\text{LHS} = \sec A\,(1-\sin A)(\sec A + \tan A) = \left(\frac{1}{\cos A}\right)(1-\sin A)\left(\frac{1}{\cos A} + \frac{\sin A}{\cos A}\right)$$

$$= \frac{(1 - \sin A)(1 + \sin A)}{\cos^2 A} = \frac{1 - \sin^2 A}{\cos^2 A}$$

$$= \frac{\cos^2 A}{\cos^2 A} = 1 = \text{RHS}$$

Example 11 : Prove that $\dfrac{\cot A - \cos A}{\cot A + \cos A} = \dfrac{\operatorname{cosec} A - 1}{\operatorname{cosec} A + 1}$

Solution : LHS = $\dfrac{\cot A - \cos A}{\cot A + \cos A} = \dfrac{\dfrac{\cos A}{\sin A} - \cos A}{\dfrac{\cos A}{\sin A} + \cos A}$

$$= \frac{\cos A\left(\dfrac{1}{\sin A} - 1\right)}{\cos A\left(\dfrac{1}{\sin A} + 1\right)} = \frac{\left(\dfrac{1}{\sin A} - 1\right)}{\left(\dfrac{1}{\sin A} + 1\right)} = \frac{\operatorname{cosec} A - 1}{\operatorname{cosec} A + 1} = \text{RHS}$$

Example 12 : Prove that $\dfrac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \dfrac{1}{\sec\theta - \tan\theta}$, using the identity $\sec^2\theta = 1 + \tan^2\theta$.

Solution : Since we will apply the identity involving $\sec\theta$ and $\tan\theta$, let us first convert the LHS (of the identity we need to prove) in terms of $\sec\theta$ and $\tan\theta$ by dividing numerator and denominator by $\cos\theta$.

$$\text{LHS} = \frac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \frac{\tan\theta - 1 + \sec\theta}{\tan\theta + 1 - \sec\theta}$$

$$= \frac{(\tan\theta + \sec\theta) - 1}{(\tan\theta - \sec\theta) + 1} = \frac{\{(\tan\theta + \sec\theta) - 1\}\,(\tan\theta - \sec\theta)}{\{(\tan\theta - \sec\theta) + 1\}\,(\tan\theta - \sec\theta)}$$

$$= \frac{(\tan^2\theta - \sec^2\theta) - (\tan\theta - \sec\theta)}{\{\tan\theta - \sec\theta + 1\}\,(\tan\theta - \sec\theta)}$$

$$= \frac{-1 - \tan\theta + \sec\theta}{(\tan\theta - \sec\theta + 1)\,(\tan\theta - \sec\theta)}$$

$$= \frac{-1}{\tan\theta - \sec\theta} = \frac{1}{\sec\theta - \tan\theta},$$

which is the RHS of the identity, we are required to prove.

EXERCISE 8.3

1. Express the trigonometric ratios sin A, sec A and tan A in terms of cot A.
2. Write all the other trigonometric ratios of $\angle$ A in terms of sec A.
3. Choose the correct option. Justify your choice.

   (i) $9\sec^2 A - 9\tan^2 A =$

   (A) 1 (B) 9 (C) 8 (D) 0

   (ii) $(1 + \tan\theta + \sec\theta)(1 + \cot\theta - \text{cosec}\,\theta) =$

   (A) 0 (B) 1 (C) 2 (D) –1

   (iii) $(\sec A + \tan A)(1 - \sin A) =$

   (A) sec A (B) sin A (C) cosec A (D) cos A

   (iv) $\frac{1 + \tan^2 A}{1 + \cot^2 A} =$

   (A) $\sec^2 A$ (B) –1 (C) $\cot^2 A$ (D) $\tan^2 A$

4. Prove the following identities, where the angles involved are acute angles for which the expressions are defined.

   (i) $(\text{cosec}\,\theta - \cot\theta)^2 = \frac{1 - \cos\theta}{1 + \cos\theta}$ (ii) $\frac{\cos A}{1 + \sin A} + \frac{1 + \sin A}{\cos A} = 2\sec A$

   (iii) $\frac{\tan\theta}{1 - \cot\theta} + \frac{\cot\theta}{1 - \tan\theta} = 1 + \sec\theta\,\text{cosec}\,\theta$

   [**Hint :** Write the expression in terms of sin θ and cos θ]

   (iv) $\frac{1 + \sec A}{\sec A} = \frac{\sin^2 A}{1 - \cos A}$ [**Hint :** Simplify LHS and RHS separately]

   (v) $\frac{\cos A - \sin A + 1}{\cos A + \sin A - 1} = \text{cosec}\,A + \cot A$, using the identity $\text{cosec}^2 A = 1 + \cot^2 A$.

   (vi) $\sqrt{\frac{1 + \sin A}{1 - \sin A}} = \sec A + \tan A$ (vii) $\frac{\sin\theta - 2\sin^3\theta}{2\cos^3\theta - \cos\theta} = \tan\theta$

   (viii) $(\sin A + \text{cosec}\,A)^2 + (\cos A + \sec A)^2 = 7 + \tan^2 A + \cot^2 A$

(ix) $(\operatorname{cosec} A - \sin A)(\sec A - \cos A) = \dfrac{1}{\tan A + \cot A}$

[**Hint :** Simplify LHS and RHS separately]

(x) $\left(\dfrac{1+\tan^2 A}{1+\cot^2 A}\right) = \left(\dfrac{1-\tan A}{1-\cot A}\right)^2 = \tan^2 A$

8.5 Summary

In this chapter, you have studied the following points :

1. In a right triangle ABC, right-angled at B,

$$\sin A = \frac{\text{side opposite to angle A}}{\text{hypotenuse}}, \quad \cos A = \frac{\text{side adjacent to angle A}}{\text{hypotenuse}}$$

$$\tan A = \frac{\text{side opposite to angle A}}{\text{side adjacent to angle A}}.$$

2. $\operatorname{cosec} A = \dfrac{1}{\sin A}$; $\sec A = \dfrac{1}{\cos A}$; $\tan A = \dfrac{1}{\cot A}$, $\tan A = \dfrac{\sin A}{\cos A}$.

3. If one of the trigonometric ratios of an acute angle is known, the remaining trigonometric ratios of the angle can be easily determined.

4. The values of trigonometric ratios for angles 0°, 30°, 45°, 60° and 90°.

5. The value of sin A or cos A never exceeds 1, whereas the value of sec A ($0° \le A < 90°$) or cosec A ($0° < A \le 90°$) is always greater than or equal to 1.

6. $\sin^2 A + \cos^2 A = 1$,

$\sec^2 A - \tan^2 A = 1$ for $0° \le A < 90°$,

$\operatorname{cosec}^2 A = 1 + \cot^2 A$ for $0° < A \le 90°$.

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SOME APPLICATIONS OF TRIGONOMETRY 9

9.1 Heights and Distances

In the previous chapter, you have studied about trigonometric ratios. In this chapter, you will be studying about some ways in which trigonometry is used in the life around you.

Let us consider Fig. 8.1 of prvious chapter, which is redrawn below in Fig. 9.1.

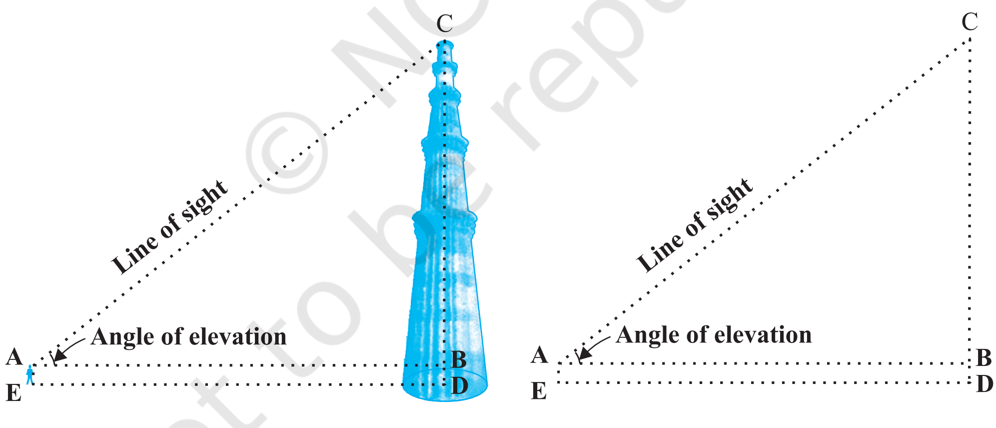


Fig. 9.1

In this figure, the line AC drawn from the eye of the student to the top of the minar is called the *line of sight*. The student is looking at the top of the minar. The angle BAC, so formed by the line of sight with the horizontal, is called the *angle of elevation* of the top of the minar from the eye of the student.

Thus, the **line of sight** is the line drawn from the eye of an observer to the point in the object viewed by the observer. The **angle of elevation** of the point viewed is the angle formed by the line of sight with the horizontal when the point being viewed is above the horizontal level, i.e., the case when we raise our head to look at the object (see Fig. 9.2).

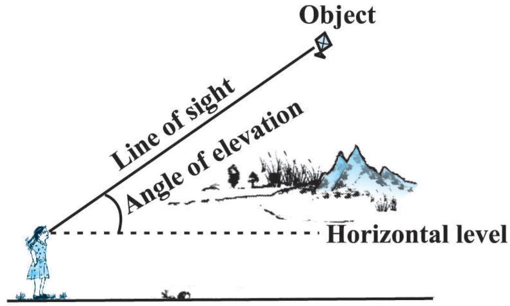


Fig. 9.2

Now, consider the situation given in Fig. 8.2. The girl sitting on the balcony is *looking down* at a flower pot placed on a stair of the temple. In this case, the line of sight is *below* the horizontal level. The angle so formed by the line of sight with the horizontal is called the *angle of depression*.

Thus, the **angle of depression** of a point on the object being viewed is the angle formed by the line of sight with the horizontal when the point is below the horizontal level, i.e., the case when we lower our head to look at the point being viewed (see Fig. 9.3).

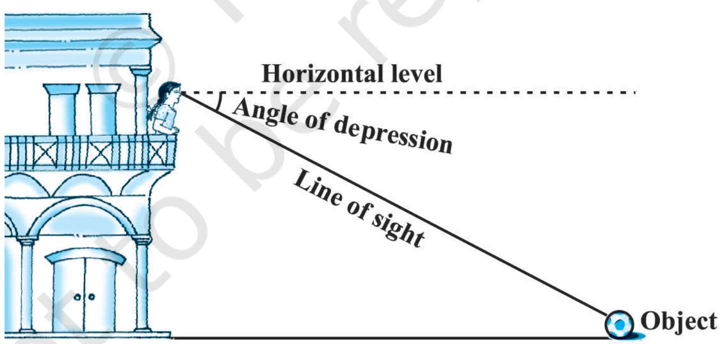


Fig. 9.3

Now, you may identify the lines of sight, and the angles so formed in Fig. 8.3. Are they angles of elevation or angles of depression?

Let us refer to Fig. 9.1 again. If you want to find the height CD of the minar without actually measuring it, what information do you need? You would need to know the following:

(i) the distance DE at which the student is standing from the foot of the minar

(ii) the angle of elevation, $\angle$ BAC, of the top of the minar

(iii) the height AE of the student.

Assuming that the above three conditions are known, how can we determine the height of the minar?

In the figure, CD = CB + BD. Here, BD = AE, which is the height of the student.

To find BC, we will use trigonometric ratios of $\angle$ BAC or $\angle$ A.

In Δ ABC, the side BC is the opposite side in relation to the known $\angle$ A. Now, which of the trigonometric ratios can we use? Which one of them has the two values that we have and the one we need to determine? Our search narrows down to using either tan A or cot A, as these ratios involve AB and BC.

Therefore, $\tan A = \frac{BC}{AB}$ or $\cot A = \frac{AB}{BC}$, which on solving would give us BC.

By adding AE to BC, you will get the height of the minar.

Now let us explain the process, we have just discussed, by solving some problems.

Example 1 : A tower stands vertically on the ground. From a point on the ground, which is 15 m away from the foot of the tower, the angle of elevation of the top of the tower is found to be 60°. Find the height of the tower.

Solution : First let us draw a simple diagram to represent the problem (see Fig. 9.4). Here AB represents the tower, CB is the distance of the point from the tower and $\angle$ ACB is the angle of elevation. We need to determine the height of the tower, i.e., AB. Also, ACB is a triangle, right-angled at B.

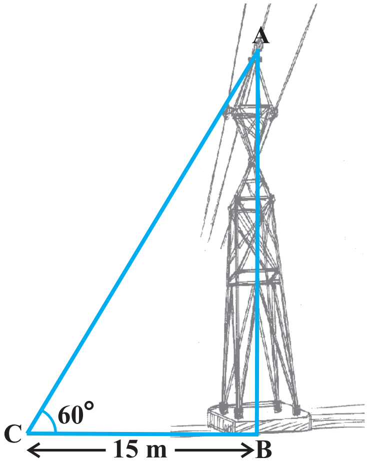


Fig. 9.4

To solve the problem, we choose the trigonometric ratio tan 60° (or cot 60°), as the ratio involves AB and BC.

Now, $$\tan 60° = \frac{AB}{BC}$$

i.e., $$\sqrt{3} = \frac{AB}{15}$$

i.e., $$AB = 15\sqrt{3}$$

Hence, the height of the tower is $15\sqrt{3}$ m.

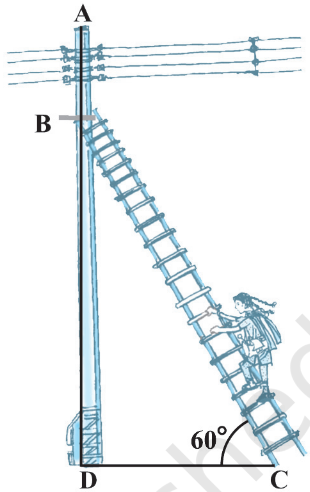


Fig. 9.5

Example 2 : An electrician has to repair an electric fault on a pole of height 5 m. She needs to reach a point 1.3m below the top of the pole to undertake the repair work (see Fig. 9.5). What should be the length of the ladder that she should use which, when inclined at an angle of 60° to the horizontal, would enable her to reach the required position? Also, how far from the foot of the pole should she place the foot of the ladder? (You may take $\sqrt{3} = 1.73$)

Solution : In Fig. 9.5, the electrician is required to reach the point B on the pole AD.

So, $\quad BD = AD - AB = (5 - 1.3)\text{m} = 3.7\text{ m}.$

Here, BC represents the ladder. We need to find its length, i.e., the hypotenuse of the right triangle BDC.

Now, can you think which trigonometic ratio should we consider?

It should be sin 60°.

So, $$\frac{BD}{BC} = \sin 60° \quad \text{or} \quad \frac{3.7}{BC} = \frac{\sqrt{3}}{2}$$

Therefore, $$BC = \frac{3.7 \times 2}{\sqrt{3}} = 4.28\text{ m (approx.)}$$

i.e., the length of the ladder should be 4.28 m.

Now, $$\frac{DC}{BD} = \cot 60° = \frac{1}{\sqrt{3}}$$

i.e., $$DC = \frac{3.7}{\sqrt{3}} = 2.14\text{ m (approx.)}$$

Therefore, she should place the foot of the ladder at a distance of 2.14 m from the pole.

Example 3 : An observer 1.5 m tall is 28.5 m away from a chimney. The angle of elevation of the top of the chimney from her eyes is 45°. What is the height of the chimney?

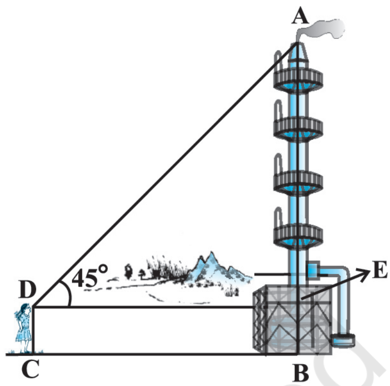


Fig. 9.6

Solution : Here, AB is the chimney, CD the observer and ∠ ADE the angle of elevation (see Fig. 9.6). In this case, ADE is a triangle, right-angled at E and we are required to find the height of the chimney.

We have $\quad AB = AE + BE = AE + 1.5$

and $\quad DE = CB = 28.5$ m

To determine AE, we choose a trigonometric ratio, which involves both AE and DE. Let us choose the tangent of the angle of elevation.

Now, $$\tan 45° = \frac{AE}{DE}$$

i.e., $$1 = \frac{AE}{28.5}$$

Therefore, $\quad AE = 28.5$

So the height of the chimney (AB) = (28.5 + 1.5) m = 30 m.

Example 4 : From a point P on the ground the angle of elevation of the top of a 10 m tall building is 30°. A flag is hoisted at the top of the building and the angle of elevation of the top of the flagstaff from P is 45°. Find the length of the flagstaff and the distance of the building from the point P. (You may take $\sqrt{3}$ = 1.732)

Solution : In Fig. 9.7, AB denotes the height of the building, BD the flagstaff and P the given point. Note that there are two right triangles PAB and PAD. We are required to find the length of the flagstaff, i.e., DB and the distance of the building from the point P, i.e., PA.

Since, we know the height of the building AB, we will first consider the right Δ PAB.

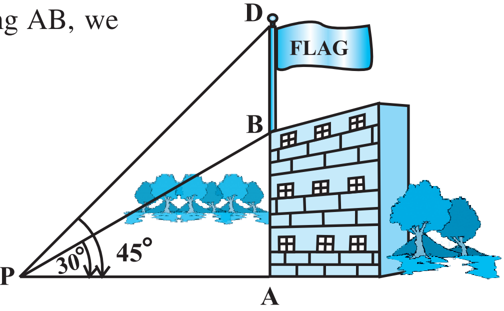


Fig. 9.7

We have $\tan 30° = \frac{AB}{AP}$

i.e., $\frac{1}{\sqrt{3}} = \frac{10}{AP}$

Therefore, $AP = 10\sqrt{3}$

i.e., the distance of the building from P is $10\sqrt{3}$ m = 17.32 m.

Next, let us suppose DB = x m. Then AD = $(10 + x)$ m.

Now, in right Δ PAD, $\tan 45° = \frac{AD}{AP} = \frac{10 + x}{10\sqrt{3}}$

Therefore, $1 = \frac{10 + x}{10\sqrt{3}}$

i.e., $x = 10\left(\sqrt{3} - 1\right) = 7.32$

So, the length of the flagstaff is 7.32 m.

Example 5 : The shadow of a tower standing on a level ground is found to be 40 m longer when the Sun's altitude is 30° than when it is 60°. Find the height of the tower.

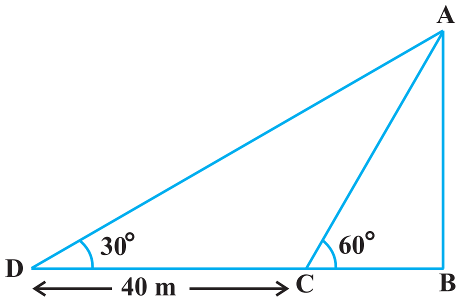


Fig. 9.8

Solution : In Fig. 9.8, AB is the tower and BC is the length of the shadow when the Sun's altitude is 60°, i.e., the angle of elevation of the top of the tower from the tip of the shadow is 60° and DB is the length of the shadow, when the angle of elevation is 30°.

Now, let AB be h m and BC be x m. According to the question, DB is 40 m longer than BC.

So, $\quad DB = (40 + x)$ m

Now, we have two right triangles ABC and ABD.

In Δ ABC, $\quad \tan 60° = \dfrac{AB}{BC}$

or, $\quad \sqrt{3} = \dfrac{h}{x} \qquad (1)$

In Δ ABD, $\quad \tan 30° = \dfrac{AB}{BD}$

i.e., $\quad \dfrac{1}{\sqrt{3}} = \dfrac{h}{x+40} \qquad (2)$

From (1), we have $\quad h = x\sqrt{3}$

Putting this value in (2), we get $\left(x\sqrt{3}\right)\sqrt{3} = x + 40$, i.e., $3x = x + 40$

i.e., $\quad x = 20$

So, $\quad h = 20\sqrt{3} \qquad$ [From (1)]

Therefore, the height of the tower is $20\sqrt{3}$ m.

Example 6 : The angles of depression of the top and the bottom of an 8 m tall building from the top of a multi-storeyed building are 30° and 45°, respectively. Find the height of the multi-storeyed building and the distance between the two buildings.

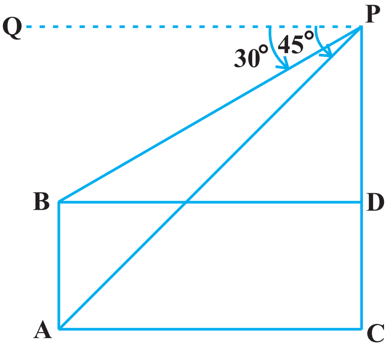


Fig. 9.9

Solution : In Fig. 9.9, PC denotes the multi-storyed building and AB denotes the 8 m tall building. We are interested to determine the height of the multi-storeyed building, i.e., PC and the distance between the two buildings, i.e., AC.

Look at the figure carefully. Observe that PB is a transversal to the parallel lines PQ and BD. Therefore, $\angle$QPB and $\angle$PBD are alternate angles, and so are equal. So $\angle$ PBD = 30°. Similarly, $\angle$ PAC = 45°.

In right Δ PBD, we have

$$\frac{PD}{BD} = \tan 30° = \frac{1}{\sqrt{3}} \quad \text{or} \quad BD = PD\sqrt{3}$$

In right Δ PAC, we have

$$\frac{PC}{AC} = \tan 45° = 1$$

i.e., $PC = AC$

Also, $PC = PD + DC$, therefore, $PD + DC = AC$.

Since, AC = BD and DC = AB = 8 m, we get PD + 8 = BD = $PD\sqrt{3}$ (Why?)

This gives $$PD = \frac{8}{\sqrt{3}-1} = \frac{8\left(\sqrt{3}+1\right)}{\left(\sqrt{3}+1\right)\left(\sqrt{3}-1\right)} = 4\left(\sqrt{3}+1\right)\text{m}.$$

So, the height of the multi-storeyed building is $\left\{4\left(\sqrt{3}+1\right)+8\right\}\text{m} = 4\left(3+\sqrt{3}\right)\text{m}$ and the distance between the two buildings is also $4\left(3+\sqrt{3}\right)\text{m}$.

Example 7 : From a point on a bridge across a river, the angles of depression of the banks on opposite sides of the river are 30° and 45°, respectively. If the bridge is at a height of 3 m from the banks, find the width of the river.

Solution : In Fig 9.10, A and B represent points on the bank on opposite sides of the river, so that AB is the width of the river. P is a point on the bridge at a height of 3 m, i.e., DP = 3 m. We are interested to determine the width of the river, which is the length of the side AB of the D APB.

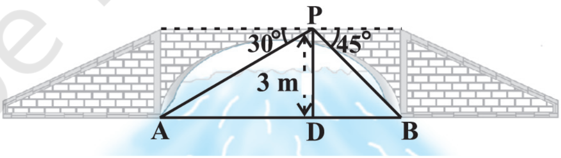


Fig. 9.10

Now, $AB = AD + DB$

In right Δ APD, ∠ A = 30°.

So, $$\tan 30° = \frac{PD}{AD}$$

i.e., $\frac{1}{\sqrt{3}} = \frac{3}{AD}$ or $AD = 3\sqrt{3}$ m

Also, in right Δ PBD, ∠ B = 45°. So, BD = PD = 3 m.

Now, $AB = BD + AD = 3 + 3\sqrt{3} = 3(1 + \sqrt{3})$ m.

Therefore, the width of the river is $3\left(\sqrt{3}+1\right)$m.

EXERCISE 9.1

1. A circus artist is climbing a 20 m long rope, which is tightly stretched and tied from the top of a vertical pole to the ground. Find the height of the pole, if the angle made by the rope with the ground level is 30° (see Fig. 9.11).

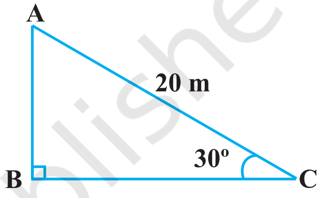


Fig. 9.11

2. A tree breaks due to storm and the broken part bends so that the top of the tree touches the ground making an angle 30° with it. The distance between the foot of the tree to the point where the top touches the ground is 8 m. Find the height of the tree.
3. A contractor plans to install two slides for the children to play in a park. For the children below the age of 5 years, she prefers to have a slide whose top is at a height of 1.5 m, and is inclined at an angle of 30° to the ground, whereas for elder children, she wants to have a steep slide at a height of 3m, and inclined at an angle of 60° to the ground. What should be the length of the slide in each case?
4. The angle of elevation of the top of a tower from a point on the ground, which is 30 m away from the foot of the tower, is 30°. Find the height of the tower.
5. A kite is flying at a height of 60 m above the ground. The string attached to the kite is temporarily tied to a point on the ground. The inclination of the string with the ground is 60°. Find the length of the string, assuming that there is no slack in the string.
6. A 1.5 m tall boy is standing at some distance from a 30 m tall building. The angle of elevation from his eyes to the top of the building increases from 30° to 60° as he walks towards the building. Find the distance he walked towards the building.
7. From a point on the ground, the angles of elevation of the bottom and the top of a transmission tower fixed at the top of a 20 m high building are 45° and 60° respectively. Find the height of the tower.

8. A statue, 1.6 m tall, stands on the top of a pedestal. From a point on the ground, the angle of elevation of the top of the statue is 60° and from the same point the angle of elevation of the top of the pedestal is 45°. Find the height of the pedestal.

9. The angle of elevation of the top of a building from the foot of the tower is 30° and the angle of elevation of the top of the tower from the foot of the building is 60°. If the tower is 50 m high, find the height of the building.

10. Two poles of equal heights are standing opposite each other on either side of the road, which is 80 m wide. From a point between them on the road, the angles of elevation of the top of the poles are 60° and 30°, respectively. Find the height of the poles and the distances of the point from the poles.

11. A TV tower stands vertically on a bank of a canal. From a point on the other bank directly opposite the tower, the angle of elevation of the top of the tower is 60°. From another point 20 m away from this point on the line joing this point to the foot of the tower, the angle of elevation of the top of the tower is 30° (see Fig. 9.12). Find the height of the tower and the width of the canal.

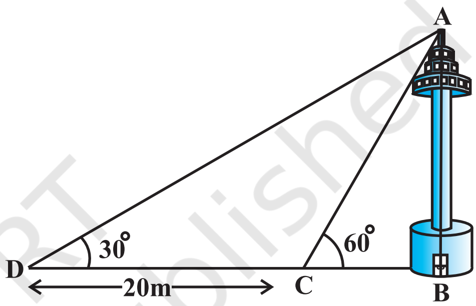


Fig. 9.12

12. From the top of a 7 m high building, the angle of elevation of the top of a cable tower is 60° and the angle of depression of its foot is 45°. Determine the height of the tower.

13. As observed from the top of a 75 m high lighthouse from the sea-level, the angles of depression of two ships are 30° and 45°. If one ship is exactly behind the other on the same side of the lighthouse, find the distance between the two ships.

14. A 1.2 m tall girl spots a balloon moving with the wind in a horizontal line at a height of 88.2 m from the ground. The angle of elevation of the balloon from the eyes of the girl at any instant is 60°. After some time, the angle of elevation reduces to 30° (see Fig. 9.13). Find the distance travelled by the balloon during the interval.

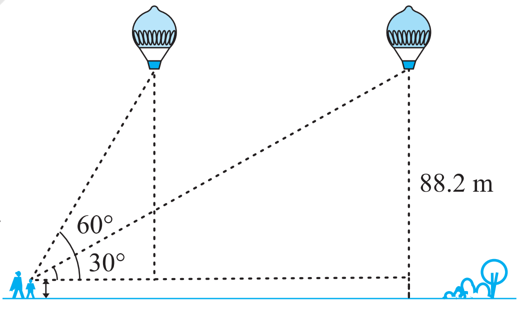


Fig. 9.13

15. A straight highway leads to the foot of a tower. A man standing at the top of the tower observes a car at an angle of depression of 30°, which is approaching the foot of the tower with a uniform speed. Six seconds later, the angle of depression of the car is found to be 60°. Find the time taken by the car to reach the foot of the tower from this point.

9.2 Summary

In this chapter, you have studied the following points :

1. (i) The **line of sight** is the line drawn from the eye of an observer to the point in the object viewed by the observer.

(ii) The **angle of elevation** of an object viewed, is the angle formed by the line of sight with the horizontal when it is above the horizontal level, i.e., the case when we raise our head to look at the object.

(iii) The **angle of depression** of an object viewed, is the angle formed by the line of sight with the horizontal when it is below the horizontal level, i.e., the case when we lower our head to look at the object.

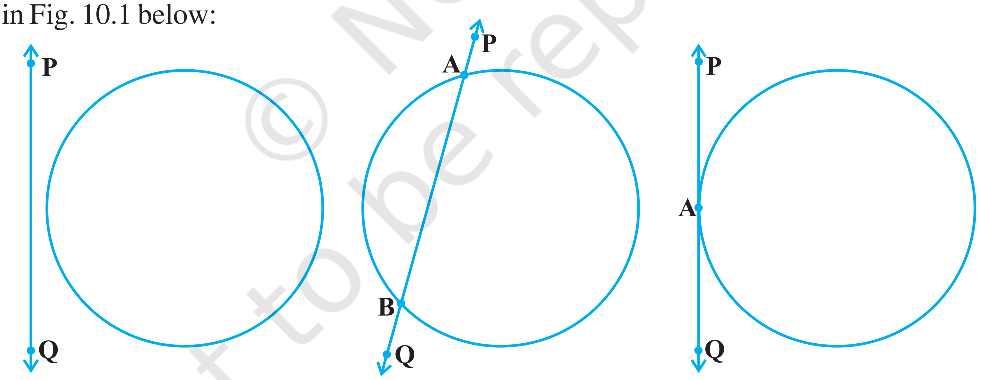
2. The height or length of an object or the distance between two distant objects can be determined with the help of trigonometric ratios.

CIRCLES 10

10.1 Introduction

You have studied in Class IX that a circle is a collection of all points in a plane which are at a constant distance (radius) from a fixed point (centre). You have also studied various terms related to a circle like chord, segment, sector, arc etc. Let us now examine the different situations that can arise when a circle and a line are given in a plane.

So, let us consider a circle and a line PQ. There can be three possibilities given in Fig. 10.1 below:



(i) (ii) (iii)

Fig. 10.1

In Fig. 10.1 (i), the line PQ and the circle have no common point. In this case, PQ is called a **non-intersecting** line with respect to the circle. In Fig. 10.1 (ii), there are two common points A and B that the line PQ and the circle have. In this case, we call the line PQ a **secant** of the circle. In Fig. 10.1 (iii), there is only one point A which is common to the line PQ and the circle. In this case, the line is called a **tangent** to the circle.

You might have seen a pulley fitted over a well which is used in taking out water from the well. Look at Fig. 10.2. Here the rope on both sides of the pulley, if considered as a ray, is like a tangent to the circle representing the pulley.

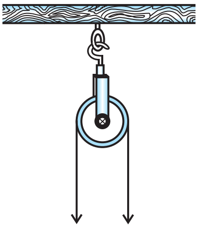


Fig. 10.2

Is there any position of the line with respect to the circle other than the types given above? You can see that there cannot be any other type of position of the line with respect to the circle. In this chapter, we will study about the existence of the tangents to a circle and also study some of their properties.

10.2 Tangent to a Circle

In the previous section, you have seen that **a tangent* to a circle is a line that intersects the circle at only one point**.

To understand the existence of the tangent to a circle at a point, let us perform the following activities:

Activity 1 : Take a circular wire and attach a straight wire AB at a point P of the circular wire so that it can rotate about the point P in a plane. Put the system on a table and gently rotate the wire AB about the point P to get different positions of the straight wire [see Fig. 10.3(i)].

In various positions, the wire intersects the circular wire at P and at another point Q_1 or Q_2 or Q_3, etc. In one position, you will see that it will intersect the circle at the point P only (see position A′B′ of AB). This shows that a tangent exists at the point P of the circle. On rotating further, you can observe that in all other positions of AB, it will intersect the circle at P and at another point, say R_1 or R_2 or R_3, etc. So, you can observe that **there is only one tangent at a point of the circle**.

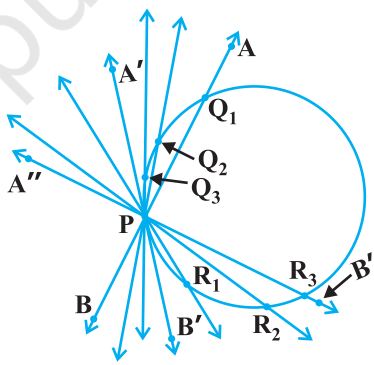


Fig. 10.3 (i)

While doing activity above, you must have observed that as the position AB moves towards the position A′ B′, the common point, say Q_1, of the line AB and the circle gradually comes nearer and nearer to the common point P. Ultimately, it coincides with the point P in the position A′B′ of A″B″. Again note, what happens if 'AB' is rotated rightwards about P? The common point R_3 gradually comes nearer and nearer to P and ultimately coincides with P. So, what we see is:

The tangent to a circle is a special case of the secant, when the two end points of its corresponding chord coincide.

*The word 'tangent' comes from the Latin word 'tangere', which means to touch and was introduced by the Danish mathematician Thomas Fineke in 1583.

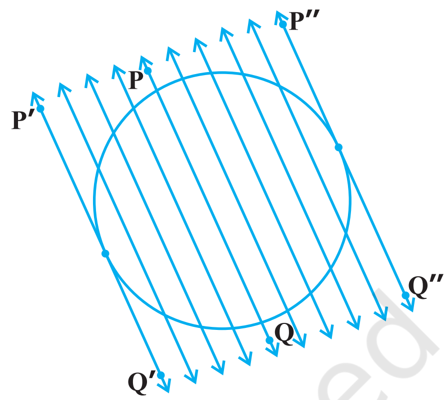


Fig. 10.3 (ii)

Activity 2 : On a paper, draw a circle and a secant PQ of the circle. Draw various lines parallel to the secant on both sides of it. You will find that after some steps, the length of the chord cut by the lines will gradually decrease, i.e., the two points of intersection of the line and the circle are coming closer and closer [see Fig. 10.3(ii)]. In one case, it becomes zero on one side of the secant and in another case, it becomes zero on the other side of the secant. See the positions P′Q′ and P″Q″ of the secant in Fig. 10.3 (ii). These are the tangents to the circle parallel to the given secant PQ. This also helps you to see that there cannot be more than two tangents parallel to a given secant.

This activity also establishes, what you must have observed, while doing Activity 1, namely, a tangent is the secant when both of the end points of the corresponding chord coincide.

The common point of the tangent and the circle is called the **point of contact** [the point A in Fig. 10.1 (iii)]and the tangent is said to **touch** the circle at the common point.

Now look around you. Have you seen a bicycle or a cart moving? Look at its wheels. All the spokes of a wheel are along its radii. Now note the position of the wheel with respect to its movement on the ground. Do you see any tangent anywhere? (See Fig. 10.4). In fact, the wheel moves along a line which is a tangent to the circle representing the wheel. Also, notice that in all positions, the radius through the point of contact with the ground appears to be at right angles to the tangent (see Fig. 10.4). We shall now prove this property of the tangent.

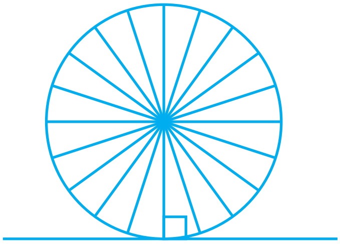


Fig. 10.4

Theorem 10.1 : *The tangent at any point of a circle is perpendicular to the radius through the point of contact.*

Proof : We are given a circle with centre O and a tangent XY to the circle at a point P. We need to prove that OP is perpendicular to XY.

Take a point Q on XY other than P and join OQ (see Fig. 10.5).

The point Q must lie outside the circle. (Why? Note that if Q lies inside the circle, XY will become a secant and not a tangent to the circle). Therefore, OQ is longer than the radius OP of the circle. That is,

$$OQ > OP.$$

Since this happens for every point on the line XY except the point P, OP is the shortest of all the distances of the point O to the points of XY. So OP is perpendicular to XY. (as shown in Theorem A1.7.)

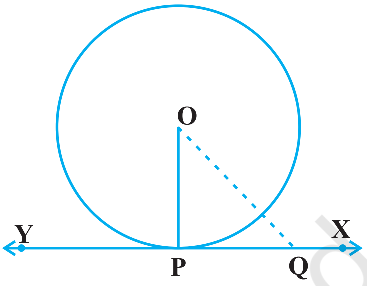


Fig. 10.5

Remarks

1. By theorem above, we can also conclude that at any point on a circle there can be one and only one tangent.

2. The line containing the radius through the point of contact is also sometimes called the 'normal' to the circle at the point.

EXERCISE 10.1

1. How many tangents can a circle have?
2. Fill in the blanks :
   (i) A tangent to a circle intersects it in ____________ point (s).
   (ii) A line intersecting a circle in two points is called a ____________ .
   (iii) A circle can have ____________ parallel tangents at the most.
   (iv) The common point of a tangent to a circle and the circle is called ____________ .
3. A tangent PQ at a point P of a circle of radius 5 cm meets a line through the centre O at a point Q so that OQ = 12 cm. Length PQ is :

   (A) 12 cm (B) 13 cm (C) 8.5 cm (D) $\sqrt{119}$ cm.
4. Draw a circle and two lines parallel to a given line such that one is a tangent and the other, a secant to the circle.

10.3 Number of Tangents from a Point on a Circle

To get an idea of the number of tangents from a point on a circle, let us perform the following activity:

Activity 3 : Draw a circle on a paper. Take a point P inside it. Can you draw a tangent to the circle through this point? You will find that all the lines through this point intersect the circle in two points. So, it is not possible to draw any tangent to a circle through a point inside it [see Fig. 10.6 (i)].

Next take a point P on the circle and draw tangents through this point. You have already observed that there is only one tangent to the circle at such a point [see Fig. 10.6 (ii)].

Finally, take a point P outside the circle and try to draw tangents to the circle from this point. What do you observe? You will find that you can draw exactly two tangents to the circle through this point [see Fig. 10.6 (iii)].

We can summarise these facts as follows:

Case 1 : There is no tangent to a circle passing through a point lying inside the circle.

Case 2 : There is one and only one tangent to a circle passing through a point lying on the circle.

Case 3 : There are exactly two tangents to a circle through a point lying outside the circle.

In Fig. 10.6 (iii), T_1 and T_2 are the points of contact of the tangents PT_1 and PT_2 respectively.

The length of the segment of the tangent from the external point P and the point of contact with the circle is called the **length of the tangent** from the point P to the circle.

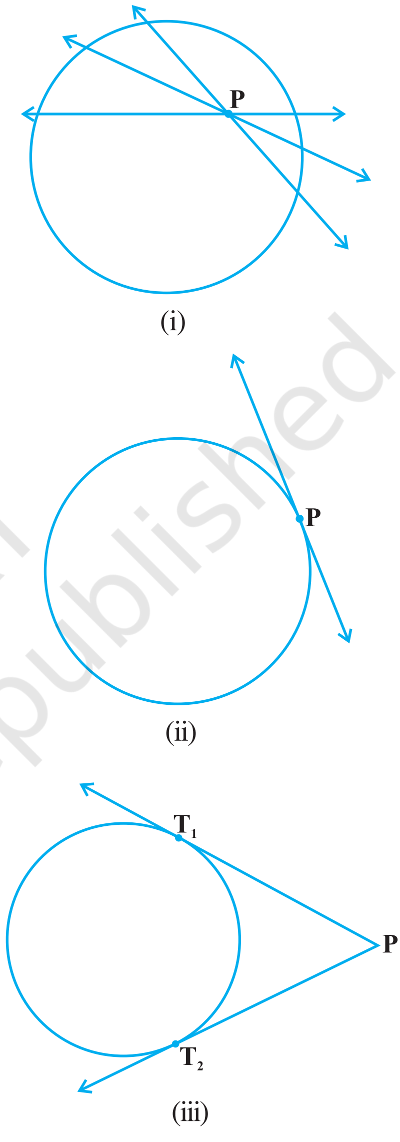


Fig. 10.6

Note that in Fig. 10.6 (iii), PT_1 and PT_2 are the lengths of the tangents from P to the circle. The lengths PT_1 and PT_2 have a common property. Can you find this? Measure PT_1 and PT_2. Are these equal? In fact, this is always so. Let us give a proof of this fact in the following theorem.

Theorem 10.2 : *The lengths of tangents drawn from an external point to a circle are equal.*

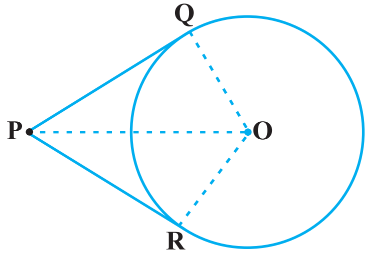


Fig. 10.7

Proof : We are given a circle with centre O, a point P lying outside the circle and two tangents PQ, PR on the circle from P (see Fig. 10.7). We are required to prove that PQ = PR.

For this, we join OP, OQ and OR. Then $\angle$ OQP and $\angle$ ORP are right angles, because these are angles between the radii and tangents, and according to Theorem 10.1 they are right angles. Now in right triangles OQP and ORP,

OQ = OR (Radii of the same circle)

OP = OP (Common)

Therefore, Δ OQP $\cong$ Δ ORP (RHS)

This gives PQ = PR (CPCT)

Remarks

1. The theorem can also be proved by using the Pythagoras Theorem as follows:

$$PQ^2 = OP^2 - OQ^2 = OP^2 - OR^2 = PR^2 \text{ (As } OQ = OR)$$

which gives PQ = PR.

2. Note also that $\angle$ OPQ = $\angle$ OPR. Therefore, OP is the angle bisector of $\angle$ QPR, i.e., the centre lies on the bisector of the angle between the two tangents.

Let us take some examples.

Example 1 : Prove that in two concentric circles, the chord of the larger circle, which touches the smaller circle, is bisected at the point of contact.

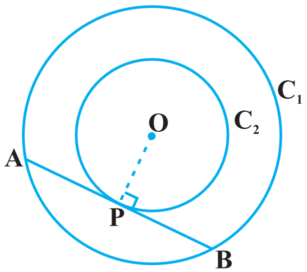


Fig. 10.8

Solution : We are given two concentric circles C_1 and C_2 with centre O and a chord AB of the larger circle C_1 which touches the smaller circle C_2 at the point P (see Fig. 10.8). We need to prove that AP = BP.

Let us join OP. Then, AB is a tangent to C_2 at P and OP is its radius. Therefore, by Theorem 10.1,

$$OP \perp AB$$

Now AB is a chord of the circle C_1 and $OP \perp AB$. Therefore, OP is the bisector of the chord AB, as the perpendicular from the centre bisects the chord,

i.e., $AP = BP$

Example 2 : Two tangents TP and TQ are drawn to a circle with centre O from an external point T. Prove that $\angle PTQ = 2 \angle OPQ$.

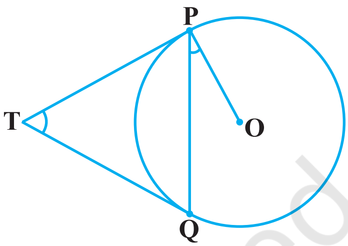


Fig. 10.9

Solution : We are given a circle with centre O, an external point T and two tangents TP and TQ to the circle, where P, Q are the points of contact (see Fig. 10.9). We need to prove that

$$\angle PTQ = 2 \angle OPQ$$

Let $\angle PTQ = \theta$

Now, by Theorem 10.2, TP = TQ. So, TPQ is an isosceles triangle.

Therefore, $\angle TPQ = \angle TQP = \frac{1}{2}(180° - \theta) = 90° - \frac{1}{2}\theta$

Also, by Theorem 10.1, $\angle OPT = 90°$

So, $\angle OPQ = \angle OPT - \angle TPQ = 90° - \left(90° - \frac{1}{2}\theta\right)$

$$= \frac{1}{2}\theta = \frac{1}{2}\angle PTQ$$

This gives $\angle PTQ = 2 \angle OPQ$

Example 3 : PQ is a chord of length 8 cm of a circle of radius 5 cm. The tangents at P and Q intersect at a point T (see Fig. 10.10). Find the length TP.

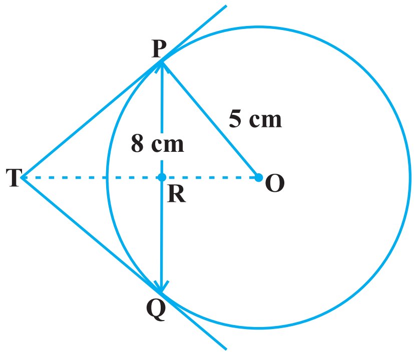


Fig. 10.10

Solution : Join OT. Let it intersect PQ at the point R. Then Δ TPQ is isosceles and TO is the angle bisector of $\angle$ PTQ. So, $OT \perp PQ$ and therefore, OT bisects PQ which gives PR = RQ = 4 cm.

Also, $OR = \sqrt{OP^2 - PR^2} = \sqrt{5^2 - 4^2}$ cm = 3 cm.

Now, $\angle$ TPR + $\angle$ RPO = 90° = $\angle$ TPR + $\angle$ PTR (Why?)

So, $\angle$ RPO = $\angle$ PTR

Therefore, right triangle TRP is similar to the right triangle PRO by AA similarity.

This gives $$\frac{TP}{PO} = \frac{RP}{RO}, \text{ i.e., } \frac{TP}{5} = \frac{4}{3} \text{ or } TP = \frac{20}{3} \text{ cm.}$$

Note : TP can also be found by using the Pythagoras Theorem, as follows:

Let TP = x and TR = y. Then

$$x^2 = y^2 + 16 \quad \text{(Taking right } \Delta \text{ PRT)} \qquad (1)$$

$$x^2 + 5^2 = (y + 3)^2 \quad \text{(Taking right } \Delta \text{ OPT)} \qquad (2)$$

Subtracting (1) from (2), we get

$$25 = 6y - 7 \quad \text{or} \quad y = \frac{32}{6} = \frac{16}{3}$$

Therefore, $$x^2 = \left(\frac{16}{3}\right)^2 + 16 = \frac{16}{9}(16 + 9) = \frac{16 \times 25}{9} \qquad \text{[From (1)]}$$

or $$x = \frac{20}{3}$$

EXERCISE 10.2

In Q.1 to 3, choose the correct option and give justification.

1. From a point Q, the length of the tangent to a circle is 24 cm and the distance of Q from the centre is 25 cm. The radius of the circle is

   (A) 7 cm (B) 12 cm

   (C) 15 cm (D) 24.5 cm

2. In Fig. 10.11, if TP and TQ are the two tangents to a circle with centre O so that $\angle$ POQ = 110°, then $\angle$ PTQ is equal to

   (A) 60° (B) 70°

   (C) 80° (D) 90°

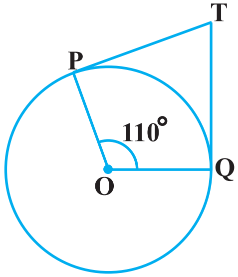


Fig. 10.11

3. If tangents PA and PB from a point P to a circle with centre O are inclined to each other at angle of 80°, then $\angle$ POA is equal to

   (A) 50° (B) 60°

   (C) 70° (D) 80°

4. Prove that the tangents drawn at the ends of a diameter of a circle are parallel.

5. Prove that the perpendicular at the point of contact to the tangent to a circle passes through the centre.

6. The length of a tangent from a point A at distance 5 cm from the centre of the circle is 4 cm. Find the radius of the circle.

7. Two concentric circles are of radii 5 cm and 3 cm. Find the length of the chord of the larger circle which touches the smaller circle.

8. A quadrilateral ABCD is drawn to circumscribe a circle (see Fig. 10.12). Prove that

$$AB + CD = AD + BC$$

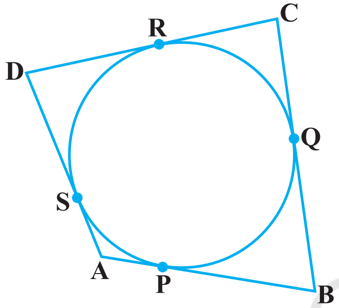


Fig. 10.12

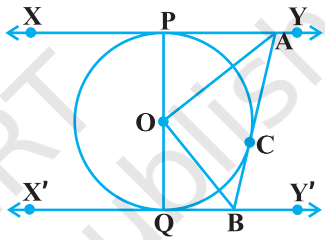


Fig. 10.13

9. In Fig. 10.13, XY and X′Y′ are two parallel tangents to a circle with centre O and another tangent AB with point of contact C intersecting XY at A and X′Y′ at B. Prove that $\angle$ AOB = 90°.

10. Prove that the angle between the two tangents drawn from an external point to a circle is supplementary to the angle subtended by the line-segment joining the points of contact at the centre.

11. Prove that the parallelogram circumscribing a circle is a rhombus.

12. A triangle ABC is drawn to circumscribe a circle of radius 4 cm such that the segments BD and DC into which BC is divided by the point of contact D are of lengths 8 cm and 6 cm respectively (see Fig. 10.14). Find the sides AB and AC.

13. Prove that opposite sides of a quadrilateral circumscribing a circle subtend supplementary angles at the centre of the circle.

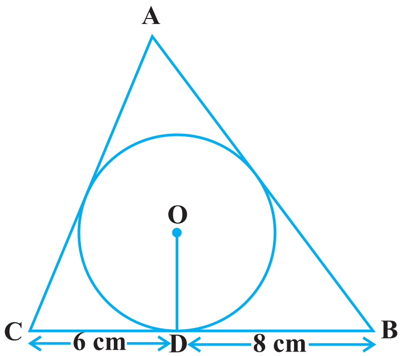


Fig. 10.14

10.4 Summary

In this chapter, you have studied the following points :

1. The meaning of a tangent to a circle.

2. The tangent to a circle is perpendicular to the radius through the point of contact.

3. The lengths of the two tangents from an external point to a circle are equal.

1062CH12

AREAS RELATED TO CIRCLES 11

11.1 Areas of Sector and Segment of a Circle

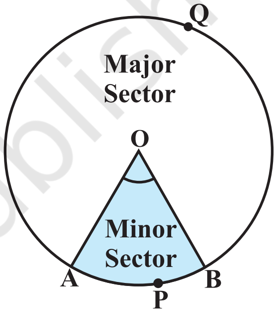


Fig. 11.1

You have already come across the terms *sector* and *segment* of a circle in your earlier classes. Recall that the portion (or part) of the circular region enclosed by two radii and the corresponding arc is called a *sector* of the circle and the portion (or part) of the circular region enclosed between a chord and the corresponding arc is called a *segment* of the circle. Thus, in Fig. 11.1, shaded region OAPB is a *sector* of the circle with centre O. $\angle$ AOB is called the *angle* of the sector. Note that in this figure, unshaded region OAQB is also a sector of the circle. For obvious reasons, OAPB is called the *minor sector* and OAQB is called the *major sector*. You can also see that angle of the major sector is $360° - \angle$ AOB.

Now, look at Fig. 11.2 in which AB is a chord of the circle with centre O. So, shaded region APB is a segment of the circle. You can also note that unshaded region AQB is another segment of the circle formed by the chord AB. For obvious reasons, APB is called the *minor segment* and AQB is called the *major segment*.

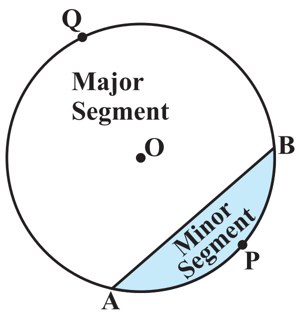


Fig. 11.2

Remark : When we write 'segment' and 'sector' we will mean the 'minor segment' and the 'minor sector' respectively, unless stated otherwise.

Now with this knowledge, let us try to find some relations (or formulae) to calculate their areas.

Let OAPB be a sector of a circle with centre O and radius r (see Fig. 11.3). Let the degree measure of $\angle$ AOB be θ.

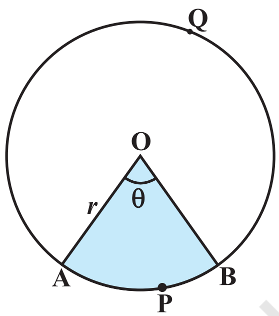


Fig. 11.3

You know that area of a circle (in fact of a circular region or disc) is πr^2.

In a way, we can consider this circular region to be a sector forming an angle of 360° (i.e., of degree measure 360) at the centre O. Now by applying the Unitary Method, we can arrive at the area of the sector OAPB as follows:

When degree measure of the angle at the centre is 360, area of the sector $= \pi r^2$

So, when the degree measure of the angle at the centre is 1, area of the sector $= \frac{\pi r^2}{360}$.

Therefore, when the degree measure of the angle at the centre is θ, area of the sector $= \frac{\pi r^2}{360} \times \theta = \frac{\theta}{360} \times \pi r^2$.

Thus, we obtain the following relation (or formula) for area of a sector of a circle:

Area of the sector of angle $\boldsymbol{\theta} = \frac{\boldsymbol{\theta}}{\mathbf{360}} \times \boldsymbol{\pi r^2}$,

where r is the radius of the circle and θ the angle of the sector in degrees.

Now, a natural question arises : Can we find the length of the arc APB corresponding to this sector? Yes. Again, by applying the Unitary Method and taking the whole length of the circle (of angle 360°) as $2\pi r$, we can obtain the required length of the arc APB as $\frac{\theta}{360} \times 2\pi r$.

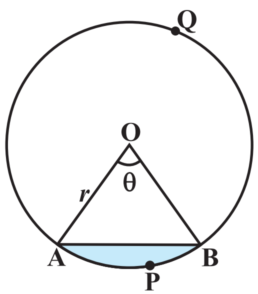


Fig. 11.4

So, **length of an arc of a sector of angle $\boldsymbol{\theta} = \frac{\boldsymbol{\theta}}{\mathbf{360}} \times \boldsymbol{2\pi r}$**.

Now let us take the case of the area of the segment APB of a circle with centre O and radius r (see Fig. 11.4). You can see that :

Area of the segment APB = Area of the sector OAPB – Area of Δ OAB

$$= \frac{\theta}{360} \times \pi r^2 - \text{area of } \Delta OAB$$

Note : From Fig. 11.3 and Fig. 11.4 respectively, you can observe that:

Area of the major sector OAQB = πr^2 – Area of the minor sector OAPB

and Area of major segment AQB = πr^2 – Area of the minor segment APB

Let us now take some examples to understand these concepts (or results).

Example 1 : Find the area of the sector of a circle with radius 4 cm and of angle 30°. Also, find the area of the corresponding major sector (Use $\pi = 3.14$).

Solution : Given sector is OAPB (see Fig. 11.5).

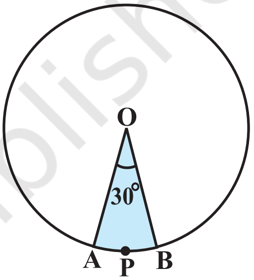


Fig. 11.5

$$\text{Area of the sector} = \frac{\theta}{360} \times \pi r^2$$

$$= \frac{30}{360} \times 3.14 \times 4 \times 4 \text{ cm}^2$$

$$= \frac{12.56}{3} \text{ cm}^2 = 4.19 \text{cm}^2 \text{ (approx.)}$$

Area of the corresponding major sector

$$= \pi r^2 - \text{area of sector OAPB}$$

$$= (3.14 \times 16 - 4.19) \text{ cm}^2$$

$$= 46.05 \text{ cm}^2 = 46.1 \text{ cm}^2 \text{(approx.)}$$

Alternatively, area of the major sector $= \dfrac{(360 - \theta)}{360} \times \pi r^2$

$$= \left(\frac{360 - 30}{360}\right) \times 3.14 \times 16 \text{ cm}^2$$

$$= \frac{330}{360} \times 3.14 \times 16 \text{cm}^2 = 46.05 \text{ cm}^2$$

$$= 46.1 \text{ cm}^2 \text{ (approx.)}$$

Example 2 : Find the area of the segment AYB shown in Fig. 11.6, if radius of the circle is 21 cm and $\angle$ AOB = 120°. (Use $\pi = \frac{22}{7}$)

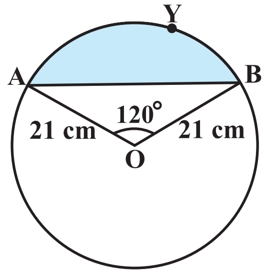


Fig. 11.6

Solution : Area of the segment AYB

= Area of sector OAYB – Area of Δ OAB (1)

Now, area of the sector OAYB = $\frac{120}{360} \times \frac{22}{7} \times 21 \times 21 \text{ cm}^2 = 462 \text{ cm}^2$ (2)

For finding the area of Δ OAB, draw OM $\perp$ AB as shown in Fig. 11.7.

Note that OA = OB. Therefore, by RHS congruence, Δ AMO $\cong$ Δ BMO.

So, M is the mid-point of AB and $\angle$ AOM = $\angle$ BOM = $\frac{1}{2} \times 120° = 60°$.

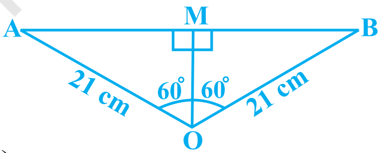


Fig. 11.7

Let OM = x cm

So, from Δ OMA, $\frac{\text{OM}}{\text{OA}} = \cos 60°$

or, $\frac{x}{21} = \frac{1}{2}$ $\left(\cos 60° = \frac{1}{2}\right)$

or, $x = \frac{21}{2}$

So, OM = $\frac{21}{2}$ cm

Also, $\frac{\text{AM}}{\text{OA}} = \sin 60° = \frac{\sqrt{3}}{2}$

So, AM = $\frac{21\sqrt{3}}{2}$ cm

Therefore, AB = 2 AM = $\frac{2 \times 21\sqrt{3}}{2}$ cm = $21\sqrt{3}$ cm

So, area of Δ OAB $= \frac{1}{2}$ AB $\times$ OM $= \frac{1}{2} \times 21\sqrt{3} \times \frac{21}{2}$ cm^2

$$= \frac{441}{4}\sqrt{3}\,\text{cm}^2 \qquad (3)$$

Therefore, area of the segment AYB $= \left(462 - \frac{441}{4}\sqrt{3}\right)$ cm^2 [From (1), (2) and (3)]

$$= \frac{21}{4}(88 - 21\sqrt{3})\,\text{cm}^2$$

EXERCISE 11.1

Unless stated otherwise, use $\pi = \frac{22}{7}$.

1. Find the area of a sector of a circle with radius 6 cm if angle of the sector is 60°.
2. Find the area of a quadrant of a circle whose circumference is 22 cm.
3. The length of the minute hand of a clock is 14 cm. Find the area swept by the minute hand in 5 minutes.
4. A chord of a circle of radius 10 cm subtends a right angle at the centre. Find the area of the corresponding : (i) minor segment (ii) major sector. (Use $\pi = 3.14$)
5. In a circle of radius 21 cm, an arc subtends an angle of 60° at the centre. Find:

   (i) the length of the arc (ii) area of the sector formed by the arc

   (iii) area of the segment formed by the corresponding chord
6. A chord of a circle of radius 15 cm subtends an angle of 60° at the centre. Find the areas of the corresponding minor and major segments of the circle.

   (Use $\pi = 3.14$ and $\sqrt{3} = 1.73$)
7. A chord of a circle of radius 12 cm subtends an angle of 120° at the centre. Find the area of the corresponding segment of the circle.

   (Use $\pi = 3.14$ and $\sqrt{3} = 1.73$)
8. A horse is tied to a peg at one corner of a square shaped grass field of side 15 m by means of a 5 m long rope (see Fig. 11.8). Find

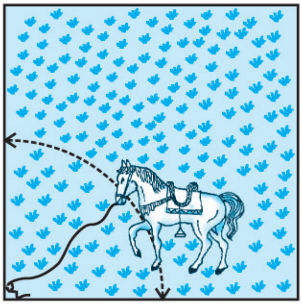


Fig. 11.8

(i) the area of that part of the field in which the horse can graze.

(ii) the increase in the grazing area if the rope were 10 m long instead of 5 m. (Use $\pi = 3.14$)

9. A brooch is made with silver wire in the form of a circle with diameter 35 mm. The wire is also used in making 5 diameters which divide the circle into 10 equal sectors as shown in Fig. 11.9. Find :

(i) the total length of the silver wire required.

(ii) the area of each sector of the brooch.

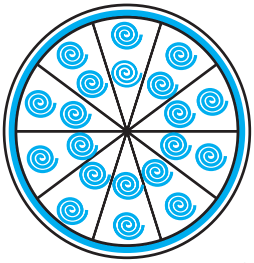


Fig. 11.9

10. An umbrella has 8 ribs which are equally spaced (see Fig. 11.10). Assuming umbrella to be a flat circle of radius 45 cm, find the area between the two consecutive ribs of the umbrella.

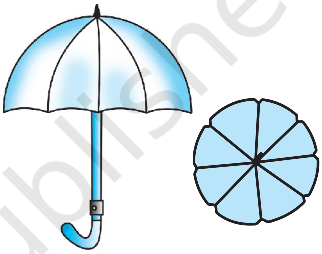


Fig. 11.10

11. A car has two wipers which do not overlap. Each wiper has a blade of length 25 cm sweeping through an angle of 115°. Find the total area cleaned at each sweep of the blades.

12. To warn ships for underwater rocks, a lighthouse spreads a red coloured light over a sector of angle 80° to a distance of 16.5 km. Find the area of the sea over which the ships are warned. (Use $\pi = 3.14$)

13. A round table cover has six equal designs as shown in Fig. 11.11. If the radius of the cover is 28 cm, find the cost of making the designs at the rate of ₹ 0.35 per cm². (Use $\sqrt{3} = 1.7$)

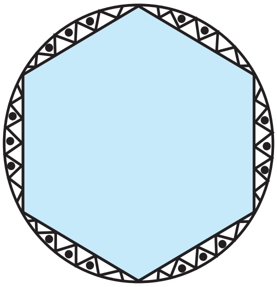


Fig. 11.11

14. Tick the correct answer in the following :

Area of a sector of angle p (in degrees) of a circle with radius R is

(A) $\frac{p}{180} \times 2\pi R$ (B) $\frac{p}{180} \times \pi R^2$ (C) $\frac{p}{360} \times 2\pi R$ (D) $\frac{p}{720} \times 2\pi R^2$

11.2 Summary

In this chapter, you have studied the following points :

1. Length of an arc of a sector of a circle with radius r and angle with degree measure θ is $\frac{\theta}{360} \times 2\pi r$.

2. Area of a sector of a circle with radius r and angle with degree measure θ is $\frac{\theta}{360} \times \pi r^2$.

3. Area of segment of a circle

= Area of the corresponding sector – Area of the corresponding triangle.

1062CH13

SURFACE AREAS AND VOLUMES 12

12.1 Introduction

From Class IX, you are familiar with some of the solids like cuboid, cone, cylinder, and sphere (see Fig. 12.1). You have also learnt how to find their surface areas and volumes.

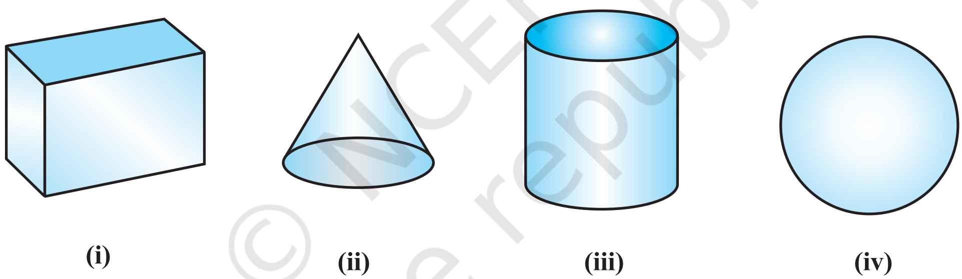


Fig. 12.1

In our day-to-day life, we come across a number of solids made up of combinations of two or more of the basic solids as shown above.

You must have seen a truck with a container fitted on its back (see Fig. 12.2), carrying oil or water from one place to another. Is it in the shape of any of the four basic solids mentioned above? You may guess that it is made of a cylinder with two hemispheres as its ends.

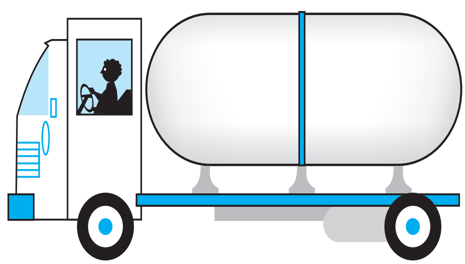


Fig. 12.2

Again, you may have seen an object like the one in Fig. 12.3. Can you name it? A test tube, right! You would have used one in your science laboratory. This tube is also a combination of a cylinder and a hemisphere. Similarly, while travelling, you may have seen some big and beautiful buildings or monuments made up of a combination of solids mentioned above.

If for some reason you wanted to find the surface areas, or volumes, or capacities of such objects, how would you do it? We cannot classify these under any of the solids you have already studied.

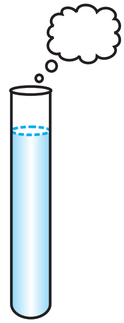


Fig. 12.3

In this chapter, you will see how to find surface areas and volumes of such objects.

12.2 Surface Area of a Combination of Solids

Let us consider the container seen in Fig. 12.2. How do we find the surface area of such a solid? Now, whenever we come across a new problem, we first try to see, if we can break it down into smaller problems, we have earlier solved. We can see that this solid is made up of a cylinder with two hemispheres stuck at either end. It would look like what we have in Fig. 12.4, after we put the pieces all together.

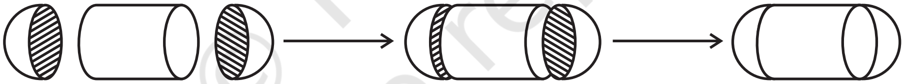


Fig. 12.4

If we consider the surface of the newly formed object, we would be able to see only the curved surfaces of the two hemispheres and the curved surface of the cylinder.

So, the *total* surface area of the new solid is the sum of the *curved* surface areas of each of the individual parts. This gives,

TSA of new solid = CSA of one hemisphere + CSA of cylinder

+ CSA of other hemisphere

where TSA, CSA stand for 'Total Surface Area' and 'Curved Surface Area' respectively.

Let us now consider another situation. Suppose we are making a toy by putting together a hemisphere and a cone. Let us see the steps that we would be going through.

First, we would take a cone and a hemisphere and bring their flat faces together. Here, of course, we would take the base radius of the cone equal to the radius of the hemisphere, for the toy is to have a smooth surface. So, the steps would be as shown in Fig. 12.5.

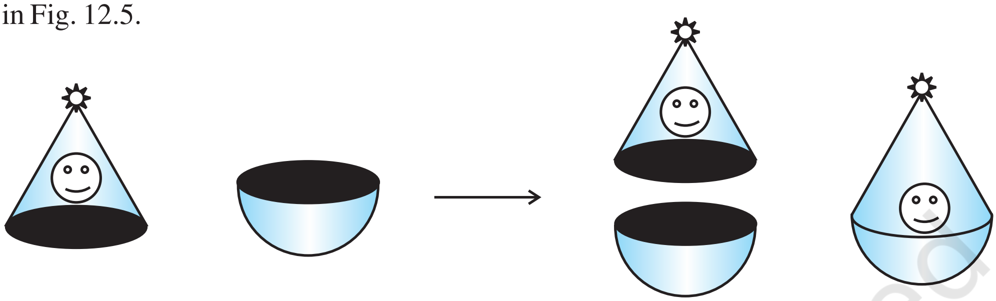


Fig. 12.5

At the end of our trial, we have got ourselves a nice round-bottomed toy. Now if we want to find how much paint we would require to colour the surface of this toy, what would we need to know? We would need to know the surface area of the toy, which consists of the CSA of the hemisphere and the CSA of the cone.

So, we can say:

Total surface area of the toy = CSA of hemisphere + CSA of cone

Now, let us consider some examples.

Example 1 : Rasheed got a playing top (*lattu*) as his birthday present, which surprisingly had no colour on it. He wanted to colour it with his crayons. The top is shaped like a cone surmounted by a hemisphere (see Fig 12.6). The entire top is 5 cm in height and the diameter of the top is 3.5 cm. Find the area he has to colour. (Take $\pi = \frac{22}{7}$)

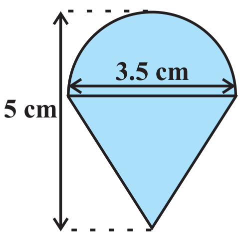


Fig. 12.6

Solution : This top is exactly like the object we have discussed in Fig. 12.5. So, we can conveniently use the result we have arrived at there. That is :

TSA of the toy = CSA of hemisphere + CSA of cone

Now, the curved surface area of the hemisphere $= \frac{1}{2}(4\pi r^2) = 2\pi r^2$

$$= \left(2 \times \frac{22}{7} \times \frac{3.5}{2} \times \frac{3.5}{2}\right) \text{cm}^2$$

Also, the height of the cone = height of the top – height (radius) of the hemispherical part

$$= \left(5 - \frac{3.5}{2}\right) \text{cm} = 3.25 \text{ cm}$$

So, the slant height of the cone $(l) = \sqrt{r^2 + h^2} = \sqrt{\left(\frac{3.5}{2}\right)^2 + (3.25)^2} \text{ cm} = 3.7 \text{ cm (approx.)}$

Therefore, CSA of cone $= \pi rl = \left(\frac{22}{7} \times \frac{3.5}{2} \times 3.7\right) \text{cm}^2$

This gives the surface area of the top as

$$= \left(2 \times \frac{22}{7} \times \frac{3.5}{2} \times \frac{3.5}{2}\right) \text{cm}^2 + \left(\frac{22}{7} \times \frac{3.5}{2} \times 3.7\right) \text{cm}^2$$

$$= \frac{22}{7} \times \frac{3.5}{2} (3.5 + 3.7) \text{ cm}^2 = \frac{11}{2} \times (3.5 + 3.7) \text{ cm}^2 = 39.6 \text{ cm}^2 \text{ (approx.)}$$

You may note that 'total surface area of the top' is *not* the sum of the total surface areas of the cone and hemisphere.

Example 2 : The decorative block shown in Fig. 12.7 is made of two solids — a cube and a hemisphere. The base of the block is a cube with edge 5 cm, and the hemisphere fixed on the top has a diameter of 4.2 cm. Find the total surface area of the block. (Take $\pi = \frac{22}{7}$)

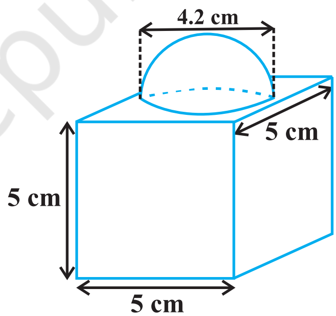


Fig. 12.7

Solution : The total surface area of the cube = 6 × (edge)^2 = 6 × 5 × 5 cm^2 = 150 cm^2.

Note that the part of the cube where the hemisphere is attached is not included in the surface area.

So, the surface area of the block = TSA of cube – base area of hemisphere + CSA of hemisphere

$$= 150 - \pi r^2 + 2\,\pi r^2 = (150 + \pi r^2) \text{ cm}^2$$

$$= 150 \text{ cm}^2 + \left(\frac{22}{7} \times \frac{4.2}{2} \times \frac{4.2}{2}\right) \text{cm}^2$$

$$= (150 + 13.86) \text{ cm}^2 = 163.86 \text{ cm}^2$$

Example 3 : A wooden toy rocket is in the shape of a cone mounted on a cylinder, as shown in Fig. 12.8. The height of the entire rocket is 26 cm, while the height of the conical part is 6 cm. The base of the conical portion has a diameter of 5 cm, while the base diameter of the cylindrical portion is 3 cm. If the conical portion is to be painted orange and the cylindrical portion yellow, find the area of the rocket painted with each of these colours. (Take $\pi = 3.14$)

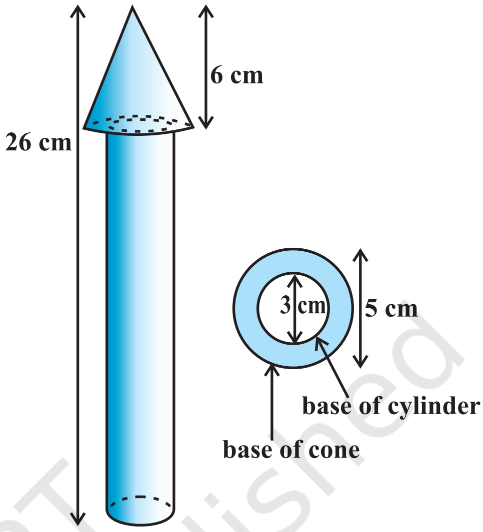


Fig. 12.8

Solution : Denote radius of cone by r, slant height of cone by l, height of cone by h, radius of cylinder by r' and height of cylinder by h'. Then $r = 2.5$ cm, $h = 6$ cm, $r' = 1.5$ cm, $h' = 26 - 6 = 20$ cm and

$$l = \sqrt{r^2 + h^2} = \sqrt{2.5^2 + 6^2} \text{ cm} = 6.5 \text{ cm}$$

Here, the conical portion has its circular base resting on the base of the cylinder, but the base of the cone is larger than the base of the cylinder. So, a part of the base of the cone (a ring) is to be painted.

So, the area to be painted orange = CSA of the cone + base area of the cone – base area of the cylinder

$$= \pi rl + \pi r^2 - \pi(r')^2$$

$$= \pi[(2.5 \times 6.5) + (2.5)^2 - (1.5)^2] \text{ cm}^2$$

$$= \pi[20.25] \text{ cm}^2 = 3.14 \times 20.25 \text{ cm}^2$$

$$= 63.585 \text{ cm}^2$$

Now, the area to be painted yellow = CSA of the cylinder + area of one base of the cylinder

$$= 2\pi r'h' + \pi(r')^2$$

$$= \pi r' (2h' + r')$$

$$= (3.14 \times 1.5) (2 \times 20 + 1.5) \text{ cm}^2$$

$$= 4.71 \times 41.5 \text{ cm}^2$$

$$= 195.465 \text{ cm}^2$$

Example 4 : Mayank made a bird-bath for his garden in the shape of a cylinder with a hemispherical depression at one end (see Fig. 12.9). The height of the cylinder is 1.45 m and its radius is 30 cm. Find the total surface area of the bird-bath. (Take $\pi = \frac{22}{7}$)

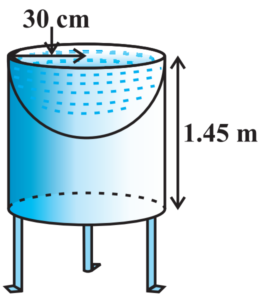


Fig. 12.9

Solution : Let h be height of the cylinder, and r the common radius of the cylinder and hemisphere. Then,

$$\begin{aligned}\text{the total surface area of the bird-bath} &= \text{CSA of cylinder} + \text{CSA of hemisphere}\\ &= 2\pi rh + 2\pi r^2 = 2\pi r(h + r)\\ &= 2 \times \frac{22}{7} \times 30(145 + 30)\ \text{cm}^2\\ &= 33000\ \text{cm}^2 = 3.3\ \text{m}^2\end{aligned}$$

EXERCISE 12.1

Unless stated otherwise, take $\pi = \frac{22}{7}$.

1. 2 cubes each of volume 64 cm^3 are joined end to end. Find the surface area of the resulting cuboid.
2. A vessel is in the form of a hollow hemisphere mounted by a hollow cylinder. The diameter of the hemisphere is 14 cm and the total height of the vessel is 13 cm. Find the inner surface area of the vessel.
3. A toy is in the form of a cone of radius 3.5 cm mounted on a hemisphere of same radius. The total height of the toy is 15.5 cm. Find the total surface area of the toy.
4. A cubical block of side 7 cm is surmounted by a hemisphere. What is the greatest diameter the hemisphere can have? Find the surface area of the solid.
5. A hemispherical depression is cut out from one face of a cubical wooden block such that the diameter l of the hemisphere is equal to the edge of the cube. Determine the surface area of the remaining solid.
6. A medicine capsule is in the shape of a cylinder with two hemispheres stuck to each of its ends (see Fig. 12.10). The length of the entire capsule is 14 mm and the diameter of the capsule is 5 mm. Find its surface area.

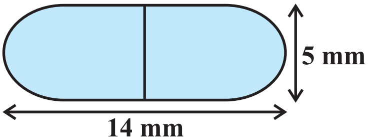


Fig. 12.10

7. A tent is in the shape of a cylinder surmounted by a conical top. If the height and diameter of the cylindrical part are 2.1 m and 4 m respectively, and the slant height of the top is 2.8 m, find the area of the canvas used for making the tent. Also, find the cost of the canvas of the tent at the rate of ₹500 per m^2. (Note that the base of the tent will not be covered with canvas.)

8. From a solid cylinder whose height is 2.4 cm and diameter 1.4 cm, a conical cavity of the same height and same diameter is hollowed out. Find the total surface area of the remaining solid to the nearest cm^2.

9. A wooden article was made by scooping out a hemisphere from each end of a solid cylinder, as shown in Fig. 12.11. If the height of the cylinder is 10 cm, and its base is of radius 3.5 cm, find the total surface area of the article.

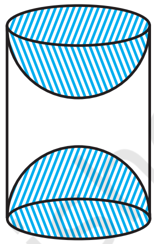


Fig. 12.11

12.3 Volume of a Combination of Solids

In the previous section, we have discussed how to find the surface area of solids made up of a combination of two basic solids. Here, we shall see how to calculate their volumes. It may be noted that in calculating the surface area, we have not added the surface areas of the two constituents, because some part of the surface area disappeared in the process of joining them. However, this will not be the case when we calculate the volume. The volume of the solid formed by joining two basic solids will actually be the sum of the volumes of the constituents, as we see in the examples below.

Example 5 : Shanta runs an industry in a shed which is in the shape of a cuboid surmounted by a half cylinder (see Fig. 12.12). If the base of the shed is of dimension 7 m × 15 m, and the height of the cuboidal portion is 8 m, find the volume of air that the shed can hold. Further, suppose the machinery in the shed occupies a total space of 300 m^3, and there are 20 workers, each of whom occupy about 0.08 m^3 space on an average. Then, how much air is in the shed? (Take $\pi = \frac{22}{7}$)

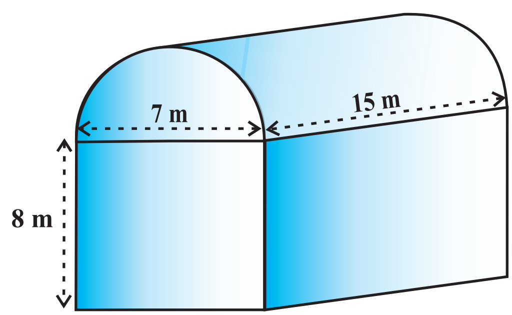


Fig. 12.12

Solution : The volume of air inside the shed (when there are no people or machinery) is given by the volume of air inside the cuboid and inside the half cylinder, taken together.

Now, the length, breadth and height of the cuboid are 15 m, 7 m and 8 m, respectively.

Also, the diameter of the half cylinder is 7 m and its height is 15 m.

So, the required volume = volume of the cuboid + $\frac{1}{2}$ volume of the cylinder

$$= \left[15 \times 7 \times 8 + \frac{1}{2} \times \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} \times 15\right] \text{m}^3 = 1128.75 \text{ m}^3$$

Next, the total space occupied by the machinery = 300 m^3

And the total space occupied by the workers = 20×0.08 m^3 = 1.6 m^3

Therefore, the volume of the air, when there are machinery and workers

$$= 1128.75 - (300.00 + 1.60) = 827.15 \text{ m}^3$$

Example 6 : A juice seller was serving his customers using glasses as shown in Fig. 12.13. The inner diameter of the cylindrical glass was 5 cm, but the bottom of the glass had a hemispherical raised portion which reduced the capacity of the glass. If the height of a glass was 10 cm, find the apparent capacity of the glass and its actual capacity. (Use $\pi = 3.14$.)

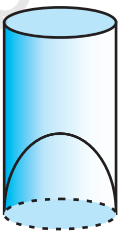


Fig. 12.13

Solution : Since the inner diameter of the glass = 5 cm and height = 10 cm,

the apparent capacity of the glass = $\pi r^2 h$

$$= 3.14 \times 2.5 \times 2.5 \times 10 \text{ cm}^3 = 196.25 \text{ cm}^3$$

But the actual capacity of the glass is less by the volume of the hemisphere at the base of the glass.

i.e., it is less by $\frac{2}{3}\pi r^3 = \frac{2}{3} \times 3.14 \times 2.5 \times 2.5 \times 2.5 \text{ cm}^3 = 32.71 \text{ cm}^3$

So, the actual capacity of the glass = apparent capacity of glass – volume of the hemisphere

$= (196.25 - 32.71)$ cm^3

$= 163.54$ cm^3

Example 7 : A solid toy is in the form of a hemisphere surmounted by a right circular cone. The height of the cone is 2 cm and the diameter of the base is 4 cm. Determine the volume of the toy. If a right circular cylinder circumscribes the toy, find the difference of the volumes of the cylinder and the toy. (Take $\pi = 3.14$)

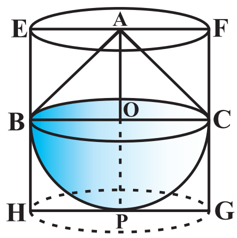


Fig. 12.14

Solution : Let BPC be the hemisphere and ABC be the cone standing on the base of the hemisphere (see Fig. 12.14). The radius BO of the hemisphere (as well as of the cone) $= \frac{1}{2} \times 4$ cm = 2 cm.

So, volume of the toy $= \frac{2}{3}\pi r^3 + \frac{1}{3}\pi r^2 h$

$$= \left[\frac{2}{3} \times 3.14 \times (2)^3 + \frac{1}{3} \times 3.14 \times (2)^2 \times 2\right] \text{cm}^3 = 25.12 \text{ cm}^3$$

Now, let the right circular cylinder EFGH circumscribe the given solid. The radius of the base of the right circular cylinder = HP = BO = 2 cm, and its height is

$$EH = AO + OP = (2 + 2) \text{ cm} = 4 \text{ cm}$$

So, the volume required = volume of the right circular cylinder – volume of the toy

$$= (3.14 \times 2^2 \times 4 - 25.12) \text{ cm}^3$$
$$= 25.12 \text{ cm}^3$$

Hence, the required difference of the two volumes = 25.12 cm^3.

EXERCISE 12.2

Unless stated otherwise, take $\pi = \frac{22}{7}$.

1. A solid is in the shape of a cone standing on a hemisphere with both their radii being equal to 1 cm and the height of the cone is equal to its radius. Find the volume of the solid in terms of π.

2. Rachel, an engineering student, was asked to make a model shaped like a cylinder with two cones attached at its two ends by using a thin aluminium sheet. The diameter of the model is 3 cm and its length is 12 cm. If each cone has a height of 2 cm, find the volume of air contained in the model that Rachel made. (Assume the outer and inner dimensions of the model to be nearly the same.)

3. A *gulab jamun,* contains sugar syrup up to about 30% of its volume. Find approximately how much syrup would be found in 45 *gulab jamuns,* each shaped like a cylinder with two hemispherical ends with length 5 cm and diameter 2.8 cm (see Fig. 12.15).

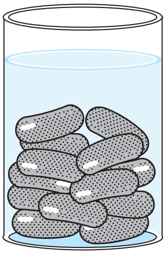


Fig. 12.15

4. A pen stand made of wood is in the shape of a cuboid with four conical depressions to hold pens. The dimensions of the cuboid are 15 cm by 10 cm by 3.5 cm. The radius of each of the depressions is 0.5 cm and the depth is 1.4 cm. Find the volume of wood in the entire stand (see Fig. 12.16).

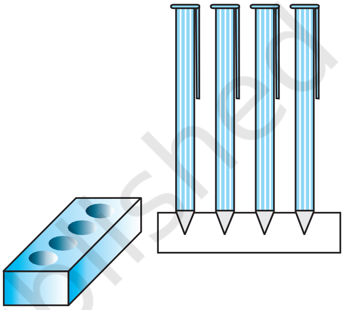


Fig. 12.16

5. A vessel is in the form of an inverted cone. Its height is 8 cm and the radius of its top, which is open, is 5 cm. It is filled with water up to the brim. When lead shots, each of which is a sphere of radius 0.5 cm are dropped into the vessel, one-fourth of the water flows out. Find the number of lead shots dropped in the vessel.
6. A solid iron pole consists of a cylinder of height 220 cm and base diameter 24 cm, which is surmounted by another cylinder of height 60 cm and radius 8 cm. Find the mass of the pole, given that 1 cm^3 of iron has approximately 8g mass. (Use $\pi = 3.14$)
7. A solid consisting of a right circular cone of height 120 cm and radius 60 cm standing on a hemisphere of radius 60 cm is placed upright in a right circular cylinder full of water such that it touches the bottom. Find the volume of water left in the cylinder, if the radius of the cylinder is 60 cm and its height is 180 cm.
8. A spherical glass vessel has a cylindrical neck 8 cm long, 2 cm in diameter; the diameter of the spherical part is 8.5 cm. By measuring the amount of water it holds, a child finds its volume to be 345 cm^3. Check whether she is correct, taking the above as the inside measurements, and $\pi = 3.14$.

12.4 Summary

In this chapter, you have studied the following points:

1. To determine the surface area of an object formed by combining any two of the basic solids, namely, cuboid, cone, cylinder, sphere and hemisphere.
2. To find the volume of objects formed by combining any two of a cuboid, cone, cylinder, sphere and hemisphere.

1062CH14

STATISTICS 13

13.1 Introduction

In Class IX, you have studied the classification of given data into ungrouped as well as grouped frequency distributions. You have also learnt to represent the data pictorially in the form of various graphs such as bar graphs, histograms (including those of varying widths) and frequency polygons. In fact, you went a step further by studying certain numerical representatives of the ungrouped data, also called measures of central tendency, namely, *mean*, *median* and *mode*. In this chapter, we shall extend the study of these three measures, i.e., mean, median and mode from ungrouped data to that of *grouped data*. We shall also discuss the concept of cumulative frequency, the cumulative frequency distribution and how to draw cumulative frequency curves, called *ogives*.

13.2 Mean of Grouped Data

The mean (or average) of observations, as we know, is the sum of the values of all the observations divided by the total number of observations. From Class IX, recall that if $x_1, x_2, \ldots, x_n$ are observations with respective frequencies $f_1, f_2, \ldots, f_n$, then this means observation x_1 occurs f_1 times, x_2 occurs f_2 times, and so on.

Now, the sum of the values of all the observations $= f_1x_1 + f_2x_2 + \ldots + f_nx_n$, and the number of observations $= f_1 + f_2 + \ldots + f_n$.

So, the mean $\overline{x}$ of the data is given by

$$\overline{x} = \frac{f_1x_1 + f_2x_2 + \cdots + f_nx_n}{f_1 + f_2 + \cdots + f_n}$$

Recall that we can write this in short form by using the Greek letter Σ (capital sigma) which means summation. That is,

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$$

which, more briefly, is written as $\bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i}$, if it is understood that i varies from 1 to n.

Let us apply this formula to find the mean in the following example.

Example 1 : The marks obtained by 30 students of Class X of a certain school in a Mathematics paper consisting of 100 marks are presented in table below. Find the mean of the marks obtained by the students.

Marks obtained (x_i)	10	20	36	40	50	56	60	70	72	80	88	92	95
Number of students (f_i)	1	1	3	4	3	2	4	4	1	1	2	3	1

Solution: Recall that to find the mean marks, we require the product of each x_i with the corresponding frequency f_i. So, let us put them in a column as shown in Table 13.1.

Table 13.1

Marks obtained (x_i)	**Number of students (f_i)**	$f_i x_i$
10	1	10
20	1	20
36	3	108
40	4	160
50	3	150
56	2	112
60	4	240
70	4	280
72	1	72
80	1	80
88	2	176
92	3	276
95	1	95
Total	$\Sigma f_i = 30$	$\Sigma f_i x_i = 1779$

Now, $$\bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{1779}{30} = 59.3$$

Therefore, the mean marks obtained is 59.3.

In most of our real life situations, data is usually so large that to make a meaningful study it needs to be condensed as grouped data. So, we need to convert given ungrouped data into grouped data and devise some method to find its mean.

Let us convert the ungrouped data of Example 1 into grouped data by forming class-intervals of width, say 15. Remember that, while allocating frequencies to each class-interval, students falling in any upper class-limit would be considered in the next class, e.g., 4 students who have obtained 40 marks would be considered in the class-interval 40-55 and not in 25-40. With this convention in our mind, let us form a grouped frequency distribution table (see Table 13.2).

Table 13.2

Class interval	10 - 25	25 - 40	40 - 55	55 - 70	70 - 85	85 - 100
Number of students	2	3	7	6	6	6

Now, for each class-interval, we require a point which would serve as the representative of the whole class. *It is assumed that the frequency of each class-interval is centred around its mid-point*. So the *mid-point* (or *class mark*) of each class can be chosen to represent the observations falling in the class. Recall that we find the mid-point of a class (or its class mark) by finding the average of its upper and lower limits. That is,

$$\textbf{Class mark} = \frac{\textbf{Upper class limit + Lower class limit}}{\textbf{2}}$$

With reference to Table 13.2, for the class 10-25, the class mark is $\frac{10+25}{2}$, i.e., 17.5. Similarly, we can find the class marks of the remaining class intervals. We put them in Table 13.3. These class marks serve as our x_i's. Now, in general, for the ith class interval, we have the frequency f_i corresponding to the class mark x_i. We can now proceed to compute the mean in the same manner as in Example 1.

Table 13.3

Class interval	Number of students (f_i)	Class mark (x_i)	$f_i x_i$
10 - 25	2	17.5	35.0
25 - 40	3	32.5	97.5
40 - 55	7	47.5	332.5
55 - 70	6	62.5	375.0
70 - 85	6	77.5	465.0
85 - 100	6	92.5	555.0
Total	$\Sigma f_i = 30$		$\Sigma f_i x_i = 1860.0$

The sum of the values in the last column gives us $\Sigma f_i x_i$. So, the mean $\overline{x}$ of the given data is given by

$$\overline{x} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{1860.0}{30} = 62$$

This new method of finding the mean is known as the **Direct Method**.

We observe that Tables 13.1 and 13.3 are using the same data and employing the same formula for the calculation of the mean but the results obtained are different. Can you think why this is so, and which one is more accurate? The difference in the two values is because of the mid-point assumption in Table 13.3, 59.3 being the exact mean, while 62 an approximate mean.

Sometimes when the numerical values of x_i and f_i are large, finding the product of x_i and f_i becomes tedious and time consuming. So, for such situations, let us think of a method of reducing these calculations.

We can do nothing with the f_i's, but we can change each x_i to a smaller number so that our calculations become easy. How do we do this? What about subtracting a fixed number from each of these x_i's? Let us try this method.

The first step is to choose one among the x_i's as the *assumed mean*, and denote it by 'a'. Also, to further reduce our calculation work, we may take 'a' to be that x_i which lies in the centre of $x_1, x_2, \ldots, x_n$. So, we can choose $a = 47.5$ or $a = 62.5$. Let us choose $a = 47.5$.

The next step is to find the difference d_i between a and each of the x_i's, that is, the **deviation** of 'a' from each of the x_i's.

i.e., $$d_i = x_i - a = x_i - 47.5$$

The third step is to find the product of d_i with the corresponding f_i, and take the sum of all the $f_i d_i$'s. The calculations are shown in Table 13.4.

Table 13.4

Class interval	**Number of students (f_i)**	**Class mark (x_i)**	**$d_i = x_i - 47.5$**	**$f_i d_i$**
10 - 25	2	17.5	–30	–60
25 - 40	3	32.5	–15	–45
40 - 55	7	47.5	0	0
55 - 70	6	62.5	15	90
70 - 85	6	77.5	30	180
85 - 100	6	92.5	45	270
Total	$\Sigma f_i = 30$			$\Sigma f_i d_i = 435$

So, from Table 13.4, the mean of the deviations, $\bar{d} = \dfrac{\Sigma f_i d_i}{\Sigma f_i}$.

Now, let us find the relation between $\bar{d}$ and $\bar{x}$.

Since in obtaining d_i, we subtracted 'a' from each x_i, so, in order to get the mean $\bar{x}$, we need to add 'a' to $\bar{d}$. This can be explained mathematically as:

Mean of deviations, $$\bar{d} = \frac{\Sigma f_i d_i}{\Sigma f_i}$$

So, $$\bar{d} = \frac{\Sigma f_i (x_i - a)}{\Sigma f_i}$$

$$= \frac{\Sigma f_i x_i}{\Sigma f_i} - \frac{\Sigma f_i a}{\Sigma f_i}$$

$$= \bar{x} - a\frac{\Sigma f_i}{\Sigma f_i}$$

$$= \bar{x} - a$$

So, $$\bar{x} = a + \bar{d}$$

i.e., $$\bar{x} = a + \frac{\Sigma f_i d_i}{\Sigma f_i}$$

Substituting the values of a, $\Sigma f_i d_i$ and Σf_i from Table 13.4, we get

$$\bar{x} = 47.5 + \frac{435}{30} = 47.5 + 14.5 = 62.$$

Therefore, the mean of the marks obtained by the students is 62.

The method discussed above is called the **Assumed Mean Method**.

Activity 1 : From the Table 13.3 find the mean by taking each of x_i (i.e., 17.5, 32.5, and so on) as 'a'. What do you observe? You will find that the mean determined in each case is the same, i.e., 62. (Why ?)

So, we can say that the value of the mean obtained does not depend on the choice of 'a'.

Observe that in Table 13.4, the values in Column 4 are all multiples of 15. So, if we divide the values in the entire Column 4 by 15, we would get smaller numbers to multiply with f_i. (Here, 15 is the class size of each class interval.)

So, let $u_i = \dfrac{x_i - a}{h}$, where a is the assumed mean and h is the class size.

Now, we calculate u_i in this way and continue as before (i.e., find $f_i u_i$ and then $\Sigma f_i u_i$). Taking h = 15, let us form Table 13.5.

Table 13.5

Class interval	f_i	x_i	$d_i = x_i - a$	$u_i = \dfrac{x_i - a}{h}$	$f_i u_i$
10 - 25	2	17.5	–30	–2	–4
25 - 40	3	32.5	–15	–1	–3
40 - 55	7	47.5	0	0	0
55 - 70	6	62.5	15	1	6
70 - 85	6	77.5	30	2	12
85 - 100	6	92.5	45	3	18
Total	$\Sigma f_i = 30$				$\Sigma f_i u_i = 29$

Let $$\bar{u} = \frac{\Sigma f_i u_i}{\Sigma f_i}$$

Here, again let us find the relation between $\bar{u}$ and $\bar{x}$.

We have, $$u_i = \frac{x_i - a}{h}$$

Therefore, $$\bar{u} = \frac{\Sigma f_i \dfrac{(x_i - a)}{h}}{\Sigma f_i} = \frac{1}{h}\left[\frac{\Sigma f_i x_i - a\Sigma f_i}{\Sigma f_i}\right]$$

$$= \frac{1}{h}\left[\frac{\Sigma f_i x_i}{\Sigma f_i} - a\frac{\Sigma f_i}{\Sigma f_i}\right]$$

$$= \frac{1}{h}[\bar{x} - a]$$

So, $$h\bar{u} = \bar{x} - a$$

i.e., $$\bar{x} = a + h\bar{u}$$

So, $$\bar{x} = a + h\left(\frac{\Sigma f_i u_i}{\Sigma f_i}\right)$$

Now, substituting the values of a, h, $\Sigma f_i u_i$ and Σf_i from Table 14.5, we get

$$\bar{x} = 47.5 + 15 \times \left(\frac{29}{30}\right)$$

$$= 47.5 + 14.5 = 62$$

So, the mean marks obtained by a student is 62.

The method discussed above is called the **Step-deviation** method.

We note that :

- the step-deviation method will be convenient to apply if all the d_i's have a common factor.
- The mean obtained by all the three methods is the same.
- The assumed mean method and step-deviation method are just simplified forms of the direct method.
- The formula $\bar{x} = a + h\bar{u}$ still holds if a and h are not as given above, but are any non-zero numbers such that $u_i = \dfrac{x_i - a}{h}$.

Let us apply these methods in another example.

Example 2 : The table below gives the percentage distribution of female teachers in the primary schools of rural areas of various states and union territories (U.T.) of India. Find the mean percentage of female teachers by all the three methods discussed in this section.

Percentage of female teachers	15 - 25	25 - 35	35 - 45	45 - 55	55 - 65	65 - 75	75 - 85
Number of States/U.T.	6	11	7	4	4	2	1

Source : *Seventh All India School Education Survey conducted by NCERT*

Solution : Let us find the class marks, x_i, of each class, and put them in a column (see Table 13.6):

Table 13.6

Percentage of female teachers	**Number of States /U.T. (f_i)**	x_i
15 - 25	6	20
25 - 35	11	30
35 - 45	7	40
45 - 55	4	50
55 - 65	4	60
65 - 75	2	70
75 - 85	1	80

Here we take $a = 50$, $h = 10$, then $d_i = x_i - 50$ and $u_i = \frac{x_i - 50}{10}$.

We now find d_i and u_i and put them in Table 13.7.

Table 13.7

Percentage of female teachers	Number of states/U.T. (f_i)	x_i	$d_i = x_i - 50$	$u_i = \frac{x_i - 50}{10}$	$f_i x_i$	$f_i d_i$	$f_i u_i$
15 - 25	6	20	–30	–3	120	–180	–18
25 - 35	11	30	–20	–2	330	–220	–22
35 - 45	7	40	–10	–1	280	–70	–7
45 - 55	4	50	0	0	200	0	0
55 - 65	4	60	10	1	240	40	4
65 - 75	2	70	20	2	140	40	4
75 - 85	1	80	30	3	80	30	3
Total	**35**				**1390**	**–360**	**–36**

From the table above, we obtain $\Sigma f_i = 35$, $\Sigma f_i x_i = 1390$,

$$\Sigma f_i d_i = -360, \quad \Sigma f_i u_i = -36.$$

Using the direct method, $\bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{1390}{35} = 39.71$

Using the assumed mean method,

$$\bar{x} = a + \frac{\Sigma f_i d_i}{\Sigma f_i} = 50 + \frac{(-360)}{35} = 39.71$$

Using the step-deviation method,

$$\bar{x} = a + \left(\frac{\Sigma f_i u_i}{\Sigma f_i}\right) \times h = 50 + \left(\frac{-36}{35}\right) \times 10 = 39.71$$

Therefore, the mean percentage of female teachers in the primary schools of rural areas is 39.71.

Remark : The result obtained by all the three methods is the same. So the choice of method to be used depends on the numerical values of x_i and f_i. If x_i and f_i are sufficiently small, then the direct method is an appropriate choice. If x_i and f_i are numerically large numbers, then we can go for the assumed mean method or step-deviation method. If the class sizes are unequal, and x_i are large numerically, we can still apply the step-deviation method by taking h to be a suitable divisor of all the d_i's.

Example 3 : The distribution below shows the number of wickets taken by bowlers in one-day cricket matches. Find the mean number of wickets by choosing a suitable method. What does the mean signify?

Number of wickets	20 - 60	60 - 100	100 - 150	150 - 250	250 - 350	350 - 450
Number of bowlers	7	5	16	12	2	3

Solution : Here, the class size varies, and the x_is are large. Let us still apply the step-deviation method with a = 200 and h = 20. Then, we obtain the data as in Table 13.8.

Table 13.8

Number of wickets taken	**Number of bowlers (f_i)**	x_i	$d_i = x_i - 200$	$u_i = \frac{d_i}{20}$	$u_i f_i$
20 - 60	7	40	–160	–8	–56
60 - 100	5	80	–120	–6	–30
100 - 150	16	125	–75	–3.75	–60
150 - 250	12	200	0	0	0
250 - 350	2	300	100	5	10
350 - 450	3	400	200	10	30
Total	**45**				**–106**

So, $\bar{u} = \frac{-106}{45}$. Therefore, $\bar{x} = 200 + 20\left(\frac{-106}{45}\right) = 200 - 47.11 = 152.89$.

This tells us that, on an average, the number of wickets taken by these 45 bowlers in one-day cricket is 152.89.

Now, let us see how well you can apply the concepts discussed in this section!

Activity 2 :

Divide the students of your class into three groups and ask each group to do one of the following activities.

1. Collect the marks obtained by all the students of your class in Mathematics in the latest examination conducted by your school. Form a grouped frequency distribution of the data obtained.
2. Collect the daily maximum temperatures recorded for a period of 30 days in your city. Present this data as a grouped frequency table.
3. Measure the heights of all the students of your class (in cm) and form a grouped frequency distribution table of this data.

After all the groups have collected the data and formed grouped frequency distribution tables, the groups should find the mean in each case by the method which they find appropriate.

EXERCISE 13.1

1. A survey was conducted by a group of students as a part of their environment awareness programme, in which they collected the following data regarding the number of plants in 20 houses in a locality. Find the mean number of plants per house.

Number of plants	0 - 2	2 - 4	4 - 6	6 - 8	8 - 10	10 - 12	12 - 14
Number of houses	1	2	1	5	6	2	3

Which method did you use for finding the mean, and why?

2. Consider the following distribution of daily wages of 50 workers of a factory.

Daily wages (in ₹)	500 - 520	520 -540	540 - 560	560 - 580	580 -600
Number of workers	12	14	8	6	10

Find the mean daily wages of the workers of the factory by using an appropriate method.

3. The following distribution shows the daily pocket allowance of children of a locality. The mean pocket allowance is Rs 18. Find the missing frequency f.

Daily pocket allowance (in ₹)	11 - 13	13 - 15	15 - 17	17 - 19	19 - 21	21 - 23	23 - 25
Number of children	7	6	9	13	f	5	4

4. Thirty women were examined in a hospital by a doctor and the number of heartbeats per minute were recorded and summarised as follows. Find the mean heartbeats per minute for these women, choosing a suitable method.

Number of heartbeats per minute	65 - 68	68 - 71	71 - 74	74 - 77	77 - 80	80 - 83	83 - 86
Number of women	2	4	3	8	7	4	2

5. In a retail market, fruit vendors were selling mangoes kept in packing boxes. These boxes contained varying number of mangoes. The following was the distribution of mangoes according to the number of boxes.

Number of mangoes	50 - 52	53 - 55	56 - 58	59 - 61	62 - 64
Number of boxes	15	110	135	115	25

Find the mean number of mangoes kept in a packing box. Which method of finding the mean did you choose?

6. The table below shows the daily expenditure on food of 25 households in a locality.

Daily expenditure (in ₹)	100 - 150	150 - 200	200 - 250	250 - 300	300 - 350
Number of households	4	5	12	2	2

Find the mean daily expenditure on food by a suitable method.

7. To find out the concentration of SO_2 in the air (in parts per million, i.e., ppm), the data was collected for 30 localities in a certain city and is presented below:

Concentration of SO_2 (in ppm)	**Frequency**
0.00 - 0.04	4
0.04 - 0.08	9
0.08 - 0.12	9
0.12 - 0.16	2
0.16 - 0.20	4
0.20 - 0.24	2

Find the mean concentration of SO_2 in the air.

8. A class teacher has the following absentee record of 40 students of a class for the whole term. Find the mean number of days a student was absent.

Number of days	0 - 6	6 - 10	10 - 14	14 - 20	20 - 28	28 - 38	38 - 40
Number of students	11	10	7	4	4	3	1

9. The following table gives the literacy rate (in percentage) of 35 cities. Find the mean literacy rate.

Literacy rate (in %)	45 - 55	55 - 65	65 - 75	75 - 85	85 - 95
Number of cities	3	10	11	8	3

13.3 Mode of Grouped Data

Recall from Class IX, a mode is that value among the observations which occurs most often, that is, the value of the observation having the maximum frequency. Further, we discussed finding the mode of ungrouped data. Here, we shall discuss ways of obtaining a mode of grouped data. It is possible that more than one value may have the same maximum frequency. In such situations, the data is said to be multimodal. Though grouped data can also be multimodal, we shall restrict ourselves to problems having a single mode only.

Let us first recall how we found the mode for ungrouped data through the following example.

Example 4 : The wickets taken by a bowler in 10 cricket matches are as follows:

2 6 4 5 0 2 1 3 2 3

Find the mode of the data.

Solution : Let us form the frequency distribution table of the given data as follows:

Number of wickets	0	1	2	3	4	5	6
Number of matches	1	1	3	2	1	1	1

Clearly, 2 is the number of wickets taken by the bowler in the maximum number (i.e., 3) of matches. So, the mode of this data is 2.

In a grouped frequency distribution, it is not possible to determine the mode by looking at the frequencies. Here, we can only locate a class with the maximum frequency, called the **modal class**. The mode is a value inside the modal class, and is given by the formula:

$$\text{Mode} = l + \left(\frac{f_1 - f_0}{2f_1 - f_0 - f_2} \right) \times h$$

where l = lower limit of the modal class,

h = size of the class interval (assuming all class sizes to be equal),

f_1 = frequency of the modal class,

f_0 = frequency of the class preceding the modal class,

f_2 = frequency of the class succeeding the modal class.

Let us consider the following examples to illustrate the use of this formula.

Example 5 : A survey conducted on 20 households in a locality by a group of students resulted in the following frequency table for the number of family members in a household:

Family size	1 - 3	3 - 5	5 - 7	7 - 9	9 - 11
Number of families	7	8	2	2	1

Find the mode of this data.

Solution : Here the maximum class frequency is 8, and the class corresponding to this frequency is 3 – 5. So, the modal class is 3 – 5.

Now

modal class = 3 – 5, lower limit (l) of modal class = 3, class size (h) = 2

frequency (f_1) of the modal class = 8,

frequency (f_0) of class preceding the modal class = 7,

frequency (f_2) of class succeeding the modal class = 2.

Now, let us substitute these values in the formula :

$$\text{Mode} = l + \left(\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right) \times h$$

$$= 3 + \left(\frac{8-7}{2 \times 8 - 7 - 2}\right) \times 2 = 3 + \frac{2}{7} = 3.286$$

Therefore, the mode of the data above is 3.286.

Example 6 : The marks distribution of 30 students in a mathematics examination are given in Table 13.3 of Example 1. Find the mode of this data. Also compare and interpret the mode and the mean.

Solution : Refer to Table 13.3 of Example 1. Since the maximum number of students (i.e., 7) have got marks in the interval 40 - 55, the modal class is 40 - 55. Therefore,

the lower limit (l) of the modal class = 40,

the class size (h) = 15,

the frequency (f_1) of modal class = 7,

the frequency (f_0) of the class preceding the modal class = 3,

the frequency (f_2) of the class succeeding the modal class = 6.

Now, using the formula:

$$\text{Mode} = l + \left(\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right) \times h,$$

we get

$$\text{Mode} = 40 + \left(\frac{7-3}{14-6-3}\right) \times 15 = 52$$

So, the mode marks is 52.

Now, from Example 1, you know that the mean marks is 62.

So, the maximum number of students obtained 52 marks, while on an average a student obtained 62 marks.

Remarks :

1. In Example 6, the mode is less than the mean. But for some other problems it may be equal or more than the mean also.

2. It depends upon the demand of the situation whether we are interested in finding the average marks obtained by the students or the average of the marks obtained by most

of the students. In the first situation, the mean is required and in the second situation, the mode is required.

Activity 3 : Continuing with the same groups as formed in Activity 2 and the situations assigned to the groups. Ask each group to find the mode of the data. They should also compare this with the mean, and interpret the meaning of both.

Remark : The mode can also be calculated for grouped data with unequal class sizes. However, we shall not be discussing it.

EXERCISE 13.2

1. The following table shows the ages of the patients admitted in a hospital during a year:

Age (in years)	5 - 15	15 - 25	25 - 35	35 - 45	45 - 55	55 - 65
Number of patients	6	11	21	23	14	5

Find the mode and the mean of the data given above. Compare and interpret the two measures of central tendency.

2. The following data gives the information on the observed lifetimes (in hours) of 225 electrical components :

Lifetimes (in hours)	0 - 20	20 - 40	40 - 60	60 - 80	80 - 100	100 - 120
Frequency	10	35	52	61	38	29

Determine the modal lifetimes of the components.

3. The following data gives the distribution of total monthly household expenditure of 200 families of a village. Find the modal monthly expenditure of the families. Also, find the mean monthly expenditure :

Expenditure (in ₹)	**Number of families**
1000 - 1500	24
1500 - 2000	40
2000 - 2500	33
2500 - 3000	28
3000 - 3500	30
3500 - 4000	22
4000 - 4500	16
4500 - 5000	7

4. The following distribution gives the state-wise teacher-student ratio in higher secondary schools of India. Find the mode and mean of this data. Interpret the two measures.

Number of students per teacher	Number of states / U.T.
15 - 20	3
20 - 25	8
25 - 30	9
30 - 35	10
35 - 40	3
40 - 45	0
45 - 50	0
50 - 55	2

5. The given distribution shows the number of runs scored by some top batsmen of the world in one-day international cricket matches.

Runs scored	Number of batsmen
3000 - 4000	4
4000 - 5000	18
5000 - 6000	9
6000 - 7000	7
7000 - 8000	6
8000 - 9000	3
9000 - 10000	1
10000 - 11000	1

Find the mode of the data.

6. A student noted the number of cars passing through a spot on a road for 100 periods each of 3 minutes and summarised it in the table given below. Find the mode of the data :

Number of cars	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60	60 - 70	70 - 80
Frequency	7	14	13	12	20	11	15	8

13.4 Median of Grouped Data

As you have studied in Class IX, the median is a measure of central tendency which gives the value of the middle-most observation in the data. Recall that for finding the median of ungrouped data, we first arrange the data values of the observations in ascending order. Then, if n is odd, the median is the $\left(\frac{n+1}{2}\right)$th observation. And, if n is even, then the median will be the average of the $\frac{n}{2}$th and the $\left(\frac{n}{2}+1\right)$th observations.

Suppose, we have to find the median of the following data, which gives the marks, out of 50, obtained by 100 students in a test :

Marks obtained	20	29	28	33	42	38	43	25
Number of students	6	28	24	15	2	4	1	20

First, we arrange the marks in ascending order and prepare a frequency table as follows :

Table 13.9

Marks obtained	**Number of students (Frequency)**
20	6
25	20
28	24
29	28
33	15
38	4
42	2
43	1
Total	**100**

Here $n = 100$, which is even. The median will be the average of the $\frac{n}{2}$ th and the $\left(\frac{n}{2}+1\right)$th observations, i.e., the 50th and 51st observations. To find these observations, we proceed as follows:

Table 13.10

Marks obtained	Number of students
20	6
upto 25	$6 + 20 = 26$
upto 28	$26 + 24 = 50$
upto 29	$50 + 28 = 78$
upto 33	$78 + 15 = 93$
upto 38	$93 + 4 = 97$
upto 42	$97 + 2 = 99$
upto 43	$99 + 1 = 100$

Now we add another column depicting this information to the frequency table above and name it as *cumulative frequency column*.

Table 13.11

Marks obtained	Number of students	Cumulative frequency
20	6	6
25	20	26
28	24	50
29	28	78
33	15	93
38	4	97
42	2	99
43	1	100

From the table above, we see that:

50th observaton is 28 (Why?)

51st observation is 29

So, $$\text{Median} = \frac{28 + 29}{2} = 28.5$$

Remark : The part of Table 13.11 consisting Column 1 and Column 3 is known as *Cumulative Frequency Table*. The median marks 28.5 conveys the information that about 50% students obtained marks less than 28.5 and another 50% students obtained marks more than 28.5.

Now, let us see how to obtain the median of grouped data, through the following situation.

Consider a grouped frequency distribution of marks obtained, out of 100, by 53 students, in a certain examination, as follows:

Table 13.12

Marks	Number of students
0 - 10	5
10 - 20	3
20 - 30	4
30 - 40	3
40 - 50	3
50 - 60	4
60 - 70	7
70 - 80	9
80 - 90	7
90 - 100	8

From the table above, try to answer the following questions:

How many students have scored marks less than 10? The answer is clearly 5.

How many students have scored less than 20 marks? Observe that the number of students who have scored less than 20 include the number of students who have scored marks from 0 - 10 as well as the number of students who have scored marks from 10 - 20. So, the total number of students with marks less than 20 is 5 + 3, i.e., 8. We say that the cumulative frequency of the class 10 -20 is 8.

Similarly, we can compute the cumulative frequencies of the other classes, i.e., the number of students with marks less than 30, less than 40, . . ., less than 100. We give them in Table 13.13 given below:

Table 13.13

Marks obtained	**Number of students (Cumulative frequency)**
Less than 10	5
Less than 20	5 + 3 = 8
Less than 30	8 + 4 = 12
Less than 40	12 + 3 = 15
Less than 50	15 + 3 = 18
Less than 60	18 + 4 = 22
Less than 70	22 + 7 = 29
Less than 80	29 + 9 = 38
Less than 90	38 + 7 = 45
Less than 100	45 + 8 = 53

The distribution given above is called the *cumulative frequency distribution of the less than type*. Here 10, 20, 30, . . . 100, are the upper limits of the respective class intervals.

We can similarly make the table for the number of students with scores, more than or equal to 0, more than or equal to 10, more than or equal to 20, and so on. From Table 13.12, we observe that all 53 students have scored marks more than or equal to 0. Since there are 5 students scoring marks in the interval 0 - 10, this means that there are 53 – 5 = 48 students getting more than or equal to 10 marks. Continuing in the same manner, we get the number of students scoring 20 or above as 48 – 3 = 45, 30 or above as 45 – 4 = 41, and so on, as shown in Table 13.14.

Table 13.14

Marks obtained	Number of students (Cumulative frequency)
More than or equal to 0	53
More than or equal to 10	53 – 5 = 48
More than or equal to 20	48 – 3 = 45
More than or equal to 30	45 – 4 = 41
More than or equal to 40	41 – 3 = 38
More than or equal to 50	38 – 3 = 35
More than or equal to 60	35 – 4 = 31
More than or equal to 70	31 – 7 = 24
More than or equal to 80	24 – 9 = 15
More than or equal to 90	15 – 7 = 8

The table above is called a *cumulative frequency distribution of the more than type*. Here 0, 10, 20, . . ., 90 give the lower limits of the respective class intervals.

Now, to find the median of grouped data, we can make use of any of these cumulative frequency distributions.

Let us combine Tables 13.12 and 13.13 to get Table 13.15 given below:

Table 13.15

Marks	Number of students (f)	Cumulative frequency (cf)
0 - 10	5	5
10 - 20	3	8
20 - 30	4	12
30 - 40	3	15
40 - 50	3	18
50 - 60	4	22
60 - 70	7	29
70 - 80	9	38
80 - 90	7	45
90 - 100	8	53

Now in a grouped data, we may not be able to find the middle observation by looking at the cumulative frequencies as the middle observation will be some value in a class interval. It is, therefore, necessary to find the value inside a class that divides the whole distribution into two halves. But which class should this be?

To find this class, we find the cumulative frequencies of all the classes and $\frac{n}{2}$. We now locate the class whose cumulative frequency is greater than (and nearest to) $\frac{n}{2}$. This is called the *median class*. In the distribution above, $n = 53$. So, $\frac{n}{2} = 26.5$. Now 60 – 70 is the class whose cumulative frequency 29 is greater than (and nearest to) $\frac{n}{2}$, i.e., 26.5.

Therefore, 60 – 70 is the **median class**.

After finding the median class, we use the following formula for calculating the median.

$$\text{Median} = l + \left(\frac{\frac{n}{2} - \text{cf}}{f}\right) \times h,$$

where l = lower limit of median class,

n = number of observations,

cf = cumulative frequency of class preceding the median class,

f = frequency of median class,

h = class size (assuming class size to be equal).

Substituting the values $\frac{n}{2} = 26.5$, $l = 60$, cf $= 22$, $f = 7$, $h = 10$

in the formula above, we get

$$\text{Median} = 60 + \left(\frac{26.5 - 22}{7}\right) \times 10$$

$$= 60 + \frac{45}{7}$$

$$= 66.4$$

So, about half the students have scored marks less than 66.4, and the other half have scored marks more than 66.4.

Example 7 : A survey regarding the heights (in cm) of 51 girls of Class X of a school was conducted and the following data was obtained:

Height (in cm)	Number of girls
Less than 140	4
Less than 145	11
Less than 150	29
Less than 155	40
Less than 160	46
Less than 165	51

Find the median height.

Solution : To calculate the median height, we need to find the class intervals and their corresponding frequencies.

The given distribution being of the *less than type*, 140, 145, 150, . . ., 165 give the upper limits of the corresponding class intervals. So, the classes should be below 140, 140 - 145, 145 - 150, . . ., 160 - 165. Observe that from the given distribution, we find that there are 4 girls with height less than 140, i.e., the frequency of class interval below 140 is 4. Now, there are 11 girls with heights less than 145 and 4 girls with height less than 140. Therefore, the number of girls with height in the interval 140 - 145 is 11 – 4 = 7. Similarly, the frequency of 145 - 150 is 29 – 11 = 18, for 150 - 155, it is 40 – 29 = 11, and so on. So, our frequency distribution table with the given cumulative frequencies becomes:

Table 13.16

Class intervals	Frequency	Cumulative frequency
Below 140	4	4
140 - 145	7	11
145 - 150	18	29
150 - 155	11	40
155 - 160	6	46
160 - 165	5	51

Now $n = 51$. So, $\frac{n}{2} = \frac{51}{2} = 25.5$. This observation lies in the class 145 - 150. Then,

l (the lower limit) = 145,

cf (the cumulative frequency of the class preceding 145 - 150) = 11,

f (the frequency of the median class 145 - 150) = 18,

h (the class size) = 5.

Using the formula, Median $= l + \left(\frac{\frac{n}{2} - \text{cf}}{f}\right) \times h$, we have

$$\text{Median} = 145 + \left(\frac{25.5 - 11}{18}\right) \times 5$$

$$= 145 + \frac{72.5}{18} = 149.03.$$

So, the median height of the girls is 149.03 cm.

This means that the height of about 50% of the girls is less than this height, and 50% are taller than this height.

Example 8 : The median of the following data is 525. Find the values of x and y, if the total frequency is 100.

Class intervals	Frequency
0 - 100	2
100 - 200	5
200 - 300	x
300 - 400	12
400 - 500	17
500 - 600	20
600 - 700	y
700 - 800	9
800 - 900	7
900 - 1000	4

Solution :

Class intervals	Frequency	Cumulative frequency
0 - 100	2	2
100 - 200	5	7
200 - 300	x	$7 + x$
300 - 400	12	$19 + x$
400 - 500	17	$36 + x$
500 - 600	20	$56 + x$
600 - 700	y	$56 + x + y$
700 - 800	9	$65 + x + y$
800 - 900	7	$72 + x + y$
900 - 1000	4	$76 + x + y$

It is given that $n = 100$

So, $76 + x + y = 100$, i.e., $x + y = 24$ (1)

The median is 525, which lies in the class 500 – 600

So, $l = 500$, $f = 20$, $\text{cf} = 36 + x$, $h = 100$

Using the formula : $$\text{Median} = l + \left(\frac{\frac{n}{2} - \text{cf}}{f}\right)h, \text{ we get}$$

$$525 = 500 + \left(\frac{50 - 36 - x}{20}\right) \times 100$$

i.e., $$525 - 500 = (14 - x) \times 5$$

i.e., $$25 = 70 - 5x$$

i.e., $$5x = 70 - 25 = 45$$

So, $$x = 9$$

Therefore, from (1), we get $9 + y = 24$

i.e., $$y = 15$$

Now, that you have studied about all the three measures of central tendency, let us discuss **which measure would be best suited for a particular requirement**.

The mean is the most frequently used measure of central tendency because it takes into account all the observations, and lies between the extremes, i.e., the largest and the smallest observations of the entire data. It also enables us to compare two or more distributions. For example, by comparing the average (mean) results of students of different schools of a particular examination, we can conclude which school has a better performance.

However, extreme values in the data affect the mean. For example, the mean of classes having frequencies more or less the same is a good representative of the data. But, if one class has frequency, say 2, and the five others have frequency 20, 25, 20, 21, 18, then the mean will certainly not reflect the way the data behaves. So, in such cases, the mean is not a good representative of the data.

In problems where individual observations are not important, and we wish to find out a 'typical' observation, the median is more appropriate, e.g., finding the typical productivity rate of workers, average wage in a country, etc. These are situations where extreme values may be there. So, rather than the mean, we take the median as a better measure of central tendency.

In situations which require establishing the most frequent value or most popular item, the mode is the best choice, e.g., to find the most popular T.V. programme being watched, the consumer item in greatest demand, the colour of the vehicle used by most of the people, etc.

Remarks :

1. There is a empirical relationship between the three measures of central tendency :

$$\textbf{3 Median = Mode + 2 Mean}$$

2. The median of grouped data with unequal class sizes can also be calculated. However, we shall not discuss it here.

EXERCISE 13.3

1. The following frequency distribution gives the monthly consumption of electricity of 68 consumers of a locality. Find the median, mean and mode of the data and compare them.

Monthly consumption (in units)	Number of consumers
65 - 85	4
85 - 105	5
105 - 125	13
125 - 145	20
145 - 165	14
165 - 185	8
185 - 205	4

2. If the median of the distribution given below is 28.5, find the values of x and y.

Class interval	Frequency
0 - 10	5
10 - 20	x
20 - 30	20
30 - 40	15
40 - 50	y
50 - 60	5
Total	60

3. A life insurance agent found the following data for distribution of ages of 100 policy holders. Calculate the median age, if policies are given only to persons having age 18 years onwards but less than 60 year.

Age (in years)	Number of policy holders
Below 20	2
Below 25	6
Below 30	24
Below 35	45
Below 40	78
Below 45	89
Below 50	92
Below 55	98
Below 60	100

4. The lengths of 40 leaves of a plant are measured correct to the nearest millimetre, and the data obtained is represented in the following table :

Length (in mm)	Number of leaves
118 - 126	3
127 - 135	5
136 - 144	9
145 - 153	12
154 - 162	5
163 - 171	4
172 - 180	2

Find the median length of the leaves.

(**Hint :** The data needs to be converted to continuous classes for finding the median, since the formula assumes continuous classes. The classes then change to 117.5 - 126.5, 126.5 - 135.5, . . ., 171.5 - 180.5.)

5. The following table gives the distribution of the life time of 400 neon lamps :

Life time (in hours)	Number of lamps
1500 - 2000	14
2000 - 2500	56
2500 - 3000	60
3000 - 3500	86
3500 - 4000	74
4000 - 4500	62
4500 - 5000	48

Find the median life time of a lamp.

6. 100 surnames were randomly picked up from a local telephone directory and the frequency distribution of the number of letters in the English alphabets in the surnames was obtained as follows:

Number of letters	1 - 4	4 - 7	7 - 10	10 - 13	13 - 16	16 - 19
Number of surnames	6	30	40	16	4	4

Determine the median number of letters in the surnames. Find the mean number of letters in the surnames? Also, find the modal size of the surnames.

7. The distribution below gives the weights of 30 students of a class. Find the median weight of the students.

Weight (in kg)	40 - 45	45 - 50	50 - 55	55 - 60	60 - 65	65 - 70	70 - 75
Number of students	2	3	8	6	6	3	2

13.5 Summary

In this chapter, you have studied the following points:

1. The mean for grouped data can be found by :

(i) the direct method : $\overline{x} = \dfrac{\Sigma f_i x_i}{\Sigma f_i}$

(ii) the assumed mean method : $\overline{x} = a + \dfrac{\Sigma f_i d_i}{\Sigma f_i}$

(iii) the step deviation method : $\bar{x} = a + \left(\frac{\Sigma f_i u_i}{\Sigma f_i}\right) \times h$,

with the assumption that the frequency of a class is centred at its mid-point, called its class mark.

2. The mode for grouped data can be found by using the formula:

$$\text{Mode} = l + \left(\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right) \times h$$

where symbols have their usual meanings.

3. The cumulative frequency of a class is the frequency obtained by adding the frequencies of all the classes preceding the given class.

4. The median for grouped data is formed by using the formula:

$$\text{Median} = l + \left(\frac{\frac{n}{2} - \text{cf}}{f}\right) \times h,$$

where symbols have their usual meanings.

A NOTE TO THE READER

For calculating mode and median for grouped data, it should be ensured that the class intervals are continuous before applying the formulae. Same condition also apply for construction of an ogive. Further, in case of ogives, the scale may not be the same on both the axes.

1062CH15

PROBABILITY 14

The theory of probabilities and the theory of errors now constitute a formidable body of great mathematical interest and of great practical importance.

– R.S. Woodward

14.1 Probability — A Theoretical Approach

Let us consider the following situation :

Suppose a coin is tossed *at random*.

When we speak of a coin, we assume it to be 'fair', that is, it is symmetrical so that there is no reason for it to come down more often on one side than the other. We call this property of the coin as being 'unbiased'. By the phrase 'random toss', we mean that the coin is allowed to fall freely without any *bias* or *interference*.

We know, in advance, that the coin can only land in one of two possible ways — either head up or tail up (we dismiss the possibility of its 'landing' on its edge, which may be possible, for example, if it falls on sand). We can reasonably assume that each outcome, head or tail, is *as likely to occur as the other.* We refer to this by saying that *the outcomes* head and tail, *are equally likely*.

For another example of equally likely outcomes, suppose we throw a die once. For us, a die will always mean a fair die. What are the possible outcomes? They are 1, 2, 3, 4, 5, 6. Each number has the same possibility of showing up. So the *equally likely outcomes* of throwing a die are 1, 2, 3, 4, 5 and 6.

Are the outcomes of every experiment equally likely? Let us see.

Suppose that a bag contains 4 red balls and 1 blue ball, and you draw a ball without looking into the bag. What are the outcomes? Are the outcomes — a red ball and a blue ball equally likely? Since there are 4 red balls and only one blue ball, you would agree that you are more likely to get a red ball than a blue ball. So, the outcomes (a red ball or a blue ball) are *not* equally likely. However, the outcome of drawing a ball of any colour from the bag is equally likely. So, all experiments do not necessarily have equally likely outcomes.

However, in this chapter, from now on, we will **assume that all the experiments have equally likely outcomes.**

In Class IX, we defined the experimental or empirical probability P(E) of an event E as

$$\text{P(E)} = \frac{\text{Number of trials in which the event happened}}{\text{Total number of trials}}$$

The empirical interpretation of probability can be applied to every event associated with an experiment which can be repeated a large number of times. The requirement of repeating an experiment has some limitations, as it may be very expensive or unfeasible in many situations. Of course, it worked well in coin tossing or die throwing experiments. But how about repeating the experiment of launching a satellite in order to compute the empirical probability of its failure during launching, or the repetition of the phenomenon of an earthquake to compute the empirical probability of a multi-storeyed building getting destroyed in an earthquake?

In experiments where we are prepared to make certain assumptions, the repetition of an experiment can be avoided, as the assumptions help in directly calculating the exact (theoretical) probability. The assumption of equally likely outcomes (which is valid in many experiments, as in the two examples above, of a coin and of a die) is one such assumption that leads us to the following definition of probability of an event.

The **theoretical probability** (also called **classical probability**) of an event E, written as P(E), is defined as

$$\mathbf{P(E)} = \frac{\textbf{Number of outcomes favourable to E}}{\textbf{Number of all possible outcomes of the experiment}},$$

where we assume that the outcomes of the experiment are *equally likely*.

We will briefly refer to theoretical probability as probability.

This definition of probability was given by Pierre Simon Laplace in 1795.

Probability theory had its origin in the 16th century when an Italian physician and mathematician J.Cardan wrote the first book on the subject, *The Book on Games of Chance.* Since its inception, the study of probability has attracted the attention of great mathematicians. James Bernoulli (1654 – 1705), A. de Moivre (1667 – 1754), and Pierre Simon Laplace are among those who made significant contributions to this field. Laplace's *Theorie Analytique des Probabilités,* 1812, is considered to be the greatest contribution by a single person to the theory of probability. In recent years, probability has been used extensively in many areas such as biology, economics, genetics, physics, sociology etc.

Pierre Simon Laplace (1749 – 1827)

Let us find the probability for some of the events associated with experiments where the equally likely assumption holds.

Example 1 : Find the probability of getting a head when a coin is tossed once. Also find the probability of getting a tail.

Solution : In the experiment of tossing a coin once, the number of possible outcomes is two — Head (H) and Tail (T). Let E be the event 'getting a head'. The number of outcomes favourable to E, (i.e., of getting a head) is 1. Therefore,

$$P(E) = P\text{ (head)} = \frac{\text{Number of outcomes favourable to E}}{\text{Number of all possible outcomes}} = \frac{1}{2}$$

Similarly, if F is the event 'getting a tail', then

$$P(F) = P(\text{tail}) = \frac{1}{2} \quad \text{(Why ?)}$$

Example 2 : A bag contains a red ball, a blue ball and a yellow ball, all the balls being of the same size. Kritika takes out a ball from the bag without looking into it. What is the probability that she takes out the

(i) yellow ball? (ii) red ball? (iii) blue ball?

Solution : Kritika takes out a ball from the bag without looking into it. So, it is equally likely that she takes out any one of them.

Let Y be the event 'the ball taken out is yellow', B be the event 'the ball taken out is blue', and R be the event 'the ball taken out is red'.

Now, the number of possible outcomes = 3.

(i) The number of outcomes favourable to the event Y = 1.

So, $$P(Y) = \frac{1}{3}$$

Similarly, (ii) $P(R) = \frac{1}{3}$ and (iii) $P(B) = \frac{1}{3}$.

Remarks :

1. An event having only one outcome of the experiment is called an *elementary event.* In Example 1, both the events E and F are elementary events. Similarly, in Example 2, all the three events, Y, B and R are elementary events.

2. In Example 1, we note that : P(E) + P(F) = 1

In Example 2, we note that : P(Y) + P(R) + P(B) = 1

Observe that **the sum of the probabilities of all the elementary events of an experiment** is 1. This is true in general also.

Example 3 : Suppose we throw a die once. (i) What is the probability of getting a number greater than 4 ? (ii) What is the probability of getting a number less than or equal to 4 ?

Solution : (i) Here, let E be the event 'getting a number greater than 4'. The number of possible outcomes is six : 1, 2, 3, 4, 5 and 6, and the outcomes favourable to E are 5 and 6. Therefore, the number of outcomes favourable to E is 2. So,

$$P(E) = P(\text{number greater than 4}) = \frac{2}{6} = \frac{1}{3}$$

(ii) Let F be the event 'getting a number less than or equal to 4'.

Number of possible outcomes = 6

Outcomes favourable to the event F are 1, 2, 3, 4.

So, the number of outcomes favourable to F is 4.

Therefore, $$P(F) = \frac{4}{6} = \frac{2}{3}$$

Are the events E and F in the example above elementary events? No, they are **not** because the event E has 2 outcomes and the event F has 4 outcomes.

Remarks : From Example 1, we note that

$$P(E) + P(F) = \frac{1}{2} + \frac{1}{2} = 1 \qquad (1)$$

where E is the event 'getting a head' and F is the event 'getting a tail'.

From (i) and (ii) of Example 3, we also get

$$P(E) + P(F) = \frac{1}{3} + \frac{2}{3} = 1 \qquad (2)$$

where E is the event 'getting a number >4' and F is the event 'getting a number ≤ 4'.

Note that getting a number *not* greater than 4 is same as getting a number less than or equal to 4, and vice versa.

In (1) and (2) above, is F not the same as 'not E'? Yes, it is. We denote the event 'not E' by $\overline{E}$.

So, P(E) + P(not E) = 1

i.e., $P(E) + P(\overline{E}) = 1$, which gives us $P(\overline{E}) = 1 - P(E)$.

In general, it is true that for an event E,

$$\mathbf{P(\overline{E}) = 1 - P(E)}$$

The event $\overline{E}$, representing 'not E', is called the **complement** of the event E. We also say that E and $\overline{E}$ are **complementary** events.

Before proceeding further, let us try to find the answers to the following questions:

(i) What is the probability of getting a number 8 in a single throw of a die?

(ii) What is the probability of getting a number less than 7 in a single throw of a die?

Let us answer (i) :

We know that there are only six possible outcomes in a single throw of a die. These outcomes are 1, 2, 3, 4, 5 and 6. Since no face of the die is marked 8, so there is no outcome favourable to 8, i.e., the number of such outcomes is zero. In other words, getting 8 in a single throw of a die, is *impossible*.

So, $$P(\text{getting } 8) = \frac{0}{6} = 0$$

That is, the probability of an event which is *impossible* to occur is 0. Such an event is called an **impossible event**.

Let us answer (ii) :

Since every face of a die is marked with a number less than 7, it is *sure* that we will always get a number less than 7 when it is thrown once. So, the number of favourable outcomes is the same as the number of all possible outcomes, which is 6.

Therefore, $\quad P(E) = P(\text{getting a number less than 7}) = \frac{6}{6} = 1$

So, the probability of an event which is *sure* (or *certain*) to occur is 1. Such an event is called a **sure event** or a **certain event**.

Note : From the definition of the probability P(E), we see that the numerator (number of outcomes favourable to the event E) is always less than or equal to the denominator (the number of all possible outcomes). Therefore,

$$\mathbf{0 \le P(E) \le 1}$$

Now, let us take an example related to playing cards. Have you seen a deck of playing cards? It consists of 52 cards which are divided into 4 suits of 13 cards each—spades (♠), hearts (♥), diamonds (♦) and clubs (♣). Clubs and spades are of black colour, while hearts and diamonds are of red colour. The cards in each suit are ace, king, queen, jack, 10, 9, 8, 7, 6, 5, 4, 3 and 2. Kings, queens and jacks are called *face cards*.

Example 4 : One card is drawn from a well-shuffled deck of 52 cards. Calculate the probability that the card will

(i) be an ace,

(ii) not be an ace.

Solution : Well-shuffling ensures *equally likely* outcomes.

(i) There are 4 aces in a deck. Let E be the event 'the card is an ace'.

The number of outcomes favourable to E = 4

The number of possible outcomes = 52 (Why ?)

Therefore, $\quad P(E) = \frac{4}{52} = \frac{1}{13}$

(ii) Let F be the event 'card drawn is not an ace'.

The number of outcomes favourable to the event F = 52 – 4 = 48 (Why?)

The number of possible outcomes = 52

Therefore, $P(F) = \frac{48}{52} = \frac{12}{13}$

Remark : Note that F is nothing but $\overline{E}$. Therefore, we can also calculate P(F) as follows: $P(F) = P(\overline{E}) = 1 - P(E) = 1 - \frac{1}{13} = \frac{12}{13}$.

Example 5 : Two players, Sangeeta and Reshma, play a tennis match. It is known that the probability of Sangeeta winning the match is 0.62. What is the probability of Reshma winning the match?

Solution : Let S and R denote the events that Sangeeta wins the match and Reshma wins the match, respectively.

The probability of Sangeeta's winning = P(S) = 0.62 (given)

The probability of Reshma's winning = P(R) = 1 – P(S)

[As the events R and S are complementary]

= 1 – 0.62 = 0.38

Example 6 : Savita and Hamida are friends. What is the probability that both will have (i) different birthdays? (ii) the same birthday? (ignoring a leap year).

Solution : Out of the two friends, one girl, say, Savita's birthday can be any day of the year. Now, Hamida's birthday can also be any day of 365 days in the year.

We assume that these 365 outcomes are equally likely.

(i) If Hamida's birthday is different from Savita's, the number of favourable outcomes for her birthday is 365 – 1 = 364

So, P (Hamida's birthday is different from Savita's birthday) = $\frac{364}{365}$

(ii) P(Savita and Hamida have the same birthday)

$$= 1 - P \text{ (both have different birthdays)}$$

$$= 1 - \frac{364}{365} \qquad [\text{Using } P(\overline{E}) = 1 - P(E)]$$

$$= \frac{1}{365}$$

Example 7 : There are 40 students in Class X of a school of whom 25 are girls and 15 are boys. The class teacher has to select one student as a class representative. She writes the name of each student on a separate card, the cards being identical. Then she puts cards in a bag and stirs them thoroughly. She then draws one card from the bag. What is the probability that the name written on the card is the name of (i) a girl? (ii) a boy?

Solution : There are 40 students, and only one name card has to be chosen.

(i) The number of all possible outcomes is 40

The number of outcomes favourable for a card with the name of a girl = 25 (Why?)

Therefore, P (card with name of a girl) = P(Girl) = $\frac{25}{40} = \frac{5}{8}$

(ii) The number of outcomes favourable for a card with the name of a boy = 15 (Why?)

Therefore, P(card with name of a boy) = P(Boy) = $\frac{15}{40} = \frac{3}{8}$

Note : We can also determine P(Boy), by taking

$$P(\text{Boy}) = 1 - P(\text{not Boy}) = 1 - P(\text{Girl}) = 1 - \frac{5}{8} = \frac{3}{8}$$

Example 8 : A box contains 3 blue, 2 white, and 4 red marbles. If a marble is drawn at *random* from the box, what is the probability that it will be

(i) white? (ii) blue? (iii) red?

Solution : Saying that a marble is drawn at random is a short way of saying that all the marbles are equally likely to be drawn. Therefore, the

number of possible outcomes = 3 +2 + 4 = 9 (Why?)

Let W denote the event 'the marble is white', B denote the event 'the marble is blue' and R denote the event 'marble is red'.

(i) The number of outcomes favourable to the event W = 2

So, $P(W) = \frac{2}{9}$

Similarly, (ii) $P(B) = \frac{3}{9} = \frac{1}{3}$ and (iii) $P(R) = \frac{4}{9}$

Note that P(W) + P(B) + P(R) = 1.

Example 9 : Harpreet tosses two different coins simultaneously (say, one is of ₹ 1 and other of ₹ 2). What is the probability that she gets *at least* one head?

Solution : We write H for 'head' and T for 'tail'. When two coins are tossed simultaneously, the possible outcomes are (H, H), (H, T), (T, H), (T, T), which are all *equally likely*. Here (H, H) means head up on the first coin (say on ₹ 1) and head up on the second coin (₹ 2). Similarly (H, T) means head up on the first coin and tail up on the second coin and so on.

The outcomes favourable to the event E, 'at least one head' are (H, H), (H, T) and (T, H). (Why?)

So, the number of outcomes favourable to E is 3.

Therefore, $\quad \text{P(E)} = \dfrac{3}{4}$

i.e., the probability that Harpreet gets at least one head is $\dfrac{3}{4}$.

Note : You can also find P(E) as follows:

$$\text{P (E)} = 1 - \text{P}(\bar{\text{E}}) = 1 - \frac{1}{4} = \frac{3}{4} \qquad \left(\text{Since } \text{P}(\bar{\text{E}}) = \text{P(no head)} = \frac{1}{4}\right)$$

Did you observe that in all the examples discussed so far, the number of possible outcomes in each experiment was finite? If not, check it now.

There are many experiments in which the outcome is any number between two given numbers, or in which the outcome is every point within a circle or rectangle, etc. Can you now count the number of all possible outcomes? As you know, this is not possible since there are infinitely many numbers between two given numbers, or there are infinitely many points within a circle. So, the definition of (theoretical) probability which you have learnt so far cannot be applied in the present form. What is the way out? To answer this, let us consider the following example :

Example 10* : In a musical chair game, the person playing the music has been advised to stop playing the music at any time within 2 minutes after she starts playing. What is the probability that the music will stop within the first half-minute after starting?

Solution : Here the possible outcomes are all the numbers between 0 and 2. This is the portion of the number line from 0 to 2 (see Fig. 14.1).

Fig. 14.1

* Not from the examination point of view.

Let E be the event that 'the music is stopped within the first half-minute'.

The outcomes favourable to E are points on the number line from 0 to $\frac{1}{2}$.

The distance from 0 to 2 is 2, while the distance from 0 to $\frac{1}{2}$ is $\frac{1}{2}$.

Since all the outcomes are equally likely, we can argue that, of the total distance of 2, the distance favourable to the event E is $\frac{1}{2}$.

So, $$P(E) = \frac{\text{Distance favourable to the event E}}{\text{Total distance in which outcomes can lie}} = \frac{\frac{1}{2}}{2} = \frac{1}{4}$$

Can we now extend the idea of Example 10 for finding the probability as the ratio of the favourable area to the total area?

Example 11* : A missing helicopter is reported to have crashed somewhere in the rectangular region shown in Fig. 14.2. What is the probability that it crashed inside the lake shown in the figure?

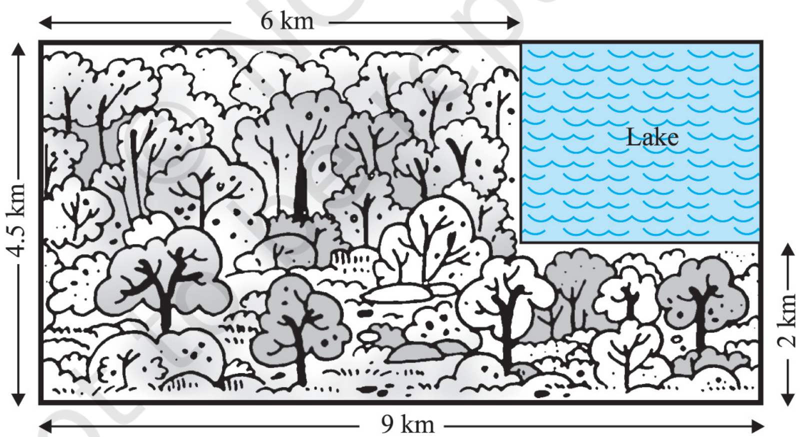


Fig. 14.2

Solution : The helicopter is equally likely to crash anywhere in the region.

Area of the entire region where the helicopter can crash

$$= (4.5 \times 9)\ \text{km}^2 = 40.5\ \text{km}^2$$

* Not from the examination point of view.

Area of the lake = (2.5 × 3) km^2 = 7.5 km^2

Therefore, P (helicopter crashed in the lake) = $\frac{7.5}{40.5} = \frac{75}{405} = \frac{5}{27}$

Example 12 : A carton consists of 100 shirts of which 88 are good, 8 have minor defects and 4 have major defects. Jimmy, a trader, will only accept the shirts which are good, but Sujatha, another trader, will only reject the shirts which have major defects. One shirt is drawn at random from the carton. What is the probability that

(i) it is acceptable to Jimmy?

(ii) it is acceptable to Sujatha?

Solution : One shirt is drawn at random from the carton of 100 shirts. Therefore, there are 100 equally likely outcomes.

(i) The number of outcomes favourable (i.e., acceptable) to Jimmy = 88 (Why?)

Therefore, P (shirt is acceptable to Jimmy) = $\frac{88}{100} = 0.88$

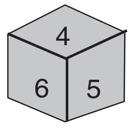
(ii) The number of outcomes favourable to Sujatha = 88 + 8 = 96 (Why?)

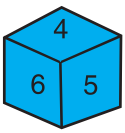
So, P (shirt is acceptable to Sujatha) = $\frac{96}{100} = 0.96$

Example 13 : Two dice, one blue and one grey, are thrown at the same time. Write down all the possible outcomes. What is the probability that the sum of the two numbers appearing on the top of the dice is

(i) 8? (ii) 13? (iii) less than or equal to 12?

Solution : When the blue die shows '1', the grey die could show any one of the numbers 1, 2, 3, 4, 5, 6. The same is true when the blue die shows '2', '3', '4', '5' or '6'. The possible outcomes of the experiment are listed in the table below; the first number in each ordered pair is the number appearing on the blue die and the second number is that on the grey die.





	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Fig. 14.3

Note that the pair (1, 4) is different from (4, 1). (Why?)

So, the number of possible outcomes = $6 \times 6 = 36$.

(i) The outcomes favourable to the event 'the sum of the two numbers is 8' denoted by E, are: (2, 6), (3, 5), (4, 4), (5, 3), (6, 2) (see Fig. 14.3)

i.e., the number of outcomes favourable to E = 5.

Hence, $$P(E) = \frac{5}{36}$$

(ii) As you can see from Fig. 14.3, there is no outcome favourable to the event F, 'the sum of two numbers is 13'.

So, $$P(F) = \frac{0}{36} = 0$$

(iii) As you can see from Fig. 14.3, all the outcomes are favourable to the event G, 'sum of two numbers ≤ 12'.

So, $$P(G) = \frac{36}{36} = 1$$

EXERCISE 14.1

1. Complete the following statements:

(i) Probability of an event E + Probability of the event 'not E' = ________ .

(ii) The probability of an event that cannot happen is ________ . Such an event is called ________ .

(iii) The probability of an event that is certain to happen is ________ . Such an event is called ________ .

(iv) The sum of the probabilities of all the elementary events of an experiment is ________ .

(v) The probability of an event is greater than or equal to ________ and less than or equal to ________ .

2. Which of the following experiments have equally likely outcomes? Explain.

(i) A driver attempts to start a car. The car starts or does not start.

(ii) A player attempts to shoot a basketball. She/he shoots or misses the shot.

(iii) A trial is made to answer a true-false question. The answer is right or wrong.

(iv) A baby is born. It is a boy or a girl.

3. Why is tossing a coin considered to be a fair way of deciding which team should get the ball at the beginning of a football game?

4. Which of the following cannot be the probability of an event?

(A) $\frac{2}{3}$ (B) -1.5 (C) 15% (D) 0.7

5. If P(E) = 0.05, what is the probability of 'not E'?

6. A bag contains lemon flavoured candies only. Malini takes out one candy without looking into the bag. What is the probability that she takes out

(i) an orange flavoured candy?

(ii) a lemon flavoured candy?

7. It is given that in a group of 3 students, the probability of 2 students not having the same birthday is 0.992. What is the probability that the 2 students have the same birthday?

8. A bag contains 3 red balls and 5 black balls. A ball is drawn at random from the bag. What is the probability that the ball drawn is (i) red ? (ii) not red?

9. A box contains 5 red marbles, 8 white marbles and 4 green marbles. One marble is taken out of the box at random. What is the probability that the marble taken out will be (i) red ? (ii) white ? (iii) not green?

10. A piggy bank contains hundred 50p coins, fifty ₹ 1 coins, twenty ₹ 2 coins and ten ₹ 5 coins. If it is equally likely that one of the coins will fall out when the bank is turned upside down, what is the probability that the coin (i) will be a 50 p coin ? (ii) will not be a ₹ 5 coin?

11. Gopi buys a fish from a shop for his aquarium. The shopkeeper takes out one fish at random from a tank containing 5 male fish and 8 female fish (see Fig. 14.4). What is the probability that the fish taken out is a male fish?

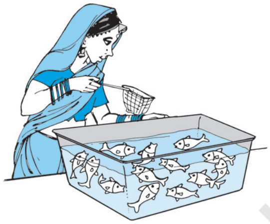


Fig. 14.4

12. A game of chance consists of spinning an arrow which comes to rest pointing at one of the numbers 1, 2, 3, 4, 5, 6, 7, 8 (see Fig. 14.5), and these are equally likely outcomes. What is the probability that it will point at

(i) 8 ?

(ii) an odd number?

(iii) a number greater than 2?

(iv) a number less than 9?

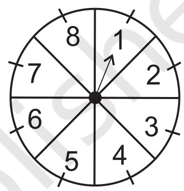


Fig. 14.5

13. A die is thrown once. Find the probability of getting

(i) a prime number; (ii) a number lying between 2 and 6; (iii) an odd number.

14. One card is drawn from a well-shuffled deck of 52 cards. Find the probability of getting

(i) a king of red colour (ii) a face card (iii) a red face card

(iv) the jack of hearts (v) a spade (vi) the queen of diamonds

15. Five cards—the ten, jack, queen, king and ace of diamonds, are well-shuffled with their face downwards. One card is then picked up at random.

(i) What is the probability that the card is the queen?

(ii) If the queen is drawn and put aside, what is the probability that the second card picked up is (a) an ace? (b) a queen?

16. 12 defective pens are accidentally mixed with 132 good ones. It is not possible to just look at a pen and tell whether or not it is defective. One pen is taken out at random from this lot. Determine the probability that the pen taken out is a good one.

17. (i) A lot of 20 bulbs contain 4 defective ones. One bulb is drawn at random from the lot. What is the probability that this bulb is defective?

(ii) Suppose the bulb drawn in (i) is not defective and is not replaced. Now one bulb is drawn at random from the rest. What is the probability that this bulb is not defective ?

18. A box contains 90 discs which are numbered from 1 to 90. If one disc is drawn at random from the box, find the probability that it bears (i) a two-digit number (ii) a perfect square number (iii) a number divisible by 5.

19. A child has a die whose six faces show the letters as given below:

A B C D E A

The die is thrown once. What is the probability of getting (i) A? (ii) D?

20*. Suppose you drop a die at random on the rectangular region shown in Fig. 14.6. What is the probability that it will land inside the circle with diameter 1m?

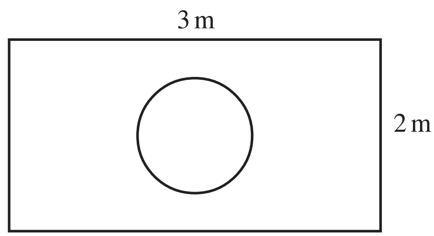


Fig. 14.6

21. A lot consists of 144 ball pens of which 20 are defective and the others are good. Nuri will buy a pen if it is good, but will not buy if it is defective. The shopkeeper draws one pen at random and gives it to her. What is the probability that

(i) She will buy it ?

(ii) She will not buy it ?

22. Refer to Example 13. (i) Complete the following table:

Event : 'Sum on 2 dice'	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$						$\frac{5}{36}$				$\frac{1}{36}$

(ii) A student argues that 'there are 11 possible outcomes 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12. Therefore, each of them has a probability $\frac{1}{11}$. Do you agree with this argument? Justify your answer.

23. A game consists of tossing a one rupee coin 3 times and noting its outcome each time. Hanif wins if all the tosses give the same result i.e., three heads or three tails, and loses otherwise. Calculate the probability that Hanif will lose the game.

24. A die is thrown twice. What is the probability that

(i) 5 will not come up either time? (ii) 5 will come up at least once?

[**Hint :** Throwing a die twice and throwing two dice simultaneously are treated as the same experiment]

* Not from the examination point of view.

25. Which of the following arguments are correct and which are not correct? Give reasons for your answer.

(i) If two coins are tossed simultaneously there are three possible outcomes—two heads, two tails or one of each. Therefore, for each of these outcomes, the probability is $\frac{1}{3}$.

(ii) If a die is thrown, there are two possible outcomes—an odd number or an even number. Therefore, the probability of getting an odd number is $\frac{1}{2}$.

14.2 Summary

In this chapter, you have studied the following points :

1. The theoretical (classical) probability of an event E, written as P(E), is defined as

$$P(E) = \frac{\text{Number of outcomes favourable to E}}{\text{Number of all possible outcomes of the experiment}}$$

where we assume that the outcomes of the experiment are equally likely.

2. The probability of a sure event (or certain event) is 1.

3. The probability of an impossible event is 0.

4. The probability of an event E is a number P(E) such that

$$0 \le P(E) \le 1$$

5. An event having only one outcome is called an elementary event. The sum of the probabilities of all the elementary events of an experiment is 1.

6. For any event E, $P(E) + P(\bar{E}) = 1$, where $\bar{E}$ stands for 'not E'. E and $\bar{E}$ are called complementary events.

A NOTE TO THE READER

The experimental or empirical probability of an event is based on what has actually happened while the theoretical probability of the event attempts to predict what will happen on the basis of certain assumptions. As the number of trials in an experiment, go on increasing we may expect the experimental and theoretical probabilities to be nearly the same.

Appendix 1

INFINITE SERIES

A.1.1 Introduction

As discussed in the Chapter 9 on Sequences and Series, a sequence $a_1, a_2, ..., a_n, ...$ having infinite number of terms is called *infinite sequence* and its indicated sum, i.e., $a_1 + a_2 + a_3 + ... + a_n + ...$ is called an *infinte series* associated with infinite sequence. This series can also be expressed in abbreviated form using the sigma notation, i.e.,

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots = \sum_{k=1}^{\infty} a_k$$

In this Chapter, we shall study about some special types of series which may be required in different problem situations.

A.1.2 Binomial Theorem for any Index

In Chapter 8, we discussed the Binomial Theorem in which the index was a positive integer. In this Section, we state a more general form of the theorem in which the index is not necessarily a whole number. It gives us a particular type of infinite series, called *Binomial Series*. We illustrate few applications, by examples.

We know the formula

$$(1 + x)^n = {}^nC_0 + {}^nC_1\, x + \ldots + {}^nC_n\, x^n$$

Here, n is non-negative integer. Observe that if we replace index n by negative integer or a fraction, then the combinations ${}^nC_r$ do not make any sense.

We now state (without proof), the Binomial Theorem, giving an infinite series in which the index is negative or a fraction and not a whole number.

Theorem The formula

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{1.2}x^2 + \frac{m(m-1)(m-2)}{1.2.3}x^3 + \ldots$$

holds whenever $|x| < 1$.

Remark 1. Note carefully the condition $|x| < 1$, i.e., $-1 < x < 1$ is necessary when m is negative integer or a fraction. For example, if we take $x = -2$ and $m = -2$, we obtain

$$(1-2)^{-2} = 1 + (-2)(-2) + \frac{(-2)(-3)}{1.2}(-2)^2 + ...$$

or $\quad 1 = 1 + 4 + 12 + \dots$

This is not possible

2. Note that there are infinite number of terms in the expansion of $(1+x)^m$, when m is a negative integer or a fraction

Consider $\quad (a+b)^m = \left[a\left(1+\frac{b}{a}\right)\right]^m = a^m\left(1+\frac{b}{a}\right)^m$

$$= a^m\left[1 + m\frac{b}{a} + \frac{m(m-1)}{1.2}\left(\frac{b}{a}\right)^2 + ...\right]$$

$$= a^m + ma^{m-1}b + \frac{m(m-1)}{1.2}a^{m-2}b^2 + ...$$

This expansion is valid when $\left|\frac{b}{a}\right| < 1$ or equivalently when $|b| < |a|$.

The general term in the expansion of $(a + b)^m$ is

$$\frac{m(m-1)(m-2)...(m-r+1)a^{m-r}b^r}{1.2.3...r}$$

We give below certain particular cases of Binomial Theorem, when we assume $|x| < 1$, these are left to students as exercises:

1. $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \dots$
2. $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \dots$
3. $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$
4. $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

Example 1 Expand $\left(1-\frac{x}{2}\right)^{-\frac{1}{2}}$, when $|x| < 2$.

Solution We have

$$\left(1-\frac{x}{2}\right)^{-\frac{1}{2}} = 1+\frac{\left(-\frac{1}{2}\right)}{1}\left(\frac{-x}{2}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1.2}\left(-\frac{x}{2}\right)^2+\ldots$$

$$= 1+\frac{x}{4}+\frac{3x^2}{32}+\ldots$$

A.1.3 Infinite Geometric Series

From Chapter 9, Section 9.5, a sequence a_1, a_2, a_3, ..., a_n is called G.P., if $\frac{a_{k+1}}{a_k} = r$ (constant) for k = 1, 2, 3, ..., n–1. Particularly, if we take $a_1 = a$, then the resulting sequence a, ar, ar^2, ..., ar^{n-1} is taken as the standard form of G.P., where a is first term and r, the common ratio of G.P.

Earlier, we have discussed the formula to find the sum of finite series $a + ar + ar^2 + \ldots + ar^{n-1}$ which is given by

$$S_n = \frac{a\left(1-r^n\right)}{1-r}.$$

In this section, we state the formula to find the sum of infinite geometric series $a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$ and illustrate the same by examples.

Let us consider the G.P. $1, \frac{2}{3}, \frac{4}{9}, \ldots$

Here $a = 1$, $r = \frac{2}{3}$. We have

$$S_n = \frac{1-\left(\frac{2}{3}\right)^n}{1-\frac{2}{3}} = 3\left[1-\left(\frac{2}{3}\right)^n\right] \quad \ldots (1)$$

Let us study the behaviour of $\left(\frac{2}{3}\right)^n$ as n becomes larger and larger.

n	1	5	10	20
$\left(\frac{2}{3}\right)^n$	0.6667	0.1316872428	0.01734152992	0.00030072866

We observe that as n becomes larger and larger, $\left(\frac{2}{3}\right)^n$ becomes closer and closer to zero. Mathematically, we say that as n becomes sufficiently large, $\left(\frac{2}{3}\right)^n$ becomes sufficiently small. In other words, as $n \to \infty, \left(\frac{2}{3}\right)^n \to 0$. Consequently, we find that the sum of infinitely many terms is given by S = 3.

Thus, for infinite geometric progression $a, ar, ar^2, ...,$ if numerical value of common ratio r is less than 1, then

$$S_n = \frac{a\left(1-r^n\right)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

In this case, $r^n \to 0$ as $n \to \infty$ since $|r|<1$ and then $\frac{ar^n}{1-r} \to 0$. Therefore,

$$S_n \to \frac{a}{1-r} \text{ as } n \to \infty .$$

Symbolically, sum to infinity of infinite geometric series is denoted by S. Thus, we have $\quad S = \frac{a}{1-r}$

For example

(i) $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... = \frac{1}{1-\frac{1}{2}} = 2$

(ii) $1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + ... = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$

Example 2 Find the sum to infinity of the G.P. ;

$$\frac{-5}{4}, \frac{5}{16}, \frac{-5}{64}, \ldots.$$

Solution Here $a = \frac{-5}{4}$ and $r = -\frac{1}{4}$. Also $|r| < 1$.

Hence, the sum to infinity is $\dfrac{\frac{-5}{4}}{1+\frac{1}{4}} = \dfrac{\frac{-5}{4}}{\frac{5}{4}} = -1$.

A.1.4 Exponential Series

Leonhard Euler (1707 – 1783), the great Swiss mathematician introduced the number e in his calculus text in 1748. The number e is useful in calculus as π in the study of the circle.

Consider the following infinite series of numbers

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots \qquad \ldots (1)$$

The sum of the series given in (1) is denoted by the number e

Let us estimate the value of the number e.

Since every term of the series (1) is positive, it is clear that its sum is also positive.

Consider the two sums

$$\frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots + \frac{1}{n!} + \ldots \qquad \ldots (2)$$

and
$$\frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \ldots. + \frac{1}{2^{n-1}} + \ldots \qquad \ldots (3)$$

Observe that

$$\frac{1}{3!} = \frac{1}{6} \text{ and } \frac{1}{2^2} = \frac{1}{4}, \text{ which gives } \frac{1}{3!} < \frac{1}{2^2}$$

$$\frac{1}{4!} = \frac{1}{24} \text{ and } \frac{1}{2^3} = \frac{1}{8}, \text{ which gives } \frac{1}{4!} < \frac{1}{2^3}$$

$$\frac{1}{5!} = \frac{1}{120} \text{ and } \frac{1}{2^4} = \frac{1}{16}, \text{ which gives } \frac{1}{5!} < \frac{1}{2^4}.$$

Therefore, by analogy, we can say that

$$\frac{1}{n!} < \frac{1}{2^{n-1}}, \text{ when } n > 2$$

We observe that each term in (2) is less than the corresponding term in (3),

Therefore $\quad \left(\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\dots+\frac{1}{n!}\right) < \left(\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\dots+\frac{1}{2^{n-1}}+\dots\right) \qquad \dots (4)$

Adding $\left(1+\frac{1}{1!}+\frac{1}{2!}\right)$ on both sides of (4), we get,

$$\left(1+\frac{1}{1!}+\frac{1}{2!}\right)+\left(\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\dots+\frac{1}{n!}+\dots\right)$$

$$< \left\{\left(1+\frac{1}{1!}+\frac{1}{2!}\right)+\left(\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\dots+\frac{1}{2^{n-1}}+\dots\right)\right\} \qquad \dots (5)$$

$$= \left\{1+\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\dots+\frac{1}{2^{n-1}}+\dots\right)\right\}$$

$$= 1+\frac{1}{1-\frac{1}{2}} = 1+2 = 3$$

Left hand side of (5) represents the series (1). Therefore $e < 3$ and also $e > 2$ and hence $2 < e < 3$.

Remark The exponential series involving variable x can be expressed as

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots+\frac{x^n}{n!}+\dots$$

Example 3 Find the coefficient of x^2 in the expansion of e^{2x+3} as a series in powers of x.

Solution In the exponential series

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$$

replacing x by $(2x + 3)$, we get

$$e^{2x+3} = 1+\frac{(2x+3)}{1!}+\frac{(2x+3)^2}{2!}+...$$

Here, the general term is $\frac{(2x+3)^n}{n!} = \frac{(3+2x)^n}{n!}$. This can be expanded by the Binomial Theorem as

$$\frac{1}{n!}\left[3^n + {}^nC_1 3^{n-1}(2x) + {}^nC_2 3^{n-2}(2x)^2 + ... + (2x)^n\right].$$

Here, the coefficient of x^2 is $\frac{{}^nC_2 3^{n-2} 2^2}{n!}$. Therefore, the coefficient of x^2 in the whole series is

$$\sum_{n=2}^{\infty}\frac{{}^nC_2 3^{n-2} 2^2}{n!} = 2\sum_{n=2}^{\infty}\frac{n(n-1)3^{n-2}}{n!}$$

$$= 2\sum_{n=2}^{\infty}\frac{3^{n-2}}{(n-2)!} \quad [\text{using } n! = n\,(n-1)\,(n-2)!]$$

$$= 2\left[1+\frac{3}{1!}+\frac{3^2}{2!}+\frac{3^3}{3!}+...\right]$$

$$= 2e^3 .$$

Thus $2e^3$ is the coefficient of x^2 in the expansion of e^{2x+3}.

Alternatively $e^{2x+3} = e^3 \,.\, e^{2x}$

$$= e^3\left[1+\frac{2x}{1!}+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+...\right]$$

Thus, the coefficient of x^2 in the expansion of e^{2x+3} is $e^3.\frac{2^2}{2!}=2e^3$

Example 4 Find the value of e^2, rounded off to one decimal place.

Solution Using the formula of exponential series involving x, we have

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+...+\frac{x^n}{n!}+...$$

Putting $x = 2$, we get

$$e^2 = 1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \frac{2^6}{6!} + \ldots$$

$$= 1 + 2 + 2 + \frac{4}{3} + \frac{2}{3} + \frac{4}{15} + \frac{4}{45} + \ldots$$

$\geq$ the sum of first seven terms ≥ 7.355.

On the other hand, we have

$$e^2 < \left(1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!}\right) + \frac{2^5}{5!}\left(1 + \frac{2}{6} + \frac{2^2}{6^2} + \frac{2^3}{6^3} + \ldots\right)$$

$$= 7 + \frac{4}{15}\left(1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \ldots\right) = 7 + \frac{4}{15}\left(\frac{1}{1 - \frac{1}{3}}\right) = 7 + \frac{2}{5} = 7.4 .$$

Thus, e^2 lies between 7.355 and 7.4. Therefore, the value of e^2, rounded off to one decimal place, is 7.4.

A.1.5 Logarithmic Series

Another very important series is logarithmic series which is also in the form of infinite series. We state the following result without proof and illustrate its application with an example.

Theorem If $|x| < 1$, then

$$\log_e(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

The series on the right hand side of the above is called the *logarithmic series*.

Note The expansion of $\log_e(1+x)$ is valid for $x = 1$. Substituting $x = 1$ in the expansion of $\log_e(1+x)$, we get

$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

Example 5 If α, β are the roots of the equation $x^2 - px + q = 0$, prove that

$$\log_e\left(1+px+qx^2\right)=\left(\alpha+\beta\right)x-\frac{\alpha^2+\beta^2}{2}x^2+\frac{\alpha^3+\beta^3}{3}x^3-...$$

Solution Right hand side $= \left[\alpha x-\frac{\alpha^2x^2}{2}+\frac{\alpha^3x^3}{3}-...\right]+\left[\beta x-\frac{\beta^2x^2}{2}+\frac{\beta^3x^3}{3}-...\right]$

$$= \log_e\left(1+\alpha x\right)+\log\left(1+\beta x\right)$$

$$= \log_e\left(1+\left(\alpha+\beta\right)x+\alpha\beta x^2\right)$$

$$= \log_e\left(1+px+qx^2\right) = \text{Left hand side.}$$

Here, we have used the facts $\alpha+\beta=p$ and $\alpha\beta=q$. We know this from the given roots of the quadratic equation. We have also assumed that both $|\alpha x|<1$ and $|\beta x|<1$.

Appendix 2

MATHEMATICAL MODELLING

A.2.1 Introduction

Much of our progress in the last few centuries has made it necessary to apply mathematical methods to real-life problems arising from different fields – be it Science, Finance, Management etc. The use of Mathematics in solving real-world problems has become widespread especially due to the increasing computational power of digital computers and computing methods, both of which have facilitated the handling of lengthy and complicated problems. The process of translation of a real-life problem into a mathematical form can give a better representation and solution of certain problems. The process of translation is called Mathematical Modelling.

Here we shall familiaries you with the steps involved in this process through examples. We shall first talk about what a mathematical model is, then we discuss the steps involved in the process of modelling.

A.2.2 Preliminaries

Mathematical modelling is an essential tool for understanding the world. In olden days the Chinese, Egyptians, Indians, Babylonians and Greeks indulged in understanding and predicting the natural phenomena through their knowledge of mathematics. The architects, artisans and craftsmen based many of their works of art on geometric prinicples.

Suppose a surveyor wants to measure the height of a tower. It is physically very difficult to measure the height using the measuring tape. So, the other option is to find out the factors that are useful to find the height. From his knowledge of trigonometry, he knows that if he has an angle of elevation and the distance of the foot of the tower to the point where he is standing, then he can calculate the height of the tower.

So, his job is now simplified to find the angle of elevation to the top of the tower and the distance from the foot of the tower to the point where he is standing. Both of which are easily measurable. Thus, if he measures the angle of elevation as 40° and the distance as 450m, then the problem can be solved as given in Example 1.

Example 1 The angle of elevation of the top of a tower from a point O on the ground, which is 450 m away from the foot of the tower, is 40°. Find the height of the tower.

Solution We shall solve this in different steps.

Step 1 We first try to understand the real problem. In the problem a tower is given and its height is to be measured. Let h denote the height. It is given that the horizontal distance of the foot of the tower from a particular point O on the ground is 450 m. Let d denotes this distance. Then d = 450m. We also know that the angle of elevation, denoted by θ, is 40°.

The real problem is to find the height h of the tower using the known distance d and the angle of elevation θ.

Step 2 The three quantities mentioned in the problem are height, distance and angle of elevation.

So we look for a relation connecting these three quantities. This is obtained by expressing it geometrically in the following way (Fig 1).

AB denotes the tower. OA gives the horizontal distance from the point O to foot of the tower. ∠AOB is the angle of elevation. Then we have

$$\tan\theta = \frac{h}{d} \text{ or } h = d\tan\theta \qquad \dots (1)$$

This is an equation connecting θ, h and d.

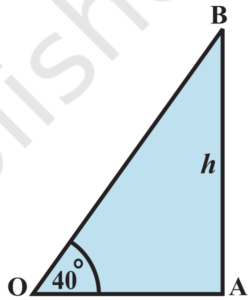


Fig 1

Step 3 We use Equation (1) to solve h. We have θ = 40°. and d = 450m. Then we get $h = \tan 40° \times 450 = 450 \times 0.839 = 377.6$m

Step 4 Thus we got that the height of the tower approximately 378m.

Let us now look at the different steps used in solving the problem. In step 1, we have studied the real problem and found that the problem involves three parameters height, distance and angle of elevation. That means in this step we have *studied the real-life problem and identified the parameters.*

In the Step 2, we used some geometry and found that the problem can be represented geometrically as given in Fig 1. Then we used the trigonometric ratio for the "tangent" function and found the relation as

$$h = d\tan\theta$$

So, in this step we formulated the problem mathematically. That means we found an equation representing the real problem.

In Step 3, we solved the mathematical problem and got that h = 377.6m. That is we found

Solution of the problem.

In the last step, we interpreted the solution of the problem and stated that the height of the tower is approximately 378m. We call this as

Interpreting the mathematical solution to the real situation

In fact these are the steps mathematicians and others use to study various real-life situations. We shall consider the question, “why is it necessary to use mathematics to solve different situations.”

Here are some of the examples where mathematics is used effectively to study various situations.

1. Proper flow of blood is essential to transmit oxygen and other nutrients to various parts of the body in humanbeings as well as in all other animals. Any constriction in the blood vessel or any change in the characteristics of blood vessels can change the flow and cause damages ranging from minor discomfort to sudden death. The problem is to find the relationship between blood flow and physiological characteristics of blood vessel.
2. In cricket a third umpire takes decision of a LBW by looking at the trajectory of a ball, simulated, assuming that the batsman is not there. Mathematical equations are arrived at, based on the known paths of balls before it hits the batsman’s leg. This simulated model is used to take decision of LBW.
3. Meteorology department makes weather predictions based on mathematical models. Some of the parameters which affect change in weather conditions are temperature, air pressure, humidity, wind speed, etc. The instruments are used to measure these parameters which include thermometers to measure temperature, barometers to measure airpressure, hygrometers to measure humidity, anemometers to measure wind speed. Once data are received from many stations around the country and feed into computers for further analysis and interpretation.
4. Department of Agriculture wants to estimate the yield of rice in India from the standing crops. Scientists identify areas of rice cultivation and find the average yield per acre by cutting and weighing crops from some representative fields. Based on some statistical techniques decisions are made on the average yield of rice.

   How do mathematicians help in solving such problems? They sit with experts in the area, for example, a physiologist in the first problem and work out a mathematical equivalent of the problem. This equivalent consists of one or more equations or inequalities etc. which are called the mathematical models. Then solve the model and interpret the solution in terms of the original problem. Before we explain the process, we shall discuss what a mathematical model is.

A mathematical model is a representation which comprehends a situation.

An interesting geometric model is illustrated in the following example.

Example 2 (Bridge Problem) Konigsberg is a town on the Pregel River, which in the 18th century was a German town, but now is Russian. Within the town are two river islands that are connected to the banks with seven bridges as shown in (Fig 2).

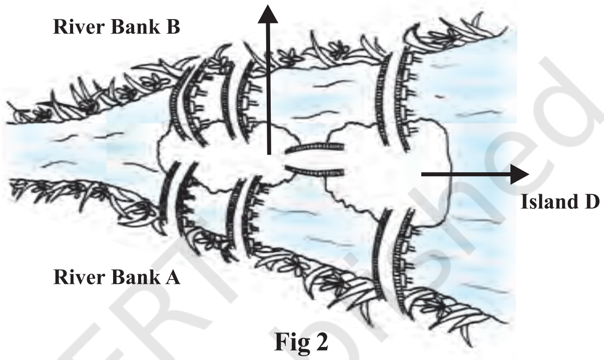


Fig 2

People tried to walk around the town in a way that only crossed each bridge once, but it proved to be difficult problem. Leonhard Euler, a Swiss mathematician in the service of the Russian empire Catherine the Great, heard about the problem. In 1736 Euler proved that the walk was not possible to do. He proved this by inventing a kind of diagram called a network, that is made up of vertices (dots where lines meet) and arcs (lines) (Fig3).

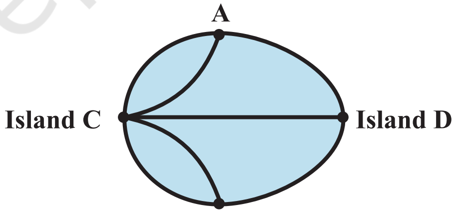


Fig 3

He used four dots (vertices) for the two river banks and the two islands. These have been marked A, B and C, D. The seven lines (arcs) are the seven bridges. You can see that 3 bridges (arcs) join to riverbank, A, and 3 join to riverbank B. 5 bridges (arcs) join to island C, and 3 join to island D. This means that all the vertices have an odd number of arcs, so they are called odd vertices (An even vertex would have to have an even number of arcs joining to it).

Remember that the problem was to travel around town crossing each bridge only once. On Euler's network this meant tracing over each arc only once, visiting all the vertices. Euler proved it could not be done because he worked out that, to have an odd vertex you would have to begin or end the trip at that vertex. (Think about it). Since there can only be one beginning and one end, there can only be two odd vertices if you are to trace over each arc only once. Since the bridge problem has 4 odd vertices, it just not possible to do!

After Euler proved his Theorem, much water has flown under the bridges in Konigsberg. In 1875, an extra bridge was built in Konigsberg, joining the land areas of river banks A and B (Fig 4). Is it possible now for the Konigsbergians to go round the city, using each bridge only once?

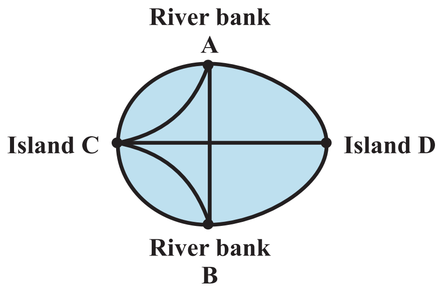


Fig 4

Here the situation will be as in Fig 4. After the addition of the new edge, both the vertices A and B have become even degree vertices. However, D and C still have odd degree. So, it is possible for the Konigsbergians to go around the city using each bridge exactly once.

The invention of networks began a new theory called graph theory which is now used in many ways, including planning and mapping railway networks (Fig 4).

A.2.3 What is Mathematical Modelling?

Here, we shall define what mathematical modelling is and illustrate the different processes involved in this through examples.

Definition Mathematical modelling is an attempt to study some part (or form) of the real-life problem in mathematical terms.

Conversion of physical situation into mathematics with some suitable conditions is known as mathematical modelling. Mathematical modelling is nothing but a technique and the pedagogy taken from fine arts and not from the basic sciences. Let us now understand the different processes involved in Mathematical Modelling. Four steps are involved in this process. As an illustrative example, we consider the modelling done to study the motion of a simple pendulum.

Understanding the problem

This involves, for example, understanding the process involved in the motion of simple pendulum. All of us are familiar with the simple pendulum. This pendulum is simply a mass (known as bob) attached to one end of a string whose other end is fixed at a point. We have studied that the motion of the simple pendulum is periodic. The period depends upon the length of the string and acceleration due to gravity. So, what we need to find is the period of oscillation. Based on this, we give a precise statement of the problem as

Statement How do we find the period of oscillation of the simple pendulum?
The next step is formulation.

Formulation Consists of two main steps.

1. Identifying the relevant factors In this, we find out what are the factors/

parameters involved in the problem. For example, in the case of pendulum, the factors are period of oscillation (T), the mass of the bob (m), effective length (l) of the pendulum which is the distance between the point of suspension to the centre of mass of the bob. Here, we consider the length of string as effective length of the pendulum and acceleration due to gravity (g), which is assumed to be constant at a place.

So, we have identified four parameters for studying the problem. Now, our purpose is to find T. For this we need to understand what are the parameters that affect the period which can be done by performing a simple experiment.

We take two metal balls of two different masses and conduct experiment with each of them attached to two strings of equal lengths. We measure the period of oscillation. We make the observation that there is no appreciable change of the period with mass. Now, we perform the same experiment on equal mass of balls but take strings of different lengths and observe that there is clear dependence of the period on the length of the pendulum.

This indicates that the mass m is not an *essential parameter* for finding period whereas the length l is an essential parameter.

This process of searching the **essential parameters** is necessary before we go to the next step.

2. Mathematical description This involves finding an equation, inequality or a geometric figure using the parameters already identified.

In the case of simple pendulum, experiments were conducted in which the values of period T were measured for different values of l. These values were plotted on a graph which resulted in a curve that resembled a parabola. It implies that the relation between T and l could be expressed

$$T^2 = kl \qquad \text{... (1)}$$

It was found that $k = \frac{4\pi^2}{g}$. This gives the equation

$$T = 2\pi\sqrt{\frac{l}{g}} \qquad \text{... (2)}$$

Equation (2) gives the mathematical formulation of the problem.

Finding the solution The mathematical formulation rarely gives the answer directly. Usually we have to do some operation which involves solving an equation, calculation or applying a theorem etc. In the case of simple pendulums the solution involves applying the formula given in Equation (2).

The period of oscillation calculated for two different pendulums having different lengths is given in Table 1

Table 1

l	225 cm	275cm
T	3.04 sec	3.36 sec

The table shows that for l = 225 cm, T = 3.04 sec and for l = 275 cm, T = 3.36 sec.

Interpretation/Validation

A mathematical model is an attempt to study, the essential characteristic of a real life problem. Many times model equations are obtained by assuming the situation in an idealised context. The model will be useful only if it explains all the facts that we would like it to explain. Otherwise, we will reject it, or else, improve it, then test it again. In other words, ***we measure the effectiveness of the model by comparing the results obtained from the mathematical model, with the known facts about the real problem. This process is called validation of the model.*** In the case of simple pendulum, we conduct some experiments on the pendulum and find out period of oscillation. The results of the experiment are given in Table 2.

Table 2

Periods obtained experimentally for four different pendulums

Mass (gms)	Length (cms)	Time (secs)
385	275	3.371
	225	3.056
230	275	3.352
	225	3.042

Now, we compare the measured values in Table 2 with the calculated values given in Table 1.

The difference in the observed values and calculated values gives the error. For example, for l = 275 cm, and mass m = 385 gm,

$$\text{error} = 3.371 - 3.36 = 0.011$$

which is small and the model is accepted.

Once we accept the model, we have to interpret the model. ***The process of describing the solution in the context of the real situation is called interpretation of the model.*** In this case, we can interpret the solution in the following way:

(a) The period is directly proportional to the square root of the length of the pendulum.

(b) It is inversely proportional to the square root of the acceleration due to gravity.

Our validation and interpretation of this model shows that the mathematical model is in good agreement with the practical (or observed) values. But we found that there is some error in the calculated result and measured result. This is because we have neglected the mass of the string and resistance of the medium. So, in such situation we look for a better model and this process continues.

This leads us to an important observation. The real world is far too complex to understand and describe completely. We just pick one or two main factors to be completely accurate that may influence the situation. Then try to obtain a simplified model which gives some information about the situation. We study the simple situation with this model expecting that we can obtain a better model of the situation.

Now, we summarise the main process involved in the modelling as

(a) Formulation (b) Solution (c) Interpretation/Validation

The next example shows how modelling can be done using the techniques of finding graphical solution of inequality.

Example 3 A farm house uses atleast 800 kg of special food daily. The special food is a mixture of corn and soyabean with the following compositions

Table 3

Material	Nutrients present per Kg Protein	Nutrients present per Kg Fibre	Cost per Kg
Corn	.09	.02	Rs 10
Soyabean	.60	.06	Rs 20

The dietary requirements of the special food stipulate atleast 30% protein and at most 5% fibre. Determine the daily minimum cost of the food mix.

Solution Step 1 Here the objective is to minimise the total daily cost of the food which is made up of corn and soyabean. So the variables (factors) that are to be considered are

x = the amount of corn

y = the amount of soyabean

z = the cost

Step 2 The last column in Table 3 indicates that z, x, y are related by the equation

$$z = 10x + 20y \qquad \ldots (1)$$

The problem is to minimise z with the following constraints:

(a) The farm used atleast 800 kg food consisting of corn and soyabean

i.e., $x + y \geq 800$ ... (2)

(b) The food should have atleast 30% protein dietary requirement in the proportion as given in the first column of Table 3. This gives

$$0.09x + 0.6y \geq 0.3\ (x + y) \qquad \text{... (3)}$$

(c) Similarly the food should have atmost 5% fibre in the proportion given in 2nd column of Table 3. This gives

$$0.02x + 0.06\ y \leq 0.05\ (x + y) \qquad \text{... (4)}$$

We simplify the constraints given in (2), (3) and (4) by grouping all the coefficients of x, y.

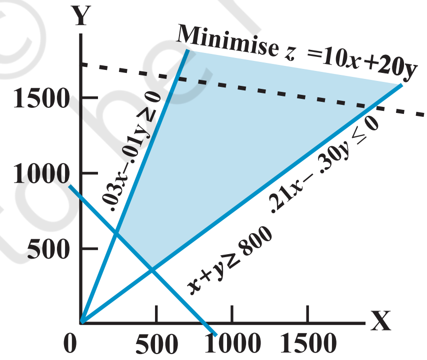
Then the problem can be restated in the following mathematical form.

Statement Minimise z subject to

$$x + y \geq 800$$
$$0.21x - .30y \leq 0$$
$$0.03x - .01y \geq 0$$

This gives the formulation of the model.

Step 3 This can be solved graphically. The shaded region in Fig 5 gives the possible solution of the equations. From the graph it is clear that the minimum value is got at the point (470.6,329.4) i.e., $x = 470.6$ and $y = 329.4$.



Optimum: $x = 470.6$ **kg**
$y = 329.4$ **kg**
$z =$ **Rs 11294**

Fig 5

This gives the value of z as $z = 10 \times 470.6 + 20 \times 329.4 = 11294$

This is the mathematical solution.

Step 4 The solution can be interpreted as saying that, "The minimum cost of the special food with corn and soyabean having the required portion of nutrient contents, protein and fibre is Rs 11294 and we obtain this minimum cost if we use 470.6 kg of corn and 329.4 kg of soyabean."

In the next example, we shall discuss how modelling is used to study the population of a country at a particular time.

Example 4 Suppose a population control unit wants to find out "how many people will be there in a certain country after 10 years"

Step 1 Formulation We first observe that the population changes with time and it increases with birth and decreases with deaths.

We want to find the population at a particular time. Let t denote the time in years. Then t takes values 0, 1, 2, ..., $t = 0$ stands for the present time, $t = 1$ stands for the next year etc. For any time t, let $p\ (t)$ denote the population in that particular year.

Suppose we want to find the population in a particular year, say $t_0 = 2006$. How will we do that. We find the population by Jan. 1st, 2005. Add the number of births in that year and subtract the number of deaths in that year. Let B(t) denote the number of births in the one year between t and $t + 1$ and D(t) denote the number of deaths between t and $t + 1$. Then we get the relation

$$\mathrm{P}\ (t + 1) = \mathrm{P}\ (t) + \mathrm{B}\ (t) - \mathrm{D}\ (t)$$

Now we make some assumptions and definitions

1. $\dfrac{\mathrm{B}\,(t)}{\mathrm{P}\,(t)}$ is called the *birth rate* for the time interval t to $t + 1$.

2. $\dfrac{\mathrm{D}\ (\mathrm{t})}{\mathrm{P}\ (\mathrm{t})}$ is called the *death rate* for the time interval t to $t + 1$.

Assumptions

1. The birth rate is the same for all intervals. Likewise, the death rate is the same for all intervals. This means that there is a constant b, called the birth rate, and a constant d, called the death rate so that, for all $t \geq 0$,

$$b = \frac{\mathrm{B}\ (t)}{\mathrm{P}\ (t)} \quad \text{and} \quad d = \frac{\mathrm{D}\ (t)}{\mathrm{P}\ (t)} \qquad \text{... (1)}$$

2. There is no migration into or out of the population; i.e., the only source of population change is birth and death.

As a result of assumptions 1 and 2, we deduce that, for $t \geq 0$,

$$
\begin{aligned}
P(t+1) &= P(t) + B(t) - D(t) \\
&= P(t) + bP(t) - dP(t) \\
&= (1 + b - d)\, P(t) \qquad \ldots (2)
\end{aligned}
$$

Setting $t = 0$ in (2) gives

$$P(1) = (1 + b - d)P(0) \qquad \ldots (3)$$

Setting $t = 1$ in Equation (2) gives

$$
\begin{aligned}
P(2) &= (1 + b - d)\, P(1) \\
&= (1 + b - d)(1 + b - d)\, P(0) \qquad \text{(Using equation 3)} \\
&= (1 + b - d)^2\, P(0)
\end{aligned}
$$

Continuing this way, we get

$$P(t) = (1 + b - d)^t\, P(0) \qquad \ldots (4)$$

for $t = 0, 1, 2, \ldots$ The constant $1 + b - d$ is often abbreviated by r and called the *growth rate or*, in more high-flown language, the *Malthusian parameter,* in honor of Robert Malthus who first brought this model to popular attention. In terms of r, Equation (4) becomes

$$P(t) = P(0)r^t \quad , \qquad t = 0, 1, 2, \ldots \qquad \ldots (5)$$

$P(t)$ is an example of an *exponential function.* Any function of the form cr^t, where c and r are constants, is an exponential function.

Equation (5) gives the mathematical formulation of the problem.

Step 2 – Solution

Suppose the current population is 250,000,000 and the rates are $b = 0.02$ and $d = 0.01$. What will the population be in 10 years? Using the formula, we calculate P(10).

$$
\begin{aligned}
P(10) &= (1.01)^{10}\,(250{,}000{,}000) \\
&= (1.104622125)\,(250{,}000{,}000) \\
&= 276{,}155{,}531.25
\end{aligned}
$$

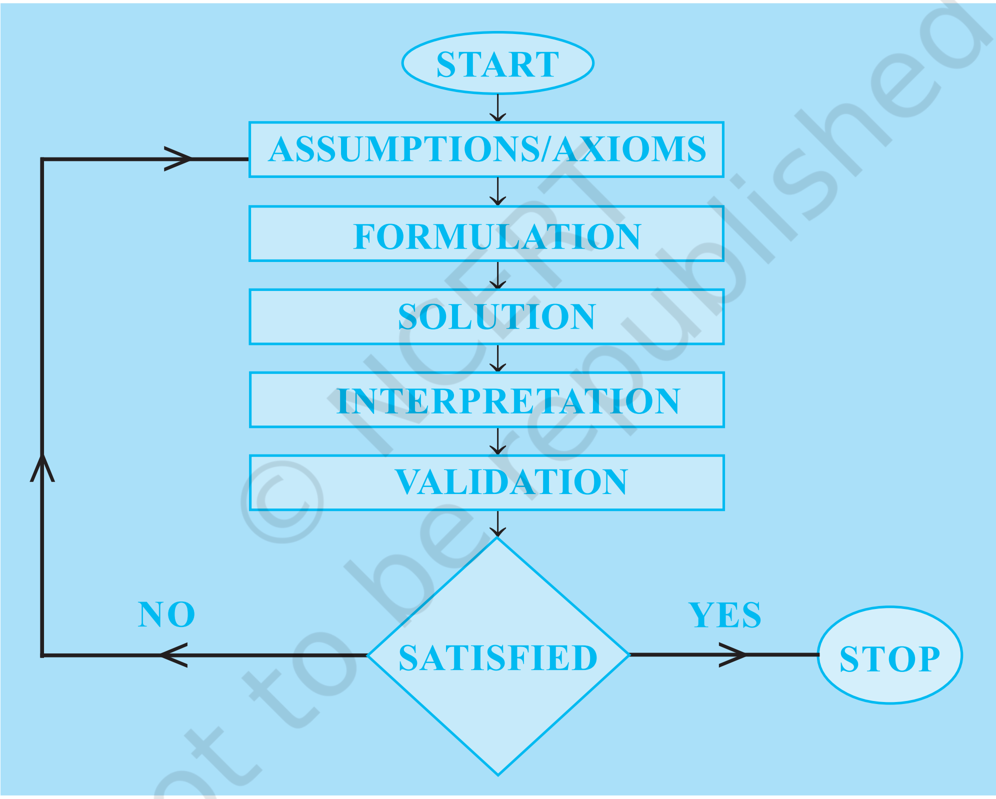
Step 3 Interpretation and Validation

Naturally, this result is absurd, since one can't have 0.25 of a person.

So, we do some approximation and conclude that the population is 276,155,531 (approximately). Here, we are not getting the exact answer because of the assumptions that we have made in our mathematical model.

The above examples show how modelling is done in variety of situations using different mathematical techniques.

Since a mathematical model is a simplified representation of a real problem, by its very nature, has built-in assumptions and approximations. Obviously, the most important question is to decide whether our model is a good one or not i.e., when the obtained results are interpreted physically whether or not the model gives reasonable answers. If a model is not accurate enough, we try to identify the sources of the shortcomings. It may happen that we need a new formulation, new mathematical manipulation and hence a new evaluation. Thus mathematical modelling can be a cycle of the modelling process as shown in the flowchart given below:



ANSWERS

EXERCISE 1.1

1. (i), (iv), (v), (vi), (vii) and (viii) are sets.

2. (i) $\in$ (ii) $\notin$ (iii) $\notin$ (vi) $\in$ (v) $\in$ (vi) $\notin$

3. (i) A $= \{-3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$ (ii) B $= \{1, 2, 3, 4, 5\}$
(iii) C $= \{17, 26, 35, 44, 53, 62, 71, 80\}$ (iv) D $= \{2, 3, 5\}$
(v) E = {T, R, I, G, O, N, M, E, Y} (vi) F = {B, E, T, R}

4. (i) $\{ x : x = 3n, n \in \mathrm{N}$ and $1 \le n \le 4 \}$ (ii) $\{ x : x = 2^n, n \in \mathrm{N}$ and $1 \le n \le 5 \}$
(iii) $\{ x : x = 5^n, n \in \mathrm{N}$ and $1 \le n \le 4 \}$ (iv) $\{ x : x$ is an even natural number$\}$
(v) $\{ x : x = n^2, n \in \mathrm{N}$ and $1 \le n \le 10 \}$

5. (i) A $= \{1, 3, 5, \ldots\}$ (ii) B $= \{0, 1, 2, 3, 4\}$
(iii) C $= \{-2, -1, 0, 1, 2\}$ (iv) D = { L, O, Y, A }
(v) E = { February, April, June, September, November }
(vi) F $= \{b, c, d, f, g, h, j\}$

6. (i) $\leftrightarrow$ (c) (ii) $\leftrightarrow$ (a) (iii) $\leftrightarrow$ (d) (iv) $\leftrightarrow$ (b)

EXERCISE 1.2

1. (i), (iii), (iv)

2. (i) Finite (ii) Infinite (iii) Finite (iv) Infinite (v) Finite

3. (i) Infinite (ii) Finite (iii) Infinite (iv) Finite (v) Infinite

4. (i) Yes (ii) No (iii) Yes (iv) No

5. (i) No (ii) Yes **6.** B= D, E = G

EXERCISE 1.3

1. (i) $\subset$ (ii) $\not\subset$ (iii) $\subset$ (iv) $\not\subset$ (v) $\not\subset$ (vi) $\subset$
(vii) $\subset$

2. (i) False (ii) True (iii) False (iv) True (v) False (vi) True

3. (i) as $\{3,4\} \in$ A, (v) as $1 \in$ A, (vii) as $\{1,2, 5\} \subset$ A,
(viii) as $3 \notin$ A, (ix) as $\phi \subset$ A, (xi) as $\phi \subset$ A,

4. (i) $\phi, \{ a \}$ (ii) $\phi, \{ a \}, \{ b \}, \{ a, b \}$
(iii) $\phi, \{ 1 \}, \{ 2 \}, \{ 3 \}, \{ 1, 2 \}, \{ 1, 3 \}, \{ 2, 3 \}, \{ 1, 2, 3 \}$ (iv) ϕ

5. (i) $(-4, 6]$ (ii) $(-12, -10)$ (iii) $[0, 7)$
(iv) $[3, 4]$

6. (i) $\{ x : x \in \mathrm{R}, -3 < x < 0 \}$ (ii) $\{ x : x \in \mathrm{R}, 6 \le x \le 12 \}$
(iii) $\{ x : x \in \mathrm{R}, 6 < x \le 12 \}$ (iv) $\{ x \ \mathrm{R} : -23 \le x < 5 \}$ **8.** (iii)

EXERCISE 1.4

1. (i) $X \cup Y = \{1, 2, 3, 5\}$ (ii) $A \cup B = \{a, b, c, e, i, o, u\}$

(iii) $A \cup B = \{x : x = 1, 2, 4, 5$ or a multiple of $3\}$

(iv) $A \cup B = \{x : 1 < x < 10, x \in N\}$ (v) $A \cup B = \{1, 2, 3\}$

2. Yes, $A \cup B = \{a, b, c\}$ **3.** B

4. (i) $\{1, 2, 3, 4, 5, 6\}$ (ii) $\{1, 2, 3, 4, 5, 6, 7, 8\}$ (iii) $\{3, 4, 5, 6, 7, 8\}$

(iv) $\{3, 4, 5, 6, 7, 8, 9, 10\}$ (v) $\{1, 2, 3, 4, 5, 6, 7, 8\}$

(vi) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ (vii) $\{3, 4, 5, 6, 7, 8, 9, 10\}$

5. (i) $X \cap Y = \{1, 3\}$ (ii) $A \cap B = \{a\}$ (iii) $\{3\}$ (iv) ϕ (v) ϕ

6. (i) $\{7, 9, 11\}$ (ii) $\{11, 13\}$ (iii) ϕ (iv) $\{11\}$

(v) ϕ (vi) $\{7, 9, 11\}$ (vii) ϕ

(viii) $\{7, 9, 11\}$ (ix) $\{7, 9, 11\}$ (x) $\{7, 9, 11, 15\}$

7. (i) B (ii) C (iii) D (iv) ϕ

(v) $\{2\}$ (vi) $\{x : x$ is an odd prime number$\}$ **8.** (iii)

9. (i) $\{3, 6, 9, 15, 18, 21\}$ (ii) $\{3, 9, 15, 18, 21\}$ (iii) $\{3, 6, 9, 12, 18, 21\}$

(iv) $\{4, 8, 16, 20\}$ (v) $\{2, 4, 8, 10, 14, 16\}$ (vi) $\{5, 10, 20\}$

(vii) $\{20\}$ (viii) $\{4, 8, 12, 16\}$ (ix) $\{2, 6, 10, 14\}$

(x) $\{5, 10, 15\}$ (xi) $\{2, 4, 6, 8, 12, 14, 16\}$ (xii) $\{5, 15, 20\}$

10. (i) $\{a, c\}$ (ii) $\{f, g\}$ (iii) $\{b, d\}$

11. Set of irrational numbers **12.** (i) F (ii) F (iii) T (iv) T

EXERCISE 1.5

1. (i) $\{5, 6, 7, 8, 9\}$ (ii) $\{1, 3, 5, 7, 9\}$ (iii) $\{7, 8, 9\}$

(iv) $\{5, 7, 9\}$ (v) $\{1, 2, 3, 4\}$ (vi) $\{1, 3, 4, 5, 6, 7, 9\}$

2. (i) $\{d, e, f, g, h\}$ (ii) $\{a, b, c, h\}$ (iii) $\{b, d, f, h\}$

(iv) $\{b, c, d, e\}$

3. (i) $\{x : x$ is an odd natural number$\}$

(ii) $\{x : x$ is an even natural number$\}$

(iii) $\{x : x \in \mathbf{N}$ and x is not a multiple of $3\}$

(iv) $\{ x : x$ is a positive composite number or $x = 1 \}$
(v) $\{ x : x$ is a positive integer which is not divisible by 3 or not divisible by $5\}$
(vi) $\{ x : x \in \mathbf{N}$ and x is not a perfect square $\}$
(vii) $\{ x : x \in \mathbf{N}$ and x is not a perfect cube $\}$
(viii) $\{ x : x \in \mathbf{N}$ and $x \neq 3 \}$ (ix) $\{ x : x \in \mathbf{N}$ and $x \neq 2 \}$
(x) $\{ x : x \in \mathbf{N}$ and $x < 7 \}$ (xi) $\{ x : x \in \mathbf{N}$ and $x \leq \frac{9}{2} \}$

6. A' is the set of all equilateral triangles.
7. (i) U (ii) A (iii) ϕ (iv) ϕ

Miscellaneous Exercise on Chapter 1

1. $A \subset B$, $A \subset C$, $B \subset C$, $D \subset A$, $D \subset B$, $D \subset C$
2. (i) False (ii) False (iii) True (iv) False (v) False
(vi) True
10. We may take $A = \{ 1, 2 \}$, $B = \{ 1, 3 \}$, $C = \{ 2, 3 \}$

EXERCISE 2.1

1. $x = 2$ and $y = 1$ **2.** The number of elements in $A \times B$ is 9.
3. $G \times H = \{(7, 5), (7, 4), (7, 2), (8, 5), (8, 4), (8, 2)\}$
$H \times G = \{(5, 7), (5, 8), (4, 7), (4, 8), (2, 7), (2, 8)\}$
4. (i) False
$P \times Q = \{(m, n), (m, m), (n, n), (n, m)\}$
(ii) True
(iii) True
5. $A \times A = \{(-1, -1), (-1, 1), (1, -1), (1, 1)\}$
$A \times A \times A = \{(-1, -1, -1), (-1, -1, 1), (-1, 1, -1), (-1, 1, 1), (1, -1, -1), (1, -1, 1), (1, 1, -1), (1, 1, 1)\}$
6. $A = \{a, b\}$, $B = \{x, y\}$
8. $A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$
$A \times B$ will have $2^4 = 16$ subsets.
9. $A = \{x, y, z\}$ and $B = \{1,2\}$
10. $A = \{-1, 0, 1\}$, remaining elements of
$A \times A$ are $(-1, -1), (-1, 1), (0, -1), (0, 0), (1, -1), (1, 0), (1, 1)$

EXERCISE 2.2

1. R = {(1, 3), (2, 6), (3, 9), (4, 12)}
Domain of R = {1, 2, 3, 4}
Range of R = {3, 6, 9, 12}
Co domain of R = {1, 2, ..., 14}

2. R = {(1, 6), (2, 7), (3, 8)}
Domain of R = {1, 2, 3}
Range of R = {6, 7, 8}

3. R = {(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)}

4. (i) R = {$(x, y) : y = x - 2$ for $x = 5, 6, 7$}
(ii) R = {(5,3), (6,4), (7,5)}. Domain of R = {5, 6, 7}, Range of R = {3, 4, 5}

5. (i) R = {(1, 1), (1,2), (1, 3), (1, 4), (1, 6), (2 4), (2, 6), (2, 2), (4, 4), (6, 6), (3, 3), (3, 6)}
(ii) Domain of R = {1, 2, 3, 4, 6}
(iii) Range of R = {1, 2, 3, 4, 6}

6. Domain of R = {0, 1, 2, 3, 4, 5,}
Range of R = {5, 6, 7, 8, 9, 10}

7. R = {(2, 8), (3, 27), (5, 125), (7, 343)}

8. No. of relations from A into B = 2^6

9. Domain of R = **Z**
Range of R = **Z**

EXERCISE 2.3

1. (i) yes, Domain = {2, 5, 8, 11, 14, 17}, Range = {1}
(ii) yes, Domain = (2, 4, 6, 8, 10, 12, 14}, Range = {1, 2, 3, 4, 5, 6, 7}
(iii) No.

2. (i) Domain = **R**, Range = $(-\infty, 0]$
(ii) Domain of function = $\{x : -3 \le x \le 3\}$
Range of function = $\{x : 0 \le x \le 3\}$

3. (i) $f(0) = -5$ (ii) $f(7) = 9$ (iii) $f(-3) = -11$

4. (i) $t(0) = 32$ (ii) $t(28) = \frac{412}{5}$ (iii) $t(-10) = 14$ (iv) 100

5. (i) Range = $(-\infty, 2)$ (ii) Range = $[2, \infty)$ (iii) Range = **R**

Miscellaneous Exercise on Chapter 2

2. 2.1 **3.** Domain of function is set of real numbers except 6 and 2.

4. Domain $= [1, \infty)$, Range $= [0, \infty)$

5. Domain $= \mathbf{R}$, Range = non-negative real numbers

6. Range $= [0, 1)$

7. $(f + g)\,x = 3x - 2$ **8.** $a = 2, b = -1$ **9.** (i) No (ii) No (iii) No

$(f - g)\,x = -x + 4$

$$\left(\frac{f}{g}\right)x = \frac{x+1}{2x-3},\quad x \neq \frac{3}{2}$$

10. (i) Yes, (ii) No **11.** No **12.** Range of $f = \{3, 5, 11, 13\}$

EXERCISE 3.1

1. (i) $\frac{5\pi}{36}$ (ii) $-\frac{19\pi}{72}$ (iii) $\frac{4\pi}{3}$ (iv) $\frac{26\pi}{9}$

2. (i) $39°\,22'\,30''$ (ii) $-229°\,5'\,27''$ (iii) $300°$ (iv) $210°$

3. 12π **4.** $12°\,36'$ **5.** $\frac{20\pi}{3}$ **6.** $5 : 4$

7. (i) $\frac{2}{15}$ (ii) $\frac{1}{5}$ (iii) $\frac{7}{25}$

EXERCISE 3.2

1. $\sin x = -\frac{\sqrt{3}}{2}, \operatorname{cosec} x = -\frac{2}{\sqrt{3}}, \sec x = -2, \tan x = \sqrt{3}, \cot x = \frac{1}{\sqrt{3}}$

2. $\operatorname{cosec} x = \frac{5}{3}, \cos x = -\frac{4}{5}, \sec x = -\frac{5}{4}, \tan x = -\frac{3}{4}, \cot x = -\frac{4}{3}$

3. $\sin x = -\frac{4}{5}, \operatorname{cosec} x = -\frac{5}{4}, \cos x = -\frac{3}{5}, \sec x = -\frac{5}{3}, \tan x = \frac{4}{3}$

4. $\sin x = -\frac{12}{13}, \operatorname{cosec} x = -\frac{13}{12}, \cos x = \frac{5}{13}, \tan x = -\frac{12}{5}, \cot x = -\frac{5}{12}$

5. $\sin x = \frac{5}{13}$, $\operatorname{cosec} x = \frac{13}{5}$, $\cos x = -\frac{12}{13}$, $\sec x = -\frac{13}{12}$, $\cot x = -\frac{12}{5}$

6. $\frac{1}{\sqrt{2}}$ **7.** 2 **8.** $\sqrt{3}$ **9.** $\frac{\sqrt{3}}{2}$ **10.** 1

EXERCISE 3.3

5. (i) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ (ii) $2-\sqrt{3}$

Miscellaneous Exercise on Chapter 3

8. $\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}, \frac{1}{2}$

9. $\frac{\sqrt{6}}{3}, -\frac{\sqrt{3}}{3}, -\sqrt{2}$

10. $\frac{\sqrt{8+2\sqrt{15}}}{4}, \frac{\sqrt{8-2\sqrt{15}}}{4}, 4+\sqrt{15}$

EXERCISE 4.1

1. $3 + i0$ **2.** $0 + i\,0$ **3.** $0 + i\,1$ **4.** $14 + 28\,i$

5. $2 - 7\,i$ **6.** $-\frac{19}{5} - \frac{21i}{10}$ **7.** $\frac{17}{3} + i\frac{5}{3}$ **8.** $-4 + i\,0$

9. $-\frac{242}{27} - 26i$ **10.** $\frac{-22}{3} - i\frac{107}{27}$ **11.** $\frac{4}{25} + i\frac{3}{25}$ **12.** $\frac{\sqrt{5}}{14} - i\frac{3}{14}$

13. $0 + i1$ **14.** $0 - i\,\frac{7\sqrt{2}}{2}$

Miscellaneous Exercise on Chapter 4

1. $2-2i$ **3.** $\dfrac{307+599i}{442}$

5. $\sqrt{2}$ **7.** (i) $\dfrac{-2}{5}$, (ii) 0 **8.** $x=3, y=-3$ **9.** 2

11. 1 **12.** 0 **14.** 4

EXERCISE 5.1

1. (i) {1, 2, 3, 4} (ii) {... – 3, – 2, –1, 0, 1,2,3,4,}

2. (i) No Solution (ii) {... – 4, – 3}

3. (i) {... – 2, – 1, 0, 1} (ii) $(-\infty, 2)$

4. (i) {–1, 0, 1, 2, 3, ...} (ii) $(-2, \infty)$

5. $(-4, \infty)$ **6.** $(-\infty, -3)$ **7.** $(-\infty, -3]$ **8.** $(-\infty, 4]$

9. $(-\infty, 6)$ **10.** $(-\infty, -6)$ **11.** $(-\infty, 2]$ **12.** $(-\infty, 120]$

13. $(4, \infty)$ **14.** $(-\infty, 2]$ **15.** $(4, \infty)$ **16.** $(-\infty, 2]$

17. $(-\infty, 3)$, 

18. $[-1, \infty)$, 

19. $(-1, \infty)$, 

20. $\left[-\dfrac{2}{7}, \infty\right)$, 

21. 35 **22.** 82

23. (5,7), (7,9) **24.** (6,8), (8,10), (10,12)

25. 9 cm **26.** Greater than or equal to 8cm but less than or equal to 22cm

Miscellaneous Exercise on Chapter 5

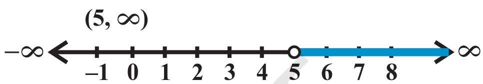
1. [2, 3] **2.** (0, 1] **3.** [– 4, 2]

4. (– 23, 2] **5.** $\left(\frac{-80}{3}, \frac{-10}{3}\right]$ **6.** $\left[1, \frac{11}{3}\right]$

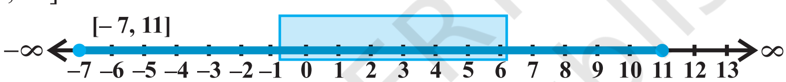
7. (–5, 5)

8. (–1, 7)

9. (5, ∞)



10. [– 7, 11]



11. Between 20°C and 25°C

12. More than 320 litres but less than 1280 litres.

13. More than 562.5 litres but less than 900 litres.

14. $9.6 \leq MA \leq 16.8$

EXERCISE 6.1

1. (i) 125, (ii) 60. **2.** 108 **3.** 5040 **4.** 336

5. 8 **6.** 20

EXERCISE 6.2

1. (i) 40320, (ii) 18 **2.** 30, No **3.** 28 **4.** 64

5. (i) 30, (ii) 15120

EXERCISE 6.3

1. 504 **2.** 4536 **3.** 60 **4.** 120, 48

5. 56 **6.** 9 **7.** (i) 3, (ii) 4 **8.** 40320

9. (i) 360, (ii) 720, (iii) 240 **10.** 33810

11. (i) 1814400, (ii) 2419200, (iii) 25401600

EXERCISE 6.4

1. 45 **2.** (i) 5, (ii) 6 **3.** 210 **4.** 40

5. 2000 **6.** 778320 **7.** 3960 **8.** 200

9. 35

Miscellaneous Exercise on Chapter 6

1. 3600 **2.** 1440 **3.** (i) 504, (ii) 588, (iii) 1632

4. 907200 **5.** 120 **6.** 50400 **7.** 420

8. ${}^{4}C_1 \times {}^{48}C_4$ **9.** 2880 **10.** ${}^{22}C_7 + {}^{22}C_{10}$ **11.** 151200

EXERCISE 7.1

1. $1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$

2. $\frac{32}{x^5} - \frac{40}{x^3} + \frac{20}{x} - 5x + \frac{5}{8}x^3 - \frac{x^5}{32}$

3. $64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$

4. $\frac{x^5}{243} + \frac{5x^3}{81} + \frac{10}{27}x + \frac{10}{9x} + \frac{5}{3x^3} + \frac{1}{x^5}$

5. $x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$

6. 884736 **7.** 11040808032 **8.** 104060401

9. 9509900499 **10.** $(1.1)^{10000} > 1000$ **11.** $8(a^3b + ab^3)$; $40\sqrt{6}$

12. $2(x^6 + 15x^4 + 15x^2 + 1)$, 198

Miscellaneous Exercise on Chapter 7

2. $396\sqrt{6}$ **3.** $2a^8 + 12a^6 - 10a^4 - 4a^2 + 2$

4. 0.9510

5. $\frac{16}{x} + \frac{8}{x^2} - \frac{32}{x^3} + \frac{16}{x^4} - 4x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - 5$

6. $27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$

EXERCISE 8.1

1. 3, 8, 15, 24, 35
2. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$
3. 2, 4, 8, 16 and 32
4. $-\frac{1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{5}{6}$ and $\frac{7}{6}$
5. 25, –125, 625, –3125, 15625
6. $\frac{3}{2}, \frac{9}{2}, \frac{21}{2}, 21$ and $\frac{75}{2}$
7. 65, 93
8. $\frac{49}{128}$
9. 729
10. $\frac{360}{23}$
11. 3, 11, 35, 107, 323; 3 + 11 + 35 + 107 + 323 + ...
12. $-1, \frac{-1}{2}, \frac{-1}{6}, \frac{-1}{24}, \frac{-1}{120}; -1 + \left(\frac{-1}{2}\right) + \left(\frac{-1}{6}\right) + \left(\frac{-1}{24}\right) + \left(\frac{-1}{120}\right) + ...$
13. 2, 2, 1, 0, –1; 2 + 2 + 1 + 0 + (–1) + ...
14. $1, 2, \frac{3}{2}, \frac{5}{3}$ and $\frac{8}{5}$

EXERCISE 8.2

1. $\frac{5}{2^{20}}, \frac{5}{2^n}$
2. 3072
4. – 2187
5. (a) 13th, (b) 12th, (c) 9th
6. ± 1
7. $\frac{1}{6}\left[1-(0.1)^{20}\right]$
8. $\frac{\sqrt{7}}{2}\left(\sqrt{3}+1\right)\left(3^{\frac{n}{2}}-1\right)$
9. $\frac{\left[1-(-a)^n\right]}{1+a}$
10. $\frac{x^3\left(1-x^{2n}\right)}{1-x^2}$
11. $22+\frac{3}{2}\left(3^{11}-1\right)$
12. $r = \frac{5}{2}$ or $\frac{2}{5}$; Terms are $\frac{2}{5}, 1, \frac{5}{2}$ or $\frac{5}{2}, 1, \frac{2}{5}$
13. 4
14. $\frac{16}{7}; 2; \frac{16}{7}\left(2^n-1\right)$
15. 2059 or 463

16. $\frac{-4}{3},\frac{-8}{3},\frac{-16}{3},\ldots$ or $4,-8,16,-32,64,\ldots$ **18.** $\frac{80}{81}\left(10^n-1\right)-\frac{8}{9}n$

19. 496 **20.** rR **21.** 3, –6, 12, –24 **26.** 9 and 27

27. $n=\frac{-1}{2}$ **30.** 120, 480, 30 (2^n) **31.** Rs 500 $(1.1)^{10}$

32. $x^2-16x+25=0$

Miscellaneous Exercise on Chapter 8

1. 4 **2.** 160; 6 **3.** ± 3 **4.** 8, 16, 32

5. 4 **11.** (i) $\frac{50}{81}\left(10^n-1\right)-\frac{5n}{9}$, (ii) $\frac{2n}{3}-\frac{2}{27}\left(1-10^{-n}\right)$

12. 1680

13. Rs 16680 **14.** Rs 39100 **15.** Rs 43690 **16.** Rs 17000; 20,000

17. Rs 5120 **18.** 25 days

EXERCISE 9.1

1. $\frac{121}{2}$ square unit.

2. (0, a), (0, – a) and $\left(-\sqrt{3}a,0\right)$ or (0, a), (0, – a), and $\left(\sqrt{3}a,0\right)$

3. (i) $\left|y_2-y_1\right|$, (ii) $\left|x_2-x_1\right|$ **4.** $\left(\frac{15}{2},0\right)$ **5.** $-\frac{1}{2}$

7. $-\sqrt{3}$ **9.** 135°

10. 1 and 2, or $\frac{1}{2}$ and 1, or – 1 and –2, or $-\frac{1}{2}$ and – 1

EXERCISE 9.2

1. $y=0$ and $x=0$ **2.** $x-2y+10=0$ **3.** $y=mx$

4. $\left(\sqrt{3}+1\right)x-\left(\sqrt{3}-1\right)y=4\left(\sqrt{3}-1\right)$ **5.** $2x+y+6=0$

6. $x-\sqrt{3}y+2\sqrt{3}=0$ **7.** $5x+3y+2=0$

8. $3x - 4y + 8 = 0$

9. $5x - y + 20 = 0$

10. $(1 + n)x + 3(1 + n)y = n + 11$

11. $x + y = 5$

12. $x + 2y - 6 = 0$, $2x + y - 6 = 0$

13. $\sqrt{3}x + y - 2 = 0$ and $\sqrt{3}x + y + 2 = 0$

14. $2x - 9y + 85 = 0$

15. $L = \frac{.192}{90}(C - 20) + 124.942$

16. 1340 litres.

18. $2kx + hy = 3kh$.

EXERCISE 9.3

1. (i) $y = -\frac{1}{7}x + 0, -\frac{1}{7}, 0$; (ii) $y = -2x + \frac{5}{3}, -2, \frac{5}{3}$; (iii) $y = 0x + 0, 0, 0$

2. (i) $\frac{x}{4} + \frac{y}{6} = 1, 4, 6;$ (ii) $\frac{x}{\frac{3}{2}} + \frac{y}{-2} = 1, \frac{3}{2}, -2;$

(iii) $y = -\frac{2}{3}$, intercept with y-axis $= -\frac{2}{3}$ and no intercept with x-axis.

3. 5 units

4. $(-2, 0)$ and $(8, 0)$

5. (i) $\frac{65}{17}$ units, (ii) $\frac{1}{\sqrt{2}}\left|\frac{p+r}{l}\right|$ units.

6. $3x - 4y + 18 = 0$

7. $y + 7x = 21$

8. 30° and 150°

9. $\frac{22}{9}$

11. $(\sqrt{3} + 2)x + (2\sqrt{3} - 1)y = 8\sqrt{3} + 1$ or $(\sqrt{3} - 2)x + (1 + 2\sqrt{3})y = -1 + 8\sqrt{3}$

12. $2x + y = 5$

13. $\left(\frac{68}{25}, -\frac{49}{25}\right)$

14. $m = \frac{1}{2}, c = \frac{5}{2}$

16. $y - x = 1, \sqrt{2}$

Miscellaneous Exercise on Chapter 9

1. (a) 3, (b) ± 2, (c) 6 or 1

2. $2x - 3y = 6, -3x + 2y = 6$

3. $\left(0, -\frac{8}{3}\right), \left(0, \frac{32}{3}\right)$

4. $\left|\cos\dfrac{\phi-\theta}{2}\right|$ **5.** $x=-\dfrac{5}{22}$ **6.** $2x-3y+18=0$

7. k^2 square units **8.** 5 **10.** $3x-y=7,\quad x+3y=9$

11. $13x+13y=6$ **13.** 1 : 2 **14.** $\dfrac{23\sqrt{5}}{18}$ units

15. The line is parallel to x - axis or parallel to y-axis

16. $x=1,\quad y=1.$ *or* $x=-4,\quad y=3$ **17.** $(-1,-4)$.

18. $\dfrac{1\pm5\sqrt{2}}{7}$ **20.** $18x+12y+11=0$

21. $\left(\dfrac{13}{5},0\right)$ **23.** $119x+102y=125$

EXERCISE 10.1

1. $x^2+y^2-4y=0$ **2.** $x^2+y^2+4x-6y-3=0$

3. $36x^2+36y^2-36x-18y+11=0$ **4.** $x^2+y^2-2x-2y=0$

5. $x^2+y^2+2ax+2by+2b^2=0$ **6.** $c(-5, 3), r=6$

7. $c(2, 4), r=\sqrt{65}$ **8.** $c(4,-5), r=\sqrt{53}$ **9.** $c\left(\dfrac{1}{4}, 0\right); r=\dfrac{1}{4}$

10. $x^2+y^2-6x-8y+15=0$ **11.** $x^2+y^2-7x+5y-14=0$

12. $x^2+y^2+4x-21=0$ & $x^2+y^2-12x+11=0$

13. $x^2+y^2-ax-by=0$ **14.** $x^2+y^2-4x-4y=5$

15. Inside the circle; since the distance of the point to the centre of the circle is less than the radius of the circle.

EXERCISE 10.2

1. F (3, 0), axis - x - axis, directrix $x=-3$, length of the Latus rectum = 12

2. F $\left(0, \dfrac{3}{2}\right)$, axis - y - axis, directrix $y=-\dfrac{3}{2}$, length of the Latus rectum = 6

3. F (–2, 0), axis - x - axis, directrix $x=2$, length of the Latus rectum = 8

4. F (0, –4), axis - y - axis, directrix $y = 4$, length of the Latus rectum = 16

5. F $(\frac{5}{2}, 0)$ axis - x - axis, directrix $x = -\frac{5}{2}$, length of the Latus rectum = 10

6. F $(0, \frac{-9}{4})$, axis - y - axis, directrix $y = \frac{9}{4}$, length of the Latus rectum = 9

7. $y^2 = 24x$
8. $x^2 = -12y$
9. $y^2 = 12x$
10. $y^2 = -8x$
11. $2y^2 = 9x$
12. $2x^2 = 25y$

EXERCISE 10.3

1. F $(\pm\sqrt{20}, 0)$; V $(\pm 6, 0)$; Major axis = 12; Minor axis = 8, $e = \frac{\sqrt{20}}{6}$,

   Latus rectum = $\frac{16}{3}$

2. F $(0, \pm\sqrt{21})$; V $(0, \pm 5)$; Major axis = 10; Minor axis = 4, $e = \frac{\sqrt{21}}{5}$;

   Latus rectum = $\frac{8}{5}$

3. F $(\pm\sqrt{7}, 0)$; V $(\pm 4, 0)$; Major axis = 8; Minor axis = 6, $e = \frac{\sqrt{7}}{4}$;

   Latus rectum = $\frac{9}{2}$

4. F $(0, \pm\sqrt{75})$; V $(0, \pm 10)$; Major axis = 20; Minor axis = 10, $e = \frac{\sqrt{3}}{2}$;

   Latus rectum = 5

5. F $(\pm\sqrt{13}, 0)$; V $(\pm 7, 0)$; Major axis = 14; Minor axis = 12, $e = \frac{\sqrt{13}}{7}$;

   Latus rectum = $\frac{72}{7}$

6. F $(0, \pm 10\sqrt{3})$; V $(0, \pm 20)$; Major axis = 40; Minor axis = 20, $e = \frac{\sqrt{3}}{2}$;

   Latus rectum = 10

7. F $(0, \pm 4\sqrt{2})$; V $(0, \pm 6)$; Major axis $=12$; Minor axis $= 4$, $e = \dfrac{2\sqrt{2}}{3}$;

Latus rectum $= \dfrac{4}{3}$

8. F$\left(0, \pm\sqrt{15}\right)$; V $(0, \pm 4)$; Major axis $= 8$; Minor axis $= 2$, $e = \dfrac{\sqrt{15}}{4}$;

Latus rectum $= \dfrac{1}{2}$

9. F $(\pm\sqrt{5}, 0)$; V $(\pm 3, 0)$; Major axis $= 6$; Minor axis $= 4$, $e = \dfrac{\sqrt{5}}{3}$;

Latus rectum $= \dfrac{8}{3}$

10. $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

11. $\dfrac{x^2}{144} + \dfrac{y^2}{169} = 1$

12. $\dfrac{x^2}{36} + \dfrac{y^2}{20} = 1$

13. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

14. $\dfrac{x^2}{1} + \dfrac{y^2}{5} = 1$

15. $\dfrac{x^2}{169} + \dfrac{y^2}{144} = 1$

16. $\dfrac{x^2}{64} + \dfrac{y^2}{100} = 1$

17. $\dfrac{x^2}{16} + \dfrac{y^2}{7} = 1$

18. $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

19. $\dfrac{x^2}{10} + \dfrac{y^2}{40} = 1$

20. $x^2 + 4y^2 = 52$ or $\dfrac{x^2}{52} + \dfrac{y^2}{13} = 1$

EXERCISE 10.4

1. Foci $(\pm 5, 0)$, Vertices $(\pm 4, 0)$; $e = \dfrac{5}{4}$; Latus rectum $= \dfrac{9}{2}$

2. Foci (0 ± 6), Vertices $(0, \pm 3)$; $e = 2$; Latus rectum $= 18$

3. Foci $(0, \pm\sqrt{13})$, Vertices $(0, \pm 2)$; $e = \dfrac{\sqrt{13}}{2}$; Latus rectum $= 9$

4. Foci $(\pm 10, 0)$, Vertices $(\pm 6, 0)$; $e = \dfrac{5}{3}$; Latus rectum $= \dfrac{64}{3}$

5. Foci $\left(0, \pm\frac{2\sqrt{14}}{\sqrt{5}}\right)$, Vertices $\left(0, \pm\frac{6}{\sqrt{5}}\right)$; $e = \frac{\sqrt{14}}{3}$; Latus rectum $= \frac{4\sqrt{5}}{3}$

6. Foci $(0, \pm\sqrt{65})$, Vertices $(0, \pm 4)$; $e = \frac{\sqrt{65}}{4}$; Latus rectum $= \frac{49}{2}$

7. $\frac{x^2}{4} - \frac{y^2}{5} = 1$ **8.** $\frac{y^2}{25} - \frac{x^2}{39} = 1$ **9.** $\frac{y^2}{9} - \frac{x^2}{16} = 1$

10. $\frac{x^2}{16} - \frac{y^2}{9} = 1$ **11.** $\frac{y^2}{25} - \frac{x^2}{144} = 1$ **12.** $\frac{x^2}{25} - \frac{y^2}{20} = 1$

13. $\frac{x^2}{4} - \frac{y^2}{12} = 1$ **14.** $\frac{x^2}{49} - \frac{9y^2}{343} = 1$ **15.** $\frac{y^2}{5} - \frac{x^2}{5} = 1$

Miscellaneous Exercise on Chapter 10

1. Focus is at the mid-point of the given diameter.

2. 2.23 m (approx.) **3.** 9.11 m (approx.) **4.** 1.56m (approx.)

5. $\frac{x^2}{81} + \frac{y^2}{9} = 1$ **6.** 18 sq units **7.** $\frac{x^2}{25} + \frac{y^2}{9} = 1$

8. $8\sqrt{3}a$

EXERCISE 11.1

1. y and z - coordinates are zero **2.** y - coordinate is zero

3. I, IV, VIII, V, VI, II, III, VII

4. (i) XY - plane (ii) $(x, y, 0)$ (iii) Eight

EXERCISE 11.2

1. (i) $2\sqrt{5}$ (ii) $\sqrt{43}$ (iii) $2\sqrt{26}$ (iv) $2\sqrt{5}$

4. $x - 2z = 0$ **5.** $9x^2 + 25y^2 + 25z^2 - 225 = 0$

Miscellaneous Exercise on Chapter 11

1. $(1, -2, 8)$ **2.** $7, \sqrt{34}, 7$ **3.** $a = -2,\ b = -\frac{16}{3},\ c = 2$

4. $x^2 + y^2 + z^2 - 2x - 7y + 2z = \frac{k^2 - 109}{2}$

EXERCISE 12.1

1. 6
2. $\left(\pi - \frac{22}{7}\right)$
3. π
4. $\frac{19}{2}$
5. $-\frac{1}{2}$
6. 5
7. $\frac{11}{4}$
8. $\frac{108}{7}$
9. b
10. 2
11. 1
12. $-\frac{1}{4}$
13. $\frac{a}{b}$
14. $\frac{a}{b}$
15. $\frac{1}{\pi}$
16. $\frac{1}{\pi}$
17. 4
18. $\frac{a+1}{b}$
19. 0
20. 1
21. 0
22. 2
23. 3, 6
24. Limit does not exist at $x = 1$
25. Limit does not exist at $x = 0$
26. Limit does not exist at $x = 0$
27. 0
28. $a = 0, b = 4$
29. $\lim_{x \to a_1} f(x) = 0$ and $\lim_{x \to a} f(x) = (a - a_1)(a - a_2) \ldots (a - a_x)$
30. $\lim_{x \to a} f(x)$ exists for all $a \neq 0$.
31. 2
32. For $\lim_{x \to 0} f(x)$ to exists, we need $m = n$; $\lim_{x \to 1} f(x)$ exists for any integral value of m and n.

EXERCISE 12.2

1. 20
2. 1
3. 99
4. (i) $3x^2$ (ii) $2x - 3$ (iii) $\frac{-2}{x^3}$ (iv) $\frac{-2}{(x-1)^2}$
6. $nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \ldots + a^{n-1}$
7. (i) $2x - a - b$ (ii) $4ax\left(ax^2 + b\right)$ (iii) $\frac{a-b}{(x-b)^2}$

8. $\dfrac{nx^n - anx^{n-1} - x^n + a^n}{(x-a)^2}$

9. (i) 2 (ii) $20x^3 - 15x^2 + 6x - 4$ (iii) $\dfrac{-3}{x^4}(5+2x)$ (iv) $15x^4 + \dfrac{24}{x^5}$

(v) $\dfrac{-12}{x^5} + \dfrac{36}{x^{10}}$ (vi) $\dfrac{-2}{(x+1)^2} - \dfrac{x(3x-2)}{(3x-1)^2}$ **10.** $-\sin x$

11. (i) $\cos 2x$ (ii) $\sec x \tan x$
(iii) $5\sec x \tan x - 4\sin x$ (iv) $-\operatorname{cosec} x \cot x$
(v) $-3\operatorname{cosec}^2 x - 5\operatorname{cosec} x \cot x$ (vi) $5\cos x + 6\sin x$
(vii) $2\sec^2 x - 7\sec x \tan x$

Miscellaneous Exercise on Chapter 12

1. (i) -1 (ii) $\dfrac{1}{x^2}$ (iii) $\cos(x+1)$ (iv) $-\sin\left(x - \dfrac{\pi}{8}\right)$ **2.** 1

3. $\dfrac{-qr}{x^2} + ps$ **4.** $2c(ax+b)(cx+d) + a(cx+d)^2$

5. $\dfrac{ad-bc}{(cx+d)^2}$ **6.** $\dfrac{-2}{(x-1)^2},\ x \neq 0,1$ **7.** $\dfrac{-(2ax+b)}{(ax^2+bx+c)^2}$

8. $\dfrac{-apx^2 - 2bpx + ar - bq}{(px^2+qx+r)^2}$ **9.** $\dfrac{apx^2 + 2bpx + bq - ar}{(ax+b)^2}$ **10.** $\dfrac{-4a}{x^5} + \dfrac{2b}{x^3} - \sin x$

11. $\dfrac{2}{\sqrt{x}}$ **12.** $na(ax+b)^{n-1}$

13. $(ax+b)^{n-1}(cx+d)^{m-1}\left[mc(ax+b) + na(cx+d)\right]$ **14.** $\cos(x+a)$

15. $-\operatorname{cosec}^3 x - \operatorname{cosec} x \cot^2 x$ **16.** $\dfrac{-1}{1+\sin x}$

17. $\dfrac{-2}{(\sin x - \cos x)^2}$ **18.** $\dfrac{2\sec x \tan x}{(\sec x + 1)^2}$ **19.** $n\sin^{n-1}x\cos x$

20. $\dfrac{bc\cos x + ad\sin x + bd}{(c + d\cos x)^2}$ **21.** $\dfrac{\cos a}{\cos^2 x}$

22. $x^3(5x\cos x + 3x\sin x + 20\sin x - 12\cos x)$

23. $-x^2\sin x - \sin x + 2x\cos x$

24. $-q\, sin\, x(ax^2 + \sin x) + (p + q\cos x)(2a\,x + \cos x)$

25. $-\tan^2 x(x + \cos x) + (x - \tan x)(1 - \sin x)$

26. $\dfrac{35 + 15x\cos x + 28\cos x + 28x\sin x - 15\sin x}{(3x + 7\cos x)^2}$

27. $\dfrac{x\cos\frac{\pi}{4}(2\sin x - x\cos x)}{\sin^2 x}$ **28.** $\dfrac{1 + \tan x - x\sec^2 x}{(1 + \tan x)^2}$

29. $(x + \sec x)(1 - \sec^2 x) + (x - \tan x).(1 + \sec x\tan x)$

30. $\dfrac{\sin x - n\,x\cos x}{\sin^{n+1} x}$

EXERCISE 13.1

1. 3 **2.** 8.4 **3.** 2.33 **4.** 7

5. 6.32 **6.** 16 **7.** 3.23 **8.** 5.1

9. 157.92 **10.** 11.28 **11.** 10.34 **12.** 7.35

EXERCISE 13.2

1. 9, 9.25 **2.** $\dfrac{n+1}{2}, \dfrac{n^2-1}{12}$ **3.** 16.5, 74.25 **4.** 19, 43.4

5. 100, 29.09 **6.** 64, 1.69 **7.** 107, 2276 **8.** 27, 132

9. 93, 105.58, 10.27 **10.** 5.55, 43.5

Miscellaneous Exercise on Chapter 13

1. 4, 8 **2.** 6, 8 **3.** 24, 12

5. (i) 10.1, 1.99 (ii) 10.2, 1.98

6. 20, 3.036

EXERCISE 14.1

1. No.

2. (i) $\{1, 2, 3, 4, 5, 6\}$ (ii) ϕ (iii) $\{3, 6\}$ (iv) $\{1, 2, 3\}$ (v) $\{6\}$

(vi) $\{3, 4, 5, 6\}$, $A \cup B = \{1, 2, 3, 4, 5, 6\}$, $A \cap B = \phi$, $B \cup C = \{3, 6\}$, $E \cap F = \{6\}$, $D \cap E = \phi$,

$A - C = \{1, 2,4,5\}$, $D - E = \{1,2,3\}$, $E \cap F' = \phi$, $F' = \{1, 2\}$

3. $A = \{(3,6), (4,5), (5, 4), (6,3), (4,6), (5,5), (6,4), (5,6), (6,5), (6,6)\}$

$B = \{(1,2), (2,2), (3, 2), (4,2), (5,2), (6,2), (2,1), (2,3), (2,4), (2,5), (2,6)\}$

$C = \{(3,6), (6,3), (5, 4), (4,5), (6,6)\}$

A and B, B and C are mutually exclusive.

4. (i) A and B; A and C; B and C; C and D (ii) A and C (iii) B and D

5. (i) "Getting at least two heads", and "getting at least two tails"

(ii) "Getting no heads", "getting exactly one head" and "getting at least two heads"

(iii) "Getting at most two tails", and "getting exactly two tails"

(iv) "Getting exactly one head" and "getting exactly two heads"

(v) "Getting exactly one tail", "getting exactly two tails", and getting exactly three tails"

Note There may be other events also as answer to the above question.

6. $A = \{(2, 1), (2,2), (2,3), (2,4), (2,5), (2,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

$B = \{(1, 1), (1,2), (1,3), (1,4), (1,5), (1,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6)\}$

$C = \{(1, 1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (3,1), (3,2), (4,1)\}$

(i) $A' = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6)\} = B$

(ii) $B' = \{(2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\} = A$

(iii) $A \cup B = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (2,1), (2,2), (2,3), (2,5), (2,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\} = S$

(iv) $A \cap B = \phi$

(v) $A - C = \{(2,4), (2,5), (2,6), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

(vi) $B \cup C = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6)\}$

(vii) $B \cap C = \{(1,1), (1,2), (1,3), (1,4), (3,1), (3,2)\}$

(viii) $A \cap B' \cap C' = \{(2,4), (2,5), (2,6), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

7. (i) True (ii) True (iii) True (iv) False (v) False (vi) False

EXERCISE 14.2

1. (a) Yes (b) Yes (c) No (d) No (e) No **2.** $\frac{3}{4}$

3. (i) $\frac{1}{2}$ (ii) $\frac{2}{3}$ (iii) $\frac{1}{6}$ (iv) 0 (v) $\frac{5}{6}$ **4.** (a) 52 (b) $\frac{1}{52}$ (c) (i) $\frac{1}{13}$ (ii) $\frac{1}{2}$

5. (i) $\frac{1}{12}$ (ii) $\frac{1}{12}$ **6.** $\frac{3}{5}$

7. Rs 4.00 gain, Rs 1.50 gain, Re 1.00 loss, Rs 3.50 loss, Rs 6.00 loss.

P (Winning Rs 4.00) $= \frac{1}{16}$, P(Winning Rs 1.50) $= \frac{1}{4}$, P (Losing Re. 1.00) $= \frac{3}{8}$

P (Losing Rs 3.50) $= \frac{1}{4}$, P (Losing Rs 6.00) $= \frac{1}{16}$.

8. (i) $\frac{1}{8}$ (ii) $\frac{3}{8}$ (iii) $\frac{1}{2}$ (iv) $\frac{7}{8}$ (v) $\frac{1}{8}$ (vi) $\frac{1}{8}$ (vii) $\frac{3}{8}$ (viii) $\frac{1}{8}$ (ix) $\frac{7}{8}$

9. $\frac{9}{11}$ **10.** (i) $\frac{6}{13}$ (ii) $\frac{7}{13}$ **11.** $\frac{1}{38760}$

12. (i) No, because $P(A \cap B)$ must be less than or equal to P(A) and P(B), (ii) Yes

13. (i) $\frac{7}{15}$ (ii) 0.5 (iii) 0.15 **14.** $\frac{4}{5}$

15. (i) $\frac{5}{8}$ (ii) $\frac{3}{8}$ **16.** No **17.** (i) 0.58 (ii) 0.52 (iii) 0.74

18. 0.6 **19.** 0.55 **20.** 0.65

21. (i) $\frac{19}{30}$ (ii) $\frac{11}{30}$ (iii) $\frac{2}{15}$

Miscellaneous Exercise on Chapter 14

1. (i) $\frac{{}^{20}C_5}{{}^{60}C_5}$ (ii) $1-\frac{{}^{30}C_5}{{}^{60}C_5}$ **2.** $\frac{{}^{13}C_3 \cdot {}^{13}C_1}{{}^{52}C_4}$

3. (i) $\frac{1}{2}$ (ii) $\frac{1}{2}$ (iii) $\frac{5}{6}$ **4.** (a) $\frac{999}{1000}$ (b) $\frac{{}^{9990}C_2}{{}^{10000}C_2}$ (c) $\frac{{}^{9990}C_{10}}{{}^{10000}C_{10}}$

5. (a) $\frac{17}{33}$ (b) $\frac{16}{33}$ **6.** $\frac{2}{3}$

7. (i) 0.88 (ii) 0.12 (iii) 0.19 (iv) 0.34 **8.** $\frac{4}{5}$

9. (i) $\frac{33}{83}$ (ii) $\frac{3}{8}$ **10.** $\frac{1}{5040}$

MATHEMATICS

Textbook for Class XI

11076

विद्ययाऽमृतमश्नुते

एन सी ई आर टी
NCERT

राष्ट्रीय शैक्षिक अनुसंधान और प्रशिक्षण परिषद्
NATIONAL COUNCIL OF EDUCATIONAL RESEARCH AND TRAINING

11076 – MATHEMATICS
Textbook for Class XI

ISBN 81-7450-486-9

First Edition
February 2006 Phalguna 1927

Reprinted
October 2006, November 2007,
December 2008, December 2009,
January 2011, February 2012,
December 2012, November 2013,
December 2014, May 2016,
December 2016, December 2017,
January 2019, August 2019,
January 2021 and November 2021

Revised Edition
November 2022 Agrahayan 1944

Reprinted
March 2024 Chaitra 1946

PD 375T SU

₹ 230.00

Printed on 80 GSM paper with NCERT watermark

Published at the Publication Division by the Secretary, National Council of Educational Research and Training, Sri Aurobindo Marg, New Delhi 110 016 and printed at Amit Printing Press, D-12 and 13, Industrial Area, Site-A, Mathura- 281 001 (U.P.)

OFFICES OF THE PUBLICATION DIVISION, NCERT

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CWC Complex
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Guwahati 781 021 **Phone : 0361-2674869**

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Cover and Layout
Arvinder Chawla

Foreword

The National Curriculum Framework (NCF), 2005, recommends that children's life at school must be linked to their life outside the school. This principle marks a departure from the legacy of bookish learning which continues to shape our system and causes a gap between the school, home and community. The syllabi and textbooks developed on the basis of NCF signify an attempt to implement this basic idea. They also attempt to discourage rote learning and the maintenance of sharp boundaries between different subject areas. We hope these measures will take us significantly further in the direction of a child-centred system of education outlined in the National Policy on Education (1986).

The success of this effort depends on the steps that school principals and teachers will take to encourage children to reflect on their own learning and to pursue imaginative activities and questions. We must recognise that given space, time and freedom, children generate new knowledge by engaging with the information passed on to them by adults. Treating the prescribed textbook as the sole basis of examination is one of the key reasons why other resources and sites of learning are ignored. Inculcating creativity and initiative is possible if we perceive and treat children as participants in learning, not as receivers of a fixed body of knowledge.

These aims imply considerable change in school routines and mode of functioning. Flexibility in the daily time-table is as necessary as rigour in implementing the annual calendar so that the required number of teaching days are actually devoted to teaching. The methods used for teaching and evaluation will also determine how effective this textbook proves for making children's life at school a happy experience, rather than a source of stress or boredom. Syllabus designers have tried to address the problem of curricular burden by restructuring and reorienting knowledge at different stages with greater consideration for child psychology and the time available for teaching. The textbook attempts to enhance this endeavour by giving higher priority and space to opportunities for contemplation and wondering, discussion in small groups, and activities requiring hands-on experience.

The National Council of Educational Research and Training (NCERT) appreciates the hard work done by the Textbook Development Committee responsible for this book. We wish to thank the Chairperson of the advisory group in Science and Mathematics, Professor J.V. Narlikar and the Chief Advisor for this book Professor P.K. Jain for guiding the work of this committee. Several teachers contributed to the development of this textbook; we are grateful to their principals for making this possible. We are indebted to the institutions and organisations which have generously permitted us to draw upon their resources, material and personnel. We are especially grateful to the members of the National Monitoring Committee, appointed by the Department of Secondary and Higher Education, Ministry of Human Resource Development under the Chairpersonship of Professor Mrinal Miri and Professor G.P. Deshpande, for their valuable time and contribution. As an organisation committed to the systemic reform and continuous improvement in the quality of its products, NCERT welcomes comments and suggestions which will enable us to undertake further revision and refinement.

Director
National Council of Educational
Research and Training

New Delhi
20 December 2005

Rationalisation of Content in the Textbooks

In view of the COVID-19 pandemic, it is imperative to reduce content load on students. The National Education Policy 2020, also emphasises reducing the content load and providing opportunities for experiential learning with creative mindset. In this background, the NCERT has undertaken the exercise to rationalise the textbooks across all classes. Learning Outcomes already developed by the NCERT across classes have been taken into consideration in this exercise.

Contents of the textbooks have been rationalised in view of the following:

- Overlapping with similar content included in other subject areas in the same class
- Similar content included in the lower or higher class in the same subject
- Difficulty level
- Content, which is easily accessible to students without much interventions from teachers and can be learned by children through self-learning or peer-learning
- Content, which is irrelevant in the present context

This present edition, is a reformatted version after carrying out the changes given above.

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Acknowledgements

The Council gratefully acknowledges the valuable contributions of the following participants of the Textbook Review Workshop: P. Bhaskar Kumar, *P.G.T.,* Jawahar Navodaya Vidyalaya, Ananthpur, (A.P.); Vinayak Bujade, *Lecturer,* Vidarbha Buniyadi Junior College, Sakkardara Chowk Nagpur, Maharashtra; Vandita Kalra, *Lecturer,* Sarvodaya Kanya Vidyalaya Vikashpuri District Centre, New Delhi; P.L. Sachdeva Deptt. of Mathematics, Indian Institute of Science, Bangalore, Karnataka; P.K.Tiwari *Assistant Commissioner (Retd.),* Kendriya Vidyalaya Sangathan; Jagdish Saran, Department of Statistics, University of Delhi; Quddus Khan, *Lecturer,* Shibli National P.G. College Azamgarh (U.P.); Sumat Kumar Jain, *Lecturer,* K.L. Jain Inter College Sasni Hathras (U.P.); R.P. Gihare, *Lecturer* (BRC), Janpad Shiksha Kendra Chicholi Distt. Betul (M.P.); Sangeeta Arora, *P.G.T.,* A.P.J. School Saket, New Delhi; P.N. Malhotra, *ADE* (Sc.), Directorate of Education, Delhi; D.R. Sharma, *P.G.T.,* J.N.V. Mungespur, Delhi; Saroj, *P.G.T.* Government Girls Sr. Secondary School, No. 1, Roop Nagar, Delhi, Manoj Kumar Thakur, *P.G.T.*, D.A.V. Public School, Rajender Nagar, Sahibabad, Ghaziabad (U.P.) and R.P. Maurya, *Reader,* DESM, NCERT, New Delhi.

Acknowledgements are due to Professor M. Chandra, *Head,* Department of Education in Science and Mathematics for her support.

The Council acknowledges the efforts of the Computer Incharge, Deepak Kapoor; Rakesh Kumar, Kamlesh Rao and Sajjad Haider Ansari, D.T.P. Operators; Kushal Pal Singh Yadav, Copy Editor and Proof Readers, Mukhtar Hussain and Kanwar Singh.

The contribution of APC–Office, administration of DESM and Publication Department is also duly acknowledged.

Contents

SUPPLEMENTARY MATERIAL

CHAPTER 8

8.6 Infinite G.P. and its Sum

G.P. of the form a, ar, ar^2, ar^3, ... is called infinite G.P. Now, to find the formulae for finding sum to infinity of a G.P., we begin with an example.

Let us consider the G.P.,

$$1, \frac{2}{3}, \frac{4}{9}, \ldots$$

Here $a = 1$, $r = \frac{2}{3}$. We have

$$S_n = \frac{1-\left(\frac{2}{3}\right)^n}{1-\frac{2}{3}} = 3\left[1-\left(\frac{2}{3}\right)^n\right]$$

Let us study the behaviour of $\left(\frac{2}{3}\right)^n$ as n becomes larger and larger:

n	*1*	*5*	*10*	*20*
$\left(\frac{2}{3}\right)^n$	0.6667	0.1316872428	0.01734152992	0.00030072866

We observe that as n becomes larger and larger, $\left(\frac{2}{3}\right)^n$ becomes closer and closer to zero. Mathematically, we say that as n becomes sufficiently large, $\left(\frac{2}{3}\right)^n$ becomes sufficiently small. In other words as $n \to \infty, \left(\frac{2}{3}\right)^n \to 0$. Consequently, we find that the sum of infinitely many terms is given by $S_\infty = 3$.

Now, for a geometric progression, a, ar, ar^2, ..., if numerical value of common ratio r is less than 1, then

$$S_n = \frac{a(1-r^n)}{(1-r)} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

In this case as $n \to \infty$, $r^n \to 0$ since $|r| < 1$. Therefore

$$S_n \to \frac{a}{1-r}$$

Symbolically sum to infinity is denoted by S_∞ or S.

Thus, we have $S = \frac{a}{1-r}$.

For examples,

(i) $$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots = \frac{1}{1-\frac{1}{2}} = 2.$$

(ii) $$1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \ldots = \frac{1}{1-\left(\frac{-1}{2}\right)} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$$

Exercise 8.3

Find the sum to infinity in each of the following Geometric Progression.

1. $1, \frac{1}{3}, \frac{1}{9}, \ldots$ (Ans. 1.5)
2. 6, 1.2, .24, ... (Ans. 7.5)
3. $5, \frac{20}{7}, \frac{80}{49}, \ldots$ (Ans. $\frac{35}{3}$)
4. $\frac{-3}{4}, \frac{3}{16}, \frac{-3}{64}, \ldots$ (Ans. $\frac{-3}{5}$)
5. Prove that $3^{\frac{1}{2}} \times 3^{\frac{1}{4}} \times 3^{\frac{1}{8}} \ldots = 3$
6. Let $x = 1 + a + a^2 + \ldots$ and $y = 1 + b + b^2 + \ldots$, where $|a| < 1$ and $|b| < 1$. Prove that

$$1 + ab + a^2b^2 + \ldots = \frac{xy}{x+y-1}$$

CHAPTER 12

12.6 Limits Involving Exponential and Logarithmic Functions

Before discussing evaluation of limits of the expressions involving exponential and logarithmic functions, we introduce these two functions stating their domain, range and also sketch their graphs roughly.

Leonhard Euler (1707–1783), the great Swiss mathematician introduced the number e whose value lies between 2 and 3. This number is useful in defining exponential function and is defined as $f(x) = e^x$, $x \in \mathbf{R}$. Its domain is **R**, range is the set of positive real numbers. The graph of exponential function, i.e., $y = e^x$ is as given in Fig.13.11.

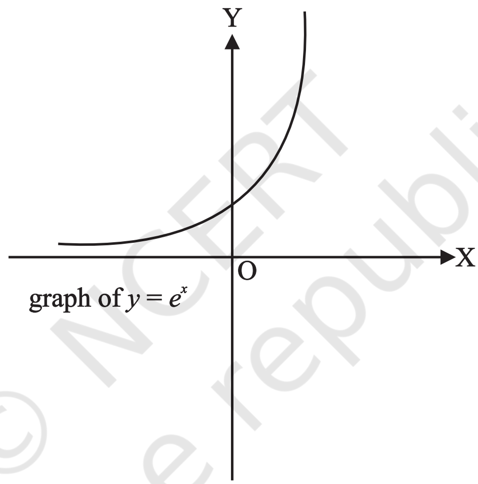


Fig. 13.11

Similarly, the logarithmic function expressed as $\log_e \mathbf{R}^+ \to \mathbf{R}$ is given by $\log_e x = y$, if and only if $e^y = x$. Its domain is $\mathbf{R}^+$ which is the set of all positive real numbers and range is **R**. The graph of logarithmic function $y = \log_e x$ is shown in Fig.13.12.

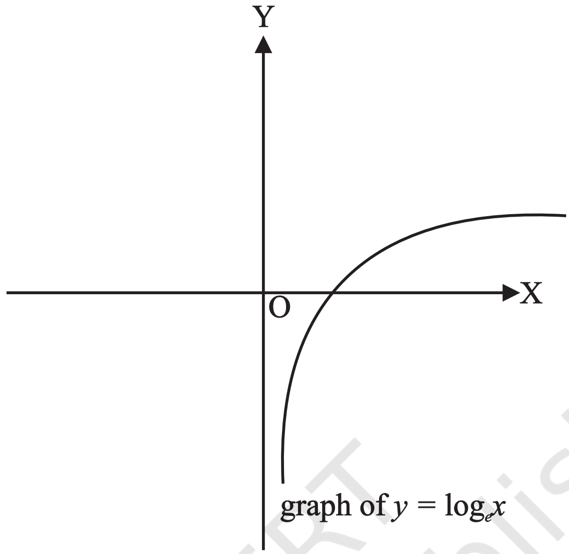


Fig. 13.12

In order to prove the result $\lim\limits_{x \to 0} \frac{e^x - 1}{x} = 1$, we make use of an inequality involving the expression $\frac{e^x - 1}{x}$ which runs as follows:

$$\frac{1}{1+|x|} \leq \frac{e^x - 1}{x} \leq 1 + (e-2)\,|x| \text{ holds for all } x \text{ in } [-1, 1] \sim \{0\}.$$

Theorem 6 Prove that $\lim\limits_{x \to 0} \frac{e^x - 1}{x} = 1$

Proof Using above inequality, we get

$$\frac{1}{1+|x|} \leq \frac{e^x - 1}{x} \leq 1 + |x|\,(e-2),\ x \text{ Î } [-1, 1] \sim \{0\}$$

Also $\lim\limits_{x \to 0} \frac{1}{1+|x|} = \frac{1}{1 + \lim\limits_{x \to 0} |x|} = \frac{1}{1+0} = 1$

and $\lim_{x \to 0} \left[1 + (e-2)|x|\right] = 1 + (e-2)\lim_{x \to 0} |x| = 1 + (e-2)0 = 1$

Therefore, by Sandwich theorem, we get

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

Theorem 7 Prove that $\lim_{x \to 0} \frac{\log_e(1+x)}{x} = 1$

Proof Let $\frac{\log_e(1+x)}{x} = y$. Then

$$\log_e(1+x) = xy$$

$$\Rightarrow 1 + x = e^{xy}$$

$$\Rightarrow \frac{e^{xy} - 1}{x} = 1$$

or

$$\frac{e^{xy} - 1}{xy} . y = 1$$

$$\Rightarrow \lim_{xy \to 0} \frac{e^{xy} - 1}{xy} \lim_{x \to 0} y = 1 \text{ (since } x \to 0 \text{ gives } xy \to 0)$$

$$\Rightarrow \lim_{x \to 0} y = 1 \left(\text{as } \lim_{xy \to 0} \frac{e^{xy} - 1}{xy} = 1 \right)$$

$$\Rightarrow \lim_{x \to 0} \frac{\log_e(1+x)}{x} = 1$$

Example 5 Compute $\lim_{x \to 0} \frac{e^{3x} - 1}{x}$

Solution We have

$$\lim_{x \to 0} \frac{e^{3x} - 1}{x} = \lim_{3x \to 0} \frac{e^{3x} - 1}{3x} \cdot 3$$

$$= 3\left(\lim_{y \to 0} \frac{e^{y} - 1}{y} \right), \quad \text{where } y = 3x$$

$$= 3.1 = 3$$

Example 6 Compute $\lim_{x \to 0} \frac{e^x - \sin x - 1}{x}$

Solution We have $\lim_{x \to 0} \frac{e^x - \sin x - 1}{x} = \lim_{x \to 0} \left[\frac{e^x - 1}{x} - \frac{\sin x}{x} \right]$

$$= \lim_{x \to 0} \frac{e^x - 1}{x} - \lim_{x \to 0} \frac{\sin x}{x} = 1 - 1 = 0$$

Example 7 Evaluate $\lim_{x \to 1} \frac{\log_e x}{x - 1}$

Solution Put $x = 1 + h$, then as $x \to 1 \Rightarrow h \to 0$. Therefore,

$$\lim_{x \to 1} \frac{\log_e x}{x - 1} = \lim_{h \to 0} \frac{\log_e (1 + h)}{h} = 1 \left(\text{since } \lim_{x \to 0} \frac{\log_e (1 + x)}{x} = 1 \right).$$

Exercise 13.2

Evaluate the following limits, if exist

1. $\lim_{x \to 0} \frac{e^{4x} - 1}{x}$ (Ans. 4)
2. $\lim_{x \to 0} \frac{e^{2+x} - e^2}{x}$ (Ans. e^2)
3. $\lim_{x \to 5} \frac{e^x - e^5}{x - 5}$ (Ans. e^5)
4. $\lim_{x \to 0} \frac{e^{\sin x} - 1}{x}$ (Ans. 1)
5. $\lim_{x \to 3} \frac{e^x - e^3}{x - 3}$ (Ans. e^3)
6. $\lim_{x \to 0} \frac{x(e^x - 1)}{1 - \cos x}$ (Ans. 2)
7. $\lim_{x \to 0} \frac{\log_e (1 + 2x)}{x}$ (Ans. 2)
8. $\lim_{x \to 0} \frac{\log (1 + x^3)}{\sin^3 x}$ (Ans. 1)

Notes

BE A STUDENT OF STUDENTS

A teacher who establishes rapport with the taught, becomes one with them, learns more from them than he teaches them. He who learns nothing from his disciples is, in my opinion, worthless. Whenever I talk with someone I learn from him. I take from him more than I give him. In this way, a true teacher regards himself as a student of his students. If you will teach your pupils with this attitude, you will benefit much from them.

Talk to Khadi Vidyalaya Students, Sevagram
Harijan Seva, 15 February 1942 (CW 75, p. 269)

USE ALL RESOURCES TO BE CONSTRUCTIVE AND CREATIVE

What we need is educationists with originality, fired with true zeal, who will think out from day to day what they are going to teach their pupils. The teacher cannot get this knowledge through musty volumes. He has to use his own faculties of observation and thinking and impart his knowledge to the children through his lips, with the help of a craft. This means a revolution in the method of teaching, a revolution in the teachers' outlook. Up till now you have been guided by inspector's reports. You wanted to do what the inspector might like, so that you might get more money yet for your institutions or higher salaries for yourselves. But the new teacher will not care for all that. He will say, '1 have done my duty to my pupil if I have made him a better man and in doing so I have used all my resources. That is enough for me'.

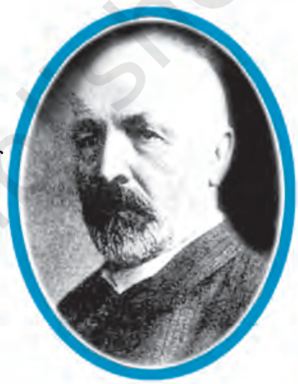
Harijan, 18 February 1939 (CW 68, pp. 374-75)

Chapter 1

SETS

❖*In these days of conflict between ancient and modern studies; there must surely be something to be said for a study which did not begin with Pythagoras and will not end with Einstein; but is the oldest and the youngest.* — *G.H. HARDY* ❖

1.1 Introduction


Georg Cantor (1845-1918)

The concept of set serves as a fundamental part of the present day mathematics. Today this concept is being used in almost every branch of mathematics. Sets are used to define the concepts of relations and functions. The study of geometry, sequences, probability, etc. requires the knowledge of sets.

The theory of sets was developed by German mathematician Georg Cantor (1845-1918). He first encountered sets while working on "problems on trigonometric series". In this Chapter, we discuss some basic definitions and operations involving sets.

1.2 Sets and their Representations

In everyday life, we often speak of collections of objects of a particular kind, such as, a pack of cards, a crowd of people, a cricket team, etc. In mathematics also, we come across collections, for example, of natural numbers, points, prime numbers, etc. More specially, we examine the following collections:

(i) Odd natural numbers less than 10, i.e., 1, 3, 5, 7, 9
(ii) The rivers of India
(iii) The vowels in the English alphabet, namely, *a, e, i, o, u*
(iv) Various kinds of triangles
(v) Prime factors of 210, namely, 2,3,5 and 7
(vi) The solution of the equation: $x^2 – 5x + 6 = 0$, viz, 2 and 3.

We note that each of the above example is a well-defined collection of objects in the sense that we can definitely decide whether a given particular object belongs to a given collection or not. For example, we can say that the river Nile does not belong to the collection of rivers of India. On the other hand, the river Ganga does belong to this colleciton.

We give below a few more examples of sets used particularly in mathematics, viz.

N : the set of all natural numbers
Z : the set of all integers
Q : the set of all rational numbers
R : the set of real numbers
$\mathbf{Z}^+$: the set of positive integers
$\mathbf{Q}^+$: the set of positive rational numbers, and
$\mathbf{R}^+$: the set of positive real numbers.

The symbols for the special sets given above will be referred to throughout this text.

Again the collection of five most renowned mathematicians of the world is not well-defined, because the criterion for determining a mathematician as most renowned may vary from person to person. Thus, it is not a well-defined collection.

We shall say that ***a set is a well-defined collection of objects.***

The following points may be noted :

(i) Objects, elements and members of a set are synonymous terms.
(ii) Sets are usually denoted by capital letters A, B, C, X, Y, Z, etc.
(iii) The elements of a set are represented by small letters *a, b, c, x, y, z,* etc.

If a is an element of a set A, we say that " a belongs to A" the Greek symbol $\in$ (epsilon) is used to denote the phrase '*belongs to*'. Thus, we write $a \in$ A. If 'b' is not an element of a set A, we write $b \notin$ A and read "b does not belong to A".

Thus, in the set V of vowels in the English alphabet, $a \in$ V but $b \notin$ V. In the set P of prime factors of 30, $3 \in$ P but $15 \notin$ P.

There are two methods of representing a set :

(i) Roster or tabular form
(ii) Set-builder form.

(i) In roster form, all the elements of a set are listed, the elements are being separated by commas and are enclosed within braces { }. For example, the set of all even positive integers less than 7 is described in roster form as {2, 4, 6}. Some more examples of representing a set in roster form are given below :

(a) The set of all natural numbers which divide 42 is {1, 2, 3, 6, 7, 14, 21, 42}.

Note In roster form, the order in which the elements are listed is immaterial. Thus, the above set can also be represented as {1, 3, 7, 21, 2, 6, 14, 42}.

(b) The set of all vowels in the English alphabet is {*a, e, i, o, u*}.

(c) The set of odd natural numbers is represented by {1, 3, 5, . . .}. The dots tell us that the list of odd numbers continue indefinitely.

Note It may be noted that while writing the set in roster form an element is not generally repeated, i.e., all the elements are taken as distinct. For example, the set of letters forming the word 'SCHOOL' is { S, C, H, O, L} or {H, O, L, C, S}. Here, the order of listing elements has no relevance.

(ii) In set-builder form, all the elements of a set possess a single common property which is not possessed by any element outside the set. For example, in the set {*a, e, i, o, u*}, all the elements possess a common property, namely, each of them is a vowel in the English alphabet, and no other letter possess this property. Denoting this set by V, we write

V = {x : x is a vowel in English alphabet}

It may be observed that we describe the element of the set by using a symbol x (any other symbol like the letters y, z, etc. could be used) which is followed by a colon " : ". After the sign of colon, we write the characteristic property possessed by the elements of the set and then enclose the whole description within braces. The above description of the set V is read as "the set of all x such that x is a vowel of the English alphabet". In this description the braces stand for "the set of all", the colon stands for "such that". For example, the set

A = {x : x is a natural number and $3 < x < 10$} is read as "the set of all x such that x is a natural number and x lies between 3 and 10." Hence, the numbers 4, 5, 6, 7, 8 and 9 are the elements of the set A.

If we denote the sets described in (*a*), (*b*) and (*c*) above in roster form by A, B, C, respectively, then A, B, C can also be represented in set-builder form as follows:

A= {x : x is a natural number which divides 42}

B= {y : y is a vowel in the English alphabet}

C= {z : z is an odd natural number}

Example 1 Write the solution set of the equation $x^2 + x - 2 = 0$ in roster form.

Solution The given equation can be written as

$$(x - 1)\ (x + 2) = 0, \text{ i. e., }\ x = 1, -2$$

Therefore, the solution set of the given equation can be written in roster form as {1, – 2}.

Example 2 Write the set {x : x is a positive integer and $x^2 < 40$} in the roster form.

Solution The required numbers are 1, 2, 3, 4, 5, 6. So, the given set in the roster form is $\{1, 2, 3, 4, 5, 6\}$.

Example 3 Write the set $A = \{1, 4, 9, 16, 25, \ldots\}$ in set-builder form.

Solution We may write the set A as

$$A = \{x : x \text{ is the square of a natural number}\}$$

Alternatively, we can write

$$A = \{x : x = n^2, \text{ where } n \in \mathbf{N}\}$$

Example 4 Write the set $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}\}$ in the set-builder form.

Solution We see that each member in the given set has the numerator one less than the denominator. Also, the numerator begin from 1 and do not exceed 6. Hence, in the set-builder form the given set is

$$\left\{x : x = \frac{n}{n+1}, \text{ where } n \text{ is a natural number and } 1 \leq n \leq 6\right\}$$

Example 5 Match each of the set on the left described in the roster form with the same set on the right described in the set-builder form :

(i) $\{P, R, I, N, C, A, L\}$ (a) $\{x : x$ is a positive integer and is a divisor of 18$\}$
(ii) $\{0\}$ (b) $\{x : x$ is an integer and $x^2 - 9 = 0\}$
(iii) $\{1, 2, 3, 6, 9, 18\}$ (c) $\{x : x$ is an integer and $x + 1 = 1\}$
(iv) $\{3, -3\}$ (d) $\{x : x$ is a letter of the word PRINCIPAL$\}$

Solution Since in (d), there are 9 letters in the word PRINCIPAL and two letters P and I are repeated, so (i) matches (d). Similarly, (ii) matches (c) as $x + 1 = 1$ implies $x = 0$. Also, 1, 2 ,3, 6, 9, 18 are all divisors of 18 and so (iii) matches (a). Finally, $x^2 - 9 = 0$ implies $x = 3, -3$ and so (iv) matches (b).

EXERCISE 1.1

1. Which of the following are sets ? Justify your answer.

(i) The collection of all the months of a year beginning with the letter J.
(ii) The collection of ten most talented writers of India.
(iii) A team of eleven best-cricket batsmen of the world.
(iv) The collection of all boys in your class.
(v) The collection of all natural numbers less than 100.
(vi) A collection of novels written by the writer Munshi Prem Chand.
(vii) The collection of all even integers.

(viii) The collection of questions in this Chapter.

(ix) A collection of most dangerous animals of the world.

2. Let A = {1, 2, 3, 4, 5, 6}. Insert the appropriate symbol $\in$ or $\notin$ in the blank spaces:

(i) 5. . .A (ii) 8 . . . A (iii) 0. . .A

(iv) 4. . . A (v) 2. . .A (vi) 10. . .A

3. Write the following sets in roster form:

(i) A = $\{x : x$ is an integer and $-3 \le x < 7\}$

(ii) B = $\{x : x$ is a natural number less than $6\}$

(iii) C = $\{x : x$ is a two-digit natural number such that the sum of its digits is $8\}$

(iv) D = $\{x : x$ is a prime number which is divisor of $60\}$

(v) E = The set of all letters in the word TRIGONOMETRY

(vi) F = The set of all letters in the word BETTER

4. Write the following sets in the set-builder form :

(i) (3, 6, 9, 12} (ii) {2,4,8,16,32} (iii) {5, 25, 125, 625}

(iv) {2, 4, 6, . . .} (v) {1,4,9, . . .,100}

5. List all the elements of the following sets :

(i) A = $\{x : x$ is an odd natural number$\}$

(ii) B = $\{x : x$ is an integer, $-\frac{1}{2} < x < \frac{9}{2}\}$

(iii) C = $\{x : x$ is an integer, $x^2 \le 4\}$

(iv) D = $\{x : x$ is a letter in the word "LOYAL"$\}$

(v) E = $\{x : x$ is a month of a year not having 31 days$\}$

(vi) F = $\{x : x$ is a consonant in the English alphabet which precedes $k\}$.

6. Match each of the set on the left in the roster form with the same set on the right described in set-builder form:

(i) {1, 2, 3, 6} (a) $\{x : x$ is a prime number and a divisor of $6\}$

(ii) {2, 3} (b) $\{x : x$ is an odd natural number less than $10\}$

(iii) {M,A,T,H,E,I,C,S} (c) $\{x : x$ is natural number and divisor of $6\}$

(iv) {1, 3, 5, 7, 9} (d) $\{x : x$ is a letter of the word MATHEMATICS$\}$.

1.3 The Empty Set

Consider the set

A = $\{ x : x$ is a student of Class XI presently studying in a school $\}$

We can go to the school and count the number of students presently studying in Class XI in the school. Thus, the set A contains a finite number of elements.

We now write another set B as follows:

B = { x : x is a student presently studying in both Classes X and XI }

We observe that a student cannot study simultaneously in both Classes X and XI. Thus, the set B contains no element at all.

Definition 1 A set which does not contain any element is called the *empty set* or the *null set* or the *void set*.

According to this definition, B is an empty set while A is not an empty set. The empty set is denoted by the symbol ϕ or { }.

We give below a few examples of empty sets.

(i) Let A = $\{x : 1 < x < 2, x$ is a natural number$\}$. Then A is the empty set, because there is no natural number between 1 and 2.

(ii) B = $\{x : x^2 - 2 = 0$ and x is rational number$\}$. Then B is the empty set because the equation $x^2 - 2 = 0$ is not satisfied by any rational value of x.

(iii) C = $\{x : x$ is an even prime number greater than 2$\}$.Then C is the empty set, because 2 is the only even prime number.

(iv) D = $\{ x : x^2 = 4, x$ is odd $\}$. Then D is the empty set, because the equation $x^2 = 4$ is not satisfied by any odd value of x.

1.4 Finite and Infinite Sets

Let A = {1, 2, 3, 4, 5}, B = $\{a, b, c, d, e, g\}$

and C = { men living presently in different parts of the world}

We observe that A contains 5 elements and B contains 6 elements. How many elements does C contain? As it is, we do not know the number of elements in C, but it is some natural number which may be quite a big number. By number of elements of a set S, we mean the number of distinct elements of the set and we denote it by n (S). If n (S) is a natural number, then S is *non-empty finite* set.

Consider the set of natural numbers. We see that the number of elements of this set is not finite since there are infinite number of natural numbers. We say that the set of natural numbers is an infinite set. The sets A, B and C given above are finite sets and n(A) = 5, n(B) = 6 and n(C) = some finite number.

Definition 2 A set which is empty or consists of a definite number of elements is called *finite* otherwise, the set is called *infinite*.

Consider some examples :

(i) Let W be the set of the days of the week. Then W is finite.

(ii) Let S be the set of solutions of the equation $x^2 - 16 = 0$. Then S is finite.

(iii) Let G be the set of points on a line. Then G is infinite.

When we represent a set in the roster form, we write all the elements of the set within braces { }. It is not possible to write all the elements of an infinite set within braces { } because the numbers of elements of such a set is not finite. So, we represent some infinite set in the roster form by writing a few elements which clearly indicate the structure of the set followed (or preceded) by three dots.

For example, {1, 2, 3 . . .} is the set of natural numbers, {1, 3, 5, 7, . . .} is the set of odd natural numbers, {. . .,–3, –2, –1, 0,1, 2 ,3, . . .} is the set of integers. All these sets are infinite.

Note All infinite sets cannot be described in the roster form. For example, the set of real numbers cannot be described in this form, because the elements of this set do not follow any particular pattern.

Example 6 State which of the following sets are finite or infinite :

(i) $\{x : x \in \text{N and } (x - 1)(x - 2) = 0\}$
(ii) $\{x : x \in \text{N and } x^2 = 4\}$
(iii) $\{x : x \in \text{N and } 2x - 1 = 0\}$
(iv) $\{x : x \in \text{N and } x \text{ is prime}\}$
(v) $\{x : x \in \text{N and } x \text{ is odd}\}$

Solution (i) Given set = {1, 2}. Hence, it is finite.
(ii) Given set = {2}. Hence, it is finite.
(iii) Given set = ϕ. Hence, it is finite.
(iv) The given set is the set of all prime numbers and since set of prime numbers is infinite. Hence the given set is infinite
(v) Since there are infinite number of odd numbers, hence, the given set is infinite.

1.5 Equal Sets

Given two sets A and B, if every element of A is also an element of B and if every element of B is also an element of A, then the sets A and B are said to be equal. Clearly, the two sets have exactly the same elements.

Definition 3 Two sets A and B are said to be *equal* if they have exactly the same elements and we write A = B. Otherwise, the sets are said to be *unequal* and we write A $\neq$ B.

We consider the following examples :

(i) Let A = {1, 2, 3, 4} and B = {3, 1, 4, 2}. Then A = B.
(ii) Let A be the set of prime numbers less than 6 and P the set of prime factors of 30. Then A and P are equal, since 2, 3 and 5 are the only prime factors of 30 and also these are less than 6.

Note A set does not change if one or more elements of the set are repeated. For example, the sets A = {1, 2, 3} and B = {2, 2, 1, 3, 3} are equal, since each element of A is in B and vice-versa. That is why we generally do not repeat any element in describing a set.

Example 7 Find the pairs of equal sets, if any, give reasons:

A = {0}, B = $\{x : x > 15 \text{ and } x < 5\}$,

C = $\{x : x - 5 = 0\}$, D = $\{x : x^2 = 25\}$,

E = $\{x : x$ is an integral positive root of the equation $x^2 - 2x - 15 = 0\}$.

Solution Since $0 \in A$ and 0 does not belong to any of the sets B, C, D and E, it follows that, $A \neq B$, $A \neq C$, $A \neq D$, $A \neq E$.

Since B = ϕ but none of the other sets are empty. Therefore $B \neq C$, $B \neq D$ and $B \neq E$. Also C = {5} but $-5 \in D$, hence $C \neq D$.

Since E = {5}, C = E. Further, D = {–5, 5} and E = {5}, we find that, $D \neq E$. Thus, the only pair of equal sets is C and E.

Example 8 Which of the following pairs of sets are equal? Justify your answer.

(i) X, the set of letters in “ALLOY” and B, the set of letters in “LOYAL”.

(ii) A = $\{n : n \in \mathbf{Z} \text{ and } n^2 \leq 4\}$ and B = $\{x : x \in \mathbf{R} \text{ and } x^2 - 3x + 2 = 0\}$.

Solution (i) We have, X = {A, L, L, O, Y}, B = {L, O, Y, A, L}. Then X and B are equal sets as repetition of elements in a set do not change a set. Thus,

X = {A, L, O, Y} = B

(ii) A = {–2, –1, 0, 1, 2}, B = {1, 2}. Since $0 \in A$ and $0 \notin B$, A and B are not equal sets.

EXERCISE 1.2

1. Which of the following are examples of the null set

(i) Set of odd natural numbers divisible by 2

(ii) Set of even prime numbers

(iii) $\{ x : x$ is a natural numbers, $x < 5$ and $x > 7 \}$

(iv) $\{ y : y$ is a point common to any two parallel lines$\}$

2. Which of the following sets are finite or infinite

(i) The set of months of a year

(ii) {1, 2, 3, . . .}

(iii) {1, 2, 3, . . .99, 100}

(iv) The set of positive integers greater than 100

(v) The set of prime numbers less than 99

3. State whether each of the following set is finite or infinite:

(i) The set of lines which are parallel to the x-axis

(ii) The set of letters in the English alphabet

(iii) The set of numbers which are multiple of 5

(iv) The set of animals living on the earth

(v) The set of circles passing through the origin (0,0)

4. In the following, state whether A = B or not:

(i) $A = \{ a, b, c, d \}$ $B = \{ d, c, b, a \}$

(ii) $A = \{ 4, 8, 12, 16 \}$ $B = \{ 8, 4, 16, 18\}$

(iii) $A = \{2, 4, 6, 8, 10\}$ $B = \{ x : x$ is positive even integer and $x \leq 10\}$

(iv) $A = \{ x : x$ is a multiple of $10\}$, $B = \{ 10, 15, 20, 25, 30, \ldots \}$

5. Are the following pair of sets equal ? Give reasons.

(i) $A = \{2, 3\}$, $B = \{x : x$ is solution of $x^2 + 5x + 6 = 0\}$

(ii) $A = \{ x : x$ is a letter in the word FOLLOW$\}$

$B = \{ y : y$ is a letter in the word WOLF$\}$

6. From the sets given below, select equal sets :

$A = \{ 2, 4, 8, 12\}$, $B = \{ 1, 2, 3, 4\}$, $C = \{ 4, 8, 12, 14\}$, $D = \{ 3, 1, 4, 2\}$

$E = \{-1, 1\}$, $F = \{ 0, a\}$, $G = \{1, -1\}$, $H = \{ 0, 1\}$

1.6 Subsets

Consider the sets : X = set of all students in your school, Y = set of all students in your class.

We note that every element of Y is also an element of X; we say that Y is a subset of X. The fact that Y is subset of X is expressed in symbols as $Y \subset X$. The symbol $\subset$ stands for 'is a subset of' or 'is contained in'.

Definition 4 A set A is said to be a subset of a set B if every element of A is also an element of B.

In other words, $A \subset B$ if whenever $a \in A$, then $a \in B$. It is often convenient to use the symbol "$\Rightarrow$" which means *implies*. Using this symbol, we can write the definiton of *subset* as follows:

$$A \subset B \text{ if } a \in A \Rightarrow a \in B$$

We read the above statement as **"***A is a subset of B if a is an element of A implies that a is also an element of B***"**. If A is not a subset of B, we write $A \not\subset B$.

We may note that for A to be a subset of B, all that is needed is that every element of A is in B. It is possible that every element of B may or may not be in A. If it so happens that every element of B is also in A, then we shall also have $B \subset A$. In this case, A and B are the same sets so that we have $A \subset B$ and $B \subset A \Leftrightarrow A = B$, where "$\Leftrightarrow$" is a symbol for two way implications, and is usually read as *if and only if* (briefly written as "iff").

It follows from the above definition that every set *A is a subset of itself,* i.e., $A \subset A$. Since the empty set ϕ has no elements, we agree to say that ϕ *is a subset of every set***.** We now consider some examples :

(i) The set **Q** of rational numbers is a subset of the set **R** of real numbes, and we write **Q** $\subset$ R.

(ii) If A is the set of all divisors of 56 and B the set of all prime divisors of 56, then B is a subset of A and we write B $\subset$ A.

(iii) Let A = {1, 3, 5} and B = {x : x is an odd natural number less than 6}. Then A $\subset$ B and B $\subset$ A and hence A = B.

(iv) Let A = { a, e, i, o, u} and B = { a, b, c, d}. Then A is not a subset of B, also B is not a subset of A.

Let A and B be two sets. If A $\subset$ B and A $\neq$ B , then A is called a *proper subset of* B and B is called *superset* of A. For example,

A = {1, 2, 3} is a proper subset of B = {1, 2, 3, 4}.

If a set A has only one element, we call it a *singleton set*. Thus,{ a } is a singleton set.

Example 9 Consider the sets

$$\phi, A = \{ 1, 3 \}, \quad B = \{1, 5, 9\}, \quad C = \{1, 3, 5, 7, 9\}.$$

Insert the symbol $\subset$ or $\not\subset$ between each of the following pair of sets:

(i) ϕ **. . .** B (ii) A . . . B (iii) A . . . C (iv) B . . . C

Solution (i) $\phi \subset$ B as ϕ is a subset of every set.

(ii) A $\not\subset$ B as 3 $\in$ A and 3 $\notin$ B

(iii) A $\subset$ C as 1, 3 $\in$ A also belongs to C

(iv) B $\subset$ C as each element of B is also an element of C.

Example 10 Let A = { a, e, i, o, u} and B = { a, b, c, d}. Is A a subset of B ? No. (Why?). Is B a subset of A? No. (Why?)

Example 11 Let A, B and C be three sets. If A $\in$ B and B $\subset$ C, is it true that A $\subset$ C?. If not, give an example.

Solution No. Let A = {1}, B = {{1}, 2} and C = {{1}, 2, 3}. Here A $\in$ B as A = {1} and B $\subset$ C. But A $\not\subset$ C as 1 $\in$ A and 1 $\notin$ C.

Note that an element of a set can never be a subset of itself.

1.6.1 *Subsets of set of real numbers*

As noted in Section 1.6, there are many important subsets of **R**. We give below the names of some of these subsets.

The set of natural numbers **N** = {1, 2, 3, 4, 5, . . .}

The set of integers **Z** = {. . ., –3, –2, –1, 0, 1, 2, 3, . . .}

The set of rational numbers **Q** = $\{ x : x = \frac{p}{q}, p, q \in$ **Z** and $q \neq 0\}$

which is read " **Q** is the set of all numbers x such that x equals the quotient $\frac{p}{q}$, where p and q are integers and q is not zero". Members of **Q** include -5 (which can be expressed as $-\frac{5}{1}$), $\frac{5}{7}$, $3\frac{1}{2}$ (which can be expressed as $\frac{7}{2}$) and $-\frac{11}{3}$.

The set of irrational numbers, denoted by **T**, is composed of all other real numbers. Thus $\mathbf{T} = \{x : x \in \mathbf{R} \text{ and } x \notin \mathbf{Q}\}$, i.e., all real numbers that are not rational. Members of **T** include $\sqrt{2}$, $\sqrt{5}$ and π.

Some of the obvious relations among these subsets are:

$\mathbf{N} \subset \mathbf{Z} \subset \mathbf{Q}$, $\mathbf{Q} \subset \mathbf{R}$, $\mathbf{T} \subset \mathbf{R}$, $\mathbf{N} \not\subset \mathbf{T}$.

1.6.2 *Intervals as subsets of R* Let $a, b \in \mathbf{R}$ and $a < b$. Then the set of real numbers $\{y : a < y < b\}$ is called an *open interval* and is denoted by (a, b). All the points between a and b belong to the open interval (a, b) but a, b themselves do not belong to this interval.

The interval which contains the end points also is called *closed interval* and is denoted by $[a, b]$. Thus

$[a, b] = \{x : a \le x \le b\}$

We can also have intervals closed at one end and open at the other, i.e.,

$[a, b) = \{x : a \le x < b\}$ is an *open interval* from a to b, including a but excluding b.

$(a, b] = \{x : a < x \le b\}$ is an *open interval* from a to b including b but excluding a.

These notations provide an alternative way of designating the subsets of set of real numbers. For example, if $A = (-3, 5)$ and $B = [-7, 9]$, then $A \subset B$. The set $[0, \infty)$ defines the set of non-negative real numbers, while set $(-\infty, 0)$ defines the set of negative real numbers. The set $(-\infty, \infty)$ describes the set of real numbers in relation to a line extending from $-\infty$ to ∞.

On real number line, various types of intervals described above as subsets of **R**, are shown in the Fig 1.1.

Fig 1.1

Here, we note that an interval contains infinitely many points.

For example, the set $\{x : x \in \mathbf{R}, -5 < x \le 7\}$, written in set-builder form, can be written in the form of interval as $(-5, 7]$ and the interval $[-3, 5)$ can be written in set-builder form as $\{x : -3 \le x < 5\}$.

The number $(b - a)$ is called the *length of any of the intervals* (a, b), $[a, b]$, $[a, b)$ or $(a, b]$.

1.7 Universal Set

Usually, in a particular context, we have to deal with the elements and subsets of a basic set which is relevant to that particular context. For example, while studying the system of numbers, we are interested in the set of natural numbers and its subsets such as the set of all prime numbers, the set of all even numbers, and so forth. This basic set is called the **"***Universal Set***".** The universal set is usually denoted by U, and all its subsets by the letters A, B, C, etc.

For example, for the set of all integers, the universal set can be the set of rational numbers or, for that matter, the set **R** of real numbers. For another example, in human population studies, the universal set consists of all the people in the world.

EXERCISE 1.3

1. Make correct statements by filling in the symbols $\subset$ or $\not\subset$ in the blank spaces :

(i) $\{2, 3, 4\} \ldots \{1, 2, 3, 4, 5\}$ (ii) $\{a, b, c\} \ldots \{b, c, d\}$

(iii) $\{x : x$ is a student of Class XI of your school$\} \ldots \{x : x$ student of your school$\}$

(iv) $\{x : x$ is a circle in the plane$\} \ldots \{x : x$ is a circle in the same plane with radius 1 unit$\}$

(v) $\{x : x$ is a triangle in a plane$\} \ldots \{x : x$ is a rectangle in the plane$\}$

(vi) $\{x : x$ is an equilateral triangle in a plane$\} \ldots \{x : x$ is a triangle in the same plane$\}$

(vii) $\{x : x$ is an even natural number$\} \ldots \{x : x$ is an integer$\}$

2. Examine whether the following statements are true or false:

(i) $\{a, b\} \not\subset \{b, c, a\}$

(ii) $\{a, e\} \subset \{x : x$ is a vowel in the English alphabet$\}$

(iii) $\{1, 2, 3\} \subset \{1, 3, 5\}$

(iv) $\{a\} \subset \{a, b, c\}$

(v) $\{a\} \in \{a, b, c\}$

(vi) $\{x : x$ is an even natural number less than 6$\} \subset \{x : x$ is a natural number which divides 36$\}$

3. Let $A = \{1, 2, \{3, 4\}, 5\}$. Which of the following statements are incorrect and why?

(i) $\{3, 4\} \subset A$ (ii) $\{3, 4\} \in A$ (iii) $\{\{3, 4\}\} \subset A$

(iv) $1 \in A$ (v) $1 \subset A$ (vi) $\{1, 2, 5\} \subset A$

(vii) $\{1, 2, 5\} \in A$ (viii) $\{1, 2, 3\} \subset A$ (ix) $\phi \in A$

(x) $\phi \subset A$ (xi) $\{\phi\} \subset A$

4. Write down all the subsets of the following sets

(i) $\{a\}$ (ii) $\{a, b\}$ (iii) $\{1, 2, 3\}$ (iv) ϕ

5. Write the following as intervals :

(i) $\{x : x \in R, -4 < x \leq 6\}$ (ii) $\{x : x \in R, -12 < x < -10\}$

(iii) $\{x : x \in R, 0 \leq x < 7\}$ (iv) $\{x : x \in R, 3 \leq x \leq 4\}$

6. Write the following intervals in set-builder form :

(i) $(-3, 0)$ (ii) $[6, 12]$ (iii) $(6, 12]$ (iv) $[-23, 5)$

7. What universal set(s) would you propose for each of the following :

(i) The set of right triangles. (ii) The set of isosceles triangles.

8. Given the sets A = {1, 3, 5}, B = {2, 4, 6} and C = {0, 2, 4, 6, 8}, which of the following may be considered as universal set (s) for all the three sets A, B and C

(i) {0, 1, 2, 3, 4, 5, 6}

(ii) ϕ

(iii) {0,1,2,3,4,5,6,7,8,9,10}

(iv) {1,2,3,4,5,6,7,8}

1.8 Venn Diagrams

Most of the relationships between sets can be represented by means of diagrams which are known as *Venn diagrams*. Venn diagrams are named after the English logician, John Venn (1834-1883). These diagrams consist of rectangles and closed curves usually circles. The universal set is represented usually by a rectangle and its subsets by circles.

In Venn diagrams, the elements of the sets are written in their respective circles (Figs 1.2 and 1.3)

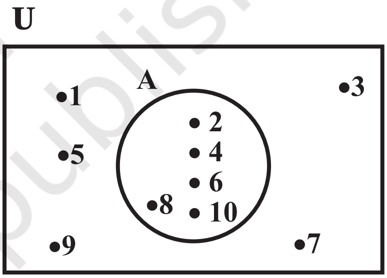


Fig 1.2

Illustration 1 In Fig 1.2, U = {1,2,3, ..., 10} is the universal set of which

A = {2,4,6,8,10} is a subset.

Illustration 2 In Fig 1.3, U = {1,2,3, ..., 10} is the universal set of which

A = {2,4,6,8,10} and B = {4, 6} are subsets, and also $B \subset A$.

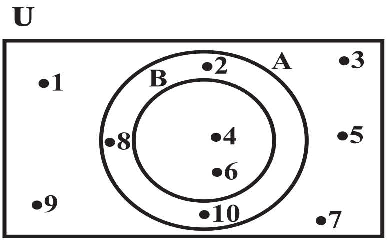


Fig 1.3

The reader will see an extensive use of the Venn diagrams when we discuss the union, intersection and difference of sets.

1.9 Operations on Sets

In earlier classes, we have learnt how to perform the operations of addition, subtraction, multiplication and division on numbers. Each one of these operations was performed on a pair of numbers to get another number. For example, when we perform the operation of addition on the pair of numbers 5 and 13, we get the number 18. Again, performing the operation of multiplication on the pair of numbers 5 and 13, we get 65.

Similarly, there are some operations which when performed on two sets give rise to another set. We will now define certain operations on sets and examine their properties. Henceforth, we will refer all our sets as subsets of some universal set.

1.9.1 ***Union of sets*** Let A and B be any two sets. The union of A and B is the set which consists of all the elements of A and all the elements of B, the common elements being taken only once. The symbol '$\cup$' is used to denote the *union*. *Symbolically, we write* $A \cup B$ *and usually read as 'A union B'*.

Example 12 Let $A = \{2, 4, 6, 8\}$ and $B = \{6, 8, 10, 12\}$. Find $A \cup B$.

Solution We have $A \cup B = \{2, 4, 6, 8, 10, 12\}$

Note that the common elements 6 and 8 have been taken only once while writing $A \cup B$.

Example 13 Let $A = \{a, e, i, o, u\}$ and $B = \{a, i, u\}$. Show that $A \cup B = A$

Solution We have, $A \cup B = \{a, e, i, o, u\} = A$.

This example illustrates that union of sets A and its subset B is the set A itself, i.e., if $B \subset A$, then $A \cup B = A$.

Example 14 Let X = {Ram, Geeta, Akbar} be the set of students of Class XI, who are in school hockey team. Let Y = {Geeta, David, Ashok} be the set of students from Class XI who are in the school football team. Find $X \cup Y$ and interpret the set.

Solution We have, $X \cup Y$ = {Ram, Geeta, Akbar, David, Ashok}. This is the set of students from Class XI who are in the hockey team or the football team or both.

Thus, we can define the union of two sets as follows:

Definition 5 The union of two sets A and B is the set C which consists of all those elements which are either in A or in B (including those which are in both). In symbols, we write. $A \cup B = \{x : x \in A \text{ or } x \in B\}$

The union of two sets can be represented by a Venn diagram as shown in Fig 1.4.

The shaded portion in Fig 1.4 represents $A \cup B$.

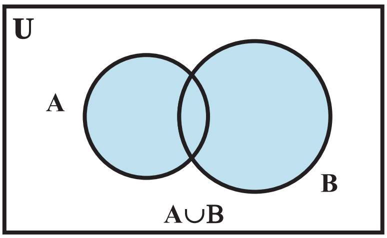


Fig 1.4

Some Properties of the Operation of Union

(i) $A \cup B = B \cup A$ (Commutative law)

(ii) $(A \cup B) \cup C = A \cup (B \cup C)$ (Associative law)

(iii) $A \cup \phi = A$ (Law of identity element, ϕ is the identity of $\cup$)

(iv) $A \cup A = A$ (Idempotent law)

(v) $U \cup A = U$ (Law of U)

1.9.2 *Intersection of sets* The intersection of sets A and B is the set of all elements which are common to both A and B. The symbol '$\cap$' is used to denote the *intersection*. The intersection of two sets A and B is the set of all those elements which belong to both A and B. Symbolically, we write $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Example 15 Consider the sets A and B of Example 12. Find $A \cap B$.

Solution We see that 6, 8 are the only elements which are common to both A and B. Hence $A \cap B = \{ 6, 8 \}$.

Example 16 Consider the sets X and Y of Example 14. Find $X \cap Y$.

Solution We see that element 'Geeta' is the only element common to both. Hence, $X \cap Y = \{\text{Geeta}\}$.

Example 17 Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $B = \{ 2, 3, 5, 7 \}$. Find $A \cap B$ and hence show that $A \cap B = B$.

Solution We have $A \cap B = \{ 2, 3, 5, 7 \} = B$. We note that $B \subset A$ and that $A \cap B = B$.

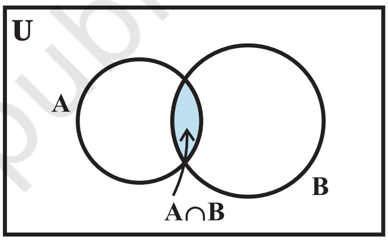


Fig 1.5

Definition 6 The intersection of two sets A and B is the set of all those elements which belong to both A and B. Symbolically, we write

$A \cap B = \{x : x \in A \text{ and } x \in B\}$

The shaded portion in Fig 1.5 indicates the intersection of A and B.

If A and B are two sets such that $A \cap B = \phi$, then A and B are called *disjoint sets*.

For example, let $A = \{ 2, 4, 6, 8 \}$ and $B = \{ 1, 3, 5, 7 \}$. Then A and B are disjoint sets, because there are no elements which are common to A and B. The disjoint sets can be represented by means of Venn diagram as shown in the Fig 1.6

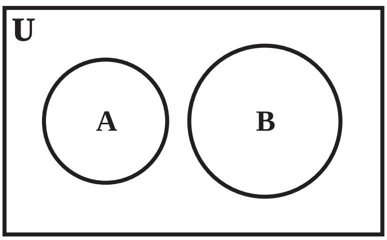


Fig 1.6

In the above diagram, A and B are disjoint sets.

Some Properties of Operation of Intersection

(i) $A \cap B = B \cap A$ (Commutative law).

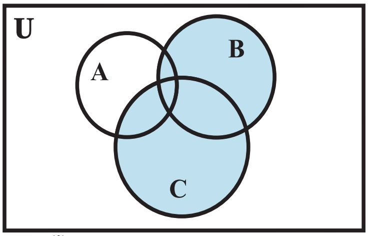
(ii) $(A \cap B) \cap C = A \cap (B \cap C)$ (Associative law).

(iii) $\phi \cap A = \phi$, $U \cap A = A$ (Law of ϕ and U).

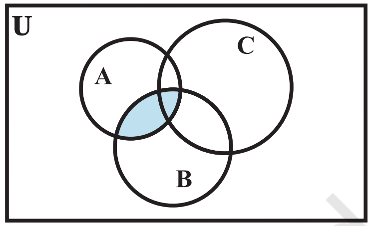
(iv) $A \cap A = A$ (Idempotent law)

(v) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive law) i. e., $\cap$ distributes over $\cup$

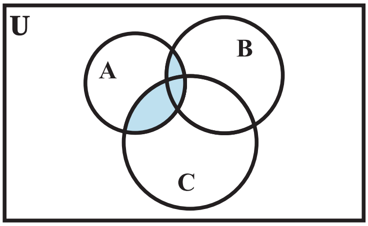
This can be seen easily from the following Venn diagrams [Figs 1.7 (i) to (v)].



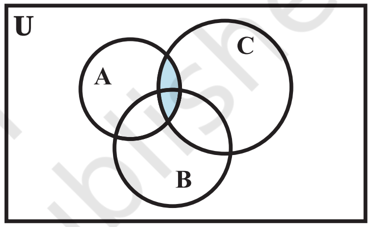
(i) **(B∪C)**



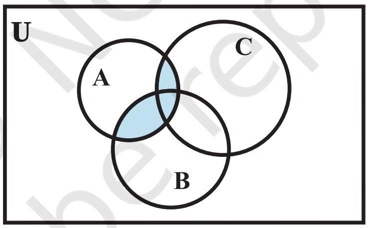
(iii) **(A∩B)**



(ii) **A∩(B∪C)**



(iv) **(A∩C)**



(v) **(A∩B)** ∪ (A∩C)

Figs 1.7 (i) to (v)

1.9.3 ***Difference of sets*** The difference of the sets A and B in this order is the set of elements which belong to A but not to B. Symbolically, we write A – B and read as "A minus B".

Example 18 Let A = { 1, 2, 3, 4, 5, 6}, B = { 2, 4, 6, 8 }. Find A – B and B – A.

Solution We have, A – B = { 1, 3, 5 }, since the elements 1, 3, 5 belong to A but not to B and B – A = { 8 }, since the element 8 belongs to B and not to A.

We note that A – B ≠ B – A.

Example 19 Let V = { a, e, i, o, u } and B = { a, i, k, u}. Find V – B and B – V

Solution We have, V – B = { e, o }, since the elements e, o belong to V but not to B and B – V = { k }, since the element k belongs to B but not to V.

We note that V – B ≠ B – V. Using the set-builder notation, we can rewrite the definition of difference as

$$A - B = \{ x : x \in A \text{ and } x \notin B \}$$

The difference of two sets A and B can be represented by Venn diagram as shown in Fig 1.8.

The shaded portion represents the difference of the two sets A and B.

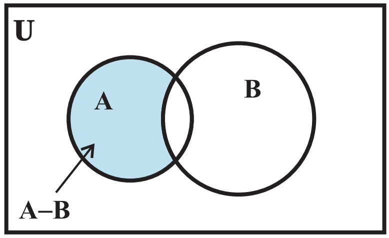


Fig 1.8

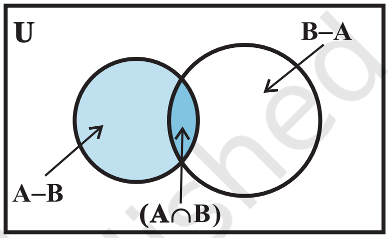


Fig 1.9

Remark The sets A – B, A ∩ B and B – A are mutually disjoint sets, i.e., the intersection of any of these two sets is the null set as shown in Fig 1.9.

EXERCISE 1.4

1. Find the union of each of the following pairs of sets :

(i) X = {1, 3, 5} Y = {1, 2, 3}

(ii) A = [a, e, i, o, u} B = {a, b, c}

(iii) A = {x : x is a natural number and multiple of 3}
B = {x : x is a natural number less than 6}

(iv) A = {x : x is a natural number and $1 < x \leq 6$ }
B = {x : x is a natural number and $6 < x < 10$ }

(v) A = {1, 2, 3}, B = φ

2. Let A = { a, b }, B = {a, b, c}. Is A ⊂ B ? What is A ∪ B ?

3. If A and B are two sets such that A ⊂ B, then what is A ∪ B ?

4. If A = {1, 2, 3, 4}, B = {3, 4, 5, 6}, C = {5, 6, 7, 8 }and D = { 7, 8, 9, 10 }; find

(i) A ∪ B (ii) A ∪ C (iii) B ∪ C (iv) B ∪ D

(v) A ∪ B ∪ C (vi) A ∪ B ∪ D (vii) B ∪ C ∪ D

5. Find the intersection of each pair of sets of question 1 above.

6. If A = { 3, 5, 7, 9, 11 }, B = {7, 9, 11, 13}, C = {11, 13, 15}and D = {15, 17}; find

(i) A ∩ B (ii) B ∩ C (iii) A ∩ C ∩ D

(iv) A ∩ C (v) B ∩ D (vi) A ∩ (B ∪ C)

(vii) A ∩ D (viii) A ∩ (B ∪ D) (ix) (A ∩ B) ∩ (B ∪ C)

(x) (A ∪ D) ∩ (B ∪ C)

7. If A = {x : x is a natural number }, B = {x : x is an even natural number}
C = {x : x is an odd natural number}andD = {x : x is a prime number }, find
(i) $A \cap B$ (ii) $A \cap C$ (iii) $A \cap D$
(iv) $B \cap C$ (v) $B \cap D$ (vi) $C \cap D$

8. Which of the following pairs of sets are disjoint
(i) {1, 2, 3, 4} and {x : x is a natural number and $4 \le x \le 6$ }
(ii) { a, e, i, o, u } and { c, d, e, f }
(iii) {x : x is an even integer } and {x : x is an odd integer}

9. If A = {3, 6, 9, 12, 15, 18, 21}, B = { 4, 8, 12, 16, 20 },
C = { 2, 4, 6, 8, 10, 12, 14, 16 }, D = {5, 10, 15, 20 }; find
(i) A – B (ii) A – C (iii) A – D (iv) B – A
(v) C – A (vi) D – A (vii) B – C (viii) B – D
(ix) C – B (x) D – B (xi) C – D (xii) D – C

10. If X= { a, b, c, d } and Y = { f, b, d, g}, find
(i) X – Y (ii) Y – X (iii) $X \cap Y$

11. If **R** is the set of real numbers and **Q** is the set of rational numbers, then what is **R** – **Q**?

12. State whether each of the following statement is true or false. Justify your answer.
(i) { 2, 3, 4, 5 } and { 3, 6} are disjoint sets.
(ii) { a, e, i, o, u } and { a, b, c, d }are disjoint sets.
(iii) { 2, 6, 10, 14 } and { 3, 7, 11, 15} are disjoint sets.
(iv) { 2, 6, 10 } and { 3, 7, 11} are disjoint sets.

1.10 Complement of a Set

Let U be the universal set which consists of all prime numbers and A be the subset of U which consists of all those prime numbers that are not divisors of 42. Thus, A = {x : $x \in$ U and x is not a divisor of 42 }. We see that $2 \in$ U but $2 \notin$ A, because 2 is divisor of 42. Similarly, $3 \in$ U but $3 \notin$ A, and $7 \in$ U but $7 \notin$ A. Now 2, 3 and 7 are the only elements of U which do not belong to A. The set of these three prime numbers, i.e., the set {2, 3, 7} is called the *Complement* of A with respect to U, and is denoted by A′. So we have A′ = {2, 3, 7}. Thus, we see that

A′ = {x : $x \in$ U and $x \notin$ A }. This leads to the following definition.

Definition 7 Let U be the universal set and A a subset of U. Then the complement of A is the set of all elements of U which are not the elements of A. Symbolically, we write A′ to denote the complement of A with respect to U. Thus,

A′ = {x : $x \in$ U and $x \notin$ A }. Obviously A′ = U – A

We note that the complement of a set A can be looked upon, alternatively, as the difference between a universal set U and the set A.

Example 20 Let U = $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and A = $\{1, 3, 5, 7, 9\}$. Find A′.

Solution We note that 2, 4, 6, 8, 10 are the only elements of U which do not belong to A. Hence $A' = \{2, 4, 6, 8, 10\}$.

Example 21 Let U be universal set of all the students of Class XI of a coeducational school and A be the set of all girls in Class XI. Find A′.

Solution Since A is the set of all girls, A′ is clearly the set of all boys in the class.

Note If A is a subset of the universal set U, then its complement A′ is also a subset of U.

Again in Example 20 above, we have $A' = \{2, 4, 6, 8, 10\}$

Hence $(A')' = \{x : x \in U \text{ and } x \notin A'\}$

$= \{1, 3, 5, 7, 9\} = A$

It is clear from the definition of the complement that for any subset of the universal set U, we have $(A')' = A$

Now, we want to find the results for $(A \cup B)'$ and $A' \cap B'$ in the followng example.

Example 22 Let U = $\{1, 2, 3, 4, 5, 6\}$, A = $\{2, 3\}$ and B = $\{3, 4, 5\}$.
Find A′, B′, $A' \cap B'$, $A \cup B$ and hence show that $(A \cup B)' = A' \cap B'$.

Solution Clearly $A' = \{1, 4, 5, 6\}$, $B' = \{1, 2, 6\}$. Hence $A' \cap B' = \{1, 6\}$

Also $A \cup B = \{2, 3, 4, 5\}$, so that $(A \cup B)' = \{1, 6\}$

$$(A \cup B)' = \{1, 6\} = A' \cap B'$$

It can be shown that the above result is true in general. If A and B are any two subsets of the universal set U, then

$(A \cup B)' = A' \cap B'$. Similarly, $(A \cap B)' = A' \cup B'$. These two results are stated in words as follows :

The complement of the union of two sets is the intersection of their complements and the complement of the intersection of two sets is the union of their complements. These are called *De Morgan's laws*. These are named after the mathematician De Morgan.

The complement A′ of a set A can be represented by a Venn diagram as shown in Fig 1.10.

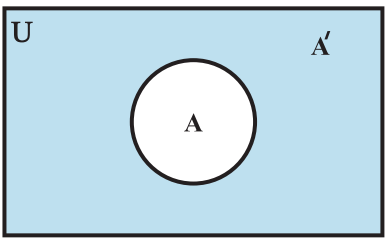


Fig 1.10

The shaded portion represents the complement of the set A.

Some Properties of Complement Sets

1. Complement laws: (i) $A \cup A' = U$ (ii) $A \cap A' = \phi$

2. De Morgan's law: (i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

3. Law of double complementation : $(A')' = A$

4. Laws of empty set and universal set $\phi' = U$ and $U' = \phi$.

These laws can be verified by using Venn diagrams.

EXERCISE 1.5

1. Let U = { 1, 2, 3, 4, 5, 6, 7, 8, 9 }, A = { 1, 2, 3, 4}, B = { 2, 4, 6, 8 } and C = { 3, 4, 5, 6 }. Find (i) A' (ii) B' (iii) $(A \cup C)'$ (iv) $(A \cup B)'$ (v) $(A')'$ (vi) $(B - C)'$

2. If U = { a, b, c, d, e, f, g, h}, find the complements of the following sets :
(i) A = {a, b, c} (ii) B = {d, e, f, g}
(iii) C = {a, c, e, g} (iv) D = { f, g, h, a}

3. Taking the set of natural numbers as the universal set, write down the complements of the following sets:
(i) {x : x is an even natural number} (ii) { x : x is an odd natural number }
(iii) {x : x is a positive multiple of 3} (iv) { x : x is a prime number }
(v) {x : x is a natural number divisible by 3 and 5}
(vi) { x : x is a perfect square } (vii) { x : x is a perfect cube}
(viii) { $x : x + 5 = 8$ } (ix) { $x : 2x + 5 = 9$}
(x) { $x : x \geq 7$ } (xi) { $x : x \in N$ and $2x + 1 > 10$ }

4. If U = {1, 2, 3, 4, 5, 6, 7, 8, 9 }, A = {2, 4, 6, 8} and B = { 2, 3, 5, 7}. Verify that
(i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

5. Draw appropriate Venn diagram for each of the following :
(i) $(A \cup B)'$, (ii) $A' \cap B'$, (iii) $(A \cap B)'$, (iv) $A' \cup B'$

6. Let U be the set of all triangles in a plane. If A is the set of all triangles with at least one angle different from 60°, what is A'?

7. Fill in the blanks to make each of the following a true statement :
(i) $A \cup A' = \ldots$ (ii) $\phi' \cap A = \ldots$
(iii) $A \cap A' = \ldots$ (iv) $U' \cap A = \ldots$

Miscellaneous Examples

Example 23 Show that the set of letters needed to spell " CATARACT " and the set of letters needed to spell " TRACT" are equal.

Solution Let X be the set of letters in "CATARACT". Then

X = { C, A, T, R }

Let Y be the set of letters in " TRACT". Then

$$Y = \{ T, R, A, C, T \} = \{ T, R, A, C \}$$

Since every element in X is in Y and every element in Y is in X. It follows that X = Y.

Example 24 List all the subsets of the set $\{ -1, 0, 1 \}$.

Solution Let $A = \{ -1, 0, 1 \}$. The subset of A having no element is the empty set ϕ. The subsets of A having one element are $\{ -1 \}, \{ 0 \}, \{ 1 \}$. The subsets of A having two elements are $\{-1, 0\}, \{-1, 1\}, \{0, 1\}$. The subset of A having three elements of A is A itself. So, all the subsets of A are ϕ, $\{-1\}$, $\{0\}$, $\{1\}$, $\{-1, 0\}$, $\{-1, 1\}$, $\{0, 1\}$ and $\{-1, 0, 1\}$.

Example 25 Show that $A \cup B = A \cap B$ implies $A = B$

Solution Let $a \in A$. Then $a \in A \cup B$. Since $A \cup B = A \cap B$, $a \in A \cap B$. So $a \in B$. Therefore, $A \subset B$. Similarly, if $b \in B$, then $b \in A \cup B$. Since
$A \cup B = A \cap B$, $b \in A \cap B$. So, $b \in A$. Therefore, $B \subset A$. Thus, $A = B$

Miscellaneous Exercise on Chapter 1

1. Decide, among the following sets, which sets are subsets of one and another:
$A = \{ x : x \in \mathbf{R}$ and x satisfy $x^2 - 8x + 12 = 0 \}$,
$B = \{ 2, 4, 6 \}$, $C = \{ 2, 4, 6, 8, \ldots \}$, $D = \{ 6 \}$.

2. In each of the following, determine whether the statement is true or false. If it is true, prove it. If it is false, give an example.

(i) If $x \in A$ and $A \in B$, then $x \in B$

(ii) If $A \subset B$ and $B \in C$, then $A \in C$

(iii) If $A \subset B$ and $B \subset C$, then $A \subset C$

(iv) If $A \not\subset B$ and $B \not\subset C$, then $A \not\subset C$

(v) If $x \in A$ and $A \not\subset B$, then $x \in B$

(vi) If $A \subset B$ and $x \notin B$, then $x \notin A$

3. Let A, B, and C be the sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$. Show that B = C.

4. Show that the following four conditions are equivalent :

(i) $A \subset B$ (ii) $A - B = \phi$ (iii) $A \cup B = B$ (iv) $A \cap B = A$

5. Show that if $A \subset B$, then $C - B \subset C - A$.

6. Show that for any sets A and B,
$A = (A \cap B) \cup (A - B)$ and $A \cup (B - A) = (A \cup B)$

7. Using properties of sets, show that
(i) $A \cup (A \cap B) = A$ (ii) $A \cap (A \cup B) = A$.

8. Show that $A \cap B = A \cap C$ need not imply $B = C$.

9. Let A and B be sets. If $A \cap X = B \cap X = \phi$ and $A \cup X = B \cup X$ for some set X, show that A = B.
(**Hints** $A = A \cap (A \cup X)$, $B = B \cap (B \cup X)$ and use Distributive law)

10. Find sets A, B and C such that $A \cap B$, $B \cap C$ and $A \cap C$ are non-empty sets and $A \cap B \cap C = \phi$.

Summary

This chapter deals with some basic definitions and operations involving sets. These are summarised below:

- A set is a well-defined collection of objects.
- A set which does not contain any element is called *empty set*.
- A set which consists of a definite number of elements is called *finite set*, otherwise, the set is called *infinite set*.
- Two sets A and B are said to be equal if they have exactly the same elements.
- A set A is said to be subset of a set B, if every element of A is also an element of B. Intervals are subsets of **R**.
- The union of two sets A and B is the set of all those elements which are either in A or in B.
- The intersection of two sets A and B is the set of all elements which are common. The difference of two sets A and B in this order is the set of elements which belong to A but not to B.
- The complement of a subset A of universal set U is the set of all elements of U which are not the elements of A.
- For any two sets A and B, $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$

Historical Note

The modern theory of sets is considered to have been originated largely by the German mathematician Georg Cantor (1845-1918). His papers on set theory appeared sometimes during 1874 to 1897. His study of set theory came when he was studying trigonometric series of the form $a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + ...$ He published in a paper in 1874 that the set of real numbers could not be put into one-to-one correspondence wih the integers. From 1879 onwards, he publishd several papers showing various properties of abstract sets.

Cantor's work was well received by another famous mathematician Richard Dedekind (1831-1916). But Kronecker (1810-1893) castigated him for regarding infinite set the same way as finite sets. Another German mathematician Gottlob Frege, at the turn of the century, presented the set theory as principles of logic. Till then the entire set theory was based on the assumption of the existence of the set of all sets. It was the famous Englih Philosopher Bertand Russell (1872-1970) who showed in 1902 that the assumption of existence of a set of all sets leads to a contradiction. This led to the famous Russell's Paradox. Paul R.Halmos writes about it in his book 'Naïve Set Theory' that "nothing contains everything".

The Russell's Paradox was not the only one which arose in set theory. Many paradoxes were produced later by several mathematicians and logicians. As a consequence of all these paradoxes, the first axiomatisation of set theory was published in 1908 by Ernst Zermelo. Another one was proposed by Abraham Fraenkel in 1922. John Von Neumann in 1925 introduced explicitly the axiom of regularity. Later in 1937 Paul Bernays gave a set of more satisfactory axiomatisation. A modification of these axioms was done by Kurt Gödel in his monograph in 1940. This was known as Von Neumann-Bernays (VNB) or Gödel-Bernays (GB) set theory.

Despite all these difficulties, Cantor's set theory is used in present day mathematics. In fact, these days most of the concepts and results in mathematics are expressed in the set theoretic language.

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Chapter **2**

RELATIONS AND FUNCTIONS

❖Mathematics is the indispensable instrument of all physical research. – BERTHELOT ❖

2.1 Introduction

G. W. Leibnitz (1646–1716)

Much of mathematics is about finding a pattern – a recognisable link between quantities that change. In our daily life, we come across many patterns that characterise relations such as brother and sister, father and son, teacher and student. In mathematics also, we come across many relations such as number m is less than number n, line l is parallel to line m, set A is a subset of set B. In all these, we notice that a relation involves pairs of objects in certain order. In this Chapter, we will learn how to link pairs of objects from two sets and then introduce relations between the two objects in the pair. Finally, we will learn about special relations which will qualify to be functions. The concept of function is very important in mathematics since it captures the idea of a mathematically precise correspondence between one quantity with the other.

2.2 Cartesian Products of Sets

Suppose A is a set of 2 colours and B is a set of 3 objects, i.e.,

$$A = \{\text{red, blue}\} \text{ and } B = \{b, c, s\},$$

where b, c and s represent a particular bag, coat and shirt, respectively.

How many pairs of coloured objects can be made from these two sets? Proceeding in a very orderly manner, we can see that there will be 6 distinct pairs as given below:

(red, b), (red, c), (red, s), (blue, b), (blue, c), (blue, s).

Thus, we get 6 distinct objects (Fig 2.1).

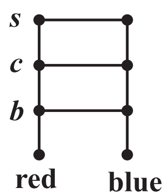


Fig 2.1

Let us recall from our earlier classes that an ordered pair of elements taken from any two sets P and Q is a pair of elements written in small brackets and grouped together in a particular order, i.e., (p,q), $p \in$ P and $q \in$ Q . This leads to the following definition:

Definition 1 Given two non-empty sets P and Q. The cartesian product P × Q is the set of all ordered pairs of elements from P and Q, i.e.,

$$P \times Q = \{ (p,q) : p \in P, q \in Q \}$$

If either P or Q is the null set, then P × Q will also be empty set, i.e., P × Q = ϕ

From the illustration given above we note that

A × B = {(red,b), (red,c), (red,s), (blue,b), (blue,c), (blue,s)}.

Again, consider the two sets:

A = {DL, MP, KA}, where DL, MP, KA represent Delhi, Madhya Pradesh and Karnataka, respectively and B = {01,02, 03}representing codes for the licence plates of vehicles issued by DL, MP and KA .

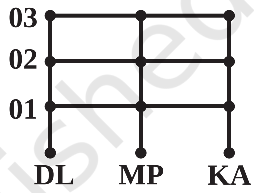


Fig 2.2

If the three states, Delhi, Madhya Pradesh and Karnataka were making codes for the licence plates of vehicles, with the restriction that the code begins with an element from set A, which are the pairs available from these sets and how many such pairs will there be (Fig 2.2)?

The available pairs are:(DL,01), (DL,02), (DL,03), (MP,01), (MP,02), (MP,03), (KA,01), (KA,02), (KA,03) and the product of set A and set B is given by

A × B = {(DL,01), (DL,02), (DL,03), (MP,01), (MP,02), (MP,03), (KA,01), (KA,02), (KA,03)}.

It can easily be seen that there will be 9 such pairs in the Cartesian product, since there are 3 elements in each of the sets A and B. This gives us 9 possible codes. Also note that the order in which these elements are paired is crucial. For example, the code (DL, 01) will not be the same as the code (01, DL).

As a final illustration, consider the two sets A= $\{a_1, a_2\}$ and B = $\{b_1, b_2, b_3, b_4\}$ (Fig 2.3).

A × B = $\{(a_1, b_1), (a_1, b_2), (a_1, b_3), (a_1, b_4), (a_2, b_1), (a_2, b_2), (a_2, b_3), (a_2, b_4)\}$.

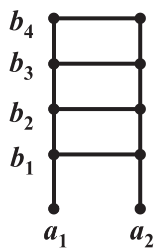


Fig 2.3

The 8 ordered pairs thus formed can represent the position of points in the plane if A and B are subsets of the set of real numbers and it is obvious that the point in the position (a_1, b_2) will be distinct from the point in the position (b_2, a_1).

Remarks

(i) Two ordered pairs are equal, if and only if the corresponding first elements are equal and the second elements are also equal.

(ii) If there are p elements in A and q elements in B, then there will be pq elements in $A \times B$, i.e., if $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$.

(iii) If A and B are non-empty sets and either A or B is an infinite set, then so is $A \times B$.

(iv) $A \times A \times A = \{(a, b, c) : a, b, c \in A\}$. Here (a, b, c) is called an *ordered triplet*.

Example 1 If $(x + 1, y - 2) = (3,1)$, find the values of x and y.

Solution Since the ordered pairs are equal, the corresponding elements are equal.

Therefore $x + 1 = 3$ and $y - 2 = 1$.

Solving we get $x = 2$ and $y = 3$.

Example 2 If $P = \{a, b, c\}$ and $Q = \{r\}$, form the sets $P \times Q$ and $Q \times P$. Are these two products equal?

Solution By the definition of the cartesian product,

$P \times Q = \{(a, r), (b, r), (c, r)\}$ and $Q \times P = \{(r, a), (r, b), (r, c)\}$

Since, by the definition of equality of ordered pairs, the pair (a, r) is not equal to the pair (r, a), we conclude that $P \times Q \neq Q \times P$.

However, the number of elements in each set will be the same.

Example 3 Let $A = \{1,2,3\}$, $B = \{3,4\}$ and $C = \{4,5,6\}$. Find

(i) $A \times (B \cap C)$ (ii) $(A \times B) \cap (A \times C)$

(iii) $A \times (B \cup C)$ (iv) $(A \times B) \cup (A \times C)$

Solution (i) By the definition of the intersection of two sets, $(B \cap C) = \{4\}$.

Therefore, $A \times (B \cap C) = \{(1,4), (2,4), (3,4)\}$.

(ii) Now $(A \times B) = \{(1,3), (1,4), (2,3), (2,4), (3,3), (3,4)\}$

and $(A \times C) = \{(1,4), (1,5), (1,6), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6)\}$

Therefore, $(A \times B) \cap (A \times C) = \{(1, 4), (2, 4), (3, 4)\}$.

(iii) Since, $(B \cup C) = \{3, 4, 5, 6\}$, we have

$A \times (B \cup C) = \{(1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,3), (3,4), (3,5), (3,6)\}$.

(iv) Using the sets $A \times B$ and $A \times C$ from part (ii) above, we obtain

$(A \times B) \cup (A \times C) = \{(1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,3), (3,4), (3,5), (3,6)\}$.

Example 4 If P = {1, 2}, form the set P × P × P.

Solution We have, $P \times P \times P = \{(1,1,1), (1,1,2), (1,2,1), (1,2,2), (2,1,1), (2,1,2), (2,2,1), (2,2,2)\}$.

Example 5 If **R** is the set of all real numbers, what do the cartesian products **R** × **R** and **R** × **R** × **R** represent?

Solution The Cartesian product **R** × **R** represents the set $\mathbf{R} \times \mathbf{R} = \{(x, y) : x, y \in \mathbf{R}\}$ which represents the *coordinates of all the points in two dimensional space* and the cartesian product **R** × **R** × **R** represents the set $\mathbf{R} \times \mathbf{R} \times \mathbf{R} = \{(x, y, z) : x, y, z \in \mathbf{R}\}$ which represents the *coordinates of all the points in three-dimensional space.*

Example 6 If $A \times B = \{(p, q), (p, r), (m, q), (m, r)\}$, find A and B.

Solution A = set of first elements = $\{p, m\}$
B = set of second elements = $\{q, r\}$.

EXERCISE 2.1

1. If $\left(\frac{x}{3}+1, y-\frac{2}{3}\right)=\left(\frac{5}{3}, \frac{1}{3}\right)$, find the values of x and y.
2. If the set A has 3 elements and the set B = {3, 4, 5}, then find the number of elements in (A×B).
3. If G = {7, 8} and H = {5, 4, 2}, find G × H and H × G.
4. State whether each of the following statements are true or false. If the statement is false, rewrite the given statement correctly.
   (i) If P = $\{m, n\}$ and Q = $\{n, m\}$, then P × Q = $\{(m, n), (n, m)\}$.
   (ii) If A and B are non-empty sets, then A × B is a non-empty set of ordered pairs (x, y) such that $x \in$ A and $y \in$ B.
   (iii) If A = {1, 2}, B = {3, 4}, then $A \times (B \cap \phi) = \phi$.
5. If A = {–1, 1}, find A × A × A.
6. If $A \times B = \{(a, x), (a, y), (b, x), (b, y)\}$. Find A and B.
7. Let A = {1, 2}, B = {1, 2, 3, 4}, C = {5, 6} and D = {5, 6, 7, 8}. Verify that
   (i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$. (ii) A × C is a subset of B × D.
8. Let A = {1, 2} and B = {3, 4}. Write A × B. How many subsets will A × B have? List them.
9. Let A and B be two sets such that $n(A) = 3$ and $n(B) = 2$. If $(x, 1), (y, 2), (z, 1)$ are in A × B, find A and B, where x, y and z are distinct elements.

10. The Cartesian product A × A has 9 elements among which are found (–1, 0) and (0,1). Find the set A and the remaining elements of A × A.

2.3 Relations

Consider the two sets P = {a, b, c} and Q = {Ali, Bhanu, Binoy, Chandra, Divya}. The cartesian product of P and Q has 15 ordered pairs which can be listed as P × Q = {(a, Ali), (a, Bhanu), (a, Binoy), ..., (c, Divya)}.

We can now obtain a subset of P × Q by introducing a relation R between the first element x and the second element y of each ordered pair (x, y) as

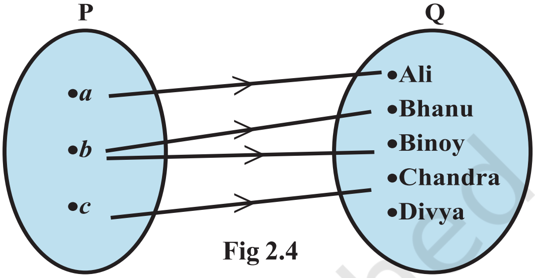


Fig 2.4

R= { (x,y): x is the first letter of the name y, $x \in$ P, $y \in$ Q}.

Then R = {(a, Ali), (b, Bhanu), (b, Binoy), (c, Chandra)}

A visual representation of this relation R (called an *arrow diagram*) is shown in Fig 2.4.

Definition 2 A relation R from a non-empty set A to a non-empty set B is a subset of the cartesian product A × B. The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in A × B. The second element is called the *image* of the first element.

Definition 3 The set of all first elements of the ordered pairs in a relation R from a set A to a set B is called the *domain* of the relation R.

Definition 4 The set of all second elements in a relation R from a set A to a set B is called the *range* of the relation R. The whole set B is called the *codomain* of the relation R. Note that range $\subset$ codomain.

Remarks (i) A *relation* may be represented algebraically either by the *Roster method* or by the *Set-builder method*.

(ii) An arrow diagram is a visual representation of a relation.

Example 7 Let A = {1, 2, 3, 4, 5, 6}. Define a relation R from A to A by

R = {(x, y) : y = x + 1 }

(i) Depict this relation using an arrow diagram.

(ii) Write down the domain, codomain and range of R.

Solution (i) By the definition of the relation,

R = {(1,2), (2,3), (3,4), (4,5), (5,6)}.

The corresponding arrow diagram is shown in Fig 2.5.

(ii) We can see that the domain $=\{1, 2, 3, 4, 5,\}$

Similarly, the range $= \{2, 3, 4, 5, 6\}$ and the codomain $= \{1, 2, 3, 4, 5, 6\}$.

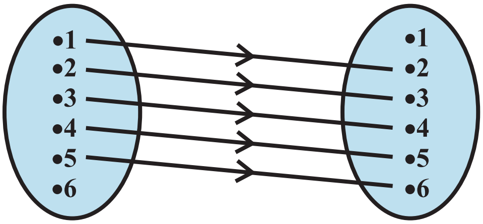


Fig 2.5

Example 8 The Fig 2.6 shows a relation between the sets P and Q. Write this relation (i) in set-builder form, (ii) in roster form. What is its domain and range?

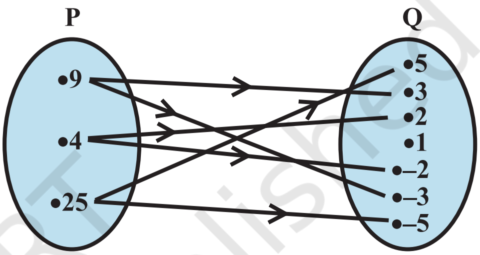


Fig 2.6

Solution It is obvious that the relation R is "x is the square of y".

(i) In set-builder form, R $= \{(x, y): x$ is the square of $y, x \in$ P, $y \in$ **Q**$\}$

(ii) In roster form, R $= \{(9, 3),$ $(9, -3), (4, 2), (4, -2), (25, 5), (25, -5)\}$

The domain of this relation is $\{4, 9, 25\}$.

The range of this relation is $\{-2, 2, -3, 3, -5, 5\}$.

Note that the element 1 is not related to any element in set P.

The set Q is the codomain of this relation.

Note The total number of relations that can be defined from a set A to a set B is the number of possible subsets of A × B. If $n(\text{A}) = p$ and $n(\text{B}) = q$, then $n(\text{A} \times \text{B}) = pq$ and the total number of relations is 2^{pq}.

Example 9 Let A $= \{1, 2\}$ and B $= \{3, 4\}$. Find the number of relations from A to B.

Solution We have,

$$\text{A} \times \text{B} = \{(1, 3), (1, 4), (2, 3), (2, 4)\}.$$

Since $n(\text{A}\times\text{B}) = 4$, the number of subsets of A×B is 2^4. Therefore, the number of relations from A into B will be 2^4.

Remark A relation R from A to A is also stated as a relation on A.

EXERCISE 2.2

1. Let A $= \{1, 2, 3,...,14\}$. Define a relation R from A to A by R $= \{(x, y) : 3x - y = 0$, where $x, y \in \text{A}\}$. Write down its domain, codomain and range.

2. Define a relation R on the set **N** of natural numbers by R = $\{(x, y) : y = x + 5$, x is a natural number less than 4; $x, y \in \mathbf{N}\}$. Depict this relationship using roster form. Write down the domain and the range.

3. A = {1, 2, 3, 5} and B = {4, 6, 9}. Define a relation R from A to B by R = $\{(x, y)$: the difference between x and y is odd; $x \in$ A, $y \in$ B$\}$. Write R in roster form.

4. The Fig2.7 shows a relationship between the sets P and Q. Write this relation

(i) in set-builder form (ii) roster form.
What is its domain and range?

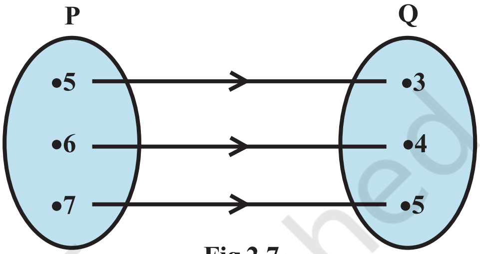


Fig 2.7

5. Let A = {1, 2, 3, 4, 6}. Let R be the relation on A defined by
$\{(a, b): a, b \in$A, b is exactly divisible by $a\}$.

(i) Write R in roster form

(ii) Find the domain of R

(iii) Find the range of R.

6. Determine the domain and range of the relation R defined by R = $\{(x, x + 5) : x \in \{0, 1, 2, 3, 4, 5\}\}$.

7. Write the relation R = $\{(x, x^3) : x$ is a prime number less than 10$\}$ in roster form.

8. Let A = $\{x, y, z\}$ and B = {1, 2}. Find the number of relations from A to B.

9. Let R be the relation on **Z** defined by R = $\{(a,b): a, b \in \mathbf{Z}, a - b$ is an integer$\}$. Find the domain and range of R.

2.4 Functions

In this Section, we study a special type of relation called *function*. It is one of the most important concepts in mathematics. We can, visualise a function as a rule, which produces new elements out of some given elements. There are many terms such as 'map' or 'mapping' used to denote a function.

Definition 5 A relation f from a set A to a set B is said to be a *function* if every element of set A has one and only one image in set B.

In other words, a function f is a relation from a non-empty set A to a non-empty set B such that the domain of f is A and no two distinct ordered pairs in f have the same first element.

If f is a function from A to B and $(a, b) \in f$, then $f(a) = b$, where b is called the *image* of a under f and a is called the *preimage* of b under f.

The function f from A to B is denoted by f: A → B.

Looking at the previous examples, we can easily see that the relation in Example 7 is not a function because the element 6 has no image.

Again, the relation in Example 8 is not a function because the elements in the domain are connected to more than one images. Similarly, the relation in Example 9 is also not a function. (*Why*?) In the examples given below, we will see many more relations some of which are functions and others are not.

Example 10 Let **N** be the set of natural numbers and the relation R be defined on N such that $R = \{(x, y) : y = 2x, x, y \in \mathbf{N}\}$.

What is the domain, codomain and range of R? Is this relation a function?

Solution The domain of R is the set of natural numbers **N**. The codomain is also **N**. The range is the set of even natural numbers.

Since every natural number n has one and only one image, this relation is a function.

Example 11 Examine each of the following relations given below and state in each case, giving reasons whether it is a function or not?

(i) $R = \{(2,1),(3,1), (4,2)\}$, (ii) $R = \{(2,2),(2,4),(3,3), (4,4)\}$

(iii) $R = \{(1,2),(2,3),(3,4), (4,5), (5,6), (6,7)\}$

Solution (i) Since 2, 3, 4 are the elements of domain of R having their unique images, this relation R is a function.

(ii) Since the same first element 2 corresponds to two different images 2 and 4, this relation is not a function.

(iii) Since every element has one and only one image, this relation is a function.

Definition 6 A function which has either R or one of its subsets as its range is called a *real valued function*. Further, if its domain is also either R or a subset of R, it is called a *real function*.

Example 12 Let **N** be the set of natural numbers. Define a real valued function $f : \mathbf{N} \to \mathbf{N}$ by $f(x) = 2x + 1$. Using this definition, complete the table given below.

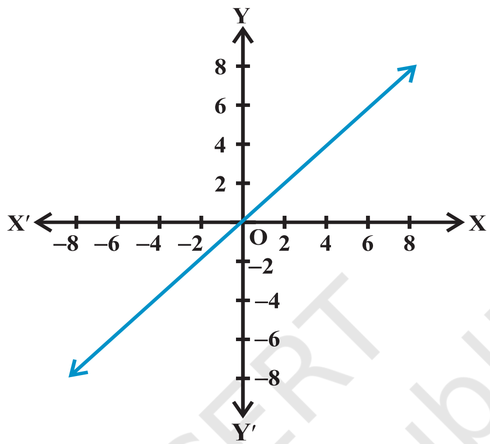
x	1	2	3	4	5	6	7
y	$f(1) = ...$	$f(2) = ...$	$f(3) = ...$	$f(4) = ...$	$f(5) = ...$	$f(6) = ...$	$f(7) = ...$

Solution The completed table is given by

x	1	2	3	4	5	6	7
y	$f(1) = 3$	$f(2) = 5$	$f(3) = 7$	$f(4) = 9$	$f(5) = 11$	$f(6) = 13$	$f(7) = 15$

2.4.1 *Some functions and their graphs*

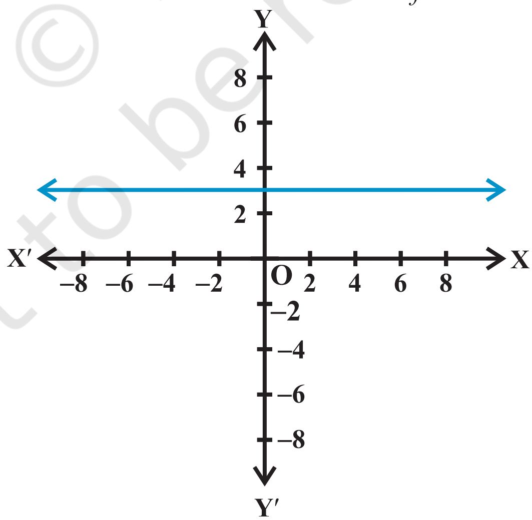
(i) **Identity function** Let **R** be the set of real numbers. Define the real valued function $f : \mathbf{R} \to \mathbf{R}$ by $y = f(x) = x$ for each $x \in \mathbf{R}$. Such a function is called the *identity function*. Here the domain and range of f are **R**. The graph is a straight line as shown in Fig 2.8. It passes through the origin.



$f(x) = x$

Fig 2.8

(ii) **Constant function** Define the function f: $\mathbf{R} \to \mathbf{R}$ by $y = f(x) = c$, $x \in \mathbf{R}$ where c is a constant and each $x \in \mathbf{R}$. Here domain of f is **R** and its range is $\{c\}$.



$f(x) = 3$

Fig 2.9

The graph is a line parallel to x-axis. For example, if $f(x)=3$ for each $x \in \mathbf{R}$, then its graph will be a line as shown in the Fig 2.9.

(iii) **Polynomial function** A function $f : \mathbf{R} \to \mathbf{R}$ is said to be *polynomial function* if for each x in $\mathbf{R}$, $y = f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$, where n is a non-negative integer and $a_0, a_1, a_2, ..., a_n \in \mathbf{R}$.

The functions defined by $f(x) = x^3 - x^2 + 2$, and $g(x) = x^4 + \sqrt{2}\, x$ are some examples of polynomial functions, whereas the function h defined by $h(x) = x^{\frac{2}{3}} + 2x$ is not a polynomial function.(*Why*?)

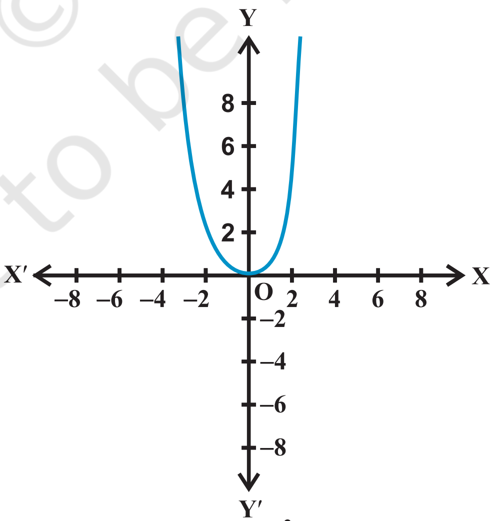
Example 13 Define the function f: $\mathbf{R} \to \mathbf{R}$ by $y = f(x) = x^2$, $x \in \mathbf{R}$. Complete the Table given below by using this definition. What is the domain and range of this function? Draw the graph of f.

x	-4	-3	-2	-1	0	1	2	3	4
$y = f(x) = x^2$									

Solution The completed Table is given below:

x	-4	-3	-2	-1	0	1	2	3	4
$y = f(x) = x^2$	16	9	4	1	0	1	4	9	16

Domain of $f = \{x : x \in \mathbf{R}\}$. Range of $f = \{x^2 : x \in \mathbf{R}\}$. The graph of f is given by Fig 2.10



$f(x) = x^2$ **Fig 2.10**

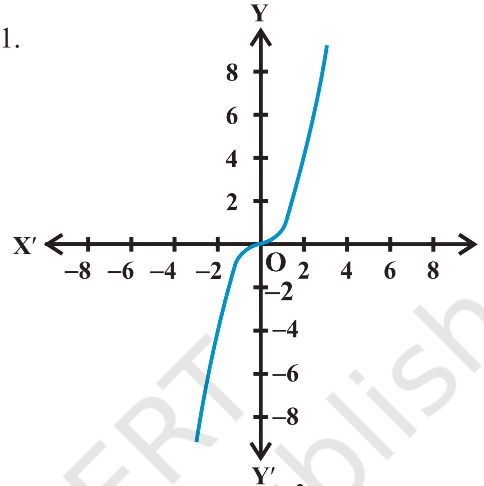
Example 14 Draw the graph of the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^3$, $x \in \mathbf{R}$.

Solution We have

$f(0) = 0$, $f(1) = 1$, $f(-1) = -1$, $f(2) = 8$, $f(-2) = -8$, $f(3) = 27$; $f(-3) = -27$, etc.

Therefore, $f = \{(x, x^3): x \in \mathbf{R}\}$.

The graph of f is given in Fig 2.11.



$f(x) = x^3$

Fig 2.11

(iv) **Rational functions** are functions of the type $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions of x defined in a domain, where $g(x) \neq 0$.

Example 15 Define the real valued function $f : \mathbf{R} - \{0\} \to \mathbf{R}$ defined by $f(x) = \frac{1}{x}$, $x \in \mathbf{R} - \{0\}$. Complete the Table given below using this definition. What is the domain and range of this function?

x	–2	–1.5	–1	–0.5	0.25	0.5	1	1.5	2
$y = \frac{1}{x}$	...	...	...	...	...	...	...	...	...

Solution The completed Table is given by

x	–2	–1.5	–1	–0.5	0.25	0.5	1	1.5	2
$y = \frac{1}{x}$	– 0.5	– 0.67	–1	– 2	4	2	1	0.67	0.5

The domain is all real numbers except 0 and its range is also all real numbers except 0. The graph of f is given in Fig 2.12.

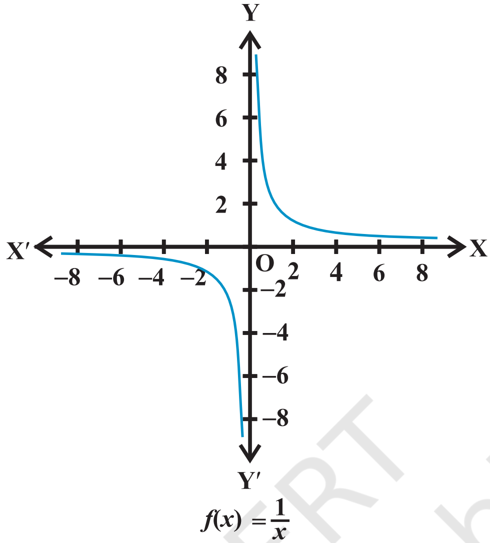


Fig 2.12

(v) **The Modulus function** The function f: $\mathbf{R}\to\mathbf{R}$ defined by $f(x) = |x|$ for each $x \in \mathbf{R}$ is called *modulus function*. For each non-negative value of x, $f(x)$ is equal to x. But for negative values of x, the value of $f(x)$ is the negative of the value of x, i.e.,

$$f(x) = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$$

The graph of the modulus function is given in Fig 2.13.

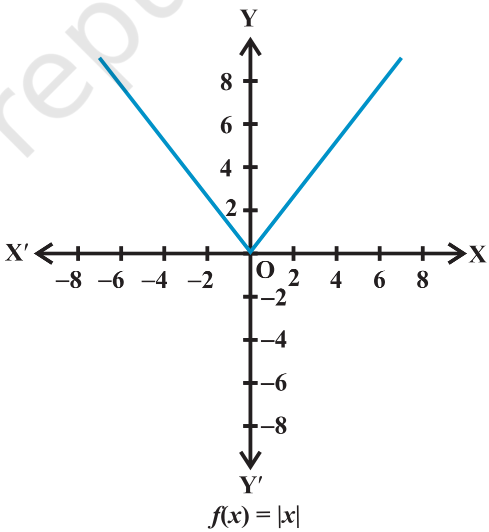
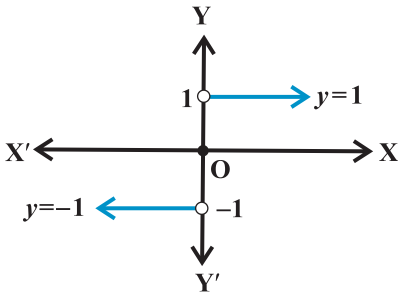


Fig 2.13

(vi) **Signum function** The function f:$\mathbf{R}\to\mathbf{R}$ defined by

$$f(x) = \begin{cases} 1, \text{if } x > 0 \\ 0, \text{if } x = 0 \\ -1, \text{if } x < 0 \end{cases}$$

is called the *signum function*. The domain of the signum function is **R** and the range is the set {–1, 0, 1}. The graph of the signum function is given by the Fig 2.14.



$$f(x) = \frac{|x|}{x}, x \neq 0 \text{ and } 0 \text{ for } x = 0$$

Fig 2.14

(vii) **Greatest integer function** The function f: **R** → **R** defined by $f(x) = [x]$, $x \in$ **R** assumes the value of the greatest integer, less than or equal to x. Such a function is called the *greatest integer function*.

From the definition of $[x]$, we can see that

$[x] = –1$ for $–1 \leq x < 0$

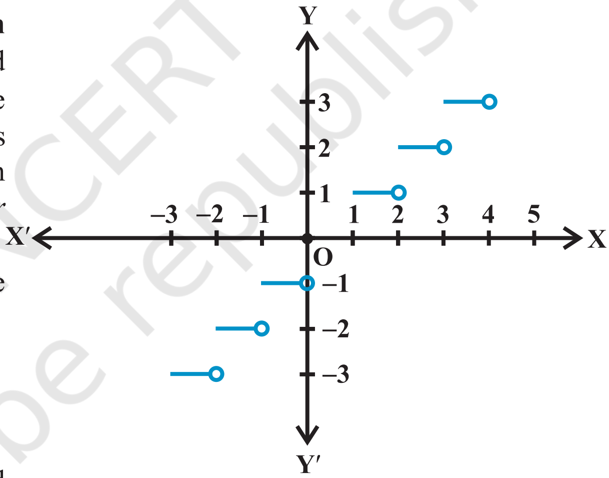
$[x] = 0$ for $0 \leq x < 1$

$[x] = 1$ for $1 \leq x < 2$

$[x] = 2$ for $2 \leq x < 3$ and

so on.

The graph of the function is shown in Fig 2.15.



$f(x) = [x]$

Fig 2.15

2.4.2 *Algebra of real functions*

In this Section, we shall learn how to add two real functions, subtract a real function from another, multiply a real function by a scalar (here by a scalar we mean a real number), multiply two real functions and divide one real function by another.

(i) **Addition of two real functions** Let f : X → **R** and g : X → **R** be any two real functions, where X ⊂ **R**. Then, we define $(f + g)$: X → **R** by

$(f + g)(x) = f(x) + g(x)$, for all $x \in$ X.

(ii) **Subtraction of a real function from another** Let $f : X \to \mathbf{R}$ and $g: X \to \mathbf{R}$ be any two real functions, where $X \subset \mathbf{R}$. Then, we define $(f - g) : X \to \mathbf{R}$ by $(f-g)(x) = f(x) - g(x)$, for all $x \in X$.

(iii) **Multiplication by a scalar** Let $f : X \to \mathbf{R}$ be a real valued function and α be a scalar. Here by scalar, we mean a real number. Then the product αf is a function from X to **R** defined by $(\alpha f)(x) = \alpha f(x)$, $x \in X$.

(iv) **Multiplication of two real functions** The product (or multiplication) of two real functions $f: X \to \mathbf{R}$ and $g: X \to \mathbf{R}$ is a function $fg: X \to \mathbf{R}$ defined by $(fg)(x) = f(x)\, g(x)$, for all $x \in X$.
This is also called *pointwise multiplication.*

(v) **Quotient of two real functions** Let f and g be two real functions defined from $X \to \mathbf{R}$, where $X \subset \mathbf{R}$. The quotient of f by g denoted by $\frac{f}{g}$ is a function defined by ,

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$, $x \in X$

Example 16 Let $f(x) = x^2$ and $g(x) = 2x + 1$ be two real functions.Find

$$(f + g)(x), (f - g)(x), (fg)(x), \left(\frac{f}{g}\right)(x).$$

Solution We have,

$$(f + g)(x) = x^2 + 2x + 1, \ (f - g)(x) = x^2 - 2x - 1,$$

$$(fg)(x) = x^2 (2x + 1) = 2x^3 + x^2, \ \left(\frac{f}{g}\right)(x) = \frac{x^2}{2x+1}, x \neq -\frac{1}{2}$$

Example 17 Let $f(x) = \sqrt{x}$ and $g(x) = x$ be two functions defined over the set of non-negative real numbers. Find $(f + g)(x)$, $(f - g)(x)$, $(fg)(x)$ and $\left(\frac{f}{g}\right)(x)$.

Solution We have

$$(f + g)(x) = \sqrt{x} + x, \ (f - g)(x) = \sqrt{x} - x,$$

$$(fg)\, x = \sqrt{x}(x) = x^{\frac{3}{2}} \text{ and } \left(\frac{f}{g}\right)(x) = \frac{\sqrt{x}}{x} = x^{-\frac{1}{2}}, x \neq 0$$

EXERCISE 2.3

1. Which of the following relations are functions? Give reasons. If it is a function, determine its domain and range.

(i) $\{(2,1), (5,1), (8,1), (11,1), (14,1), (17,1)\}$

(ii) $\{(2,1), (4,2), (6,3), (8,4), (10,5), (12,6), (14,7)\}$

(iii) $\{(1,3), (1,5), (2,5)\}$.

2. Find the domain and range of the following real functions:

(i) $f(x) = -|x|$ (ii) $f(x) = \sqrt{9-x^2}$.

3. A function f is defined by $f(x) = 2x - 5$. Write down the values of

(i) $f(0)$, (ii) $f(7)$, (iii) $f(-3)$.

4. The function 't' which maps temperature in degree Celsius into temperature in degree Fahrenheit is defined by $t(\text{C}) = \dfrac{9\text{C}}{5} + 32$.

Find (i) $t(0)$ (ii) $t(28)$ (iii) $t(-10)$ (iv) The value of C, when $t(\text{C}) = 212$.

5. Find the range of each of the following functions.

(i) $f(x) = 2 - 3x$, $x \in \mathbf{R}$, $x > 0$.

(ii) $f(x) = x^2 + 2$, x is a real number.

(iii) $f(x) = x$, x is a real number.

Miscellaneous Examples

Example 18 Let **R** be the set of real numbers. Define the real function

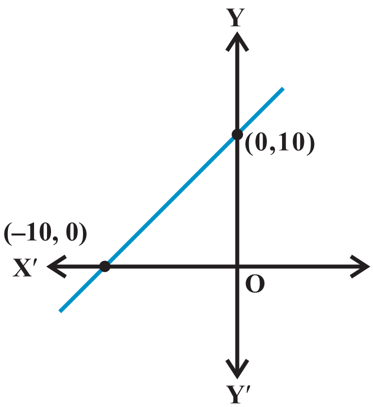
$$f: \mathbf{R} \rightarrow \mathbf{R} \text{ by } f(x) = x + 10$$

and sketch the graph of this function.

Solution Here $f(0) = 10$, $f(1) = 11$, $f(2) = 12$, ..., $f(10) = 20$, etc., and

$f(-1) = 9$, $f(-2) = 8$, ..., $f(-10) = 0$ and so on.

Therefore, shape of the graph of the given function assumes the form as shown in Fig 2.16.



$f(x) = x + 10$

Fig 2.16

Remark The function f defined by $f(x) = mx + c$, $x \in \mathbf{R}$, is called *linear function*, where m and c are constants. Above function is an example of a *linear function*.

Example 19 Let R be a relation from **Q** to **Q** defined by R = $\{(a,b): a,b \in \mathbf{Q}$ and $a - b \in \mathbf{Z}\}$. Show that

(i) $(a,a) \in$ R for all $a \in \mathbf{Q}$

(ii) $(a,b) \in$ R implies that $(b, a) \in$ R

(iii) $(a,b) \in$ R and $(b,c) \in$ R implies that $(a,c) \in$R

Solution (i) Since, $a - a = 0 \in \mathbf{Z}$, if follows that $(a, a) \in$ R.

(ii) $(a,b) \in$ R implies that $a - b \in \mathbf{Z}$. So, $b - a \in \mathbf{Z}$. Therefore, $(b, a) \in$ R

(iii) (a, b) and $(b, c) \in$ R implies that $a - b \in \mathbf{Z}$. $b - c \in \mathbf{Z}$. So, $a - c = (a - b) + (b - c) \in \mathbf{Z}$. Therefore, $(a,c) \in$ R

Example 20 Let $f = \{(1,1), (2,3), (0,-1), (-1,-3)\}$ be a linear function from **Z** into **Z**. Find $f(x)$.

Solution Since f is a linear function, $f(x) = mx + c$. Also, since $(1, 1), (0, -1) \in$ R, $f(1) = m + c = 1$ and $f(0) = c = -1$. This gives $m = 2$ and $f(x) = 2x - 1$.

Example 21 Find the domain of the function $f(x) = \dfrac{x^2 + 3x + 5}{x^2 - 5x + 4}$

Solution Since $x^2 - 5x + 4 = (x - 4)(x - 1)$, the function f is defined for all real numbers except at $x = 4$ and $x = 1$. Hence the domain of f is $\mathbf{R} - \{1, 4\}$.

Example 22 The function f is defined by

$$f(x) = \begin{cases} 1 - x, & x < 0 \\ 1 & , x = 0 \\ x + 1, & x > 0 \end{cases}$$

Draw the graph of $f(x)$.

Solution Here, $f(x) = 1 - x, x < 0$, this gives

$f(-4) = 1 - (-4) = 5;$

$f(-3) = 1 - (-3) = 4,$

$f(-2) = 1 - (-2) = 3$

$f(-1) = 1 - (-1) = 2;$ etc,

and $f(1) = 2, f(2) = 3, f(3) = 4$

$f(4) = 5$ and so on for $f(x) = x + 1, x > 0$.

Thus, the graph of f is as shown in Fig 2.17

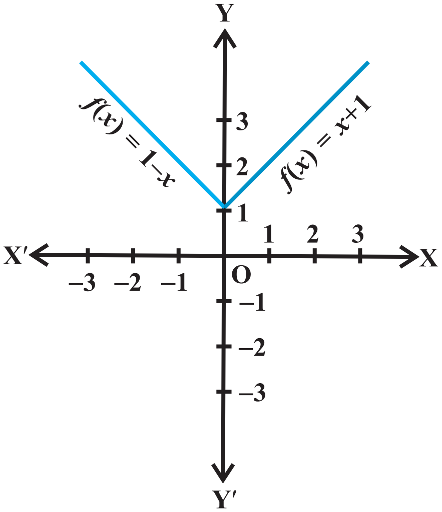


Fig 2.17

Miscellaneous Exercise on Chapter 2

1. The relation f is defined by $f(x) = \begin{cases} x^2, & 0 \le x \le 3 \\ 3x, & 3 \le x \le 10 \end{cases}$

The relation g is defined by $g(x) = \begin{cases} x^2, & 0 \le x \le 2 \\ 3x, & 2 \le x \le 10 \end{cases}$

Show that f is a function and g is not a function.

2. If $f(x) = x^2$, find $\dfrac{f(1.1) - f(1)}{(1.1 - 1)}$.

3. Find the domain of the function $f(x) = \dfrac{x^2 + 2x + 1}{x^2 - 8x + 12}$.

4. Find the domain and the range of the real function f defined by $f(x) = \sqrt{(x-1)}$.

5. Find the domain and the range of the real function f defined by $f(x) = |x - 1|$.

6. Let $f = \left\{\left(x, \dfrac{x^2}{1+x^2}\right) : x \in \mathbf{R}\right\}$ be a function from **R** into **R**. Determine the range of f.

7. Let f, g : **R**→**R** be defined, respectively by $f(x) = x + 1$, $g(x) = 2x - 3$. Find $f + g$, $f - g$ and $\dfrac{f}{g}$.

8. Let $f = \{(1,1), (2,3), (0,-1), (-1, -3)\}$ be a function from **Z** to **Z** defined by $f(x) = ax + b$, for some integers a, b. Determine a, b.

9. Let R be a relation from **N** to **N** defined by R = $\{(a, b) : a, b \in \mathbf{N}$ and $a = b^2\}$. Are the following true?

(i) $(a,a) \in$ R, for all $a \in \mathbf{N}$ (ii) $(a,b) \in$ R, implies $(b,a) \in$ R

(iii) $(a,b) \in$ R, $(b,c) \in$ R implies $(a,c) \in$ R.

Justify your answer in each case.

10. Let A = {1,2,3,4}, B = {1,5,9,11,15,16} and $f = \{(1,5), (2,9), (3,1), (4,5), (2,11)\}$ Are the following true?

(i) f is a relation from A to B (ii) f is a function from A to B.

Justify your answer in each case.

11. Let f be the subset of $\mathbf{Z} \times \mathbf{Z}$ *defined by* $f = \{(ab, a + b) : a, b \in \mathbf{Z}\}$. Is f a function from $\mathbf{Z}$ to $\mathbf{Z}$? Justify your answer.

12. Let A = {9,10,11,12,13} and let f : A$\rightarrow$**N** be defined by $f(n)$ = the highest prime factor of n. Find the range of f.

Summary

In this Chapter, we studied about relations and functions. The main features of this Chapter are as follows:

- ***Ordered pair*** A pair of elements grouped together in a particular order.
- ***Cartesian product*** A × B of two sets A and B is given by

  $A \times B = \{(a, b): a \in A, b \in B\}$

  In particular $\mathbf{R} \times \mathbf{R} = \{(x, y): x, y \in \mathbf{R}\}$

  and $\mathbf{R} \times \mathbf{R} \times \mathbf{R} = \{(x, y, z): x, y, z \in \mathbf{R}\}$
- If $(a, b) = (x, y)$, then $a = x$ and $b = y$.
- If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$.
- $A \times \phi = \phi$
- In general, $A \times B \neq B \times A$.
- ***Relation*** A relation R from a set A to a set B is a subset of the cartesian product A × B obtained by describing a relationship between the first element x and the second element y of the ordered pairs in A × B.
- The ***image*** of an element x under a relation R is given by y, where $(x, y) \in$ R,
- The ***domain*** of R is the set of all first elements of the ordered pairs in a relation R.
- The ***range*** of the relation R is the set of all second elements of the ordered pairs in a relation R.
- ***Function*** A function f from a set A to a set B is a specific type of relation for which every element x of set A has one and only one image y in set B.

  We write f: A$\rightarrow$B, where $f(x) = y$.
- A is the domain and B is the codomain of f.

- The range of the function is the set of images.
- A real function has the set of real numbers or one of its subsets both as its domain and as its range.
- ***Algebra of functions*** For functions $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$, we have

$$(f + g)(x) = f(x) + g(x),\ x \in X$$

$$(f - g)(x) = f(x) - g(x),\ x \in X$$

$$(f.g)(x) = f(x).g(x),\ x \in X$$

$$(kf)(x) = k(f(x)),\ x \in X, \text{ where } k \text{ is a real number.}$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},\ x \in X,\ g(x) \neq 0$$

Historical Note

The word FUNCTION first appears in a Latin manuscript "Methodus tangentium inversa, seu de fuctionibus" written by Gottfried Wilhelm Leibnitz (1646-1716) in 1673; Leibnitz used the word in the non-analytical sense. He considered a function in terms of "mathematical job" – the "employee" being just a curve.

On July 5, 1698, Johan Bernoulli, in a letter to Leibnitz, for the first time deliberately assigned a specialised use of the term *function* in the analytical sense. At the end of that month, Leibnitz replied showing his approval.

Function is found in English in 1779 in Chambers' Cyclopaedia: "The term function is used in algebra, for an analytical expression any way compounded of a variable quantity, and of numbers, or constant quantities".

Chapter 3

TRIGONOMETRIC FUNCTIONS

❖*A mathematician knows how to solve a problem, he can not solve it. – MILNE* ❖

3.1 Introduction

The word 'trigonometry' is derived from the Greek words '*trigon*' and '*metron*' and it means 'measuring the sides of a triangle'. The subject was originally developed to solve geometric problems involving triangles. It was studied by sea captains for navigation, surveyor to map out the new lands, by engineers and others. Currently, trigonometry is used in many areas such as the science of seismology, designing electric circuits, describing the state of an atom, predicting the heights of tides in the ocean, analysing a musical tone and in many other areas.

Arya Bhatt (476-550)

In earlier classes, we have studied the trigonometric ratios of acute angles as the ratio of the sides of a right angled triangle. We have also studied the trigonometric identities and application of trigonometric ratios in solving the problems related to heights and distances. In this Chapter, we will generalise the concept of trigonometric ratios to trigonometric functions and study their properties.

3.2 Angles

Angle is a measure of rotation of a given ray about its initial point. The original ray is

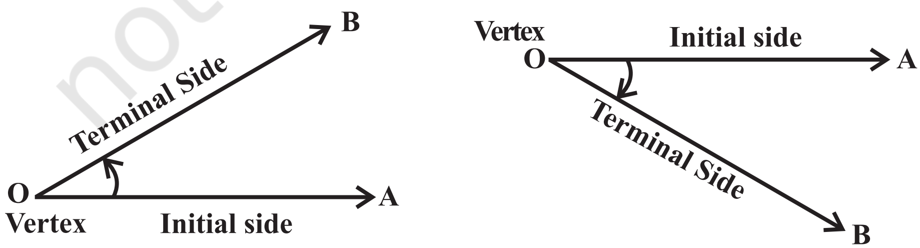


Fig 3.1

called the *initial side* and the final position of the ray after rotation is called the *terminal side* of the angle. The point of rotation is called the *vertex*. If the direction of rotation is anticlockwise, the angle is said to be positive and if the direction of rotation is clockwise, then the angle is *negative* (Fig 3.1).

The measure of an angle is the amount of rotation performed to get the terminal side from the initial side. There are several units for measuring angles. The definition of an angle suggests a unit, viz. *one complete revolution* from the position of the initial side as indicated in Fig 3.2.

Fig 3.2

This is often convenient for large angles. For example, we can say that a rapidly spinning wheel is making an angle of say 15 revolution per second. We shall describe two other units of measurement of an angle which are most commonly used, viz. degree measure and radian measure.

3.2.1 *Degree measure*

If a rotation from the initial side to terminal side is $\left(\frac{1}{360}\right)^{\text{th}}$ of a revolution, the angle is said to have a measure of one *degree*, written as 1°. A degree is divided into 60 minutes, and a minute is divided into 60 seconds . One sixtieth of a degree is called a *minute*, written as 1′, and one sixtieth of a minute is called a *second*, written as 1″.

Thus, $1° = 60′$, $1′ = 60″$

Some of the angles whose measures are 360°,180°, 270°, 420°, – 30°, – 420° are shown in Fig 3.3.

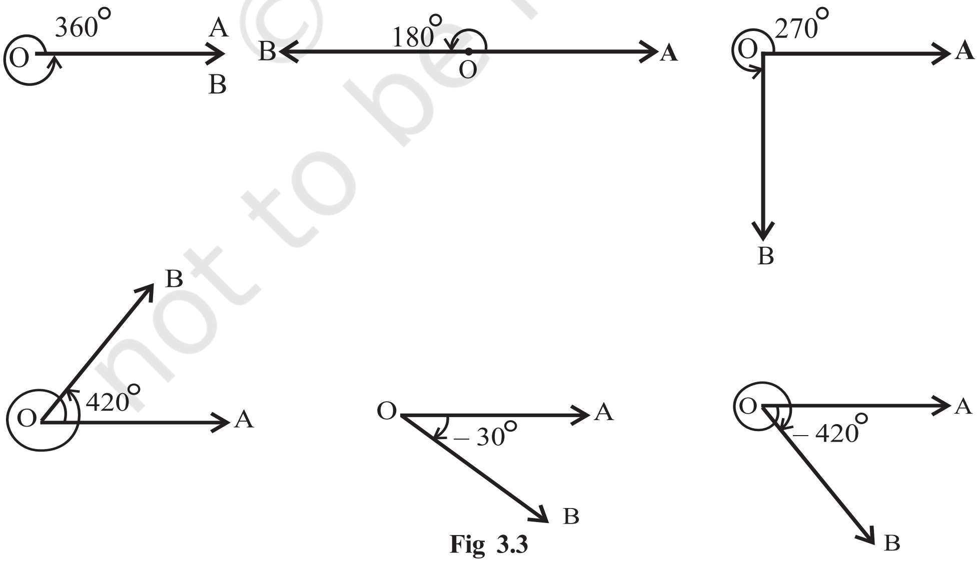


Fig 3.3

3.2.2 *Radian measure* There is another unit for measurement of an angle, called the *radian* measure. Angle subtended at the centre by an arc of length 1 unit in a unit circle (circle of radius 1 unit) is said to have a measure of 1 radian. In the Fig 3.4(i) to (iv), OA is the initial side and OB is the terminal side. The figures show the angles whose measures are 1 radian, –1 radian, $1\frac{1}{2}$ radian and $-1\frac{1}{2}$ radian.

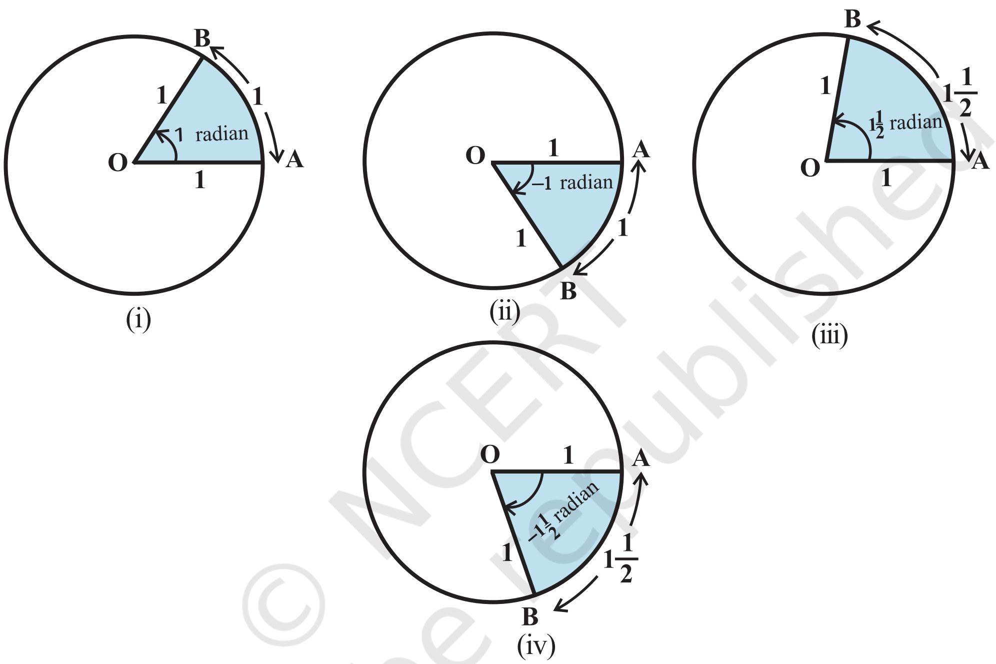


Fig 3.4 (i) to (iv)

We know that the circumference of a circle of radius 1 unit is 2π. Thus, one complete revolution of the initial side subtends an angle of 2π radian.

More generally, in a circle of radius r, an arc of length r will subtend an angle of 1 radian. It is well-known that equal arcs of a circle subtend equal angle at the centre. Since in a circle of radius r, an arc of length r subtends an angle whose measure is 1 radian, an arc of length l will subtend an angle whose measure is $\frac{l}{r}$ radian. Thus, if in a circle of radius r, an arc of length l subtends an angle θ radian at the centre, we have

$$\theta = \frac{l}{r} \text{ or } l = r\,\theta.$$

3.2.3 *Relation between radian and real numbers*

Consider the unit circle with centre O. Let A be any point on the circle. Consider OA as initial side of an angle. Then the length of an arc of the circle will give the radian measure of the angle which the arc will subtend at the centre of the circle. Consider the line PAQ which is tangent to the circle at A. Let the point A represent the real number zero, AP represents positive real number and AQ represents negative real numbers (Fig 3.5). If we rope the line AP in the anticlockwise direction along the circle, and AQ in the clockwise direction, then every real number will correspond to a radian measure and conversely. Thus, radian measures and real numbers can be considered as one and the same.

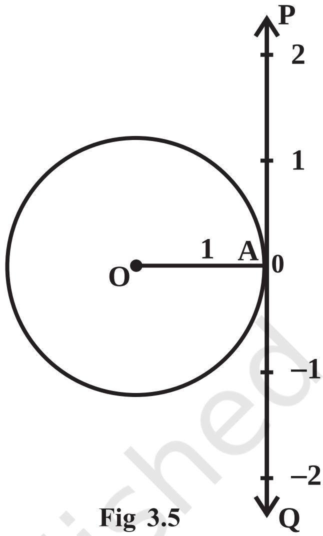


Fig 3.5

3.2.4 *Relation between degree and radian*

Since a circle subtends at the centre an angle whose radian measure is 2π and its degree measure is 360°, it follows that

$$2\pi \text{ radian} = 360° \quad \text{or} \quad \pi \text{ radian} = 180°$$

The above relation enables us to express a radian measure in terms of degree measure and a degree measure in terms of radian measure. Using approximate value of π as $\frac{22}{7}$, we have

$$1 \text{ radian} = \frac{180°}{\pi} = 57°\,16' \text{ approximately.}$$

Also $$1° = \frac{\pi}{180} \text{ radian} = 0.01746 \text{ radian approximately.}$$

The relation between degree measures and radian measure of some common angles are given in the following table:

Degree	30°	45°	60°	90°	180°	270°	360°
Radian	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Notational Convention

Since angles are measured either in degrees or in radians, we adopt the convention that whenever we write angle θ°, we mean the angle whose degree measure is θ and whenever we write angle β, we mean the angle whose radian measure is β.

Note that when an angle is expressed in radians, the word 'radian' is frequently omitted. Thus, $\pi = 180°$ and $\frac{\pi}{4} = 45°$ are written with the understanding that π and $\frac{\pi}{4}$ are radian measures. Thus, we can say that

$$\text{Radian measure} = \frac{\pi}{180} \times \text{Degree measure}$$

$$\text{Degree measure} = \frac{180}{\pi} \times \text{Radian measure}$$

Example 1 Convert $40°\ 20'$ into radian measure.

Solution We know that 180° = π radian.

Hence $\quad 40°\ 20' = 40\frac{1}{3}$ degree $= \frac{\pi}{180} \times \frac{121}{3}$ radian $= \frac{121\pi}{540}$ radian.

Therefore $\quad 40°\ 20' = \frac{121\pi}{540}$ radian.

Example 2 Convert 6 radians into degree measure.

Solution We know that π radian = 180°.

Hence $\quad 6 \text{ radians} = \frac{180}{\pi} \times 6 \text{ degree} = \frac{1080 \times 7}{22} \text{degree}$

$$= 343\frac{7}{11} \text{degree} \quad = 343° + \frac{7 \times 60}{11} \text{ minute} \quad [\text{as } 1° = 60']$$

$$= 343° + 38' + \frac{2}{11} \text{ minute} \quad [\text{as } 1' = 60'']$$

$$= 343° + 38' + 10.9'' \quad = 343°38'\ 11'' \text{ approximately.}$$

Hence $\quad$ 6 radians = 343° 38′ 11″ approximately.

Example 3 Find the radius of the circle in which a central angle of 60° intercepts an arc of length 37.4 cm (use $\pi = \frac{22}{7}$).

Solution Here $l = 37.4$ cm and $\theta = 60° = \frac{60\pi}{180}$ radian $= \frac{\pi}{3}$

Hence, by $r = \frac{l}{\theta}$, we have

$$r = \frac{37.4 \times 3}{\pi} = \frac{37.4 \times 3 \times 7}{22} = 35.7 \text{ cm}$$

Example 4 The minute hand of a watch is 1.5 cm long. How far does its tip move in 40 minutes? (Use $\pi = 3.14$).

Solution In 60 minutes, the minute hand of a watch completes one revolution. Therefore, in 40 minutes, the minute hand turns through $\frac{2}{3}$ of a revolution. Therefore, $\theta = \frac{2}{3} \times 360°$ or $\frac{4\pi}{3}$ radian. Hence, the required distance travelled is given by

$$l = r\,\theta = 1.5 \times \frac{4\pi}{3} \text{ cm} = 2\pi \text{ cm} = 2 \times 3.14 \text{ cm} = 6.28 \text{ cm}.$$

Example 5 If the arcs of the same lengths in two circles subtend angles 65°and 110° at the centre, find the ratio of their radii.

Solution Let r_1 and r_2 be the radii of the two circles. Given that

$$\theta_1 = 65° = \frac{\pi}{180} \times 65 = \frac{13\pi}{36} \text{ radian}$$

and $$\theta_2 = 110° = \frac{\pi}{180} \times 110 = \frac{22\pi}{36} \text{ radian}$$

Let l be the length of each of the arc. Then $l = r_1\theta_1 = r_2\theta_2$, which gives

$$\frac{13\pi}{36} \times r_1 = \frac{22\pi}{36} \times r_2, \text{ i.e., } \frac{r_1}{r_2} = \frac{22}{13}$$

Hence $r_1 : r_2 = 22 : 13$.

EXERCISE 3.1

1. Find the radian measures corresponding to the following degree measures:

(i) 25° (ii) – 47°30′ (iii) 240° (iv) 520°

2. Find the degree measures corresponding to the following radian measures (Use $\pi = \frac{22}{7}$).

(i) $\frac{11}{16}$ (ii) -4 (iii) $\frac{5\pi}{3}$ (iv) $\frac{7\pi}{6}$

3. A wheel makes 360 revolutions in one minute. Through how many radians does it turn in one second?

4. Find the degree measure of the angle subtended at the centre of a circle of radius 100 cm by an arc of length 22 cm (Use $\pi = \frac{22}{7}$).

5. In a circle of diameter 40 cm, the length of a chord is 20 cm. Find the length of minor arc of the chord.

6. If in two circles, arcs of the same length subtend angles 60° and 75° at the centre, find the ratio of their radii.

7. Find the angle in radian through which a pendulum swings if its length is 75 cm and th e tip describes an arc of length

(i) 10 cm (ii) 15 cm (iii) 21 cm

3.3 Trigonometric Functions

In earlier classes, we have studied trigonometric ratios for acute angles as the ratio of sides of a right angled triangle. We will now extend the definition of trigonometric ratios to any angle in terms of radian measure and study them as trigonometric functions.

Consider a unit circle with centre at origin of the coordinate axes. Let P (a, b) be any point on the circle with angle AOP $= x$ radian, i.e., length of arc AP $= x$ (Fig 3.6).

We define $\cos x = a$ and $\sin x = b$ Since ΔOMP is a right triangle, we have

$$OM^2 + MP^2 = OP^2 \text{ or } a^2 + b^2 = 1$$

Thus, for every point on the unit circle, we have

$$a^2 + b^2 = 1 \text{ or } \cos^2 x + \sin^2 x = 1$$

Since one complete revolution subtends an angle of 2π radian at the centre of the circle, $\angle AOB = \frac{\pi}{2}$,

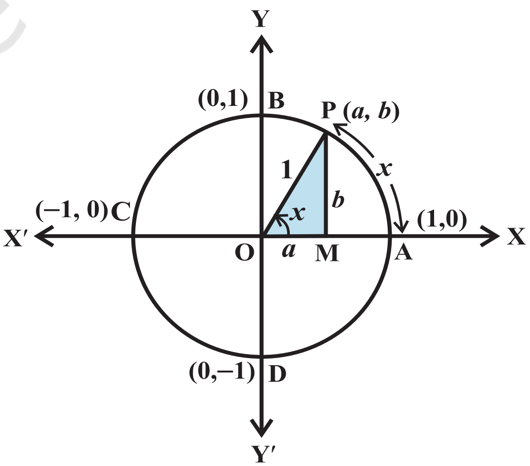


Fig 3.6

$\angle AOC = \pi$ and $\angle AOD = \frac{3\pi}{2}$. All angles which are integral multiples of $\frac{\pi}{2}$ are called *quadrantal angles*. The coordinates of the points A, B, C and D are, respectively, (1, 0), (0, 1), (–1, 0) and (0, –1). Therefore, for quadrantal angles, we have

$$\cos 0° = 1 \qquad \sin 0° = 0,$$

$$\cos \frac{\pi}{2} = 0 \qquad \sin \frac{\pi}{2} = 1$$

$$\cos\pi = -1 \qquad \sin\pi = 0$$

$$\cos \frac{3\pi}{2} = 0 \qquad \sin \frac{3\pi}{2} = -1$$

$$\cos 2\pi = 1 \qquad \sin 2\pi = 0$$

Now, if we take one complete revolution from the point P, we again come back to same point P. Thus, we also observe that if x increases (or decreases) by any integral multiple of 2π, the values of sine and cosine functions do not change. Thus,

$$\sin (2n\pi + x) = \sin x,\ n \in \mathbf{Z},\ \cos (2n\pi + x) = \cos x,\ n \in \mathbf{Z}$$

Further, $\sin x = 0$, if $x = 0, \pm \pi, \pm 2\pi, \pm 3\pi, ...$, i.e., when x is an integral multiple of π and $\cos x = 0$, if $x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, ...$ i.e., $\cos x$ vanishes when x is an odd multiple of $\frac{\pi}{2}$. Thus

sin x = 0 implies $x = n\pi$, where n is any integer

cos x = 0 implies $x = (2n + 1)\frac{\pi}{2}$, where n is any integer.

We now define other trigonometric functions in terms of sine and cosine functions:

$$\operatorname{cosec} x = \frac{1}{\sin x},\ x \neq n\pi, \text{ where } n \text{ is any integer.}$$

$$\sec x = \frac{1}{\cos x},\ x \neq (2n + 1)\frac{\pi}{2}, \text{ where } n \text{ is any integer.}$$

$$\tan x = \frac{\sin x}{\cos x},\ x \neq (2n + 1)\frac{\pi}{2}, \text{ where } n \text{ is any integer.}$$

$$\cot x = \frac{\cos x}{\sin x},\ x \neq n\pi, \text{ where } n \text{ is any integer.}$$

We have shown that for all real x, $\sin^2 x + \cos^2 x = 1$

It follows that

$$1 + \tan^2 x = \sec^2 x \qquad \text{(why?)}$$

$$1 + \cot^2 x = \text{cosec}^2 x \qquad \text{(why?)}$$

In earlier classes, we have discussed the values of trigonometric ratios for 0°, 30°, 45°, 60° and 90°. The values of trigonometric functions for these angles are same as that of trigonometric ratios studied in earlier classes. Thus, we have the following table:

	0°	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	– 1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	– 1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined	0	not defined	0

The values of cosec x, sec x and cot x are the reciprocal of the values of sin x, cos x and tan x, respectively.

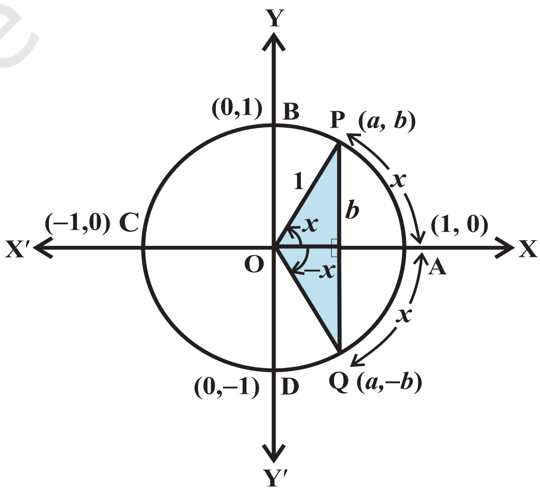


Fig 3.7

3.3.1 *Sign of trigonometric functions*

Let P (a, b) be a point on the unit circle with centre at the origin such that $\angle AOP = x$. If $\angle AOQ = -x$, then the coordinates of the point Q will be $(a, -b)$ (Fig 3.7). Therefore

$$\cos(-x) = \cos x$$

and $\sin(-x) = -\sin x$

Since for every point P (a, b) on the unit circle, $-1 \le a \le 1$ and

$-1 \leq b \leq 1$, we have $-1 \leq \cos x \leq 1$ and $-1 \leq \sin x \leq 1$ for all x. We have learnt in previous classes that in the first quadrant ($0 < x < \frac{\pi}{2}$) a and b are both positive, in the second quadrant ($\frac{\pi}{2} < x < \pi$) a is negative and b is positive, in the third quadrant ($\pi < x < \frac{3\pi}{2}$) a and b are both negative and in the fourth quadrant ($\frac{3\pi}{2} < x < 2\pi$) a is positive and b is negative. Therefore, $\sin x$ is positive for $0 < x < \pi$, and negative for $\pi < x < 2\pi$. Similarly, $\cos x$ is positive for $0 < x < \frac{\pi}{2}$, negative for $\frac{\pi}{2} < x < \frac{3\pi}{2}$ and also positive for $\frac{3\pi}{2} < x < 2\pi$. Likewise, we can find the signs of other trigonometric functions in different quadrants. In fact, we have the following table.

	I	II	III	IV
$\sin x$	+	+	–	–
$\cos x$	+	–	–	+
$\tan x$	+	–	+	–
$\operatorname{cosec} x$	+	+	–	–
$\sec x$	+	–	–	+
$\cot x$	+	–	+	–

3.3.2 ***Domain and range of trigonometric functions*** From the definition of sine and cosine functions, we observe that they are defined for all real numbers. Further, we observe that for each real number x,

$$-1 \leq \sin x \leq 1 \text{ and } -1 \leq \cos x \leq 1$$

Thus, domain of $y = \sin x$ and $y = \cos x$ is the set of all real numbers and range is the interval $[-1, 1]$, i.e., $-1 \leq y \leq 1$.

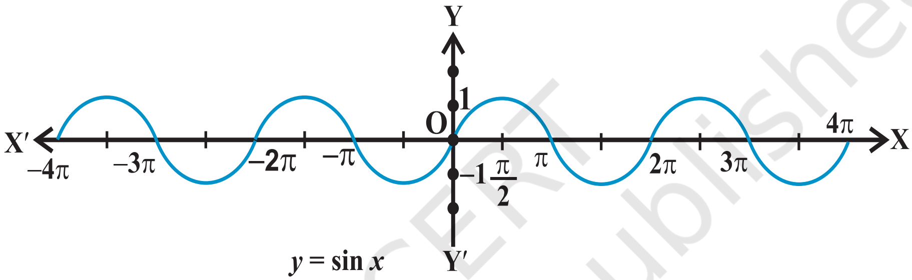
Since cosec $x = \dfrac{1}{\sin x}$, the domain of y = cosec x is the set $\{x : x \in \mathbf{R}$ and $x \neq n\pi, n \in \mathbf{Z}\}$ and range is the set $\{y : y \in \mathbf{R}, y \geq 1 \text{ or } y \leq -1\}$. Similarly, the domain of y = sec x is the set $\{x : x \in \mathbf{R}$ and $x \neq (2n + 1)\dfrac{\pi}{2}, n \in \mathbf{Z}\}$ and range is the set $\{y : y \in \mathbf{R}, y \leq -1 \text{ or } y \geq 1\}$. The domain of y = tan x is the set $\{x : x \in \mathbf{R}$ and $x \neq (2n + 1)\dfrac{\pi}{2}, n \in \mathbf{Z}\}$ and range is the set of all real numbers. The domain of y = cot x is the set $\{x : x \in \mathbf{R}$ and $x \neq n\pi, n \in \mathbf{Z}\}$ and the range is the set of all real numbers.

We further observe that in the first quadrant, as x increases from 0 to $\dfrac{\pi}{2}$, sin x increases from 0 to 1, as x increases from $\dfrac{\pi}{2}$ to π, sin x decreases from 1 to 0. In the third quadrant, as x increases from π to $\dfrac{3\pi}{2}$, sin x decreases from 0 to –1and finally, in the fourth quadrant, sin x increases from –1 to 0 as x increases from $\dfrac{3\pi}{2}$ to 2π. Similarly, we can discuss the behaviour of other trigonometric functions. In fact, we have the following table:

	I quadrant	II quadrant	III quadrant	IV quadrant
sin	increases from 0 to 1	decreases from 1 to 0	decreases from 0 to –1	increases from –1 to 0
cos	decreases from 1 to 0	decreases from 0 to – 1	increases from –1 to 0	increases from 0 to 1
tan	increases from 0 to ∞	increases from –∞to 0	increases from 0 to ∞	increases from –∞to 0
cot	decreases from ∞ to 0	decreases from 0 to–∞	decreases from ∞ to 0	decreases from 0to –∞
sec	increases from 1 to ∞	increases from –∞to–1	decreases from –1to–∞	decreases from ∞ to 1
cosec	decreases from ∞ to 1	increases from 1 to ∞	increases from –∞to–1	decreases from–1to–∞

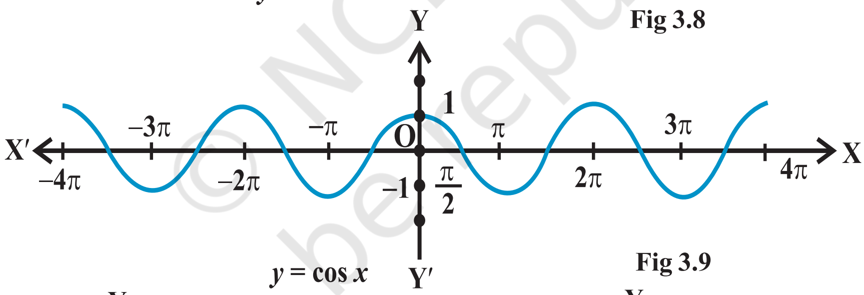
Remark In the above table, the statement tan x increases from 0 to ∞ (infinity) for $0 < x < \dfrac{\pi}{2}$ simply means that tan x increases as x increases for $0 < x < \dfrac{\pi}{2}$ and assumes arbitraily large positive values as x approaches to $\frac{\pi}{2}$. Similarly, to say that cosec x decreases from –1 to $-\infty$ (minus infinity) in the fourth quadrant means that cosec x decreases for $x \in (\frac{3\pi}{2}, 2\pi)$ and assumes arbitrarily large negative values as x approaches to 2π. The symbols ∞ and $-\infty$ simply specify certain types of behaviour of functions and variables.

We have already seen that values of sin x and cos x repeats after an interval of 2π. Hence, values of cosec x and sec x will also repeat after an interval of 2π. We



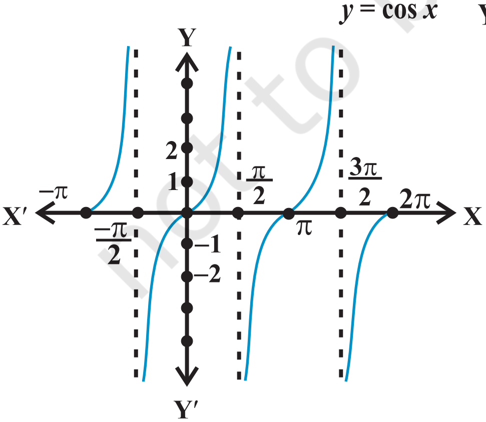
$y = \sin x$

Fig 3.8

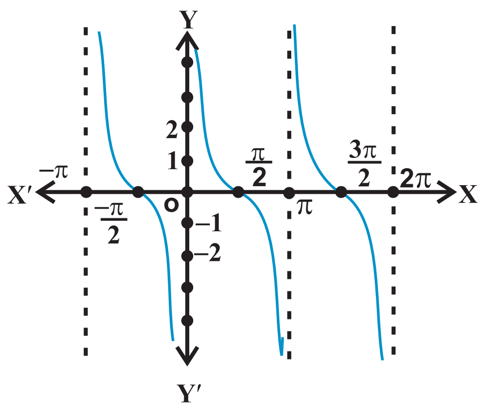


$y = \cos x$

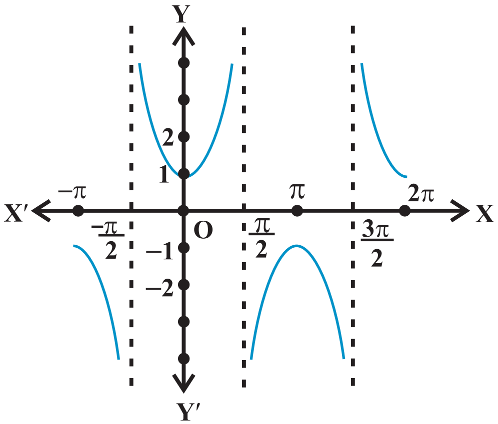
Fig 3.9



$y = \tan x$ **Fig 3.10**

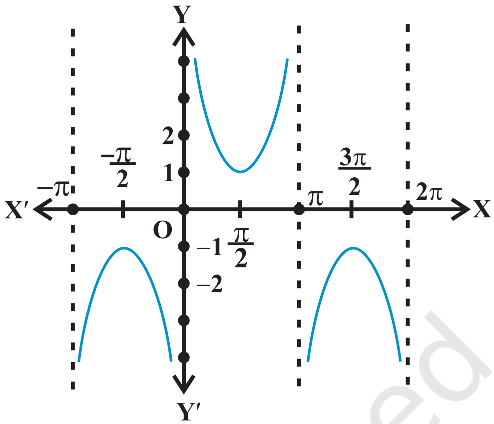


$y = \cot x$ **Fig 3.11**



$y = \sec x$

Fig 3.12



$y = \text{cosec}\, x$

Fig 3.13

shall see in the next section that $\tan(\pi + x) = \tan x$. Hence, values of $\tan x$ will repeat after an interval of π. Since $\cot x$ is reciprocal of $\tan x$, its values will also repeat after an interval of π. Using this knowledge and behaviour of trigonometic functions, we can sketch the graph of these functions. The graph of these functions are given above:

Example 6 If $\cos x = -\frac{3}{5}$, x lies in the third quadrant, find the values of other five trigonometric functions.

Solution Since $\cos x = -\frac{3}{5}$, we have $\sec x = -\frac{5}{3}$

Now $\sin^2 x + \cos^2 x = 1$, i.e., $\sin^2 x = 1 - \cos^2 x$

or $\sin^2 x = 1 - \frac{9}{25} = \frac{16}{25}$

Hence $\sin x = \pm \frac{4}{5}$

Since x lies in third quadrant, $\sin x$ is negative. Therefore

$$\sin x = -\frac{4}{5}$$

which also gives

$$\text{cosec}\, x = -\frac{5}{4}$$

Further, we have

$$\tan x = \frac{\sin x}{\cos x} = \frac{4}{3} \text{ and } \cot x = \frac{\cos x}{\sin x} = \frac{3}{4}.$$

Example 7 If $\cot x = -\frac{5}{12}$, x lies in second quadrant, find the values of other five trigonometric functions.

Solution Since $\cot x = -\frac{5}{12}$, we have $\tan x = -\frac{12}{5}$

Now $\sec^2 x = 1 + \tan^2 x = 1 + \frac{144}{25} = \frac{169}{25}$

Hence $\sec x = \pm \frac{13}{5}$

Since x lies in second quadrant, $\sec x$ will be negative. Therefore

$$\sec x = -\frac{13}{5},$$

which also gives

$$\cos x = -\frac{5}{13}$$

Further, we have

$$\sin x = \tan x \cos x = \left(-\frac{12}{5}\right) \times \left(-\frac{5}{13}\right) = \frac{12}{13}$$

and $\operatorname{cosec} x = \frac{1}{\sin x} = \frac{13}{12}.$

Example 8 Find the value of $\sin \frac{31\pi}{3}$.

Solution We know that values of $\sin x$ repeats after an interval of 2π. Therefore

$$\sin \frac{31\pi}{3} = \sin \left(10\pi + \frac{\pi}{3}\right) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

Example 9 Find the value of cos (–1710°).

Solution We know that values of cos x repeats after an interval of 2π or 360°. Therefore, cos (–1710°) = cos (–1710° + 5 × 360°)

= cos (–1710° + 1800°) = cos 90° = 0.

EXERCISE 3.2

Find the values of other five trigonometric functions in Exercises 1 to 5.

1. $\cos x = -\frac{1}{2}$, x lies in third quadrant.
2. $\sin x = \frac{3}{5}$, x lies in second quadrant.
3. $\cot x = \frac{3}{4}$, x lies in third quadrant.
4. $\sec x = \frac{13}{5}$, x lies in fourth quadrant.
5. $\tan x = -\frac{5}{12}$, x lies in second quadrant.

Find the values of the trigonometric functions in Exercises 6 to 10.

6. sin 765°
7. cosec (– 1410°)
8. $\tan \frac{19\pi}{3}$
9. $\sin (-\frac{11\pi}{3})$
10. $\cot (-\frac{15\pi}{4})$

3.4 Trigonometric Functions of Sum and Difference of Two Angles

In this Section, we shall derive expressions for trigonometric functions of the sum and difference of two numbers (angles) and related expressions. The basic results in this connection are called *trigonometric identities*. We have seen that

1. **sin (– x) = – sin x**
2. **cos (– x) = cos x**

We shall now prove some more results:

3. $\cos(x + y) = \cos x \cos y - \sin x \sin y$

Consider the unit circle with centre at the origin. Let x be the angle P_4OP_1 and y be the angle P_1OP_2. Then $(x + y)$ is the angle P_4OP_2. Also let $(-y)$ be the angle P_4OP_3. Therefore, P_1, P_2, P_3 and P_4 will have the coordinates $P_1(\cos x, \sin x)$, $P_2[\cos(x + y), \sin(x + y)]$, $P_3[\cos(-y), \sin(-y)]$ and $P_4(1, 0)$ (Fig 3.14).

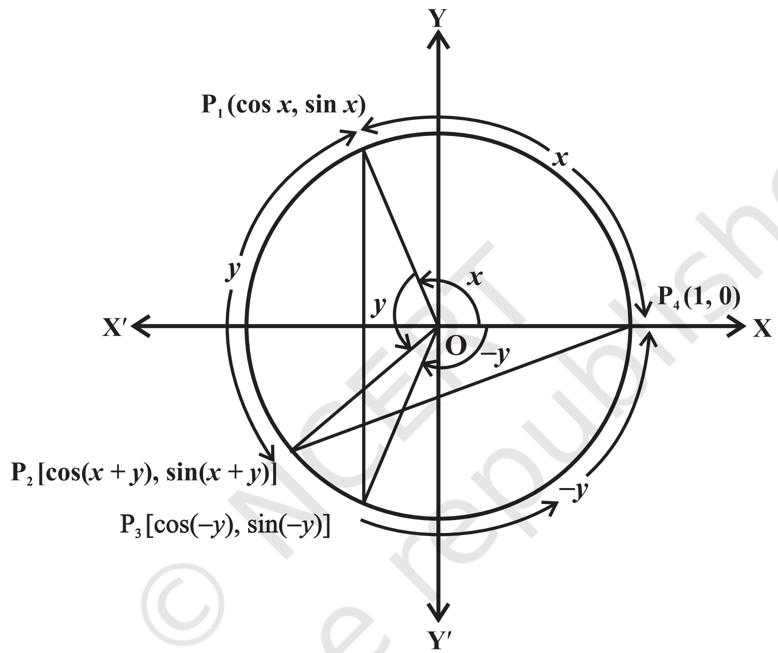


Fig 3.14

Consider the triangles P_1OP_3 and P_2OP_4. They are congruent (Why?). Therefore, P_1P_3 and P_2P_4 are equal. By using distance formula, we get

$$P_1P_3^2 = [\cos x - \cos(-y)]^2 + [\sin x - \sin(-y)]^2$$
$$= (\cos x - \cos y)^2 + (\sin x + \sin y)^2$$
$$= \cos^2 x + \cos^2 y - 2\cos x \cos y + \sin^2 x + \sin^2 y + 2\sin x \sin y$$
$$= 2 - 2(\cos x \cos y - \sin x \sin y) \qquad \text{(Why?)}$$

Also, $$P_2P_4^2 = [1 - \cos(x + y)]^2 + [0 - \sin(x + y)]^2$$
$$= 1 - 2\cos(x + y) + \cos^2(x + y) + \sin^2(x + y)$$
$$= 2 - 2\cos(x + y)$$

Since $P_1P_3 = P_2P_4$, we have $P_1P_3^2 = P_2P_4^2$.
Therefore, $2 - 2 (\cos x \cos y - \sin x \sin y) = 2 - 2 \cos (x + y)$.

Hence $\cos (x + y) = \cos x \cos y - \sin x \sin y$

4. $\cos (x - y) = \cos x \cos y + \sin x \sin y$
Replacing y by $-y$ in identity 3, we get

$$\cos (x + (-y)) = \cos x \cos (-y) - \sin x \sin (-y)$$
or $$\cos (x - y) = \cos x \cos y + \sin x \sin y$$

5. $\cos \left(\frac{\pi}{2} - x\right) = \sin x$

If we replace x by $\frac{\pi}{2}$ and y by x in Identity (4), we get

$$\cos \left(\frac{\pi}{2} - x\right) = \cos \frac{\pi}{2} \cos x + \sin \frac{\pi}{2} \sin x = \sin x.$$

6. $\sin \left(\frac{\pi}{2} - x\right) = \cos x$

Using the Identity 5, we have

$$\sin \left(\frac{\pi}{2} - x\right) = \cos \left[\frac{\pi}{2} - \left(\frac{\pi}{2} - x\right)\right] = \cos x.$$

7. $\sin (x + y) = \sin x \cos y + \cos x \sin y$
We know that

$$\sin (x + y) = \cos \left(\frac{\pi}{2} - (x + y)\right) = \cos \left(\left(\frac{\pi}{2} - x\right) - y\right)$$

$$= \cos \left(\frac{\pi}{2} - x\right) \cos y + \sin \left(\frac{\pi}{2} - x\right) \sin y$$

$$= \sin x \cos y + \cos x \sin y$$

8. $\sin (x - y) = \sin x \cos y - \cos x \sin y$
If we replace y by $-y$, in the Identity 7, we get the result.

9. By taking suitable values of x and y in the identities 3, 4, 7 and 8, we get the following results:

$$\cos \left(\frac{\pi}{2} + x\right) = -\sin x \qquad \sin \left(\frac{\pi}{2} + x\right) = \cos x$$

$$\cos (\pi - x) = -\cos x \qquad \sin (\pi - x) = \sin x$$

$$\cos(\pi + x) = -\cos x \qquad \sin(\pi + x) = -\sin x$$
$$\cos(2\pi - x) = \cos x \qquad \sin(2\pi - x) = -\sin x$$

Similar results for tan x, cot x, sec x and cosec x can be obtained from the results of sin x and cos x.

10. If none of the angles x, y and $(x + y)$ is an odd multiple of $\frac{\pi}{2}$, then

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Since none of the x, y and $(x + y)$ is an odd multiple of $\frac{\pi}{2}$, it follows that cos x, cos y and cos $(x + y)$ are non-zero. Now

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x\cos y + \cos x\sin y}{\cos x\cos y - \sin x\sin y}.$$

Dividing numerator and denominator by cos x cos y, we have

$$\tan(x+y) = \frac{\dfrac{\sin x\cos y}{\cos x\cos y} + \dfrac{\cos x\sin y}{\cos x\cos y}}{\dfrac{\cos x\cos y}{\cos x\cos y} - \dfrac{\sin x\sin y}{\cos x\cos y}}$$

$$= \frac{\tan x + \tan y}{1 - \tan x\tan y}$$

11. $$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x\tan y}$$

If we replace y by $-y$ in Identity 10, we get

$$\tan(x-y) = \tan[x + (-y)]$$

$$= \frac{\tan x + \tan(-y)}{1 - \tan x\tan(-y)} = \frac{\tan x - \tan y}{1 + \tan x\tan y}$$

12. If none of the angles x, y and $(x + y)$ is a multiple of π, then

$$\cot(x+y) = \frac{\cot x\cot y - 1}{\cot y + \cot x}$$

Since, none of the x, y and $(x + y)$ is multiple of π, we find that sin x sin y and sin $(x + y)$ are non-zero. Now,

$$\cot(x+y) = \frac{\cos(x+y)}{\sin(x+y)} = \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y + \cos x \sin y}$$

Dividing numerator and denominator by sin x sin y, we have

$$\cot(x+y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}$$

13. $\cot(x-y) = \dfrac{\cot x \cot y + 1}{\cot y - \cot x}$ if none of angles x, y and $x–y$ is a multiple of π

If we replace y by $–y$ in identity 12, we get the result

14. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$

We know that

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

Replacing y by x, we get

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= \cos^2 x - (1 - \cos^2 x) = 2\cos^2 x - 1$$

Again, $\cos 2x = \cos^2 x - \sin^2 x$
$$= 1 - \sin^2 x - \sin^2 x = 1 - 2\sin^2 x.$$

We have $\cos 2x = \cos^2 x - \sin^2 x = \dfrac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x}$

Dividing numerator and denominator by $\cos^2 x$, we get

$$\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}, \quad x \neq n\pi + \frac{\pi}{2}, \text{ where n is an integer}$$

15. $\sin 2x = 2\sin x \cos x = \dfrac{2\tan x}{1+\tan^2 x}$ $x \neq n\pi + \dfrac{\pi}{2}$, where n is an integer

We have

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

Replacing y by x, we get $\sin 2x = 2\sin x \cos x$.

Again $\sin 2x = \dfrac{2\sin x \cos x}{\cos^2 x + \sin^2 x}$

Dividing each term by $\cos^2 x$, we get

$$\sin 2x = \frac{2\tan x}{1+\tan^2 x}$$

16. $\mathbf{\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}}$ if $2x \neq n\pi + \dfrac{\pi}{2}$, where n is an integer

We know that

$$\tan (x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Replacing y by x, we get $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

17. $\mathbf{\sin 3x = 3 \sin x - 4 \sin^3 x}$

We have,

$$\begin{aligned}\sin 3x &= \sin (2x + x)\\ &= \sin 2x \cos x + \cos 2x \sin x\\ &= 2 \sin x \cos x \cos x + (1 - 2\sin^2 x) \sin x\\ &= 2 \sin x (1 - \sin^2 x) + \sin x - 2 \sin^3 x\\ &= 2 \sin x - 2 \sin^3 x + \sin x - 2 \sin^3 x\\ &= 3 \sin x - 4 \sin^3 x\end{aligned}$$

18. $\mathbf{\cos 3x = 4 \cos^3 x - 3 \cos x}$

We have,

$$\begin{aligned}\cos 3x &= \cos (2x + x)\\ &= \cos 2x \cos x - \sin 2x \sin x\\ &= (2\cos^2 x - 1) \cos x - 2\sin x \cos x \sin x\\ &= (2\cos^2 x - 1) \cos x - 2\cos x (1 - \cos^2 x)\\ &= 2\cos^3 x - \cos x - 2\cos x + 2 \cos^3 x\\ &= 4\cos^3 x - 3\cos x.\end{aligned}$$

19. $\mathbf{\tan 3x = \dfrac{3\tan x - \tan^3 x}{1-3\tan^2 x}}$ if $3x \neq n\pi + \dfrac{\pi}{2}$, where n is an integer

We have $\tan 3x = \tan (2x + x)$

$$= \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x} = \frac{\dfrac{2\tan x}{1-\tan^2 x} + \tan x}{1 - \dfrac{2\tan x \,.\, \tan x}{1-\tan^2 x}}$$

$$=\frac{2\tan x+\tan x-\tan^3 x}{1-\tan^2 x-2\tan^2 x}=\frac{3\tan x-\tan^3 x}{1-3\tan^2 x}$$

20. (i) $\cos x+\cos y=2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$

(ii) $\cos x-\cos y=-2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$

(iii) $\sin x+\sin y=2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$

(iv) $\sin x-\sin y=2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$

We know that

$$\cos(x+y)=\cos x\cos y-\sin x\sin y \quad \dots (1)$$

and $$\cos(x-y)=\cos x\cos y+\sin x\sin y \quad \dots (2)$$

Adding and subtracting (1) and (2), we get

$$\cos(x+y)+\cos(x-y)=2\cos x\cos y \quad \dots (3)$$

and $$\cos(x+y)-\cos(x-y)=-2\sin x\sin y \quad \dots (4)$$

Further $$\sin(x+y)=\sin x\cos y+\cos x\sin y \quad \dots (5)$$

and $$\sin(x-y)=\sin x\cos y-\cos x\sin y \quad \dots (6)$$

Adding and subtracting (5) and (6), we get

$$\sin(x+y)+\sin(x-y)=2\sin x\cos y \quad \dots (7)$$

$$\sin(x+y)-\sin(x-y)=2\cos x\sin y \quad \dots (8)$$

Let $x+y=\theta$ and $x-y=\phi$. Therefore

$$x=\left(\frac{\theta+\phi}{2}\right)\text{ and } y=\left(\frac{\theta-\phi}{2}\right)$$

Substituting the values of x and y in (3), (4), (7) and (8), we get

$$\cos\theta+\cos\phi=2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$

$$\cos\theta-\cos\phi=-2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

$$\sin\theta+\sin\phi=2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$

$$\sin\theta - \sin\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

Since θ and ϕ can take any real values, we can replace θ by x and ϕ by y. Thus, we get

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}\ ;\ \cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2},$$

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}\ ;\ \sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}.$$

Remark As a part of identities given in 20, we can prove the following results:

21. (i) $\mathbf{2\cos x\cos y = \cos(x+y) + \cos(x-y)}$

(ii) $\mathbf{-2\sin x\sin y = \cos(x+y) - \cos(x-y)}$

(iii) $\mathbf{2\sin x\cos y = \sin(x+y) + \sin(x-y)}$

(iv) $\mathbf{2\cos x\sin y = \sin(x+y) - \sin(x-y)}$.

Example 10 Prove that

$$3\sin\frac{\pi}{6}\sec\frac{\pi}{3} - 4\sin\frac{5\pi}{6}\cot\frac{\pi}{4} = 1$$

Solution We have

$$\text{L.H.S.} = 3\sin\frac{\pi}{6}\sec\frac{\pi}{3} - 4\sin\frac{5\pi}{6}\cot\frac{\pi}{4}$$

$$= 3 \times \frac{1}{2} \times 2 - 4\sin\left(\pi - \frac{\pi}{6}\right) \times 1 = 3 - 4\sin\frac{\pi}{6}$$

$$= 3 - 4 \times \frac{1}{2} = 1 = \text{R.H.S.}$$

Example 11 Find the value of sin 15°.

Solution We have

$$\sin 15° = \sin(45° - 30°)$$

$$= \sin 45° \cos 30° - \cos 45° \sin 30°$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Example 12 Find the value of $\tan\dfrac{13\pi}{12}$.

Solution We have

$$\tan\frac{13\pi}{12}=\tan\left(\pi+\frac{\pi}{12}\right)=\tan\frac{\pi}{12}=\tan\left(\frac{\pi}{4}-\frac{\pi}{6}\right)$$

$$=\frac{\tan\frac{\pi}{4}-\tan\frac{\pi}{6}}{1+\tan\frac{\pi}{4}\tan\frac{\pi}{6}}=\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}-1}{\sqrt{3}+1}=2-\sqrt{3}$$

Example 13 Prove that

$$\frac{\sin(x+y)}{\sin(x-y)}=\frac{\tan x+\tan y}{\tan x-\tan y}.$$

Solution We have

$$\text{L.H.S.}\quad=\frac{\sin(x+y)}{\sin(x-y)}=\frac{\sin x\cos y+\cos x\sin y}{\sin x\cos y-\cos x\sin y}$$

Dividing the numerator and denominator by $\cos x\cos y$, we get

$$\frac{\sin(x+y)}{\sin(x-y)}=\frac{\tan x+\tan y}{\tan x-\tan y}.$$

Example 14 Show that

$$\tan 3x\tan 2x\tan x=\tan 3x-\tan 2x-\tan x$$

Solution We know that $3x = 2x + x$

Therefore, $\tan 3x = \tan(2x + x)$

or $$\tan 3x=\frac{\tan 2x+\tan x}{1-\tan 2x\tan x}$$

or $\tan 3x - \tan 3x\tan 2x\tan x = \tan 2x + \tan x$

or $\tan 3x - \tan 2x - \tan x = \tan 3x\tan 2x\tan x$

or $\tan 3x\tan 2x\tan x = \tan 3x - \tan 2x - \tan x$.

Example 15 Prove that

$$\cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)=\sqrt{2}\cos x$$

Solution Using the Identity 20(i), we have

$$\text{L.H.S.} = \cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)$$

$$= 2\cos\left(\frac{\frac{\pi}{4}+x+\frac{\pi}{4}-x}{2}\right)\cos\left(\frac{\frac{\pi}{4}+x-(\frac{\pi}{4}-x)}{2}\right)$$

$$= 2\cos\frac{\pi}{4}\cos x = 2\times\frac{1}{\sqrt{2}}\cos x = \sqrt{2}\cos x = \text{R.H.S.}$$

Example 16 Prove that $\dfrac{\cos 7x+\cos 5x}{\sin 7x-\sin 5x} = \cot x$

Solution Using the Identities 20 (i) and 20 (iv), we get

$$\text{L.H.S.} = \frac{2\cos\dfrac{7x+5x}{2}\cos\dfrac{7x-5x}{2}}{2\cos\dfrac{7x+5x}{2}\sin\dfrac{7x-5x}{2}} = \frac{\cos x}{\sin x} = \cot x = \text{R.H.S.}$$

Example 17 Prove that $= \dfrac{\sin 5x-2\sin 3x+\sin x}{\cos 5x-\cos x} = \tan x$

Solution We have

$$\text{L.H.S.} = \frac{\sin 5x-2\sin 3x+\sin x}{\cos 5x-\cos x} = \frac{\sin 5x+\sin x-2\sin 3x}{\cos 5x-\cos x}$$

$$= \frac{2\sin 3x\cos 2x-2\sin 3x}{-2\sin 3x\sin 2x} = -\frac{\sin 3x\,(\cos 2x-1)}{\sin 3x\sin 2x}$$

$$= \frac{1-\cos 2x}{\sin 2x} = \frac{2\sin^2 x}{2\sin x\cos x} = \tan x = \text{R.H.S.}$$

EXERCISE 3.3

Prove that:

1. $\sin^2\frac{\pi}{6}+\cos^2\frac{\pi}{3}-\tan^2\frac{\pi}{4}=-\frac{1}{2}$

2. $2\sin^2\frac{\pi}{6}+\operatorname{cosec}^2\frac{7\pi}{6}\cos^2\frac{\pi}{3}=\frac{3}{2}$

3. $\cot^2\frac{\pi}{6}+\operatorname{cosec}\frac{5\pi}{6}+3\tan^2\frac{\pi}{6}=6$

4. $2\sin^2\frac{3\pi}{4}+2\cos^2\frac{\pi}{4}+2\sec^2\frac{\pi}{3}=10$

5. Find the value of:

   (i) sin 75° (ii) tan 15°

Prove the following:

6. $\cos\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-y\right)-\sin\left(\frac{\pi}{4}-x\right)\sin\left(\frac{\pi}{4}-y\right)=\sin(x+y)$

7. $\dfrac{\tan\left(\frac{\pi}{4}+x\right)}{\tan\left(\frac{\pi}{4}-x\right)}=\left(\dfrac{1+\tan x}{1-\tan x}\right)^2$

8. $\dfrac{\cos(\pi+x)\cos(-x)}{\sin(\pi-x)\cos\left(\frac{\pi}{2}+x\right)}=\cot^2 x$

9. $\cos\left(\frac{3\pi}{2}+x\right)\cos(2\pi+x)\left[\cot\left(\frac{3\pi}{2}-x\right)+\cot(2\pi+x)\right]=1$

10. $\sin(n+1)x\sin(n+2)x+\cos(n+1)x\cos(n+2)x=\cos x$

11. $\cos\left(\frac{3\pi}{4}+x\right)-\cos\left(\frac{3\pi}{4}-x\right)=-\sqrt{2}\sin x$

12. $\sin^2 6x-\sin^2 4x=\sin 2x\sin 10x$

13. $\cos^2 2x-\cos^2 6x=\sin 4x\sin 8x$

14. $\sin 2x+2\sin 4x+\sin 6x=4\cos^2 x\sin 4x$

15. $\cot 4x(\sin 5x+\sin 3x)=\cot x(\sin 5x-\sin 3x)$

16. $\dfrac{\cos 9x-\cos 5x}{\sin 17x-\sin 3x}=-\dfrac{\sin 2x}{\cos 10x}$

17. $\dfrac{\sin 5x+\sin 3x}{\cos 5x+\cos 3x}=\tan 4x$

18. $\dfrac{\sin x-\sin y}{\cos x+\cos y}=\tan\dfrac{x-y}{2}$

19. $\dfrac{\sin x+\sin 3x}{\cos x+\cos 3x}=\tan 2x$

20. $\dfrac{\sin x-\sin 3x}{\sin^2 x-\cos^2 x}=2\sin x$

21. $\dfrac{\cos 4x+\cos 3x+\cos 2x}{\sin 4x+\sin 3x+\sin 2x}=\cot 3x$

22. $\cot x \cot 2x - \cot 2x \cot 3x - \cot 3x \cot x = 1$

23. $\tan 4x = \dfrac{4\tan x\,(1-\tan^2 x)}{1-6\tan^2 x+\tan^4 x}$ **24.** $\cos 4x = 1 - 8\sin^2 x \cos^2 x$

25. $\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$

Miscellaneous Examples

Example 18 If $\sin x = \dfrac{3}{5}$, $\cos y = -\dfrac{12}{13}$, where x and y both lie in second quadrant, find the value of $\sin(x + y)$.

Solution We know that

$$\sin(x + y) = \sin x \cos y + \cos x \sin y \qquad \text{... (1)}$$

Now $\cos^2 x = 1 - \sin^2 x = 1 - \dfrac{9}{25} = \dfrac{16}{25}$

Therefore $\cos x = \pm\dfrac{4}{5}$.

Since x lies in second quadrant, $\cos x$ is negative.

Hence $\cos x = -\dfrac{4}{5}$

Now $\sin^2 y = 1 - \cos^2 y = 1 - \dfrac{144}{169} = \dfrac{25}{169}$

i.e. $\sin y = \pm\dfrac{5}{13}$.

Since y lies in second quadrant, hence $\sin y$ is positive. Therefore, $\sin y = \dfrac{5}{13}$. Substituting the values of $\sin x$, $\sin y$, $\cos x$ and $\cos y$ in (1), we get

$$\sin(x + y) = \frac{3}{5}\times\left(-\frac{12}{13}\right)+\left(-\frac{4}{5}\right)\times\frac{5}{13} = -\frac{36}{65}-\frac{20}{65}=-\frac{56}{65}.$$

Example 19 Prove that

$$\cos 2x \cos\frac{x}{2} - \cos 3x \cos\frac{9x}{2} = \sin 5x \sin\frac{5x}{2}.$$

Solution We have

L.H.S. $= \frac{1}{2}\left[2\cos 2x \cos\frac{x}{2} - 2\cos\frac{9x}{2}\cos 3x\right]$

$$= \frac{1}{2}\left[\cos\left(2x+\frac{x}{2}\right)+\cos\left(2x-\frac{x}{2}\right)-\cos\left(\frac{9x}{2}+3x\right)-\cos\left(\frac{9x}{2}-3x\right)\right]$$

$$= \frac{1}{2}\left[\cos\frac{5x}{2}+\cos\frac{3x}{2}-\cos\frac{15x}{2}-\cos\frac{3x}{2}\right] = \frac{1}{2}\left[\cos\frac{5x}{2}-\cos\frac{15x}{2}\right]$$

$$= \frac{1}{2}\left[-2\sin\left\{\frac{\frac{5x}{2}+\frac{15x}{2}}{2}\right\}\sin\left\{\frac{\frac{5x}{2}-\frac{15x}{2}}{2}\right\}\right]$$

$$= -\sin 5x \sin\left(-\frac{5x}{2}\right) = \sin 5x \sin\frac{5x}{2} = \text{R.H.S.}$$

Example 20 Find the value of $\tan\frac{\pi}{8}$.

Solution Let $x = \frac{\pi}{8}$. Then $2x = \frac{\pi}{4}$.

Now $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$

or $\tan\frac{\pi}{4} = \frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$

Let $y = \tan\frac{\pi}{8}$. Then $1 = \frac{2y}{1-y^2}$

or $y^2 + 2y - 1 = 0$

Therefore $y = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$

Since $\frac{\pi}{8}$ lies in the first quadrant, $y = \tan\frac{\pi}{8}$ is positve. Hence

$$\tan\frac{\pi}{8} = \sqrt{2} - 1.$$

Example 21 If $\tan x = \frac{3}{4}, \pi < x < \frac{3\pi}{2}$, find the value of $\sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$.

Solution Since $\pi < x < \frac{3\pi}{2}$, $\cos x$ is negative.

Also $$\frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}.$$

Therefore, $\sin\frac{x}{2}$ is positive and $\cos\frac{x}{2}$ is negative.

Now $$\sec^2 x = 1 + \tan^2 x = 1 + \frac{9}{16} = \frac{25}{16}$$

Therefore $$\cos^2 x = \frac{16}{25} \text{ or } \cos x = -\frac{4}{5} \quad \text{(Why?)}$$

Now $$2\sin^2\frac{x}{2} = 1 - \cos x = 1 + \frac{4}{5} = \frac{9}{5}.$$

Therefore $$\sin^2\frac{x}{2} = \frac{9}{10}$$

or $$\sin\frac{x}{2} = \frac{3}{\sqrt{10}} \quad \text{(Why?)}$$

Again $$2\cos^2\frac{x}{2} = 1 + \cos x = 1 - \frac{4}{5} = \frac{1}{5}$$

Therefore $$\cos^2\frac{x}{2} = \frac{1}{10}$$

or $$\cos\frac{x}{2} = -\frac{1}{\sqrt{10}} \quad \text{(Why?)}$$

Hence $$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{3}{\sqrt{10}} \times \left(\frac{-\sqrt{10}}{1}\right) = -3.$$

Example 22 Prove that $\cos^2 x + \cos^2\left(x+\frac{\pi}{3}\right) + \cos^2\left(x-\frac{\pi}{3}\right) = \frac{3}{2}$.

Solution We have

$$\text{L.H.S.} = \frac{1+\cos 2x}{2} + \frac{1+\cos\left(2x+\frac{2\pi}{3}\right)}{2} + \frac{1+\cos\left(2x-\frac{2\pi}{3}\right)}{2}.$$

$$= \frac{1}{2}\left[3+\cos 2x+\cos\left(2x+\frac{2\pi}{3}\right)+\cos\left(2x-\frac{2\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[3+\cos 2x+2\cos 2x\cos\frac{2\pi}{3}\right]$$

$$= \frac{1}{2}\left[3+\cos 2x+2\cos 2x\cos\left(\pi-\frac{\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[3+\cos 2x-2\cos 2x\cos\frac{\pi}{3}\right]$$

$$= \frac{1}{2}\left[3+\cos 2x-\cos 2x\right] = \frac{3}{2} = \text{R.H.S.}$$

Miscellaneous Exercise on Chapter 3

Prove that:

1. $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}=0$

2. $(\sin 3x + \sin x)\sin x + (\cos 3x - \cos x)\cos x = 0$

3. $(\cos x + \cos y)^2 + (\sin x - \sin y)^2 = 4\cos^2\frac{x+y}{2}$

4. $(\cos x - \cos y)^2 + (\sin x - \sin y)^2 = 4 \sin^2 \frac{x-y}{2}$

5. $\sin x + \sin 3x + \sin 5x + \sin 7x = 4 \cos x \cos 2x \sin 4x$

6. $\frac{(\sin 7x + \sin 5x) + (\sin 9x + \sin 3x)}{(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)} = \tan 6x$

7. $\sin 3x + \sin 2x - \sin x = 4\sin x \cos \frac{x}{2} \cos \frac{3x}{2}$

Find $\sin \frac{x}{2}$, $\cos \frac{x}{2}$ and $\tan \frac{x}{2}$ in each of the following :

8. $\tan x = -\frac{4}{3}$, x in quadrant II

9. $\cos x = -\frac{1}{3}$, x in quadrant III

10. $\sin x = \frac{1}{4}$, x in quadrant II

Summary

- If in a circle of radius r, an arc of length l subtends an angle of θ radians, then $l = r\,\theta$
- Radian measure $= \frac{\pi}{180} \times$ Degree measure
- Degree measure $= \frac{180}{\pi} \times$ Radian measure
- $\cos^2 x + \sin^2 x = 1$
- $1 + \tan^2 x = \sec^2 x$
- $1 + \cot^2 x = \operatorname{cosec}^2 x$
- $\cos (2n\pi + x) = \cos x$
- $\sin (2n\pi + x) = \sin x$
- $\sin (-x) = -\sin x$
- $\cos (-x) = \cos x$
- $\cos (x + y) = \cos x \cos y - \sin x \sin y$
- $\cos (x - y) = \cos x \cos y + \sin x \sin y$
- $\cos \left(\frac{\pi}{2} - x\right) = \sin x$

- $\sin\left(\frac{\pi}{2} - x\right) = \cos x$
- $\sin(x + y) = \sin x \cos y + \cos x \sin y$
- $\sin(x - y) = \sin x \cos y - \cos x \sin y$
- $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$ $\quad\quad$ $\sin\left(\frac{\pi}{2} + x\right) = \cos x$

  $\cos(\pi - x) = -\cos x$ $\quad\quad$ $\sin(\pi - x) = \sin x$

  $\cos(\pi + x) = -\cos x$ $\quad\quad$ $\sin(\pi + x) = -\sin x$

  $\cos(2\pi - x) = \cos x$ $\quad\quad$ $\sin(2\pi - x) = -\sin x$

- If none of the angles x, y and $(x \pm y)$ is an odd multiple of $\frac{\pi}{2}$, then

  $$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

- $$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$
- If none of the angles x, y and $(x \pm y)$ is a multiple of π, then

  $$\cot(x + y) = \frac{\cot x \cot y - 1}{\cot y + \cot x}$$

- $$\cot(x - y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}$$
- $$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$$
- $$\sin 2x = 2\sin x \cos x = \frac{2\tan x}{1 + \tan^2 x}$$
- $$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$
- $\sin 3x = 3\sin x - 4\sin^3 x$
- $\cos 3x = 4\cos^3 x - 3\cos x$

- $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

- (i) $\cos x + \cos y = 2\cos\dfrac{x+y}{2}\cos\dfrac{x-y}{2}$

  (ii) $\cos x - \cos y = -2\sin\dfrac{x+y}{2}\sin\dfrac{x-y}{2}$

  (iii) $\sin x + \sin y = 2\sin\dfrac{x+y}{2}\cos\dfrac{x-y}{2}$

  (iv) $\sin x - \sin y = 2\cos\dfrac{x+y}{2}\sin\dfrac{x-y}{2}$

- (i) $2\cos x \cos y = \cos(x+y) + \cos(x-y)$

  (ii) $-2\sin x \sin y = \cos(x+y) - \cos(x-y)$

  (iii) $2\sin x \cos y = \sin(x+y) + \sin(x-y)$

  (iv) $2\cos x \sin y = \sin(x+y) - \sin(x-y)$.

Historical Note

The study of trigonometry was first started in India. The ancient Indian Mathematicians, Aryabhatta (476), Brahmagupta (598), Bhaskara I (600) and Bhaskara II (1114) got important results. All this knowledge first went from India to middle-east and from there to Europe. The Greeks had also started the study of trigonometry but their approach was so clumsy that when the Indian approach became known, it was immediately adopted throughout the world.

In India, the predecessor of the modern trigonometric functions, known as the sine of an angle, and the introduction of the sine function represents the main contribution of the *siddhantas* (Sanskrit astronomical works) to the history of mathematics.

Bhaskara I (about 600) gave formulae to find the values of sine functions for angles more than 90°. A sixteenth century Malayalam work *Yuktibhasa* (period) contains a proof for the expansion of sin (A + B). Exact expression for sines or cosines of 18°, 36°, 54°, 72°, etc., are given by Bhaskara II.

The symbols $\sin^{-1} x$, $\cos^{-1} x$, etc., for arc sin x, arc cos x, etc., were suggested by the astronomer Sir John F.W. Hersehel (1813) The names of Thales (about 600 B.C.) is invariably associated with height and distance problems. He is credited with the determination of the height of a great pyramid in Egypt by measuring shadows of the pyramid and an auxiliary staff (or gnomon) of known height, and comparing the ratios:

$$\frac{\mathrm{H}}{\mathrm{S}} = \frac{h}{s} = \tan\ (\text{sun's altitude})$$

Thales is also said to have calculated the distance of a ship at sea through the proportionality of sides of similar triangles. Problems on height and distance using the similarity property are also found in ancient Indian works.

Chapter 4

COMPLEX NUMBERS AND QUADRATIC EQUATIONS

❖ *Mathematics is the Queen of Sciences and Arithmetic is the Queen of Mathematics. – GAUSS* ❖

4.1 Introduction

In earlier classes, we have studied linear equations in one and two variables and quadratic equations in one variable. We have seen that the equation $x^2 + 1 = 0$ has no real solution as $x^2 + 1 = 0$ gives $x^2 = -1$ and square of every real number is non-negative. So, we need to extend the real number system to a larger system so that we can find the solution of the equation $x^2 = -1$. In fact, the main objective is to solve the equation $ax^2 + bx + c = 0$, where $D = b^2 - 4ac < 0$, which is not possible in the system of real numbers.

W. R. Hamilton (1805-1865)

4.2 Complex Numbers

Let us denote $\sqrt{-1}$ by the symbol i. Then, we have $i^2 = -1$. This means that i is a solution of the equation $x^2 + 1 = 0$.

A number of the form $a + ib$, where a and b are real numbers, is defined to be a complex number. For example, $2 + i3$, $(-1) + i\sqrt{3}$, $4 + i\left(\frac{-1}{11}\right)$ are complex numbers.

For the complex number $z = a + ib$, a is called the *real part*, denoted by Re z and b is called the *imaginary part* denoted by Im z *of the complex number* z. For example, if $z = 2 + i5$, then Re $z = 2$ and Im $z = 5$.

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are equal if $a = c$ and $b = d$.

Example 1 If $4x + i(3x - y) = 3 + i(-6)$, where x and y are real numbers, then find the values of x and y.

Solution We have

$$4x + i(3x - y) = 3 + i(-6) \quad \dots (1)$$

Equating the real and the imaginary parts of (1), we get

$$4x = 3,\ 3x - y = -6,$$

which, on solving simultaneously, give $x = \frac{3}{4}$ and $y = \frac{33}{4}$.

4.3 Algebra of Complex Numbers

In this Section, we shall develop the algebra of complex numbers.

4.3.1 ***Addition of two complex numbers*** Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers. Then, the sum $z_1 + z_2$ is defined as follows:

$z_1 + z_2 = (a + c) + i(b + d)$, which is again a complex number.

For example, $(2 + i3) + (-6 + i5) = (2 - 6) + i(3 + 5) = -4 + i\,8$

The addition of complex numbers satisfy the following properties:

(i) *The closure law* The sum of two complex numbers is a complex number, i.e., $z_1 + z_2$ is a complex number for all complex numbers z_1 and z_2.

(ii) *The commutative law* For any two complex numbers z_1 and z_2, $z_1 + z_2 = z_2 + z_1$

(iii) *The associative law* For any three complex numbers z_1, z_2, z_3, $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.

(iv) *The existence of additive identity* There exists the complex number $0 + i\,0$ (denoted as 0), called the *additive identity* or the *zero complex number,* such that, for every complex number z, $z + 0 = z$.

(v) *The existence of additive inverse* To every complex number $z = a + ib$, we have the complex number $-a + i(-b)$ (denoted as $-z$), called the *additive inverse* or *negative of z*. We observe that $z + (-z) = 0$ (the additive identity).

4.3.2 ***Difference of two complex numbers*** Given any two complex numbers z_1 and z_2, the difference $z_1 - z_2$ is defined as follows:

$$z_1 - z_2 = z_1 + (-z_2).$$

For example, $(6 + 3i) - (2 - i) = (6 + 3i) + (-2 + i) = 4 + 4i$

and $(2 - i) - (6 + 3i) = (2 - i) + (-6 - 3i) = -4 - 4i$

4.3.3 ***Multiplication of two complex numbers*** Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers. Then, the product $z_1 z_2$ is defined as follows:

$$z_1 z_2 = (ac - bd) + i(ad + bc)$$

For example, $(3 + i5)(2 + i6) = (3 \times 2 - 5 \times 6) + i(3 \times 6 + 5 \times 2) = -24 + i28$

The multiplication of complex numbers possesses the following properties, which we state without proofs.

(i) **The closure law** The product of two complex numbers is a complex number, the product $z_1 z_2$ is a complex number for all complex numbers z_1 and z_2.

(ii) **The commutative law** For any two complex numbers z_1 and z_2,

$$z_1 z_2 = z_2 z_1.$$

(iii) **The associative law** For any three complex numbers z_1, z_2, z_3,

$$(z_1 z_2) z_3 = z_1 (z_2 z_3).$$

(iv) **The existence of multiplicative identity** There exists the complex number $1 + i\,0$ (denoted as 1), called the *multiplicative identity* such that $z.1 = z$, for every complex number z.

(v) **The existence of multiplicative inverse** For every non-zero complex number $z = a + ib$ or $a + bi (a \neq 0, b \neq 0)$, we have the complex number

$\dfrac{a}{a^2+b^2} + i\dfrac{-b}{a^2+b^2}$ (denoted by $\dfrac{1}{z}$ or z^{-1}), called the *multiplicative inverse* of z such that

$z.\dfrac{1}{z} = 1$ (the multiplicative identity).

(vi) **The distributive law** For any three complex numbers z_1, z_2, z_3,

(a) $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$

(b) $(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$

4.3.4 ***Division of two complex numbers*** Given any two complex numbers z_1 and z_2, where $z_2 \neq 0$, the quotient $\dfrac{z_1}{z_2}$ is defined by

$$\frac{z_1}{z_2} = z_1 \frac{1}{z_2}$$

For example, let $z_1 = 6 + 3i$ and $z_2 = 2 - i$

Then $$\frac{z_1}{z_2} = \left((6+3i) \times \frac{1}{2-i}\right) = (6+3i)\left(\frac{2}{2^2+(-1)^2} + i\frac{-(-1)}{2^2+(-1)^2}\right)$$

$$= (6+3i)\left(\frac{2+i}{5}\right) = \frac{1}{5}\left[12-3+i(6+6)\right] = \frac{1}{5}(9+12i)$$

4.3.5 *Power of i* we know that

$$i^3 = i^2 i = (-1)\, i = -i\,, \qquad i^4 = \left(i^2\right)^2 = (-1)^2 = 1$$

$$i^5 = \left(i^2\right)^2 i = (-1)^2\, i = i\,, \qquad i^6 = \left(i^2\right)^3 = (-1)^3 = -1\,, \text{ etc.}$$

Also, we have $\quad i^{-1} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{-1} = -i, \quad i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1,$

$$i^{-3} = \frac{1}{i^3} = \frac{1}{-i} \times \frac{i}{i} = \frac{i}{1} = i, \quad i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1$$

In general, for any integer k, $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$

4.3.6 *The square roots of a negative real number*

Note that $i^2 = -1$ and $(-i)^2 = i^2 = -1$

Therefore, the square roots of -1 are $i, -i$. However, by the symbol $\sqrt{-1}$, we would mean i only.

Now, we can see that i and $-i$ both are the solutions of the equation $x^2 + 1 = 0$ or $x^2 = -1$.

Similarly $\qquad \left(\sqrt{3}\,i\right)^2 = \left(\sqrt{3}\right)^2 i^2 = 3\,(-1) = -3$

$$\left(-\sqrt{3}\,i\right)^2 = \left(-\sqrt{3}\right)^2 i^2 = -3$$

Therefore, the square roots of -3 are $\sqrt{3}\,i$ and $-\sqrt{3}\,i$.

Again, the symbol $\sqrt{-3}$ is meant to represent $\sqrt{3}\,i$ only, i.e., $\sqrt{-3} = \sqrt{3}\,i$.

Generally, if a is a positive real number, $\sqrt{-a} = \sqrt{a}\,\sqrt{-1} = \sqrt{a}\,i$,

We already know that $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ for all positive real number a and b. This result also holds true when either $a > 0, b < 0$ or $a < 0, b > 0$. What if $a < 0, b < 0$? Let us examine.

Note that

$i^2 = \sqrt{-1}\,\sqrt{-1} = \sqrt{(-1)(-1)}$ (by assuming $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ for all real numbers)

$= \sqrt{1} = 1$, which is a contradiction to the fact that $i^2 = -1$.

Therefore, $\sqrt{a} \times \sqrt{b} \neq \sqrt{ab}$ if both a and b are negative real numbers.

Further, if any of a and b is zero, then, clearly, $\sqrt{a} \times \sqrt{b} = \sqrt{ab} = 0$.

4.3.7 *Identities* We prove the following identity

$$(z_1 + z_2)^2 = z_1^2 + z_2^2 + 2z_1z_2,$$ for all complex numbers z_1 and z_2.

Proof We have, $(z_1 + z_2)^2 = (z_1 + z_2)(z_1 + z_2)$,

$= (z_1 + z_2)\, z_1 + (z_1 + z_2)\, z_2$ (Distributive law)

$= z_1^2 + z_2z_1 + z_1z_2 + z_2^2$ (Distributive law)

$= z_1^2 + z_1z_2 + z_1z_2 + z_2^2$ (Commutative law of multiplication)

$= z_1^2 + 2z_1z_2 + z_2^2$

Similarly, we can prove the following identities:

(i) $(z_1 - z_2)^2 = z_1^2 - 2z_1z_2 + z_2^2$

(ii) $(z_1 + z_2)^3 = z_1^3 + 3z_1^2z_2 + 3z_1z_2^2 + z_2^3$

(iii) $(z_1 - z_2)^3 = z_1^3 - 3z_1^2z_2 + 3z_1z_2^2 - z_2^3$

(iv) $z_1^2 - z_2^2 = (z_1 + z_2)(z_1 - z_2)$

In fact, many other identities which are true for all real numbers, can be proved to be true for all complex numbers.

Example 2 Express the following in the form of $a + bi$:

(i) $(-5i)\left(\frac{1}{8}i\right)$ (ii) $(-i)(2i)\left(-\frac{1}{8}i\right)^3$

Solution (i) $(-5i)\left(\frac{1}{8}i\right) = \frac{-5}{8}i^2 = \frac{-5}{8}(-1) = \frac{5}{8} = \frac{5}{8} + i0$

(ii) $(-i)(2i)\left(-\frac{1}{8}i\right)^3 = 2 \times \frac{1}{8 \times 8 \times 8} \times i^5 = \frac{1}{256}\left(i^2\right)^2\, i = \frac{1}{256}i$.

Example 3 Express $(5 - 3i)^3$ in the form $a + ib$.

Solution We have, $(5 - 3i)^3 = 5^3 - 3 \times 5^2 \times (3i) + 3 \times 5\,(3i)^2 - (3i)^3$

$$= 125 - 225i - 135 + 27i = -10 - 198i.$$

Example 4 Express $\left(-\sqrt{3}+\sqrt{-2}\right)\left(2\sqrt{3}-i\right)$ in the form of $a + ib$

Solution We have, $\left(-\sqrt{3}+\sqrt{-2}\right)\left(2\sqrt{3}-i\right) = \left(-\sqrt{3}+\sqrt{2}\,i\right)\left(2\sqrt{3}-i\right)$

$$= -6+\sqrt{3}i+2\sqrt{6}i-\sqrt{2}\,i^2 = \left(-6+\sqrt{2}\right)+\sqrt{3}\left(1+2\sqrt{2}\right)i$$

4.4 The Modulus and the Conjugate of a Complex Number

Let $z = a + ib$ be a complex number. Then, the modulus of z, denoted by $|z|$, is defined to be the non-negative real number $\sqrt{a^2+b^2}$, i.e., $|z| = \sqrt{a^2+b^2}$ and the conjugate of z, denoted as $\bar{z}$, is the complex number $a - ib$, i.e., $\bar{z} = a - ib$.

For example, $\quad |3+i| = \sqrt{3^2+1^2} = \sqrt{10}$, $|2-5i| = \sqrt{2^2+(-5)^2} = \sqrt{29}$,

and $\quad \overline{3+i} = 3-i$, $\overline{2-5i} = 2+5i$, $\overline{-3i-5} = 3i-5$

Observe that the multiplicative inverse of the non-zero complex number z is given by

$$z^{-1} = \frac{1}{a+ib} = \frac{a}{a^2+b^2}+i\frac{-b}{a^2+b^2} = \frac{a-ib}{a^2+b^2} = \frac{\bar{z}}{|z|^2}$$

or $\quad z\,\bar{z} = |z|^2$

Furthermore, the following results can easily be derived.

For any two compex numbers z_1 and z_2, we have

(i) $|z_1 z_2| = |z_1||z_2|$ (ii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ provided $|z_2| \neq 0$

(iii) $\overline{z_1 z_2} = \bar{z}_1\,\bar{z}_2$ (iv) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$ (v) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$ provided $z_2 \neq 0$.

Example 5 Find the multiplicative inverse of $2 - 3i$.

Solution Let $z = 2 - 3i$

Then $\bar{z} = 2 + 3i$ and $|z|^2 = 2^2 + (-3)^2 = 13$

Therefore, the multiplicative inverse of $2-3i$ is given by

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{2+3i}{13} = \frac{2}{13} + \frac{3}{13}i$$

The above working can be reproduced in the following manner also,

$$z^{-1} = \frac{1}{2-3i} = \frac{2+3i}{(2-3i)(2+3i)}$$

$$= \frac{2+3i}{2^2-(3i)^2} = \frac{2+3i}{13} = \frac{2}{13} + \frac{3}{13}i$$

Example 6 Express the following in the form $a + ib$

(i) $\frac{5+\sqrt{2}i}{1-\sqrt{2}i}$ (ii) i^{-35}

Solution (i) We have, $\frac{5+\sqrt{2}i}{1-\sqrt{2}i} = \frac{5+\sqrt{2}i}{1-\sqrt{2}i} \times \frac{1+\sqrt{2}i}{1+\sqrt{2}i} = \frac{5+5\sqrt{2}i+\sqrt{2}i-2}{1-\left(\sqrt{2}i\right)^2}$

$$= \frac{3+6\sqrt{2}i}{1+2} = \frac{3(1+2\sqrt{2}i)}{3} = 1+2\sqrt{2}i.$$

(ii) $i^{-35} = \frac{1}{i^{35}} = \frac{1}{\left(i^2\right)^{17} i} = \frac{1}{-i} \times \frac{i}{i} = \frac{i}{-i^2} = i$

EXERCISE 4.1

Express each of the complex number given in the Exercises 1 to 10 in the form $a + ib$.

1. $(5i)\left(-\frac{3}{5}i\right)$ **2.** $i^9 + i^{19}$ **3.** i^{-39}

4. $3(7 + i7) + i\,(7 + i7)$ **5.** $(1 – i) – (–1 + i6)$

6. $\left(\frac{1}{5}+i\frac{2}{5}\right)-\left(4+i\frac{5}{2}\right)$ **7.** $\left[\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i\frac{1}{3}\right)\right]-\left(-\frac{4}{3}+i\right)$

8. $(1 – i)^4$ **9.** $\left(\frac{1}{3}+3i\right)^3$ **10.** $\left(-2-\frac{1}{3}i\right)^3$

Find the multiplicative inverse of each of the complex numbers given in the Exercises 11 to 13.

11. $4 – 3i$ **12.** $\sqrt{5}+3i$ **13.** $– i$

14. Express the following expression in the form of $a + ib$:

$$\frac{\left(3+i\sqrt{5}\right)\left(3-i\sqrt{5}\right)}{\left(\sqrt{3}+\sqrt{2}\,i\right)-\left(\sqrt{3}-i\sqrt{2}\right)}$$

4.5 Argand Plane and Polar Representation

We already know that corresponding to each ordered pair of real numbers (x, y), we get a unique point in the XY-plane and vice-versa with reference to a set of mutually perpendicular lines known as the x-axis and the y-axis. The complex number $x + iy$ which corresponds to the ordered pair (x, y) can be represented geometrically as the unique point P(x, y) in the XY-plane and vice-versa.

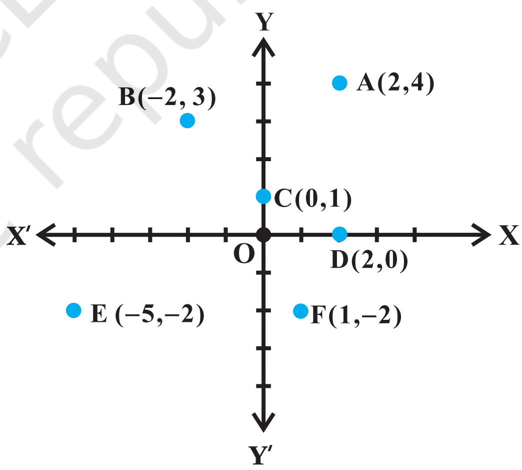


Fig 4.1

Some complex numbers such as $2 + 4i, – 2 + 3i, 0 + 1i, 2 + 0i, – 5 –2i$ and $1 – 2i$ which correspond to the ordered pairs (2, 4), (– 2, 3), (0, 1), (2, 0), (–5, –2), and (1, – 2), respectively, have been represented geometrically by the points A, B, C, D, E, and F, respectively in the Fig 4.1.

The plane having a complex number assigned to each of its point is called the *complex plane* or the *Argand plane*.

Obviously, in the Argand plane, the modulus of the complex number $x + iy = \sqrt{x^2 + y^2}$ is the distance between the point P(x, y) and the origin O (0, 0) (Fig 4.2). The points on the x-axis corresponds to the complex numbers of the form $a + i\,0$ and the points on the y-axis corresponds to the complex numbers of the form

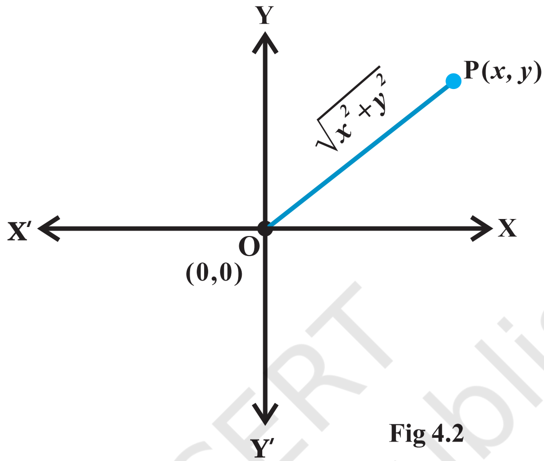


Fig 4.2

$0 + i\,b$. The x-axis and y-axis in the Argand plane are called, respectively, the *real axis* and the *imaginary axis*.

The representation of a complex number $z = x + iy$ and its conjugate $z = x - iy$ in the Argand plane are, respectively, the points P (x, y) and Q (x, $-y$).

Geometrically, the point (x, $-y$) is the mirror image of the point (x, y) on the real axis (Fig 4.3).

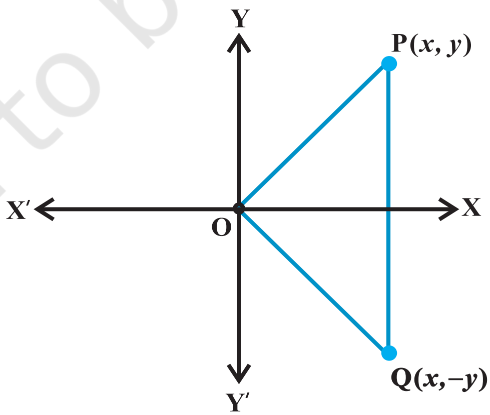


Fig 4.3

Miscellaneous Examples

Example 7 Find the conjugate of $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$.

Solution We have , $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$

$$= \frac{6+9i-4i+6}{2-i+4i+2} = \frac{12+5i}{4+3i}\times\frac{4-3i}{4-3i}$$

$$= \frac{48-36i+20i+15}{16+9} = \frac{63-16i}{25} = \frac{63}{25}-\frac{16}{25}i$$

Therefore, conjugate of $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$ is $\dfrac{63}{25}+\dfrac{16}{25}i$.

Example 8 If $x + iy = \dfrac{a+ib}{a-ib}$, prove that $x^2 + y^2 = 1$.

Solution We have,

$$x + iy = \frac{(a+ib)(a+ib)}{(a-ib)(a+ib)} = \frac{a^2-b^2+2abi}{a^2+b^2} = \frac{a^2-b^2}{a^2+b^2}+\frac{2ab}{a^2+b^2}i$$

So that, $x - iy = \dfrac{a^2-b^2}{a^2+b^2}-\dfrac{2ab}{a^2+b^2}i$

Therefore,

$$x^2 + y^2 = (x + iy)(x - iy) = \frac{(a^2-b^2)^2}{(a^2+b^2)^2}+\frac{4a^2b^2}{(a^2+b^2)^2} = \frac{(a^2+b^2)^2}{(a^2+b^2)^2} = 1$$

Miscellaneous Exercise on Chapter 4

1. Evaluate: $\left[i^{18}+\left(\dfrac{1}{i}\right)^{25}\right]^3$.

2. For any two complex numbers z_1 and z_2, prove that
$\text{Re}\,(z_1 z_2) = \text{Re}\, z_1\, \text{Re}\, z_2 - \text{Im}z_1\, \text{Im}z_2$.

3. Reduce $\left(\frac{1}{1-4i}-\frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$ to the standard form .

4. If $x-iy=\sqrt{\frac{a-ib}{c-id}}$ prove that $\left(x^2+y^2\right)^2=\frac{a^2+b^2}{c^2+d^2}$.

5. If $z_1 = 2 - i$, $z_2 = 1 + i$, find $\left|\frac{z_1+z_2+1}{z_1-z_2+1}\right|$.

6. If $a + ib = \frac{(x+i)^2}{2x^2+1}$, prove that $a^2 + b^2 = \frac{(x^2+1)^2}{\left(2x^2+1\right)^2}$.

7. Let $z_1 = 2 - i$, $z_2 = -2 + i$. Find

(i) $\text{Re}\left(\frac{z_1 z_2}{\bar{z}_1}\right)$, (ii) $\text{Im}\left(\frac{1}{z_1\bar{z}_1}\right)$.

8. Find the real numbers x and y if $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$.

9. Find the modulus of $\frac{1+i}{1-i}-\frac{1-i}{1+i}$.

10. If $(x + iy)^3 = u + iv$, then show that $\frac{u}{x}+\frac{v}{y}=4(x^2-y^2)$.

11. If α and β are different complex numbers with $|\beta|=1$, then find $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|$.

12. Find the number of non-zero integral solutions of the equation $|1-i|^x=2^x$.

13. If $(a + ib)(c + id)(e + if)(g + ih) = A + iB$, then show that

$(a^2 + b^2)(c^2 + d^2)(e^2 + f^2)(g^2 + h^2) = A^2 + B^2$

14. If $\left(\frac{1+i}{1-i}\right)^m=1$, then find the least positive integral value of m.

Summary

- A number of the form $a + ib$, where a and b are real numbers, is called a *complex number*, a is called the *real part* and b is called the *imaginary* part of the complex number.
- Let $z_1 = a + ib$ and $z_2 = c + id$. Then
  (i) $z_1 + z_2 = (a + c) + i\,(b + d)$
  (ii) $z_1 z_2 = (ac - bd) + i\,(ad + bc)$
- For any non-zero complex number $z = a + ib$ $(a \neq 0, b \neq 0)$, there exists the complex number $\frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2}$, denoted by $\frac{1}{z}$ or z^{-1}, called the *multiplicative inverse* of z such that $(a + ib)\left(\frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2}\right) = 1 + i0$ $=1$
- For any integer k, $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$
- The conjugate of the complex number $z = a + ib$, denoted by $\bar{z}$, is given by $\bar{z} = a - ib$.

Historical Note

The fact that square root of a negative number does not exist in the real number system was recognised by the Greeks. But the credit goes to the Indian mathematician *Mahavira* (850) who first stated this difficulty clearly. "He mentions in his work '*Ganitasara Sangraha*' as in the nature of things a negative (quantity) is not a square (quantity)', it has, therefore, no square root". *Bhaskara*, another Indian mathematician, also writes in his work *Bijaganita*, written in 1150. "There is no square root of a negative quantity, for it is not a square." *Cardan* (1545) considered the problem of solving

$$x + y = 10,\ xy = 40.$$

He obtained $x = 5 + \sqrt{-15}$ and $y = 5 - \sqrt{-15}$ as the solution of it, which was discarded by him by saying that these numbers are 'useless'. *Albert Girard* (about 1625) accepted square root of negative numbers and said that this will enable us to get as many roots as the degree of the polynomial equation. *Euler* was the first to introduce the symbol i for $\sqrt{-1}$ and *W.R. Hamilton* (about 1830) regarded the complex number $a + ib$ as an ordered pair of real numbers (a, b) thus giving it a purely mathematical definition and avoiding use of the so called '*imaginary numbers*'.

Chapter 5

LINEAR INEQUALITIES

❖ *Mathematics is the art of saying many things in many different ways. – MAXWELL* ❖

5.1 Introduction

In earlier classes, we have studied equations in one variable and two variables and also solved some statement problems by translating them in the form of equations. Now a natural question arises: 'Is it always possible to translate a statement problem in the form of an equation? For example, the height of all the students in your class is less than 160 cm. Your classroom can occupy atmost 60 tables or chairs or both. Here we get certain statements involving a sign '<' (less than), '>' (greater than), '≤' (less than or equal) and ≥ (greater than or equal) which are known as *inequalities*.

In this Chapter, we will study linear inequalities in one and two variables. The study of inequalities is very useful in solving problems in the field of science, mathematics, statistics, economics, psychology, etc.

5.2 Inequalities

Let us consider the following situations:

(i) Ravi goes to market with ₹ 200 to buy rice, which is available in packets of 1kg. The price of one packet of rice is ₹ 30. If x denotes the number of packets of rice, which he buys, then the total amount spent by him is ₹ $30x$. Since, he has to buy rice in packets only, he may not be able to spend the entire amount of ₹ 200. (Why?) Hence

$$30x < 200 \qquad \ldots (1)$$

Clearly the statement (i) is not an equation as it does not involve the sign of equality.

(ii) Reshma has ₹ 120 and wants to buy some registers and pens. The cost of one register is ₹ 40 and that of a pen is ₹ 20. In this case, if x denotes the number of registers and y, the number of pens which Reshma buys, then the total amount spent by her is ₹ $(40x + 20y)$ and we have

$$40x + 20y \leq 120 \qquad \ldots (2)$$

Since in this case the total amount spent may be upto ₹ 120. Note that the statement (2) consists of two statements

$$40x + 20y < 120 \quad \text{... (3)}$$

and $$40x + 20y = 120 \quad \text{... (4)}$$

Statement (3) is not an equation, i.e., it is an inequality while statement (4) is an equation.

Definition 1 Two real numbers or two algebraic expressions related by the symbol '<', '>', '≤' or '≥' form an *inequality*.

Statements such as (1), (2) and (3) above are inequalities.

$3 < 5$; $7 > 5$ are the examples of *numerical inequalities* while

$x < 5$; $y > 2$; $x \geq 3$, $y \leq 4$ are some examples of *literal inequalities*.

$3 < 5 < 7$ (read as 5 is greater than 3 and less than 7), $3 \leq x < 5$ (read as x is greater than or equal to 3 and less than 5) and $2 < y \leq 4$ are the examples of *double inequalities*.

Some more examples of inequalities are:

$ax + b < 0$ ... (5)

$ax + b > 0$ ... (6)

$ax + b \leq 0$ ... (7)

$ax + b \geq 0$ ... (8)

$ax + by < c$ ... (9)

$ax + by > c$ ... (10)

$ax + by \leq c$ ... (11)

$ax + by \geq c$ ... (12)

$ax^2 + bx + c \leq 0$ ... (13)

$ax^2 + bx + c > 0$ ... (14)

Inequalities (5), (6), (9), (10) and (14) are *strict inequalities* while inequalities (7), (8), (11), (12), and (13) are *slack inequalities*. Inequalities from (5) to (8) are *linear inequalities* in one variable x when $a \neq 0$, while inequalities from (9) to (12) *are linear inequalities in two variables x and y* when $a \neq 0$, $b \neq 0$.

Inequalities (13) and (14) are not linear *(in fact, these are quadratic inequalities in one variable x when a* $\neq 0$).

In this Chapter, we shall confine ourselves to the study of linear inequalities in one and two variables only.

5.3 Algebraic Solutions of Linear Inequalities in One Variable and their Graphical Representation

Let us consider the inequality (1) of Section 6.2, viz, $30x < 200$
Note that here x denotes the number of packets of rice.

Obviously, x cannot be a negative integer or a fraction. Left hand side (L.H.S.) of this inequality is $30x$ and right hand side (RHS) is 200. Therefore, we have

For $x = 0$, L.H.S. = 30 (0) = 0 < 200 (R.H.S.), which is true.

For $x = 1$, L.H.S. = 30 (1) = 30 < 200 (R.H.S.), which is true.

For $x = 2$, L.H.S. = 30 (2) = 60 < 200, which is true.

For $x = 3$, L.H.S. = 30 (3) = 90 < 200, which is true.

For $x = 4$, L.H.S. = 30 (4) = 120 < 200, which is true.

For $x = 5$, L.H.S. = 30 (5) = 150 < 200, which is true.

For $x = 6$, L.H.S. = 30 (6) = 180 < 200, which is true.

For $x = 7$, L.H.S. = 30 (7) = 210 < 200, which is false.

In the above situation, we find that the values of x, which makes the above inequality a true statement, are 0,1,2,3,4,5,6. These values of x, which make above inequality a true statement, are called *solutions* of inequality and the set {0,1,2,3,4,5,6} is called its *solution set*.

Thus, any solution of an inequality in one variable is a value of the variable which makes it a true statement.

We have found the solutions of the above inequality by *trial and error* method which is not very efficient. Obviously, this method is time consuming and sometimes not feasible. We must have some better or systematic techniques for solving inequalities. Before that we should go through some more properties of numerical inequalities and follow them as rules while solving the inequalities.

You will recall that while solving linear equations, we followed the following rules:

Rule 1 Equal numbers may be added to (or subtracted from) both sides of an equation.

Rule 2 Both sides of an equation may be multiplied (or divided) by the same non-zero number.

In the case of solving inequalities, we again follow the same rules except with a difference that in Rule 2, the sign of inequality is reversed (i.e., ‘$<$‘ becomes ‘$>$’, $\leq$’ becomes ‘$\geq$’ and so on) whenever we multiply (or divide) both sides of an inequality by a negative number. It is evident from the facts that

$$3 > 2 \text{ while } -3 < -2,$$
$$-8 < -7 \text{ while } (-8)(-2) > (-7)(-2) \text{, i.e., } 16 > 14.$$

Thus, we state the following rules for solving an inequality:

Rule 1 Equal numbers may be added to (or subtracted from) both sides of an inequality without affecting the sign of inequality.

Rule 2 Both sides of an inequality can be multiplied (or divided) by the same positive number. But when both sides are multiplied or divided by a negative number, then the sign of inequality is *reversed.*

Now, let us consider some examples.

Example 1 Solve $30\,x < 200$ when

(i) x is a natural number, (ii) x is an integer.

Solution We are given $30\,x < 200$

or $\frac{30x}{30} < \frac{200}{30}$ (Rule 2), i.e., $x < 20 / 3$.

(i) When x is a natural number, in this case the following values of x make the statement true.

1, 2, 3, 4, 5, 6.

The solution set of the inequality is $\{1,2,3,4,5,6\}$.

(ii) When x is an integer, the solutions of the given inequality are

$..., -3, -2, -1, 0, 1, 2, 3, 4, 5, 6$

The solution set of the inequality is $\{...,-3, -2,-1, 0, 1, 2, 3, 4, 5, 6\}$

Example 2 Solve $5x - 3 < 3x + 1$ when

(i) x is an integer, (ii) x is a real number.

Solution We have, $5x -3 < 3x + 1$

or $5x -3 + 3 < 3x +1 +3$ (Rule 1)

or $5x < 3x +4$

or $5x - 3x < 3x + 4 - 3x$ (Rule 1)

or $2x < 4$

or $x < 2$ (Rule 2)

(i) When x is an integer, the solutions of the given inequality are

$..., -4, -3, -2, -1, 0, 1$

(ii) When x is a real number, the solutions of the inequality are given by $x < 2$, i.e., all real numbers x which are less than 2. Therefore, the solution set of the inequality is $x \in (-\infty, 2)$.

We have considered solutions of inequalities in the set of natural numbers, set of integers and in the set of real numbers. Henceforth, unless stated otherwise, we shall solve the inequalities in this Chapter in the set of real numbers.

Example 3 Solve $4x + 3 < 6x + 7$.

Solution We have, $4x + 3 < 6x + 7$

or $4x - 6x < 6x + 4 - 6x$

or $-2x < 4$ or $x > -2$

i.e., all the real numbers which are greater than –2, are the solutions of the given inequality. Hence, the solution set is $(-2, \infty)$.

Example 4 Solve $\frac{5-2x}{3} \leq \frac{x}{6} - 5$.

Solution We have

$$\frac{5-2x}{3} \leq \frac{x}{6} - 5$$

or $2(5 - 2x) \leq x - 30.$

or $10 - 4x \leq x - 30$

or $-5x \leq -40$, i.e., $x \geq 8$

Thus, all real numbers x which are greater than or equal to 8 are the solutions of the given inequality, i.e., $x \in [8, \infty)$.

Example 5 Solve $7x + 3 < 5x + 9$. Show the graph of the solutions on number line.

Solution We have $7x + 3 < 5x + 9$ or

$2x < 6$ or $x < 3$

The graphical representation of the solutions are given in Fig 5.1.

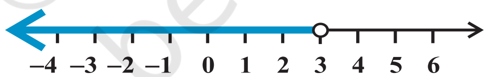


Fig 5.1

Example 6 Solve $\frac{3x-4}{2} \geq \frac{x+1}{4} - 1$. Show the graph of the solutions on number line.

Solution We have

$$\frac{3x-4}{2} \geq \frac{x+1}{4} - 1$$

or $$\frac{3x-4}{2} \geq \frac{x-3}{4}$$

or $2(3x - 4) \geq (x - 3)$

or $\qquad 6x - 8 \geq x - 3$

or $\qquad 5x \geq 5$ or $x \geq 1$

The graphical representation of solutions is given in Fig 5.2.

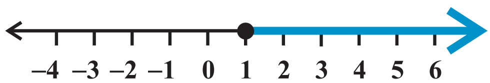


Fig 5.2

Example 7 The marks obtained by a student of Class XI in first and second terminal examination are 62 and 48, respectively. Find the minimum marks he should get in the annual examination to have an average of at least 60 marks.

Solution Let x be the marks obtained by student in the annual examination. Then

$$\frac{62+48+x}{3} \geq 60$$

or $\qquad 110 + x \geq 180$

or $\qquad x \geq 70$

Thus, the student must obtain a minimum of 70 marks to get an average of at least 60 marks.

Example 8 Find all pairs of consecutive odd natural numbers, both of which are larger than 10, such that their sum is less than 40.

Solution Let x be the smaller of the two consecutive odd natural number, so that the other one is x +2. Then, we should have

$x > 10 \qquad$ ... (1)

and $\quad x + (x + 2) < 40 \qquad$ ... (2)

Solving (2), we get

$2x + 2 < 40$

i.e., $\quad x < 19 \qquad$ ... (3)

From (1) and (3), we get

$10 < x < 19$

Since x is an odd number, x can take the values 11, 13, 15, and 17. So, the required possible pairs will be

(11, 13), (13, 15), (15, 17), (17, 19)

EXERCISE 5.1

1. Solve $24x < 100$, when
(i) x is a natural number. (ii) x is an integer.

2. Solve $-12x > 30$, when
(i) x is a natural number. (ii) x is an integer.

3. Solve $5x - 3 < 7$, when
(i) x is an integer. (ii) x is a real number.

4. Solve $3x + 8 > 2$, when
(i) x is an integer. (ii) x is a real number.

Solve the inequalities in Exercises 5 to 16 for real x.

5. $4x + 3 < 5x + 7$

6. $3x - 7 > 5x - 1$

7. $3(x - 1) \leq 2(x - 3)$

8. $3(2 - x) \geq 2(1 - x)$

9. $x + \frac{x}{2} + \frac{x}{3} < 11$

10. $\frac{x}{3} > \frac{x}{2} + 1$

11. $\frac{3(x-2)}{5} \leq \frac{5(2-x)}{3}$

12. $\frac{1}{2}\left(\frac{3x}{5} + 4\right) \geq \frac{1}{3}(x - 6)$

13. $2(2x + 3) - 10 < 6(x - 2)$

14. $37 - (3x + 5) \geq 9x - 8(x - 3)$

15. $\frac{x}{4} < \frac{(5x-2)}{3} - \frac{(7x-3)}{5}$

16. $\frac{(2x-1)}{3} \geq \frac{(3x-2)}{4} - \frac{(2-x)}{5}$

Solve the inequalities in Exercises 17 to 20 and show the graph of the solution in each case on number line

17. $3x - 2 < 2x + 1$

18. $5x - 3 \geq 3x - 5$

19. $3(1 - x) < 2(x + 4)$

20. $\frac{x}{2} \geq \frac{(5x-2)}{3} - \frac{(7x-3)}{5}$

21. Ravi obtained 70 and 75 marks in first two unit test. Find the minimum marks he should get in the third test to have an average of at least 60 marks.

22. To receive Grade 'A' in a course, one must obtain an average of 90 marks or more in five examinations (each of 100 marks). If Sunita's marks in first four examinations are 87, 92, 94 and 95, find minimum marks that Sunita must obtain in fifth examination to get grade 'A' in the course.

23. Find all pairs of consecutive odd positive integers both of which are smaller than 10 such that their sum is more than 11.

24. Find all pairs of consecutive even positive integers, both of which are larger than 5 such that their sum is less than 23.

25. The longest side of a triangle is 3 times the shortest side and the third side is 2 cm shorter than the longest side. If the perimeter of the triangle is at least 61 cm, find the minimum length of the shortest side.

26. A man wants to cut three lengths from a single piece of board of length 91cm. The second length is to be 3cm longer than the shortest and the third length is to be twice as long as the shortest. What are the possible lengths of the shortest board if the third piece is to be at least 5cm longer than the second?
[***Hint:*** If x is the length of the shortest board, then x , $(x + 3)$ and $2x$ are the lengths of the second and third piece, respectively. Thus, $x + (x + 3) + 2x \le 91$ and $2x \ge (x + 3) + 5$].

Miscellaneous Examples

Example 9 Solve $-8 \le 5x - 3 < 7$.

Solution In this case, we have two inequalities, $-8 \le 5x - 3$ and $5x - 3 < 7$, which we will solve simultaneously. We have $-8 \le 5x - 3 < 7$

or $\quad -5 \le 5x < 10 \quad$ or $\quad -1 \le x < 2$

Example 10 Solve $-5 \le \dfrac{5-3x}{2} \le 8$.

Solution We have $\quad -5 \le \dfrac{5-3x}{2} \le 8$

or $\quad -10 \le 5 - 3x \le 16 \quad$ or $\quad -15 \le -3x \le 11$

or $\quad 5 \ge x \ge -\dfrac{11}{3}$

which can be written as $\dfrac{-11}{3} \le x \le 5$

Example 11 Solve the system of inequalities:

$$3x - 7 < 5 + x \qquad \text{... (1)}$$

$$11 - 5x \le 1 \qquad \text{... (2)}$$

and represent the solutions on the number line.

Solution From inequality (1), we have

$$3x - 7 < 5 + x$$

or $\quad x < 6 \qquad$ ... (3)

Also, from inequality (2), we have

$$11 - 5x \le 1$$

or $\quad -5x \le -10 \quad$ i.e., $x \ge 2 \qquad$ ... (4)

If we draw the graph of inequalities (3) and (4) on the number line, we see that the values of x, which are common to both, are shown by bold line in Fig 5.3.

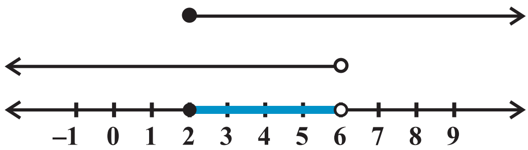


Fig 5.3

Thus, solution of the system are real numbers x lying between 2 and 6 including 2, i.e.,

$$2 \leq x < 6$$

Example 12 In an experiment, a solution of hydrochloric acid is to be kept between 30° and 35° Celsius. What is the range of temperature in degree Fahrenheit if conversion formula is given by $C = \frac{5}{9}(F - 32)$, where C and F represent temperature in degree Celsius and degree Fahrenheit, respectively.

Solution It is given that $30 < C < 35$.

Putting $\quad C = \frac{5}{9}(F - 32)$, we get

$$30 < \frac{5}{9}(F - 32) < 35,$$

or $\quad \frac{9}{5} \times (30) < (F - 32) < \frac{9}{5} \times (35)$

or $\quad 54 < (F - 32) < 63$

or $\quad 86 < F < 95.$

Thus, the required range of temperature is between 86° F and 95° F.

Example 13 A manufacturer has 600 litres of a 12% solution of acid. How many litres of a 30% acid solution must be added to it so that acid content in the resulting mixture will be more than 15% but less than 18%?

Solution Let x litres of 30% acid solution is required to be added. Then

Total mixture = $(x + 600)$ litres

Therefore $\quad 30\%\, x + 12\% \text{ of } 600 > 15\% \text{ of } (x + 600)$

and $\quad 30\%\, x + 12\% \text{ of } 600 < 18\% \text{ of } (x + 600)$

or $\quad \frac{30x}{100} + \frac{12}{100}(600) > \frac{15}{100}(x + 600)$

and $\frac{30x}{100} + \frac{12}{100}(600) < \frac{18}{100}(x + 600)$

or $30x + 7200 > 15x + 9000$

and $30x + 7200 < 18x + 10800$

or $15x > 1800$ and $12x < 3600$

or $x > 120$ and $x < 300$,

i.e. $120 < x < 300$

Thus, the number of litres of the 30% solution of acid will have to be more than 120 litres but less than 300 litres.

Miscellaneous Exercise on Chapter 5

Solve the inequalities in Exercises 1 to 6.

1. $2 \le 3x - 4 \le 5$

2. $6 \le -3(2x - 4) < 12$

3. $-3 \le 4 - \frac{7x}{2} \le 18$

4. $-15 < \frac{3(x-2)}{5} \le 0$

5. $-12 < 4 - \frac{3x}{-5} \le 2$

6. $7 \le \frac{(3x+11)}{2} \le 11$.

Solve the inequalities in Exercises 7 to 10 and represent the solution graphically on number line.

7. $5x + 1 > -24$, $5x - 1 < 24$

8. $2(x - 1) < x + 5$, $3(x + 2) > 2 - x$

9. $3x - 7 > 2(x - 6)$, $6 - x > 11 - 2x$

10. $5(2x - 7) - 3(2x + 3) \le 0$, $2x + 19 \le 6x + 47$.

11. A solution is to be kept between 68° F and 77° F. What is the range in temperature in degree Celsius (C) if the Celsius / Fahrenheit (F) conversion formula is given by

$$F = \frac{9}{5}C + 32 ?$$

12. A solution of 8% boric acid is to be diluted by adding a 2% boric acid solution to it. The resulting mixture is to be more than 4% but less than 6% boric acid. If we have 640 litres of the 8% solution, how many litres of the 2% solution will have to be added?

13. How many litres of water will have to be added to 1125 litres of the 45% solution of acid so that the resulting mixture will contain more than 25% but less than 30% acid content?

14. IQ of a person is given by the formula

$$IQ = \frac{MA}{CA} \times 100,$$

where MA is mental age and CA is chronological age. If $80 \leq IQ \leq 140$ for a group of 12 years old children, find the range of their mental age.

Summary

- Two real numbers or two algebraic expressions related by the symbols $<, >, \leq$ or $\geq$ form an inequality.
- Equal numbers may be added to (or subtracted from) both sides of an inequality.
- Both sides of an inequality can be multiplied (or divided) by the same positive number. But when both sides are multiplied (or divided) by a negative number, then the inequality is reversed.
- The values of x, which make an inequality a true statement, are called *solutions of the inequality*.
- To represent $x < a$ (or $x > a$) on a number line, put a circle on the number a and dark line to the left (or right) of the number a.
- To represent $x \leq a$ (or $x \geq a$) on a number line, put a dark circle on the number a and dark the line to the left (or right) of the number x.

Chapter 6

PERMUTATIONS AND COMBINATIONS

❖ *Every body of discovery is mathematical in form because there is no other guidance we can have – DARWIN* ❖

6.1 Introduction

Jacob Bernoulli (1654-1705)

Suppose you have a suitcase with a number lock. The number lock has 4 wheels each labelled with 10 digits from 0 to 9. The lock can be opened if 4 specific digits are arranged in a particular sequence with no repetition. Some how, you have forgotten this specific sequence of digits. You remember only the first digit which is 7. In order to open the lock, how many sequences of 3-digits you may have to check with? To answer this question, you may, immediately, start listing all possible arrangements of 9 remaining digits taken 3 at a time. But, this method will be tedious, because the number of possible sequences may be large. Here, in this Chapter, we shall learn some basic counting techniques which will enable us to answer this question without actually listing 3-digit arrangements. In fact, these techniques will be useful in determining the number of different ways of arranging and selecting objects without actually listing them. As a first step, we shall examine a principle which is most fundamental to the learning of these techniques.

6.2 Fundamental Principle of Counting

Let us consider the following problem. Mohan has 3 pants and 2 shirts. How many different pairs of a pant and a shirt, can he dress up with? There are 3 ways in which a pant can be chosen, because there are 3 pants available. Similarly, a shirt can be chosen in 2 ways. For every choice of a pant, there are 2 choices of a shirt. Therefore, there are $3 \times 2 = 6$ pairs of a pant and a shirt.

Let us name the three pants as P_1, P_2, P_3 and the two shirts as S_1, S_2. Then, these six possibilities can be illustrated in the Fig. 6.1.

Let us consider another problem of the same type.

Sabnam has 2 school bags, 3 tiffin boxes and 2 water bottles. In how many ways can she carry these items (choosing one each).

A school bag can be chosen in 2 different ways. After a school bag is chosen, a tiffin box can be chosen in 3 different ways. Hence, there are $2 \times 3 = 6$ pairs of school bag and a tiffin box. For each of these pairs a water bottle can be chosen in 2 different ways.

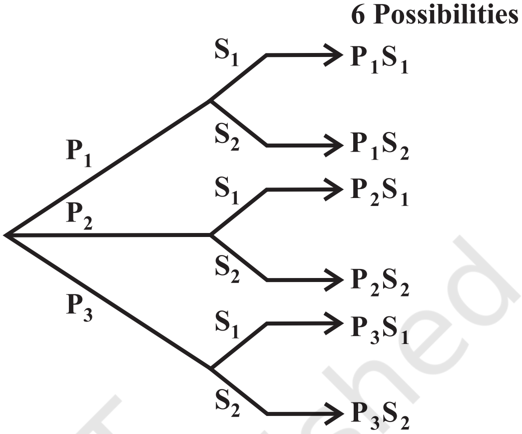


Fig 6.1

Hence, there are $6 \times 2 = 12$ different ways in which, Sabnam can carry these items to school. If we name the 2 school bags as B_1, B_2, the three tiffin boxes as T_1, T_2, T_3 and the two water bottles as W_1, W_2, these possibilities can be illustrated in the Fig. 6.2.

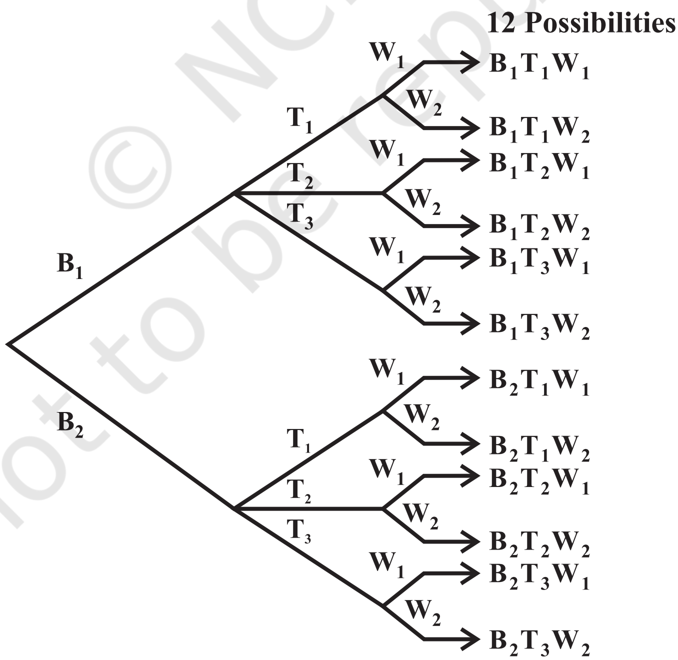


Fig 6.2

In fact, the problems of the above types are solved by applying the following principle known as the *fundamental principle of counting*, or, simply, the *multiplication principle*, which states that

"If an event can occur in m different ways, following which another event can occur in n different ways, then the total number of occurrence of the events in the given order is m×n."

The above principle can be generalised for any finite number of events. For example, for 3 events, the principle is as follows:

'If an event can occur in m different ways, following which another event can occur in n different ways, following which a third event can occur in p different ways, then the total number of occurrence to 'the events in the given order is $m \times n \times p$."

In the first problem, the required number of ways of wearing a pant and a shirt was the number of different ways of the occurence of the following events in succession:

(i) the event of choosing a pant
(ii) the event of choosing a shirt.

In the second problem, the required number of ways was the number of different ways of the occurence of the following events in succession:

(i) the event of choosing a school bag
(ii) the event of choosing a tiffin box
(iii) the event of choosing a water bottle.

Here, in both the cases, the events in each problem could occur in various possible orders. But, we have to choose any one of the possible orders and count the number of different ways of the occurence of the events in this chosen order.

Example 1 Find the number of 4 letter words, with or without meaning, which can be formed out of the letters of the word ROSE, where the repetition of the letters is not allowed.

Solution There are as many words as there are ways of filling in 4 vacant places $\square\ \square\ \square\ \square$ by the 4 letters, keeping in mind that the repetition is not allowed. The first place can be filled in 4 different ways by anyone of the 4 letters R,O,S,E. Following which, the second place can be filled in by anyone of the remaining 3 letters in 3 different ways, following which the third place can be filled in 2 different ways; following which, the fourth place can be filled in 1 way. Thus, the number of ways in which the 4 places can be filled, by the multiplication principle, is $4 \times 3 \times 2 \times 1 = 24$. Hence, the required number of words is 24.

> **Note** If the repetition of the letters was allowed, how many words can be formed? One can easily understand that each of the 4 vacant places can be filled in succession in 4 different ways. Hence, the required number of words = $4 \times 4 \times 4 \times 4 = 256$.

Example 2 Given 4 flags of different colours, how many different signals can be generated, if a signal requires the use of 2 flags one below the other?

Solution There will be as many signals as there are ways of filling in 2 vacant places in succession by the 4 flags of different colours. The upper vacant place can be filled in 4 different ways by anyone of the 4 flags; following which, the lower vacant place can be filled in 3 different ways by anyone of the remaining 3 different flags. Hence, by the multiplication principle, the required number of signals = $4 \times 3 = 12$.

Example 3 How many 2 digit even numbers can be formed from the digits 1, 2, 3, 4, 5 if the digits can be repeated?

Solution There will be as many ways as there are ways of filling 2 vacant places in succession by the five given digits. Here, in this case, we start filling in unit's place, because the options for this place are 2 and 4 only and this can be done in 2 ways; following which the ten's place can be filled by any of the 5 digits in 5 different ways as the digits can be repeated. Therefore, by the multiplication principle, the required number of two digits even numbers is 2×5, i.e., 10.

Example 4 Find the number of different signals that can be generated by arranging at least 2 flags in order (one below the other) on a vertical staff, if five different flags are available.

Solution A signal can consist of either 2 flags, 3 flags, 4 flags or 5 flags. Now, let us count the possible number of signals consisting of 2 flags, 3 flags, 4 flags and 5 flags separately and then add the respective numbers.

There will be as many 2 flag signals as there are ways of filling in 2 vacant places in succession by the 5 flags available. By Multiplication rule, the number of ways is $5 \times 4 = 20$.

Similarly, there will be as many 3 flag signals as there are ways of filling in 3 vacant places in succession by the 5 flags.

The number of ways is $5 \times 4 \times 3 = 60$.

Continuing the same way, we find that

The number of 4 flag signals $= 5 \times 4 \times 3 \times 2 = 120$

and the number of 5 flag signals $= 5 \times 4 \times 3 \times 2 \times 1 = 120$

Therefore, the required no of signals $= 20 + 60 + 120 + 120 = 320$.

EXERCISE 6.1

1. How many 3-digit numbers can be formed from the digits 1, 2, 3, 4 and 5 assuming that

(i) repetition of the digits is allowed?

(ii) repetition of the digits is not allowed?

2. How many 3-digit even numbers can be formed from the digits 1, 2, 3, 4, 5, 6 if the digits can be repeated?

3. How many 4-letter code can be formed using the first 10 letters of the English alphabet, if no letter can be repeated?

4. How many 5-digit telephone numbers can be constructed using the digits 0 to 9 if each number starts with 67 and no digit appears more than once?

5. A coin is tossed 3 times and the outcomes are recorded. How many possible outcomes are there?

6. Given 5 flags of different colours, how many different signals can be generated if each signal requires the use of 2 flags, one below the other?

6.3 Permutations

In Example 1 of the previous Section, we are actually counting the different possible arrangements of the letters such as ROSE, REOS, ..., etc. Here, in this list, each arrangement is different from other. In other words, the order of writing the letters is important. Each arrangement is called a *permutation of* 4 *different letters taken all at a time*. Now, if we have to determine the number of 3-letter words, with or without meaning, which can be formed out of the letters of the word NUMBER, where the repetition of the letters is not allowed, we need to count the arrangements NUM, NMU, MUN, NUB, ..., etc. Here, we are counting the permutations of 6 different letters taken 3 at a time. The required number of words $= 6 \times 5 \times 4 = 120$ (by using multiplication principle).

If the repetition of the letters was allowed, the required number of words would be $6 \times 6 \times 6 = 216$.

Definition 1 A permutation is an arrangement in a definite order of a number of objects taken some or all at a time.

In the following sub-section, we shall obtain the formula needed to answer these questions immediately.

6.3.1 *Permutations when all the objects are distinct*

Theorem 1 The number of permutations of n different objects taken r at a time, where $0 < r \le n$ and the objects do not repeat is $n(n-1)(n-2)\ldots(n-r+1)$, which is denoted by ${}^nP_r$.

Proof There will be as many permutations as there are ways of filling in r vacant places $\square\ \square\ \square \ldots \square$ by

$\leftarrow r$ vacant places $\rightarrow$

the n objects. The first place can be filled in n ways; following which, the second place can be filled in $(n-1)$ ways, following which the third place can be filled in $(n-2)$ ways,..., the rth place can be filled in $(n-(r-1))$ ways. Therefore, the number of ways of filling in r vacant places in succession is $n(n-1)(n-2)\ldots(n-(r-1))$ or $n(n-1)(n-2)\ldots(n-r+1)$

This expression for ${}^nP_r$ is cumbersome and we need a notation which will help to reduce the size of this expression. The symbol $n!$ (read as factorial n or n factorial) comes to our rescue. In the following text we will learn what actually $n!$ means.

6.3.2 ***Factorial notation*** The notation $n!$ represents the product of first n natural numbers, i.e., the product $1 \times 2 \times 3 \times \ldots \times (n-1) \times n$ is denoted as $n!$. We read this symbol as 'n factorial'. Thus, $1 \times 2 \times 3 \times 4 \ldots \times (n-1) \times n = n!$

$$1 = 1!$$
$$1 \times 2 = 2!$$
$$1 \times 2 \times 3 = 3!$$
$$1 \times 2 \times 3 \times 4 = 4! \text{ and so on.}$$

We define $0! = 1$

We can write $5! = 5 \times 4! = 5 \times 4 \times 3! = 5 \times 4 \times 3 \times 2!$
$$= 5 \times 4 \times 3 \times 2 \times 1!$$

Clearly, for a natural number n

$$n! = n(n-1)!$$
$$= n(n-1)(n-2)! \qquad [\text{provided } (n \ge 2)]$$
$$= n(n-1)(n-2)(n-3)! \qquad [\text{provided } (n \ge 3)]$$

and so on.

Example 5 Evaluate (i) 5 ! (ii) 7 ! (iii) 7 ! – 5!

Solution (i) $5\,! = 1 \times 2 \times 3 \times 4 \times 5 = 120$

(ii) $7\,! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 = 5040$

and (iii) $7\,! - 5! = 5040 - 120 = 4920$.

Example 6 Compute (i) $\dfrac{7!}{5!}$ (ii) $\dfrac{12!}{(10!)\,(2!)}$

Solution (i) We have $\dfrac{7!}{5!} = \dfrac{7\times6\times5!}{5!} = 7 \times 6 = 42$

and (ii) $\dfrac{12!}{(10!)\,(2!)} = \dfrac{12\times11\times(10!)}{(10!)\times(2)} = 6 \times 11 = 66$.

Example 7 Evaluate $\dfrac{n!}{r!(n-r)!}$, when $n = 5$, $r = 2$.

Solution We have to evaluate $\dfrac{5!}{2!(5-2)!}$ (since $n = 5$, $r = 2$)

We have $\dfrac{5!}{2\,!(5-2)!} = \dfrac{5!}{2!\times3!} = \dfrac{5\times4}{2} = 10$.

Example 8 If $\dfrac{1}{8!} + \dfrac{1}{9!} = \dfrac{x}{10!}$, find x.

Solution We have $\dfrac{1}{8!} + \dfrac{1}{9\times8!} = \dfrac{x}{10\times9\times8!}$

Therefore $1+\dfrac{1}{9} = \dfrac{x}{10\times9}$ or $\dfrac{10}{9} = \dfrac{x}{10\times9}$

So $x = 100$.

EXERCISE 6.2

1. Evaluate

(i) 8 ! (ii) 4 ! – 3 !

2. Is 3 ! + 4 ! = 7 ! ? **3.** Compute $\frac{8!}{6!\times 2!}$ **4.** If $\frac{1}{6!}+\frac{1}{7!}=\frac{x}{8!}$, find x

5. Evaluate $\frac{n!}{(n-r)!}$, when

(i) $n = 6, r = 2$ (ii) $n = 9, r = 5$.

6.3.3 *Derivation of the formula for ^nP_r*

$$^nP_r = \frac{n!}{(n-r)!},\ 0 \le r \le n$$

Let us now go back to the stage where we had determined the following formula:

$$^nP_r = n\ (n-1)\ (n-2)\ .\ .\ .\ \ (n-r+1)$$

Multiplying numerator and denomirator by $(n-r)\ (n-r-1)\ .\ .\ .\ 3 \times 2 \times 1$, we get

$$^nP_r = \frac{n(n-1)\,(n-2)...(n-r+1)(n-r)(n-r-1)...3\times 2\times 1}{(n-r)(n-r-1)...3\times 2\times 1} = \frac{n!}{(n-r)!},$$

Thus $\quad ^nP_r = \frac{n!}{(n-r)!}$, where $0 < r \le n$

This is a much more convenient expression for ^nP_r than the previous one.

In particular, when $r = n$, $^nP_n = \frac{n!}{0!} = n!$

Counting permutations is merely counting the number of ways in which some or all objects at a time are rearranged. Arranging no object at all is the same as leaving behind all the objects and we know that there is only one way of doing so. Thus, we can have

$$^nP_0 = 1 = \frac{n!}{n!} = \frac{n!}{(n-0)!} \qquad \text{... (1)}$$

Therefore, the formula (1) is applicable for $r = 0$ also.

Thus $\quad ^nP_r = \frac{n!}{(n-r)!},\ 0 \le r \le n$.

Theorem 2 The number of permutations of n different objects taken r at a time, where repetition is allowed, is n^r.

Proof is very similar to that of Theorem 1 and is left for the reader to arrive at.

Here, we are solving some of the problems of the pervious Section using the formula for ${}^nP_r$ to illustrate its usefulness.

In Example 1, the required number of words = ${}^4P_4 = 4! = 24$. Here repetition is not allowed. If repetition is allowed, the required number of words would be $4^4 = 256$.

The number of 3-letter words which can be formed by the letters of the word NUMBER = ${}^6P_3 = \frac{6!}{3!} = 4 \times 5 \times 6 = 120$. Here, in this case also, the repetition is not allowed. If the repetition is allowed,the required number of words would be $6^3 = 216$.

The number of ways in which a Chairman and a Vice-Chairman can be chosen from amongst a group of 12 persons assuming that one person can not hold more than one position, clearly ${}^{12}P_2 = \frac{12!}{10!} = 11 \times 12 = 132$.

6.3.4 ***Permutations when all the objects are not distinct objects*** Suppose we have to find the number of ways of rearranging the letters of the word ROOT. In this case, the letters of the word are not all different. There are 2 Os, which are of the same kind. Let us treat, temporarily, the 2 Os as different, say, O_1 and O_2. The number of permutations of 4-different letters, in this case, taken all at a time is 4!. Consider one of these permutations say, RO_1O_2T. Corresponding to this permutation,we have 2 ! permutations RO_1O_2T and RO_2O_1T which will be exactly the same permutation if O_1 and O_2 are not treated as different, i.e., if O_1 and O_2 are the same O at both places.

Therefore, the required number of permutations = $\frac{4!}{2!} = 3 \times 4 = 12$.

Permutations when O_1, O_2 are different. — Permutations when O_1, O_2 are the same O.

$$\left.\begin{matrix} RO_1O_2T \\ RO_2O_1T \end{matrix}\right] \longrightarrow \text{R O O T}$$

$$\left.\begin{matrix} TO_1O_2R \\ TO_2O_1R \end{matrix}\right] \longrightarrow \text{T O O R}$$

$$\left.\begin{matrix} R\,O_1\,T\,O_2 \\ R\,O_2\,T\,O_1 \end{matrix}\right] \longrightarrow R\,O\,T\,O$$

$$\left.\begin{matrix} T\,O_1\,R\,O_2 \\ T\,O_2\,R\,O_1 \end{matrix}\right] \longrightarrow T\,O\,R\,O$$

$$\left.\begin{matrix} R\,T\,O_1\,O_2 \\ R\,T\,O_2\,O_1 \end{matrix}\right] \longrightarrow R\,T\,O\,O$$

$$\left.\begin{matrix} T\,R\,O_1\,O_2 \\ T\,R\,O_2\,O_1 \end{matrix}\right] \longrightarrow T\,R\,O\,O$$

$$\left.\begin{matrix} O_1\,O_2\,R\,T \\ O_2\,O_1\,T\,R \end{matrix}\right] \longrightarrow O\,O\,R\,T$$

$$\left.\begin{matrix} O_1\,R\,O_2\,T \\ O_2\,R\,O_1\,T \end{matrix}\right] \longrightarrow O\,R\,O\,T$$

$$\left.\begin{matrix} O_1\,T\,O_2\,R \\ O_2\,T\,O_1\,R \end{matrix}\right] \longrightarrow O\,T\,O\,R$$

$$\left.\begin{matrix} O_1\,R\,T\,O_2 \\ O_2\,R\,T\,O_1 \end{matrix}\right] \longrightarrow O\,R\,T\,O$$

$$\left.\begin{matrix} O_1\,T\,R\,O_2 \\ O_2\,T\,R\,O_1 \end{matrix}\right] \longrightarrow O\,T\,R\,O$$

$$\left.\begin{matrix} O_1\,O_2\,T\,R \\ O_2\,O_1\,T\,R \end{matrix}\right] \longrightarrow O\,O\,T\,R$$

Let us now find the number of ways of rearranging the letters of the word INSTITUTE. In this case there are 9 letters, in which I appears 2 times and T appears 3 times.

Temporarily, let us treat these letters different and name them as I_1, I_2, T_1, T_2, T_3. The number of permutations of 9 different letters, in this case, taken all at a time is 9 !. Consider one such permutation, say, I_1 NT_1 SI_2 T_2 U E T_3. Here if I_1, I_2 are not same and T_1, T_2, T_3 are not same, then I_1, I_2 can be arranged in 2! ways and T_1, T_2, T_3 can be arranged in 3! ways. Therefore, 2! × 3! permutations will be just the same permutation corresponding to this chosen permutation $I_1NT_1SI_2T_2UET_3$. Hence, total number of different permutations will be $\frac{9!}{2!\,3!}$

We can state (without proof) the following theorems:

Theorem 3 The number of permutations of n objects, where p objects are of the same kind and rest are all different $= \frac{n!}{p!}$.

In fact, we have a more general theorem.

Theorem 4 The number of permutations of n objects, where p_1 objects are of one kind, p_2 are of second kind, ..., p_k are of k^{th} kind and the rest, if any, are of different kind is $\frac{n!}{p_1!\,p_2!\ldots\,p_k!}$.

Example 9 Find the number of permutations of the letters of the word ALLAHABAD.

Solution Here, there are 9 objects (letters) of which there are 4A's, 2 L's and rest are all different.

Therefore, the required number of arrangements $= \frac{9!}{4!\,2!} = \frac{5\times6\times7\times8\times9}{2} = 7560$

Example 10 How many 4-digit numbers can be formed by using the digits 1 to 9 if repetition of digits is not allowed?

Solution Here order matters for example 1234 and 1324 are two different numbers. Therefore, there will be as many 4 digit numbers as there are permutations of 9 different digits taken 4 at a time.

Therefore, the required 4 digit numbers $= {}^9P_4 = \frac{9!}{(9-4)!} = \frac{9!}{5!} = 9 \times 8 \times 7 \times 6 = 3024.$

Example 11 How many numbers lying between 100 and 1000 can be formed with the digits 0, 1, 2, 3, 4, 5, if the repetition of the digits is not allowed?

Solution Every number between 100 and 1000 is a 3-digit number. We, first, have to count the permutations of 6 digits taken 3 at a time. This number would be ${}^6P_3$. But, these permutations will include those also where 0 is at the 100's place. For example, 092, 042, . . ., etc are such numbers which are actually 2-digit numbers and hence the number of such numbers has to be subtracted from ${}^6P_3$ to get the required number. To get the number of such numbers, we fix 0 at the 100's place and rearrange the remaining 5 digits taking 2 at a time. This number is ${}^5P_2$. So

The required number $= {}^6P_3 - {}^5P_2 = \dfrac{6!}{3!} - \dfrac{5!}{3!}$

$= 4 \times 5 \times 6 - 4 \times 5 = 100$

Example 12 Find the value of n such that

(i) ${}^nP_5 = 42\ {}^nP_3,\ n > 4$ (ii) $\dfrac{{}^nP_4}{{}^{n-1}P_4} = \dfrac{5}{3},\ n > 4$

Solution (i) Given that

${}^nP_5 = 42\ {}^nP_3$

or $n\ (n - 1)\ (n - 2)\ (n - 3)\ (n - 4) = 42\ n(n - 1)\ (n - 2)$

Since $n > 4$ so $n(n - 1)\ (n - 2) \neq 0$

Therefore, by dividing both sides by $n(n - 1)\ (n - 2)$, we get

$(n - 3\ (n - 4) = 42$

or $n^2 - 7n - 30 = 0$

or $n^2 - 10n + 3n - 30$

or $(n - 10)\ (n + 3) = 0$

or $n - 10 = 0$ or $n + 3 = 0$

or $n = 10$ or $n = -3$

As n cannot be negative, so $n = 10$.

(ii) Given that $\dfrac{{}^nP_4}{{}^{n-1}P_4} = \dfrac{5}{3}$

Therefore $3n\ (n - 1)\ (n - 2)\ (n - 3) = 5(n - 1)\ (n - 2)\ (n - 3)\ (n - 4)$

or $3n = 5\ (n - 4)$ [as $(n - 1)\ (n - 2)\ (n - 3) \neq 0,\ n > 4$]

or $n = 10$.

Example 13 Find r, if $5\ {}^4P_r = 6\ {}^5P_{r-1}$.

Solution We have $5\ {}^4P_r = 6\ {}^5P_{r-1}$

or $$5 \times \frac{4!}{(4-r)!} = 6 \times \frac{5!}{(5-r+1)!}$$

or $$\frac{5!}{(4-r)!} = \frac{6 \times 5!}{(5-r+1)(5-r)(5-r-1)!}$$

or $(6 - r)(5 - r) = 6$

or $r^2 - 11r + 24 = 0$

or $r^2 - 8r - 3r + 24 = 0$

or $(r - 8)(r - 3) = 0$

or $r = 8$ or $r = 3$.

Hence $r = 8, 3$.

Example 14 Find the number of different 8-letter arrangements that can be made from the letters of the word DAUGHTER so that

(i) all vowels occur together (ii) all vowels do not occur together.

Solution (i) There are 8 different letters in the word DAUGHTER, in which there are 3 vowels, namely, A, U and E. Since the vowels have to occur together, we can for the time being, assume them as a single object (AUE). This single object together with 5 remaining letters (objects) will be counted as 6 objects. Then we count permutations of these 6 objects taken all at a time. This number would be ${}^6P_6 = 6!$. Corresponding to each of these permutations, we shall have 3! permutations of the three vowels A, U, E taken all at a time . Hence, by the multiplication principle the required number of permutations = 6 ! × 3 ! = 4320.

(ii) If we have to count those permutations in which all vowels are never together, we first have to find all possible arrangments of 8 letters taken all at a time, which can be done in 8! ways. Then, we have to subtract from this number, the number of permutations in which the vowels are always together.

Therefore, the required number $8! - 6! \times 3! = 6!\,(7\times 8 - 6)$

$= 2 \times 6!\,(28 - 3)$

$= 50 \times 6! = 50 \times 720 = 36000$

Example 15 In how many ways can 4 red, 3 yellow and 2 green discs be arranged in a row if the discs of the same colour are indistinguishable ?

Solution Total number of discs are 4 + 3 + 2 = 9. Out of 9 discs, 4 are of the first kind (red), 3 are of the second kind (yellow) and 2 are of the third kind (green).

Therefore, the number of arrangements $\frac{9!}{4!\,3!\,2!} = 1260$.

Example 16 Find the number of arrangements of the letters of the word INDEPENDENCE. In how many of these arrangements,

(i) do the words start with P
(ii) do all the vowels always occur together
(iii) do the vowels never occur together
(iv) do the words begin with I and end in P?

Solution There are 12 letters, of which N appears 3 times, E appears 4 times and D appears 2 times and the rest are all different. Therefore

The required number of arrangements $= \frac{12!}{3!\,4!\,2!} = 1663200$

(i) Let us fix P at the extreme left position, we, then, count the arrangements of the remaining 11 letters. Therefore, the required number of words starting with P

$$= \frac{11!}{3!\,2!\,4!} = 138600.$$

(ii) There are 5 vowels in the given word, which are 4 Es and 1 I. Since, they have to always occur together, we treat them as a single object $\boxed{\text{EEEEI}}$ for the time being. This single object together with 7 remaining objects will account for 8 objects. These 8 objects, in which there are 3Ns and 2 Ds, can be rearranged in $\frac{8!}{3!\,2!}$ ways. Corresponding to each of these arrangements, the 5 vowels E, E, E, E and I can be rearranged in $\frac{5!}{4!}$ ways. Therefore, by multiplication principle, the required number of arrangements

$$= \frac{8!}{3!\,2!} \times \frac{5!}{4!} = 16800$$

(iii) The required number of arrangements
= the total number of arrangements (without any restriction) – the number of arrangements where all the vowels occur together.

$$= 1663200 - 16800 = 1646400$$

(iv) Let us fix I and P at the extreme ends (I at the left end and P at the right end). We are left with 10 letters.

Hence, the required number of arrangements

$$= \frac{10!}{3!\,2!\,4!} = 12600$$

EXERCISE 6.3

1. How many 3-digit numbers can be formed by using the digits 1 to 9 if no digit is repeated?

2. How many 4-digit numbers are there with no digit repeated?

3. How many 3-digit even numbers can be made using the digits 1, 2, 3, 4, 6, 7, if no digit is repeated?

4. Find the number of 4-digit numbers that can be formed using the digits 1, 2, 3, 4, 5 if no digit is repeated. How many of these will be even?

5. From a committee of 8 persons, in how many ways can we choose a chairman and a vice chairman assuming one person can not hold more than one position?

6. Find n if ${}^{n-1}P_3 : {}^{n}P_4 = 1 : 9$.

7. Find r if (i) ${}^5P_r = 2\,{}^6P_{r-1}$ (ii) ${}^5P_r = {}^6P_{r-1}$.

8. How many words, with or without meaning, can be formed using all the letters of the word EQUATION, using each letter exactly once?

9. How many words, with or without meaning can be made from the letters of the word MONDAY, assuming that no letter is repeated, if.

(i) 4 letters are used at a time, (ii) all letters are used at a time,

(iii) all letters are used but first letter is a vowel?

10. In how many of the distinct permutations of the letters in MISSISSIPPI do the four I's not come together?

11. In how many ways can the letters of the word PERMUTATIONS be arranged if the

(i) words start with P and end with S, (ii) vowels are all together,

(iii) there are always 4 letters between P and S?

6.4 Combinations

Let us now assume that there is a group of 3 lawn tennis players X, Y, Z. A team consisting of 2 players is to be formed. In how many ways can we do so? Is the team of X and Y different from the team of Y and X ? Here, order is not important. In fact, there are only 3 possible ways in which the team could be constructed.

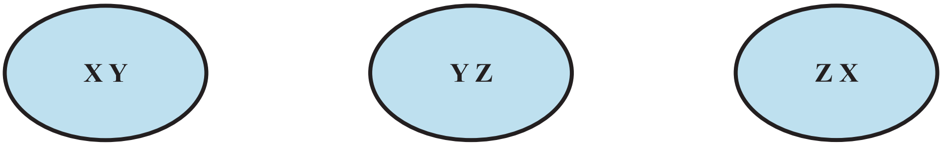


Fig. 6.3

These are XY, YZ and ZX (Fig 6.3).

Here, each selection is called a *combination of 3 different objects taken 2 at a time*. In a combination, the order is not important.

Now consider some more illustrations.

Twelve persons meet in a room and each shakes hand with all the others. How do we determine the number of hand shakes. X shaking hands with Y and Y with X will not be two different hand shakes. Here, order is not important. There will be as many hand shakes as there are combinations of 12 different things taken 2 at a time.

Seven points lie on a circle. How many chords can be drawn by joining these points pairwise? There will be as many chords as there are combinations of 7 different things taken 2 at a time.

Now, we obtain the formula for finding the number of combinations of n different objects taken r at a time, denoted by ${}^nC_r$..

Suppose we have 4 different objects A, B, C and D. Taking 2 at a time, if we have to make combinations, these will be AB, AC, AD, BC, BD, CD. Here, AB and BA are the same combination as order does not alter the combination. This is why we have not included BA, CA, DA, CB, DB and DC in this list. There are as many as 6 combinations of 4 different objects taken 2 at a time, i.e., ${}^4C_2 = 6$.

Corresponding to each combination in the list, we can arrive at 2! permutations as 2 objects in each combination can be rearranged in 2! ways. Hence, the number of permutations $= {}^4C_2 \times 2!$.

On the other hand, the number of permutations of 4 different things taken 2 at a time $= {}^4P_2$.

Therefore $\quad \mathbf{{}^4P_2 = {}^4C_2 \times 2!} \quad$ or $\quad \dfrac{4!}{(4-2)!\,2!} = {}^4C_2$

Now, let us suppose that we have 5 different objects A, B, C, D, E. Taking 3 at a time, if we have to make combinations, these will be ABC, ABD, ABE, BCD, BCE, CDE, ACE, ACD, ADE, BDE. Corresponding to each of these ${}^5C_3$ combinations, there are 3! permutations, because, the three objects in each combination can be rearranged in 3 ! ways. Therefore, the total of permutations = ${}^5C_3 \times 3!$

Therefore $\quad {}^5\mathbf{P}_3 = {}^5\mathbf{C}_3 \times \mathbf{3!} \quad$ or $\quad \dfrac{5!}{(5-3)!\ 3!} = {}^5C_3$

These examples suggest the following theorem showing relationship between permutaion and combination:

Theorem 5 ${}^nP_r = {}^nC_r\ r!$, $0 < r \le n$.

Proof Corresponding to each combination of ${}^nC_r$, we have r ! permutations, because r objects in every combination can be rearranged in r ! ways.

Hence, the total number of permutations of n different things taken r at a time is ${}^nC_r \times r!$. On the other hand, it is ${}^nP_r$. Thus

$$ {}^nP_r = {}^nC_r \times r!,\ 0 < r \le n \cdot $$

Remarks **1.** From above $\dfrac{n!}{(n-r)!} = {}^nC_r \times r!$, i.e., ${}^nC_r = \dfrac{n!}{r!(n-r)!}$.

In particular, if $r = n$, ${}^nC_n = \dfrac{n!}{n!\ 0!} = 1$.

2. We define ${}^nC_0 = 1$, i.e., the number of combinations of n different things taken nothing at all is considered to be 1. Counting combinations is merely counting the number of ways in which some or all objects at a time are selected. Selecting nothing at all is the same as leaving behind all the objects and we know that there is only one way of doing so. This way we define ${}^nC_0 = 1$.

3. As $\dfrac{n!}{0!(n-0)!} = 1 = {}^nC_0$, the formula ${}^nC_r = \dfrac{n!}{r!(n-r)!}$ is applicable for $r = 0$ also.

Hence

$$ {}^nC_r = \frac{n!}{r!\ (n-r)!},\ 0 \le r \le n. $$

4. $\quad {}^nC_{n-r} = \dfrac{n!}{(n-r)!(n-(n-r))!} = \dfrac{n!}{(n-r)!r!} = {}^nC_r$, i.e., selecting r objects out of n objects is same as rejecting $(n - r)$ objects.

5. ${}^nC_a = {}^nC_b \Rightarrow a = b$ or $a = n - b$, i.e., $n = a + b$

Theorem 6 ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

Proof We have ${}^nC_r + {}^nC_{r-1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!}$

$$= \frac{n!}{r\times(r-1)!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)\,(n-r)!}$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!}{(r-1)!(n-r)!}\times\frac{n-r+1+r}{r(n-r+1)} = \frac{(n+1)!}{r!(n+1-r)!} = {}^{n+1}C_r$$

Example 17 If ${}^nC_9 = {}^nC_8$, find ${}^nC_{17}$.

Solution We have ${}^nC_9 = {}^nC_8$

i.e., $$\frac{n!}{9!(n-9)!} = \frac{n!}{(n-8)!\,8!}$$

or $\frac{1}{9} = \frac{1}{n-8}$ or $n - 8 = 9$ or $n = 17$

Therefore ${}^nC_{17} = {}^{17}C_{17} = 1$.

Example 18 A committee of 3 persons is to be constituted from a group of 2 men and 3 women. In how many ways can this be done? How many of these committees would consist of 1 man and 2 women?

Solution Here, order does not matter. Therefore, we need to count combinations. There will be as many committees as there are combinations of 5 different persons taken 3 at a time. Hence, the required number of ways = ${}^5C_3 = \frac{5!}{3!\,2!} = \frac{4\times5}{2} = 10$.

Now, 1 man can be selected from 2 men in ${}^2C_1$ ways and 2 women can be selected from 3 women in ${}^3C_2$ ways. Therefore, the required number of committees

$$= {}^2C_1 \times {}^3C_2 = \frac{2!}{1!\ 1!} \times \frac{3!}{2!\ 1!} = 6.$$

Example 19 What is the number of ways of choosing 4 cards from a pack of 52 playing cards? In how many of these

(i) four cards are of the same suit,
(ii) four cards belong to four different suits,
(iii) are face cards,
(iv) two are red cards and two are black cards,
(v) cards are of the same colour?

Solution There will be as many ways of choosing 4 cards from 52 cards as there are combinations of 52 different things, taken 4 at a time. Therefore

The required number of ways $= {}^{52}C_4 = \frac{52!}{4!\ 48!} = \frac{49 \times 50 \times 51 \times 52}{2 \times 3 \times 4}$

$= 270725$

(i) There are four suits: diamond, club, spade, heart and there are 13 cards of each suit. Therefore, there are ${}^{13}C_4$ ways of choosing 4 diamonds. Similarly, there are ${}^{13}C_4$ ways of choosing 4 clubs, ${}^{13}C_4$ ways of choosing 4 spades and ${}^{13}C_4$ ways of choosing 4 hearts. Therefore

The required number of ways $= {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4$.

$$= 4 \times \frac{13!}{4!\ 9!} = 2860$$

(ii) There are13 cards in each suit.

Therefore, there are ${}^{13}C_1$ ways of choosing 1 card from 13 cards of diamond, ${}^{13}C_1$ ways of choosing 1 card from 13 cards of hearts, ${}^{13}C_1$ ways of choosing 1 card from 13 cards of clubs, ${}^{13}C_1$ ways of choosing 1 card from 13 cards of spades. Hence, by multiplication principle, the required number of ways

$= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 = 13^4$

(iii) There are 12 face cards and 4 are to be selected out of these 12 cards. This can be done in ${}^{12}C_4$ ways. Therefore, the required number of ways $= \frac{12!}{4!\ 8!} = 495$.

(iv) There are 26 red cards and 26 black cards. Therefore, the required number of ways = ${}^{26}C_2 \times {}^{26}C_2$

$$= \left(\frac{26!}{2!\ 24!}\right)^2 = (325)^2 = 105625$$

(v) 4 red cards can be selected out of 26 red cards in ${}^{26}C_4$ ways. 4 black cards can be selected out of 26 black cards in ${}^{26}C_4$ways.

Therefore, the required number of ways = ${}^{26}C_4 + {}^{26}C_4$

$$= 2\times\frac{26!}{4!\ 22!} = 29900.$$

EXERCISE 6.4

1. If ${}^{n}C_8 = {}^{n}C_2$, find ${}^{n}C_2$.

2. Determine n if

(i) ${}^{2n}C_3 : {}^{n}C_3 = 12 : 1$ (ii) ${}^{2n}C_3 : {}^{n}C_3 = 11 : 1$

3. How many chords can be drawn through 21 points on a circle?

4. In how many ways can a team of 3 boys and 3 girls be selected from 5 boys and 4 girls?

5. Find the number of ways of selecting 9 balls from 6 red balls, 5 white balls and 5 blue balls if each selection consists of 3 balls of each colour.

6. Determine the number of 5 card combinations out of a deck of 52 cards if there is exactly one ace in each combination.

7. In how many ways can one select a cricket team of eleven from 17 players in which only 5 players can bowl if each cricket team of 11 must include exactly 4 bowlers?

8. A bag contains 5 black and 6 red balls. Determine the number of ways in which 2 black and 3 red balls can be selected.

9. In how many ways can a student choose a programme of 5 courses if 9 courses are available and 2 specific courses are compulsory for every student?

Miscellaneous Examples

Example 20 How many words, with or without meaning, each of 3 vowels and 2 consonants can be formed from the letters of the word INVOLUTE ?

Solution In the word INVOLUTE, there are 4 vowels, namely, I,O,E,Uand 4 consonants, namely, N, V, L and T.

The number of ways of selecting 3 vowels out of 4 = ${}^4C_3 = 4$.
The number of ways of selecting 2 consonants out of 4 = ${}^4C_2 = 6$.

Therefore, the number of combinations of 3 vowels and 2 consonants is $4 \times 6 = 24$.

Now, each of these 24 combinations has 5 letters which can be arranged among themselves in 5 ! ways. Therefore, the required number of different words is $24 \times 5! = 2880$.

Example 21 A group consists of 4 girls and 7 boys. In how many ways can a team of 5 members be selected if the team has (i) no girl ? (ii) at least one boy and one girl ? (iii) at least 3 girls ?

Solution (i) Since, the team will not include any girl, therefore, only boys are to be selected. 5 boys out of 7 boys can be selected in ${}^7C_5$ ways. Therefore, the required

number of ways $= {}^7C_5 = \frac{7!}{5!\ 2!} = \frac{6 \times 7}{2} = 21$

(ii) Since, at least one boy and one girl are to be there in every team. Therefore, the team can consist of

(a) 1 boy and 4 girls (b) 2 boys and 3 girls
(c) 3 boys and 2 girls (d) 4 boys and 1 girl.

1 boy and 4 girls can be selected in ${}^7C_1 \times {}^4C_4$ ways.
2 boys and 3 girls can be selected in ${}^7C_2 \times {}^4C_3$ ways.
3 boys and 2 girls can be selected in ${}^7C_3 \times {}^4C_2$ ways.
4 boys and 1 girl can be selected in ${}^7C_4 \times {}^4C_1$ ways.

Therefore, the required number of ways

$= {}^7C_1 \times {}^4C_4 + {}^7C_2 \times {}^4C_3 + {}^7C_3 \times {}^4C_2 + {}^7C_4 \times {}^4C_1$
$= 7 + 84 + 210 + 140 = 441$

(iii) Since, the team has to consist of at least 3 girls, the team can consist of
(a) 3 girls and 2 boys, or (b) 4 girls and 1 boy.

Note that the team cannot have all 5 girls, because, the group has only 4 girls.

3 girls and 2 boys can be selected in ${}^4C_3 \times {}^7C_2$ ways.
4 girls and 1 boy can be selected in ${}^4C_4 \times {}^7C_1$ ways.

Therefore, the required number of ways

$= {}^4C_3 \times {}^7C_2 + {}^4C_4 \times {}^7C_1 = 84 + 7 = 91$

Example 22 Find the number of words with or without meaning which can be made using all the letters of the word AGAIN. If these words are written as in a dictionary, what will be the 50th word?

Solution There are 5 letters in the word AGAIN, in which A appears 2 times. Therefore, the required number of words $= \frac{5!}{2!} = 60$.

To get the number of words starting with A, we fix the letter A at the extreme left position, we then rearrange the remaining 4 letters taken all at a time. There will be as many arrangements of these 4 letters taken 4 at a time as there are permutations of 4 different things taken 4 at a time. Hence, the number of words starting with A $= 4! = 24$. Then, starting with G, the number of words $= \frac{4!}{2!} = 12$ as after placing G at the extreme left position, we are left with the letters A, A, I and N. Similarly, there are 12 words starting with the next letter I. Total number of words so far obtained $= 24 + 12 + 12 = 48$.

The 49th word is NAAGI. The 50th word is NAAIG.

Example 23 How many numbers greater than 1000000 can be formed by using the digits 1, 2, 0, 2, 4, 2, 4?

Solution Since, 1000000 is a 7-digit number and the number of digits to be used is also 7. Therefore, the numbers to be counted will be 7-digit only. Also, the numbers have to be greater than 1000000, so they can begin either with 1, 2 or 4.

The number of numbers beginning with 1 $= \frac{6!}{3!\,2!} = \frac{4 \times 5 \times 6}{2} = 60$, as when 1 is fixed at the extreme left position, the remaining digits to be rearranged will be 0, 2, 2, 2, 4, 4, in which there are 3, 2*s* and 2, 4*s*.

Total numbers begining with 2

$$= \frac{6!}{2!\ 2!} = \frac{3 \times 4 \times 5 \times 6}{2} = 180$$

and total numbers begining with 4 $= \frac{6!}{3!} = 4 \times 5 \times 6 = 120$

Therefore, the required number of numbers $= 60 + 180 + 120 = 360$.

Alternative Method

The number of 7-digit arrangements, clearly, $\frac{7!}{3!\ 2!} = 420$. But, this will include those numbers also, which have 0 at the extreme left position. The number of such arrangements $\frac{6!}{3!\ 2!}$ (by fixing 0 at the extreme left position) = 60.

Therefore, the required number of numbers = 420 – 60 = 360.

Note If one or more than one digits given in the list is repeated, it will be understood that in any number, the digits can be used as many times as is given in the list, e.g., in the above example 1 and 0 can be used only once whereas 2 and 4 can be used 3 times and 2 times, respectively.

Example 24 In how many ways can 5 girls and 3 boys be seated in a row so that no two boys are together?

Solution Let us first seat the 5 girls. This can be done in 5! ways. For each such arrangement, the three boys can be seated only at the cross marked places.

$$\times\ G \times G \times G \times G \times G \times.$$

There are 6 cross marked places and the three boys can be seated in ${}^6P_3$ ways. Hence, by multiplication principle, the total number of ways

$$= 5! \times {}^6P_3 = 5! \times \frac{6!}{3!}$$

$$= 4 \times 5 \times 2 \times 3 \times 4 \times 5 \times 6 = 14400.$$

Miscellaneous Exercise on Chapter 6

1. How many words, with or without meaning, each of 2 vowels and 3 consonants can be formed from the letters of the word DAUGHTER ?
2. How many words, with or without meaning, can be formed using all the letters of the word EQUATION at a time so that the vowels and consonants occur together?
3. A committee of 7 has to be formed from 9 boys and 4 girls. In how many ways can this be done when the committee consists of:

   (i) exactly 3 girls ? (ii) atleast 3 girls ? (iii) atmost 3 girls ?
4. If the different permutations of all the letter of the word EXAMINATION are

listed as in a dictionary, how many words are there in this list before the first word starting with E ?

5. How many 6-digit numbers can be formed from the digits 0, 1, 3, 5, 7 and 9 which are divisible by 10 and no digit is repeated ?
6. The English alphabet has 5 vowels and 21 consonants. How many words with two different vowels and 2 different consonants can be formed from the alphabet ?
7. In an examination, a question paper consists of 12 questions divided into two parts i.e., Part I and Part II, containing 5 and 7 questions, respectively. A student is required to attempt 8 questions in all, selecting at least 3 from each part. In how many ways can a student select the questions ?
8. Determine the number of 5-card combinations out of a deck of 52 cards if each selection of 5 cards has exactly one king.
9. It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible ?
10. From a class of 25 students, 10 are to be chosen for an excursion party. There are 3 students who decide that either all of them will join or none of them will join. In how many ways can the excursion party be chosen ?
11. In how many ways can the letters of the word ASSASSINATION be arranged so that all the S's are together ?

Summary

- *Fundamental principle of counting* If an event can occur in m different ways, following which another event can occur in n different ways, then the total number of occurrence of the events in the given order is $m \times n$.
- The number of permutations of n different things taken r at a time, where repetition is not allowed, is denoted by ${}^nP_r$ and is given by ${}^nP_r = \frac{n!}{(n-r)!}$, where $0 \le r \le n$.
- $n! = 1 \times 2 \times 3 \times ... \times n$
- $n! = n \times (n-1)\,!$
- The number of permutations of n different things, taken r at a time, where repeatition is allowed, is n^r.
- The number of permutations of n objects taken all at a time, where p_1 objects

are of first kind, p_2 objects are of the second kind, ..., p_k objects are of the k^{th} kind and rest, if any, are all different is $\frac{n!}{p_1!\,p_2!\ldots p_k!}$.

- The number of combinations of n different things taken r at a time, denoted by ${}^nC_r$, is given by ${}^nC_r = \frac{n!}{r!(n-r)!}$, $0 \le r \le n$.

Historical Note

The concepts of permutations and combinations can be traced back to the advent of Jainism in India and perhaps even earlier. The credit, however, goes to the Jains who treated its subject matter as a self-contained topic in mathematics, under the name *Vikalpa*.

Among the Jains, *Mahavira*, (around 850) is perhaps the world's first mathematician credited with providing the general formulae for permutations and combinations.

In the 6th century B.C., *Sushruta,* in his medicinal work, *Sushruta Samhita*, asserts that 63 combinations can be made out of 6 different tastes, taken one at a time, two at a time, etc. *Pingala*, a Sanskrit scholar around third century B.C., gives the method of determining the number of combinations of a given number of letters, taken one at a time, two at a time, etc. in his work *Chhanda Sutra*. *Bhaskaracharya* (born 1114) treated the subject matter of permutations and combinations under the name *Anka Pasha* in his famous work *Lilavati.* In addition to the general formulae for ${}^nC_r$ and ${}^nP_r$ already provided by *Mahavira, Bhaskaracharya* gives several important theorems and results concerning the subject.

Outside India, the subject matter of permutations and combinations had its humble beginnings in China in the famous book I–King (Book of changes). It is difficult to give the approximate time of this work, since in 213 B.C., the emperor had ordered all books and manuscripts in the country to be burnt which fortunately was not completely carried out. Greeks and later Latin writers also did some scattered work on the theory of permutations and combinations.

Some Arabic and Hebrew writers used the concepts of permutations and combinations in studying astronomy. *Rabbi ben Ezra,* for instance, determined the number of combinations of known planets taken two at a time, three at a time and so on. This was around 1140. It appears that *Rabbi ben Ezra* did not know the formula for ${}^nC_r$. However, he was aware that ${}^nC_r = {}^nC_{n-r}$ for specific values n and r. In 1321, *Levi Ben Gerson*, another Hebrew writer came up with the formulae for ${}^nP_r$, ${}^nP_n$ *and the general formula for* ${}^nC_r$.

The first book which gives a complete treatment of the subject matter of permutations and combinations is Ars Conjectandi written by a Swiss, *Jacob Bernoulli* (1654 – 1705), posthumously published in 1713. This book contains essentially the theory of permutations and combinations as is known today.

Chapter 7

BINOMIAL THEOREM

❖ *Mathematics is a most exact science and its conclusions are capable of absolute proofs. – C.P. STEINMETZ* ❖

7.1 Introduction

Blaise Pascal (1623-1662)

In earlier classes, we have learnt how to find the squares and cubes of binomials like $a + b$ and $a – b$. Using them, we could evaluate the numerical values of numbers like $(98)^2 = (100 – 2)^2$, $(999)^3 = (1000 – 1)^3$, etc. However, for higher powers like $(98)^5$, $(101)^6$, etc., the calculations become difficult by using repeated multiplication. This difficulty was overcome by a theorem known as binomial theorem. It gives an easier way to expand $(a + b)^n$, where n is an integer or a rational number. In this Chapter, we study binomial theorem for positive integral indices only.

7.2 Binomial Theorem for Positive Integral Indices

Let us have a look at the following identities done earlier:

$$(a+ b)^0 = 1 \qquad\qquad a + b \neq 0$$
$$(a+ b)^1 = a + b$$
$$(a+ b)^2 = a^2 + 2ab + b^2$$
$$(a+ b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+ b)^4 = (a + b)^3 (a + b) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

In these expansions, we observe that

(i) The total number of terms in the expansion is one more than the index. For example, in the expansion of $(a + b)^2$, number of terms is 3 whereas the index of $(a + b)^2$ is 2.

(ii) Powers of the first quantity 'a' go on decreasing by 1 whereas the powers of the second quantity 'b' increase by 1, in the successive terms.

(iii) In each term of the expansion, the sum of the indices of a and b is the same and is equal to the index of $a + b$.

We now arrange the coefficients in these expansions as follows (Fig 7.1):

Index				**Coefficients**				
0				**1**				
1			**1**		**1**			
2		**1**		**2**		**1**		
3								
3	**1**		**3**		**3**		**1**	
4	**1**	**4**		**6**		**4**		**1**

Fig 7.1

Do we observe any pattern in this table that will help us to write the next row? Yes we do. It can be seen that the addition of 1's in the row for index 1 gives rise to 2 in the row for index 2. The addition of 1, 2 and 2, 1 in the row for index 2, gives rise to 3 and 3 in the row for index 3 and so on. Also, 1 is present at the beginning and at the end of each row. This can be continued till any index of our interest.

We can extend the pattern given in Fig 7.2 by writing a few more rows.

Index				**Coefficients**				
0				**1**				
1			**1**		**1**			
2		**1**		**2**		**1**		
3	**1**		**3**		**3**		**1**	
4	**1**	**4**		**6**		**4**		**1**

Fig 7.2

Pascal's Triangle

The structure given in Fig 7.2 looks like a triangle with 1 at the top vertex and running down the two slanting sides. This array of numbers is known as *Pascal's triangle*, after the name of French mathematician Blaise Pascal. It is also known as *Meru Prastara* by Pingla.

Expansions for the higher powers of a binomial are also possible by using Pascal's triangle. Let us expand $(2x + 3y)^5$ by using Pascal's triangle. The row for index 5 is

$$1 \qquad 5 \qquad 10 \qquad 10 \qquad 5 \qquad 1$$

Using this row and our observations (i), (ii) and (iii), we get

$$\begin{aligned}(2x + 3y)^5 &= (2x)^5 + 5(2x)^4 (3y) + 10(2x)^3 (3y)^2 + 10 (2x)^2 (3y)^3 + 5(2x)(3y)^4 + (3y)^5 \\ &= 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5.\end{aligned}$$

Now, if we want to find the expansion of $(2x + 3y)^{12}$, we are first required to get the row for index 12. This can be done by writing all the rows of the Pascal's triangle till index 12. This is a slightly lengthy process. The process, as you observe, will become more difficult, if we need the expansions involving still larger powers.

We thus try to find a rule that will help us to find the expansion of the binomial for any power without writing all the rows of the Pascal's triangle, that come before the row of the desired index.

For this, we make use of the concept of combinations studied earlier to rewrite the numbers in the Pascal's triangle. We know that $^nC_r = \dfrac{n!}{r!(n-r)!}$, $0 \le r \le n$ and n is a non-negative integer. Also, $^nC_0 = 1 = {^nC_n}$

The Pascal's triangle can now be rewritten as (Fig 7.3)

Index	**Coefficients**
0	$^0C_0$ (=1)
1	$^1C_0$ (=1) $^1C_1$ (=1)
2	$^2C_0$ (=1) $^2C_1$ (=2) $^2C_2$ (=1)
3	$^3C_0$ (=1) $^3C_1$ (=3) $^3C_2$ (=3) $^3C_3$ (=1)
4	$^4C_0$ (=1) $^4C_1$ (=4) $^4C_2$ (=6) $^4C_3$ (=4) $^4C_4$ (=1)
5	$^5C_0$ (=1) $^5C_1$ (=5) $^5C_2$ (=10) $^5C_3$ (=10) $^5C_4$ (=5) $^5C_5$ (=1)

Fig 7.3 Pascal's triangle

Observing this pattern, we can now write the row of the Pascal's triangle for any index without writing the earlier rows. For example, for the index 7 the row would be

$$^7C_0 \quad ^7C_1 \quad ^7C_2 \quad ^7C_3 \quad ^7C_4 \quad ^7C_5 \quad ^7C_6 \quad ^7C_7.$$

Thus, using this row and the observations (i), (ii) and (iii), we have

$$(a + b)^7 = {^7C_0}\, a^7 + 7C_1 a^6 b + {^7C_2} a^5 b^2 + {^7C_3} a^4 b^3 + 7C_4 a^3 b^4 + {^7C_5} a^2 b^5 + {^7C_6} ab^6 + {^7C_7} b^7$$

An expansion of a binomial to any positive integral index say n can now be visualised using these observations. We are now in a position to write the expansion of a binomial to any positive integral index.

7.2.1 *Binomial theorem for any positive integer n,*

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + ...+ {}^nC_{n-1}a.b^{n-1} + {}^nC_nb^n$$

Proof The proof is obtained by applying principle of mathematical induction.

Let the given statement be

$$P(n) : (a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + ...+ {}^nC_{n-1}a.b^{n-1} + {}^nC_nb^n$$

For $n = 1$, we have

$$P(1) : (a+b)^1 = {}^1C_0a^1 + {}^1C_1b^1 = a + b$$

Thus, P (1) is true.

Suppose P (k) is true for some positive integer k, i.e.

$$(a+b)^k = {}^kC_0a^k + {}^kC_1a^{k-1}b + {}^kC_2a^{k-2}b^2 + ...+ {}^kC_kb^k \quad ... (1)$$

We shall prove that P(k + 1) is also true, i.e.,

$$(a+b)^{k+1} = {}^{k+1}C_0\,a^{k+1} + {}^{k+1}C_1\,a^kb + {}^{k+1}C_2\,a^{k-1}b^2 + ...+ {}^{k+1}C_{k+1}\,b^{k+1}$$

Now, $(a+b)^{k+1} = (a+b)(a+b)^k$

$$= (a+b)({}^kC_0\,a^k + {}^kC_1a^{k-1}\,b + {}^kC_2\,a^{k-2}\,b^2 + ...+ {}^kC_{k-1}\,ab^{k-1} + {}^kC_k\,b^k) \quad \text{[from (1)]}$$

$$= {}^kC_0\,a^{k+1} + {}^kC_1\,a^kb + {}^kC_2a^{k-1}b^2 + ...+ {}^kC_{k-1}\,a^2b^{k-1} + {}^kC_k\,ab^k + {}^kC_0\,a^kb$$
$$+ {}^kC_1a^{k-1}b^2 + {}^kC_2a^{k-2}b^3 + ...+ {}^kC_{k-1}ab^k + {}^kC_kb^{k+1}$$

[by actual multiplication]

$$= {}^kC_0a^{k+1} + ({}^kC_1 + {}^kC_0)\,a^kb + ({}^kC_2 + {}^kC_1)a^{k-1}b^2 + ...$$
$$+ ({}^kC_k + {}^kC_{k-1})\,ab^k + {}^kC_kb^{k+1} \quad \text{[grouping like terms]}$$

$$= {}^{k+1}C_0a^{k+1} + {}^{k+1}C_1a^kb + {}^{k+1}C_2\,a^{k-1}b^2 + ...+ {}^{k+1}C_kab^k + {}^{k+1}C_{k+1}\,b^{k+1}$$

(by using ${}^{k+1}C_0 = 1$, ${}^kC_r + {}^kC_{r-1} = {}^{k+1}C_r$ and ${}^kC_k = 1 = {}^{k+1}C_{k+1}$)

Thus, it has been proved that P (k + 1) is true whenever P(k) is true. Therefore, by principle of mathematical induction, P(n) is true for every positive integer n.

We illustrate this theorem by expanding $(x + 2)^6$:

$$(x+2)^6 = {}^6C_0x^6 + {}^6C_1x^5.2 + {}^6C_2x^42^2 + {}^6C_3x^3.2^3 + {}^6C_4x^2.2^4 + {}^6C_5x.2^5 + {}^6C_6.2^6.$$
$$= x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64$$

Thus $(x+2)^6 = x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64$.

Observations

1. The notation $\sum_{k=0}^{n} {}^{n}C_k\, a^{n-k} b^k$ stands for

${}^{n}C_0 a^n b^0 + {}^{n}C_1 a^{n-1} b^1 + ... + {}^{n}C_r a^{n-r} b^r + ... + {}^{n}C_n a^{n-n} b^n$, where $b^0 = 1 = a^{n-n}$.
Hence the theorem can also be stated as

$$(a+b)^n = \sum_{k=0}^{n} {}^{n}C_k\, a^{n-k} b^k .$$

2. The coefficients ${}^{n}C_r$ occuring in the binomial theorem are known as binomial coefficients.

3. There are $(n+1)$ terms in the expansion of $(a+b)^n$, i.e., one more than the index.

4. In the successive terms of the expansion the index of a goes on decreasing by unity. It is n in the first term, $(n–1)$ in the second term, and so on ending with zero in the last term. At the same time the index of b increases by unity, starting with zero in the first term, 1 in the second and so on ending with n in the last term.

5. In the expansion of $(a+b)^n$, the sum of the indices of a and b is $n + 0 = n$ in the first term, $(n – 1) + 1 = n$ in the second term and so on $0 + n = n$ in the last term. Thus, it can be seen that the sum of the indices of a and b is n in every term of the expansion.

7.2.2 ***Some special cases*** In the expansion of $(a + b)^n$,

(i) Taking $a = x$ and $b = – y$, we obtain

$$\begin{aligned}(x - y)^n &= [x + (-y)]^n \\ &= {}^{n}C_0 x^n + {}^{n}C_1 x^{n-1}(-y) + {}^{n}C_2 x^{n-2}(-y)^2 + {}^{n}C_3 x^{n-3}(-y)^3 + ... + {}^{n}C_n (-y)^n \\ &= {}^{n}C_0 x^n - {}^{n}C_1 x^{n-1} y + {}^{n}C_2 x^{n-2} y^2 - {}^{n}C_3 x^{n-3} y^3 + ... + (-1)^n\, {}^{n}C_n\, y^n\end{aligned}$$

Thus $(x-y)^n = {}^{n}C_0 x^n - {}^{n}C_1 x^{n-1} y + {}^{n}C_2 x^{n-2} y^2 + ... + (-1)^n\, {}^{n}C_n\, y^n$

Using this, we have

$$\begin{aligned}(x-2y)^5 &= {}^{5}C_0 x^5 - {}^{5}C_1 x^4 (2y) + {}^{5}C_2 x^3 (2y)^2 - {}^{5}C_3 x^2 (2y)^3 + {}^{5}C_4 x (2y)^4 - {}^{5}C_5 (2y)^5 \\ &= x^5 - 10x^4 y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5.\end{aligned}$$

(ii) Taking $a = 1$, $b = x$, we obtain

$$\begin{aligned}(1 + x)^n &= {}^{n}C_0 (1)^n + {}^{n}C_1 (1)^{n-1} x + {}^{n}C_2 (1)^{n-2} x^2 + ... + {}^{n}C_n x^n \\ &= {}^{n}C_0 + {}^{n}C_1 x + {}^{n}C_2 x^2 + {}^{n}C_3 x^3 + ... + {}^{n}C_n x^n\end{aligned}$$

Thus $(1 + x)^n = {}^{n}C_0 + {}^{n}C_1 x + {}^{n}C_2 x^2 + {}^{n}C_3 x^3 + ... + {}^{n}C_n x^n$

In particular, for $x = 1$, we have

$$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n.$$

(iii) Taking $a = 1$, $b = -x$, we obtain

$$(1-x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 - \ldots + (-1)^n\, {}^nC_nx^n$$

In particular, for $x = 1$, we get

$$0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - \ldots + (-1)^n\, {}^nC_n$$

Example 1 Expand $\left(x^2 + \dfrac{3}{x}\right)^4$, $x \neq 0$

Solution By using binomial theorem, we have

$$\left(x^2 + \frac{3}{x}\right)^4 = {}^4C_0(x^2)^4 + {}^4C_1(x^2)^3\left(\frac{3}{x}\right) + {}^4C_2(x^2)^2\left(\frac{3}{x}\right)^2 + {}^4C_3(x^2)\left(\frac{3}{x}\right)^3 + {}^4C_4\left(\frac{3}{x}\right)^4$$

$$= x^8 + 4.x^6 \cdot \frac{3}{x} + 6.x^4 \cdot \frac{9}{x^2} + 4.x^2.\frac{27}{x^3} + \frac{81}{x^4}$$

$$= x^8 + 12x^5 + 54x^2 + \frac{108}{x} + \frac{81}{x^4}.$$

Example 2 Compute $(98)^5$.

Solution We express 98 as the sum or difference of two numbers whose powers are easier to calculate, and then use Binomial Theorem.

Write 98 = 100 – 2

Therefore, $(98)^5 = (100 - 2)^5$

$$= {}^5C_0\,(100)^5 - {}^5C_1\,(100)^4.2 + {}^5C_2\,(100)^3 2^2$$
$$- {}^5C_3\,(100)^2\,(2)^3 + {}^5C_4\,(100)\,(2)^4 - {}^5C_5\,(2)^5$$
$$= 10000000000 - 5 \times 100000000 \times 2 + 10 \times 1000000 \times 4 - 10 \times 10000 \times 8 + 5 \times 100 \times 16 - 32$$
$$= 10040008000 - 1000800032 = 9039207968.$$

Example 3 Which is larger $(1.01)^{1000000}$ or 10,000?

Solution Splitting 1.01 and using binomial theorem to write the first few terms we have

$$(1.01)^{1000000} = (1 + 0.01)^{1000000}$$
$$= {}^{1000000}C_0 + {}^{1000000}C_1(0.01) + \text{other positive terms}$$
$$= 1 + 1000000 \times 0.01 + \text{other positive terms}$$
$$= 1 + 10000 + \text{other positive terms}$$
$$> 10000$$

Hence $(1.01)^{1000000} > 10000$

Example 4 Using binomial theorem, prove that $6^n–5n$ always leaves remainder 1 when divided by 25.

Solution For two numbers a and b if we can find numbers q and r such that $a = bq + r$, then we say that b divides a with q as quotient and r as remainder. Thus, in order to show that $6^n – 5n$ leaves remainder 1 when divided by 25, we prove that $6^n – 5n = 25k + 1$, where k is some natural number.

We have

$$(1 + a)^n = {}^nC_0 + {}^nC_1a + {}^nC_2a^2 + ... + {}^nC_na^n$$

For $a = 5$, we get

$$(1 + 5)^n = {}^nC_0 + {}^nC_15 + {}^nC_25^2 + ... + {}^nC_n5^n$$

i.e. $(6)^n = 1 + 5n + 5^2.{}^nC_2 + 5^3.{}^nC_3 + ... + 5^n$

i.e. $6^n – 5n = 1+5^2 ({}^nC_2 + {}^nC_35 + ... + 5^{n-2})$

or $6^n – 5n = 1+ 25 ({}^nC_2 + 5 .{}^nC_3 + ... + 5^{n-2})$

or $6^n – 5n = 25k+1$ where $k = {}^nC_2 + 5 .{}^nC_3 + ... + 5^{n–2}$.

This shows that when divided by 25, $6^n – 5n$ leaves remainder 1.

EXERCISE 7.1

Expand each of the expressions in Exercises 1 to 5.

1. $(1–2x)^5$ **2.** $\left(\dfrac{2}{x}-\dfrac{x}{2}\right)^5$ **3.** $(2x – 3)^6$

4. $\left(\dfrac{x}{3}+\dfrac{1}{x}\right)^5$ **5.** $\left(x+\dfrac{1}{x}\right)^6$

Using binomial theorem, evaluate each of the following:

6. $(96)^3$ **7.** $(102)^5$ **8.** $(101)^4$

9. $(99)^5$

10. Using Binomial Theorem, indicate which number is larger $(1.1)^{10000}$ or 1000.

11. Find $(a + b)^4 - (a - b)^4$. Hence, evaluate $(\sqrt{3}+\sqrt{2})^4 - (\sqrt{3}-\sqrt{2})^4$.

12. Find $(x + 1)^6 + (x - 1)^6$. Hence or otherwise evaluate $(\sqrt{2}+1)^6 + (\sqrt{2}-1)^6$.

13. Show that $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer.

14. Prove that $\displaystyle\sum_{r=0}^{n} 3^r\ {}^nC_r = 4^n$.

Miscellaneous Exercise on Chapter 7

1. If a and b are distinct integers, prove that $a - b$ is a factor of $a^n - b^n$, whenever n is a positive integer.

[**Hint** write $a^n = (a - b + b)^n$ and expand]

2. Evaluate $\left(\sqrt{3}+\sqrt{2}\right)^6 - \left(\sqrt{3}-\sqrt{2}\right)^6$.

3. Find the value of $\left(a^2+\sqrt{a^2-1}\right)^4 + \left(a^2-\sqrt{a^2-1}\right)^4$.

4. Find an approximation of $(0.99)^5$ using the first three terms of its expansion.

5. Expand using Binomial Theorem $\left(1+\dfrac{x}{2}-\dfrac{2}{x}\right)^4, x \neq 0$.

6. Find the expansion of $(3x^2 - 2ax + 3a^2)^3$ using binomial theorem.

Summary

- The expansion of a binomial for any positive integral n is given by Binomial Theorem, which is $(a + b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_{n-1}a.b^{n-1} + {}^nC_nb^n$.
- The coefficients of the expansions are arranged in an array. This array is called *Pascal's triangle*.

Historical Note

The ancient Indian mathematicians knew about the coefficients in the expansions of $(x + y)^n$, $0 \leq n \leq 7$. The arrangement of these coefficients was in the form of a diagram called *Meru-Prastara*, provided by Pingla in his book *Chhanda shastra* (200B.C.). This triangular arrangement is also found in the work of Chinese mathematician Chu-shi-kie in 1303. The term binomial coefficients was first introduced by the German mathematician, Michael Stipel (1486-1567) in approximately 1544. Bombelli (1572) also gave the coefficients in the expansion of $(a + b)^n$, for $n = 1,2 ...,7$ and Oughtred (1631) gave them for $n = 1, 2,..., 10$. The arithmetic triangle, popularly known as *Pascal's triangle* and similar to the *Meru-Prastara* of Pingla was constructed by the French mathematician Blaise Pascal (1623-1662) in 1665.

The present form of the binomial theorem for integral values of n appeared in *Trate du triange arithmetic*, written by Pascal and published posthumously in 1665.

Chapter **8**

SEQUENCES AND SERIES

❖*Natural numbers are the product of human spirit. – DEDEKIND* ❖

8.1 Introduction

Fibonacci
(1175-1250)

In mathematics, the word, "*sequence*" is used in much the same way as it is in ordinary English. When we say that a collection of objects is listed in a sequence, we usually mean that the collection is ordered in such a way that it has an identified first member, second member, third member and so on. For example, population of human beings or bacteria at different times form a sequence. The amount of money deposited in a bank, over a number of years form a sequence. Depreciated values of certain commodity occur in a sequence. Sequences have important applications in several spheres of human activities.

Sequences, following specific patterns are called *progressions*. In previous class, we have studied about *arithmetic progression* (A.P). In this Chapter, besides discussing more about A.P.; *arithmetic mean, geometric mean, relationship between A.M. and G.M., special series in forms of sum to n terms of consecutive natural numbers, sum to n terms of squares of natural numbers and sum to n terms of cubes of natural numbers* will also be studied.

8.2 Sequences

Let us consider the following examples:

Assume that there is a generation gap of 30 years, we are asked to find the number of ancestors, i.e., parents, grandparents, great grandparents, etc. that a person might have over 300 years.

Here, the total number of generations $= \dfrac{300}{30} = 10$

The number of person's ancestors for the first, second, third, …, tenth generations are 2, 4, 8, 16, 32, …, 1024. These numbers form what we call a *sequence*.

Consider the successive quotients that we obtain in the division of 10 by 3 at different steps of division. In this process we get 3,3.3,3.33,3.333, ... and so on. These quotients also form a sequence. The various numbers occurring in a sequence are called its *terms*. We denote the terms of a sequence by $a_1, a_2, a_3, ..., a_n$, …, etc., the subscripts denote the position of the term. The n^{th} term is the number at the n^{th} position of the sequence and is denoted by a_n. The n^{th} term is also called the *general term* of the sequence.

Thus, the terms of the sequence of person's ancestors mentioned above are:

$$a_1 = 2, a_2 = 4, a_3 = 8, ..., a_{10} = 1024.$$

Similarly, in the example of successive quotients

$$a_1 = 3, a_2 = 3.3, a_3 = 3.33, \ldots, a_6 = 3.33333, \text{etc.}$$

A sequence containing finite number of terms is called a *finite sequence*. For example, sequence of ancestors is a finite sequence since it contains 10 terms (a fixed number).

A sequence is called *infinite*, if it is not a finite sequence. For example, the sequence of successive quotients mentioned above is an *infinite sequence*, infinite in the sense that it never ends.

Often, it is possible to express the rule, which yields the various terms of a sequence in terms of algebraic formula. Consider for instance, the sequence of even natural numbers 2, 4, 6, …

Here $a_1 = 2 = 2 \times 1 \qquad a_2 = 4 = 2 \times 2$

$a_3 = 6 = 2 \times 3 \qquad a_4 = 8 = 2 \times 4$

....

....

$a_{23} = 46 = 2 \times 23$, $a_{24} = 48 = 2 \times 24$, and so on.

In fact, we see that the n^{th} term of this sequence can be written as $a_n = 2n$, where n is a natural number. Similarly, in the sequence of odd natural numbers 1,3,5, …, the n^{th} term is given by the formula, $a_n = 2n - 1$, where n is a natural number.

In some cases, an arrangement of numbers such as 1, 1, 2, 3, 5, 8,.. has no visible pattern, but the sequence is generated by the recurrence relation given by

$$a_1 = a_2 = 1$$
$$a_3 = a_1 + a_2$$
$$a_n = a_{n-2} + a_{n-1}, n > 2$$

This sequence is called *Fibonacci sequence*.

In the sequence of primes 2,3,5,7,…, we find that there is no formula for the n^{th} prime. Such sequence can only be described by verbal description.

In every sequence, we should not expect that its terms will necessarily be given by a specific formula. However, we expect a theoretical scheme or a rule for generating the terms $a_1, a_2, a_3,\ldots,a_n,\ldots$ in succession.

In view of the above, *a sequence can be regarded as a function whose domain is the set of natural numbers or some subset of it. Sometimes, we use the functional notation a(n) for* a_n.

8.3 Series

Let $a_1, a_2, a_3,\ldots,a_n$, be a given sequence. Then, the expression

$$a_1 + a_2 + a_3 +,\ldots + a_n + \ldots$$

is called the *series associated with the given sequence* .The series is finite or infinite according as the given sequence is finite or infinite. Series are often represented in compact form, called *sigma notation*, using the Greek letter $\sum$ (sigma) as means of indicating the summation involved. Thus, the series $a_1 + a_2 + a_3 + \ldots + a_n$ is abbreviated as $\sum_{k=1}^{n} a_k$.

Remark When the series is used, it refers to the indicated sum not to the sum itself. For example, 1 + 3 + 5 + 7 is a finite series with four terms. When we use the phrase "*sum of a series*," we will mean the number that results from adding the terms, the sum of the series is 16.

We now consider some examples.

Example 1 Write the first three terms in each of the following sequences defined by the following:

(i) $a_n = 2n + 5$, (ii) $a_n = \dfrac{n-3}{4}$.

Solution (i) Here $a_n = 2n + 5$

Substituting $n = 1, 2, 3$, we get

$$a_1 = 2(1) + 5 = 7,\ a_2 = 9,\ a_3 = 11$$

Therefore, the required terms are 7, 9 and 11.

(ii) Here $a_n = \dfrac{n-3}{4}$. Thus, $a_1 = \dfrac{1-3}{4} = -\dfrac{1}{2},\ a_2 = -\dfrac{1}{4},\ a_3 = 0$

Hence, the first three terms are $-\frac{1}{2}, -\frac{1}{4}$ and 0.

Example 2 What is the 20th term of the sequence defined by

$$a_n = (n-1)(2-n)(3+n)\ ?$$

Solution Putting $n = 20$, we obtain

$$a_{20} = (20-1)(2-20)(3+20)$$
$$= 19 \times (-18) \times (23) = -7866.$$

Example 3 Let the sequence a_n be defined as follows:

$$a_1 = 1,\ \ a_n = a_{n-1} + 2 \text{ for } n \geq 2.$$

Find first five terms and write corresponding series.

Solution We have

$$a_1 = 1, a_2 = a_1 + 2 = 1 + 2 = 3, a_3 = a_2 + 2 = 3 + 2 = 5,$$
$$a_4 = a_3 + 2 = 5 + 2 = 7, a_5 = a_4 + 2 = 7 + 2 = 9.$$

Hence, the first five terms of the sequence are 1,3,5,7 and 9. The corresponding series is 1 + 3 + 5 + 7 + 9 +...

EXERCISE 8.1

Write the first five terms of each of the sequences in Exercises 1 to 6 whose n^{th} terms are:

1. $a_n = n(n+2)$ **2.** $a_n = \frac{n}{n+1}$ **3.** $a_n = 2^n$

4. $a_n = \frac{2n-3}{6}$ **5.** $a_n = (-1)^{n-1}\, 5^{n+1}$ **6.** $a_n = n\frac{n^2+5}{4}$.

Find the indicated terms in each of the sequences in Exercises 7 to 10 whose n^{th} terms are:

7. $a_n = 4n - 3;\ a_{17}, a_{24}$ **8.** $a_n = \frac{n^2}{2^n}; a_7$

9. $a_n = (-1)^{n-1} n^3;\ a_9$ **10.** $a_n = \frac{n(n-2)}{n+3}; a_{20}$.

Write the first five terms of each of the sequences in Exercises 11 to 13 and obtain the corresponding series:

11. $a_1 = 3, a_n = 3a_{n-1} + 2$ for all $n > 1$ **12.** $a_1 = -1, a_n = \frac{a_{n-1}}{n}, n \geq 2$

13. $a_1 = a_2 = 2, a_n = a_{n-1} - 1, n > 2$

14. The Fibonacci sequence is defined by

$$1 = a_1 = a_2 \text{ and } a_n = a_{n-1} + a_{n-2}, n > 2.$$

Find $\frac{a_{n+1}}{a_n}$, for $n = 1, 2, 3, 4, 5$

8.4 Geometric Progression (G. P.)

Let us consider the following sequences:

(i) 2,4,8,16,..., (ii) $\frac{1}{9}, \frac{-1}{27}, \frac{1}{81}, \frac{-1}{243} \ldots$ (iii) .01,.0001,.000001,...

In each of these sequences, how their terms progress? We note that each term, except the first progresses in a definite order.

In (i), we have $a_1 = 2, \frac{a_2}{a_1} = 2, \frac{a_3}{a_2} = 2, \frac{a_4}{a_3} = 2$ and so on.

In (ii), we observe, $a_1 = \frac{1}{9}, \frac{a_2}{a_1} = -\frac{1}{3}, \frac{a_3}{a_2} = -\frac{1}{3}, \frac{a_4}{a_3} = -\frac{1}{3}$ and so on.

Similarly, state how do the terms in (iii) progress? It is observed that in each case, every term except the first term bears a constant ratio to the term immediately preceding it. In (i), this constant ratio is 2; in (ii), it is $-\frac{1}{3}$ and in (iii), the constant ratio is 0.01. Such sequences are called *geometric sequence* or *geometric progression* abbreviated as G.P.

A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is called *geometric progression*, if each term is non-zero and $\frac{a_{k+1}}{a_k} = r$ (constant), for $k \geq 1$.

By letting $a_1 = a$, we obtain a geometric progression, $a, ar, ar^2, ar^3, \ldots.$, where a is called the *first term* and r is called the *common ratio* of the G.P. Common ratio in geometric progression (i), (ii) and (iii) above are 2, $-\frac{1}{3}$ and 0.01, respectively.

As in case of arithmetic progression, the problem of finding the n^{th} term or sum of n terms of a geometric progression containing a large number of terms would be difficult without the use of the formulae which we shall develop in the next Section. We shall use the following notations with these formulae:

a = the first term, r = the common ratio, l = the last term,

n = the numbers of terms,

S_n = the sum of first n terms.

8.4.1 General term of a G.P. Let us consider a G.P. with first non-zero term 'a' and common ratio 'r'. Write a few terms of it. The second term is obtained by multiplying a by r, thus $a_2 = ar$. Similarly, third term is obtained by multiplying a_2 by r. Thus, $a_3 = a_2 r = ar^2$, and so on.

We write below these and few more terms.

1st term $= a_1 = a = ar^{1-1}$, 2nd term $= a_2 = ar = ar^{2-1}$, 3rd term $= a_3 = ar^2 = ar^{3-1}$

4th term $= a_4 = ar^3 = ar^{4-1}$, 5th term $= a_5 = ar^4 = ar^{5-1}$

Do you see a pattern? What will be 16th term?

$$a_{16} = ar^{16-1} = ar^{15}$$

Therefore, the pattern suggests that the n^{th} term of a G.P. is given by $a_n = ar^{n-1}$.

Thus, a, G.P. can be written as $a, ar, ar^2, ar^3, \ldots ar^{n-1}$; $a, ar, ar^2, \ldots, ar^{n-1} \ldots$; according as G.P. is *finite* or *infinite*, respectively.

The series $a + ar + ar^2 + \ldots + ar^{n-1}$ or $a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$ are called *finite* or *infinite geometric series*, respectively.

8.4.2. Sum to n terms of a G.P. Let the first term of a G.P. be a and the common ratio be r. Let us denote by S_n the sum to first n terms of G.P. Then

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1} \qquad \ldots (1)$$

Case 1 If $r = 1$, we have $S_n = a + a + a + \ldots + a$ (n terms) $= na$

Case 2 If $r \neq 1$, multiplying (1) by r, we have

$$rS_n = ar + ar^2 + ar^3 + \ldots + ar^n \qquad \ldots (2)$$

Subtracting (2) from (1), we get $(1 - r) S_n = a - ar^n = a(1 - r^n)$

This gives or $S_n = \dfrac{a(r^n - 1)}{r - 1}$

Example 4 Find the 10th and n^{th} terms of the G.P. 5, 25,125,… .

Solution Here $a = 5$ and $r = 5$. Thus, $a_{10} = 5(5)^{10-1} = 5(5)^9 = 5^{10}$

and $a_n = ar^{n-1} = 5(5)^{n-1} = 5^n$.

Example 5 Which term of the G.P., 2,8,32, ... up to n terms is 131072?

Solution Let 131072 be the n^{th} term of the given G.P. Here $a = 2$ and $r = 4$.

Therefore $131072 = a_n = 2(4)^{n-1}$ or $65536 = 4^{n-1}$

This gives $4^8 = 4^{n-1}$.

So that $n - 1 = 8$, i.e., $n = 9$. Hence, 131072 is the 9^{th} term of the G.P.

Example 6 In a G.P., the 3^{rd} term is 24 and the 6^{th} term is 192.Find the 10^{th} term.

Solution Here, $a_3 = ar^2 = 24$ ... (1)

and $a_6 = ar^5 = 192$ ... (2)

Dividing (2) by (1), we get $r = 2$. Substituting $r = 2$ in (1), we get $a = 6$.

Hence $a_{10} = 6\,(2)^9 = 3072$.

Example 7 Find the sum of first n terms and the sum of first 5 terms of the geometric series $1+\frac{2}{3}+\frac{4}{9}+...$

Solution Here $a = 1$ and $r = \frac{2}{3}$. Therefore

$$S_n = \frac{a\,(1-r^n)}{1-r} = \frac{\left[1-\left(\frac{2}{3}\right)^n\right]}{1-\frac{2}{3}} = 3\left[1-\left(\frac{2}{3}\right)^n\right]$$

In particular, $$S_5 = 3\left[1-\left(\frac{2}{3}\right)^5\right] = 3\times\frac{211}{243} = \frac{211}{81}.$$

Example 8 How many terms of the G.P. $3,\frac{3}{2},\frac{3}{4},...$ are needed to give the sum $\frac{3069}{512}$?

Solution Let n be the number of terms needed. Given that $a = 3$, $r = \frac{1}{2}$ and $S_n = \frac{3069}{512}$

Since $$S_n = \frac{a\,(1-r^n)}{1-r}$$

Therefore $$\frac{3069}{512}=\frac{3(1-\frac{1}{2^n})}{1-\frac{1}{2}}=6\left(1-\frac{1}{2^n}\right)$$

or $$\frac{3069}{3072} = 1-\frac{1}{2^n}$$

or $$\frac{1}{2^n} = 1-\frac{3069}{3072}=\frac{3}{3072}=\frac{1}{1024}$$

or $2^n = 1024 = 2^{10}$, which gives $n = 10$.

Example 9 The sum of first three terms of a G.P. is $\frac{13}{12}$ and their product is – 1. Find the common ratio and the terms.

Solution Let $\frac{a}{r}$, a, ar be the first three terms of the G.P. Then

$$\frac{a}{r} + ar + a = \frac{13}{12} \qquad \text{... (1)}$$

and $$\left(\frac{a}{r}\right)(a)\,(ar) = -1 \qquad \text{... (2)}$$

From (2), we get $a^3 = -1$, i.e., $a = -1$ (considering only real roots)

Substituting $a = -1$ in (1), we have

$$-\frac{1}{r}-1-r=\frac{13}{12} \text{ or } 12r^2 + 25r + 12 = 0.$$

This is a quadratic in r, solving, we get $r=-\frac{3}{4}$ or $-\frac{4}{3}$.

Thus, the three terms of G.P. are : $\frac{4}{3},-1,\frac{3}{4}$ for $r = \frac{-3}{4}$ and $\frac{3}{4},-1,\ \frac{4}{3}$ for $r = \frac{-4}{3}$,

Example10 Find the sum of the sequence 7, 77, 777, 7777, ... to n terms.

Solution This is not a G.P., however, we can relate it to a G.P. by writing the terms as

$$S_n = 7 + 77 + 777 + 7777 + \dots \text{ to } n \text{ terms}$$

$$= \frac{7}{9}[9+99+999+9999+ \text{...to } n \text{ term}]$$

$$= \frac{7}{9}[(10-1)+(10^2-1)+(10^3-1)+(10^4-1)+...n \text{ terms}]$$

$$= \frac{7}{9}[(10+10^2+10^3+...n \text{ terms})-(1+1+1+...n \text{ terms})]$$

$$= \frac{7}{9}\left[\frac{10(10^n-1)}{10-1}-n\right] = \frac{7}{9}\left[\frac{10\,(10^n-1)}{9}-n\right].$$

Example 11 A person has 2 parents, 4 grandparents, 8 great grandparents, and so on. Find the number of his ancestors during the ten generations preceding his own.

Solution Here $a = 2$, $r = 2$ and $n = 10$

Using the sum formula $\quad S_n = \dfrac{a\,(r^n-1)}{r-1}$

We have $\quad S_{10} = 2(2^{10} - 1) = 2046$

Hence, the number of ancestors preceding the person is 2046.

8.4.3 *Geometric Mean (G.M.)* The geometric mean of two positive numbers a and b is the number $\sqrt{ab}$. Therefore, the geometric mean of 2 and 8 is 4. We observe that the three numbers 2,4,8 are consecutive terms of a G.P. This leads to a generalisation of the concept of geometric means of two numbers.

Given any two positive numbers a and b, we can insert as many numbers as we like between them to make the resulting sequence in a G.P.

Let $G_1, G_2,..., G_n$ be n numbers between positive numbers a and b such that $a,G_1,G_2,G_3,...,G_n,b$ is a G.P. Thus, b being the $(n + 2)^{\text{th}}$ term,we have

$$b = ar^{n+1}, \quad \text{or} \quad r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}.$$

Hence $\quad G_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, \; G_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}, \; G_3 = ar^3 = a\left(\frac{b}{a}\right)^{\frac{3}{n+1}},$

$$G_n = ar^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Example 12 Insert three numbers between 1 and 256 so that the resulting sequence is a G.P.

Solution Let G_1, G_2, G_3 be three numbers between 1 and 256 such that $1, G_1, G_2, G_3, 256$ is a G.P.

Therefore $256 = r^4$ giving $r = \pm 4$ (Taking real roots only)

For $r = 4$, we have $G_1 = ar = 4$, $G_2 = ar^2 = 16$, $G_3 = ar^3 = 64$

Similarly, for $r = -4$, numbers are $-4, 16$ and -64.

Hence, we can insert 4, 16, 64 between 1 and 256 so that the resulting sequences are in G.P.

8.5 Relationship Between A.M. and G.M.

Let A and G be A.M. and G.M. of two given positive real numbers a and b, respectively. Then

$$\mathrm{A} = \frac{a+b}{2} \quad \text{and } \mathrm{G} = \sqrt{ab}$$

Thus, we have

$$\mathrm{A} - \mathrm{G} = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2}$$

$$= \frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2} \geq 0 \qquad \ldots (1)$$

From (1), we obtain the relationship $\mathrm{A} \geq \mathrm{G}$.

Example 13 If A.M. and G.M. of two positive numbers a and b are 10 and 8, respectively, find the numbers.

Solution Given that $\quad \text{A.M.} = \frac{a+b}{2} = 10 \qquad \ldots (1)$

and $\quad \text{G.M.} = \sqrt{ab} = 8 \qquad \ldots (2)$

From (1) and (2), we get

$$a + b = 20 \qquad \ldots (3)$$

$$ab = 64 \qquad \ldots (4)$$

Putting the value of a and b from (3), (4) in the identity $(a - b)^2 = (a + b)^2 - 4ab$, we get

$$(a - b)^2 = 400 - 256 = 144$$

or $\quad a - b = \pm 12 \qquad \ldots (5)$

Solving (3) and (5), we obtain

$$a = 4,\ b = 16 \text{ or } a = 16,\ b = 4$$

Thus, the numbers a and b are 4, 16 or 16, 4 respectively.

EXERCISE 8.2

1. Find the 20th and nth terms of the G.P. $\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \ldots$
2. Find the 12th term of a G.P. whose 8th term is 192 and the common ratio is 2.
3. The 5th, 8th and 11th terms of a G.P. are p, q and s, respectively. Show that $q^2 = ps$.
4. The 4th term of a G.P. is square of its second term, and the first term is – 3. Determine its 7th term.
5. Which term of the following sequences:

   (a) $2, 2\sqrt{2}, 4, \ldots$ is 128 ? (b) $\sqrt{3}, 3, 3\sqrt{3}, \ldots$ is 729 ?

   (c) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ is $\frac{1}{19683}$?

6. For what values of x, the numbers $-\frac{2}{7}, x, -\frac{7}{2}$ are in G.P.?

Find the sum to indicated number of terms in each of the geometric progressions in Exercises 7 to 10:

7. 0.15, 0.015, 0.0015, ... 20 terms.
8. $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, \ldots$ n terms.
9. $1, -a, a^2, -a^3, \ldots$ n terms (if $a \neq -1$).
10. $x^3, x^5, x^7, \ldots$ n terms (if $x \neq \pm 1$).
11. Evaluate $\sum_{k=1}^{11} (2 + 3^k)$.
12. The sum of first three terms of a G.P. is $\frac{39}{10}$ and their product is 1. Find the common ratio and the terms.
13. How many terms of G.P. $3, 3^2, 3^3, \ldots$ are needed to give the sum 120?
14. The sum of first three terms of a G.P. is 16 and the sum of the next three terms is 128. Determine the first term, the common ratio and the sum to n terms of the G.P.
15. Given a G.P. with $a = 729$ and 7th term 64, determine S_7.

16. Find a G.P. for which sum of the first two terms is – 4 and the fifth term is 4 times the third term.

17. If the 4th, 10th and 16th terms of a G.P. are x, y and z, respectively. Prove that x, y, z are in G.P.

18. Find the sum to n terms of the sequence, 8, 88, 888, 8888… .

19. Find the sum of the products of the corresponding terms of the sequences 2, 4, 8, 16, 32 and 128, 32, 8, 2, $\frac{1}{2}$.

20. Show that the products of the corresponding terms of the sequences $a, ar, ar^2, \ldots ar^{n-1}$ and A, AR, AR^2, … AR^{n-1} form a G.P, and find the common ratio.

21. Find four numbers forming a geometric progression in which the third term is greater than the first term by 9, and the second term is greater than the 4th by 18.

22. If the p^{th}, q^{th} and r^{th} terms of a G.P. are a, b and c, respectively. Prove that

$$a^{q-r}\, b^{r-p} c^{p-q} = 1.$$

23. If the first and the n^{th} term of a G.P. are a and b, respectively, and if P is the product of n terms, prove that $P^2 = (ab)^n$.

24. Show that the ratio of the sum of first n terms of a G.P. to the sum of terms from $(n + 1)^{th}$ to $(2n)^{th}$ term is $\frac{1}{r^n}$.

25. If a, b, c and d are in G.P. show that

$(a^2 + b^2 + c^2)\,(b^2 + c^2 + d^2) = (ab + bc + cd)^2$.

26. Insert two numbers between 3 and 81 so that the resulting sequence is G.P.

27. Find the value of n so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be the geometric mean between a and b.

28. The sum of two numbers is 6 times their geometric mean, show that numbers are in the ratio $\left(3+2\sqrt{2}\right):\left(3-2\sqrt{2}\right)$.

29. If A and G be A.M. and G.M., respectively between two positive numbers, prove that the numbers are $A \pm \sqrt{(A+G)(A-G)}$.

30. The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of 2nd hour, 4th hour and n^{th} hour ?

31. What will Rs 500 amounts to in 10 years after its deposit in a bank which pays annual interest rate of 10% compounded annually?

32. If A.M. and G.M. of roots of a quadratic equation are 8 and 5, respectively, then obtain the quadratic equation.

Miscellaneous Examples

Example 14 If a, b, c, d and p are different real numbers such that $(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)\,p + (b^2 + c^2 + d^2) \le 0$, then show that a, b, c and d are in G.P.

Solution Given that

$$(a^2 + b^2 + c^2)\,p^2 - 2\,(ab + bc + cd)\,p + (b^2 + c^2 + d^2) \le 0 \qquad \ldots (1)$$

But L.H.S.

$$= (a^2p^2 - 2abp + b^2) + (b^2p^2 - 2bcp + c^2) + (c^2p^2 - 2cdp + d^2),$$

which gives $(ap - b)^2 + (bp - c)^2 + (cp - d)^2 \ \ge 0$ ... (2)

Since the sum of squares of real numbers is non negative, therefore, from (1) and (2), we have, $(ap - b)^2 + (bp - c)^2 + (cp - d)^2 = 0$

or $ap - b = 0,\ bp - c = 0,\ cp - d = 0$

This implies that $\dfrac{b}{a} = \dfrac{c}{b} = \dfrac{d}{c} = p$

Hence a, b, c and d are in G.P.

Miscellaneous Exercise On Chapter 8

1. If f is a function satisfying $f(x + y) = f(x)\,f(y)$ for all $x, y \in \mathbf{N}$ such that $f(1) = 3$ and $\sum_{x=1}^{n} f(x) = 120$, find the value of n.

2. The sum of some terms of G.P. is 315 whose first term and the common ratio are 5 and 2, respectively. Find the last term and the number of terms.

3. The first term of a G.P. is 1. The sum of the third term and fifth term is 90. Find the common ratio of G.P.

4. The sum of three numbers in G.P. is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an arithmetic progression. Find the numbers.

5. A G.P. consists of an even number of terms. If the sum of all the terms is 5 times the sum of terms occupying odd places, then find its common ratio.

6. If $\frac{a+bx}{a-bx}=\frac{b+cx}{b-cx}=\frac{c+dx}{c-dx}(x\neq 0)$, then show that *a, b, c* and *d* are in G.P.

7. Let S be the sum, P the product and R the sum of reciprocals of *n* terms in a G.P. Prove that $P^2R^n = S^n$.

8. If *a, b, c, d* are in G.P, prove that $(a^n + b^n), (b^n + c^n), (c^n + d^n)$ are in G.P.

9. If *a* and *b* are the roots of $x^2 - 3x + p = 0$ and *c, d* are roots of $x^2 - 12x + q = 0$, where *a, b, c, d* form a G.P. Prove that $(q + p) : (q - p) = 17:15$.

10. The ratio of the A.M. and G.M. of two positive numbers *a* and *b, is m : n*. Show that $a:b=\left(m+\sqrt{m^2-n^2}\right):\left(m-\sqrt{m^2-n^2}\right)$.

11. Find the sum of the following series up to *n* terms:
(i) 5 + 55 +555 + … (ii) .6 +. 66 +. 666+…

12. Find the 20^{th} term of the series $2 \times 4 + 4 \times 6 + 6 \times 8 + ... + n$ terms.

13. A farmer buys a used tractor for Rs 12000. He pays Rs 6000 cash and agrees to pay the balance in annual instalments of Rs 500 plus 12% interest on the unpaid amount. How much will the tractor cost him?

14. Shamshad Ali buys a scooter for Rs 22000. He pays Rs 4000 cash and agrees to pay the balance in annual instalment of Rs 1000 plus 10% interest on the unpaid amount. How much will the scooter cost him?

15. A person writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 50 paise to mail one letter. Find the amount spent on the postage when 8^{th} set of letter is mailed.

16. A man deposited Rs 10000 in a bank at the rate of 5% simple interest annually. Find the amount in 15^{th} year since he deposited the amount and also calculate the total amount after 20 years.

17. A manufacturer reckons that the value of a machine, which costs him Rs. 15625, will depreciate each year by 20%. Find the estimated value at the end of 5 years.

18. 150 workers were engaged to finish a job in a certain number of days. 4 workers dropped out on second day, 4 more workers dropped out on third day and so on. It took 8 more days to finish the work. Find the number of days in which the work was completed.

Summary

- By a *sequence*, we mean an arrangement of number in definite order according to some rule. Also, we define a sequence as a function whose domain is the set of natural numbers or some subsets of the type $\{1, 2, 3, ...k\}$. A sequence containing a finite number of terms is called a *finite sequence*. A sequence is called *infinite* if it is not a finite sequence.
- Let $a_1, a_2, a_3, ...$ be the sequence, then the sum expressed as $a_1 + a_2 + a_3 + ...$ is called *series*. A series is called *finite series* if it has got finite number of terms.
- A sequence is said to be a *geometric progression* or *G.P.*, if the ratio of any term to its preceding term is same throughout. This constant factor is called the *common ratio*. Usually, we denote the first term of a G.P. by a and its common ratio by r. The general or the n^{th} term of G.P. is given by $a_n = ar^{n-1}$. The sum S_n of the first n terms of G.P. is given by

$$S_n = \frac{a(r^n - 1)}{r-1} \text{ or } \frac{a(1-r^n)}{1-r}, \text{ if } r \neq 1$$

- The geometric mean (G.M.) of any two positive numbers a and b is given by $\sqrt{ab}$ i.e., the sequence a, G, b is G.P.

Historical Note

Evidence is found that Babylonians, some 4000 years ago, knew of arithmetic and geometric sequences. According to Boethius (510), arithmetic and geometric sequences were known to early Greek writers. Among the Indian mathematician, Aryabhatta (476) was the first to give the formula for the sum of squares and cubes of natural numbers in his famous work Aryabhatiyam, written around 499. He also gave the formula for finding the sum to n terms of an arithmetic sequence starting with p^{th} term. Noted Indian mathematicians Brahmgupta

(598), Mahavira (850) and Bhaskara (1114-1185) also considered the sum of squares and cubes. Another specific type of sequence having important applications in mathematics, called *Fibonacci sequence*, was discovered by Italian mathematician Leonardo Fibonacci (1170-1250). Seventeenth century witnessed the classification of series into specific forms. In 1671 James Gregory used the term infinite series in connection with infinite sequence. It was only through the rigorous development of algebraic and set theoretic tools that the concepts related to sequence and series could be formulated suitably.

Chapter 9

STRAIGHT LINES

❖ *Geometry, as a logical system, is a means and even the most powerful means to make children feel the strength of the human spirit that is of their own spirit. – H. FREUDENTHAL* ❖

9.1 Introduction

We are familiar with two-dimensional *coordinate geometry* from earlier classes. Mainly, it is a combination of *algebra* and *geometry*. A systematic study of geometry by the use of algebra was first carried out by celebrated French philosopher and mathematician René Descartes, in his book 'La Géométry, published in 1637. This book introduced the notion of the equation of a curve and related analytical methods into the study of geometry. The resulting combination of analysis and geometry is referred now as *analytical geometry*. In the earlier classes, we initiated the study of coordinate geometry, where we studied about coordinate axes, coordinate plane, plotting of points in a plane, distance between two points, section formulae, etc. All these concepts are the basics of coordinate geometry.

René Descartes (1596 -1650)

Let us have a brief recall of coordinate geometry done in earlier classes. To recapitulate, the location of the points (6, – 4) and (3, 0) in the XY-plane is shown in Fig 9.1.

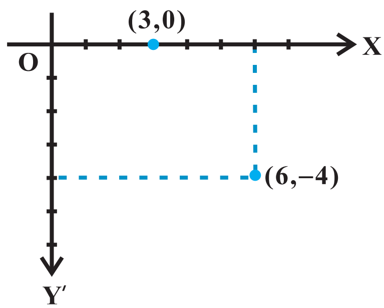


Fig 9.1

We may note that the point (6, – 4) is at 6 units distance from the y-axis measured along the positive x-axis and at 4 units distance from the x-axis measured along the negative y-axis. Similarly, the point (3, 0) is at 3 units distance from the y-axis measured along the positive x-axis and has zero distance from the x-axis.

We also studied there following important formulae:

I. Distance between the points P (x_1, y_1) and Q (x_2, y_2) is

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

For example, distance between the points (6, – 4) and (3, 0) is

$$\sqrt{(3-6)^2 + (0+4)^2} = \sqrt{9+16} = 5 \text{ units.}$$

II. The coordinates of a point dividing the line segment joining the points (x_1, y_1) and (x_2, y_2) internally, in the ratio $m: n$ are $\left(\dfrac{m x_2 + n x_1}{m+n}, \dfrac{m y_2 + n y_1}{m+n}\right)$.

For example, the coordinates of the point which divides the line segment joining A (1, –3) and B (–3, 9) internally, in the ratio 1: 3 are given by $x = \dfrac{1.(-3)+3.1}{1+3} = 0$

and $y = \dfrac{1.9 + 3.(-3)}{1+3} = 0.$

III. In particular, if $m = n$, the coordinates of the mid-point of the line segment joining the points (x_1, y_1) and (x_2, y_2) are $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

IV. Area of the triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$\frac{1}{2}\left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \right|.$$

For example, the area of the triangle, whose vertices are (4, 4), (3, – 2) and (– 3, 16) is

$$\frac{1}{2}\left| 4(-2-16) + 3(16-4) + (-3)(4+2) \right| = \frac{|-54|}{2} = 27.$$

Remark If the area of the triangle ABC is zero, then three points A, B and C lie on a line, i.e., they are collinear.

In the this Chapter, we shall continue the study of coordinate geometry to study properties of the simplest geometric figure – *straight line*. Despite its simplicity, the line is a vital concept of geometry and enters into our daily experiences in numerous interesting and useful ways. Main focus is on representing the line algebraically, for which *slope* is most essential.

9.2 Slope of a Line

A line in a coordinate plane forms two angles with the x-axis, which are supplementary.

The angle (say) θ made by the line l with positive direction of x-axis and measured anti clockwise is called the *inclination of the line*. Obviously $0° \le \theta \le 180°$ (Fig 9.2).

We observe that lines parallel to x-axis, or coinciding with x-axis, have inclination of 0°. The inclination of a vertical line (parallel to or coinciding with y-axis) is 90°.

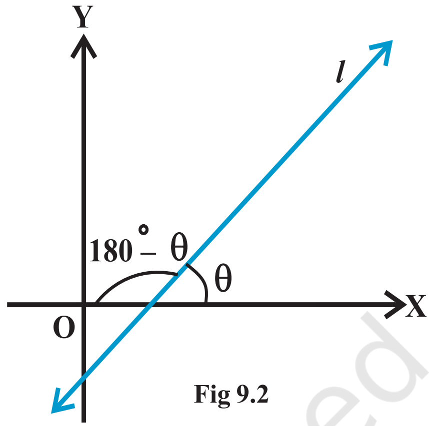


Fig 9.2

Definition 1 If θ is the inclination of a line l, then tan θ is called the *slope* or *gradient* of the line l.

The slope of a line whose inclination is 90° is not defined.

The slope of a line is denoted by m.

Thus, $m = \tan \theta$, $\theta \ne 90°$

It may be observed that the slope of x-axis is zero and slope of y-axis is not defined.

9.2.1 *Slope of a line when coordinates of any two points on the line are given*

We know that a line is completely determined when we are given two points on it. Hence, we proceed to find the slope of a line in terms of the coordinates of two points on the line.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on non-vertical line l whose inclination is θ. Obviously, $x_1 \ne x_2$, otherwise the line will become perpendicular to x-axis and its slope will not be defined. The inclination of the line l may be acute or obtuse. Let us take these two cases.

Draw perpendicular QR to x-axis and PM perpendicular to RQ as shown in Figs. 9.3 (i) and (ii).

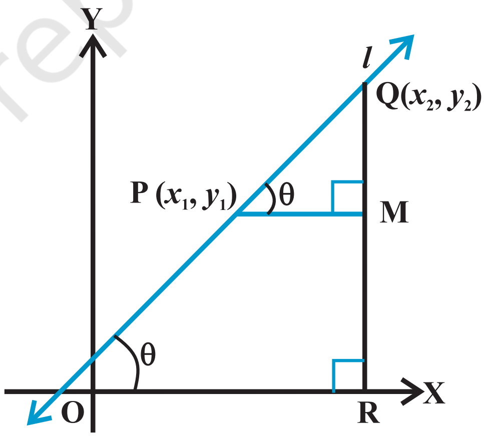


Fig 9. 3 (i)

Case 1 When angle θ is acute:

In Fig 9.3 (i), $\angle MPQ = \theta$. ... (1)

Therefore, slope of line $l = m = \tan \theta$.

But in ΔMPQ, we have $\tan\theta = \dfrac{MQ}{MP} = \dfrac{y_2 - y_1}{x_2 - x_1}$. ... (2)

From equations (1) and (2), we have

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Case II When angle θ is obtuse:
In Fig 9.3 (ii), we have
$\angle MPQ = 180° - \theta$.

Therefore, $\theta = 180° - \angle MPQ$.

Now, slope of the line l

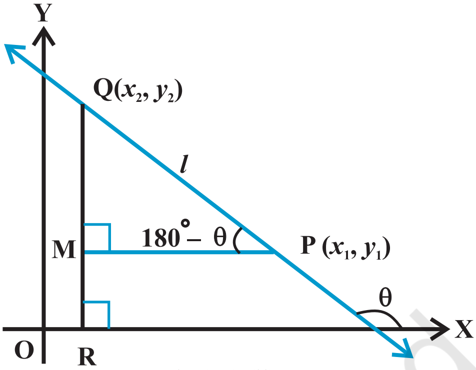


Fig 9. 3 (ii)

$$m = \tan\theta$$
$$= \tan(180° - \angle MPQ) = -\tan\angle MPQ$$
$$= -\frac{MQ}{MP} = -\frac{y_2 - y_1}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Consequently, we see that in both the cases the slope m of the line through the points (x_1, y_1) and (x_2, y_2) is given by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

9.2.2 ***Conditions for parallelism and perpendicularity of lines in terms of their slopes*** In a coordinate plane, suppose that non-vertical *lines* l_1 and l_2 have slopes m_1 *and* m_2, respectively. Let their inclinations be α and β, respectively.

If the line l_1 is parallel to l_2 (Fig 9.4), then their inclinations are equal, i.e.,

$$\alpha = \beta, \text{ and hence, } \tan\alpha = \tan\beta$$

Therefore $m_1 = m_2$, i.e., their slopes are equal.

Conversely, if the slope of two lines l_1 and l_2 is same, i.e.,

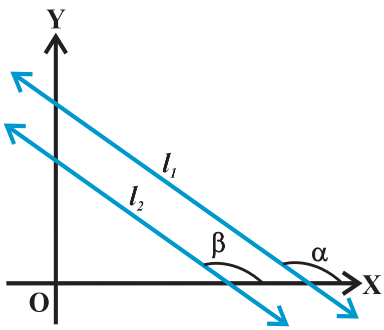


Fig 9. 4

$$m_1 = m_2.$$

Then $\tan\alpha = \tan\beta$.

By the property of tangent function (between 0° and 180°), α = β.

Therefore, the lines are parallel.

Hence, two non vertical lines l_1 and l_2 are parallel if and only if their slopes are equal.

If the lines l_1 and l_2 are perpendicular (Fig 9.5), then $\beta = \alpha + 90°$.

Therefore, $\tan \beta = \tan (\alpha + 90°)$

$$= -\cot \alpha = -\frac{1}{\tan \alpha}$$

i.e., $m_2 = -\frac{1}{m_1}$ or $m_1 \, m_2 = -1$

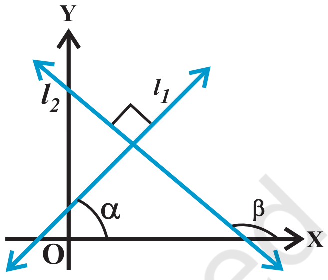


Fig 9. 5

Conversely, if $m_1 \, m_2 = -1$, i.e., $\tan \alpha \tan \beta = -1$.

Then $\tan \alpha = -\cot \beta = \tan (\beta + 90°)$ or $\tan (\beta - 90°)$

Therefore, α and β differ by 90°.

Thus, lines l_1 and l_2 are perpendicular to each other.

Hence, *two non-vertical lines are perpendicular to each other if and only if their slopes are negative reciprocals of each other,*

i.e., $m_2 = -\frac{1}{m_1}$ or, $m_1 \, m_2 = -1$.

Let us consider the following example.

Example 1 Find the slope of the lines:

(a) Passing through the points (3, – 2) and (–1, 4),

(b) Passing through the points (3, – 2) and (7, – 2),

(c) Passing through the points (3, – 2) and (3, 4),

(d) Making inclination of 60° with the positive direction of x-axis.

Solution (a) The slope of the line through (3, – 2) and (– 1, 4) is

$$m = \frac{4-(-2)}{-1-3} = \frac{6}{-4} = -\frac{3}{2}.$$

(b) The slope of the line through the points (3, – 2) and (7, – 2) is

$$m = \frac{-2-(-2)}{7-3} = \frac{0}{4} = 0.$$

(c) The slope of the line through the points (3, – 2) and (3, 4) is

$$m = \frac{4-(-2)}{3-3} = \frac{6}{0}, \text{ which is not defined.}$$

(d) Here inclination of the line $\alpha = 60°$. Therefore, slope of the line is

$m = \tan 60° = \sqrt{3}$.

9.2.3 ***Angle between two lines*** When we think about more than one line in a plane, then we find that these lines are either intersecting or parallel. Here we will discuss the angle between two lines in terms of their slopes.

Let L_1 and L_2 be two non-vertical lines with slopes m_1 and m_2, respectively. If α_1 and α_2 are the inclinations of lines L_1 and L_2, respectively. Then

$$m_1 = \tan\alpha_1 \text{ and } m_2 = \tan\alpha_2.$$

We know that when two lines intersect each other, they make two pairs of vertically opposite angles such that sum of any two adjacent angles is 180°. Let θ and ϕ be the adjacent angles between the lines L_1 and L_2 (Fig 9.6). Then

$$\theta = \alpha_2 - \alpha_1 \text{ and } \alpha_1, \alpha_2 \neq 90°.$$

Therefore $\tan\theta = \tan(\alpha_2 - \alpha_1) = \dfrac{\tan\alpha_2 - \tan\alpha_1}{1+\tan\alpha_1\tan\alpha_2} = \dfrac{m_2 - m_1}{1+m_1 m_2}$ (as $1 + m_1 m_2 \neq 0$)

and $\phi = 180° - \theta$ so that

$$\tan\phi = \tan(180° - \theta) = -\tan\theta = -\frac{m_2 - m_1}{1+m_1 m_2}, \text{ as } 1 + m_1 m_2 \neq 0$$

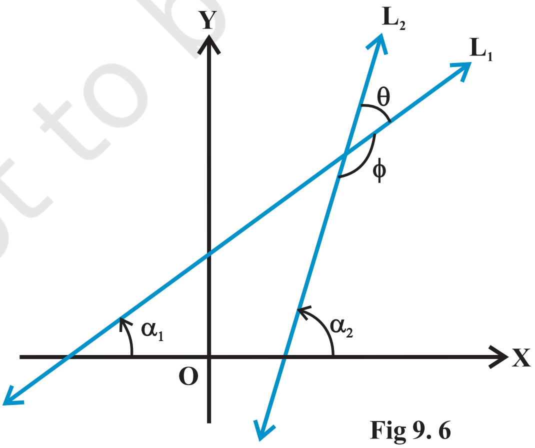


Fig 9. 6

Now, there arise two cases:

Case I If $\frac{m_2 - m_1}{1 + m_1 m_2}$ is positive, then tan θ will be positive and tan ϕ will be negative, which means θ will be acute and ϕ will be obtuse.

Case II If $\frac{m_2 - m_1}{1 + m_1 m_2}$ is negative, then tan θ will be negative and tan ϕ will be positive, which means that θ will be obtuse and ϕ will be acute.

Thus, the acute angle (say θ) between lines L_1 and L_2 with slopes m_1 and m_2, respectively, is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|, \text{ as } \quad 1 + m_1 m_2 \neq 0 \qquad \text{... (1)}$$

The obtuse angle (say ϕ) can be found by using $\phi = 180^0 - \theta$.

Example 2 If the angle between two lines is $\frac{\pi}{4}$ and slope of one of the lines is $\frac{1}{2}$, find the slope of the other line.

Solution We know that the acute angle θ between two lines with slopes m_1 and m_2 is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| \qquad \text{... (1)}$$

Let $m_1 = \frac{1}{2}$, $m_2 = m$ and $\theta = \frac{\pi}{4}$.

Now, putting these values in (1), we get

$$\tan\frac{\pi}{4} = \left|\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m}\right| \quad \text{or} \quad 1 = \left|\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m}\right|,$$

which gives

$$\frac{m - \frac{1}{2}}{1 + \frac{1}{2}\,m} = 1 \quad \text{or} \quad \frac{m - \frac{1}{2}}{1 + \frac{1}{2}\,m} = -1.$$

Therefore $m = 3$ or $m = -\frac{1}{3}$.

Hence, slope of the other line is 3 or $-\frac{1}{3}$. Fig 9.7 explains the reason of two answers.

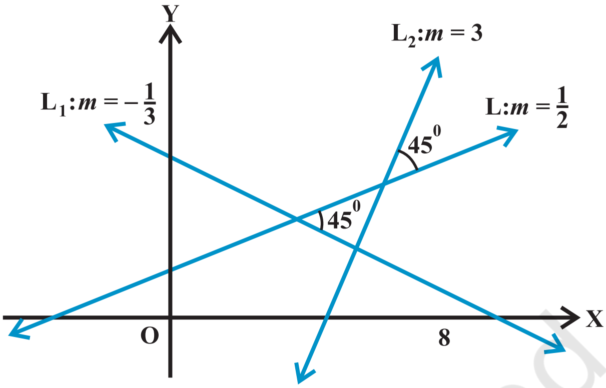


Fig 9.7

Example 3 Line through the points (–2, 6) and (4, 8) is perpendicular to the line through the points (8, 12) and (x, 24). Find the value of x.

Solution Slope of the line through the points (– 2, 6) and (4, 8) is

$$m_1 = \frac{8-6}{4-(-2)} = \frac{2}{6} = \frac{1}{3}$$

Slope of the line through the points (8, 12) and (x, 24) is

$$m_2 = \frac{24-12}{x-8} = \frac{12}{x-8}$$

Since two lines are perpendicular,

$m_1 m_2 = -1$, which gives

$$\frac{1}{3} \times \frac{12}{x-8} = -1 \text{ or } x = 4.$$

EXERCISE 9.1

1. Draw a quadrilateral in the Cartesian plane, whose vertices are (– 4, 5), (0, 7), (5, – 5) and (– 4, –2). Also, find its area.
2. The base of an equilateral triangle with side $2a$ lies along the y-axis such that the mid-point of the base is at the origin. Find vertices of the triangle.

3. Find the distance between P (x_1, y_1) and Q (x_2, y_2) when : (i) PQ is parallel to the y-axis, (ii) PQ is parallel to the x-axis.
4. Find a point on the x-axis, which is equidistant from the points (7, 6) and (3, 4).
5. Find the slope of a line, which passes through the origin, and the mid-point of the line segment joining the points P (0, – 4) and B (8, 0).
6. Without using the Pythagoras theorem, show that the points (4, 4), (3, 5) and (–1, –1) are the vertices of a right angled triangle.
7. Find the slope of the line, which makes an angle of 30° with the positive direction of y-axis measured anticlockwise.
8. Without using distance formula, show that points (– 2, – 1), (4, 0), (3, 3) and (–3, 2) are the vertices of a parallelogram.
9. Find the angle between the x-axis and the line joining the points (3,–1) and (4,–2).
10. The slope of a line is double of the slope of another line. If tangent of the angle between them is $\frac{1}{3}$, find the slopes of the lines.
11. A line passes through (x_1, y_1) and (h, k). If slope of the line is m, show that $k - y_1 = m\,(h - x_1)$.

9.3 Various Forms of the Equation of a Line

We know that every line in a plane contains infinitely many points on it. This relationship between line and points leads us to find the solution of the following problem:

How can we say that a given point lies on the given line? Its answer may be that for a given line we should have a definite condition on the points lying on the line. Suppose P (x, y) is an arbitrary point in the XY-plane and L is the given line. For the equation of L, we wish to construct a *statement* or *condition* for the point P that is true, when P is on L, otherwise false. Of course the statement is merely an algebraic equation involving the variables x and y. Now, we will discuss the equation of a line under different conditions.

9.3.1 *Horizontal and vertical lines* If a horizontal line L is at a distance a from the x-axis then ordinate of every point lying on the line is either a or $-a$ [Fig 9.8 (a)]. Therefore, equation of the line L is either $y = a$ or $y = -a$. Choice of sign will depend upon the position of the line according as the line is above or below the y-axis. Similarly, the equation of a vertical line at a distance b from the y-axis is either $x = b$ or $x = -b$ [Fig 9.8(b)].

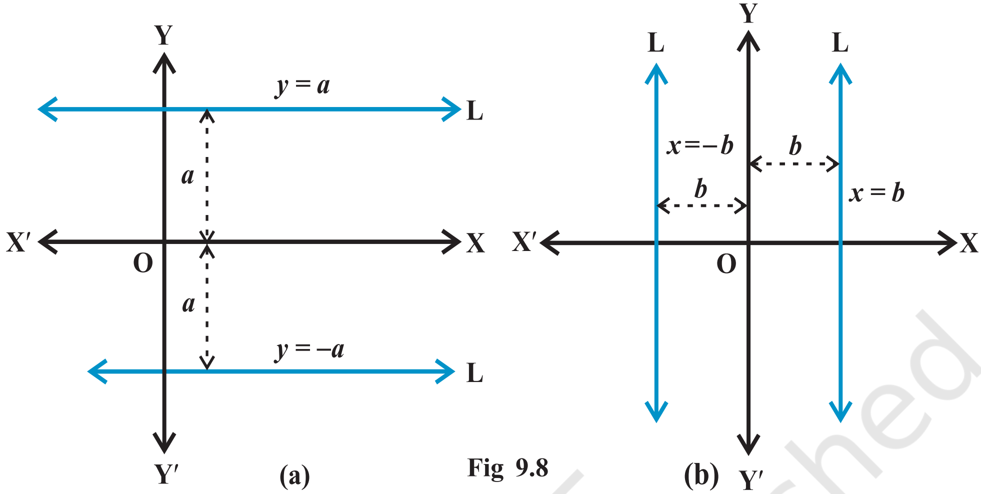


Fig 9.8

Example 4 Find the equations of the lines parallel to axes and passing through (– 2, 3).

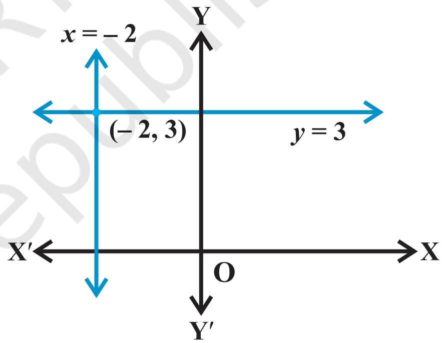


Fig 9.9

Solution Position of the lines is shown in the Fig 9.9. The y-coordinate of every point on the line parallel to x-axis is 3, therefore, equation of the line parallel to x-axis and passing through (– 2, 3) is $y = 3$. Similarly, equation of the line parallel to y-axis and passing through (– 2, 3) is $x = – 2$.

9.3.2 *Point-slope form*

Suppose that $P_0(x_0, y_0)$ is a fixed point on a non-vertical line L, whose slope is m. Let P (x, y) be an arbitrary point on L (Fig 9.10).

Then, by the definition, the slope of L is given by

$$m = \frac{y - y_0}{x - x_0}, \text{ i.e., } y - y_0 = m(x - x_0) \quad ...(1)$$

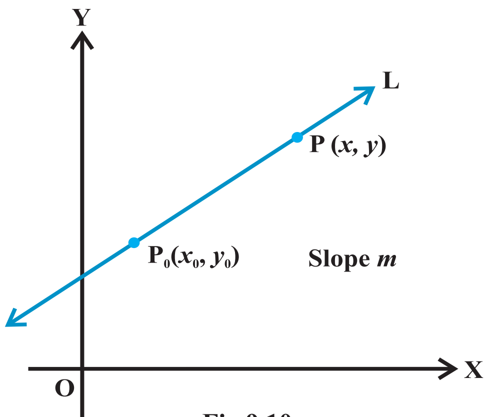


Fig 9.10

Since the point $P_0(x_0, y_0)$ along with all points (x, y) on L satisfies (1) and no other point in the plane satisfies (1). Equation (1) is indeed the equation for the given line L.

Thus, the point (x, y) lies on the line with slope m through the fixed point (x_0, y_0), if and only if, its coordinates satisfy the equation

$$y - y_0 = m\,(x - x_0)$$

Example 5 Find the equation of the line through $(-2, 3)$ with slope -4.

Solution Here $m = -4$ and given point (x_0, y_0) is $(-2, 3)$.

By slope-intercept form formula (1) above, equation of the given line is

$y - 3 = -4\,(x + 2)$ or $4x + y + 5 = 0$, which is the required equation.

9.3.3 ***Two-point form*** Let the line L passes through two given points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. Let $P(x, y)$ be a general point on L (Fig 9.11).

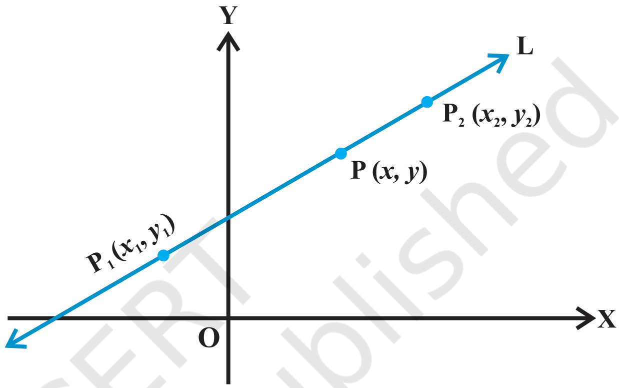


Fig 9.11

The three points P_1, P_2 and P are collinear, therefore, we have

slope of P_1P = slope of P_1P_2

i.e., $\dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$, or $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.

Thus, equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \qquad \text{... (2)}$$

Example 6 Write the equation of the line through the points $(1, -1)$ and $(3, 5)$.

Solution Here $x_1 = 1$, $y_1 = -1$, $x_2 = 3$ and $y_2 = 5$. Using two-point form (2) above for the equation of the line, we have

$$y - (-1) = \frac{5 - (-1)}{3 - 1}(x - 1)$$

or $\qquad -3x + y + 4 = 0$, which is the required equation.

9.3.4 ***Slope-intercept form*** Sometimes a line is known to us with its slope and an intercept on one of the axes. We will now find equations of such lines.

Case I Suppose a line L with slope m cuts the y-axis at a distance c from the origin (Fig 9.12). The distance c is called the *y-intercept* of the line L. Obviously, coordinates of the point where the line meet the y-axis are $(0, c)$. Thus, L has slope m and passes through a fixed point $(0, c)$. Therefore, by point-slope form, the equation of L is

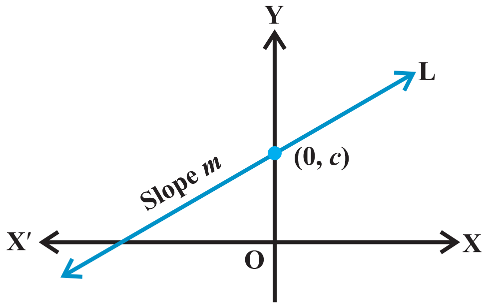


Fig 9.12

$$y - c = m(x - 0) \quad \text{or} \quad y = mx + c$$

Thus, the point (x, y) on the line with slope m and y-intercept c lies on the line if and only if

$$y = mx + c \qquad \text{...(3)}$$

Note that the value of c will be positive or negative according as the intercept is made on the positive or negative side of the y-axis, respectively.

Case II Suppose line L with slope m makes x-intercept d. Then equation of L is

$$y = m(x - d) \qquad \text{... (4)}$$

Students may derive this equation themselves by the same method as in Case I.

Example 7 Write the equation of the lines for which $\tan\theta = \frac{1}{2}$, where θ is the inclination of the line and (i) y-intercept is $-\frac{3}{2}$ (ii) x-intercept is 4.

Solution (i) Here, slope of the line is $m = \tan\theta = \frac{1}{2}$ and y - intercept $c = -\frac{3}{2}$.

Therefore, by slope-intercept form (3) above, the equation of the line is

$$y = \frac{1}{2}x - \frac{3}{2} \text{ or } 2y - x + 3 = 0,$$

which is the required equation.

(ii) Here, we have $m = \tan\theta = \frac{1}{2}$ and $d = 4$.

Therefore, by slope-intercept form (4) above, the equation of the line is

$$y = \frac{1}{2}(x - 4) \text{ or } 2y - x + 4 = 0,$$

which is the required equation.

9.3.5 *Intercept - form* Suppose a line L makes x-intercept a and y-intercept b on the axes. Obviously L meets x-axis at the point $(a, 0)$ and y-axis at the point $(0, b)$ (Fig .9.13). By two-point form of the equation of the line, we have

$$y - 0 = \frac{b-0}{0-a}(x-a) \quad \text{or} \quad ay = -bx + ab\,,$$

i.e.,
$$\frac{x}{a} + \frac{y}{b} = 1.$$

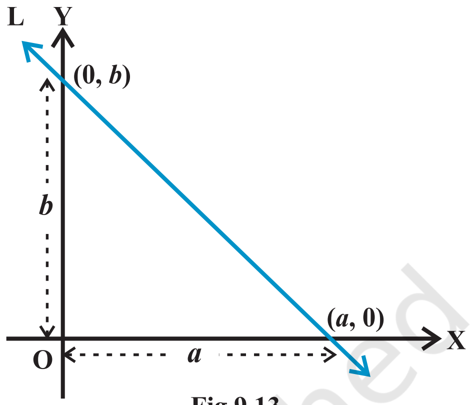


Fig 9.13

Thus, equation of the line making intercepts a and b on x-and y-axis, respectively, is

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad \text{... (5)}$$

Example 8 Find the equation of the line, which makes intercepts –3 and 2 on the x- and y-axes respectively.

Solution Here $a = -3$ and $b = 2$. By intercept form (5) above, equation of the line is

$$\frac{x}{-3} + \frac{y}{2} = 1 \quad \text{or} \quad 2x - 3y + 6 = 0.$$

Any equation of the form Ax + By + C = 0, where A and B are not zero simultaneously is called *general linear equation* or *general equation of a line*.

EXERCISE 9.2

In Exercises 1 to 8, find the equation of the line which satisfy the given conditions:

1. Write the equations for the x-and y-axes.
2. Passing through the point (– 4, 3) with slope $\frac{1}{2}$.
3. Passing through (0, 0) with slope m.
4. Passing through $(2, 2\sqrt{3})$ and inclined with the x-axis at an angle of 75°.
5. Intersecting the x-axis at a distance of 3 units to the left of origin with slope –2.
6. Intersecting the y-axis at a distance of 2 units above the origin and making an angle of 30° with positive direction of the x-axis.

7. Passing through the points (–1, 1) and (2, – 4).
8. The vertices of Δ PQR are P (2, 1), Q (–2, 3) and R (4, 5). Find equation of the median through the vertex R.
9. Find the equation of the line passing through (–3, 5) and perpendicular to the line through the points (2, 5) and (–3, 6).
10. A line perpendicular to the line segment joining the points (1, 0) and (2, 3) divides it in the ratio 1: *n*. Find the equation of the line.
11. Find the equation of a line that cuts off equal intercepts on the coordinate axes and passes through the point (2, 3).
12. Find equation of the line passing through the point (2, 2) and cutting off intercepts on the axes whose sum is 9.
13. Find equation of the line through the point (0, 2) making an angle $\frac{2\pi}{3}$ with the positive *x*-axis. Also, find the equation of line parallel to it and crossing the *y*-axis at a distance of 2 units below the origin.
14. The perpendicular from the origin to a line meets it at the point (–2, 9), find the equation of the line.
15. The length L (in centimetre) of a copper rod is a linear function of its Celsius temperature C. In an experiment, if L = 124.942 when C = 20 and L= 125.134 when C = 110, express L in terms of C.
16. The owner of a milk store finds that, he can sell 980 litres of milk each week at Rs 14/litre and 1220 litres of milk each week at Rs 16/litre. Assuming a linear relationship between selling price and demand, how many litres could he sell weekly at Rs 17/litre?
17. P (*a*, *b*) is the mid-point of a line segment between axes. Show that equation of the line is $\frac{x}{a}+\frac{y}{b}=2$.
18. Point R (*h*, *k*) divides a line segment between the axes in the ratio 1: 2. Find equation of the line.
19. By using the concept of equation of a line, prove that the three points (3, 0), (– 2, – 2) and (8, 2) are collinear.

9.4 Distance of a Point From a Line

The distance of a point from a line is the length of the perpendicular drawn from the point to the line. Let L : A*x* + B*y* + C = 0 be a line, whose distance from the point P (x_1, y_1) is *d*. Draw a perpendicular PM from the point P to the line L (Fig 9.14). If the

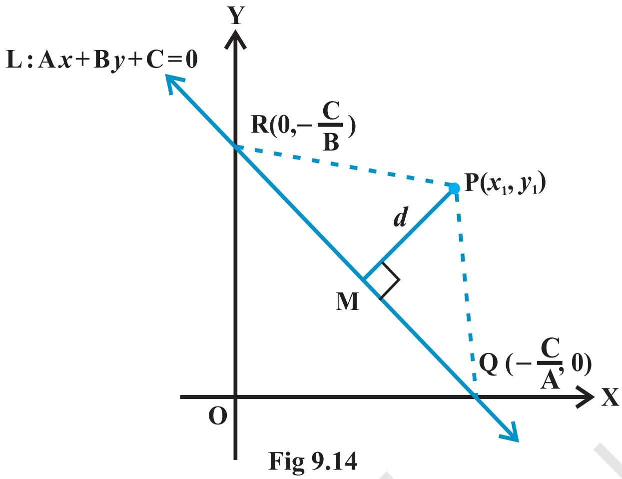


Fig 9.14

line meets the x-and y-axes at the points Q and R, respectively. Then, coordinates of the points are $Q\left(-\frac{C}{A},\ 0\right)$ and $R\left(0,\ -\frac{C}{B}\right)$. Thus, the area of the triangle PQR is given by

$$\text{area }(\Delta PQR) = \frac{1}{2}PM.QR\text{, which gives } PM = \frac{2\ \text{area }(\Delta PQR)}{QR} \qquad \ldots (1)$$

Also, area $(\Delta PQR) = \frac{1}{2}\left| x_1\left(0+\frac{C}{B}\right) + \left(-\frac{C}{A}\right)\left(-\frac{C}{B} - y_1\right) + 0\left(y_1 - 0\right)\right|$

$$= \frac{1}{2}\left| x_1\frac{C}{B} + y_1\frac{C}{A} + \frac{C^2}{AB}\right|$$

or $\quad 2\text{ area }(\Delta PQR) = \left|\frac{C}{AB}\right|.\ \left|Ax_1 + By_1 + C\right|$, and

$$QR = \sqrt{\left(0+\frac{C}{A}\right)^2 + \left(\frac{C}{B} - 0\right)^2} = \left|\frac{C}{AB}\right|\ \sqrt{A^2+B^2}$$

Substituting the values of area (ΔPQR) and QR in (1), we get

$$PM = \frac{\left|Ax_1 + By_1 + C\right|}{\sqrt{A^2+B^2}}$$

or $$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

Thus, the perpendicular distance (d) of a line $Ax + By + C = 0$ from a point (x_1, y_1) is given by

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

9.4.1 ***Distance between two parallel lines*** We know that slopes of two parallel lines are equal. Therefore, two parallel lines can be taken in the form

$$y = mx + c_1 \qquad \text{... (1)}$$

and $$y = mx + c_2 \qquad \text{... (2)}$$

Line (1) will intersect x-axis at the point $A\left(-\frac{c_1}{m},\ 0\right)$ as shown in Fig 9.15.

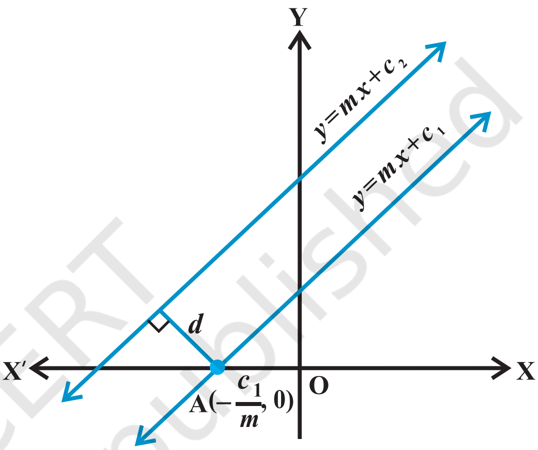


Fig 9.15

Distance between two lines is equal to the length of the perpendicular from point A to line (2). Therefore, distance between the lines (1) and (2) is

$$\frac{\left|(-m)\left(-\frac{c_1}{m}\right) + (-c_2)\right|}{\sqrt{1+m^2}} \quad \text{or} \quad d = \frac{|c_1 - c_2|}{\sqrt{1+m^2}}.$$

Thus, the distance d between two parallel lines $y = mx + c_1$ and $y = mx + c_2$ is given by

$$d = \frac{|c_1 - c_2|}{\sqrt{1+m^2}}.$$

If lines are given in general form, i.e., $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$,

then above formula will take the form $$d = \frac{|C_1 - C_2|}{\sqrt{A^2 + B^2}}$$

Students can derive it themselves.

Example 9 Find the distance of the point (3, – 5) from the line $3x - 4y - 26 = 0$.

Solution Given line is $\quad 3x - 4y - 26 = 0 \qquad \ldots (1)$

Comparing (1) with general equation of line $Ax + By + C = 0$, we get

$$A = 3,\ B = -4 \text{ and } C = -26.$$

Given point is $(x_1, y_1) = (3, -5)$. The distance of the given point from given line is

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \frac{|3.3 + (-4)(-5) - 26|}{\sqrt{3^2 + (-4)^2}} = \frac{3}{5}.$$

Example 10 Find the distance between the parallel lines $3x - 4y + 7 = 0$ and

$$3x - 4y + 5 = 0$$

Solution Here $A = 3$, $B = -4$, $C_1 = 7$ and $C_2 = 5$. Therefore, the required distance is

$$d = \frac{|7 - 5|}{\sqrt{3^2 + (-4)^2}} = \frac{2}{5}.$$

EXERCISE 9.3

1. Reduce the following equations into slope - intercept form and find their slopes and the y - intercepts.
   (i) $x + 7y = 0$, (ii) $6x + 3y - 5 = 0$, (iii) $y = 0$.
2. Reduce the following equations into intercept form and find their intercepts on the axes.
   (i) $3x + 2y - 12 = 0$, (ii) $4x - 3y = 6$, (iii) $3y + 2 = 0$.
3. Find the distance of the point (–1, 1) from the line $12(x + 6) = 5(y - 2)$.
4. Find the points on the x-axis, whose distances from the line $\frac{x}{3} + \frac{y}{4} = 1$ are 4 units.
5. Find the distance between parallel lines
   (i) $15x + 8y - 34 = 0$ and $15x + 8y + 31 = 0$ (ii) $l(x + y) + p = 0$ and $l(x + y) - r = 0$.
6. Find equation of the line parallel to the line $3x - 4y + 2 = 0$ and passing through the point (–2, 3).
7. Find equation of the line perpendicular to the line $x - 7y + 5 = 0$ and having x intercept 3.
8. Find angles between the lines $\sqrt{3}x + y = 1$ and $x + \sqrt{3}y = 1$.
9. The line through the points $(h, 3)$ and $(4, 1)$ intersects the line $7x - 9y - 19 = 0$. at right angle. Find the value of h.

10. Prove that the line through the point (x_1, y_1) and parallel to the line $Ax + By + C = 0$ is

$$A(x - x_1) + B(y - y_1) = 0.$$

11. Two lines passing through the point (2, 3) intersects each other at an angle of 60^o. If slope of one line is 2, find equation of the other line.

12. Find the equation of the right bisector of the line segment joining the points (3, 4) and (–1, 2).

13. Find the coordinates of the foot of perpendicular from the point (–1, 3) to the line $3x - 4y - 16 = 0$.

14. The perpendicular from the origin to the line $y = mx + c$ meets it at the point (–1, 2). Find the values of m and c.

15. If p and q are the lengths of perpendiculars from the origin to the lines $x\cos\theta - y\sin\theta = k\cos 2\theta$ and $x \sec\theta + y \operatorname{cosec}\theta = k$, respectively, prove that $p^2 + 4q^2 = k^2$.

16. In the triangle ABC with vertices A (2, 3), B (4, –1) and C (1, 2), find the equation and length of altitude from the vertex A.

17. If p is the length of perpendicular from the origin to the line whose intercepts on the axes are a and b, then show that $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

Miscellaneous Examples

Example 11 If the lines $2x + y - 3 = 0$, $5x + ky - 3 = 0$ and $3x - y - 2 = 0$ are concurrent, find the value of k.

Solution Three lines are said to be concurrent, if they pass through a common point, i.e., point of intersection of any two lines lies on the third line. Here given lines are

$$2x + y - 3 = 0 \quad \text{... (1)}$$

$$5x + ky - 3 = 0 \quad \text{... (2)}$$

$$3x - y - 2 = 0 \quad \text{... (3)}$$

Solving (1) and (3) by cross-multiplication method, we get

$$\frac{x}{-2-3} = \frac{y}{-9+4} = \frac{1}{-2-3} \quad \text{or} \quad x = 1,\ y = 1.$$

Therefore, the point of intersection of two lines is (1, 1). Since above three lines are concurrent, the point (1, 1) will satisfy equation (2) so that

$$5.1 + k.1 - 3 = 0 \text{ or } k = -2.$$

Example 12 Find the distance of the line $4x - y = 0$ from the point P (4, 1) measured along the line making an angle of 135° with the positive x-axis.

Solution Given line is $4x - y = 0$ ... (1)

In order to find the distance of the line (1) from the point P (4, 1) along another line, we have to find the point of intersection of both the lines. For this purpose, we will first find the equation of the second line (Fig 9.16). Slope of second line is tan 135° = –1. Equation of the line with slope – 1 through the point P (4, 1) is

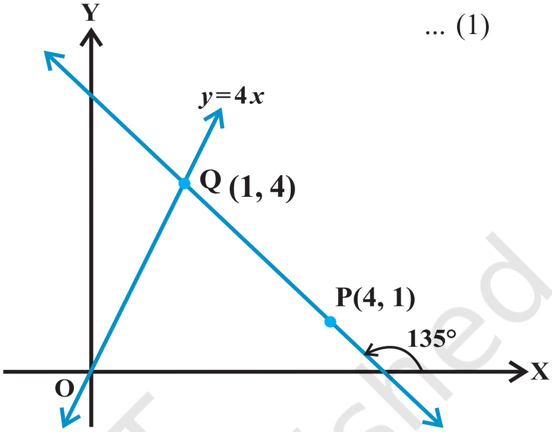


Fig 9.16

$$y - 1 = - 1 (x - 4) \text{ or } x + y - 5 = 0 \quad \text{... (2)}$$

Solving (1) and (2), we get $x = 1$ and $y = 4$ so that point of intersection of the two lines is Q (1, 4). Now, distance of line (1) from the point P (4, 1) along the line (2)

= the distance between the points P (4, 1) and Q (1, 4).

$$= \sqrt{(1-4)^2 + (4-1)^2} = 3\sqrt{2} \text{ units.}$$

Example 13 Assuming that straight lines work as the plane mirror for a point, find the image of the point (1, 2) in the line $x - 3y + 4 = 0$.

Solution Let Q (h, k) is the image of the point P (1, 2) in the line

$$x - 3y + 4 = 0 \quad \text{... (1)}$$

Therefore, the line (1) is the perpendicular bisector of line segment PQ (Fig 9.17).

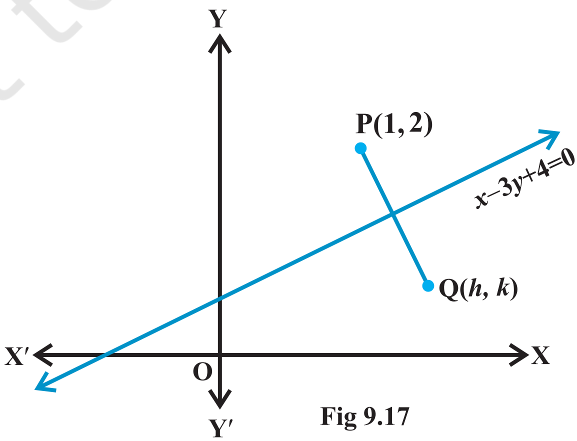


Fig 9.17

Hence Slope of line PQ $= \dfrac{-1}{\text{Slope of line } x-3y+4=0}$,

so that $\dfrac{k-2}{h-1} = \dfrac{-1}{\frac{1}{3}}$ or $3h+k=5$ ... (2)

and the mid-point of PQ, i.e., point $\left(\dfrac{h+1}{2}, \dfrac{k+2}{2}\right)$ will satisfy the equation (1) so that

$$\frac{h+1}{2} - 3\left(\frac{k+2}{2}\right) + 4 = 0 \text{ or } h - 3k = -3 \qquad \text{... (3)}$$

Solving (2) and (3), we get $h = \dfrac{6}{5}$ and $k = \dfrac{7}{5}$.

Hence, the image of the point (1, 2) in the line (1) is $\left(\dfrac{6}{5}, \dfrac{7}{5}\right)$.

Example 14 Show that the area of the triangle formed by the lines

$y = m_1x + c_1$, $y = m_2x + c_2$ and $x = 0$ is $\dfrac{(c_1 - c_2)^2}{2|m_1 - m_2|}$.

Solution Given lines are

$y = m_1 x + c_1$ ... (1)

$y = m_2 x + c_2$ ... (2)

$x = 0$ ... (3)

We know that line $y = mx + c$ meets the line $x = 0$ (y-axis) at the point $(0, c)$. Therefore, two vertices of the triangle formed by lines (1) to (3) are P $(0, c_1)$ and Q $(0, c_2)$ (Fig 9.18). Third vertex can be obtained by solving equations (1) and (2). Solving (1) and (2), we get

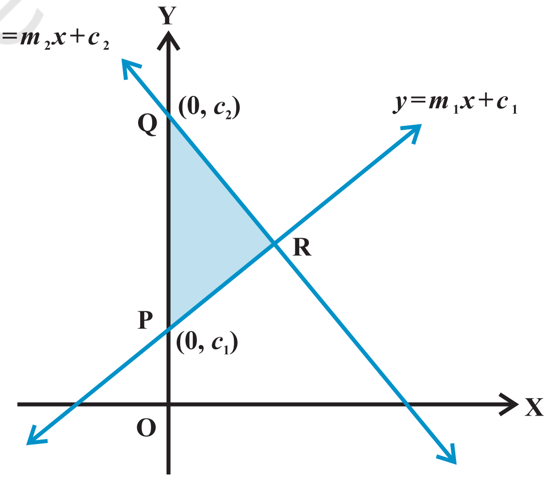


Fig 9.18

$$x = \frac{(c_2 - c_1)}{(m_1 - m_2)} \text{ and } y = \frac{(m_1 c_2 - m_2 c_1)}{(m_1 - m_2)}$$

Therefore, third vertex of the triangle is R $\left(\frac{(c_2 - c_1)}{(m_1 - m_2)}, \frac{(m_1 c_2 - m_2 c_1)}{(m_1 - m_2)}\right)$.

Now, the area of the triangle is

$$= \frac{1}{2}\left| 0 \left(\frac{m_1 c_2 - m_2 c_1}{m_1 - m_2} - c_2\right) + \frac{c_2 - c_1}{m_1 - m_2}(c_2 - c_1) + 0 \left(c_1 - \frac{m_1 c_2 - m_2 c_1}{m_1 - m_2}\right)\right| = \frac{(c_2 - c_1)^2}{2|m_1 - m_2|}$$

Example 15 A line is such that its segment between the lines $5x - y + 4 = 0$ and $3x + 4y - 4 = 0$ is bisected at the point (1, 5). Obtain its equation.

Solution Given lines are

$5x - y + 4 = 0$ ... (1)

$3x + 4y - 4 = 0$ ... (2)

Let the required line intersects the lines (1) and (2) at the points, (α_1, β_1) and (α_2, β_2), respectively (Fig 9.19). Therefore

$5\alpha_1 - \beta_1 + 4 = 0$ and

$3\alpha_2 + 4\beta_2 - 4 = 0$

or $\beta_1 = 5\alpha_1 + 4$ and $\beta_2 = \dfrac{4 - 3\alpha_2}{4}$.

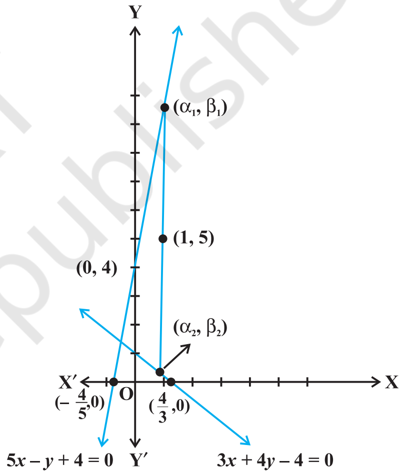


Fig 9.19

We are given that the mid point of the segment of the required line between (α_1, β_1) and (α_2, β_2) is (1, 5). Therefore

$$\frac{\alpha_1 + \alpha_2}{2} = 1 \text{ and } \frac{\beta_1 + \beta_2}{2} = 5,$$

or
$$\alpha_1 + \alpha_2 = 2 \text{ and } \frac{5\alpha_1 + 4 + \dfrac{4 - 3\alpha_2}{4}}{2} = 5,$$

or $\alpha_1 + \alpha_2 = 2$ and $20\alpha_1 - 3\alpha_2 = 20$ ... (3)

Solving equations in (3) for α_1 and α_2, we get

$$\alpha_1 = \frac{26}{23} \text{ and } \alpha_2 = \frac{20}{23} \text{ and hence, } \beta_1 = 5.\frac{26}{23} + 4 = \frac{222}{23}.$$

Equation of the required line passing through (1, 5) and (α_1, β_1) is

$$y - 5 = \frac{\beta_1 - 5}{\alpha_1 - 1}(x - 1) \text{ or } y - 5 = \frac{\frac{222}{23} - 5}{\frac{26}{23} - 1}(x - 1)$$

or $\qquad 107x - 3y - 92 = 0,$

which is the equation of required line.

Example 16 Show that the path of a moving point such that its distances from two lines $3x - 2y = 5$ and $3x + 2y = 5$ are equal is a straight line.

Solution Given lines are

$$3x - 2y = 5 \qquad \dots (1)$$

and

$$3x + 2y = 5 \qquad \dots (2)$$

Let (h, k) is any point, whose distances from the lines (1) and (2) are equal. Therefore

$$\frac{|3h - 2k - 5|}{\sqrt{9+4}} = \frac{|3h + 2k - 5|}{\sqrt{9+4}} \text{ or } |3h - 2k - 5| = |3h + 2k - 5|,$$

which gives $3h - 2k - 5 = 3h + 2k - 5$ or $-(3h - 2k - 5) = 3h + 2k - 5$.

Solving these two relations we get $k = 0$ or $h = \frac{5}{3}$. Thus, the point (h, k) satisfies the equations $y = 0$ or $x = \frac{5}{3}$, which represent straight lines. Hence, path of the point equidistant from the lines (1) and (2) is a straight line.

Miscellaneous Exercise on Chapter 9

1. Find the values of k for which the line $(k-3)\, x - (4 - k^2)\, y + k^2 - 7k + 6 = 0$ is
   (a) Parallel to the x-axis,
   (b) Parallel to the y-axis,
   (c) Passing through the origin.

2. Find the equations of the lines, which cut-off intercepts on the axes whose sum and product are 1 and – 6, respectively.

3. What are the points on the y-axis whose distance from the line $\frac{x}{3}+\frac{y}{4}=1$ is 4 units.

4. Find perpendicular distance from the origin to the line joining the points $(\cos\theta, \sin\theta)$ and $(\cos\phi, \sin\phi)$.

5. Find the equation of the line parallel to y-axis and drawn through the point of intersection of the lines $x - 7y + 5 = 0$ and $3x + y = 0$.

6. Find the equation of a line drawn perpendicular to the line $\frac{x}{4}+\frac{y}{6}=1$ through the point, where it meets the y-axis.

7. Find the area of the triangle formed by the lines $y - x = 0$, $x + y = 0$ and $x - k = 0$.

8. Find the value of p so that the three lines $3x + y - 2 = 0$, $px + 2y - 3 = 0$ and $2x - y - 3 = 0$ may intersect at one point.

9. If three lines whose equations are $y = m_1x + c_1$, $y = m_2x + c_2$ and $y = m_3x + c_3$ are concurrent, then show that $m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0$.

10. Find the equation of the lines through the point (3, 2) which make an angle of 45° with the line $x - 2y = 3$.

11. Find the equation of the line passing through the point of intersection of the lines $4x + 7y - 3 = 0$ and $2x - 3y + 1 = 0$ that has equal intercepts on the axes.

12. Show that the equation of the line passing through the origin and making an angle θ with the line $y = mx + c$ is $\frac{y}{x} = \frac{m \pm \tan ,}{1 \mp m \tan ,}$.

13. In what ratio, the line joining (–1, 1) and (5, 7) is divided by the line $x + y = 4$?

14. Find the distance of the line $4x + 7y + 5 = 0$ from the point (1, 2) along the line $2x - y = 0$.

15. Find the direction in which a straight line must be drawn through the point (–1, 2) so that its point of intersection with the line $x + y = 4$ may be at a distance of 3 units from this point.

16. The hypotenuse of a right angled triangle has its ends at the points (1, 3) and (– 4, 1). Find an equation of the legs (perpendicular sides) of the triangle which are parallel to the axes.

17. Find the image of the point (3, 8) with respect to the line $x + 3y = 7$ assuming the line to be a plane mirror.

18. If the lines $y = 3x + 1$ and $2y = x + 3$ are equally inclined to the line $y = mx + 4$, find the value of m.

19. If sum of the perpendicular distances of a variable point P (x, y) from the lines $x + y - 5 = 0$ and $3x - 2y + 7 = 0$ is always 10. Show that P must move on a line.

20. Find equation of the line which is equidistant from parallel lines $9x + 6y - 7 = 0$ and $3x + 2y + 6 = 0$.

21. A ray of light passing through the point (1, 2) reflects on the x-axis at point A and the reflected ray passes through the point (5, 3). Find the coordinates of A.

22. Prove that the product of the lengths of the perpendiculars drawn from the points $\left(\sqrt{a^2-b^2},0\right)$ and $\left(-\sqrt{a^2-b^2},0\right)$ to the line $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$ is b^2.

23. A person standing at the junction (crossing) of two straight paths represented by the equations $2x - 3y + 4 = 0$ and $3x + 4y - 5 = 0$ wants to reach the path whose equation is $6x - 7y + 8 = 0$ in the least time. Find equation of the path that he should follow.

Summary

- *Slope* (m) of a non-vertical line passing through the points (x_1, y_1) and (x_2, y_2) is given by $m=\frac{y_2-y_1}{x_2-x_1}=\frac{y_1-y_2}{x_1-x_2}$, $x_1 \neq x_2$.
- If a line makes an angle á with the positive direction of x-axis, then the slope of the line is given by $m = \tan\alpha$, $\alpha \neq 90°$.
- Slope of horizontal line is zero and slope of vertical line is undefined.
- An acute angle (say θ) between lines L_1 and L_2 with slopes m_1 and m_2 is given by $\tan\theta=\left|\frac{m_2-m_1}{1+m_1 m_2}\right|, 1+m_1 m_2 \neq 0$.
- Two lines are *parallel* if and only if their slopes are equal.
- Two lines are *perpendicular* if and only if product of their slopes is –1.
- Three points A, B and C are collinear, if and only if slope of AB = slope of BC.
- Equation of the horizontal line having distance a from the x-axis is either $y = a$ or $y = -a$.
- Equation of the vertical line having distance b from the y-axis is either $x = b$ or $x = -b$.
- The point (x, y) lies on the line with slope m and through the fixed point (x_o, y_o), if and only if its coordinates satisfy the equation $y - y_o = m(x - x_o)$.
- Equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1).$$

- The point (x, y) on the line with slope m and y-intercept c lies on the line if and only if $y = mx + c$.
- If a line with slope m makes x-intercept d. Then equation of the line is $y = m(x - d)$.
- Equation of a line making intercepts a and b on the x-and y-axis, respectively, is $\frac{x}{a} + \frac{y}{b} = 1$.
- Any equation of the form Ax + By + C = 0, with A and B are not zero, simultaneously, is called the *general linear equation* or *general equation of a line*.
- The perpendicular distance (d) of a line Ax + By+ C = 0 from a point (x_1, y_1) is given by $d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$.
- Distance between the parallel lines $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$, is given by $d = \frac{|C_1 - C_2|}{\sqrt{A^2 + B^2}}$.

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Chapter 10

CONIC SECTIONS

❖*Let the relation of knowledge to real life be very visible to your pupils and let them understand how by knowledge the world could be transformed. – BERTRAND RUSSELL* ❖

10.1 Introduction

Apollonius
(262 B.C. -190 B.C.)

In the preceding Chapter 10, we have studied various forms of the equations of a line. In this Chapter, we shall study about some other curves, viz., circles, ellipses, parabolas and hyperbolas. The names parabola and hyperbola are given by Apollonius. These curves are in fact, known as *conic sections* or more commonly *conics* because they can be obtained as intersections of a plane with a double napped right circular cone. These curves have a very wide range of applications in fields such as planetary motion, design of telescopes and antennas, reflectors in flashlights and automobile headlights, etc. Now, in the subsequent sections we will see how the intersection of a plane with a double napped right circular cone results in different types of curves.

10.2 Sections of a Cone

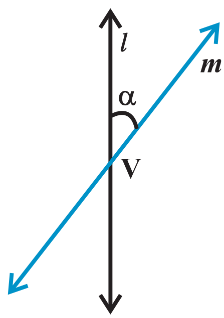


Fig 10. 1

Let l be a fixed vertical line and m be another line intersecting it at a fixed point V and inclined to it at an angle α (Fig10.1).

Suppose we rotate the line m around the line l in such a way that the angle α remains constant. Then the surface generated is a double-napped right circular hollow cone herein after referred as

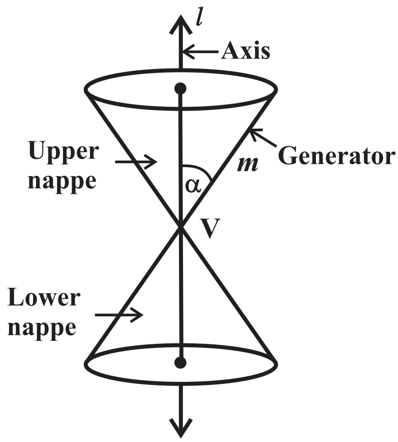


Fig 10. 2

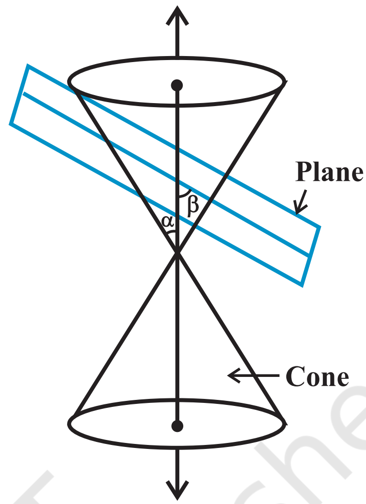


Fig 10. 3

cone and extending indefinitely far in both directions (Fig10.2).

The point V is called the *vertex*; the line l is the *axis* of the cone. The rotating line m is called a *generator* of the cone. The *vertex* separates the cone into two parts called *nappes*.

If we take the intersection of a plane with a cone, the section so obtained is called a *conic section*. Thus, conic sections are the curves obtained by intersecting a right circular cone by a plane.

We obtain different kinds of conic sections depending on the position of the intersecting plane with respect to the cone and by the angle made by it with the vertical axis of the cone. Let β be the angle made by the intersecting plane with the vertical axis of the cone (Fig10.3).

The intersection of the plane with the cone can take place either at the vertex of the cone or at any other part of the nappe either below or above the vertex.

10.2.1 ***Circle, ellipse, parabola and hyperbola*** When the plane cuts the nappe (other than the vertex) of the cone, we have the following situations:

(a) When $\beta = 90^o$, the section is a *circle* (Fig10.4).

(b) When $\alpha < \beta < 90^o$, the section is an *ellipse* (Fig10.5).

(c) When $\beta = \alpha$; the section is a *parabola* (Fig10.6).

(In each of the above three situations, the plane cuts entirely across one nappe of the cone).

(d) When $0 \leq \beta < \alpha$; the plane cuts through both the nappes and the curves of intersection is a *hyperbola* (Fig10.7).

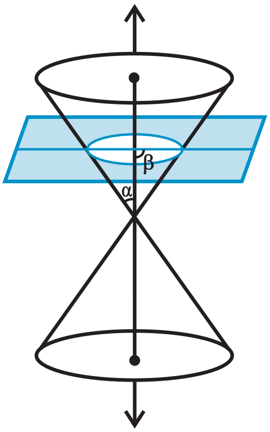


Fig 10. 4

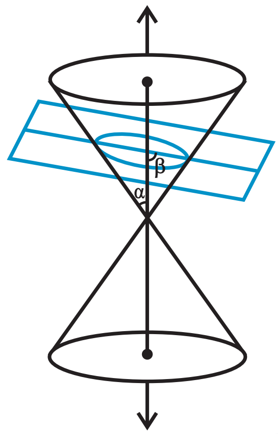


Fig 10. 5

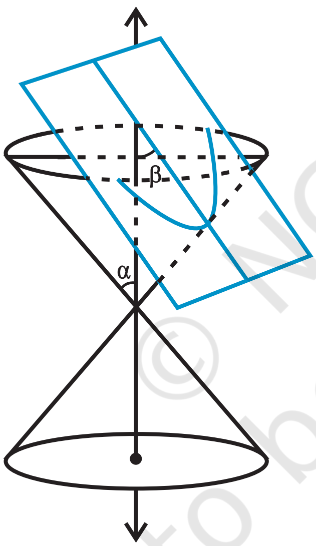


Fig 10. 6

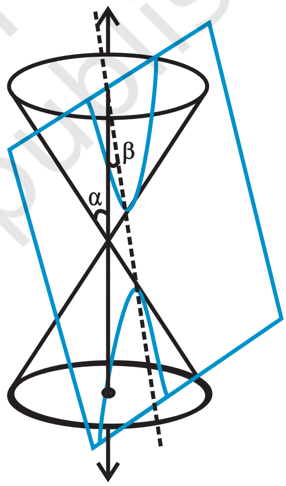


Fig 10. 7

10.2.2 *Degenerated conic sections*

When the plane cuts at the vertex of the cone, we have the following different cases:

(a) When $\alpha < \beta \leq 90^{\circ}$, then the section is a point (Fig10.8).

(b) When $\beta = \alpha$, the plane contains a generator of the cone and the section is a straight line (Fig10.9).

It is the degenerated case of a parabola.

(c) When $0 \leq \beta < \alpha$, the section is a pair of intersecting straight lines (Fig10.10). It is the degenerated case of a *hyperbola*.

In the following sections, we shall obtain the equations of each of these conic sections in standard form by defining them based on geometric properties.

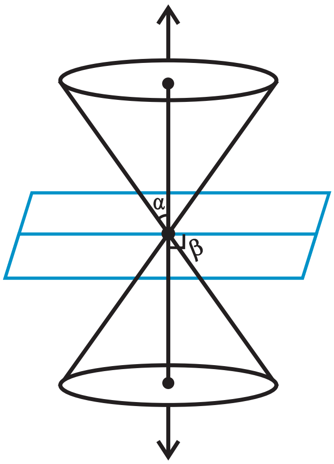


Fig 10. 8

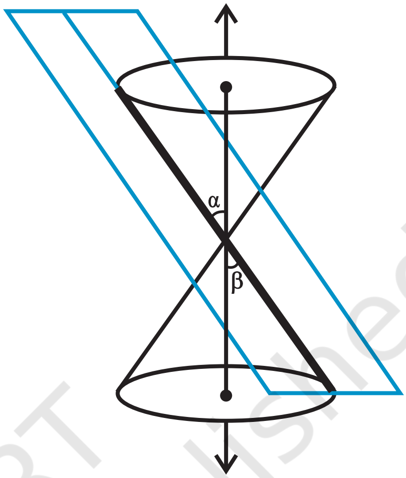
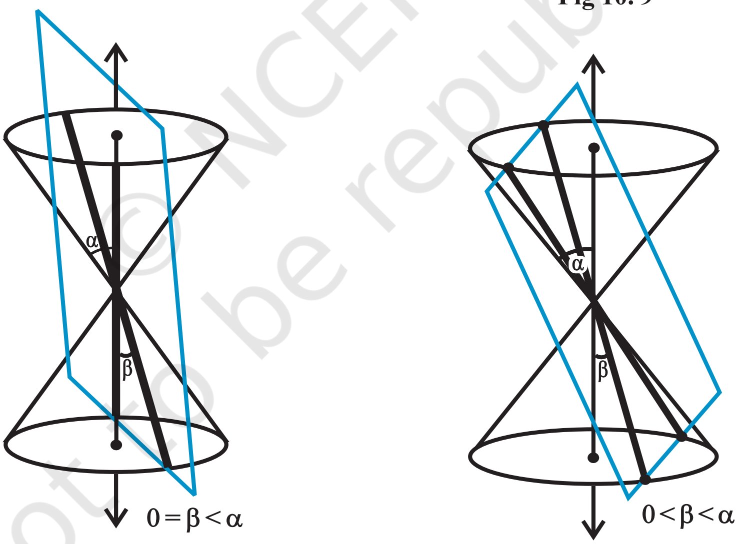


Fig 10. 9



(a) **Fig 10. 10** **(b)**

10.3 Circle

Definition 1 A circle is the set of all points in a plane that are equidistant from a fixed point in the plane.

The fixed point is called the *centre of the circle* and the distance from the centre to a point on the circle is called the *radius* of the circle (Fig 10.11).

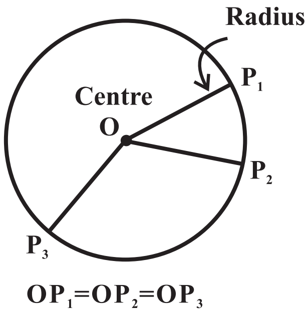


Fig 10. 11

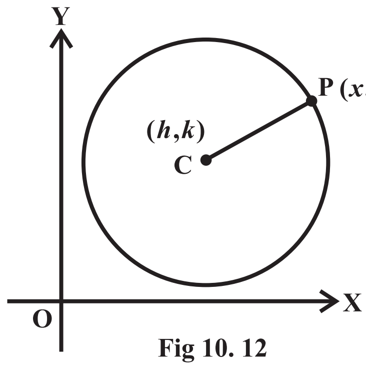


Fig 10. 12

The equation of the circle is simplest if the centre of the circle is at the origin. However, we derive below the equation of the circle with a given centre and radius (Fig 10.12).

Given C (h, k) be the centre and r the radius of circle. Let P(x, y) be any point on the circle (Fig10.12). Then, by the definition, $| \text{CP} | = r$. By the distance formula, we have

$$\sqrt{(x-h)^2+(y-k)^2}=r$$

i.e. $$(x-h)^2+(y-k)^2=r^2$$

This is the required equation of the circle with centre at (h,k) and radius r .

Example 1 Find an equation of the circle with centre at (0,0) and radius r.

Solution Here $h = k = 0$. Therefore, the equation of the circle is $x^2 + y^2 = r^2$.

Example 2 Find the equation of the circle with centre (–3, 2) and radius 4.

Solution Here $h = –3$, $k = 2$ and $r = 4$. Therefore, the equation of the required circle is

$$(x + 3)^2 + (y – 2)^2 = 16$$

Example 3 Find the centre and the radius of the circle $x^2 + y^2 + 8x + 10y – 8 = 0$

Solution The given equation is

$$(x^2 + 8x) + (y^2 + 10y) = 8$$

Now, completing the squares within the parenthesis, we get

$$(x^2 + 8x + 16) + (y^2 + 10y + 25) = 8 + 16 + 25$$

i.e. $$(x + 4)^2 + (y + 5)^2 = 49$$

i.e. $$\{x – (– 4)\}^2 + \{y – (–5)\}^2 = 7^2$$

Therefore, the given circle has centre at (– 4, –5) and radius 7.

Example 4 Find the equation of the circle which passes through the points (2, – 2), and (3,4) and whose centre lies on the line $x + y = 2$.

Solution Let the equation of the circle be $(x - h)^2 + (y - k)^2 = r^2$.

Since the circle passes through (2, – 2) and (3,4), we have

$$(2 - h)^2 + (-2 - k)^2 = r^2 \quad \text{... (1)}$$

and $$(3 - h)^2 + (4 - k)^2 = r^2 \quad \text{... (2)}$$

Also since the centre lies on the line $x + y = 2$, we have

$$h + k = 2 \quad \text{... (3)}$$

Solving the equations (1), (2) and (3), we get

$$h = 0.7, \quad k = 1.3 \quad \text{and } r^2 = 12.58$$

Hence, the equation of the required circle is

$$(x - 0.7)^2 + (y - 1.3)^2 = 12.58.$$

EXERCISE 10.1

In each of the following Exercises 1 to 5, find the equation of the circle with

1. centre (0,2) and radius 2
2. centre (–2,3) and radius 4
3. centre $\left(\frac{1}{2}, \frac{1}{4}\right)$ and radius $\frac{1}{12}$
4. centre (1,1) and radius $\sqrt{2}$
5. centre $(-a, -b)$ and radius $\sqrt{a^2 - b^2}$.

In each of the following Exercises 6 to 9, find the centre and radius of the circles.

6. $(x + 5)^2 + (y - 3)^2 = 36$
7. $x^2 + y^2 - 4x - 8y - 45 = 0$
8. $x^2 + y^2 - 8x + 10y - 12 = 0$
9. $2x^2 + 2y^2 - x = 0$
10. Find the equation of the circle passing through the points (4,1) and (6,5) and whose centre is on the line $4x + y = 16$.
11. Find the equation of the circle passing through the points (2,3) and (–1,1) and whose centre is on the line $x - 3y - 11 = 0$.
12. Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point (2,3).
13. Find the equation of the circle passing through (0,0) and making intercepts a and b on the coordinate axes.
14. Find the equation of a circle with centre (2,2) and passes through the point (4,5).
15. Does the point (–2.5, 3.5) lie inside, outside or on the circle $x^2 + y^2 = 25$?

10.4 Parabola

Definition 2 A parabola is the set of all points in a plane that are equidistant from a fixed line and a fixed point (not on the line) in the plane.

The fixed line is called the *directrix* of the parabola and the fixed point F is called the *focus* (Fig 10.13). ('Para' means 'for' and 'bola' means 'throwing', i.e., the shape described when you throw a ball in the air).

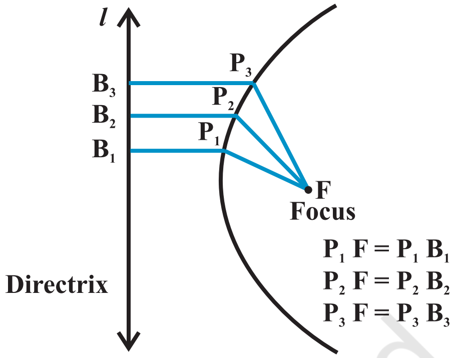


Fig 10. 13

> **Note** If the fixed point lies on the fixed line, then the set of points in the plane, which are equidistant from the fixed point and the fixed line is the straight line through the fixed point and perpendicular to the fixed line. We call this straight line as *degenerate case* of the parabola.

A line through the focus and perpendicular to the *directrix* is called the *axis* of the parabola. The point of intersection of parabola with the axis is called the vertex of the parabola (Fig10.14).

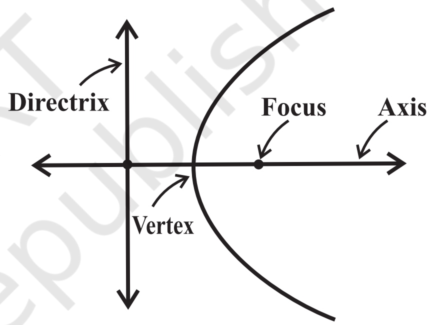
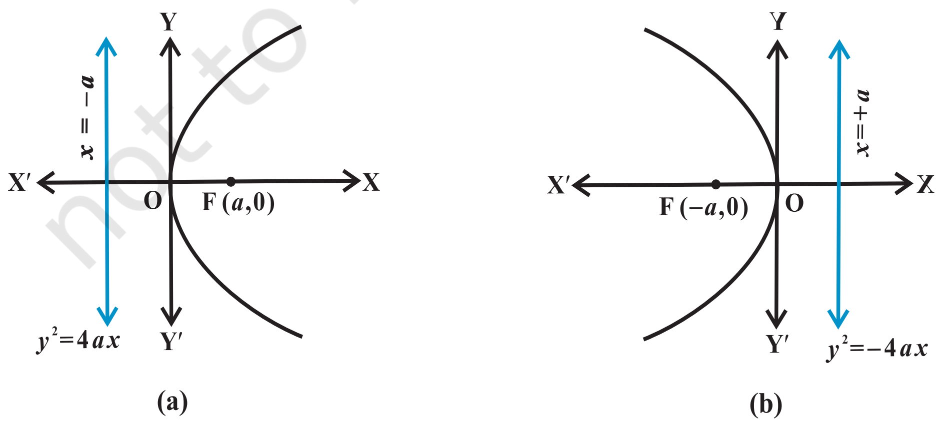
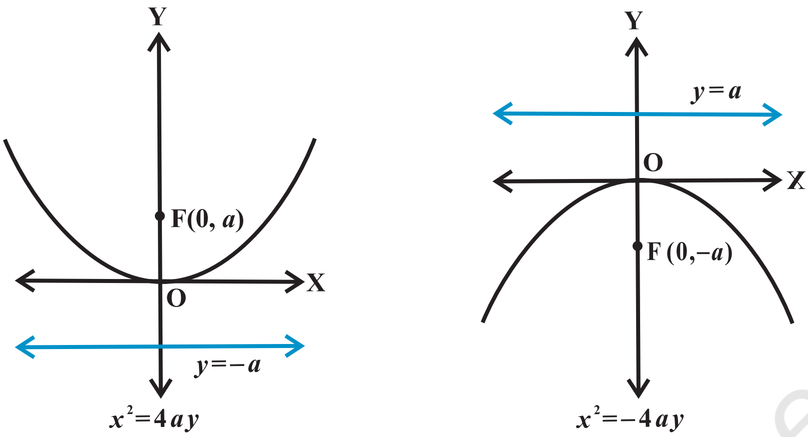


Fig 10.14

10.4.1 *Standard equations of parabola*

The equation of a *parabola* is simplest if the vertex is at the origin and the axis of symmetry is along the x-axis or y-axis. The four possible such orientations of parabola are shown below in Fig10.15 (a) to (d).





(c) **Fig 10.15 (a) to (d)** (d)

We will derive the equation for the parabola shown above in Fig 10.15 (a) with *focus* at (a, 0) $a > 0$; and directricx $x = -a$ as below:

Let F be the *focus* and l the *directrix*. Let FM be perpendicular to the *directrix* and bisect FM at the point O. Produce MO to X. By the definition of parabola, the mid-point O is on the parabola and is called the *vertex* of the parabola. Take O as origin, OX the x-axis and OY perpendicular to it as the y-axis. Let the distance from the directrix to the focus be $2a$. Then, the coordinates of the *focus* are (a, 0), and the equation of the *directrix* is $x + a = 0$ as in Fig10.16. Let P(x, y) be any point on the parabola such that

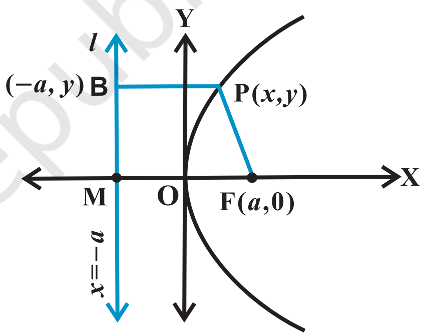


Fig 10.16

PF = PB, ... (1)

where PB is perpendicular to l. The coordinates of B are ($-a$, y). By the distance formula, we have

$$PF = \sqrt{(x-a)^2 + y^2} \text{ and } PB = \sqrt{(x+a)^2}$$

Since PF = PB, we have

$$\sqrt{(x-a)^2 + y^2} = \sqrt{(x+a)^2}$$

i.e. $(x - a)^2 + y^2 = (x + a)^2$

or $x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$

or $y^2 = 4ax$ ($a > 0$).

Hence, any point on the parabola satisfies

$$y^2 = 4ax. \quad \text{... (2)}$$

Conversely, let P(x, y) satisfy the equation (2)

$$\text{PF} = \sqrt{(x-a)^2 + y^2} = \sqrt{(x-a)^2 + 4ax}$$

$$= \sqrt{(x+a)^2} = \text{PB} \quad \text{... (3)}$$

and so P(x,y) lies on the parabola.

Thus, from (2) and (3) we have proved that the equation to the parabola with vertex at the origin, focus at (a,0) and directrix $x = -a$ is $y^2 = 4ax$.

Discussion In equation (2), since $a > 0$, x can assume any positive value or zero but no negative value and the curve extends indefinitely far into the first and the fourth quadrants. The axis of the parabola is the positive x-axis.

Similarly, we can derive the equations of the parabolas in:

Fig 11.15 (b) as $y^2 = -4ax$,
Fig 11.15 (c) as $x^2 = 4ay$,
Fig 11.15 (d) as $x^2 = -4ay$,

These four equations are known as *standard equations* of parabolas.

Note The standard equations of parabolas have focus on one of the coordinate axis; vertex at the *origin* and thereby the directrix is parallel to the other coordinate axis. However, the study of the equations of parabolas with focus at any point and any line as directrix is beyond the scope here.

From the standard equations of the parabolas, Fig10.15, we have the following observations:

1. Parabola is symmetric with respect to the axis of the parabola.If the equation has a y^2 term, then the axis of symmetry is along the x-axis and if the equation has an x^2 term, then the axis of symmetry is along the y-axis.

2. When the axis of symmetry is along the x-axis the parabola opens to the
   (a) right if the coefficient of x is positive,
   (b) left if the coefficient of x is negative.

3. When the axis of symmetry is along the y-axis the parabola opens
   (c) upwards if the coefficient of y is positive.
   (d) downwards if the coefficient of y is negative.

10.4.2 *Latus rectum*

Definition 3 Latus rectum of a parabola is a line segment perpendicular to the axis of the parabola, through the focus and whose end points lie on the parabola (Fig10.17).

To find the Length of the latus rectum of the parabola $y^2 = 4ax$ (Fig 10.18).

By the definition of the parabola, AF = AC.

But $\quad$ AC = FM = $2a$

Hence $\quad$ AF = $2a$.

And since the parabola is symmetric with respect to x-axis AF = FB and so

AB = Length of the latus rectum = $4a$.

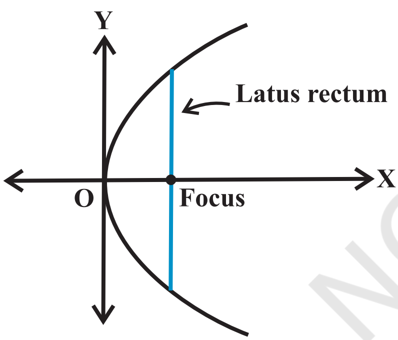


Fig 10.17

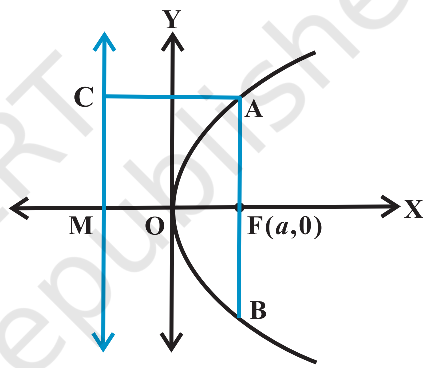


Fig 10.18

Example 5 Find the coordinates of the focus, axis, the equation of the directrix and latus rectum of the parabola $y^2 = 8x$.

Solution The given equation involves y^2, so the axis of symmetry is along the x-axis.

The coefficient of x is positive so the parabola opens to the right. Comparing with the given equation $y^2 = 4ax$, we find that $a = 2$.

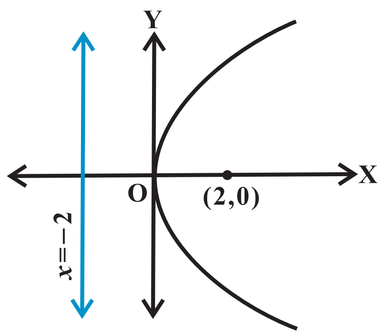


Fig 10.19

Thus, the focus of the parabola is (2, 0) and the equation of the directrix of the parabola is $x = -2$ (Fig 10.19).

Length of the latus rectum is $4a = 4 \times 2 = 8$.

Example 6 Find the equation of the parabola with focus (2,0) and directrix $x = -2$.

Solution Since the focus (2,0) lies on the x-axis, the x-axis itself is the axis of the parabola. Hence the equation of the parabola is of the form either $y^2 = 4ax$ or $y^2 = -4ax$. Since the directrix is $x = -2$ and the focus is (2,0), the parabola is to be of the form $y^2 = 4ax$ with $a = 2$. Hence the required equation is

$$y^2 = 4(2)x = 8x$$

Example 7 Find the equation of the parabola with vertex at (0, 0) and focus at (0, 2).

Solution Since the vertex is at (0,0) and the focus is at (0,2) which lies on y-axis, the y-axis is the axis of the parabola. Therefore, equation of the parabola is of the form $x^2 = 4ay$. thus, we have

$$x^2 = 4(2)y, \text{ i.e., } x^2 = 8y.$$

Example 8 Find the equation of the parabola which is symmetric about the y-axis, and passes through the point (2,–3).

Solution Since the parabola is symmetric about y-axis and has its vertex at the origin, the equation is of the form $x^2 = 4ay$ or $x^2 = -4ay$, where the sign depends on whether the parabola opens upwards or downwards. But the parabola passes through (2,–3) which lies in the fourth quadrant, it must open downwards. Thus the equation is of the form $x^2 = -4ay$.

Since the parabola passes through (2,–3), we have

$$2^2 = -4a(-3), \text{ i.e., } a = \frac{1}{3}$$

Therefore, the equation of the parabola is

$$x^2 = -4\left(\frac{1}{3}\right)y, \text{ i.e., } 3x^2 = -4y.$$

EXERCISE 10.2

In each of the following Exercises 1 to 6, find the coordinates of the focus, axis of the parabola, the equation of the directrix and the length of the latus rectum.

1. $y^2 = 12x$ **2.** $x^2 = 6y$ **3.** $y^2 = -8x$

4. $x^2 = -16y$ **5.** $y^2 = 10x$ **6.** $x^2 = -9y$

In each of the Exercises 7 to 12, find the equation of the parabola that satisfies the given conditions:

7. Focus (6,0); directrix $x = -6$

8. Focus (0,–3); directrix $y = 3$

9. Vertex (0,0); focus (3,0)

10. Vertex (0,0); focus (–2,0)

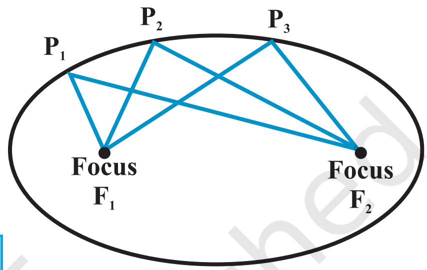
11. Vertex (0,0) passing through (2,3) and axis is along x-axis.

12. Vertex (0,0), passing through (5,2) and symmetric with respect to y-axis.

10.5 Ellipse

Definition 4 An *ellipse* is the set of all points in a plane, the sum of whose distances from two fixed points in the plane is a constant.

The two fixed points are called the *foci* (plural of '*focus*') of the ellipse (Fig10.20).



$P_1F_1 + P_1F_2 = P_2F_1 + P_2F_2 = P_3F_1 + P_3F_2$

Fig 10.20

Note The constant which is the sum of the distances of a point on the ellipse from the two fixed points is always greater than the distance between the two fixed points.

The mid point of the line segment joining the foci is called the *centre* of the ellipse. The line segment through the foci of the ellipse is called the *major axis* and the line segment through the centre and perpendicular to the major axis is called the *minor axis*. The end points of the major axis are called the *vertices* of the ellipse(Fig 10.21).

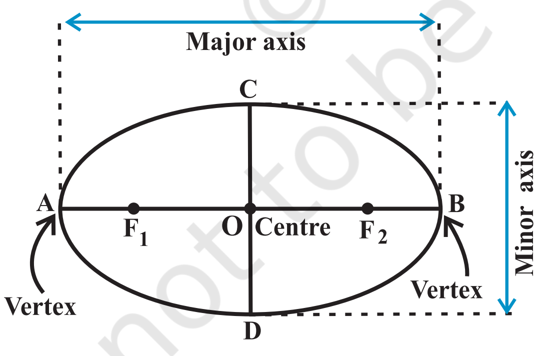


Fig 10.21

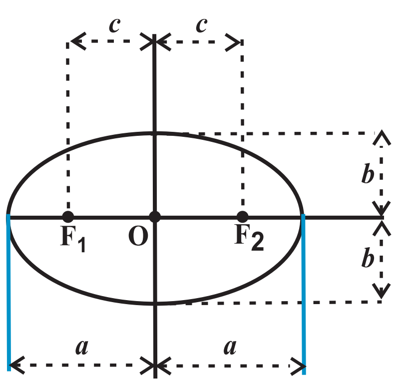


Fig 10.22

We denote the length of the major axis by $2a$, the length of the minor axis by $2b$ and the distance between the foci by $2c$. Thus, the length of the semi major axis is a and semi-minor axis is b (Fig10.22).

10.5.1 *Relationship between semi-major axis, semi-minor axis and the distance of the focus from the centre of the ellipse (Fig 10.23).*

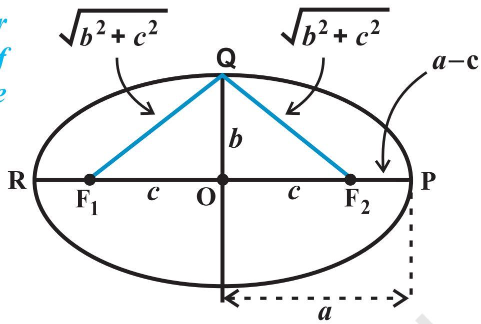


Fig 10.23

Take a point P at one end of the major axis.

Sum of the distances of the point P to the foci is $F_1P + F_2P = F_1O + OP + F_2P$

(Since, $F_1P = F_1O + OP$)

$= c + a + a - c = 2a$

Take a point Q at one end of the minor axis.
Sum of the distances from the point Q to the foci is

$$F_1Q + F_2Q = \sqrt{b^2 + c^2} + \sqrt{b^2 + c^2} = 2\sqrt{b^2 + c^2}$$

Since both P and Q lies on the ellipse.

By the definition of ellipse, we have

$$2\sqrt{b^2 + c^2} = 2a, \text{ i.e.,} \qquad a = \sqrt{b^2 + c^2}$$

or $$a^2 = b^2 + c^2, \text{ i.e., } c = \sqrt{a^2 - b^2}.$$

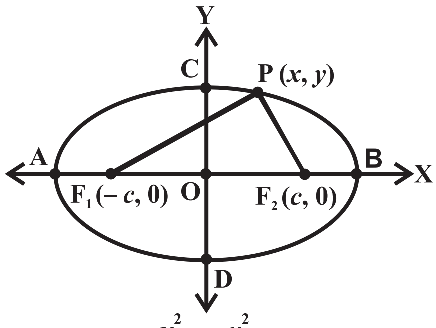
10.5.2 *Eccentricity*

Definition 5 The eccentricity of an ellipse is the ratio of the distances from the centre of the ellipse to one of the foci and to one of the vertices of the ellipse (eccentricity is denoted by *e*) i.e., $e = \frac{c}{a}$.

Then since the focus is at a distance of *c* from the centre, in terms of the eccentricity the focus is at a distance of *ae* from the centre.

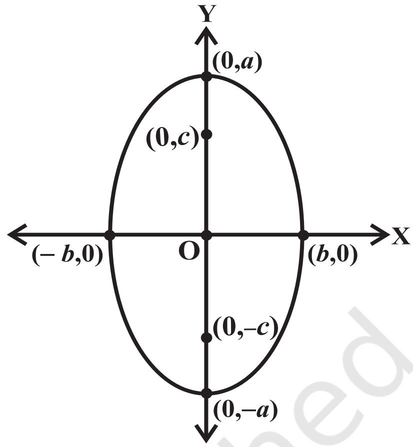
10.5.3 ***Standard equations of an ellipse*** The equation of an ellipse is simplest if the centre of the ellipse is at the origin and the foci are on the *x*-axis or *y*-axis. The two such possible orientations are shown in Fig 10.24.

We will derive the equation for the ellipse shown above in Fig 10.24 (a) with foci on the *x*-axis.



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

(a)



$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

(b)

Fig 10.24

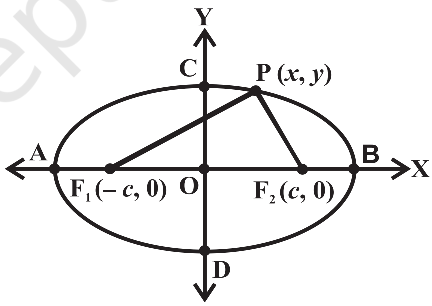
Let F_1 and F_2 be the foci and O be the mid-point of the line segment F_1F_2. Let O be the origin and the line from O through F_2 be the positive x-axis and that through F_1 as the negative x-axis. Let, the line through O perpendicular to the x-axis be the y-axis. Let the coordinates of F_1 be $(-\,c, 0)$ and F_2 be $(c, 0)$ (Fig 10.25).

Let $P(x, y)$ be any point on the ellipse such that the sum of the distances from P to the two foci be $2a$ so given

$PF_1 + PF_2 = 2a$. ... (1)

Using the distance formula, we have

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$$



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Fig 10.25

i.e., $\sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2}$

Squaring both sides, we get

$$(x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$

which on simplification gives

$$\sqrt{(x-c)^2+y^2} = a - \frac{c}{a}x$$

Squaring again and simplifying, we get

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

i.e., $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{(Since } c^2 = a^2 - b^2\text{)}$$

Hence any point on the ellipse satisfies

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad \text{... (2)}$$

Conversely, let P (x, y) satisfy the equation (2) with $0 < c < a$. Then

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

Therefore, $$\text{PF}_1 = \sqrt{(x+c)^2+y^2}$$

$$= \sqrt{(x+c)^2 + b^2\left(\frac{a^2-x^2}{a^2}\right)}$$

$$= \sqrt{(x+c)^2+(a^2-c^2)\left(\frac{a^2-x^2}{a^2}\right)} \quad \text{(since } b^2 = a^2 - c^2\text{)}$$

$$= \sqrt{\left(a+\frac{cx}{a}\right)^2} = a + \frac{c}{a}x$$

Similarly $$\text{PF}_2 = a - \frac{c}{a}x$$

Hence $$\text{PF}_1 + \text{PF}_2 = a + \frac{c}{a}x + a - \frac{c}{a}x = 2a \qquad \text{... (3)}$$

So, any point that satisfies $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, satisfies the geometric condition and so P(x, y) lies on the ellipse.

Hence from (2) and (3), we proved that the equation of an ellipse with centre of the origin and major axis along the x-axis is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Discussion From the equation of the ellipse obtained above, it follows that for every point P (x, y) on the ellipse, we have

$$\frac{x^2}{a^2} = 1 - \frac{y^2}{b^2} \leq 1, \text{ i.e., } x^2 \leq a^2, \text{ so } -a \leq x \leq a.$$

Therefore, the ellipse lies between the lines $x = -a$ and $x = a$ and touches these lines.

Similarly, the ellipse lies between the lines $y = -b$ and $y = b$ and touches these lines.

Similarly, we can derive the equation of the ellipse in Fig 10.24 (b) as $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.

These two equations are known as *standard equations* of the ellipses.

Note The standard equations of ellipses have centre at the origin and the major and minor axis are coordinate axes. However, the study of the ellipses with centre at any other point, and any line through the centre as major and the minor axes passing through the centre and perpendicular to major axis are beyond the scope here.

From the standard equations of the ellipses (Fig10.24), we have the following observations:

1. Ellipse is symmetric with respect to both the coordinate axes since if (x, y) is a point on the ellipse, then $(-x, y)$, $(x, -y)$ and $(-x, -y)$ are also points on the ellipse.

2. The foci always lie on the major axis. The major axis can be determined by finding the intercepts on the axes of symmetry. That is, major axis is along the x-axis if the coefficient of x^2 has the larger denominator and it is along the y-axis if the coefficient of y^2 has the larger denominator.

10.5.4 *Latus rectum*

Definition 6 Latus rectum of an ellipse is a line segment perpendicular to the major axis through any of the foci and whose end points lie on the ellipse (Fig 10.26).

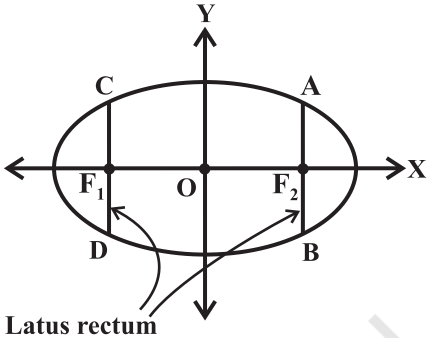


Fig 10. 26

To find the length of the latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Let the length of AF_2 be l.

Then the coordinates of A are (c, l), i.e., (ae, l)

Since A lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we have

$$\frac{(ae)^2}{a^2}+\frac{l^2}{b^2}=1$$

$$\Rightarrow l^2 = b^2(1-e^2)$$

But $$e^2=\frac{c^2}{a^2}=\frac{a^2-b^2}{a^2}=1-\frac{b^2}{a^2}$$

Therefore $$l^2=\frac{b^4}{a^2}\text{, i.e., } l=\frac{b^2}{a}$$

Since the ellipse is symmetric with respect to y-axis (of course, it is symmetric w.r.t. both the coordinate axes), $AF_2 = F_2B$ and so length of the latus rectum is $\frac{2b^2}{a}$.

Example 9 Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the latus rectum of the ellipse

$$\frac{x^2}{25}+\frac{y^2}{9}=1$$

Solution Since denominator of $\frac{x^2}{25}$ is larger than the denominator of $\frac{y^2}{9}$, the major axis is along the x-axis. Comparing the given equation with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

$$a = 5 \text{ and } b = 3. \text{ Also}$$

$$c=\sqrt{a^2-b^2}=\sqrt{25-9}=4$$

Therefore, the coordinates of the foci are (– 4,0) and (4,0), vertices are (– 5, 0) and (5, 0). Length of the major axis is 10 units length of the minor axis $2b$ is 6 units and the eccentricity is $\frac{4}{5}$ and latus rectum is $\frac{2b^2}{a}=\frac{18}{5}$.

Example 10 Find the coordinates of the foci, the vertices, the lengths of major and minor axes and the eccentricity of the ellipse $9x^2 + 4y^2 = 36$.

Solution The given equation of the ellipse can be written in standard form as

$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

Since the denominator of $\frac{y^2}{9}$ is larger than the denominator of $\frac{x^2}{4}$, the major axis is along the y-axis. Comparing the given equation with the standard equation

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1, \text{ we have } b = 2 \text{ and } a = 3.$$

Also $\qquad c=\sqrt{a^2-b^2} = \sqrt{9-4}=\sqrt{5}$

and $\qquad e=\frac{c}{a}=\frac{\sqrt{5}}{3}$

Hence the foci are $(0, \sqrt{5})$ and $(0, -\sqrt{5})$, vertices are (0,3) and (0, –3), length of the major axis is 6 units, the length of the minor axis is 4 units and the eccentricity of the ellipse is $\frac{\sqrt{5}}{3}$.

Example 11 Find the equation of the ellipse whose vertices are (± 13, 0) and foci are (± 5, 0).

Solution Since the vertices are on x-axis, the equation will be of the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{ where } a \text{ is the semi-major axis.}$$

Given that $a = 13$, $c = \pm 5$.
Therefore, from the relation $c^2 = a^2 - b^2$, we get

$$25 = 169 - b^2, \text{ i.e., } b = 12$$

Hence the equation of the ellipse is $\frac{x^2}{169} + \frac{y^2}{144} = 1$.

Example 12 Find the equation of the ellipse, whose length of the major axis is 20 and foci are $(0, \pm 5)$.

Solution Since the foci are on y-axis, the major axis is along the y-axis. So, equation of the ellipse is of the form $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.

Given that

$$a = \text{semi-major axis} = \frac{20}{2} = 10$$

and the relation $c^2 = a^2 - b^2$ gives

$$5^2 = 10^2 - b^2 \text{ i.e., } b^2 = 75$$

Therefore, the equation of the ellipse is

$$\frac{x^2}{75} + \frac{y^2}{100} = 1$$

Example 13 Find the equation of the ellipse, with major axis along the x-axis and passing through the points (4, 3) and (– 1,4).

Solution The standard form of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Since the points (4, 3) and (–1, 4) lie on the ellipse, we have

$$\frac{16}{a^2} + \frac{9}{b^2} = 1 \qquad \text{... (1)}$$

and

$$\frac{1}{a^2} + \frac{16}{b^2} = 1 \qquad \text{....(2)}$$

Solving equations (1) and (2), we find that $a^2 = \frac{247}{7}$ and $b^2 = \frac{247}{15}$.

Hence the required equation is

$$\frac{x^2}{\left(\frac{247}{7}\right)}+\frac{y^2}{\frac{247}{15}}=1 \text{, i.e., } 7x^2 + 15y^2 = 247.$$

EXERCISE 10.3

In each of the Exercises 1 to 9, find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse.

1. $\frac{x^2}{36}+\frac{y^2}{16}=1$ **2.** $\frac{x^2}{4}+\frac{y^2}{25}=1$ **3.** $\frac{x^2}{16}+\frac{y^2}{9}=1$

4. $\frac{x^2}{25}+\frac{y^2}{100}=1$ **5.** $\frac{x^2}{49}+\frac{y^2}{36}=1$ **6.** $\frac{x^2}{100}+\frac{y^2}{400}=1$

7. $36x^2 + 4y^2 = 144$ **8.** $16x^2 + y^2 = 16$ **9.** $4x^2 + 9y^2 = 36$

In each of the following Exercises 10 to 20, find the equation for the ellipse that satisfies the given conditions:

10. Vertices ($\pm$ 5, 0), foci ($\pm$ 4, 0)

11. Vertices (0, $\pm$ 13), foci (0, $\pm$ 5)

12. Vertices ($\pm$ 6, 0), foci ($\pm$ 4, 0)

13. Ends of major axis ($\pm$ 3, 0), ends of minor axis (0, $\pm$ 2)

14. Ends of major axis (0, $\pm\sqrt{5}$), ends of minor axis ($\pm$ 1, 0)

15. Length of major axis 26, foci ($\pm$ 5, 0)

16. Length of minor axis 16, foci (0, $\pm$ 6).

17. Foci ($\pm$ 3, 0), $a = 4$

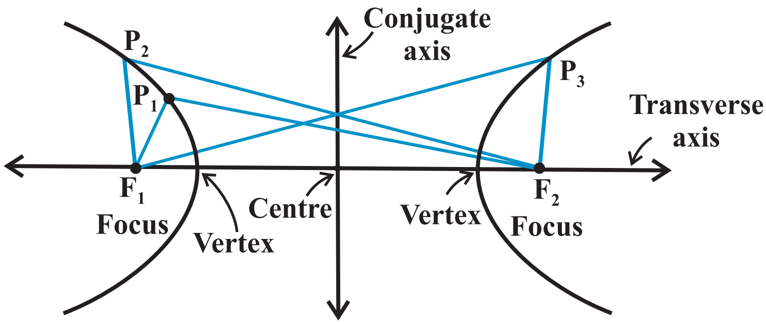
18. $b = 3$, $c = 4$, centre at the origin; foci on the x axis.

19. Centre at (0,0), major axis on the y-axis and passes through the points (3, 2) and (1,6).

20. Major axis on the x-axis and passes through the points (4,3) and (6,2).

10.6 Hyperbola

Definition 7 A hyperbola is the set of all points in a plane, the difference of whose distances from two fixed points in the plane is a constant.



$$P_1F_2 - P_1F_1 = P_2F_2 - P_2F_1 = P_3F_1 - P_3F_2$$

Fig 10.27

The term "*difference*" that is used in the definition means the distance to the farther point minus the distance to the closer point. The two fixed points are called the foci of the hyperbola. The mid-point of the line segment joining the foci is called the *centre of the hyperbola*. The line through the foci is called the *transverse axis* and the line through the centre and perpendicular to the transverse axis is called the *conjugate axis*. The points at which the hyperbola intersects the transverse axis are called the *vertices of the hyperbola* (Fig 10.27).

We denote the distance between the two foci by $2c$, the distance between two vertices (the length of the transverse axis) by $2a$ and we define the quantity b as

$$b = \sqrt{c^2 - a^2}$$

Also $2b$ is the length of the conjugate axis (Fig 10.28).

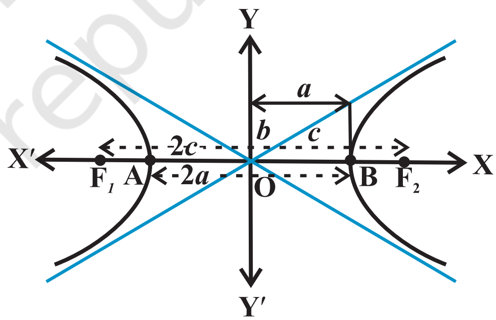


Fig 10.28

To find the constant $P_1F_2 - P_1F_1$:

By taking the point P at A and B in the Fig 10.28, we have

$BF_1 - BF_2 = AF_2 - AF_1$ (by the definition of the hyperbola)

$BA + AF_1 - BF_2 = AB + BF_2 - AF_1$

i.e., $AF_1 = BF_2$

So that, $BF_1 - BF_2 = BA + AF_1 - BF_2 = BA = 2a$

10.6.1 *Eccentricity*

Definition 8 Just like an ellipse, the ratio $e = \dfrac{c}{a}$ is called the *eccentricity of the hyperbola*. Since $c \geq a$, the eccentricity is never less than one. In terms of the eccentricity, the foci are at a distance of ae from the centre.

10.6.2 *Standard equation of Hyperbola* The equation of a hyperbola is simplest if the centre of the hyperbola is at the origin and the foci are on the x-axis or y-axis. The two such possible orientations are shown in Fig10.29.

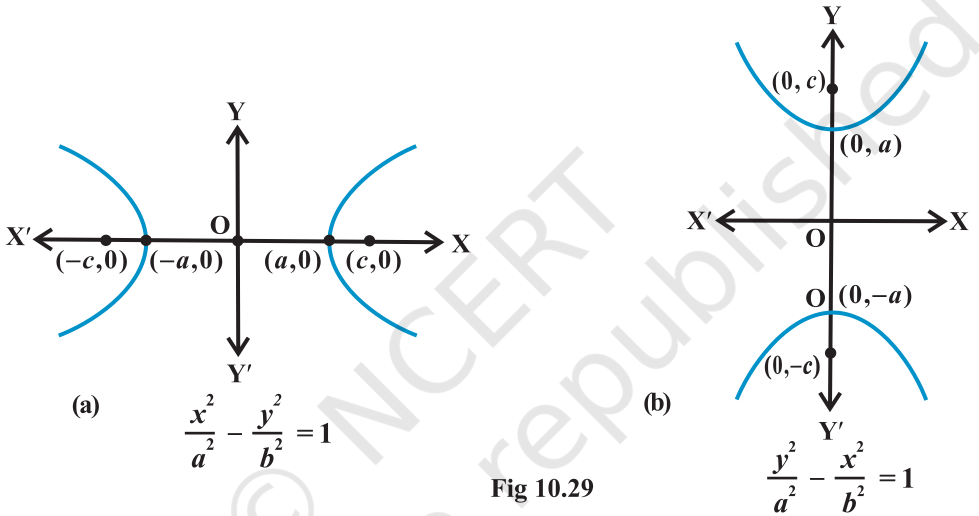


Fig 10.29

We will derive the equation for the hyperbola shown in Fig 10.29(a) with *foci* on the x-axis.

Let F_1 and F_2 be the foci and O be the mid-point of the line segment F_1F_2. Let O be the origin and the line through O through F_2 be the positive x-axis and that through F_1 as the negative x-axis. The line through O perpendicular to the x-axis be the y-axis. Let the coordinates of F_1 be $(-c,0)$ and F_2 be $(c,0)$ (Fig 10.30).

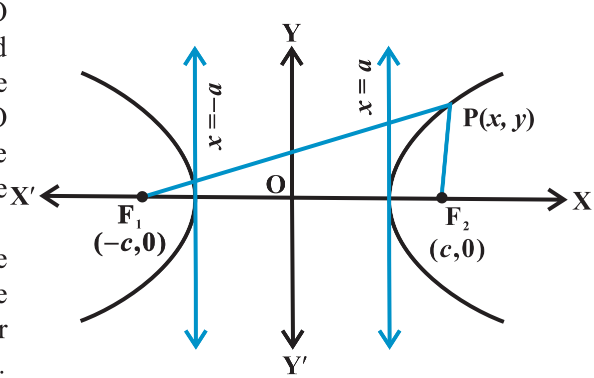


Fig 10.30

Let $P(x, y)$ be any point on the hyperbola such that the difference of the distances from P to the farther point minus the closer point be $2a$. So given, $PF_1 - PF_2 = 2a$

Using the distance formula, we have

$$\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=2a$$

i.e.,
$$\sqrt{(x+c)^2+y^2}=2a+\sqrt{(x-c)^2+y^2}$$

Squaring both side, we get

$$(x+c)^2+y^2=4a^2+4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$

and on simplifying, we get

$$\frac{cx}{a}-a=\sqrt{(x-c)^2+y^2}$$

On squaring again and further simplifying, we get

$$\frac{x^2}{a^2}-\frac{y^2}{c^2-a^2}=1$$

i.e.,
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad (\text{Since } c^2-a^2=b^2)$$

Hence any point on the hyperbola satisfies $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Conversely, let P(x, y) satisfy the above equation with $0 < a < c$. Then

$$y^2 = b^2\left(\frac{x^2-a^2}{a^2}\right)$$

Therefore,
$$\text{PF}_1 = +\sqrt{(x+c)^2+y^2}$$

$$= +\sqrt{(x+c)^2+b^2\left(\frac{x^2-a^2}{a^2}\right)} = a+\frac{c}{a}x$$

Similarly,
$$\text{PF}_2 = a-\frac{a}{c}x$$

In hyperbola $c > a$; and since P is to the right of the line $x = a$, $x > a$, $\frac{c}{a}x > a$. Therefore,

$a-\frac{c}{a}x$ becomes negative. Thus, $\text{PF}_2 = \frac{c}{a}x - a$.

Therefore $$\mathrm{PF}_1 - \mathrm{PF}_2 = a + \frac{c}{a}x - \frac{cx}{a} + a = 2a$$

Also, note that if P is to the left of the line $x = -a$, then

$$\mathrm{PF}_1 = -\left(a + \frac{c}{a}x\right), \ \mathrm{PF}_2 = a - \frac{c}{a}x.$$

In that case P $\mathrm{F}_2 - \mathrm{PF}_1 = 2a$. So, any point that satisfies $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, lies on the hyperbola.

Thus, we proved that the equation of hyperbola with origin (0,0) and transverse axis along x-axis is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Note A hyperbola in which $a = b$ is called an *equilateral hyperbola*.

Discussion From the equation of the hyperbola we have obtained, it follows that, we have for every point (x, y) on the hyperbola, $\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geq 1$.

i.e, $\left|\frac{x}{a}\right| \geq 1$, i.e., $x \leq -a$ or $x \geq a$. Therefore, no portion of the curve lies between the lines $x = +a$ and $x = -a$, (i.e. no real intercept on the conjugate axis).

Similarly, we can derive the equation of the hyperbola in Fig 11.31 (b) as $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

These two equations are known as the *standard equations* of *hyperbolas*.

Note The standard equations of hyperbolas have transverse and conjugate axes as the coordinate axes and the centre at the origin. However, there are hyperbolas with any two perpendicular lines as transverse and conjugate axes, but the study of such cases will be dealt in higher classes.

From the standard equations of hyperbolas (Fig10.27), we have the following observations:

1. Hyperbola is symmetric with respect to both the axes, since if (x, y) is a point on the hyperbola, then $(-x, y)$, $(x, -y)$ and $(-x, -y)$ are also points on the hyperbola.

2. The foci are always on the transverse axis. It is the positive term whose denominator gives the transverse axis. For example, $\frac{x^2}{9}-\frac{y^2}{16}=1$ has transverse axis along x-axis of length 6, while $\frac{y^2}{25}-\frac{x^2}{16}=1$ has transverse axis along y-axis of length 10.

10.6.3 *Latus rectum*

Definition 9 Latus rectum of hyperbola is a line segment perpendicular to the transverse axis through any of the foci and whose end points lie on the hyperbola.

As in ellipse, it is easy to show that the length of the latus rectum in hyperbola is $\frac{2b^2}{a}$.

Example 14 Find the coordinates of the foci and the vertices, the eccentricity,the length of the latus rectum of the hyperbolas:

(i) $\frac{x^2}{9}-\frac{y^2}{16}=1$, (ii) $y^2 - 16x^2 = 16$

Solution **(i) Comparing the equation $\frac{x^2}{9}-\frac{y^2}{16}=1$ with the standard equation**

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Here, $a = 3$, $b = 4$ and $c = \sqrt{a^2+b^2}=\sqrt{9+16}=5$

Therefore, the coordinates of the foci are $(\pm 5, 0)$ and that of vertices are $(\pm 3, 0)$.Also,

The eccentricity $e = \frac{c}{a}=\frac{5}{3}$. The latus rectum $=\frac{2b^2}{a}=\frac{32}{3}$

(ii) Dividing the equation by 16 on both sides, we have $\frac{y^2}{16}-\frac{x^2}{1}=1$

Comparing the equation with the standard equation $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$, we find that

$a = 4$, $b = 1$ and $c=\sqrt{a^2+b^2}=\sqrt{16+1}=\sqrt{17}$.

Therefore, the coordinates of the foci are $(0, \pm\sqrt{17})$ and that of the vertices are $(0, \pm 4)$. Also,

The eccentricity $e = \frac{c}{a} = \frac{\sqrt{17}}{4}$. The latus rectum $= \frac{2b^2}{a} = \frac{1}{2}$.

Example 15 Find the equation of the hyperbola with foci $(0, \pm 3)$ and vertices $(0, \pm \frac{\sqrt{11}}{2})$.

Solution Since the foci is on y-axis, the equation of the hyperbola is of the form

$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Since vertices are $(0, \pm \frac{\sqrt{11}}{2})$, $a = \frac{\sqrt{11}}{2}$

Also, since foci are $(0, \pm 3)$; $c = 3$ and $b^2 = c^2 - a^2 = \frac{25}{4}$.

Therefore, the equation of the hyperbola is

$$\frac{y^2}{\left(\frac{11}{4}\right)} - \frac{x^2}{\left(\frac{25}{4}\right)} = 1, \text{ i.e., } 100y^2 - 44x^2 = 275.$$

Example 16 Find the equation of the hyperbola where foci are $(0, \pm 12)$ and the length of the latus rectum is 36.

Solution Since foci are $(0, \pm 12)$, it follows that $c = 12$.

Length of the latus rectum $= \frac{2b^2}{a} = 36$ or $b^2 = 18a$

Therefore $\quad c^2 = a^2 + b^2$; gives

$$144 = a^2 + 18a$$

i.e., $\quad a^2 + 18a - 144 = 0,$

So $\quad a = -24, 6.$

Since a cannot be negative, we take $a = 6$ and so $b^2 = 108$.

Therefore, the equation of the required hyperbola is $\frac{y^2}{36} - \frac{x^2}{108} = 1$, i.e., $3y^2 - x^2 = 108$

EXERCISE 10.4

In each of the Exercises 1 to 6, find the coordinates of the foci and the vertices, the eccentricity and the length of the latus rectum of the hyperbolas.

1. $\frac{x^2}{16} - \frac{y^2}{9} = 1$ **2.** $\frac{y^2}{9} - \frac{x^2}{27} = 1$ **3.** $9y^2 - 4x^2 = 36$

4. $16x^2 - 9y^2 = 576$ **5.** $5y^2 - 9x^2 = 36$ **6.** $49y^2 - 16x^2 = 784$.

In each of the Exercises 7 to 15, find the equations of the hyperbola satisfying the given conditions.

7. Vertices $(\pm 2, 0)$, foci $(\pm 3, 0)$
8. Vertices $(0, \pm 5)$, foci $(0, \pm 8)$
9. Vertices $(0, \pm 3)$, foci $(0, \pm 5)$
10. Foci $(\pm 5, 0)$, the transverse axis is of length 8.
11. Foci $(0, \pm 13)$, the conjugate axis is of length 24.
12. Foci $(\pm 3\sqrt{5}, 0)$, the latus rectum is of length 8.
13. Foci $(\pm 4, 0)$, the latus rectum is of length 12
14. vertices $(\pm 7, 0)$, $e = \frac{4}{3}$.
15. Foci $(0, \pm \sqrt{10})$, passing through (2,3)

Miscellaneous Examples

Example 17 The focus of a parabolic mirror as shown in Fig 10.31 is at a distance of 5 cm from its vertex. If the mirror is 45 cm deep, find the distance AB (Fig 10.31).

Solution Since the distance from the focus to the vertex is 5 cm. We have, $a = 5$. If the origin is taken at the vertex and the axis of the mirror lies along the positive x-axis, the equation of the parabolic section is

$$y^2 = 4\,(5)\,x = 20\,x$$

Note that $x = 45$. Thus

$$y^2 = 900$$

Therefore $y = \pm 30$

Hence AB = $2y = 2 \times 30 = 60$ cm.

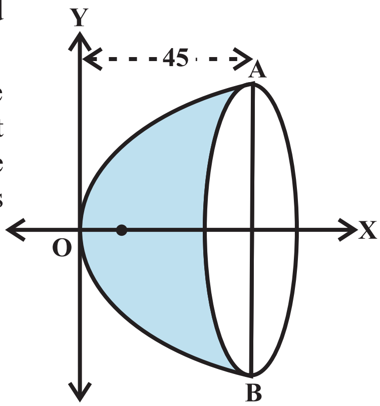


Fig 10.31

Example 18 A beam is supported at its ends by supports which are 12 metres apart. Since the load is concentrated at its centre, there is a deflection of 3 cm at the centre and the deflected beam is in the shape of a parabola. How far from the centre is the deflection 1 cm?

Solution Let the vertex be at the lowest point and the axis vertical. Let the coordinate axis be chosen as shown in Fig 10.32.

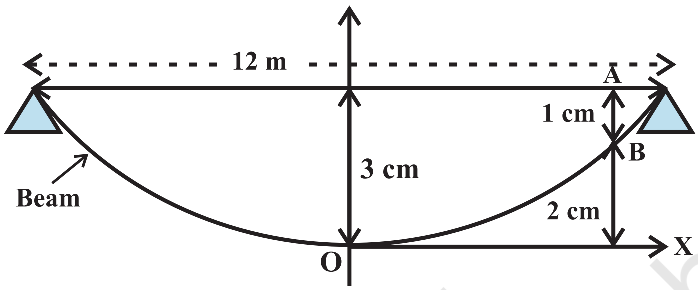


Fig 10.32

The equation of the parabola takes the form $x^2 = 4ay$. Since it passes through $\left(6, \frac{3}{100}\right)$, we have $(6)^2 = 4a\left(\frac{3}{100}\right)$, i.e., $a = \frac{36\times100}{12} = 300$ m

Let AB be the deflection of the beam which is $\frac{1}{100}$ m. Coordinates of B are $\left(x, \frac{2}{100}\right)$.

Therefore $\qquad x^2 = 4 \times 300 \times \frac{2}{100} = 24$

i.e. $\qquad x = \sqrt{24} = 2\sqrt{6}$ metres

Example 19 A rod AB of length 15 cm rests in between two coordinate axes in such a way that the end point A lies on x-axis and end point B lies on y-axis. A point P(x, y) is taken on the rod in such a way that AP = 6 cm. Show that the locus of P is an ellipse.

Solution Let AB be the rod making an angle θ with OX as shown in Fig 10.33 and P (x, y) the point on it such that AP = 6 cm.

Since $\qquad$ AB = 15 cm, we have

$\qquad$ PB = 9 cm.

From P draw PQ and PR perpendiculars on y-axis and x-axis, respectively.

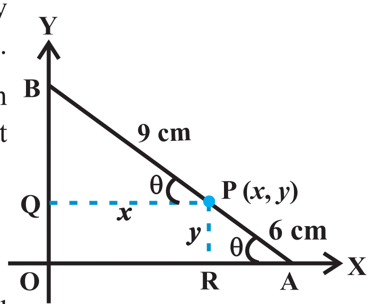


Fig 10.33

From Δ PBQ, $\cos\theta = \dfrac{x}{9}$

From Δ PRA, $\sin\theta = \dfrac{y}{6}$

Since $\cos^2\theta + \sin^2\theta = 1$

$$\left(\frac{x}{9}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$$

or
$$\frac{x^2}{81} + \frac{y^2}{36} = 1$$

Thus the locus of P is an ellipse.

Miscellaneous Exercise on Chapter 10

1. If a parabolic reflector is 20 cm in diameter and 5 cm deep, find the focus.

2. An arch is in the form of a parabola with its axis vertical. The arch is 10 m high and 5 m wide at the base. How wide is it 2 m from the vertex of the parabola?

3. The cable of a uniformly loaded suspension bridge hangs in the form of a parabola. The roadway which is horizontal and 100 m long is supported by vertical wires attached to the cable, the longest wire being 30 m and the shortest being 6 m. Find the length of a supporting wire attached to the roadway 18 m from the middle.

4. An arch is in the form of a semi-ellipse. It is 8 m wide and 2 m high at the centre. Find the height of the arch at a point 1.5 m from one end.

5. A rod of length 12 cm moves with its ends always touching the coordinate axes. Determine the equation of the locus of a point P on the rod, which is 3 cm from the end in contact with the x-axis.

6. Find the area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latus rectum.

7. A man running a racecourse notes that the sum of the distances from the two flag posts from him is always 10 m and the distance between the flag posts is 8 m. Find the equation of the posts traced by the man.

8. An equilateral triangle is inscribed in the parabola $y^2 = 4\,ax$, where one vertex is at the vertex of the parabola. Find the length of the side of the triangle.

Summary

In this Chapter the following concepts and generalisations are studied.

- A circle is the set of all points in a plane that are equidistant from a fixed point in the plane.
- The equation of a circle with centre (h, k) and the radius r is

$$(x - h)^2 + (y - k)^2 = r^2.$$

- A parabola is the set of all points in a plane that are equidistant from a fixed line and a fixed point in the plane.
- The equation of the parabola with focus at $(a, 0)$ $a > 0$ and directrix $x = -a$ is

$$y^2 = 4ax.$$

- Latus rectum of a parabola is a line segment perpendicular to the axis of the parabola, through the focus and whose end points lie on the parabola.
- Length of the latus rectum of the parabola $y^2 = 4ax$ is $4a$.
- An *ellipse* is the set of all points in a plane, the sum of whose distances from two fixed points in the plane is a constant.
- The equation of an ellipse with foci on the x-axis is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.
- Latus rectum of an ellipse is a line segment perpendicular to the major axis through any of the foci and whose end points lie on the ellipse.
- Length of the latus rectum of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $\dfrac{2b^2}{a}$.
- The eccentricity of an ellipse is the ratio between the distances from the centre of the ellipse to one of the foci and to one of the vertices of the ellipse.
- A hyperbola is the set of all points in a plane, the difference of whose distances from two fixed points in the plane is a constant.
- The equation of a hyperbola with foci on the x-axis is : $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

- Latus rectum of hyperbola is a line segment perpendicular to the transverse axis through any of the foci and whose end points lie on the hyperbola.
- Length of the latus rectum of the hyperbola : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is : $\frac{2b^2}{a}$.
- The eccentricity of a hyperbola is the ratio of the distances from the centre of the hyperbola to one of the foci and to one of the vertices of the hyperbola.

Historical Note

Geometry is one of the most ancient branches of mathematics. The Greek geometers investigated the properties of many curves that have theoretical and practical importance. Euclid wrote his treatise on geometry around 300 B.C. He was the first who organised the geometric figures based on certain axioms suggested by physical considerations. Geometry as initially studied by the ancient Indians and Greeks, who made essentially no use of the process of algebra. The synthetic approach to the subject of geometry as given by Euclid and in *Sulbasutras*, etc., was continued for some 1300 years. In the 200 B.C., Apollonius wrote a book called '*The Conic*' which was all about conic sections with many important discoveries that have remained unsurpassed for eighteen centuries.

Modern analytic geometry is called '*Cartesian*' after the name of Rene Descartes (1596-1650) whose relevant 'La Geometrie' was published in 1637. But the fundamental principle and method of analytical geometry were already discovered by Pierre de Fermat (1601-1665). Unfortunately, Fermats treatise on the subject, entitled *Ad Locus Planos et So LIDOS Isagoge* (Introduction to Plane and Solid Loci) was published only posthumously in 1679. So, Descartes came to be regarded as the unique inventor of the analytical geometry.

Isaac Barrow avoided using cartesian method. Newton used method of undetermined coefficients to find equations of curves. He used several types of coordinates including polar and bipolar. Leibnitz used the terms '*abscissa*', 'ordinate' and 'coordinate'. L' Hospital (about 1700) wrote an important textbook on analytical geometry.

Clairaut (1729) was the first to give the distance formula although in clumsy form. He also gave the intercept form of the linear equation. Cramer (1750) made formal use of the two axes and gave the equation of a circle as

$$(y-a)^2+(b-x)^2=r$$

He gave the best exposition of the analytical geometry of his time. Monge (1781) gave the modern 'point-slope' form of equation of a line as

$$y-y'=a\,(x-x')$$

and the condition of perpendicularity of two lines as $aa'+1=0$.

S.F. Lacroix (1765–1843) was a prolific textbook writer, but his contributions to analytical geometry are found scattered. He gave the 'two-point' form of equation of a line as

$$y-\beta=\frac{\beta'-\beta}{\alpha'-\alpha}(x-\alpha)$$

and the length of the perpendicular from (α, β) on $y = ax + b$ as $\dfrac{(\beta-a-b)}{\sqrt{1+a^2}}$.

His formula for finding angle between two lines was $\tan\theta=\left(\dfrac{a'-a}{1+aa'}\right)$. It is, of course, surprising that one has to wait for more than 150 years after the invention of analytical geometry before finding such essential basic formula. In 1818, C. Lame, a civil engineer, gave $mE + m'E' = 0$ as the curve passing through the points of intersection of two loci $E = 0$ and $E' = 0$.

Many important discoveries, both in Mathematics and Science, have been linked to the conic sections. The Greeks particularly Archimedes (287–212 B.C.) and Apollonius (200 B.C.) studied conic sections for their own beauty. These curves are important tools for present day exploration of outer space and also for research into behaviour of atomic particles.

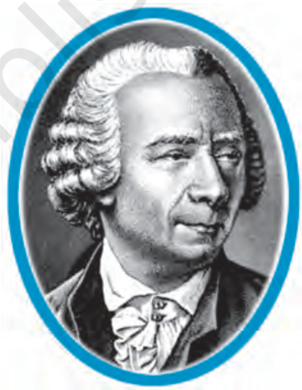
Chapter 11

INTRODUCTION TO THREE DIMENSIONAL GEOMETRY

❖ *Mathematics is both the queen and the hand-maiden of all sciences – E.T. BELL* ❖

11.1 Introduction

You may recall that to locate the position of a point in a plane, we need two intersecting mutually perpendicular lines in the plane. These lines are called the *coordinate axes* and the two numbers are called the *coordinates of the point with respect to the axes*. In actual life, we do not have to deal with points lying in a plane only. For example, consider the position of a ball thrown in space at different points of time or the position of an aeroplane as it flies from one place to another at different times during its flight.



Leonhard Euler (1707-1783)

Similarly, if we were to locate the position of the lowest tip of an electric bulb hanging from the ceiling of a room or the position of the central tip of the ceiling fan in a room, we will not only require the perpendicular distances of the point to be located from two perpendicular walls of the room but also the height of the point from the floor of the room. Therefore, we need not only two but three numbers representing the perpendicular distances of the point from three mutually perpendicular planes, namely the floor of the room and two adjacent walls of the room. The three numbers representing the three distances are called the *coordinates of the point with reference to the three coordinate planes*. So, a point in space has three coordinates. In this Chapter, we shall study the basic concepts of geometry in three dimensional space.*

* For various activities in three dimensional geometry one may refer to the Book, "***A Hand Book for designing Mathematics Laboratory in Schools***", NCERT, 2005.

11.2 Coordinate Axes and Coordinate Planes in Three Dimensional Space

Consider three planes intersecting at a point O such that these three planes are mutually perpendicular to each other (Fig 11.1). These three planes intersect along the lines $X'OX$, $Y'OY$ and $Z'OZ$, called the x, y and z-*axes*, respectively. We may note that these lines are mutually perpendicular to each other. These lines constitute the *rectangular coordinate system*. The planes XOY, YOZ and ZOX, called, respectively the XY-*plane*, YZ-*plane* and the ZX-*plane*, are known as the three coordinate planes. We take the XOY plane as the plane of the paper and the line $Z'OZ$ as perpendicular to the plane XOY. If the plane of the paper is considered as horizontal, then the line $Z'OZ$ will be vertical. The distances measured from XY-plane upwards in the direction of OZ are taken as positive and those measured downwards in the direction of OZ' are taken as negative. Similarly, the distance measured to the right of ZX-plane along OY are taken as positive, to the left of ZX-plane and along OY' as negative, in front of the YZ-plane along OX as positive and to the back of it along OX' as negative. The point O is called the *origin* of the coordinate system. The three coordinate planes divide the space into eight parts known as *octants*. These octants could be named as XOYZ, $X'OYZ$, $X'OY'Z$, $XOY'Z$, $XOYZ'$, $X'OYZ'$, $X'OY'Z'$ and $XOY'Z'$. and denoted by I, II, III, ..., VIII , respectively.

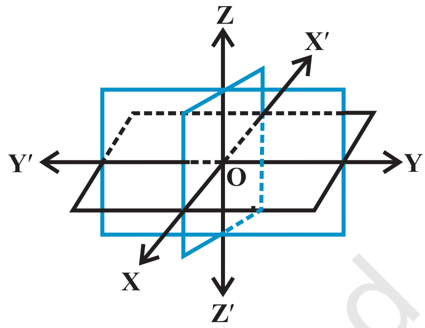


Fig 11.1

11.3 Coordinates of a Point in Space

Having chosen a fixed coordinate system in the space, consisting of coordinate axes, coordinate planes and the origin, we now explain, as to how, given a point in the space, we associate with it three coordinates (x,y,z) and conversely, given a triplet of three numbers (x, y, z), how, we locate a point in the space.

Given a point P in space, we drop a perpendicular PM on the XY-plane with M as the foot of this perpendicular (Fig 11.2). Then, from the point M, we draw a perpendicular ML to the x-axis, meeting it at L. Let OL be x, LM be y and MP be z. Then x,y and z are called the x, y and z *coordinates*, respectively, of the point P in the space. In Fig 11.2, we may note that the point P (x, y, z) lies in the octant XOYZ and so all x, y, z are positive. If P was in any other octant, the signs of x, y and z would change

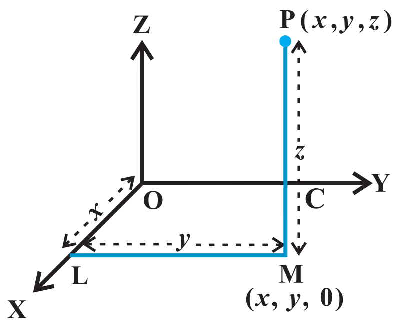


Fig 11.2

accordingly. Thus, to each point P in the space there corresponds an ordered triplet (x, y, z) of real numbers.

Conversely, given any triplet (x, y, z), we would first fix the point L on the x-axis corresponding to x, then locate the point M in the XY-plane such that (x, y) are the coordinates of the point M in the XY-plane. Note that LM is perpendicular to the x-axis or is parallel to the y-axis. Having reached the point M, we draw a perpendicular MP to the XY-plane and locate on it the point P corresponding to z. The point P so obtained has then the coordinates (x, y, z). Thus, there is a one to one correspondence between the points in space and ordered triplet (x, y, z) of real numbers.

Alternatively, through the point P in the space, we draw three planes parallel to the coordinate planes, meeting the x-axis, y-axis and z-axis in the points A, B and C, respectively (Fig 11.3). Let OA = x, OB = y and OC = z. Then, the point P will have the coordinates x, y and z and we write P (x, y, z). Conversely, given x, y and z, we locate the three points A, B and C on the three coordinate axes. Through the points A, B and C we draw planes parallel to the YZ-plane, ZX-plane and XY-plane, respectively. The point of interesection of these three planes, namely, ADPF, BDPE and CEPF is obviously the point P, corresponding to the ordered triplet (x, y, z). We observe that if P (x, y, z) is any point in the space, then x, y and z are perpendicular distances from YZ, ZX and XY planes, respectively.

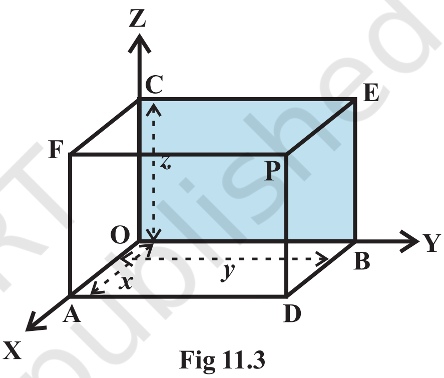


Fig 11.3

> **Note** The coordinates of the origin O are (0,0,0). The coordinates of any point on the x-axis will be as $(x,0,0)$ and the coordinates of any point in the YZ-plane will be as $(0, y, z)$.

Remark The sign of the coordinates of a point determine the octant in which the point lies. The following table shows the signs of the coordinates in eight octants.

Table 11.1

Octants / Coordinates	I	II	III	IV	V	VI	VII	VIII
x	+	–	–	+	+	–	–	+
y	+	+	–	–	+	+	–	–
z	+	+	+	+	–	–	–	–

Example 1 In Fig 11.3, if P is (2,4,5), find the coordinates of F.

Solution For the point F, the distance measured along OY is zero. Therefore, the coordinates of F are (2,0,5).

Example 2 Find the octant in which the points (–3,1,2) and (–3,1,– 2) lie.

Solution From the Table 11.1, the point (–3,1, 2) lies in second octant and the point (–3, 1, – 2) lies in octant VI.

EXERCISE 11.1

1. A point is on the x-axis. What are its y-coordinate and z-coordinates?

2. A point is in the XZ-plane. What can you say about its y-coordinate?

3. Name the octants in which the following points lie:

(1, 2, 3), (4, –2, 3), (4, –2, –5), (4, 2, –5), (– 4, 2, –5), (– 4, 2, 5), (–3, –1, 6) (– 2, – 4, –7).

4. Fill in the blanks:

(i) The x-axis and y-axis taken together determine a plane known as_______.

(ii) The coordinates of points in the XY-plane are of the form _______.

(iii) Coordinate planes divide the space into ______ octants.

11.4 Distance between Two Points

We have studied about the distance between two points in two-dimensional coordinate system. Let us now extend this study to three-dimensional system.

Let P(x_1, y_1, z_1) and Q (x_2, y_2, z_2) be two points referred to a system of rectangular axes OX, OY and OZ. Through the points P and Q draw planes parallel to the coordinate planes so as to form a rectangular parallelopiped with one diagonal PQ (Fig 11.4).

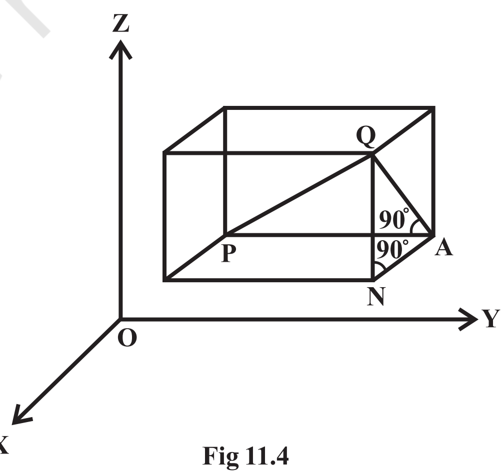


Fig 11.4

Now, since ∠PAQ is a right angle, it follows that, in triangle PAQ,

$$PQ^2 = PA^2 + AQ^2 \quad \ldots (1)$$

Also, triangle ANQ is right angle triangle with ∠ANQ a right angle.

Therefore $\quad AQ^2 = AN^2 + NQ^2 \qquad \dots (2)$

From (1) and (2), we have

$$PQ^2 = PA^2 + AN^2 + NQ^2$$

Now $\quad PA = y_2 - y_1,\ AN = x_2 - x_1$ and $NQ = z_2 - z_1$

Hence $\quad PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

Therefore $\quad PQ = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

This gives us the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2).

In particular, if $x_1 = y_1 = z_1 = 0$, i.e., point P is origin O, then $OQ = \sqrt{x_2^2+y_2^2+z_2^2}$, which gives the distance between the origin O and any point Q (x_2, y_2, z_2).

Example 3 Find the distance between the points P(1, –3, 4) and Q (– 4, 1, 2).

Solution The distance PQ between the points P (1,–3, 4) and Q (– 4, 1, 2) is

$$PQ = \sqrt{(-4-1)^2 + (1+3)^2 + (2-4)^2}$$

$$= \sqrt{25+16+4}$$

$$= \sqrt{45} = 3\sqrt{5} \text{ units}$$

Example 4 Show that the points P (–2, 3, 5), Q (1, 2, 3) and R (7, 0, –1) are collinear.

Solution We know that points are said to be collinear if they lie on a line.

Now, $\quad PQ = \sqrt{(1+2)^2 + (2-3)^2 + (3-5)^2} = \sqrt{9+1+4} = \sqrt{14}$

$\quad QR = \sqrt{(7-1)^2 + (0-2)^2 + (-1-3)^2} = \sqrt{36+4+16} = \sqrt{56} = 2\sqrt{14}$

and $\quad PR = \sqrt{(7+2)^2 + (0-3)^2 + (-1-5)^2} = \sqrt{81+9+36} = \sqrt{126} = 3\sqrt{14}$

Thus, PQ + QR = PR. Hence, P, Q and R are collinear.

Example 5 Are the points A (3, 6, 9), B (10, 20, 30) and C (25, – 41, 5), the vertices of a right angled triangle?

Solution By the distance formula, we have

$$AB^2 = (10-3)^2 + (20-6)^2 + (30-9)^2$$

$$= 49 + 196 + 441 = 686$$

$$BC^2 = (25-10)^2 + (-41-20)^2 + (5-30)^2$$

$$= 225 + 3721 + 625 = 4571$$

$$CA^2 = (3-25)^2 + (6+41)^2 + (9-5)^2$$
$$= 484 + 2209 + 16 = 2709$$

We find that $CA^2 + AB^2 \neq BC^2$.

Hence, the triangle ABC is not a right angled triangle.

Example 6 Find the equation of set of points P such that $PA^2 + PB^2 = 2k^2$, where A and B are the points (3, 4, 5) and (–1, 3, –7), respectively.

Solution Let the coordinates of point P be (x, y, z).

Here $PA^2 = (x-3)^2 + (y-4)^2 + (z-5)^2$

$PB^2 = (x+1)^2 + (y-3)^2 + (z+7)^2$

By the given condition $PA^2 + PB^2 = 2k^2$, we have

$$(x-3)^2 + (y-4)^2 + (z-5)^2 + (x+1)^2 + (y-3)^2 + (z+7)^2 = 2k^2$$

i.e., $2x^2 + 2y^2 + 2z^2 - 4x - 14y + 4z = 2k^2 - 109$.

EXERCISE 11.2

1. Find the distance between the following pairs of points:
   (i) (2, 3, 5) and (4, 3, 1)
   (ii) (–3, 7, 2) and (2, 4, –1)
   (iii) (–1, 3, – 4) and (1, –3, 4)
   (iv) (2, –1, 3) and (–2, 1, 3).
2. Show that the points (–2, 3, 5), (1, 2, 3) and (7, 0, –1) are collinear.
3. Verify the following:
   (i) (0, 7, –10), (1, 6, – 6) and (4, 9, – 6) are the vertices of an isosceles triangle.
   (ii) (0, 7, 10), (–1, 6, 6) and (– 4, 9, 6) are the vertices of a right angled triangle.
   (iii) (–1, 2, 1), (1, –2, 5), (4, –7, 8) and (2, –3, 4) are the vertices of a parallelogram.
4. Find the equation of the set of points which are equidistant from the points (1, 2, 3) and (3, 2, –1).
5. Find the equation of the set of points P, the sum of whose distances from A (4, 0, 0) and B (– 4, 0, 0) is equal to 10.

Miscellaneous Examples

Example 7 Show that the points A (1, 2, 3), B (–1, –2, –1), C (2, 3, 2) and D (4, 7, 6) are the vertices of a parallelogram ABCD, but it is not a rectangle.

Solution To show ABCD is a parallelogram we need to show opposite side are equal Note that.

$$AB = \sqrt{(-1-1)^2+(-2-2)^2+(-1-3)^2} = \sqrt{4+16+16} = 6$$

$$BC = \sqrt{(2+1)^2+(3+2)^2+(2+1)^2} = \sqrt{9+25+9} = \sqrt{43}$$

$$CD = \sqrt{(4-2)^2+(7-3)^2+(6-2)^2} = \sqrt{4+16+16} = 6$$

$$DA = \sqrt{(1-4)^2+(2-7)^2+(3-6)^2} = \sqrt{9+25+9}=\sqrt{43}$$

Since AB = CD and BC = AD, ABCD is a parallelogram.

Now, it is required to prove that ABCD is not a rectangle. For this, we show that diagonals AC and BD are unequal. We have

$$AC = \sqrt{(2-1)^2+(3-2)^2+(2-3)^2} = \sqrt{1+1+1}=\sqrt{3}$$

$$BD = \sqrt{(4+1)^2+(7+2)^2+(6+1)^2} = \sqrt{25+81+49}=\sqrt{155}.$$

Since AC ≠ BD, ABCD is not a rectangle.

> **Note** We can also show that ABCD is a parallelogram, using the property that diagonals AC and BD bisect each other.

Example 8 Find the equation of the set of the points P such that its distances from the points A (3, 4, –5) and B (– 2, 1, 4) are equal.

Solution If P (x, y, z) be any point such that PA = PB.

Now $\sqrt{(x-3)^2+(y-4)^2+(z+5)^2} = \sqrt{(x+2)^2+(y-1)^2+(z-4)^2}$

or $(x-3)^2+(y-4)^2+(z+5)^2 = (x+2)^2+(y-1)^2+(z-4)^2$

or $10x + 6y - 18z - 29 = 0$.

Example 9 The centroid of a triangle ABC is at the point (1, 1, 1). If the coordinates of A and B are (3, –5, 7) and (–1, 7, – 6), respectively, find the coordinates of the point C.

Solution Let the coordinates of C be (x, y, z) and the coordinates of the centroid G be (1, 1, 1). Then

$$\frac{x+3-1}{3}=1, \text{ i.e., } x=1; \ \frac{y-5+7}{3}=1, \text{ i.e., } y=1; \ \frac{z+7-6}{3}=1, \text{ i.e., } z=2.$$

Hence, coordinates of C are (1, 1, 2).

Miscellaneous Exercise on Chapter 11

1. Three vertices of a parallelogram ABCD are A(3, – 1, 2), B (1, 2, – 4) and C (– 1, 1, 2). Find the coordinates of the fourth vertex.

2. Find the lengths of the medians of the triangle with vertices A (0, 0, 6), B (0,4, 0) and (6, 0, 0).

3. If the origin is the centroid of the triangle PQR with vertices P ($2a$, 2, 6), Q (– 4, $3b$, –10) and R(8, 14, $2c$), then find the values of a, b and c.

4. If A and B be the points (3, 4, 5) and (–1, 3, –7), respectively, find the equation of the set of points P such that $PA^2 + PB^2 = k^2$, where k is a constant.

Summary

- In three dimensions, the coordinate axes of a rectangular Cartesian coordinate system are three mutually perpendicular lines. The axes are called the x, y and z-axes.
- The three planes determined by the pair of axes are the coordinate planes, called XY, YZ and ZX-*planes*.
- The three coordinate planes divide the space into eight parts known as *octants*.
- The coordinates of a point P in three dimensional geometry is always written in the form of triplet like (x, y, z). Here x, y and z are the distances from the YZ, ZX and XY-planes.
- (i) Any point on x-axis is of the form (x, 0, 0)

  (ii) Any point on y-axis is of the form (0, y, 0)

  (iii) Any point on z-axis is of the form (0, 0, z).
- Distance between two points P(x_1, y_1, z_1) and Q (x_2, y_2, z_2) is given by

$$PQ=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Historical Note

Rene' Descartes (1596–1650), the father of analytical geometry, essentially dealt with plane geometry only in 1637. The same is true of his co-inventor Pierre Fermat (1601-1665) and La Hire (1640-1718). Although suggestions for the three dimensional coordinate geometry can be found in their works but no details. Descartes had the idea of coordinates in three dimensions but did not develop it. J.Bernoulli (1667-1748) in a letter of 1715 to Leibnitz introduced the three coordinate planes which we use today. It was Antoinne Parent (1666-1716), who gave a systematic development of analytical solid geometry for the first time in a paper presented to the French Academy in 1700.

L.Euler (1707-1783) took up systematically the three dimensional coordinate geometry, in Chapter 5 of the appendix to the second volume of his "Introduction to Geometry" in 1748.

It was not until the middle of the nineteenth century that geometry was extended to more than three dimensions, the well-known application of which is in the Space-Time Continuum of Einstein's Theory of Relativity.

Chapter 12

LIMITS AND DERIVATIVES

❖ *With the Calculus as a key, Mathematics can be successfully applied to the explanation of the course of Nature – WHITEHEAD* ❖

12.1 Introduction

Sir Issac Newton (1642-1727)

This chapter is an introduction to Calculus. Calculus is that branch of mathematics which mainly deals with the study of change in the value of a function as the points in the domain change. First, we give an intuitive idea of derivative (without actually defining it). Then we give a naive definition of limit and study some algebra of limits. Then we come back to a definition of derivative and study some algebra of derivatives. We also obtain derivatives of certain standard functions.

12.2 Intuitive Idea of Derivatives

Physical experiments have confirmed that the body dropped from a tall cliff covers a distance of $4.9t^2$ metres in t seconds, i.e., distance s in metres covered by the body as a function of time t in seconds is given by $s = 4.9t^2$.

The adjoining Table 13.1 gives the distance travelled in metres at various intervals of time in seconds of a body dropped from a tall cliff.

The objective is to find the veloctiy of the body at time $t = 2$ seconds from this data. One way to approach this problem is to find the average velocity for various intervals of time ending at $t = 2$ seconds and hope that these throw some light on the velocity at $t = 2$ seconds.

Average velocity between $t = t_1$ and $t = t_2$ equals distance travelled between $t = t_1$ and $t = t_2$ seconds divided by $(t_2 - t_1)$. Hence the average velocity in the first two seconds

$$= \frac{\text{Distance travelled between } t_2 = 2 \text{ and } t_1 = 0}{\text{Time interval } (t_2 - t_1)}$$

$$= \frac{(19.6 - 0)\,m}{(2 - 0)\,s} = 9.8\,m/s\,.$$

Similarly, the average velocity between $t = 1$ and $t = 2$ is

$$\frac{(19.6 - 4.9)\,m}{(2 - 1)\,s} = 14.7\ m/s$$

Likewise we compute the average velocitiy between $t = t_1$ and $t = 2$ for various t_1. The following Table 13.2 gives the average velocity (v), $t = t_1$ seconds and $t = 2$ seconds.

Table 12.1

t	s
0	0
1	4.9
1.5	11.025
1.8	15.876
1.9	17.689
1.95	18.63225
2	19.6
2.05	20.59225
2.1	21.609
2.2	23.716
2.5	30.625
3	44.1
4	78.4

Table 12.2

t_1	0	1	1.5	1.8	1.9	1.95	1.99
v	9.8	14.7	17.15	18.62	19.11	19.355	19.551

From Table 12.2, we observe that the average velocity is gradually increasing. As we make the time intervals ending at $t = 2$ smaller, we see that we get a better idea of the velocity at $t = 2$. Hoping that nothing really dramatic happens between 1.99 seconds and 2 seconds, we conclude that the average velocity at $t = 2$ seconds is just above $19.551 m/s$.

This conclusion is somewhat strengthened by the following set of computation. Compute the average velocities for various time intervals starting at $t = 2$ seconds. As before the average velocity v between $t = 2$ seconds and $t = t_2$ seconds is

$$= \frac{\text{Distance travelled between 2 seconds and } t_2 \text{ seconds}}{t_2 - 2}$$

$$= \frac{\text{Distance travelled in } t_2 \text{ seconds } - \text{ Distance travelled in 2 seconds}}{t_2 - 2}$$

$$= \frac{\text{Distance travelled in } t_2 \text{ seconds } - 19.6}{t_2 - 2}$$

The following Table 12.3 gives the average velocity v in metres per second between $t = 2$ seconds and t_2 seconds.

Table 12.3

t_2	4	3	2.5	2.2	2.1	2.05	2.01
v	29.4	24.5	22.05	20.58	20.09	19.845	19.649

Here again we note that if we take smaller time intervals starting at $t = 2$, we get better idea of the velocity at $t = 2$.

In the first set of computations, what we have done is to find average velocities in increasing time intervals ending at $t = 2$ and then hope that nothing dramatic happens just before $t = 2$. In the second set of computations, we have found the average velocities decreasing in time intervals ending at $t = 2$ and then hope that nothing dramatic happens just after $t = 2$. Purely on the physical grounds, both these sequences of average velocities must approach a common limit. We can safely conclude that the velocity of the body at $t = 2$ is between 19.551*m/s* and 19.649 *m/s*. Technically, we say that the instantaneous velocity at $t = 2$ is between 19.551 *m/s* and 19.649 *m/s*. As is well-known, *velocity is the rate of change of displacement*. Hence what we have accomplished is the following. From the given data of distance covered at various time instants we have estimated the rate of change of the distance at a given instant of time. We say that the *derivative* of the distance function $s = 4.9t^2$ at $t = 2$ is between 19.551 and 19.649.

An alternate way of viewing this limiting process is shown in Fig 12.1. This is a plot of distance s of the body from the top of the cliff versus the time t elapsed. In the limit as the sequence of time intervals $h_1, h_2, ...,$ approaches zero, the sequence of average velocities approaches the same limit as does the sequence of ratios

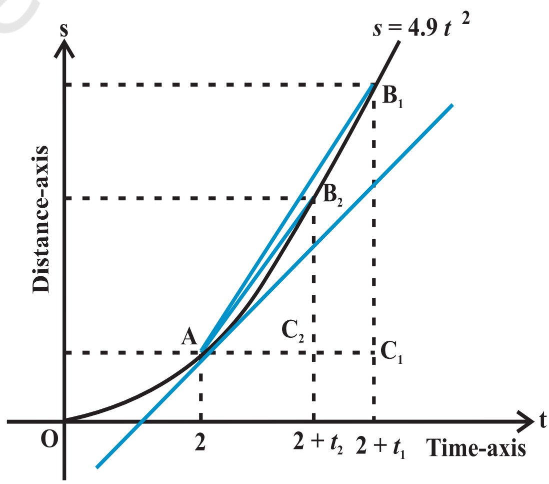


Fig 12.1

$$\frac{C_1B_1}{AC_1}, \frac{C_2B_2}{AC_2}, \frac{C_3B_3}{AC_3}, \ldots$$

where $C_1B_1 = s_1 - s_0$ is the distance travelled by the body in the time interval $h_1 = AC_1$, etc. From the Fig 12.1 it is safe to conclude that this latter sequence approaches the slope of the tangent to the curve at point A. In other words, the instantaneous velocity $v(t)$ of a body at time $t = 2$ is equal to the slope of the tangent of the curve $s = 4.9t^2$ at $t = 2$.

12.3 Limits

The above discussion clearly points towards the fact that we need to understand limiting process in greater clarity. We study a few illustrative examples to gain some familiarity with the concept of limits.

Consider the function $f(x) = x^2$. Observe that as x takes values very close to 0, the value of $f(x)$ also moves towards 0 (See Fig 2.10 Chapter 2). We say

$$\lim_{x \to 0} f(x) = 0$$

(to be read as limit of $f(x)$ as x tends to zero equals zero). The limit of $f(x)$ as x tends to zero is to be thought of as the value $f(x)$ should assume at $x = 0$.

In general as $x \to a$, $f(x) \to l$, then l is called *limit of the function* $f(x)$ which is symbolically written as $\lim_{x \to a} f(x) = l$.

Consider the following function $g(x) = |x|$, $x \neq 0$. Observe that $g(0)$ is not defined. Computing the value of $g(x)$ for values of x very near to 0, we see that the value of $g(x)$ moves towards 0. So, $\lim_{x \to 0} g(x) = 0$. This is intuitively clear from the graph of $y = |x|$ for $x \neq 0$. (See Fig 2.13, Chapter 2).

Consider the following function.

$$h(x) = \frac{x^2 - 4}{x - 2}, \ x \neq 2.$$

Compute the value of $h(x)$ for values of x very near to 2 (but not at 2). Convince yourself that all these values are near to 4. This is somewhat strengthened by considering the graph of the function $y = h(x)$ given here (Fig 12.2).

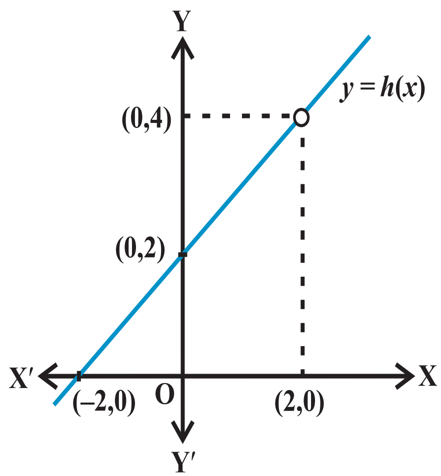


Fig 12.2

In all these illustrations the value which the function should assume at a given point $x = a$ did not really depend on how is x tending to a. Note that there are essentially two ways x could approach a number a either from left or from right, i.e., all the values of x near a could be less than a or could be greater than a. This naturally leads to two limits – the *right hand limit* and the *left hand limit*. *Right hand limit* of a function $f(x)$ is that value of $f(x)$ which is dictated by the values of $f(x)$ when x tends to a from the right. Similarly, the *left hand limit*. To illustrate this, consider the function

$$f(x) = \begin{cases} 1, & x \le 0 \\ 2, & x > 0 \end{cases}$$

Graph of this function is shown in the Fig 12.3. It is clear that the value of f at 0 dictated by values of $f(x)$ with $x \le 0$ equals 1, i.e., the left hand limit of $f(x)$ at 0 is

$$\lim_{x \to 0^-} f(x) = 1 .$$

Similarly, the value of f at 0 dictated by values of $f(x)$ with $x > 0$ equals 2, i.e., the right hand limit of $f(x)$ at 0 is

$$\lim_{x \to 0^+} f(x) = 2 .$$

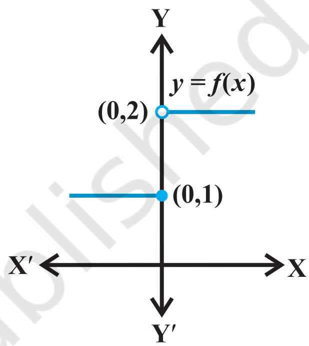


Fig 12.3

In this case the right and left hand limits are different, and hence we say that the limit of $f(x)$ as x tends to zero does not exist (even though the function is defined at 0).

Summary

We say $\lim_{x \to a^-} f(x)$ is the expected value of f at $x = a$ given the values of f near x to the left of a. This value is called the *left hand limit* of f at a.

We say $\lim_{x \to a^+} f(x)$ is the expected value of f at $x = a$ given the values of f near x to the right of a. This value is called the *right hand limit* of $f(x)$ at a.

If the right and left hand limits coincide, we call that common value as the limit of $f(x)$ at $x = a$ and denote it by $\lim_{x \to a} f(x)$.

Illustration 1 Consider the function $f(x) = x + 10$. We want to find the limit of this function at $x = 5$. Let us compute the value of the function $f(x)$ for x very near to 5. Some of the points near and to the left of 5 are 4.9, 4.95, 4.99, 4.995. . ., etc. Values of the function at these points are tabulated below. Similarly, the real number 5.001, 5.01, 5.1 are also points near and to the right of 5. Values of the function at these points are also given in the Table 12.4.

Table 12.4

x	4.9	4.95	4.99	4.995	5.001	5.01	5.1
$f(x)$	14.9	14.95	14.99	14.995	15.001	15.01	15.1

From the Table 12.4, we deduce that value of $f(x)$ at $x = 5$ should be greater than 14.995 and less than 15.001 assuming nothing dramatic happens between $x = 4.995$ and 5.001. It is reasonable to assume that the value of the $f(x)$ at $x = 5$ as dictated by the numbers to the left of 5 is 15, i.e.,

$$\lim_{x \to 5^-} f(x) = 15.$$

Similarly, when x approaches 5 from the right, $f(x)$ should be taking value 15, i.e.,

$$\lim_{x \to 5^+} f(x) = 15.$$

Hence, it is likely that the left hand limit of $f(x)$ and the right hand limit of $f(x)$ are both equal to 15. Thus,

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = \lim_{x \to 5} f(x) = 15.$$

This conclusion about the limit being equal to 15 is somewhat strengthened by seeing the graph of this function which is given in Fig 2.16, Chapter 2. In this figure, we note that as x approaches 5 from either right or left, the graph of the function $f(x) = x + 10$ approaches the point (5, 15).

We observe that the value of the function at $x = 5$ also happens to be equal to 15.

Illustration 2 Consider the function $f(x) = x^3$. Let us try to find the limit of this function at $x = 1$. Proceeding as in the previous case, we tabulate the value of $f(x)$ at x near 1. This is given in the Table 12.5.

Table 12.5

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.729	0.970299	0.997002999	1.003003001	1.030301	1.331

From this table, we deduce that value of $f(x)$ at $x = 1$ should be greater than 0.997002999 and less than 1.003003001 assuming nothing dramatic happens between

$x = 0.999$ and 1.001. It is reasonable to assume that the value of the $f(x)$ at $x = 1$ as dictated by the numbers to the left of 1 is 1, i.e.,

$$\lim_{x \to 1^-} f(x) = 1 .$$

Similarly, when x approaches 1 from the right, $f(x)$ should be taking value 1, i.e.,

$$\lim_{x \to 1^+} f(x) = 1 .$$

Hence, it is likely that the left hand limit of $f(x)$ and the right hand limit of $f(x)$ are both equal to 1. Thus,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = 1 .$$

This conclusion about the limit being equal to 1 is somewhat strengthened by seeing the graph of this function which is given in Fig 2.11, Chapter 2. In this figure, we note that as x approaches 1 from either right or left, the graph of the function $f(x) = x^3$ approaches the point (1, 1).

We observe, again, that the value of the function at $x = 1$ also happens to be equal to 1.

Illustration 3 Consider the function $f(x) = 3x$. Let us try to find the limit of this function at $x = 2$. The following Table 12.6 is now self-explanatory.

Table 12.6

x	1.9	1.95	1.99	1.999	2.001	2.01	2.1
$f(x)$	5.7	5.85	5.97	5.997	6.003	6.03	6.3

As before we observe that as x approaches 2 from either left or right, the value of $f(x)$ seem to approach 6. We record this as

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2} f(x) = 6$$

Its graph shown in Fig 12.4 strengthens this fact.

Here again we note that the value of the function at $x = 2$ coincides with the limit at $x = 2$.

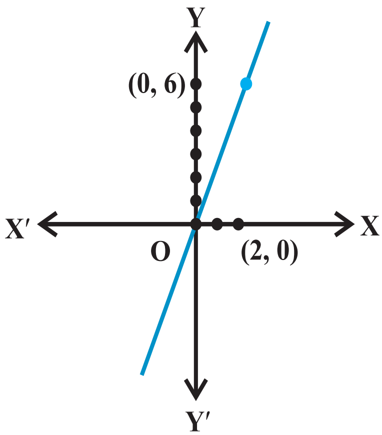


Fig 12.4

Illustration 4 Consider the constant function $f(x) = 3$. Let us try to find its limit at $x = 2$. This function being the constant function takes the same value (3, in this case) everywhere, i.e., its value at points close to 2 is 3. Hence

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2} f(x) = 3$$

Graph of $f(x) = 3$ is anyway the line parallel to x-axis passing through (0, 3) and is shown in Fig 2.9, Chapter 2. From this also it is clear that the required limit is 3. In fact, it is easily observed that $\lim_{x \to a} f(x) = 3$ for any real number a.

Illustration 5 Consider the function $f(x) = x^2 + x$. We want to find $\lim_{x \to 1} f(x)$. We tabulate the values of $f(x)$ near $x = 1$ in Table 12.7.

Table 12.7

x	0.9	0.99	0.999	1.01	1.1	1.2
$f(x)$	1.71	1.9701	1.997001	2.0301	2.31	2.64

From this it is reasonable to deduce that

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = 2.$$

From the graph of $f(x) = x^2 + x$ shown in the Fig 12.5, it is clear that as x approaches 1, the graph approaches (1, 2).

Here, again we observe that the

$$\lim_{x \to 1} f(x) = f(1)$$

Now, convince yourself of the following three facts:

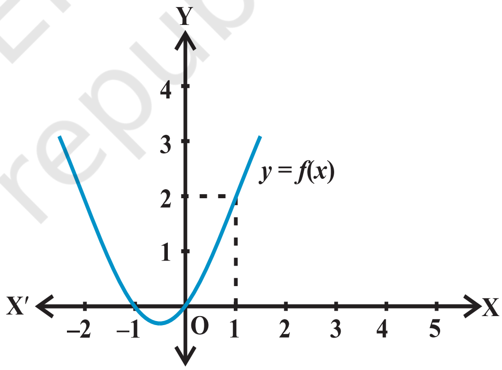


Fig 12.5

$$\lim_{x \to 1} x^2 = 1, \quad \lim_{x \to 1} x = 1 \text{ and } \lim_{x \to 1} x + 1 = 2$$

Then $$\lim_{x \to 1} x^2 + \lim_{x \to 1} x = 1 + 1 = 2 = \lim_{x \to 1} \left[x^2 + x\right].$$

Also $$\lim_{x \to 1} x . \lim_{x \to 1} (x + 1) = 1.2 = 2 = \lim_{x \to 1} \left[x(x+1)\right] = \lim_{x \to 1} \left[x^2 + x\right].$$

Illustration 6 Consider the function $f(x) = \sin x$. We are interested in $\lim\limits_{x \to \frac{\pi}{2}} \sin x$, where the angle is measured in radians.

Here, we tabulate the (approximate) value of $f(x)$ near $\frac{\pi}{2}$ (Table 12.8). From this, we may deduce that

$$\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^+} f(x) = \lim_{x \to \frac{\pi}{2}} f(x) = 1.$$

Further, this is supported by the graph of $f(x) = \sin x$ which is given in the Fig 3.8 (Chapter 3). In this case too, we observe that $\lim\limits_{x \to \frac{\pi}{2}} \sin x = 1$.

Table 12.8

x	$\frac{\pi}{2} - 0.1$	$\frac{\pi}{2} - 0.01$	$\frac{\pi}{2} + 0.01$	$\frac{\pi}{2} + 0.1$
$f(x)$	0.9950	0.9999	0.9999	0.9950

Illustration 7 Consider the function $f(x) = x + \cos x$. We want to find the $\lim\limits_{x \to 0} f(x)$.

Here we tabulate the (approximate) value of $f(x)$ near 0 (Table 12.9).

Table 12.9

x	– 0.1	– 0.01	– 0.001	0.001	0.01	0.1
$f(x)$	0.9850	0.98995	0.9989995	1.0009995	1.00995	1.0950

From the Table 13.9, we may deduce that

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0} f(x) = 1$$

In this case too, we observe that $\lim\limits_{x \to 0} f(x) = f(0) = 1$.

Now, can you convince yourself that

$$\lim_{x \to 0} [x + \cos x] = \lim_{x \to 0} x + \lim_{x \to 0} \cos x \text{ is indeed true?}$$

Illustration 8 Consider the function $f(x)=\frac{1}{x^2}$ for $x>0$. We want to know $\lim_{x\to 0} f(x)$.

Here, observe that the domain of the function is given to be all positive real numbers. Hence, when we tabulate the values of $f(x)$, it does not make sense to talk of x approaching 0 from the left. Below we tabulate the values of the function for positive x close to 0 (in this table n denotes any positive integer).

From the Table 12.10 given below, we see that as x tends to 0, $f(x)$ becomes larger and larger. What we mean here is that the value of $f(x)$ may be made larger than any given number.

Table 12.10

x	1	0.1	0.01	10^{-n}
$f(x)$	1	100	10000	10^{2n}

Mathematically, we say

$$\lim_{x\to 0} f(x)=+\infty$$

We also remark that we will not come across such limits in this course.

Illustration 9 We want to find $\lim_{x\to 0} f(x)$, where

$$f(x)=\begin{cases} x-2, & x<0 \\ 0\ , & x=0 \\ x+2, & x>0 \end{cases}$$

As usual we make a table of x near 0 with $f(x)$. Observe that for negative values of x we need to evaluate $x – 2$ and for positive values, we need to evaluate $x + 2$.

Table 12.11

x	– 0.1	– 0.01	– 0.001	0.001	0.01	0.1
$f(x)$	– 2.1	– 2.01	– 2.001	2.001	2.01	2.1

From the first three entries of the Table 12.11, we deduce that the value of the function is decreasing to –2 and hence.

$$\lim_{x\to 0^-} f(x)=-2$$

From the last three entires of the table we deduce that the value of the function is increasing from 2 and hence

$$\lim_{x \to 0^+} f(x) = 2$$

Since the left and right hand limits at 0 do not coincide, we say that the limit of the function at 0 does not exist.

Graph of this function is given in the Fig12.6. Here, we remark that the value of the function at $x = 0$ is well defined and is, indeed, equal to 0, but the limit of the function at $x = 0$ is not even defined.

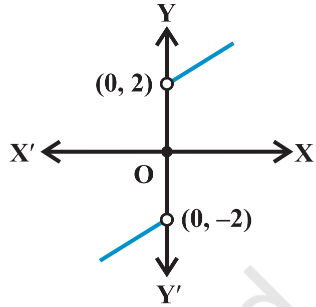


Fig 12.6

Illustration 10 As a final illustration, we find $\lim_{x \to 1} f(x)$, where

$$f(x) = \begin{cases} x+2 & x \neq 1 \\ 0 & x = 1 \end{cases}$$

Table 12.12

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	2.9	2.99	2.999	3.001	3.01	3.1

As usual we tabulate the values of $f(x)$ for x near 1. From the values of $f(x)$ for x less than 1, it seems that the function should take value 3 at $x = 1$., i.e.,

$$\lim_{x \to 1^-} f(x) = 3.$$

Similarly, the value of $f(x)$ should be 3 as dictated by values of $f(x)$ at x greater than 1. i.e.

$$\lim_{x \to 1^+} f(x) = 3.$$

But then the left and right hand limits coincide and hence

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = 3.$$

Graph of function given in Fig 12.7 strengthens our deduction about the limit. Here, we

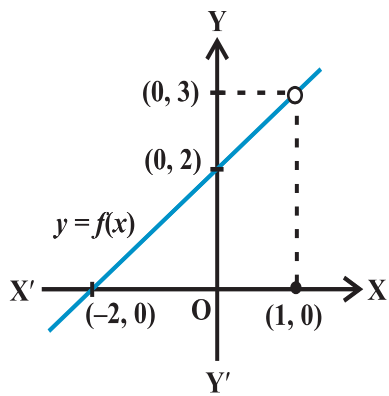


Fig 12.7

note that in general, at a given point the value of the function and its limit may be different (even when both are defined).

12.3.1 *Algebra of limits* In the above illustrations, we have observed that the limiting process respects addition, subtraction, multiplication and division as long as the limits and functions under consideration are well defined. This is not a coincidence. In fact, below we formalise these as a theorem without proof.

Theorem 1 Let f and g be two functions such that both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then

(i) Limit of sum of two functions is sum of the limits of the functions, i.e.,

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x).$$

(ii) Limit of difference of two functions is difference of the limits of the functions, i.e.,

$$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x).$$

(iii) Limit of product of two functions is product of the limits of the functions, i.e.,

$$\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) . \lim_{x \to a} g(x).$$

(iv) Limit of quotient of two functions is quotient of the limits of the functions (whenever the denominator is non zero), i.e.,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

Note In particular as a special case of (iii), when g is the constant function such that $g(x) = \lambda$, for some real number λ, we have

$$\lim_{x \to a} [(\lambda . f)(x)] = \lambda . \lim_{x \to a} f(x).$$

In the next two subsections, we illustrate how to exploit this theorem to evaluate limits of special types of functions.

12.3.2 *Limits of polynomials and rational functions* A function f is said to be a polynomial function of degree n $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, where a_is are real numbers such that $a_n \neq 0$ for some natural number n.

We know that $\lim_{x \to a} x = a$. Hence

$$\lim_{x \to a} x^2 = \lim_{x \to a} (x.x) = \lim_{x \to a} x. \lim_{x \to a} x = a. a = a^2$$

An easy exercise in induction on n tells us that

$$\lim_{x \to a} x^n = a^n$$

Now, let $f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$ be a polynomial function. Thinking of each of $a_0, a_1 \ x, a_2 x^2, ..., a_n \ x^n$ as a function, we have

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} \left[a_0 + a_1 x + a_2 x^2 + ... + a_n x^n \right] \\
&= \lim_{x \to a} a_0 + \lim_{x \to a} a_1 x + \lim_{x \to a} a_2 x^2 + ... + \lim_{x \to a} a_n x^n \\
&= a_0 + a_1 \lim_{x \to a} x + a_2 \lim_{x \to a} x^2 + ... + a_n \lim_{x \to a} x^n \\
&= a_0 + a_1 a + a_2 a^2 + ... + a_n a^n \\
&= f(a)
\end{aligned}$$

(Make sure that you understand the justification for each step in the above!)

A function f is said to be a rational function, if $f(x) = \dfrac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomials such that $h(x) \neq 0$. Then

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{g(x)}{h(x)} = \frac{\lim\limits_{x \to a} g(x)}{\lim\limits_{x \to a} h(x)} = \frac{g(a)}{h(a)}$$

However, if $h(a) = 0$, there are two scenarios – (i) when $g(a) \neq 0$ and (ii) when $g(a) = 0$. In the former case we say that the limit does not exist. In the latter case we can write $g(x) = (x - a)^k g_1(x)$, where k is the maximum of powers of $(x - a)$ in $g(x)$ Similarly, $h(x) = (x - a)^l h_1(x)$ as $h(a) = 0$. Now, if $k > l$, we have

$$\lim_{x \to a} f(x) = \frac{\lim\limits_{x \to a} g(x)}{\lim\limits_{x \to a} h(x)} = \frac{\lim\limits_{x \to a} (x - a)^k g_1(x)}{\lim\limits_{x \to a} (x - a)^l h_1(x)}$$

$$= \frac{\lim_{x\to a}(x-a)^{(k-l)} g_1(x)}{\lim_{x\to a} h_1(x)} = \frac{0.g_1(a)}{h_1(a)} = 0$$

If $k < l$, the limit is not defined.

Example 1 Find the limits: (i) $\lim_{x\to 1}\left[x^3 - x^2 + 1\right]$ (ii) $\lim_{x\to 3}\left[x(x+1)\right]$

(iii) $\lim_{x\to -1}\left[1 + x + x^2 + ... + x^{10}\right]$.

Solution The required limits are all limits of some polynomial functions. Hence the limits are the values of the function at the prescribed points. We have

(i) $\lim_{x\to 1}\ [x^3 - x^2 + 1] = 1^3 - 1^2 + 1 = 1$

(ii) $\lim_{x\to 3}\left[x(x+1)\right] = 3(3+1) = 3(4) = 12$

(iii) $\lim_{x\to -1}\left[1 + x + x^2 + ... + x^{10}\right] = 1 + (-1) + (-1)^2 + ... + (-1)^{10}$

$$= 1 - 1 + 1 ... + 1 = 1.$$

Example 2 Find the limits:

(i) $\lim_{x\to 1}\left[\frac{x^2+1}{x+100}\right]$ (ii) $\lim_{x\to 2}\left[\frac{x^3-4x^2+4x}{x^2-4}\right]$

(iii) $\lim_{x\to 2}\left[\frac{x^2-4}{x^3-4x^2+4x}\right]$ (iv) $\lim_{x\to 2}\left[\frac{x^3-2x^2}{x^2-5x+6}\right]$

(v) $\lim_{x\to 1}\left[\frac{x-2}{x^2-x} - \frac{1}{x^3-3x^2+2x}\right]$.

Solution All the functions under consideration are rational functions. Hence, we first evaluate these functions at the prescribed points. If this is of the form $\frac{0}{0}$, we try to rewrite the function cancelling the factors which are causing the limit to be of the form $\frac{0}{0}$.

(i) We have $\lim_{x \to 1} \frac{x^2 + 1}{x + 100} = \frac{1^2 + 1}{1 + 100} = \frac{2}{101}$

(ii) Evaluating the function at 2, it is of the form $\frac{0}{0}$.

Hence $\lim_{x \to 2} \frac{x^3 - 4x^2 + 4x}{x^2 - 4} = \lim_{x \to 2} \frac{x(x-2)^2}{(x+2)(x-2)}$

$$= \lim_{x \to 2} \frac{x(x-2)}{(x+2)} \qquad \text{as } x \neq 2$$

$$= \frac{2(2-2)}{2+2} = \frac{0}{4} = 0.$$

(iii) Evaluating the function at 2, we get it of the form $\frac{0}{0}$.

Hence $\lim_{x \to 2} \frac{x^2 - 4}{x^3 - 4x^2 + 4x} = \lim_{x \to 2} \frac{(x+2)(x-2)}{x(x-2)^2}$

$$= \lim_{x \to 2} \frac{(x+2)}{x(x-2)} = \frac{2+2}{2(2-2)} = \frac{4}{0}$$

which is not defined.

(iv) Evaluating the function at 2, we get it of the form $\frac{0}{0}$.

Hence $\lim_{x \to 2} \frac{x^3 - 2x^2}{x^2 - 5x + 6} = \lim_{x \to 2} \frac{x^2(x-2)}{(x-2)(x-3)}$

$$= \lim_{x \to 2} \frac{x^2}{(x-3)} = \frac{(2)^2}{2-3} = \frac{4}{-1} = -4.$$

(v) First, we rewrite the function as a rational function.

$$\left[\frac{x-2}{x^2-x}-\frac{1}{x^3-3x^2+2x}\right]=\left[\frac{x-2}{x(x-1)}-\frac{1}{x\left(x^2-3x+2\right)}\right]$$

$$=\left[\frac{x-2}{x(x-1)}-\frac{1}{x(x-1)(x-2)}\right]$$

$$=\left[\frac{x^2-4x+4-1}{x(x-1)(x-2)}\right]$$

$$=\frac{x^2-4x+3}{x(x-1)(x-2)}$$

Evaluating the function at 1, we get it of the form $\frac{0}{0}$.

Hence $$\lim_{x\to 1}\left[\frac{x^2-2}{x^2-x}-\frac{1}{x^3-3x^2+2x}\right]=\lim_{x\to 1}\frac{x^2-4x+3}{x(x-1)(x-2)}$$

$$=\lim_{x\to 1}\frac{(x-3)(x-1)}{x(x-1)(x-2)}$$

$$=\lim_{x\to 1}\frac{x-3}{x(x-2)}=\frac{1-3}{1(1-2)}=2.$$

We remark that we could cancel the term $(x-1)$ in the above evaluation because $x\neq 1$.

Evaluation of an important limit which will be used in the sequel is given as a theorem below.

Theorem 2 For any positive integer *n*,

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}.$$

Remark The expression in the above theorem for the limit is true even if n is any rational number and a is positive.

Proof Dividing $(x^n - a^n)$ by $(x - a)$, we see that

$$x^n - a^n = (x-a)\,(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + x\,a^{n-2} + a^{n-1})$$

Thus, $\lim\limits_{x\to a}\dfrac{x^n - a^n}{x - a} = \lim\limits_{x\to a}(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + x\,a^{n-2} + a^{n-1})$

$$= a^{n-1} + a\,a^{n-2} + \ldots + a^{n-2}(a) + a^{n-1}$$

$$= a^{n-1} + a^{n-1} + \ldots + a^{n-1} + a^{n-1} \quad (n \text{ terms})$$

$$= na^{n-1}$$

Example 3 Evaluate:

(i) $\lim\limits_{x\to 1}\dfrac{x^{15} - 1}{x^{10} - 1}$ (ii) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x} - 1}{x}$

Solution (i) We have

$$\lim_{x\to 1}\frac{x^{15} - 1}{x^{10} - 1} = \lim_{x\to 1}\left[\frac{x^{15} - 1}{x - 1} \div \frac{x^{10} - 1}{x - 1}\right]$$

$$= \lim_{x\to 1}\left[\frac{x^{15} - 1}{x - 1}\right] \div \lim_{x\to 1}\left[\frac{x^{10} - 1}{x - 1}\right]$$

$$= 15\,(1)^{14} \div 10(1)^9 \quad \text{(by the theorem above)}$$

$$= 15 \div 10 = \frac{3}{2}$$

(ii) Put $y = 1 + x$, so that $y \to 1$ as $x \to 0$.

Then $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x} - 1}{x} = \lim\limits_{y\to 1}\dfrac{\sqrt{y} - 1}{y - 1}$

$$= \lim_{y\to 1}\frac{y^{\frac{1}{2}} - 1^{\frac{1}{2}}}{y - 1}$$

$$= \frac{1}{2}(1)^{\frac{1}{2}-1} \quad \text{(by the remark above)} \quad = \frac{1}{2}$$

12.4 Limits of Trigonometric Functions

The following facts (stated as theorems) about functions in general come in handy in calculating limits of some trigonometric functions.

Theorem 3 Let f and g be two real valued functions with the same domain such that $f(x) \leq g(x)$ for all x in the domain of definition, For some a, if both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$. This is illustrated in Fig 12.8.

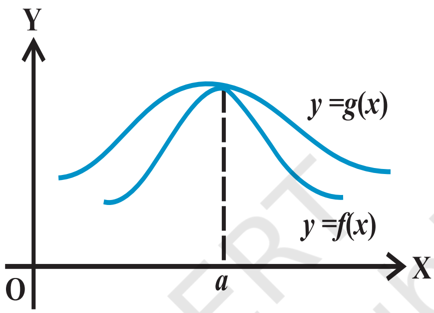


Fig 12.8

Theorem 4 (Sandwich Theorem) Let f, g and h be real functions such that $f(x) \leq g(x) \leq h(x)$ for all x in the common domain of definition. For some real number a, if $\lim_{x \to a} f(x) = l = \lim_{x \to a} h(x)$, then $\lim_{x \to a} g(x) = l$. This is illustrated in Fig 12.9.

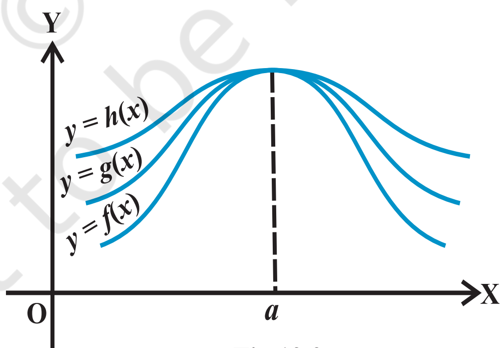


Fig 12.9

Given below is a beautiful geometric proof of the following important inequality relating trigonometric functions.

$$\cos x < \frac{\sin x}{x} < 1 \quad \text{for } 0 < |x| < \frac{\pi}{2} \qquad (*)$$

Proof We know that sin $(-x) = -\sin x$ and $\cos(-x) = \cos x$. Hence, it is sufficient to prove the inequality for $0 < x < \frac{\pi}{2}$.

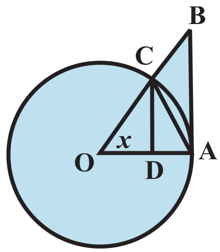


Fig 12.10

In the Fig 12.10, O is the centre of the unit circle such that the angle AOC is x radians and $0 < x < \frac{\pi}{2}$. Line segments B A and CD are perpendiculars to OA. Further, join AC. Then

Area of ΔOAC < Area of sector OAC < Area of Δ OAB.

i.e., $\frac{1}{2}\text{OA.CD} < \frac{x}{2\pi}.\pi.(\text{OA})^2 < \frac{1}{2}\text{OA.AB}$.

i.e., CD < x . OA < AB.

From Δ OCD,

$\sin x = \frac{\text{CD}}{\text{OA}}$ (since OC = OA) and hence CD = OA sin x. Also $\tan x = \frac{\text{AB}}{\text{OA}}$ and

hence AB = OA. tan x. Thus

OA sin x < OA. x < OA. tan x.

Since length OA is positive, we have

$\sin x < x < \tan x$.

Since $0 < x < \frac{\pi}{2}$, sin$x$ is positive and thus by dividing throughout by sin x, we have

$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$. Taking reciprocals throughout, we have

$$\cos x < \frac{\sin x}{x} < 1$$

which complete the proof.

Theorem 5 The following are two important limits.

(i) $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$. (ii) $\lim\limits_{x\to 0}\frac{1-\cos x}{x} = 0$.

Proof (i) The inequality in (*) says that the function $\frac{\sin x}{x}$ is sandwiched between the function cos x and the constant function which takes value 1.

Further, since $\lim\limits_{x \to 0} \cos x = 1$, we see that the proof of (i) of the theorem is complete by sandwich theorem.

To prove (ii), we recall the trigonometric identity $1 - \cos x = 2 \sin^2\left(\frac{x}{2}\right)$.

Then

$$\lim_{x \to 0} \frac{1-\cos x}{x} = \lim_{x \to 0} \frac{2\sin^2\left(\frac{x}{2}\right)}{x} = \lim_{x \to 0} \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}.\sin\left(\frac{x}{2}\right)$$

$$= \lim_{x \to 0} \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}.\lim_{x \to 0} \sin\left(\frac{x}{2}\right) = 1.0 = 0$$

Observe that we have implicitly used the fact that $x \to 0$ is equivalent to $\frac{x}{2} \to 0$. This may be justified by putting $y = \frac{x}{2}$.

Example 4 Evaluate: (i) $\lim\limits_{x \to 0} \frac{\sin 4x}{\sin 2x}$ (ii) $\lim\limits_{x \to 0} \frac{\tan x}{x}$

Solution (i) $\lim\limits_{x \to 0} \frac{\sin 4x}{\sin 2x} = \lim\limits_{x \to 0} \left[\frac{\sin 4x}{4x}.\frac{2x}{\sin 2x}.2\right]$

$$= 2.\lim_{x \to 0} \left[\frac{\sin 4x}{4x}\right] \div \left[\frac{\sin 2x}{2x}\right]$$

$$= 2.\lim_{4x \to 0} \left[\frac{\sin 4x}{4x}\right] \div \lim_{2x \to 0} \left[\frac{\sin 2x}{2x}\right]$$

$= 2.1.1 = 2$ (as $x \to 0$, $4x \to 0$ and $2x \to 0$)

(ii) We have $\lim\limits_{x \to 0} \frac{\tan x}{x} = \lim\limits_{x \to 0} \frac{\sin x}{x \cos x} = \lim\limits_{x \to 0} \frac{\sin x}{x} . \lim\limits_{x \to 0} \frac{1}{\cos x} = 1.1 = 1$

A general rule that needs to be kept in mind while evaluating limits is the following.

Say, given that the limit $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ exists and we want to evaluate this. First we check the value of $f(a)$ and $g(a)$. If both are 0, then we see if we can get the factor which is causing the terms to vanish, i.e., see if we can write $f(x) = f_1(x) f_2(x)$ so that $f_1(a) = 0$ and $f_2(a) \neq 0$. Similarly, we write $g(x) = g_1(x) g_2(x)$, where $g_1(a) = 0$ and $g_2(a) \neq 0$. Cancel out the common factors from $f(x)$ and $g(x)$ (if possible) and write

$$\frac{f(x)}{g(x)} = \frac{p(x)}{q(x)}, \text{ where } q(x) \neq 0.$$

Then
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{p(a)}{q(a)}.$$

EXERCISE 12.1

Evaluate the following limits in Exercises 1 to 22.

1. $\lim\limits_{x \to 3} x + 3$ **2.** $\lim\limits_{x \to \pi} \left(x - \frac{22}{7}\right)$ **3.** $\lim\limits_{r \to 1} \pi r^2$

4. $\lim\limits_{x \to 4} \frac{4x + 3}{x - 2}$ **5.** $\lim\limits_{x \to -1} \frac{x^{10} + x^5 + 1}{x - 1}$ **6.** $\lim\limits_{x \to 0} \frac{(x+1)^5 - 1}{x}$

7. $\lim\limits_{x \to 2} \frac{3x^2 - x - 10}{x^2 - 4}$ **8.** $\lim\limits_{x \to 3} \frac{x^4 - 81}{2x^2 - 5x - 3}$ **9.** $\lim\limits_{x \to 0} \frac{ax + b}{cx + 1}$

10. $\lim\limits_{z \to 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1}$ **11.** $\lim\limits_{x \to 1} \frac{ax^2 + bx + c}{cx^2 + bx + a}, a + b + c \neq 0$

12. $\lim\limits_{x \to -2} \frac{\frac{1}{x} + \frac{1}{2}}{x + 2}$ **13.** $\lim\limits_{x \to 0} \frac{\sin ax}{bx}$ **14.** $\lim\limits_{x \to 0} \frac{\sin ax}{\sin bx}, a, b \neq 0$

15. $\lim\limits_{x \to \pi} \dfrac{\sin(\pi - x)}{\pi(\pi - x)}$

16. $\lim\limits_{x \to 0} \dfrac{\cos x}{\pi - x}$

17. $\lim\limits_{x \to 0} \dfrac{\cos 2x - 1}{\cos x - 1}$

18. $\lim\limits_{x \to 0} \dfrac{ax + x\cos x}{b \sin x}$

19. $\lim\limits_{x \to 0} x \sec x$

20. $\lim\limits_{x \to 0} \dfrac{\sin ax + bx}{ax + \sin bx}\ a, b, a + b \neq 0,$

21. $\lim\limits_{x \to 0} (\operatorname{cosec} x - \cot x)$

22. $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\tan 2x}{x - \dfrac{\pi}{2}}$

23. Find $\lim\limits_{x \to 0} f(x)$ and $\lim\limits_{x \to 1} f(x)$, where $f(x) = \begin{cases} 2x + 3, & x \leq 0 \\ 3(x + 1), & x > 0 \end{cases}$

24. Find $\lim\limits_{x \to 1} f(x)$, where $f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ -x^2 - 1, & x > 1 \end{cases}$

25. Evaluate $\lim\limits_{x \to 0} f(x)$, where $f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

26. Find $\lim\limits_{x \to 0} f(x)$, where $f(x) = \begin{cases} \dfrac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

27. Find $\lim\limits_{x \to 5} f(x)$, where $f(x) = |x| - 5$

28. Suppose $f(x) = \begin{cases} a + bx, & x < 1 \\ 4, & x = 1 \\ b - ax, & x > 1 \end{cases}$

and if $\lim\limits_{x \to 1} f(x) = f(1)$ what are possible values of a and b?

29. Let a_1, a_2, . . ., a_n be fixed real numbers and define a function

$f(x) = (x - a_1)(x - a_2)...(x - a_n)$.

What is $\lim_{x \to a_1} f(x)$? For some $a \neq a_1, a_2, ..., a_n$, compute $\lim_{x \to a} f(x)$.

30. If $f(x) = \begin{cases} |x| + 1, & x < 0 \\ 0, & x = 0 \\ |x| - 1, & x > 0 \end{cases}$.

For what value (s) of a does $\lim_{x \to a} f(x)$ exists?

31. If the function $f(x)$ satisfies $\lim_{x \to 1} \frac{f(x) - 2}{x^2 - 1} = \pi$, evaluate $\lim_{x \to 1} f(x)$.

32. If $f(x) = \begin{cases} mx^2 + n, & x < 0 \\ nx + m, & 0 \leq x \leq 1 \\ nx^3 + m, & x > 1 \end{cases}$. For what integers m and n does both $\lim_{x \to 0} f(x)$ and $\lim_{x \to 1} f(x)$ exist?

12.5 Derivatives

We have seen in the Section 13.2, that by knowing the position of a body at various time intervals it is possible to find the rate at which the position of the body is changing. It is of very general interest to know a certain parameter at various instants of time and try to finding the rate at which it is changing. There are several real life situations where such a process needs to be carried out. For instance, people maintaining a reservoir need to know when will a reservoir overflow knowing the depth of the water at several instances of time, Rocket Scientists need to compute the precise velocity with which the satellite needs to be shot out from the rocket knowing the height of the rocket at various times. Financial institutions need to predict the changes in the value of a particular stock knowing its present value. In these, and many such cases it is desirable to know how a particular parameter is changing with respect to some other parameter. The heart of the matter is derivative of a function at a given point in its domain of definition.

Definition 1 *Suppose f is a real valued function and a is a point in its domain of definition. The derivative of f at a is defined by*

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists. Derivative of f (x) at a is denoted by f′(a).

Observe that $f'(a)$ quantifies the change in $f(x)$ at a with respect to x.

Example 5 Find the derivative at $x = 2$ of the function $f(x) = 3x$.

Solution We have

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{3(2+h) - 3(2)}{h}$$

$$= \lim_{h \to 0} \frac{6 + 3h - 6}{h} = \lim_{h \to 0} \frac{3h}{h} = \lim_{h \to 0} 3 = 3.$$

The derivative of the function $3x$ at $x = 2$ is 3.

Example 6 Find the derivative of the function $f(x) = 2x^2 + 3x - 5$ at $x = -1$. Also prove that $f'(0) + 3f'(-1) = 0$.

Solution We first find the derivatives of $f(x)$ at $x = -1$ and at $x = 0$. We have

$$f'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$$

$$= \lim_{h \to 0} \frac{\left[2(-1+h)^2 + 3(-1+h) - 5\right] - \left[2(-1)^2 + 3(-1) - 5\right]}{h}$$

$$= \lim_{h \to 0} \frac{2h^2 - h}{h} = \lim_{h \to 0} (2h - 1) = 2(0) - 1 = -1$$

and
$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{\left[2(0+h)^2 + 3(0+h) - 5\right] - \left[2(0)^2 + 3(0) - 5\right]}{h}$$

$$= \lim_{h\to 0}\frac{2h^2+3h}{h} = \lim_{h\to 0}(2h+3) = 2(0)+3 = 3$$

Clearly $\quad f'(0)+3f'(-1)=0$

Remark At this stage note that evaluating derivative at a point involves effective use of various rules, limits are subjected to. The following illustrates this.

Example 7 Find the derivative of sin x at $x = 0$.

Solution Let $f(x) = \sin x$. Then

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(0+h)-\sin(0)}{h} = \lim_{h\to 0}\frac{\sin h}{h} = 1$$

Example 8 Find the derivative of $f(x) = 3$ at $x = 0$ and at $x = 3$.

Solution Since the derivative measures the change in function, intuitively it is clear that the derivative of the constant function must be zero at every point. This is indeed, supported by the following computation.

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{3-3}{h} = \lim_{h\to 0}\frac{0}{h} = 0.$$

Similarly $\quad f'(3) = \lim_{h\to 0}\frac{f(3+h)-f(3)}{h} = \lim_{h\to 0}\frac{3-3}{h} = 0.$

We now present a geometric interpretation of derivative of a function at a point. Let $y = f(x)$ be a function and let $P = (a, f(a))$ and $Q = (a + h, f(a + h)$ be two points close to each other on the graph of this function. The Fig 12.11 is now self explanatory.

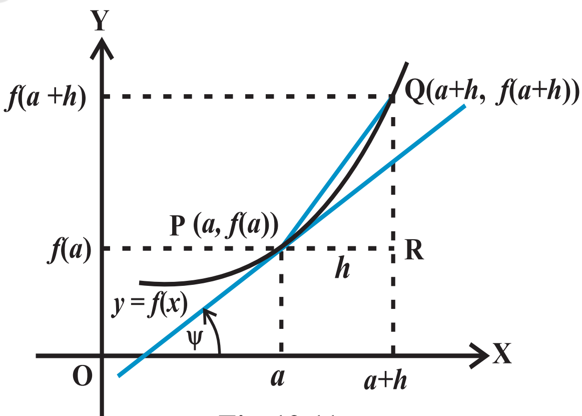


Fig 12.11

We know that $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

From the triangle PQR, it is clear that the ratio whose limit we are taking is precisely equal to tan(QPR) which is the slope of the chord PQ. In the limiting process, as h tends to 0, the point Q tends to P and we have

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{Q \to P} \frac{QR}{PR}$$

This is equivalent to the fact that the chord PQ tends to the tangent at P of the curve $y = f(x)$. Thus the limit turns out to be equal to the slope of the tangent. Hence

$$f'(a) = \tan \psi .$$

For a given function f we can find the derivative at every point. If the derivative exists at every point, it defines a new function called the derivative of f. Formally, we define derivative of a function as follows.

Definition 2 *Suppose f is a real valued function, the function defined by*

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

wherever the limit exists is defined to be the derivative of f at x and is denoted by $f'(x)$. This definition of derivative is also called the first principle of derivative.

Thus $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Clearly the domain of definition of $f'(x)$ is wherever the above limit exists. There are different notations for derivative of a function. Sometimes $f'(x)$ is denoted by $\dfrac{d}{dx}(f(x))$ or if $y = f(x)$, it is denoted by $\dfrac{dy}{dx}$. This is referred to as derivative of $f(x)$ or y with respect to x. It is also denoted by D $(f(x))$. Further, derivative of f at $x = a$ is also denoted by $\left.\dfrac{d}{dx} f(x)\right|_a$ or $\left.\dfrac{df}{dx}\right|_a$ or even $\left(\dfrac{df}{dx}\right)_{x=a}$.

Example 9 Find the derivative of $f(x) = 10x$.

Solution Since $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{10(x+h) - 10(x)}{h}$$

$$= \lim_{h \to 0} \frac{10h}{h} = \lim_{h \to 0} (10) = 10.$$

Example 10 Find the derivative of $f(x) = x^2$.

Solution We have, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{(x+h)^2 - (x)^2}{h} = \lim_{h \to 0} (h + 2x) = 2x$$

Example 11 Find the derivative of the constant function $f(x) = a$ for a fixed real number a.

Solution We have, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{a - a}{h} = \lim_{h \to 0} \frac{0}{h} = 0 \text{ as } h \neq 0$$

Example 12 Find the derivative of $f(x) = \frac{1}{x}$

Solution We have $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\frac{1}{(x+h)} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{x - (x+h)}{x(x+h)} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{-h}{x(x+h)} \right] = \lim_{h \to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

12.5.1 ***Algebra of derivative of functions*** Since the very definition of derivatives involve limits in a rather direct fashion, we expect the rules for derivatives to follow closely that of limits. We collect these in the following theorem.

Theorem 5 Let f and g be two functions such that their derivatives are defined in a common domain. Then

(i) Derivative of sum of two functions is sum of the derivatives of the functions.

$$\frac{d}{dx}\left[f(x)+g(x)\right]=\frac{d}{dx}f(x)+\frac{d}{dx}g(x).$$

(ii) Derivative of difference of two functions is difference of the derivatives of the functions.

$$\frac{d}{dx}\left[f(x)-g(x)\right]=\frac{d}{dx}f(x)-\frac{d}{dx}g(x).$$

(iii) Derivative of product of two functions is given by the following *product rule.*

$$\frac{d}{dx}\left[f(x)\,.\,g(x)\right]=\frac{d}{dx}f(x).g(x)+f(x).\frac{d}{dx}g(x)$$

(iv) Derivative of quotient of two functions is given by the following *quotient rule* (whenever the denominator is non–zero).

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{\frac{d}{dx}f(x).g(x)-f(x)\frac{d}{dx}g(x)}{\left(g(x)\right)^2}$$

The proofs of these follow essentially from the analogous theorem for limits. We will not prove these here. As in the case of limits this theorem tells us how to compute derivatives of special types of functions. The last two statements in the theorem may be restated in the following fashion which aids in recalling them easily:

Let $u=f(x)$ and $v=g(x)$. Then

$$(uv)' = u'v+uv'$$

This is referred to a Leibnitz rule for differentiating product of functions or the product rule. Similarly, the quotient rule is

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Now, let us tackle derivatives of some standard functions.

It is easy to see that the derivative of the function $f(x) = x$ is the constant function 1. This is because $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{x+h-x}{h}$

$$= \lim_{h\to 0} 1 = 1 .$$

We use this and the above theorem to compute the derivative of $f(x) = 10x = x + ... + x$ (ten terms). By (*i*) of the above theorem

$$\begin{aligned}\frac{df(x)}{dx} &= \frac{d}{dx}\left(x+\ldots+x\right) \text{ (ten terms)}\\ &= \frac{d}{dx}x+\ldots+\frac{d}{dx}x \quad \text{(ten terms)}\\ &= 1+\ldots+1 \text{ (ten terms)} = 10.\end{aligned}$$

We note that this limit may be evaluated using product rule too. Write $f(x) = 10x = uv$, where u is the constant function taking value 10 everywhere and $v(x) = x$. Here, $f(x) = 10x = uv$ we know that the derivative of u equals 0. Also derivative of $v(x) = x$ equals 1. Thus by the product rule we have

$$f'(x) = (10x)' = (uv)' = u'v + uv' = 0.x + 10.1 = 10$$

On similar lines the derivative of $f(x) = x^2$ may be evaluated. We have $f(x) = x^2 = x\,.x$ and hence

$$\begin{aligned}\frac{df}{dx} &= \frac{d}{dx}(x.x) = \frac{d}{dx}(x).x + x.\frac{d}{dx}(x)\\ &= 1.x + x.1 = 2x .\end{aligned}$$

More generally, we have the following theorem.

Theorem 6 Derivative of $f(x) = x^n$ is nx^{n-1} for any positive integer n.

Proof By definition of the derivative function, we have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} .$$

Binomial theorem tells that $(x + h)^n = \left({}^nC_0\right)x^n + \left({}^nC_1\right)x^{n-1}h + ... + \left({}^nC_n\right)h^n$ and hence $(x + h)^n - x^n = h(nx^{n-1} + ... + h^{n-1})$. Thus

$$\frac{df(x)}{dx} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0}\frac{h\left(nx^{n-1} + ... + h^{n-1}\right)}{h}$$

$$= \lim_{h\to 0}\left(nx^{n-1} + ... + h^{n-1}\right) = nx^{n-1}.$$

Alternatively, we may also prove this by induction on n and the product rule as follows. The result is true for $n = 1$, which has been proved earlier. We have

$$\frac{d}{dx}\left(x^n\right) = \frac{d}{dx}\left(x.x^{n-1}\right)$$

$$= \frac{d}{dx}(x).\left(x^{n-1}\right) + x.\frac{d}{dx}\left(x^{n-1}\right) \text{ (by product rule)}$$

$$= 1.x^{n-1} + x.\left((n-1)x^{n-2}\right) \text{ (by induction hypothesis)}$$

$$= x^{n-1} + (n-1)x^{n-1} = nx^{n-1}.$$

Remark The above theorem is true for all powers of x, i.e., n can be any real number (but we will not prove it here).

12.5.2 ***Derivative of polynomials and trigonometric functions*** We start with the following theorem which tells us the derivative of a polynomial function.

Theorem 7 Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ be a polynomial function, where a_is are all real numbers and $a_n \neq 0$. Then, the derivative function is given by

$$\frac{df(x)}{dx} = na_n x^{n-1} + (n-1)a_{n-1}x^{x-2} + ... + 2a_2 x + a_1.$$

Proof of this theorem is just putting together part (i) of Theorem 5 and Theorem 6.

Example 13 Compute the derivative of $6x^{100} - x^{55} + x$.

Solution A direct application of the above theorem tells that the derivative of the above function is $600x^{99} - 55x^{54} + 1$.

Example 14 Find the derivative of $f(x) = 1 + x + x^2 + x^3 + ... + x^{50}$ at $x = 1$.

Solution A direct application of the above Theorem 6 tells that the derivative of the above function is $1 + 2x + 3x^2 + . . . + 50x^{49}$. At $x = 1$ the value of this function equals

$$1 + 2(1) + 3(1)^2 + .. . + 50(1)^{49} = 1 + 2 + 3 + . . . + 50 = \frac{(50)(51)}{2} = 1275.$$

Example 15 Find the derivative of $f(x) = \dfrac{x+1}{x}$

Solution Clearly this function is defined everywhere except at $x = 0$. We use the quotient rule with $u = x + 1$ and $v = x$. Hence $u' = 1$ and $v' = 1$. Therefore

$$\frac{df(x)}{dx} = \frac{d}{dx}\left(\frac{x+1}{x}\right) = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2} = \frac{1(x)-(x+1)1}{x^2} = -\frac{1}{x^2}$$

Example 16 Compute the derivative of $\sin x$.

Solution Let $f(x) = \sin x$. Then

$$\frac{df(x)}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} \quad \text{(using formula for sin A – sin B)}$$

$$= \lim_{h\to 0}\cos\left(x+\frac{h}{2}\right).\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}} = \cos x.1 = \cos x.$$

Example 17 Compute the derivative of $\tan x$.

Solution Let $f(x) = \tan x$. Then

$$\frac{df(x)}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\tan(x+h)-\tan(x)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}\right]$$

$$= \lim_{h\to 0}\left[\frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h\cos(x+h)\cos x}\right]$$

$$= \lim_{h\to 0}\frac{\sin(x+h-x)}{h\cos(x+h)\cos x} \text{ (using formula for sin (A + B))}$$

$$= \lim_{h\to 0}\frac{\sin h}{h}.\lim_{h\to 0}\frac{1}{\cos(x+h)\cos x}$$

$$= 1.\frac{1}{\cos^2 x} = \sec^2 x .$$

Example 18 Compute the derivative of $f(x) = \sin^2 x$.

Solution We use the Leibniz product rule to evaluate this.

$$\frac{df(x)}{dx} = \frac{d}{dx}(\sin x \sin x)$$

$$= (\sin x)' \sin x + \sin x (\sin x)'$$

$$= (\cos x)\sin x + \sin x(\cos x)$$

$$= 2\sin x \cos x = \sin 2x \cdot$$

EXERCISE 12.2

1. Find the derivative of $x^2 - 2$ at $x = 10$.
2. Find the derivative of x at $x = 1$.
3. Find the derivative of $99x$ at $x = 100$.
4. Find the derivative of the following functions from first principle.

   (i) $x^3 - 27$ (ii) $(x-1)(x-2)$

   (iii) $\frac{1}{x^2}$ (iv) $\frac{x+1}{x-1}$
5. For the function

$$f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \ldots + \frac{x^2}{2} + x + 1 .$$

Prove that $f'(1)=100 f'(0)$.

6. Find the derivative of $x^n + ax^{n-1} + a^2x^{n-2} + \ldots + a^{n-1}x + a^n$ for some fixed real number a.

7. For some constants a and b, find the derivative of

(i) $(x-a)(x-b)$ (ii) $(ax^2+b)^2$ (iii) $\dfrac{x-a}{x-b}$

8. Find the derivative of $\dfrac{x^n - a^n}{x-a}$ for some constant a.

9. Find the derivative of

(i) $2x - \dfrac{3}{4}$ (ii) $(5x^3+3x-1)(x-1)$

(iii) $x^{-3}(5+3x)$ (iv) $x^5(3-6x^{-9})$

(v) $x^{-4}(3-4x^{-5})$ (vi) $\dfrac{2}{x+1} - \dfrac{x^2}{3x-1}$

10. Find the derivative of $\cos x$ from first principle.

11. Find the derivative of the following functions:

(i) $\sin x \cos x$ (ii) $\sec x$ (iii) $5\sec x + 4\cos x$

(iv) $\text{cosec } x$ (v) $3\cot x + 5\,\text{cosec}\, x$

(vi) $5\sin x - 6\cos x + 7$ (vii) $2\tan x - 7\sec x$

Miscellaneous Examples

Example 19 Find the derivative of f from the first principle, where f is given by

(i) $f(x) = \dfrac{2x+3}{x-2}$ (ii) $f(x) = x + \dfrac{1}{x}$

Solution (i) Note that function is not defined at $x = 2$. But, we have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\dfrac{2(x+h)+3}{x+h-2} - \dfrac{2x+3}{x-2}}{h}$$

$$= \lim_{h\to 0} \frac{(2x+2h+3)(x-2)-(2x+3)(x+h-2)}{h(x-2)(x+h-2)}$$

$$= \lim_{h\to 0} \frac{(2x+3)(x-2)+2h(x-2)-(2x+3)(x-2)-h(2x+3)}{h(x-2)(x+h-2)}$$

$$= \lim_{h\to 0} \frac{-7}{(x-2)\,(x+h-2)} = -\frac{7}{(x-2)^2}$$

Again, note that the function f' is also not defined at $x = 2$.

(ii) The function is not defined at $x = 0$. But, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\left(x+h+\frac{1}{x+h}\right)-\left(x+\frac{1}{x}\right)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[h+\frac{1}{x+h}-\frac{1}{x}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[h+\frac{x-x-h}{x(x+h)}\right] = \lim_{h\to 0} \frac{1}{h}\left[h\left(1-\frac{1}{x(x+h)}\right)\right]$$

$$= \lim_{h\to 0}\left[1-\frac{1}{x(x+h)}\right] = 1-\frac{1}{x^2}$$

Again, note that the function f' is not defined at $x = 0$.

Example 20 Find the derivative of $f(x)$ from the first principle, where $f(x)$ is

(i) $\sin x + \cos x$ (ii) $x \sin x$

Solution (i) we have $f'(x) = \dfrac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{\sin(x+h)+\cos(x+h)-\sin x-\cos x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x\,\cos h+\cos x\,\sin h+\cos x\cos h-\sin x\,\sin h-\sin x-\cos x}{h}$$

$$= \lim_{h \to 0} \frac{\sin h(\cos x - \sin x) + \sin x(\cos h - 1) + \cos x(\cos h - 1)}{h}$$

$$= \lim_{h \to 0} \frac{\sin h}{h}(\cos x - \sin x) + \lim_{h \to 0} \sin x \frac{(\cos h - 1)}{h} \quad + \lim_{h \to 0} \cos x \frac{(\cos h - 1)}{h}$$

$$= \cos x - \sin x$$

(ii) $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)\sin(x+h) - x\sin x}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)(\sin x \cos h + \sin h \cos x) - x\sin x}{h}$$

$$= \lim_{h \to 0} \frac{x\sin x(\cos h - 1) + x\cos x \sin h + h(\sin x \cos h + \sin h \cos x)}{h}$$

$$= \lim_{h \to 0} \frac{x\sin x(\cos h - 1)}{h} + \lim_{h \to 0} x\cos x \frac{\sin h}{h} + \lim_{h \to 0}(\sin x \cos h + \sin h \cos x)$$

$$= x\cos x + \sin x$$

Example 21 Compute derivative of

(i) $f(x) = \sin 2x$ (ii) $g(x) = \cot x$

Solution (i) Recall the trigonometric formula $\sin 2x = 2 \sin x \cos x$. Thus

$$\frac{df(x)}{dx} = \frac{d}{dx}(2\sin x \cos x) = 2\frac{d}{dx}(\sin x \cos x)$$

$$= 2\left[(\sin x)' \cos x + \sin x (\cos x)'\right]$$

$$= 2\left[(\cos x)\cos x + \sin x(-\sin x)\right]$$

$$= 2\left(\cos^2 x - \sin^2 x\right)$$

(ii) By definition, $g(x) = \cot x = \dfrac{\cos x}{\sin x}$. We use the quotient rule on this function wherever it is defined. $\dfrac{dg}{dx} = \dfrac{d}{dx}(\cot x) = \dfrac{d}{dx}\left(\dfrac{\cos x}{\sin x}\right)$

$$= \frac{(\cos x)'(\sin x) - (\cos x)(\sin x)'}{(\sin x)^2}$$

$$= \frac{(-\sin x)(\sin x) - (\cos x)(\cos x)}{(\sin x)^2}$$

$$= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\operatorname{cosec}^2 x$$

Alternatively, this may be computed by noting that $\cot x = \frac{1}{\tan x}$. Here, we use the fact that the derivative of *tan x* is *sec*^{2} *x* which we saw in Example 17 and also that the derivative of the constant function is 0.

$$\frac{dg}{dx} = \frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{1}{\tan x}\right)$$

$$= \frac{(1)'(\tan x) - (1)(\tan x)'}{(\tan x)^2}$$

$$= \frac{(0)(\tan x) - (\sec x)^2}{(\tan x)^2}$$

$$= \frac{-\sec^2 x}{\tan^2 x} = -\operatorname{cosec}^2 x$$

Example 22 Find the derivative of

(i) $\frac{x^5 - \cos x}{\sin x}$ (ii) $\frac{x + \cos x}{\tan x}$

Solution (i) Let $h(x) = \frac{x^5 - \cos x}{\sin x}$. We use the quotient rule on this function wherever it is defined.

$$h'(x) = \frac{(x^5 - \cos x)' \sin x - (x^5 - \cos x)(\sin x)'}{(\sin x)^2}$$

$$= \frac{(5x^4 + \sin x)\sin x - (x^5 - \cos x)\cos x}{\sin^2 x}$$

$$= \frac{-x^5 \cos x + 5x^4 \sin x + 1}{(\sin x)^2}$$

(ii) We use quotient rule on the function $\frac{x + \cos x}{\tan x}$ wherever it is defined.

$$h'(x) = \frac{(x + \cos x)' \tan x - (x + \cos x)(\tan x)'}{(\tan x)^2}$$

$$= \frac{(1 - \sin x)\tan x - (x + \cos x)\sec^2 x}{(\tan x)^2}$$

Miscellaneous Exercise on Chapter 12

1. Find the derivative of the following functions from first principle:

(i) $-x$ (ii) $(-x)^{-1}$ (iii) $\sin(x + 1)$ (iv) $\cos\left(x - \frac{\pi}{8}\right)$

Find the derivative of the following functions (it is to be understood that *a, b, c, d, p, q, r* and *s* are fixed non-zero constants and *m* and *n* are integers):

2. $(x + a)$ **3.** $(px + q)\left(\frac{r}{x} + s\right)$ **4.** $(ax + b)(cx + d)^2$

5. $\frac{ax + b}{cx + d}$ **6.** $\frac{1 + \frac{1}{x}}{1 - \frac{1}{x}}$ **7.** $\frac{1}{ax^2 + bx + c}$

8. $\frac{ax + b}{px^2 + qx + r}$ **9.** $\frac{px^2 + qx + r}{ax + b}$ **10.** $\frac{a}{x^4} - \frac{b}{x^2} + \cos x$

11. $4\sqrt{x} - 2$ **12.** $(ax + b)^n$ **13.** $(ax + b)^n (cx + d)^m$

14. $\sin(x + a)$ **15.** $\operatorname{cosec} x \cot x$ **16.** $\frac{\cos x}{1 + \sin x}$

17. $\dfrac{\sin x+\cos x}{\sin x-\cos x}$ **18.** $\dfrac{\sec x-1}{\sec x+1}$ **19.** $\sin^n x$

20. $\dfrac{a+b\sin x}{c+d\cos x}$ **21.** $\dfrac{\sin(x+a)}{\cos x}$ **22.** $x^4(5\sin x-3\cos x)$

23. $\left(x^2+1\right)\cos x$ **24.** $\left(ax^2+\sin x\right)\left(p+q\cos x\right)$

25. $(x+\cos x)\,(x-\tan x)$ **26.** $\dfrac{4x+5\sin x}{3x+7\cos x}$ **27.** $\dfrac{x^2\cos\left(\dfrac{\pi}{4}\right)}{\sin x}$

28. $\dfrac{x}{1+\tan x}$ **29.** $(x+\sec x)\,(x-\tan x)$ **30.** $\dfrac{x}{\sin^n x}$

Summary

- The expected value of the function as dictated by the points to the left of a point defines the left hand limit of the function at that point. Similarly the right hand limit.
- Limit of a function at a point is the common value of the left and right hand limits, if they coincide.
- For a function f and a real number a, $\lim\limits_{x\to a} f(x)$ and $f(a)$ may not be same (In fact, one may be defined and not the other one).
- For functions f and g the following holds:

$$\lim_{x\to a}\left[f(x)\pm g(x)\right]=\lim_{x\to a} f(x)\pm\lim_{x\to a} g(x)$$

$$\lim_{x\to a}\left[f(x).g(x)\right]=\lim_{x\to a} f(x).\lim_{x\to a} g(x)$$

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right]=\frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$$

- Following are some of the standard limits

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

- The derivative of a function f at a is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- Derivative of a function f at any point x is defined by

$$f'(x) = \frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- For functions u and v the following holds:

$$(u \pm v)' = u' \pm v'$$

$$(uv)' = u'v + uv'$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$ provided all are defined.

- Following are some of the standard derivatives.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

Historical Note

In the history of mathematics two names are prominent to share the credit for inventing calculus, Issac Newton (1642 – 1727) and G.W. Leibnitz (1646 – 1717). Both of them independently invented calculus around the seventeenth century. After the advent of calculus many mathematicians contributed for further development of calculus. The rigorous concept is mainly attributed to the great mathematicians, A.L. Cauchy, J.L.Lagrange and Karl Weierstrass. Cauchy gave the foundation of calculus as we have now generally accepted in our textbooks. Cauchy used D' Alembert's limit concept to define the derivative of a function. Starting with definition of a limit, Cauchy gave examples such as the limit of $\frac{\sin\alpha}{\alpha}$ for $\alpha = 0$. He wrote $\frac{\Delta y}{\Delta x} = \frac{f(x+i)-f(x)}{i}$, and called the limit for $i \to 0$, the "function derive'e, y' for $f'(x)$".

Before 1900, it was thought that calculus is quite difficult to teach. So calculus became beyond the reach of youngsters. But just in 1900, John Perry and others in England started propagating the view that essential ideas and methods of calculus were simple and could be taught even in schools. F.L. Griffin, pioneered the teaching of calculus to first year students. This was regarded as one of the most daring act in those days.

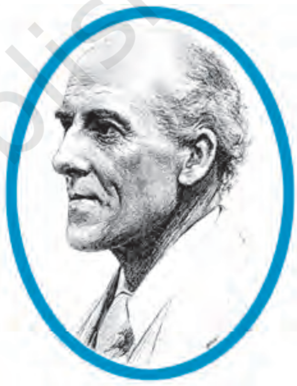
Today not only the mathematics but many other subjects such as Physics, Chemistry, Economics and Biological Sciences are enjoying the fruits of calculus.

Chapter 13

STATISTICS

❖ *"Statistics may be rightly called the science of averages and their estimates." – A.L.BOWLEY & A.L. BODDINGTON* ❖

13.1 Introduction



Karl Pearson (1857-1936)

We know that statistics deals with data collected for specific purposes. We can make decisions about the data by analysing and interpreting it. In earlier classes, we have studied methods of representing data graphically and in tabular form. This representation reveals certain salient features or characteristics of the data. We have also studied the methods of finding a representative value for the given data. This value is called the measure of central tendency. Recall mean (arithmetic mean), median and mode are three measures of central tendency. A *measure of central tendency* gives us a rough idea where data points are centred. But, in order to make better interpretation from the data, we should also have an idea how the data are scattered or how much they are bunched around a measure of central tendency.

Consider now the runs scored by two batsmen in their last ten matches as follows:

Batsman A : 30, 91, 0, 64, 42, 80, 30, 5, 117, 71
Batsman B : 53, 46, 48, 50, 53, 53, 58, 60, 57, 52

Clearly, the mean and median of the data are

	Batsman A	Batsman B
Mean	53	53
Median	53	53

Recall that, we calculate the mean of a data (denoted by $\bar{x}$) by dividing the sum of the observations by the number of observations, i.e.,

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Also, the median is obtained by first arranging the data in ascending or descending order and applying the following rule.

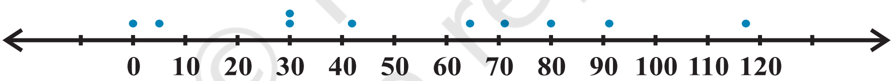
If the number of observations is odd, then the median is $\left(\frac{n+1}{2}\right)^{\text{th}}$ observation.

If the number of observations is even, then median is the mean of $\left(\frac{n}{2}\right)^{\text{th}}$ and $\left(\frac{n}{2}+1\right)^{\text{th}}$ observations.

We find that the mean and median of the runs scored by both the batsmen A and B are same i.e., 53. Can we say that the performance of two players is same? Clearly No, because the variability in the scores of batsman A is from 0 (minimum) to 117 (maximum). Whereas, the range of the runs scored by batsman B is from 46 to 60.

Let us now plot the above scores as dots on a number line. We find the following diagrams:

For batsman A



For batsman B

Fig 13.1

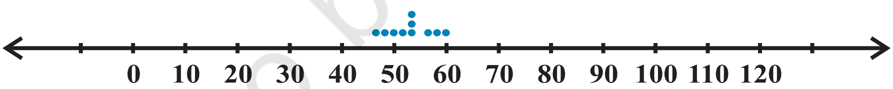


Fig 13.2

We can see that the dots corresponding to batsman B are close to each other and are clustering around the measure of central tendency (mean and median), while those corresponding to batsman A are scattered or more spread out.

Thus, the measures of central tendency are not sufficient to give complete information about a given data. Variability is another factor which is required to be studied under statistics. Like '*measures of central tendency*' we want to have a single number to describe variability. This single number is called a '*measure of dispersion*'. In this Chapter, we shall learn some of the important measures of dispersion and their methods of calculation for ungrouped and grouped data.

13.2 Measures of Dispersion

The dispersion or scatter in a data is measured on the basis of the observations and the types of the measure of central tendency, used there. There are following measures of dispersion:

(i) Range, (ii) Quartile deviation, (iii) Mean deviation, (iv) Standard deviation.

In this Chapter, we shall study all of these measures of dispersion except the quartile deviation.

13.3 Range

Recall that, in the example of runs scored by two batsmen A and B, we had some idea of variability in the scores on the basis of minimum and maximum runs in each series. To obtain a single number for this, we find the difference of maximum and minimum values of each series. This difference is called the 'Range' of the data.

In case of batsman A, Range = 117 – 0 = 117 and for batsman B, Range = 60 – 46 = 14. Clearly, Range of A > Range of B. Therefore, the scores are scattered or dispersed in case of A while for B these are close to each other.

Thus, Range of a series = Maximum value – Minimum value.

The range of data gives us a rough idea of variability or scatter but does not tell about the dispersion of the data from a measure of central tendency. For this purpose, we need some other measure of variability. Clearly, such measure must depend upon the difference (or deviation) of the values from the central tendency.

The important measures of dispersion, which depend upon the deviations of the observations from a central tendency are mean deviation and standard deviation. Let us discuss them in detail.

13.4 Mean Deviation

Recall that the deviation of an observation x from a fixed value *'a'* is the difference $x - a$. In order to find the dispersion of values of x from a central value *'a'*, we find the deviations about a. An absolute measure of dispersion is the mean of these deviations. To find the mean, we must obtain the sum of the deviations. But, we know that a measure of central tendency lies between the maximum and the minimum values of the set of observations. Therefore, some of the deviations will be negative and some positive. Thus, the sum of deviations may vanish. Moreover, the sum of the deviations from mean ($\overline{x}$) is zero.

Also $$\text{Mean of deviations} = \frac{\text{Sum of deviations}}{\text{Number of observations}} = \frac{0}{n} = 0$$

Thus, finding the mean of deviations about mean is not of any use for us, as far as the measure of dispersion is concerned.

Remember that, in finding a suitable measure of dispersion, we require the distance of each value from a central tendency or a fixed number 'a'. Recall, that the absolute value of the difference of two numbers gives the distance between the numbers when represented on a number line. Thus, to find the measure of dispersion from a fixed number 'a' we may take the mean of the absolute values of the deviations from the central value. This mean is called the '*mean deviation*'. Thus mean deviation about a central value 'a' is the mean of the absolute values of the deviations of the observations from 'a'. The mean deviation from 'a' is denoted as M.D. (a). Therefore,

$$\text{M.D.}(a) = \frac{\text{Sum of absolute values of deviations from '}a\text{'}}{\text{Number of observations}}.$$

Remark Mean deviation may be obtained from any measure of central tendency. However, mean deviation from mean and median are commonly used in statistical studies.

Let us now learn how to calculate mean deviation about mean and mean deviation about median for various types of data

13.4.1 ***Mean deviation for ungrouped data*** Let n observations be $x_1, x_2, x_3, ..., x_n$. The following steps are involved in the calculation of mean deviation about mean or median:

Step 1 Calculate the measure of central tendency about which we are to find the mean deviation. Let it be 'a'.

Step 2 Find the deviation of each x_i from a, i.e., $x_1 - a, x_2 - a, x_3 - a, . . . , x_n - a$

Step 3 Find the absolute values of the deviations, i.e., drop the minus sign (–), if it is there, i.e., $|x_1 - a|, |x_2 - a|, |x_3 - a|, ..., |x_n - a|$

Step 4 Find the mean of the absolute values of the deviations. This mean is the mean deviation about a, i.e.,

$$\text{M.D.}(a) = \frac{\sum_{i=1}^{n} |x_i - a|}{n}$$

Thus $$\text{M.D.}(\bar{x}) = \frac{1}{n}\sum_{i=1}^{n} |x_i - \bar{x}|, \text{ where } \bar{x} = \text{Mean}$$

and $$\text{M.D.}(\text{M}) = \frac{1}{n}\sum_{i=1}^{n} |x_i - \text{M}|, \text{ where M = Median}$$

Note In this Chapter, we shall use the symbol M to denote median unless stated otherwise.Let us now illustrate the steps of the above method in following examples.

Example 1 Find the mean deviation about the mean for the following data:

6, 7, 10, 12, 13, 4, 8, 12

Solution We proceed step-wise and get the following:

Step 1 Mean of the given data is

$$\bar{x} = \frac{6+7+10+12+13+4+8+12}{8} = \frac{72}{8} = 9$$

Step 2 The deviations of the respective observations from the mean $\bar{x}$, i.e., $x_i - \bar{x}$ are

6 – 9, 7 – 9, 10 – 9, 12 – 9, 13 – 9, 4 – 9, 8 – 9, 12 – 9,

or –3, –2, 1, 3, 4, –5, –1, 3

Step 3 The absolute values of the deviations, i.e., $|x_i - \bar{x}|$ are

3, 2, 1, 3, 4, 5, 1, 3

Step 4 The required mean deviation about the mean is

$$\text{M.D.}\ (\bar{x}) = \frac{\sum_{i=1}^{8}|x_i - \bar{x}|}{8}$$

$$= \frac{3+2+1+3+4+5+1+3}{8} = \frac{22}{8} = 2.75$$

Note Instead of carrying out the steps every time, we can carry on calculation, step-wise without referring to steps.

Example 2 Find the mean deviation about the mean for the following data :

12, 3, 18, 17, 4, 9, 17, 19, 20, 15, 8, 17, 2, 3, 16, 11, 3, 1, 0, 5

Solution We have to first find the mean ($\bar{x}$) of the given data

$$\bar{x} = \frac{1}{20}\sum_{i=1}^{20} x_i = \frac{200}{20} = 10$$

The respective absolute values of the deviations from mean, i.e., $|x_i - \bar{x}|$ are

2, 7, 8, 7, 6, 1, 7, 9, 10, 5, 2, 7, 8, 7, 6, 1, 7, 9, 10, 5

Therefore $$\sum_{i=1}^{20} |x_i - \bar{x}| = 124$$

and $$\text{M.D.}\ (\bar{x}) = \frac{124}{20} = 6.2$$

Example 3 Find the mean deviation about the median for the following data:

3, 9, 5, 3, 12, 10, 18, 4, 7, 19, 21.

Solution Here the number of observations is 11 which is odd. Arranging the data into ascending order, we have 3, 3, 4, 5, 7, 9, 10, 12, 18, 19, 21

Now $$\text{Median} = \left(\frac{11 + 1}{2}\right)^{\text{th}} \text{ or } 6^{\text{th}} \text{ observation} = 9$$

The absolute values of the respective deviations from the median, i.e., $|x_i - \text{M}|$ are

6, 6, 5, 4, 2, 0, 1, 3, 9, 10, 12

Therefore $$\sum_{i=1}^{11} |x_i - \text{M}| = 58$$

and $$\text{M.D.}\,(\text{M}) = \frac{1}{11}\sum_{i=1}^{11} |x_i - \text{M}| = \frac{1}{11} \times 58 = 5.27$$

13.4.2 *Mean deviation for grouped data* We know that data can be grouped into two ways :

(a) Discrete frequency distribution,

(b) Continuous frequency distribution.

Let us discuss the method of finding mean deviation for both types of the data.

(a) Discrete frequency distribution Let the given data consist of n distinct values $x_1, x_2, ..., x_n$ occurring with frequencies $f_1, f_2, ..., f_n$ respectively. This data can be represented in the tabular form as given below, and is called *discrete frequency distribution*:

$$\begin{array}{lllll} x : x_1 & x_2 & x_3 & \ldots & x_n \\ f : f_1 & f_2 & f_3 & \ldots & f_n \end{array}$$

(i) Mean deviation about mean

First of all we find the mean $\bar{x}$ of the given data by using the formula

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i f_i}{\sum_{i=1}^{n} f_i} = \frac{1}{N} \sum_{i=1}^{n} x_i f_i ,$$

where $\sum_{i=1}^{n} x_i f_i$ denotes the sum of the products of observations x_i with their respective frequencies f_i and $N = \sum_{i=1}^{n} f_i$ is the sum of the frequencies.

Then, we find the deviations of observations x_i from the mean $\bar{x}$ and take their absolute values, i.e., $|x_i - \bar{x}|$ for all $i = 1, 2, ..., n$.

After this, find the mean of the absolute values of the deviations, which is the required mean deviation about the mean. Thus

$$\text{M.D.}(\bar{x}) = \frac{\sum_{i=1}^{n} f_i |x_i - \bar{x}|}{\sum_{i=1}^{n} f_i} = \frac{1}{N} \sum_{i=1}^{n} f_i |x_i - \bar{x}|$$

(ii) Mean deviation about median To find mean deviation about median, we find the median of the given discrete frequency distribution. For this the observations are arranged in ascending order. After this the cumulative frequencies are obtained. Then, we identify the observation whose cumulative frequency is equal to or just greater than $\frac{N}{2}$, where N is the sum of frequencies. This value of the observation lies in the middle of the data, therefore, it is the required median. After finding median, we obtain the mean of the absolute values of the deviations from median.Thus,

$$\text{M.D.(M)} = \frac{1}{N} \sum_{i=1}^{n} f_i |x_i - M|$$

Example 4 Find mean deviation about the mean for the following data :

x_i	2	5	6	8	10	12
f_i	2	8	10	7	8	5

Solution Let us make a Table 13.1 of the given data and append other columns after calculations.

Table 13.1

x_i	f_i	$f_i x_i$	$\lvert x_i - \bar{x}\rvert$	$f_i \lvert x_i - \bar{x}\rvert$
2	2	4	5.5	11
5	8	40	2.5	20
6	10	60	1.5	15
8	7	56	0.5	3.5
10	8	80	2.5	20
12	5	60	4.5	22.5
	40	300		92

$$N = \sum_{i=1}^{6} f_i = 40, \quad \sum_{i=1}^{6} f_i x_i = 300, \quad \sum_{i=1}^{6} f_i \lvert x_i - \bar{x}\rvert = 92$$

Therefore $$\bar{x} = \frac{1}{N}\sum_{i=1}^{6} f_i x_i = \frac{1}{40} \times 300 = 7.5$$

and $$\text{M.D. } (\bar{x}) = \frac{1}{N}\sum_{i=1}^{6} f_i \, \lvert x_i - \bar{x}\rvert = \frac{1}{40} \times 92 = 2.3$$

Example 5 Find the mean deviation about the median for the following data:

x_i	3	6	9	12	13	15	21	22
f_i	3	4	5	2	4	5	4	3

Solution The given observations are already in ascending order. Adding a row corresponding to cumulative frequencies to the given data, we get (Table 13.2).

Table 13.2

x_i	3	6	9	12	13	15	21	22
f_i	3	4	5	2	4	5	4	3
$c.f.$	3	7	12	14	18	23	27	30

Now, N=30 which is even.

Median is the mean of the 15th and 16th observations. Both of these observations lie in the cumulative frequency 18, for which the corresponding observation is 13.

Therefore, Median M $= \dfrac{15^{\text{th}} \text{ observation} + 16^{\text{th}} \text{ observation}}{2} = \dfrac{13+13}{2} = 13$

Now, absolute values of the deviations from median, i.e., $|x_i - \text{M}|$ are shown in Table 13.3.

Table 13.3

$\lvert x_i - \text{M}\rvert$	10	7	4	1	0	2	8	9
f_{i}	3	4	5	2	4	5	4	3
$f_{\text{i}}\lvert x_i - \text{M}\rvert$	30	28	20	2	0	10	32	27

We have $\displaystyle\sum_{i=1}^{8} f_i = 30$ and $\displaystyle\sum_{i=1}^{8} f_i\,|x_i - \text{M}| = 149$

Therefore $$\text{M. D. (M)} = \frac{1}{\text{N}} \sum_{i=1}^{8} f_i |x_i - \text{M}|$$

$$= \frac{1}{30} \times 149 = 4.97.$$

(b) Continuous frequency distribution A continuous frequency distribution is a series in which the data are classified into different class-intervals without gaps alongwith their respective frequencies.

For example, marks obtained by 100 students are presented in a continuous frequency distribution as follows :

Marks obtained	0-10	10-20	20-30	30-40	40-50	50-60
Number of Students	12	18	27	20	17	6

(i) Mean deviation about mean While calculating the mean of a continuous frequency distribution, we had made the assumption that the frequency in each class is centred at its mid-point. Here also, we write the mid-point of each given class and proceed further as for a discrete frequency distribution to find the mean deviation.

Let us take the following example.

Example 6 Find the mean deviation about the mean for the following data.

Marks obtained	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Number of students	2	3	8	14	8	3	2

Solution We make the following Table 13.4 from the given data :

Table 13.4

Marks obtained	Number of students f_i	Mid-points x_i	$f_i x_i$	$\lvert x_i - \bar{x}\rvert$	$f_i\lvert x_i - \bar{x}\rvert$
10-20	2	15	30	30	60
20-30	3	25	75	20	60
30-40	8	35	280	10	80
40-50	14	45	630	0	0
50-60	8	55	440	10	80
60-70	3	65	195	20	60
70-80	2	75	150	30	60
	40		1800		400

Here $$N = \sum_{i=1}^{7} f_i = 40, \sum_{i=1}^{7} f_i x_i = 1800, \sum_{i=1}^{7} f_i \lvert x_i - \bar{x}\rvert = 400$$

Therefore $$\bar{x} = \frac{1}{N}\sum_{i=1}^{7} f_i x_i = \frac{1800}{40} = 45$$

and $$\text{M.D.}(\bar{x}) = \frac{1}{N}\sum_{i=1}^{7} f_i \lvert x_i - \bar{x}\rvert = \frac{1}{40} \times 400 = 10$$

Shortcut method for calculating mean deviation about mean We can avoid the tedious calculations of computing $\bar{x}$ by following step-deviation method. Recall that in this method, we take an assumed mean which is in the middle or just close to it in the data. Then deviations of the observations (or mid-points of classes) are taken from the assumed mean. This is nothing but the shifting of origin from zero to the assumed mean on the number line, as shown in Fig 13.3

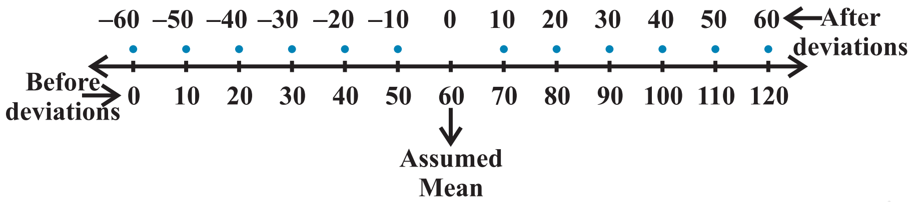


Fig 13.3

If there is a common factor of all the deviations, we divide them by this common factor to further simplify the deviations. These are known as step-deviations. The process of taking step-deviations is the change of scale on the number line as shown in Fig 13.4

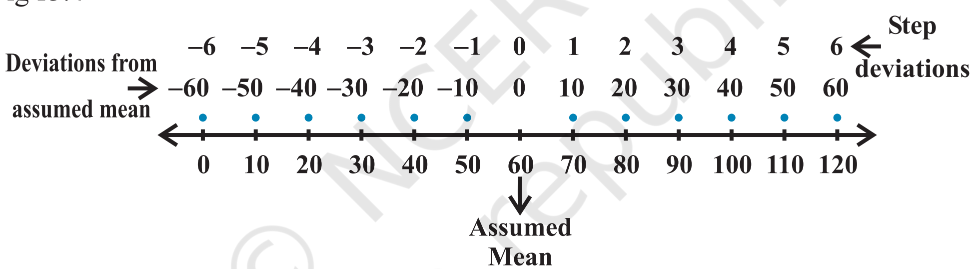


Fig 13.4

The deviations and step-deviations reduce the size of the observations, so that the computations viz. multiplication, etc., become simpler. Let, the new variable be denoted by $d_i = \dfrac{x_i - a}{h}$, where 'a' is the assumed mean and h is the common factor. Then, the mean $\bar{x}$ by step-deviation method is given by

$$\bar{x} = a + \frac{\sum_{i=1}^{n} f_i d_i}{\text{N}} \times h$$

Let us take the data of Example 6 and find the mean deviation by using step-deviation method.

Take the assumed mean $a = 45$ and $h = 10$, and form the following Table 13.5.

Table 13.5

Marks obtained	Number of students f_i	Mid-points x_i	$d_i = \frac{x_i - 45}{10}$	$f_i d_i$	$\lvert x_i - \bar{x} \rvert$	$f_i \lvert x_i - \bar{x} \rvert$
10-20	2	15	– 3	– 6	30	60
20-30	3	25	– 2	– 6	20	60
30-40	8	35	– 1	– 8	10	80
40-50	14	45	0	0	0	0
50-60	8	55	1	8	10	80
60-70	3	65	2	6	20	60
70-80	2	75	3	6	30	60
	40			0		400

Therefore

$$\bar{x} = a + \frac{\sum_{i=1}^{7} f_i d_i}{N} \times h$$

$$= 45 + \frac{0}{40} \times 10 = 45$$

and

$$\text{M.D. } (\bar{x}) = \frac{1}{N} \sum_{i=1}^{7} f_i \lvert x_i - \bar{x} \rvert = \frac{400}{40} = 10$$

Note The step deviation method is applied to compute $\bar{x}$. Rest of the procedure is same.

(ii) Mean deviation about median The process of finding the mean deviation about median for a continuous frequency distribution is similar as we did for mean deviation about the mean. The only difference lies in the replacement of the mean by median while taking deviations.

Let us recall the process of finding median for a continuous frequency distribution.

The data is first arranged in ascending order. Then, the median of continuous frequency distribution is obtained by first identifying the class in which median lies (median class) and then applying the formula

$$\text{Median} = l + \frac{\frac{N}{2} - C}{f} \times h$$

where median class is the class interval whose cumulative frequency is just greater than or equal to $\frac{N}{2}$, N is the sum of frequencies, l, f, h and C are, respectively the lower limit, the frequency, the width of the median class and C the cumulative frequency of the class just preceding the median class. After finding the median, the absolute values of the deviations of mid-point x_i of each class from the median i.e., $|x_i - M|$ are obtained.

Then $\quad \text{M.D. (M)} = \frac{1}{N} \sum_{i=1}^{n} f_i |x_i - M|$

The process is illustrated in the following example:

Example 7 Calculate the mean deviation about median for the following data :

Class	0-10	10-20	20-30	30-40	40-50	50-60
Frequency	6	7	15	16	4	2

Solution Form the following Table 13.6 from the given data :

Table 13.6

Class	Frequency	Cumulative frequency	Mid-points	$\|x_i - \text{Med.}\|$	$f_i \|x_i - \text{Med.}\|$
	f_i	$(c.f.)$	x_i		
0-10	6	6	5	23	138
10-20	7	13	15	13	91
20-30	15	28	25	3	45
30-40	16	44	35	7	112
40-50	4	48	45	17	68
50-60	2	50	55	27	54
	50				508

The class interval containing $\frac{N}{2}^{th}$ or 25th item is 20-30. Therefore, 20–30 is the median class. We know that

$$\text{Median} = l + \frac{\frac{N}{2} - C}{f} \times h$$

Here $l = 20$, $C = 13$, $f = 15$, $h = 10$ and $N = 50$

Therefore, $\qquad \text{Median} = 20 + \frac{25-13}{15} \times 10 = 20 + 8 = 28$

Thus, Mean deviation about median is given by

$$\text{M.D. (M)} = \frac{1}{N}\sum_{i=1}^{6} f_i |x_i - M| = \frac{1}{50} \times 508 = 10.16$$

EXERCISE 13.1

Find the mean deviation about the mean for the data in Exercises 1 and 2.

1. 4, 7, 8, 9, 10, 12, 13, 17

2. 38, 70, 48, 40, 42, 55, 63, 46, 54, 44

Find the mean deviation about the median for the data in Exercises 3 and 4.

3. 13, 17, 16, 14, 11, 13, 10, 16, 11, 18, 12, 17

4. 36, 72, 46, 42, 60, 45, 53, 46, 51, 49

Find the mean deviation about the mean for the data in Exercises 5 and 6.

5.

x_i	5	10	15	20	25
f_i	7	4	6	3	5

6.

x_i	10	30	50	70	90
f_i	4	24	28	16	8

Find the mean deviation about the median for the data in Exercises 7 and 8.

7.

x_i	5	7	9	10	12	15
f_i	8	6	2	2	2	6

8.

x_i	15	21	27	30	35
f_i	3	5	6	7	8

Find the mean deviation about the mean for the data in Exercises 9 and 10.

9.

Income per day in ₹	0-100	100-200	200-300	300-400	400-500	500-600	600-700	700-800
Number of persons	4	8	9	10	7	5	4	3

10.

Height in cms	95-105	105-115	115-125	125-135	135-145	145-155
Number of boys	9	13	26	30	12	10

11. Find the mean deviation about median for the following data :

Marks	0-10	10-20	20-30	30-40	40-50	50-60
Number of Girls	6	8	14	16	4	2

12. Calculate the mean deviation about median age for the age distribution of 100 persons given below:

Age (in years)	16-20	21-25	26-30	31-35	36-40	41-45	46-50	51-55
Number	5	6	12	14	26	12	16	9

[**Hint** Convert the given data into continuous frequency distribution by subtracting 0.5 from the lower limit and adding 0.5 to the upper limit of each class interval]

13.4.3 ***Limitations of mean deviation*** In a series, where the degree of variability is very high, the median is not a representative central tendency. Thus, the mean deviation about median calculated for such series can not be fully relied.

The sum of the deviations from the mean (minus signs ignored) is more than the sum of the deviations from median. Therefore, the mean deviation about the mean is not very scientific.Thus, in many cases, mean deviation may give unsatisfactory results. Also mean deviation is calculated on the basis of absolute values of the deviations and therefore, cannot be subjected to further algebraic treatment. This implies that we must have some other measure of dispersion. Standard deviation is such a measure of dispersion.

13.5 Variance and Standard Deviation

Recall that while calculating mean deviation about mean or median, the absolute values of the deviations were taken. The absolute values were taken to give meaning to the mean deviation, otherwise the deviations may cancel among themselves.

Another way to overcome this difficulty which arose due to the signs of deviations, is to take squares of all the deviations. Obviously all these squares of deviations are non-negative. Let $x_1, x_2, x_3, ..., x_n$ be n observations and $\bar{x}$ be their mean. Then

$$(x_1-\bar{x})^2 + (x_2 - \bar{x})^2 + ...+ (x_n-\bar{x})^2 = \sum_{i=1}^{n} (x_i-\bar{x})^2 .$$

If this sum is zero, then each $(x_i - \bar{x})$ has to be zero. This implies that there is no dispersion at all as all observations are equal to the mean $\bar{x}$.

If $\sum_{i=1}^{n}(x_i - \bar{x})^2$ is small , this indicates that the observations $x_1, x_2, x_3,...,x_n$ are close to the mean $\bar{x}$ and therefore, there is a lower degree of dispersion. On the contrary, if this sum is large, there is a higher degree of dispersion of the observations from the mean $\bar{x}$. Can we thus say that the sum $\sum_{i=1}^{n}(x_i - \bar{x})^2$ is a reasonable indicator of the degree of dispersion or scatter?

Let us take the set A of six observations 5, 15, 25, 35, 45, 55. The mean of the observations is $\bar{x}$ = 30. The sum of squares of deviations from $\bar{x}$ for this set is

$$\sum_{i=1}^{6}(x_i - \bar{x})^2 = (5–30)^2 + (15–30)^2 + (25–30)^2 + (35–30)^2 + (45–30)^2 +(55–30)^2$$

$$= 625 + 225 + 25 + 25 + 225 + 625 = 1750$$

Let us now take another set B of 31 observations 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45. The mean of these observations is $\bar{y} = 30$

Note that both the sets A and B of observations have a mean of 30.

Now, the sum of squares of deviations of observations for set B from the mean $\bar{y}$ is given by

$$\sum_{i=1}^{31}(y_i - \bar{y})^2 = (15–30)^2 +(16–30)^2 + (17–30)^2 + ...+ (44–30)^2 +(45–30)^2$$

$$= (–15)^2 +(–14)^2 + ...+ (–1)^2 + 0^2 + 1^2 + 2^2 + 3^2 + ...+ 14^2 + 15^2$$

$$= 2 [15^2 + 14^2 + ... + 1^2]$$

$$= 2 \times \frac{15 \times (15+1)\,(30+1)}{6} = 5 \times 16 \times 31 = 2480$$

(Because sum of squares of first n natural numbers = $\frac{n\,(n+1)\,(2n+1)}{6}$. Here n = 15)

If $\sum_{i=1}^{n}(x_i - \bar{x})^2$ is simply our measure of dispersion or scatter about mean, we will tend to say that the set A of six observations has a lesser dispersion about the mean than the set B of 31 observations, even though the observations in set A are more scattered from the mean (the range of deviations being from –25 to 25) than in the set B (where the range of deviations is from –15 to 15).

This is also clear from the following diagrams.

For the set A, we have

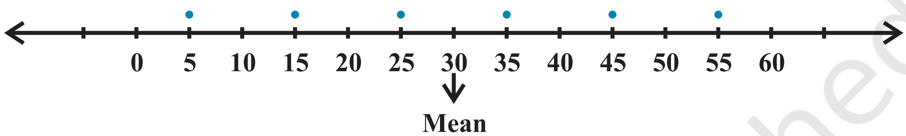


Fig 13.5

For the set B, we have

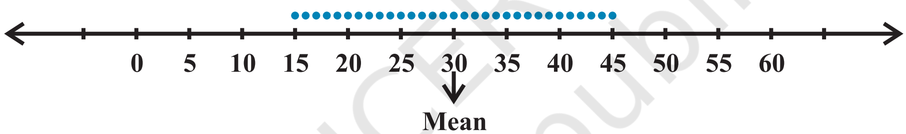


Fig 13.6

Thus, we can say that the sum of squares of deviations from the mean is not a proper measure of dispersion. To overcome this difficulty we take the mean of the squares of the deviations, i.e., we take $\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$. In case of the set A, we have

Mean $= \frac{1}{6} \times 1750 = 291.67$ and in case of the set B, it is $\frac{1}{31} \times 2480 = 80$.

This indicates that the scatter or dispersion is more in set A than the scatter or dispersion in set B, which confirms with the geometrical representation of the two sets.

Thus, we can take $\frac{1}{n}\sum(x_i - \bar{x})^2$ as a quantity which leads to a proper measure of dispersion. This number, i.e., mean of the squares of the deviations from mean is called the ***variance*** and is denoted by σ^2 (read as sigma square). Therefore, the variance of n observations $x_1, x_2, \ldots, x_n$ is given by

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

13.5.1 ***Standard Deviation*** In the calculation of variance, we find that the units of individual observations x_i and the unit of their mean $\bar{x}$ are different from that of variance, since variance involves the sum of squares of $(x_i - \bar{x})$. For this reason, the proper measure of dispersion about the mean of a set of observations is expressed as positive square-root of the variance and is called *standard deviation*. Therefore, the standard deviation, usually denoted by σ, is given by

$$\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2} \quad \text{... (1)}$$

Let us take the following example to illustrate the calculation of variance and hence, standard deviation of ungrouped data.

Example 8 Find the variance of the following data:

6, 8, 10, 12, 14, 16, 18, 20, 22, 24

Solution From the given data we can form the following Table 13.7. The mean is calculated by step-deviation method taking 14 as assumed mean. The number of observations is $n = 10$

Table 13.7

x_i	$d_i = \frac{x_i - 14}{2}$	Deviations from mean $(x_i - \bar{x})$	$(x_i - \bar{x})$
6	–4	–9	81
8	–3	–7	49
10	–2	–5	25
12	–1	–3	9
14	0	–1	1
16	1	1	1
18	2	3	9
20	3	5	25
22	4	7	49
24	5	9	81
	5		330

Therefore Mean $\bar{x}$ = assumed mean + $\dfrac{\sum_{i=1}^{n} d_i}{n} \times h$ = $14 + \dfrac{5}{10} \times 2 = 15$

and Variance (σ^2) = $\dfrac{1}{n}\sum_{i=1}^{10}(x_i - \bar{x})^2 = \dfrac{1}{10} \times 330 = 33$

Thus Standard deviation (σ) = $\sqrt{33} = 5.74$

13.5.2 *Standard deviation of a discrete frequency distribution* Let the given discrete frequency distribution be

$$x: \quad x_1, \; x_2, \; x_3, \ldots, x_n$$
$$f: \quad f_1, \; f_2, \; f_3, \ldots, f_n$$

In this case standard deviation $(\sigma) = \sqrt{\dfrac{1}{N}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2}$ ... (2)

where $N = \sum_{i=1}^{n} f_i$.

Let us take up following example.

Example 9 Find the variance and standard deviation for the following data:

x_i	4	8	11	17	20	24	32
f_i	3	5	9	5	4	3	1

Solution Presenting the data in tabular form (Table 13.8), we get

Table 13.8

x_i	f_i	$f_i x_i$	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
4	3	12	–10	100	300
8	5	40	–6	36	180
11	9	99	–3	9	81
17	5	85	3	9	45
20	4	80	6	36	144
24	3	72	10	100	300
32	1	32	18	324	324
	30	420			1374

$$\mathrm{N} = 30,\ \sum_{i=1}^{7} f_i x_i = 420,\ \sum_{i=1}^{7} f_i \left(x_i - \bar{x}\right)^2 = 1374$$

Therefore $\bar{x} = \dfrac{\sum_{i=1}^{7} f_i x_i}{\mathrm{N}} = \dfrac{1}{30} \times 420 = 14$

Hence variance $(\sigma^2) = \dfrac{1}{\mathrm{N}} \sum_{i=1}^{7} f_i (x_i - \bar{x})^2$

$$= \frac{1}{30} \times 1374 = 45.8$$

and Standard deviation $(\sigma) = \sqrt{45.8} = 6.77$

13.5.3 ***Standard deviation of a continuous frequency distribution*** The given continuous frequency distribution can be represented as a discrete frequency distribution by replacing each class by its mid-point. Then, the standard deviation is calculated by the technique adopted in the case of a discrete frequency distribution.

If there is a frequency distribution of n classes each class defined by its mid-point x_i with frequency f_i, the standard deviation will be obtained by the formula

$$\sigma = \sqrt{\frac{1}{\mathrm{N}} \sum_{i=1}^{n} f_i (x_i - \bar{x})^2},$$

where $\bar{x}$ is the mean of the distribution and $\mathrm{N} = \sum_{i=1}^{n} f_i$.

Another formula for standard deviation We know that

$$\text{Variance } (\sigma^2) = \frac{1}{\mathrm{N}} \sum_{i=1}^{n} f_i (x_i - \bar{x})^2 = \frac{1}{\mathrm{N}} \sum_{i=1}^{n} f_i (x_i^2 + \bar{x}^2 - 2\bar{x}\, x_i)$$

$$= \frac{1}{\mathrm{N}} \left[\sum_{i=1}^{n} f_i x_i^2 + \sum_{i=1}^{n} \bar{x}^2 f_i - \sum_{i=1}^{n} 2\bar{x}\, f_i x_i \right]$$

$$= \frac{1}{\mathrm{N}} \left[\sum_{i=1}^{n} f_i x_i^2 + \bar{x}^2 \sum_{i=1}^{n} f_i - 2\bar{x} \sum_{i=1}^{n} x_i f_i \right]$$

$$= \frac{1}{N} \sum_{i=1}^{n} f_i x_i^2 + \bar{x}^2 N - 2\bar{x} . N \bar{x} \quad \left[\text{Here } \frac{1}{N}\sum_{i=1}^{n} x_i f_i = \bar{x} \text{ or } \sum_{i=1}^{n} x_i f_i = N\bar{x} \right]$$

$$= \frac{1}{N} \sum_{i=1}^{n} f_i x_i^2 + \bar{x}^2 - 2\bar{x}^2 = \frac{1}{N} \sum_{i=1}^{n} f_i x_i^2 - \bar{x}^2$$

or $$\sigma^2 = \frac{1}{N} \sum_{i=1}^{n} f_i x_i^2 - \left(\frac{\sum_{i=1}^{n} f_i x_i}{N} \right)^2 = \frac{1}{N^2} \left[N \sum_{i=1}^{n} f_i x_i^2 - \left(\sum_{i=1}^{n} f_i x_i \right)^2 \right]$$

Thus, standard deviation $$(\sigma) = \frac{1}{N} \sqrt{N \sum_{i=1}^{n} f_i x_i^2 - \left(\sum_{i=1}^{n} f_i x_i \right)^2} \quad \text{... (3)}$$

Example 10 Calculate the mean, variance and standard deviation for the following distribution :

Class	30-40	40-50	50-60	60-70	70-80	80-90	90-100
Frequency	3	7	12	15	8	3	2

Solution From the given data, we construct the following Table 13.9.

Table 13.9

Class	Frequency (f_i)	Mid-point (x_i)	$f_i x_i$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
30-40	3	35	105	729	2187
40-50	7	45	315	289	2023
50-60	12	55	660	49	588
60-70	15	65	975	9	135
70-80	8	75	600	169	1352
80-90	3	85	255	529	1587
90-100	2	95	190	1089	2178
	50		3100		10050

Thus Mean $\bar{x} = \frac{1}{N}\sum_{i=1}^{7} f_i x_i = \frac{3100}{50} = 62$

Variance $(\sigma^2) = \frac{1}{N}\sum_{i=1}^{7} f_i (x_i - \bar{x})^2$

$= \frac{1}{50} \times 10050 = 201$

and Standard deviation $(\sigma) = \sqrt{201} = 14.18$

Example 11 Find the standard deviation for the following data :

x_i	3	8	13	18	23
f_i	7	10	15	10	6

Solution Let us form the following Table 13.10:

Table 13.10

x_i	f_i	$f_i x_i$	x_i^2	$f_i x_i^2$
3	7	21	9	63
8	10	80	64	640
13	15	195	169	2535
18	10	180	324	3240
23	6	138	529	3174
	48	614		9652

Now, by formula (3), we have

$$\sigma = \frac{1}{N}\sqrt{N\sum f_i x_i^2 - \left(\sum f_i x_i\right)^2}$$

$$= \frac{1}{48}\sqrt{48 \times 9652 - (614)^2}$$

$$= \frac{1}{48}\sqrt{463296 - 376996}$$

$$= \frac{1}{48} \times 293.77 = 6.12$$

Therefore, Standard deviation $(\sigma) = 6.12$

13.5.4. ***Shortcut method to find variance and standard deviation*** Sometimes the values of x_i in a discrete distribution or the mid points x_i of different classes in a continuous distribution are large and so the calculation of mean and variance becomes tedious and time consuming. By using step-deviation method, it is possible to simplify the procedure.

Let the assumed mean be 'A' and the scale be reduced to $\frac{1}{h}$ times (h being the width of class-intervals). Let the step-deviations or the new values be y_i.

i.e. $$y_i = \frac{x_i - \text{A}}{h} \quad \text{or} \quad x_i = \text{A} + hy_i \qquad \text{... (1)}$$

We know that $$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\text{N}} \qquad \text{... (2)}$$

Replacing x_i from (1) in (2), we get

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i(\text{A} + hy_i)}{\text{N}}$$

$$= \frac{1}{\text{N}}\left(\sum_{i=1}^{n} f_i\text{A} + \sum_{i=1}^{n} h\, f_i y_i\right) = \frac{1}{\text{N}}\left(\text{A}\sum_{i=1}^{n} f_i + h\sum_{i=1}^{n} f_i y_i\right)$$

$$= \text{A} \cdot \frac{\text{N}}{\text{N}} + h\frac{\sum_{i=1}^{n} f_i y_i}{\text{N}} \qquad \left(\text{because} \sum_{i=1}^{n} f_i = \text{N}\right)$$

Thus $$\bar{x} = \text{A} + h\,\bar{y} \qquad \text{... (3)}$$

Now Variance of the variable x, $\sigma_x^2 = \frac{1}{\text{N}}\sum_{i=1}^{n} f_i(x_i - \bar{x})^2$

$$= \frac{1}{\text{N}}\sum_{i=1}^{n} f_i\,(\text{A} + hy_i - \text{A} - h\,\bar{y})^2 \qquad \text{(Using (1) and (3))}$$

$$= \frac{1}{N}\sum_{i=1}^{n} f_i\, h^2 (y_i - \bar{y})^2$$

$$= \frac{h^2}{N}\sum_{i=1}^{n} f_i (y_i - \bar{y})^2 = h^2 \times \text{variance of the variable } y_i$$

i.e. $\sigma_x^{\ 2} = h^2 \sigma_y^{\ 2}$

or $\sigma_x = h\sigma_y$ ... (4)

From (3) and (4), we have

$$\sigma_x = \frac{h}{N}\sqrt{N\sum_{i=1}^{n} f_i y_i^{\ 2} - \left(\sum_{i=1}^{n} f_i y_i\right)^2} \quad \text{... (5)}$$

Let us solve Example 11 by the short-cut method and using formula (5)

Examples 12 Calculate mean, variance and standard deviation for the following distribution.

Classes	30-40	40-50	50-60	60-70	70-80	80-90	90-100
Frequency	3	7	12	15	8	3	2

Solution Let the assumed mean A = 65. Here h = 10

We obtain the following Table 13.11 from the given data :

Table 13.11

Class	Frequency f_i	Mid-point x_i	$y_i = \frac{x_i - 65}{10}$	$y_i^{\ 2}$	$f_i\, y_i$	$f_i\, y_i^{\ 2}$
30-40	3	35	– 3	9	– 9	27
40-50	7	45	– 2	4	– 14	28
50-60	12	55	– 1	1	– 12	12
60-70	15	65	0	0	0	0
70-80	8	75	1	1	8	8
80-90	3	85	2	4	6	12
9 0-100	2	95	3	9	6	18
	N=50				– 15	105

Therefore $\qquad \bar{x} = A + \frac{\sum f_i y_i}{50} \times h = 65 - \frac{15}{50} \times 10 = 62$

Variance $\qquad \sigma^2 = \frac{h^2}{N^2}\left[N\Sigma f_i y_i^2 - \left(\Sigma f_i y_i\right)^2\right]$

$$= \frac{(10)^2}{(50)^2}\left[50 \times 105 - (-15)^2\right]$$

$$= \frac{1}{25}[5250 - 225] = 201$$

and standard deviation $(\sigma) = \sqrt{201} = 14.18$

EXERCISE 13.2

Find the mean and variance for each of the data in Exercises 1 to 5.

1. 6, 7, 10, 12, 13, 4, 8, 12

2. First n natural numbers

3. First 10 multiples of 3

4.

x_i	6	10	14	18	24	28	30
f_i	2	4	7	12	8	4	3

5.

x_i	92	93	97	98	102	104	109
f_i	3	2	3	2	6	3	3

6. Find the mean and standard deviation using short-cut method.

x_i	60	61	62	63	64	65	66	67	68
f_i	2	1	12	29	25	12	10	4	5

Find the mean and variance for the following frequency distributions in Exercises 7 and 8.

7.

Classes	0-30	30-60	60-90	90-120	120-150	150-180	180-210
Frequencies	2	3	5	10	3	5	2

8.

Classes	0-10	10-20	20-30	30-40	40-50
Frequencies	5	8	15	16	6

9. Find the mean, variance and standard deviation using short-cut method

Height in cms	70-75	75-80	80-85	85-90	90-95	95-100	100-105	105-110	110-115
No. of children	3	4	7	7	15	9	6	6	3

10. The diameters of circles (in mm) drawn in a design are given below:

Diameters	33-36	37-40	41-44	45-48	49-52
No. of circles	15	17	21	22	25

Calculate the standard deviation and mean diameter of the circles.

[**Hint** First make the data continuous by making the classes as 32.5-36.5, 36.5-40.5, 40.5-44.5, 44.5 - 48.5, 48.5 - 52.5 and then proceed.]

Miscellaneous Examples

Example 13 The variance of 20 observations is 5. If each observation is multiplied by 2, find the new variance of the resulting observations.

Solution Let the observations be $x_1, x_2, ..., x_{20}$ and $\bar{x}$ be their mean. Given that variance = 5 and n = 20. We know that

$$\text{Variance } \left(\sigma^2\right) = \frac{1}{n}\sum_{i=1}^{20}(x_i - \bar{x})^2 \text{ , i.e., } 5 = \frac{1}{20}\sum_{i=1}^{20}(x_i - \bar{x})^2$$

or
$$\sum_{i=1}^{20}(x_i - \bar{x})^2 = 100$$

If each observation is multiplied by 2, and the new resulting observations are y_i, then

$$y_i = 2x_i \text{ i.e., } x_i = \frac{1}{2}y_i \qquad \text{... (1)}$$

Therefore $$\bar{y} = \frac{1}{n}\sum_{i=1}^{20} y_i = \frac{1}{20}\sum_{i=1}^{20} 2x_i = 2\,.\,\frac{1}{20}\sum_{i=1}^{20} x_i$$

i.e. $$\bar{y} = 2\,\bar{x} \quad \text{or} \quad \bar{x} = \frac{1}{2}\bar{y}$$

Substituting the values of x_i and $\bar{x}$ in (1), we get

$$\sum_{i=1}^{20}\left(\frac{1}{2}y_i - \frac{1}{2}\bar{y}\right)^2 = 100, \text{ i.e., } \sum_{i=1}^{20}(y_i - \bar{y})^2 = 400$$

Thus the variance of new observations $= \frac{1}{20} \times 400 = 20 = 2^2 \times 5$

Note The reader may note that if each observation is multiplied by a constant k, the variance of the resulting observations becomes k^2 times the original variance.

Example14 The mean of 5 observations is 4.4 and their variance is 8.24. If three of the observations are 1, 2 and 6, find the other two observations.

Solution Let the other two observations be x and y.
Therefore, the series is 1, 2, 6, x, y.

Now Mean $\bar{x} = 4.4 = \dfrac{1+2+6+x+y}{5}$

or $22 = 9 + x + y$

Therefore $x + y = 13$ ... (1)

Also variance $= 8.24 = \dfrac{1}{n}\sum_{i=1}^{5}(x_i - \bar{x})^2$

i.e. $$8.24 = \frac{1}{5}\left[(3.4)^2 + (2.4)^2 + (1.6)^2 + x^2 + y^2 - 2\times 4.4\,(x+y) + 2\times(4.4)^2\right]$$

or $41.20 = 11.56 + 5.76 + 2.56 + x^2 + y^2 - 8.8 \times 13 + 38.72$

Therefore $x^2 + y^2 = 97$ ... (2)

But from (1), we have

$x^2 + y^2 + 2xy = 169$ ... (3)

From (2) and (3), we have

$2xy = 72$ ... (4)

Subtracting (4) from (2), we get

$$x^2 + y^2 - 2xy = 97 - 72 \text{ i.e. } (x - y)^2 = 25$$

or $$x - y = \pm 5 \quad \text{... (5)}$$

So, from (1) and (5), we get

$x = 9, y = 4$ when $x - y = 5$

or $x = 4, y = 9$ when $x - y = -5$

Thus, the remaining observations are 4 and 9.

Example 15 If each of the observation $x_1, x_2, ..., x_n$ is increased by 'a', where a is a negative or positive number, show that the variance remains unchanged.

Solution Let $\bar{x}$ be the mean of $x_1, x_2, ..., x_n$. Then the variance is given by

$$\sigma_1^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

If 'a is added to each observation, the new observations will be

$$y_i = x_i + a \quad \text{... (1)}$$

Let the mean of the new observations be $\bar{y}$. Then

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{1}{n}\sum_{i=1}^{n}(x_i + a)$$

$$= \frac{1}{n}\left[\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} a\right] = \frac{1}{n}\sum_{i=1}^{n} x_i + \frac{na}{n} = \bar{x} + a$$

i.e. $$\bar{y} = \bar{x} + a \quad \text{... (2)}$$

Thus, the variance of the new observations

$$\sigma_2^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i + a - \bar{x} - a)^2 \quad \text{[Using (1) and (2)]}$$

$$= \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sigma_1^2$$

Thus, the variance of the new observations is same as that of the original observations.

Note We may note that adding (or subtracting) a positive number to (or from) each observation of a group does not affect the variance.

Example 16 The mean and standard deviation of 100 observations were calculated as 40 and 5.1, respectively by a student who took by mistake 50 instead of 40 for one observation. What are the correct mean and standard deviation?

Solution Given that number of observations (n) = 100

Incorrect mean ($\bar{x}$) = 40,

Incorrect standard deviation (σ) = 5.1

We know that $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

i.e. $40 = \frac{1}{100}\sum_{i=1}^{100} x_i$ or $\sum_{i=1}^{100} x_i = 4000$

i.e. Incorrect sum of observations = 4000

Thus the correct sum of observations = Incorrect sum – 50 + 40

= 4000 – 50 + 40 = 3990

Hence Correct mean $= \frac{\text{correct sum}}{100} = \frac{3990}{100} = 39.9$

Also Standard deviation $\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \frac{1}{n^2}\left(\sum_{i=1}^{n} x_i\right)^2}$

$= \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - (\bar{x})^2}$

i.e. $5.1 = \sqrt{\frac{1}{100} \times \text{Incorrect} \sum_{i=1}^{n} x_i^2 - (40)^2}$

or $26.01 = \frac{1}{100} \times \text{Incorrect} \sum_{i=1}^{n} x_i^2 - 1600$

Therefore $\text{Incorrect} \sum_{i=1}^{n} x_i^2 = 100\,(26.01 + 1600) = 162601$

Now $\text{Correct} \sum_{i=1}^{n} x_i^2 = \text{Incorrect} \sum_{i=1}^{n} x_i^2 - (50)^2 + (40)^2$

= 162601 – 2500 + 1600 = 161701

Therefore Correct standard deviation

$$= \sqrt{\frac{\text{Correct } \sum x_i^2}{n} - (\text{Correct mean})^2}$$

$$= \sqrt{\frac{161701}{100} - (39.9)^2}$$

$$= \sqrt{1617.01 - 1592.01} = \sqrt{25} = 5$$

Miscellaneous Exercise On Chapter 13

1. The mean and variance of eight observations are 9 and 9.25, respectively. If six of the observations are 6, 7, 10, 12, 12 and 13, find the remaining two observations.

2. The mean and variance of 7 observations are 8 and 16, respectively. If five of the observations are 2, 4, 10, 12, 14. Find the remaining two observations.

3. The mean and standard deviation of six observations are 8 and 4, respectively. If each observation is multiplied by 3, find the new mean and new standard deviation of the resulting observations.

4. Given that $\bar{x}$ is the mean and σ^2 is the variance of n observations $x_1, x_2, ..., x_n$. Prove that the mean and variance of the observations $ax_1, ax_2, ax_3, ..., ax_n$ are $a\bar{x}$ and $a^2\sigma^2$, respectively, $(a \neq 0)$.

5. The mean and standard deviation of 20 observations are found to be 10 and 2, respectively. On rechecking, it was found that an observation 8 was incorrect. Calculate the correct mean and standard deviation in each of the following cases:
(i) If wrong item is omitted. (ii) If it is replaced by 12.

6. The mean and standard deviation of a group of 100 observations were found to be 20 and 3, respectively. Later on it was found that three observations were incorrect, which were recorded as 21, 21 and 18. Find the mean and standard deviation if the incorrect observations are omitted.

Summary

- **Measures of dispersion** Range, Quartile deviation, mean deviation, variance, standard deviation are measures of dispersion.
Range = Maximum Value – Minimum Value
- **Mean deviation for ungrouped data**

$$\text{M.D.}(\bar{x}) = \frac{\sum |x_i - \bar{x}|}{n}, \qquad \text{M.D. (M)} = \frac{\sum |x_i - \text{M}|}{n}$$

- **Mean deviation for grouped data**

$$\text{M.D.}(\bar{x}) = \frac{\sum f_i |x_i \quad \bar{x}|}{\text{N}}, \qquad \text{M.D.}(\text{M}) = \frac{\sum f_i |x_i \quad \text{M}|}{\text{N}}, \text{ where N} = \sum f_i$$

- **Variance and standard deviation for ungrouped data**

$$\sigma^2 = \frac{1}{n} \sum (x_i - \bar{x})^2, \qquad \sigma = \sqrt{\frac{1}{n} \sum (x_i - \bar{x})^2}$$

- **Variance and standard deviation of a discrete frequency distribution**

$$\sigma^2 = \frac{1}{\text{N}} \sum f_i (x_i - \bar{x})^2, \qquad \sigma = \sqrt{\frac{1}{\text{N}} \sum f_i (x_i - \bar{x})^2}$$

- **Variance and standard deviation of a continuous frequency distribution**

$$\sigma^2 = \frac{1}{\text{N}} \sum f_i (x_i - \bar{x})^2, \qquad \sigma = \frac{1}{\text{N}} \sqrt{\text{N}\sum f_i x_i^2 - \left(\sum f_i x_i\right)^2}$$

- **Shortcut method to find variance and standard deviation.**

$$\sigma^2 = \frac{h^2}{\text{N}^2} \left[\text{N}\sum f_i y_i^2 - \left(\sum f_i y_i\right)^2\right], \sigma = \frac{h}{\text{N}} \sqrt{\text{N}\sum f_i y_i^2 - \left(\sum f_i y_i\right)^2},$$

where $y_i = \dfrac{x_i - \text{A}}{h}$

Historical Note

'Statistics' is derived from the Latin word 'status' which means a political state. This suggests that statistics is as old as human civilisation. In the year 3050 B.C., perhaps the first census was held in Egypt. In India also, about 2000 years ago, we had an efficient system of collecting administrative statistics, particularly, during the regime of Chandra Gupta Maurya (324-300 B.C.). The system of collecting data related to births and deaths is mentioned in Kautilya's *Arthshastra* (around 300 B.C.) A detailed account of administrative surveys conducted during Akbar's regime is given in *Ain-I-Akbari* written by Abul Fazl.

Captain John Graunt of London (1620-1674) is known as father of vital statistics due to his studies on statistics of births and deaths. Jacob Bernoulli (1654-1705) stated the Law of Large numbers in his book "Ars Conjectandi', published in 1713.

The theoretical development of statistics came during the mid seventeenth century and continued after that with the introduction of theory of games and chance (i.e., probability). Francis Galton (1822-1921), an Englishman, pioneered the use of statistical methods, in the field of Biometry. Karl Pearson (1857-1936) contributed a lot to the development of statistical studies with his discovery of *Chi square test* and foundation of *statistical laboratory* in England (1911). Sir Ronald A. Fisher (1890-1962), known as the Father of modern statistics, applied it to various diversified fields such as Genetics, Biometry, Education, Agriculture, etc.

Chapter 14

PROBABILITY

❖ *Where a mathematical reasoning can be had, it is as great a folly to make use of any other, as to grope for a thing in the dark, when you have a candle in your hand. – JOHN ARBUTHNOT* ❖

14.1 Event

We have studied about random experiment and sample space associated with an experiment. The sample space serves as an universal set for all questions concerned with the experiment.

Consider the experiment of tossing a coin two times. An associated sample space is S = {HH, HT, TH, TT}.

Now suppose that we are interested in those outcomes which correspond to the occurrence of exactly one head. We find that HT and TH are the only elements of S corresponding to the occurrence of this happening (event). These two elements form the set E = { HT, TH}

We know that the set E is a subset of the sample space S . Similarly, we find the following correspondence between events and subsets of S.

Description of events	Corresponding subset of 'S'
Number of tails is exactly 2	A = {TT}
Number of tails is atleast one	B = {HT, TH, TT}
Number of heads is atmost one	C = {HT, TH, TT}
Second toss is not head	D = { HT, TT}
Number of tails is atmost two	S = {HH, HT, TH, TT}
Number of tails is more than two	ϕ

The above discussion suggests that a subset of sample space is associated with an event and an event is associated with a subset of sample space. In the light of this we define an event as follows.

Definition Any subset E of a sample space S is called *an event*.

14.1.1 ***Occurrence of an event*** Consider the experiment of throwing a die. Let E denotes the event " a number less than 4 appears". If actually '1' had appeared on the die then we say that event E has occurred. As a matter of fact if outcomes are 2 or 3, we say that event E has occurred

Thus, the event E of a sample space S is said to have occurred if the outcome ω of the experiment is such that $\omega \in$ E. If the outcome ω is such that $\omega \notin$ E, we say that the event E has not occurred.

14.1.2 ***Types of events*** Events can be classified into various types on the basis of the elements they have.

1. Impossible and Sure Events The empty set ϕ and the sample space S describe events. In fact ϕ is called an *impossible event* and S, i.e., the whole sample space is called the *sure event*.

To understand these let us consider the experiment of rolling a die. The associated sample space is

$$S = \{1, 2, 3, 4, 5, 6\}$$

Let E be the event " the number appears on the die is a multiple of 7". Can you write the subset associated with the event E?

Clearly no outcome satisfies the condition given in the event, i.e., no element of the sample space ensures the occurrence of the event E. Thus, we say that the empty set only correspond to the event E. In other words we can say that it is impossible to have a multiple of 7 on the upper face of the die. Thus, the event E = ϕ is an impossible event.

Now let us take up another event F "the number turns up is odd or even". Clearly F = {1, 2, 3, 4, 5, 6,} = S, i.e., all outcomes of the experiment ensure the occurrence of the event F. Thus, the event F = S is a sure event.

2. Simple Event If an event E has only one sample point of a sample space, it is called a *simple* (or *elementary*) *event*.

In a sample space containing n distinct elements, there are exactly n simple events.

For example in the experiment of tossing two coins, a sample space is

$$S=\{HH, HT, TH, TT\}$$

There are four simple events corresponding to this sample space. These are

$$E_1= \{HH\},\ E_2=\{HT\},\ E_3= \{ TH\} \text{ and } E_4=\{TT\}.$$

3. Compound Event If an event has more than one sample point, it is called a *Compound event*.

For example, in the experiment of "tossing a coin thrice" the events

E: 'Exactly one head appeared'

F: 'Atleast one head appeared'

G: 'Atmost one head appeared' etc.

are all compound events. The subsets of S associated with these events are

E={HTT,THT,TTH}

F={HTT,THT, TTH, HHT, HTH, THH, HHH}

G= {TTT, THT, HTT, TTH}

Each of the above subsets contain more than one sample point, hence they are all compound events.

14.1.3 ***Algebra of events*** In the Chapter on Sets, we have studied about different ways of combining two or more sets, viz, union, intersection, difference, complement of a set etc. Like-wise we can combine two or more events by using the analogous set notations.

Let A, B, C be events associated with an experiment whose sample space is S.

1. Complementary Event For every event A, there corresponds another event A' called the complementary event to A. It is also called the *event 'not* A'.

For example, take the experiment 'of tossing three coins'. An associated sample space is

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Let A={HTH, HHT, THH} be the event 'only one tail appears'

Clearly for the outcome HTT, the event A has not occurred. But we may say that the event 'not A' has occurred. Thus, with every outcome which is not in A, we say that 'not A' occurs.

Thus the complementary event 'not A' to the event A is

$$A' = \{HHH, HTT, THT, TTH, TTT\}$$

or $$A' = \{\omega : \omega \in S \text{ and } \omega \notin A\} = S - A.$$

2. The Event 'A or B' Recall that union of two sets A and B denoted by $A \cup B$ contains all those elements which are either in A or in B or in both.

When the sets A and B are two events associated with a sample space, then '$A \cup B$' is the event 'either A or B or both'. This event '$A \cup B$' is also called 'A or B'.

Therefore $$\text{Event 'A or B'} = A \cup B$$
$$= \{\omega : \omega \in A \text{ or } \omega \in B\}$$

3. The Event 'A and B' We know that intersection of two sets $A \cap B$ is the set of those elements which are common to both A and B. i.e., which belong to both 'A and B'.

If A and B are two events, then the set $A \cap B$ denotes the event 'A and B'.

Thus, $A \cap B = \{\omega : \omega \in A \text{ and } \omega \in B\}$

For example, in the experiment of 'throwing a die twice' Let A be the event 'score on the first throw is six' and B is the event 'sum of two scores is atleast 11' then

$$A = \{(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}, \text{ and } B = \{(5,6), (6,5), (6,6)\}$$

so $A \cap B = \{(6,5), (6,6)\}$

Note that the set $A \cap B = \{(6,5), (6,6)\}$ may represent the event 'the score on the first throw is six and the sum of the scores is atleast 11'.

4. The Event 'A but not B' We know that A–B is the set of all those elements which are in A but not in B. Therefore, the set A–B may denote the event 'A but not B'.We know that

$$A - B = A \cap B'$$

Example 1 Consider the experiment of rolling a die. Let A be the event 'getting a prime number', B be the event 'getting an odd number'. Write the sets representing the events (i) Aor B (ii) A and B (iii) A but not B (iv) 'not A'.

Solution Here $S = \{1, 2, 3, 4, 5, 6\}$, $A = \{2, 3, 5\}$ and $B = \{1, 3, 5\}$

Obviously

(i) 'A or B' = $A \cup B = \{1, 2, 3, 5\}$

(ii) 'A and B' = $A \cap B = \{3,5\}$

(iii) 'A but not B' = $A - B = \{2\}$

(iv) 'not A' = $A' = \{1,4,6\}$

14.1.4 *Mutually exclusive events* In the experiment of rolling a die, a sample space is $S = \{1, 2, 3, 4, 5, 6\}$. Consider events, A 'an odd number appears' and B 'an even number appears'

Clearly the event A excludes the event B and vice versa. In other words, there is no outcome which ensures the occurrence of events A and B simultaneously. Here

$A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$

Clearly $A \cap B = \phi$, i.e., A and B are disjoint sets.

In general, two events A and B are called *mutually exclusive* events if the occurrence of any one of them excludes the occurrence of the other event, i.e., if they can not occur simultaneously. In this case the sets A and B are disjoint.

Again in the experiment of rolling a die, consider the events A 'an odd number appears' and event B 'a number less than 4 appears'

Obviously A = {1, 3, 5} and B = {1, 2, 3}

Now $3 \in A$ as well as $3 \in B$

Therefore, A and B are not mutually exclusive events.

Remark Simple events of a sample space are always mutually exclusive.

14.1.5 *Exhaustive events* Consider the experiment of throwing a die. We have S = {1, 2, 3, 4, 5, 6}. Let us define the following events

A: 'a number less than 4 appears',
B: 'a number greater than 2 but less than 5 appears'
and C: 'a number greater than 4 appears'.

Then A = {1, 2, 3}, B = {3,4} and C = {5, 6}. We observe that

$$A \cup B \cup C = \{1, 2, 3\} \cup \{3, 4\} \cup \{5, 6\} = S.$$

Such events A, B and C are called exhaustive events. In general, if $E_1, E_2, ..., E_n$ are n events of a sample space S and if

$$E_1 \cup E_2 \cup E_3 \cup ... \cup E_n = \bigcup_{i=1}^{n} E_i = S$$

then $E_1, E_2, ..., E_n$ are called *exhaustive events*.In other words, events $E_1, E_2, ..., E_n$ are said to be exhaustive if atleast one of them necessarily occurs whenever the experiment is performed.

Further, if $E_i \cap E_j = \phi$ for $i \neq j$ i.e., events E_i and E_j are pairwise disjoint and $\bigcup_{i=1}^{n} E_i = S$, then events $E_1, E_2, ..., E_n$ are called *mutually exclusive and exhaustive events*.

We now consider some examples.

Example 2 Two dice are thrown and the sum of the numbers which come up on the dice is noted. Let us consider the following events associated with this experiment

A: 'the sum is even'.
B: 'the sum is a multiple of 3'.
C: 'the sum is less than 4'.
D: 'the sum is greater than 11'.

Which pairs of these events are mutually exclusive?

Solution There are 36 elements in the sample space S = $\{(x, y): x, y = 1, 2, 3, 4, 5, 6\}$. Then

A = {(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (2, 6), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (4, 6), (5, 1), (5, 3), (5, 5), (6, 2), (6, 4), (6, 6)}

B = {(1, 2), (2, 1), (1, 5), (5, 1), (3, 3), (2, 4), (4, 2), (3, 6), (6, 3), (4, 5), (5, 4), (6, 6)}

C = {(1, 1), (2, 1), (1, 2)} and D = {(6, 6)}

We find that

$$A \cap B = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1), (6, 6)\} \neq \phi$$

Therefore, A and B are not mutually exclusive events.

Similarly $A \cap C \neq \phi$, $A \cap D \neq \phi$, $B \cap C \neq \phi$ and $B \cap D \neq \phi$.

Thus, the pairs of events, (A, C), (A, D), (B, C), (B, D) are not mutually exclusive events.

Also $C \cap D = \phi$ and so C and D are mutually exclusive events.

Example 3 A coin is tossed three times, consider the following events.

A: 'No head appears', B: 'Exactly one head appears' and C: 'Atleast two heads appear'.

Do they form a set of mutually exclusive and exhaustive events?

Solution The sample space of the experiment is

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

and A = {TTT}, B = {HTT, THT, TTH}, C = {HHT, HTH, THH, HHH}

Now

$A \cup B \cup C$ = {TTT, HTT, THT, TTH, HHT, HTH, THH, HHH} = S

Therefore, A, B and C are exhaustive events.

Also, $A \cap B = \phi$, $A \cap C = \phi$ and $B \cap C = \phi$

Therefore, the events are pair-wise disjoint, i.e., they are mutually exclusive.

Hence, A, B and C form a set of mutually exclusive and exhaustive events.

EXERCISE 14.1

1. A die is rolled. Let E be the event "die shows 4" and F be the event "die shows even number". Are E and F mutually exclusive?

2. A die is thrown. Describe the following events:

(i) A: a number less than 7 (ii) B: a number greater than 7

(iii) C: a multiple of 3 (iv) D: a number less than 4

(v) E: an even number greater than 4 (vi) F: a number not less than 3

Also find $A \cup B$, $A \cap B$, $B \cup C$, $E \cap F$, $D \cap E$, $A - C$, $D - E$, $E \cap F'$, F'

3. An experiment involves rolling a pair of dice and recording the numbers that come up. Describe the following events:

A: the sum is greater than 8, B: 2 occurs on either die

C: the sum is at least 7 and a multiple of 3.

Which pairs of these events are mutually exclusive?

4. Three coins are tossed once. Let A denote the event 'three heads show", B denote the event "two heads and one tail show", C denote the event" three tails show and D denote the event 'a head shows on the first coin". Which events are

(i) mutually exclusive? (ii) simple? (iii) Compound?

5. Three coins are tossed. Describe

(i) Two events which are mutually exclusive.

(ii) Three events which are mutually exclusive and exhaustive.

(iii) Two events, which are not mutually exclusive.

(iv) Two events which are mutually exclusive but not exhaustive.

(v) Three events which are mutually exclusive but not exhaustive.

6. Two dice are thrown. The events A, B and C are as follows:

A: getting an even number on the first die.

B: getting an odd number on the first die.

C: getting the sum of the numbers on the dice ≤ 5.

Describe the events

(i) A' (ii) not B (iii) A or B

(iv) A and B (v) A but not C (vi) B or C

(vii) B and C (viii) $A \cap B' \cap C'$

7. Refer to question 6 above, state true or false: (give reason for your answer)

(i) A and B are mutually exclusive

(ii) A and B are mutually exclusive and exhaustive

(iii) $A = B'$

(iv) A and C are mutually exclusive

(v) A and B' are mutually exclusive.

(vi) A', B', C are mutually exclusive and exhaustive.

14.2 Axiomatic Approach to Probability

In earlier sections, we have considered random experiments, sample space and events associated with these experiments. In our day to day life we use many words about the chances of occurrence of events. Probability theory attempts to quantify these chances of occurrence or non occurrence of events.

In earlier classes, we have studied some methods of assigning probability to an event associated with an experiment having known the number of total outcomes.

Axiomatic approach is another way of describing probability of an event. In this approach some axioms or rules are depicted to assign probabilities.

Let S be the sample space of a random experiment. The probability P is a real valued function whose domain is the power set of S and range is the interval [0,1] satisfying the following axioms

(i) For any event E, $P(E) \geq 0$ (ii) $P(S) = 1$

(iii) If E and F are mutually exclusive events, then $P(E \cup F) = P(E) + P(F)$.

It follows from (iii) that $P(\phi) = 0$. To prove this, we take $F = \phi$ and note that E and ϕ are disjoint events. Therefore, from axiom (iii), we get

$P(E \cup \phi) = P(E) + P(\phi)$ or $P(E) = P(E) + P(\phi)$ i.e. $P(\phi) = 0$.

Let S be a sample space containing outcomes $\omega_1, \omega_2, ..., \omega_n$, i.e.,

$$S = \{\omega_1, \omega_2, ..., \omega_n\}$$

It follows from the axiomatic definition of probability that

(i) $0 \leq P(\omega_i) \leq 1$ for each $\omega_i \in S$

(ii) $P(\omega_1) + P(\omega_2) + ... + P(\omega_n) = 1$

(iii) For any event A, $P(A) = \sum P(\omega_i)$, $\omega_i \in A$.

Note It may be noted that the singleton $\{\omega_i\}$ is called elementary event and for notational convenience, we write $P(\omega_i)$ for $P(\{\omega_i\})$.

For example, in 'a coin tossing' experiment we can assign the number $\frac{1}{2}$ to each of the outcomes H and T.

i.e. $$P(H) = \frac{1}{2} \text{ and } P(T) = \frac{1}{2}$$

(1)

Clearly this assignment satisfies both the conditions i.e., each number is neither less than zero nor greater than 1 and

$$P(H) + P(T) = \frac{1}{2} + \frac{1}{2} = 1$$

Therefore, in this case we can say that probability of H $= \frac{1}{2}$, and probability of T $= \frac{1}{2}$

If we take $P(H) = \frac{1}{4}$ and $P(T) = \frac{3}{4}$ ... (2)

Does this assignment satisfy the conditions of axiomatic approach?

Yes, in this case, probability of H = $\frac{1}{4}$ and probability of T = $\frac{3}{4}$.

We find that both the assignments (1) and (2) are valid for probability of H and T.

In fact, we can assign the numbers p and $(1 - p)$ to both the outcomes such that $0 \le p \le 1$ and $P(H) + P(T) = p + (1 - p) = 1$

This assignment, too, satisfies both conditions of the axiomatic approach of probability. Hence, we can say that there are many ways (rather infinite) to assign probabilities to outcomes of an experiment. We now consider some examples.

Example 4 Let a sample space be S = $\{\omega_1, \omega_2, ..., \omega_6\}$.Which of the following assignments of probabilities to each outcome are valid?

Outcomes	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
(a)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
(b)	1	0	0	0	0	0
(c)	$\frac{1}{8}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{4}$	$-\frac{1}{3}$
(d)	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{3}{2}$
(e)	0.1	0.2	0.3	0.4	0.5	0.6

Solution (a) Condition (i): Each of the number $p(\omega_i)$ is positive and less than one.

Condition (ii): Sum of probabilities

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$$

Therefore, the assignment is valid

(b) Condition (i): Each of the number $p(\omega_i)$ is either 0 or 1.

Condition (ii) Sum of the probabilities = 1 + 0 + 0 + 0 + 0 + 0 = 1

Therefore, the assignment is valid

(c) Condition (i) Two of the probabilities $p(\omega_5)$ and $p(\omega_6)$ are negative, the assignment is not valid

(d) Since $p(\omega_6) = \frac{3}{2} > 1$, the assignment is not valid

(e) Since, sum of probabilities = 0.1 + 0.2 + 0.3 + 0.4 + 0.5 + 0.6 = 2.1, the assignment is not valid.

14.2.1 ***Probability of an event*** Let S be a sample space associated with the experiment 'examining three consecutive pens produced by a machine and classified as Good (non-defective) and bad (defective)'. We may get 0, 1, 2 or 3 defective pens as result of this examination.

A sample space associated with this experiment is

$$S = \{BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG\},$$

where B stands for a defective or bad pen and G for a non – defective or good pen.

Let the probabilities assigned to the outcomes be as follows

Sample point:	BBB	BBG	BGB	GBB	BGG	GBG	GGB	GGG
Probability:	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

Let event A: there is exactly one defective pen and event B: there are atleast two defective pens.

Hence A = {BGG, GBG, GGB} and B = {BBG, BGB, GBB, BBB}

Now $\quad P(A) = \sum P(\omega_i), \forall \omega_i \in A$

$$= P(BGG) + P(GBG) + P(GGB) = \frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$$

and $\quad P(B) = \sum P(\omega_i), \forall \omega_i \in B$

$$= P(BBG) + P(BGB) + P(GBB) + P(BBB) = \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{4}{8}=\frac{1}{2}$$

Let us consider another experiment of 'tossing a coin "twice"

The sample space of this experiment is S = {HH, HT, TH, TT}

Let the following probabilities be assigned to the outcomes

$$P(HH) = \frac{1}{4},\ P(HT) = \frac{1}{7},\ P(TH) = \frac{2}{7},\ P(TT) = \frac{9}{28}$$

Clearly this assignment satisfies the conditions of axiomatic approach. Now, let us find the probability of the event E: 'Both the tosses yield the same result'.

Here $\quad E = \{HH, TT\}$

Now $\quad P(E) = \Sigma\, P(w_i)$, for all $w_i \in E$

$$= P(HH) + P(TT) = \frac{1}{4} + \frac{9}{28} = \frac{4}{7}$$

For the event F: 'exactly two heads', we have F = {HH}

and $$P(F) = P(HH) = \frac{1}{4}$$

14.2.2 ***Probabilities of equally likely outcomes*** Let a sample space of an experiment be

$$S = \{\omega_1, \omega_2, ..., \omega_n\}.$$

Let all the outcomes are equally likely to occur, i.e., the chance of occurrence of each simple event must be same.

i.e. $$P(\omega_i) = p, \text{ for all } \omega_i \in S \text{ where } 0 \le p \le 1$$

Since $$\sum_{i=1}^{n} P(\omega_i) = 1 \text{ i.e., } p + p + ... + p \; (n \text{ times}) = 1$$

or $$np = 1 \text{ i.e., } p = \frac{1}{n}$$

Let S be a sample space and E be an event, such that $n(S) = n$ and $n(E) = m$. If each out come is equally likely, then it follows that

$$P(E) = \frac{m}{n} = \frac{\text{Number of outcomes favourable to E}}{\text{Total possible outcomes}}$$

14.2.3 ***Probability of the event 'A or B'*** Let us now find the probability of event 'A or B', i.e., P (A ∪ B)

Let A = {HHT, HTH, THH} and B = {HTH, THH, HHH} be two events associated with 'tossing of a coin thrice'

Clearly A ∪ B = {HHT, HTH, THH, HHH}

Now $$P(A \cup B) = P(HHT) + P(HTH) + P(THH) + P(HHH)$$

If all the outcomes are equally likely, then

$$P(A \cup B) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

Also $$P(A) = P(HHT) + P(HTH) + P(THH) = \frac{3}{8}$$

and $\quad P(B) = P(HTH) + P(THH) + P(HHH) = \frac{3}{8}$

Therefore $\quad P(A) + P(B) = \frac{3}{8}+\frac{3}{8}=\frac{6}{8}$

It is clear that $\quad P(A \cup B) \neq P(A) + P(B)$

The points HTH and THH are common to both A and B. In the computation of P(A) + P(B) the probabilities of points HTH and THH, i.e., the elements of $A \cap B$ are included twice. Thus to get the probability $P(A \cup B)$ we have to subtract the probabilities of the sample points in $A \cap B$ from P(A) + P(B)

i.e. $\quad P(A \cup B) = P(A) + P(B) - \sum P(\omega_i), \forall \omega_i \in A \cap B$

$$= P(A) + P(B) - P(A \cap B)$$

Thus we observe that, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

In general, if A and B are any two events associated with a random experiment, then by the definition of probability of an event, we have

$$P(A \cup B) = \sum p(\omega_i), \forall \omega_i \in A \cup B.$$

Since $\quad A \cup B = (A-B) \cup (A \cap B) \cup (B-A)$,

we have

$P(A \cup B) = \left[\sum P(\omega_i) \forall \omega_i \in (A-B)\right] + \left[\sum P(\omega_i) \forall \omega_i \in A \cap B\right] + \left[\sum P(\omega_i) \forall \omega_i \in B - A\right]$

(because A–B, $A \cap B$ and B – A are mutually exclusive) ... (1)

Also $P(A) + P(B) = \left[\sum p(\omega_i)\ \forall \omega_i \in A\right] + \left[\sum p(\omega_i)\ \forall \omega_i \in B\right]$

$= \left[\sum P(\omega_i) \forall \omega_i \in (A-B) \cup (A \cap B)\right] + \left[\sum P(\omega_i) \forall \omega_i \in (B-A) \cup (A \cap B)\right]$

$= \left[\sum P(\omega_i) \forall \omega_i \in (A-B)\right] + \left[\sum P(\omega_i) \forall \omega_i \in (A \cap B)\right] + \left[\sum P(\omega_i) \forall \omega_i \in (B-A)\right] +$

$\left[\sum P(\omega_i) \forall \omega_i \in (A \cap B)\right]$

$= P(A \cup B) + \left[\sum P(\omega_i) \forall \omega_i \in A \cap B\right]$ [using (1)]

$= P(A \cup B) + P(A \cap B)$.

Hence $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Alternatively, it can also be proved as follows:

$A \cup B = A \cup (B - A)$, where A and B – A are mutually exclusive,

and $\quad B = (A \cap B) \cup (B - A)$, where $A \cap B$ and B – A are mutually exclusive.

Using Axiom (iii) of probability, we get

$$P(A \cup B) = P(A) + P(B - A) \qquad \ldots (2)$$

and $$P(B) = P(A \cap B) + P(B - A) \qquad \ldots (3)$$

Subtracting (3) from (2) gives

$$P(A \cup B) - P(B) = P(A) - P(A \cap B)$$

or $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

The above result can further be verified by observing the Venn Diagram (Fig 14.1)

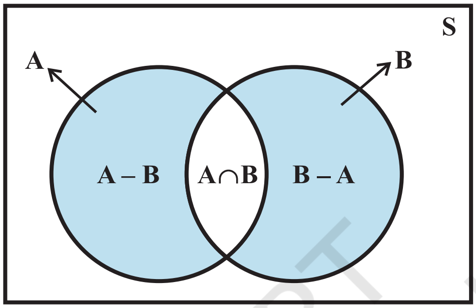


Fig 14.1

If A and B are disjoint sets, i.e., they are mutually exclusive events, then $A \cap B = \phi$

Therefore $\quad P(A \cap B) = P(\phi) = 0$

Thus, for mutually exclusive events A and B, we have

$$P(A \cup B) = P(A) + P(B),$$

which is Axiom (iii) of probability.

14.2.4 ***Probability of event 'not A'*** Consider the event A = {2, 4, 6, 8} associated with the experiment of drawing a card from a deck of ten cards numbered from 1 to 10. Clearly the sample space is S = {1, 2, 3, ...,10}

If all the outcomes 1, 2, ...,10 are considered to be equally likely, then the probability of each outcome is $\frac{1}{10}$

Now $\quad P(A) = P(2) + P(4) + P(6) + P(8)$

$$= \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{4}{10} = \frac{2}{5}$$

Also event 'not A' = $A' = \{1, 3, 5, 7, 9, 10\}$

Now $\quad P(A') = P(1) + P(3) + P(5) + P(7) + P(9) + P(10)$

$$= \frac{6}{10} = \frac{3}{5}$$

Thus, $\qquad P(A') = \frac{3}{5} = 1 - \frac{2}{5} = 1 - P(A)$

Also, we know that A' and A are mutually exclusive and exhaustive events i.e.,

$A \cap A' = \phi$ and $A \cup A' = S$

or $\qquad P(A \cup A') = P(S)$

Now $\qquad P(A) + P(A') = 1, \qquad$ by using axioms (ii) and (iii).

or $\qquad P(A') = P(\text{not } A) = 1 - P(A)$

We now consider some examples and exercises having equally likely outcomes unless stated otherwise.

Example 5 One card is drawn from a well shuffled deck of 52 cards. If each outcome is equally likely, calculate the probability that the card will be

(i) a diamond
(ii) not an ace
(iii) a black card (i.e., a club or, a spade)
(iv) not a diamond
(v) not a black card.

Solution When a card is drawn from a well shuffled deck of 52 cards, the number of possible outcomes is 52.

(i) Let A be the event 'the card drawn is a diamond'

Clearly the number of elements in set A is 13.

Therefore, $\quad P(A) = \frac{13}{52} = \frac{1}{4}$

i.e. probability of a diamond card $= \frac{1}{4}$

(ii) We assume that the event 'Card drawn is an ace' is B

Therefore 'Card drawn is not an ace' should be B'.

We know that $P(B') = 1 - P(B) = 1 - \frac{4}{52} = 1 - \frac{1}{13} = \frac{12}{13}$

(iii) Let C denote the event 'card drawn is black card'

Therefore, number of elements in the set C = 26

i.e. $\quad P(C) = \frac{26}{52} = \frac{1}{2}$

Thus, probability of a black card $= \frac{1}{2}$.

(iv) We assumed in (i) above that A is the event 'card drawn is a diamond', so the event 'card drawn is not a diamond' may be denoted as A' or 'not A'

Now P(not A) = 1 – P(A) $= 1 - \frac{1}{4} = \frac{3}{4}$

(v) The event 'card drawn is not a black card' may be denoted as C′ or 'not C'.

We know that P(not C) = 1 – P(C) $= 1 - \frac{1}{2} = \frac{1}{2}$

Therefore, probability of not a black card $= \frac{1}{2}$

Example 6 A bag contains 9 discs of which 4 are red, 3 are blue and 2 are yellow. The discs are similar in shape and size. A disc is drawn at random from the bag. Calculate the probability that it will be (i) red, (ii) yellow, (iii) blue, (iv) not blue, (v) either red or blue.

Solution There are 9 discs in all so the total number of possible outcomes is 9.
Let the events A, B, C be defined as

A: 'the disc drawn is red'

B: 'the disc drawn is yellow'

C: 'the disc drawn is blue'.

(i) The number of red discs = 4, i.e., n (A) = 4

Hence $P(A) = \frac{4}{9}$

(ii) The number of yellow discs = 2, i.e., n (B) = 2

Therefore, $P(B) = \frac{2}{9}$

(iii) The number of blue discs = 3, i.e., n(C) = 3

Therefore, $P(C) = \frac{3}{9} = \frac{1}{3}$

(iv) Clearly the event 'not blue' is 'not C'. We know that P(not C) = 1 – P(C)

Therefore $\qquad P(\text{not } C) = 1 - \frac{1}{3} = \frac{2}{3}$

(v) The event 'either red or blue' may be described by the set 'A or C'

Since, A and C are mutually exclusive events, we have

$$P(A \text{ or } C) = P(A \cup C) = P(A) + P(C) = \frac{4}{9} + \frac{1}{3} = \frac{7}{9}$$

Example 7 Two students Anil and Ashima appeared in an examination. The probability that Anil will qualify the examination is 0.05 and that Ashima will qualify the examination is 0.10. The probability that both will qualify the examination is 0.02. Find the probability that

(a) Both Anil and Ashima will not qualify the examination.

(b) Atleast one of them will not qualify the examination and

(c) Only one of them will qualify the examination.

Solution Let E and F denote the events that Anil and Ashima will qualify the examination, respectively. Given that

$$P(E) = 0.05,\ P(F) = 0.10 \text{ and } P(E \cap F) = 0.02.$$

Then

(a) The event 'both Anil and Ashima will not qualify the examination' may be expressed as $E' \cap F'$.

Since, E' is 'not E', i.e., Anil will not qualify the examination and F' is 'not F', i.e., Ashima will not qualify the examination.

Also $\qquad E' \cap F' = (E \cup F)'$ (by Demorgan's Law)

Now $\qquad P(E \cup F) = P(E) + P(F) - P(E \cap F)$

or $\qquad P(E \cup F) = 0.05 + 0.10 - 0.02 = 0.13$

Therefore $P(E' \cap F') = P(E \cup F)' = 1 - P(E \cup F) = 1 - 0.13 = 0.87$

(b) P (atleast one of them will not qualify)

$= 1 -$ P(both of them will qualify)

$= 1 - 0.02 = 0.98$

(c) The event only one of them will qualify the examination is same as the event either (Anil will qualify, and Ashima will not qualify) or (Anil will not qualify and Ashima will qualify) i.e., $E \cap F'$ or $E' \cap F$, where $E \cap F'$ and $E' \cap F$ are mutually exclusive.

Therefore, P(only one of them will qualify) $= P(E \cap F' \text{ or } E' \cap F)$

$$= P(E \cap F') + P(E' \cap F) = P(E) - P(E \cap F) + P(F) - P(E \cap F)$$

$$= 0.05 - 0.02 + 0.10 - 0.02 = 0.11$$

Example 8 A committee of two persons is selected from two men and two women. What is the probability that the committee will have (a) no man? (b) one man? (c) two men?

Solution The total number of persons = 2 + 2 = 4. Out of these four person, two can be selected in ${}^4C_2$ ways.

(a) No men in the committee of two means there will be two women in the committee. Out of two women, two can be selected in ${}^2C_2 = 1$ way.

Therefore $$P(\text{no man}) = \frac{{}^2C_2}{{}^4C_2} = \frac{1 \times 2 \times 1}{4 \times 3} = \frac{1}{6}$$

(b) One man in the committee means that there is one woman. One man out of 2 can be selected in ${}^2C_1$ ways and one woman out of 2 can be selected in ${}^2C_1$ ways. Together they can be selected in ${}^2C_1 \times {}^2C_1$ ways.

Therefore $$P(\text{One man}) = \frac{{}^2C_1 \times {}^2C_1}{{}^4C_2} = \frac{2 \times 2}{2 \times 3} = \frac{2}{3}$$

(c) Two men can be selected in ${}^2C_2$ way.

Hence $$P(\text{Two men}) = \frac{{}^2C_2}{{}^4C_2} = \frac{1}{{}^4C_2} = \frac{1}{6}$$

EXERCISE 14.2

1. Which of the following can not be valid assignment of probabilities for outcomes of sample Space S $= \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7\}$

Assignment	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
(a)	0.1	0.01	0.05	0.03	0.01	0.2	0.6
(b)	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$
(c)	0.1	0.2	0.3	0.4	0.5	0.6	0.7
(d)	– 0.1	0.2	0.3	0.4	– 0.2	0.1	0.3
(e)	$\frac{1}{14}$	$\frac{2}{14}$	$\frac{3}{14}$	$\frac{4}{14}$	$\frac{5}{14}$	$\frac{6}{14}$	$\frac{15}{14}$

2. A coin is tossed twice, what is the probability that atleast one tail occurs?

3. A die is thrown, find the probability of following events:

(i) A prime number will appear,

(ii) A number greater than or equal to 3 will appear,

(iii) A number less than or equal to one will appear,

(iv) A number more than 6 will appear,

(v) A number less than 6 will appear.

4. A card is selected from a pack of 52 cards.

(a) How many points are there in the sample space?

(b) Calculate the probability that the card is an ace of spades.

(c) Calculate the probability that the card is (i) an ace (ii) black card.

5. A fair coin with 1 marked on one face and 6 on the other and a fair die are both tossed. find the probability that the sum of numbers that turn up is (i) 3 (ii) 12

6. There are four men and six women on the city council. If one council member is selected for a committee at random, how likely is it that it is a woman?

7. A fair coin is tossed four times, and a person win Re 1 for each head and lose Rs 1.50 for each tail that turns up.
From the sample space calculate how many different amounts of money you can have after four tosses and the probability of having each of these amounts.

8. Three coins are tossed once. Find the probability of getting

(i) 3 heads	(ii) 2 heads	(iii) atleast 2 heads
(iv) atmost 2 heads	(v) no head	(vi) 3 tails
(vii) exactly two tails	(viii) no tail	(ix) atmost two tails

9. If $\frac{2}{11}$ is the probability of an event, what is the probability of the event 'not A'.

10. A letter is chosen at random from the word 'ASSASSINATION'. Find the probability that letter is (i) a vowel (ii) a consonant

11. In a lottery, a person choses six different natural numbers at random from 1 to 20, and if these six numbers match with the six numbers already fixed by the lottery committee, he wins the prize. What is the probability of winning the prize in the game? [**Hint** order of the numbers is not important.]

12. Check whether the following probabilities P(A) and P(B) are consistently defined

(i) $P(A) = 0.5$, $P(B) = 0.7$, $P(A \cap B) = 0.6$

(ii) $P(A) = 0.5$, $P(B) = 0.4$, $P(A \cup B) = 0.8$

13. Fill in the blanks in following table:

	P(A)	**P(B)**	**P(A ∩ B)**	**P(A ∪ B)**
(i)	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{15}$	. . .
(ii)	0.35	. . .	0.25	0.6
(iii)	0.5	0.35	. . .	0.7

14. Given $P(A) = \frac{3}{5}$ and $P(B) = \frac{1}{5}$. Find P(A or B), if A and B are mutually exclusive events.

15. If E and F are events such that $P(E) = \frac{1}{4}$, $P(F) = \frac{1}{2}$ and $P(E \text{ and } F) = \frac{1}{8}$, find (i) P(E or F), (ii) P(not E and not F).

16. Events E and F are such that P(not E or not F) = 0.25, State whether E and F are mutually exclusive.

17. A and B are events such that P(A) = 0.42, P(B) = 0.48 and P(A and B) = 0.16. Determine (i) P(not A), (ii) P(not B) and (iii) P(A or B)

18. In Class XI of a school 40% of the students study Mathematics and 30% study Biology. 10% of the class study both Mathematics and Biology. If a student is selected at random from the class, find the probability that he will be studying Mathematics or Biology.

19. In an entrance test that is graded on the basis of two examinations, the probability of a randomly chosen student passing the first examination is 0.8 and the probability of passing the second examination is 0.7. The probability of passing atleast one of them is 0.95. What is the probability of passing both?

20. The probability that a student will pass the final examination in both English and Hindi is 0.5 and the probability of passing neither is 0.1. If the probability of passing the English examination is 0.75, what is the probability of passing the Hindi examination?

21. In a class of 60 students, 30 opted for NCC, 32 opted for NSS and 24 opted for both NCC and NSS. If one of these students is selected at random, find the probability that

(i) The student opted for NCC or NSS.
(ii) The student has opted neither NCC nor NSS.
(iii) The student has opted NSS but not NCC.

Miscellaneous Examples

Example 9 On her vacations Veena visits four cities (A, B, C and D) in a random order. What is the probability that she visits

(i) A before B? (ii) A before B and B before C?
(iii) A first and B last? (iv) A either first or second?
(v) A just before B?

Solution The number of arrangements (orders) in which Veena can visit four cities A, B, C, or D is 4! i.e., 24.Therefore, n (S) = 24.
Since the number of elements in the sample space of the experiment is 24 all of these outcomes are considered to be equally likely. A sample space for the experiment is

S = {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB
BACD, BADC, BDAC, BDCA, BCAD, BCDA
CABD, CADB, CBDA, CBAD, CDAB, CDBA
DABC, DACB, DBCA, DBAC, DCAB, DCBA}

(i) Let the event 'she visits A before B' be denoted by E

Therefore, E = {ABCD, CABD, DABC, ABDC, CADB, DACB
ACBD, ACDB, ADBC, CDAB, DCAB, ADCB}

Thus $$\mathrm{P}(\mathrm{E})=\frac{n(\mathrm{E})}{n(\mathrm{S})}=\frac{12}{24}=\frac{1}{2}$$

(ii) Let the event 'Veena visits A before B and B before C' be denoted by F.
Here F = {ABCD, DABC, ABDC, ADBC}

Therefore, $$\mathrm{P}(\mathrm{F})=\frac{n(\mathrm{F})}{n(\mathrm{S})}=\frac{4}{24}=\frac{1}{6}$$

Students are advised to find the probability in case of (iii), (iv) and (v).

Example 10 Find the probability that when a hand of 7 cards is drawn from a well shuffled deck of 52 cards, it contains (i) all Kings (ii) 3 Kings (iii) atleast 3 Kings.

Solution Total number of possible hands = ${}^{52}C_7$

(i) Number of hands with 4 Kings = ${}^{4}C_4 \times {}^{48}C_3$ (other 3 cards must be chosen from the rest 48 cards)

Hence P (a hand will have 4 Kings) = $\dfrac{{}^{4}C_4 \times {}^{48}C_3}{{}^{52}C_7} = \dfrac{1}{7735}$

(ii) Number of hands with 3 Kings and 4 non-King cards = ${}^{4}C_3 \times {}^{48}C_4$

Therefore P (3 Kings) = $\dfrac{{}^{4}C_3 \times {}^{48}C_4}{{}^{52}C_7} = \dfrac{9}{1547}$

(iii) P(atleast 3 King) = P(3 Kings or 4 Kings)
= P(3 Kings) + P(4 Kings)

$$= \frac{9}{1547} + \frac{1}{7735} = \frac{46}{7735}$$

Example 11 If A, B, C are three events associated with a random experiment, prove that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C)$$
$$- P(B \cap C) + P(A \cap B \cap C)$$

Solution Consider E = B ∪ C so that

$$P(A \cup B \cup C) = P(A \cup E)$$
$$= P(A) + P(E) - P(A \cap E) \quad \text{... (1)}$$

Now

$$P(E) = P(B \cup C)$$
$$= P(B) + P(C) - P(B \cap C) \quad \text{... (2)}$$

Also $A \cap E = A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ [using distribution property of intersection of sets over the union]. Thus

$$P(A \cap E) = P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)]$$

$$= P(A \cap B) + P(A \cap C) - P[A \cap B \cap C] \quad \text{... (3)}$$

Using (2) and (3) in (1), we get

$$P[A \cup B \cup C] = P(A) + P(B) + P(C) - P(B \cap C)$$
$$- P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$$

Example 12 In a relay race there are five teams A, B, C, D and E.

(a) What is the probability that A, B and C finish first, second and third, respectively.

(b) What is the probability that A, B and C are first three to finish (in any order) (Assume that all finishing orders are equally likely)

Solution If we consider the sample space consisting of all finishing orders in the first three places, we will have ${}^5P_3$, i.e., $\frac{5!}{(5-3)!} = 5 \times 4 \times 3 = 60$ sample points, each with a probability of $\frac{1}{60}$.

(a) A, B and C finish first, second and third, respectively. There is only one finishing order for this, i.e., ABC.

Thus P(A, B and C finish first, second and third respectively) = $\frac{1}{60}$.

(b) A, B and C are the first three finishers. There will be 3! arrangements for A, B and C. Therefore, the sample points corresponding to this event will be 3! in number.

So P (A, B and C are first three to finish) $= \frac{3!}{60} = \frac{6}{60} = \frac{1}{10}$

Miscellaneous Exercise on Chapter 14

1. A box contains 10 red marbles, 20 blue marbles and 30 green marbles. 5 marbles are drawn from the box, what is the probability that
(i) all will be blue? (ii) atleast one will be green?

2. 4 cards are drawn from a well – shuffled deck of 52 cards. What is the probability of obtaining 3 diamonds and one spade?

3. A die has two faces each with number '1', three faces each with number '2' and one face with number '3'. If die is rolled once, determine

(i) P(2) (ii) P(1 or 3) (iii) P(not 3)

4. In a certain lottery 10,000 tickets are sold and ten equal prizes are awarded. What is the probability of not getting a prize if you buy (a) one ticket (b) two tickets (c) 10 tickets.

5. Out of 100 students, two sections of 40 and 60 are formed. If you and your friend are among the 100 students, what is the probability that

(a) you both enter the same section?

(b) you both enter the different sections?

6. Three letters are dictated to three persons and an envelope is addressed to each of them, the letters are inserted into the envelopes at random so that each envelope contains exactly one letter. Find the probability that at least one letter is in its proper envelope.

7. A and B are two events such that $P(A) = 0.54$, $P(B) = 0.69$ and $P(A \cap B) = 0.35$. Find (i) $P(A \cup B)$ (ii) $P(A' \cap B')$ (iii) $P(A \cap B')$ (iv) $P(B \cap A')$

8. From the employees of a company, 5 persons are selected to represent them in the managing committee of the company. Particulars of five persons are as follows:

S. No.	Name	Sex	Age in years
1.	Harish	M	30
2.	Rohan	M	33
3.	Sheetal	F	46
4.	Alis	F	28
5.	Salim	M	41

A person is selected at random from this group to act as a spokesperson. What is the probability that the spokesperson will be either male or over 35 years?

9. If 4-digit numbers greater than 5,000 are randomly formed from the digits 0, 1, 3, 5, and 7, what is the probability of forming a number divisible by 5 when, (i) the digits are repeated? (ii) the repetition of digits is not allowed?

10. The number lock of a suitcase has 4 wheels, each labelled with ten digits i.e., from 0 to 9. The lock opens with a sequence of four digits with no repeats. What is the probability of a person getting the right sequence to open the suitcase?

Summary

In this Chapter, we studied about the axiomatic approach of probability. The main features of this Chapter are as follows:

- ***Event***: A subset of the sample space
- ***Impossible event*** : The empty set
- ***Sure event***: The whole sample space
- ***Complementary event or 'not event'*** : The set A′ or S – A
- ***Event* A *or* B**: The set $A \cup B$
- ***Event* A *and* B**: The set $A \cap B$
- ***Event* A *and* not B**: The set A – B
- ***Mutually exclusive event***: A and B are mutually exclusive if $A \cap B = \phi$
- ***Exhaustive and mutually exclusive events***: Events $E_1, E_2, ..., E_n$ are mutually exclusive and exhaustive if $E_1 \cup E_2 \cup ... \cup E_n = S$ and $E_i \cap E_j = \phi \ \forall \ i \neq j$
- ***Probability***: Number P (ω_i) associated with sample point ω_i such that

  (i) $0 \leq P(\omega_i) \leq 1$ (ii) $\sum P(\omega_i)$ for all $\omega_i \in S = 1$

  (iii) $P(A) = \sum P(\omega_i)$ for all $\omega_i \in A$. The number $P(\omega_i)$ is called *probability of the outcome* ω_i.
- ***Equally likely outcomes***: All outcomes with equal probability
- ***Probability of an event***: For a finite sample space with equally likely outcomes

  Probability of an event $P(A) = \frac{n(A)}{n(S)}$, where $n(A)$ = number of elements in the set A, $n(S)$ = number of elements in the set S.
- If A and B are any two events, then

  P(A or B) = P(A) + P(B) – P(A and B)

  equivalently, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- If A and B are mutually exclusive, then P(A or B) = P(A) + P(B)
- If A is any event, then

  P(not A) = 1 – P(A)

Historical Note

Probability theory like many other branches of mathematics, evolved out of practical consideration. It had its origin in the 16th century when an Italian physician and mathematician Jerome Cardan (1501–1576) wrote the first book on the subject "Book on Games of Chance" (Biber de Ludo Aleae). It was published in 1663 after his death.

In 1654, a gambler Chevalier de Metre approached the well known French Philosopher and Mathematician Blaise Pascal (1623–1662) for certain dice problem. Pascal became interested in these problems and discussed with famous French Mathematician Pierre de Fermat (1601–1665). Both Pascal and Fermat solved the problem independently. Besides, Pascal and Fermat, outstanding contributions to probability theory were also made by Christian Huygenes (1629–1665), a Dutchman, J. Bernoulli (1654–1705), De Moivre (1667–1754), a Frenchman Pierre Laplace (1749–1827), the Russian P.L Chebyshev (1821–1897), A. A Markov (1856–1922) and A. N Kolmogorove (1903–1987). Kolmogorov is credited with the axiomatic theory of probability. His book 'Foundations of Probability' published in 1933, introduces probability as a set function and is considered a classic.

Appendix 1

PROOFS IN MATHEMATICS

❖ *Proofs are to Mathematics what calligraphy is to poetry.*
Mathematical works do consist of proofs just as
poems do consist of characters.
— VLADIMIR ARNOLD ❖

A.1.1 Introduction

In Classes IX, X and XI, we have learnt about the concepts of a statement, compound statement, negation, converse and contrapositive of a statement; axioms, conjectures, theorems and deductive reasoning.

Here, we will discuss various methods of proving mathematical propositions.

A.1.2 What is a Proof?

Proof of a mathematical statement consists of sequence of statements, each statement being justified with a definition or an axiom or a proposition that is previously established by the method of deduction using only the allowed logical rules.

Thus, each proof is a chain of deductive arguments each of which has its premises and conclusions. Many a times, we prove a proposition directly from what is given in the proposition. But some times it is easier to prove an equivalent proposition rather than proving the proposition itself. This leads to, two ways of proving a proposition directly or indirectly and the proofs obtained are called direct proof and indirect proof and further each has three different ways of proving which is discussed below.

Direct Proof It is the proof of a proposition in which we directly start the proof with what is given in the proposition.

(i) Straight forward approach It is a chain of arguments which leads directly from what is given or assumed, with the help of axioms, definitions or already proved theorems, to what is to be proved using rules of logic.

Consider the following example:

Example 1 Show that if $x^2 - 5x + 6 = 0$, then $x = 3$ or $x = 2$.

Solution $x^2 - 5x + 6 = 0$ (given)

$\Rightarrow$ $(x-3)(x-2)=0$ (replacing an expression by an equal/equivalent expression)

$\Rightarrow$ $x-3=0$ or $x-2=0$ (from the established theorem $ab=0 \Rightarrow$ either $a=0$ or $b=0$, for a, b in **R**)

$\Rightarrow$ $x-3+3=0+3$ or $x-2+2=0+2$ (adding equal quantities on either side of the equation does not alter the nature of the equation)

$\Rightarrow$ $x+0=3$ or $x+0=2$ (using the identity property of integers under addition)

$\Rightarrow$ $x=3$ or $x=2$ (using the identity property of integers under addition)

Hence, $x^2-5x+6=0$ implies $x=3$ or $x=2$.

Explanation Let p be the given statement "$x^2-5x+6=0$" and q be the conclusion statement "$x=3$ or $x=2$".

From the statement p, we deduced the statement r: "$(x-3)(x-2)=0$" by replacing the expression x^2-5x+6 in the statement p by another expression $(x-3)(x-2)$ which is equal to x^2-5x+6.

There arise two questions:

(i) How does the expression $(x-3)(x-2)$ is equal to the expression x^2-5x+6?

(ii) How can we replace an expression with another expression which is equal to the former?

The first one is proved in earlier classes by factorization, i.e.,

$x^2-5x+6 = x^2-3x-2x+6 = x(x-3)-2(x-3) = (x-3)(x-2)$.

The second one is by valid form of argumentation (rules of logic)

Next this statement r becomes premises or given and deduce the statement s "$x-3=0$ or $x-2=0$" and the reasons are given in the brackets.

This process continues till we reach the conclusion.

The symbolic equivalent of the argument is to prove by deduction that $p \Rightarrow q$ is true.

Starting with p, we deduce $p \Rightarrow r \Rightarrow s \Rightarrow \ldots \Rightarrow q$. This implies that "$p \Rightarrow q$" is true.

Example 2 Prove that the function $f: \mathbf{R} \rightarrow \mathbf{R}$ defined by $f(x)=2x+5$ is one-one.

Solution Note that a function f is one-one if

$f(x_1)=f(x_2) \Rightarrow x_1=x_2$ (definition of one-one function)

Now, given that $f(x_1)=f(x_2)$, i.e., $2x_1+5=2x_2+5$

$\Rightarrow$ $2x_1+5-5=2x_2+5-5$ (adding the same quantity on both sides)

$$\Rightarrow \qquad 2x_1 + 0 = 2x_2 + 0$$

$$\Rightarrow \qquad 2x_1 = 2x_2 \text{ (using additive identity of real number)}$$

$$\Rightarrow \qquad \frac{2}{2}x_1 = \frac{2}{2}x_2 \text{ (dividing by the same non zero quantity)}$$

$$\Rightarrow \qquad x_1 = x_2$$

Hence, the given function is one-one.

(ii) Mathematical Induction

Mathematical induction, is a strategy, of proving a proposition which is deductive in nature. The whole basis of proof of this method depends on the following axiom:

For a given subset S of **N**, if

(i) the natural number $1 \in S$ and

(ii) the natural number $k + 1 \in S$ whenever $k \in S$, then $S = \mathbf{N}$.

According to the principle of mathematical induction, if a statement "S(n) is true for $n = 1$" (or for some starting point j), and if "S(n) is true for $n = k$" implies that "S(n) is true for $n = k + 1$" (whatever integer $k \geq j$ may be), then the statement is true for any positive integer n, for all $n \geq j$.

We now consider some examples.

Example 3 Show that if

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \text{ then } A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$$

Solution We have

$$P(n) : A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$$

We note that

$$P(1) : A^1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Therefore, P(1) is true.

Assume that P(k) is true, i.e.,

$$P(k) : A^k = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$$

We want to prove that P(k + 1) is true whenever P(k) is true, i.e.,

$$P(k+1) : A^{k+1} = \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

Now $$A^{k+1} = A^k \,.\, A$$

Since P(k) is true, we have

$$A^{k+1} = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos k\theta \cos\theta - \sin k\theta \sin\theta & \cos k\theta \sin\theta + \sin k\theta \cos\theta \\ -\sin k\theta \cos\theta - \cos k\theta \sin\theta & -\sin k\theta \sin\theta + \cos k\theta \cos\theta \end{bmatrix}$$

(by matrix multiplication)

$$= \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, P(n) is true for all $n \geq 1$ (by the principle of mathematical induction).

(iii) Proof by cases or by exhaustion

This method of proving a statement $p \Rightarrow q$ is possible only when p can be split into several cases, r, s, t (say) so that $p = r \vee s \vee t$ (where "$\vee$" is the symbol for "OR").

If the conditionals $r \Rightarrow q$;

$s \Rightarrow q$;

and $t \Rightarrow q$

are proved, then $(r \vee s \vee t) \Rightarrow q$, is proved and so $p \Rightarrow q$ is proved.

The method consists of examining every possible case of the hypothesis. It is practically convenient only when the number of possible cases are few.

Example 4 Show that in any triangle ABC,

$$a = b \cos C + c \cos B$$

Solution Let p be the statement "ABC is any triangle" and q be the statement

"$a = b \cos C + c \cos B$"

Let ABC be a triangle. From A draw AD a perpendicular to BC (BC produced if necessary).

As we know that any triangle has to be either acute or obtuse or right angled, we can split p into three statements r, s and t, where

r : ABC is an acute angled triangle with $\angle$ C is acute.

s : ABC is an obtuse angled triangle with $\angle$ C is obtuse.

t : ABC is a right angled triangle with $\angle$ C is right angle.

Hence, we prove the theorem by three cases.

Case (i) When $\angle$ C is acute (Fig. A1.1).

From the right angled triangle ADB,

$$\frac{BD}{AB} = \cos B$$

i.e. $$BD = AB \cos B$$
$$= c \cos B$$

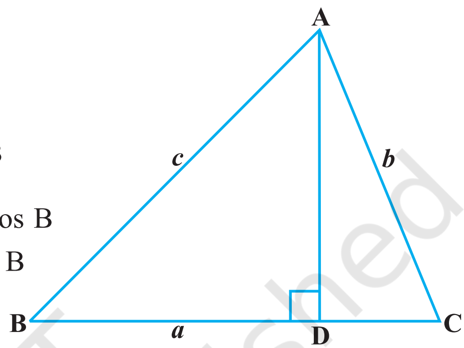


Fig A1.1

From the right angled triangle ADC,

$$\frac{CD}{AC} = \cos C$$

i.e. $$CD = AC \cos C$$
$$= b \cos C$$

Now $$a = BD + CD$$
$$= c \cos B + b \cos C \qquad \text{... (1)}$$

Case (ii) When $\angle$ C is obtuse (Fig A1.2).

From the right angled triangle ADB,

$$\frac{BD}{AB} = \cos B$$

i.e. $$BD = AB \cos B$$
$$= c \cos B$$

From the right angled triangle ADC,

$$\frac{CD}{AC} = \cos \angle ACD$$
$$= \cos (180° - C)$$
$$= -\cos C$$

i.e. $$CD = -AC \cos C$$
$$= -b \cos C$$

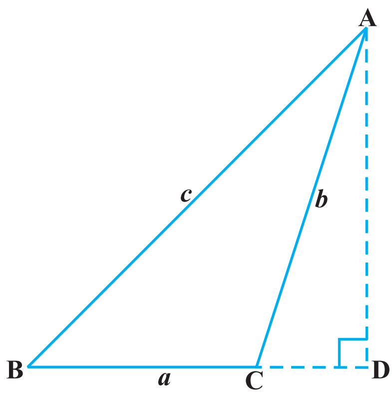


Fig A1.2

Now $a = \text{BC} = \text{BD} - \text{CD}$

i.e. $a = c \cos \text{B} - (-b \cos \text{C})$

$$a = c \cos \text{B} + b \cos \text{C} \quad \ldots (2)$$

Case (iii) When $\angle$ C is a right angle (Fig A1.3).

From the right angled triangle ACB,

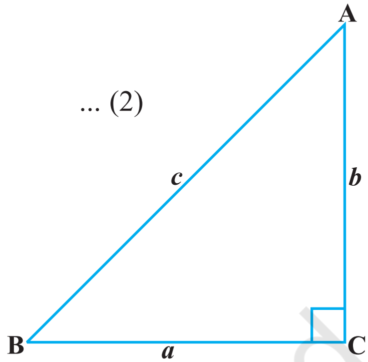


Fig A1.3

$$\frac{\text{BC}}{\text{AB}} = \cos \text{B}$$

i.e. $\text{BC} = \text{AB} \cos \text{B}$

$$a = c \cos \text{B},$$

and $b \cos \text{C} = b \cos 90^0 = 0.$

Thus, we may write $a = 0 + c \cos \text{B}$

$$= b \cos \text{C} + c \cos \text{B} \quad \ldots (3)$$

From (1), (2) and (3). We assert that for any triangle ABC,

$$a = b \cos \text{C} + c \cos \text{B}$$

By case (i), $r \Rightarrow q$ is proved.

By case (ii), $s \Rightarrow q$ is proved.

By case (iii), $t \Rightarrow q$ is proved.

Hence, from the proof by cases, $(r \vee s \vee t) \Rightarrow q$ is proved, i.e., $p \Rightarrow q$ is proved.

Indirect Proof Instead of proving the given proposition directly, we establish the proof of the proposition through proving a proposition which is equivalent to the given proposition.

(i) **Proof by contradiction** (*Reductio Ad Absurdum*) : Here, we start with the assumption that the given statement is false. By rules of logic, we arrive at a conclusion contradicting the assumption and hence it is inferred that the assumption is wrong and hence the given statement is true.

Let us illustrate this method by an example.

Example 5 Show that the set of all prime numbers is infinite.

Solution Let P be the set of all prime numbers. We take the negation of the statement "the set of all prime numbers is infinite", i.e., we assume the set of all prime numbers to be finite. Hence, we can list all the prime numbers as $P_1, P_2, P_3, \ldots, P_k$ (say). Note that we have assumed that there is no prime number other than $P_1, P_2, P_3, \ldots, P_k$.

Now consider $N = (P_1 P_2 P_3 \ldots P_k) + 1 \ldots (1)$

N is not in the list as N is larger than any of the numbers in the list.

N is either prime or composite.

If N is a prime, then by (1), there exists a prime number which is not listed.

On the other hand, if N is composite, it should have a prime divisor. But none of the numbers in the list can divide N, because they all leave the remainder 1. Hence, the prime divisor should be other than the one in the list.

Thus, in both the cases whether N is a prime or a composite, we ended up with contradiction to the fact that we have listed all the prime numbers.

Hence, our assumption that set of all prime numbers is finite is false.

Thus, the set of all prime numbers is infinite.

Note Observe that the above proof also uses the method of proof by cases.

(ii) Proof by using contrapositive statement of the given statement

Instead of proving the conditional $p \Rightarrow q$, we prove its equivalent, i.e., $\sim q \Rightarrow \sim p$. (students can verify).

The contrapositive of a conditional can be formed by interchanging the conclusion and the hypothesis and negating both.

Example 6 Prove that the function $f : \mathbf{R} \rightarrow \mathbf{R}$ defined by $f(x) = 2x + 5$ is one-one.

Solution A function is one-one if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Using this we have to show that "$2x_1 + 5 = 2x_2 + 5$" $\Rightarrow$ "$x_1 = x_2$". This is of the form $p \Rightarrow q$, where, p is $2x_1 + 5 = 2x_2 + 5$ and $q : x_1 = x_2$. We have proved this in Example 2 of "direct method".

We can also prove the same by using contrapositive of the statement. Now contrapositive of this statement is $\sim q \Rightarrow \sim p$, i.e., contrapositive of "if $f(x_1) = f(x_2)$, then $x_1 = x_2$" is "if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$".

Now $\qquad x_1 \neq x_2$

$\Rightarrow \qquad 2x_1 \neq 2x_2$

$\Rightarrow \qquad 2x_1 + 5 \neq 2x_2 + 5$

$\Rightarrow \qquad f(x_1) \neq f(x_2)$.

Since "$\sim q \Rightarrow \sim p$", is equivalent to "$p \Rightarrow q$" the proof is complete.

Example 7 Show that "if a matrix A is invertible, then A is non singular".

Solution Writing the above statement in symbolic form, we have
$p \Rightarrow q$, where, p is "matrix A is invertible" and q is "A is non singular"

Instead of proving the given statement, we prove its contrapositive statement, i.e., if A is not a non singular matrix, then the matrix A is not invertible.

If A is not a non singular matrix, then it means the matrix A is singular, i.e.,

$$|A| = 0$$

Then $$A^{-1} = \frac{adj\ A}{|A|} \text{ does not exist as } |A| = 0$$

Hence, A is not invertible.

Thus, we have proved that if A is not a non singular matrix, then A is not invertible. i.e., $\sim q \Rightarrow \sim p$.

Hence, if a matrix A is invertible, then A is non singular.

(iii) Proof by a counter example

In the history of Mathematics, there are occasions when all attempts to find a valid proof of a statement fail and the uncertainty of the truth value of the statement remains unresolved.

In such a situation, it is beneficial, if we find an example to falsify the statement. The example to disprove the statement is called a *counter example*. Since the disproof of a proposition $p \Rightarrow q$ is merely a proof of the proposition $\sim (p \Rightarrow q)$. Hence, this is also a method of proof.

Example 8 For each $n, 2^{2^n} + 1$ is a prime $(n \in \mathbf{N})$.

This was once thought to be true on the basis that

$$2^{2^1} + 1 = 2^2 + 1 = 5 \text{ is a prime.}$$

$$2^{2^2} + 1 = 2^4 + 1 = 17 \text{ is a prime.}$$

$$2^{2^3} + 1 = 2^8 + 1 = 257 \text{ is a prime.}$$

However, at first sight the generalisation looks to be correct. But, eventually it was shown that $$2^{2^5} + 1 = 2^{32} + 1 = 4294967297$$

which is not a prime since 4294967297 = 641 × 6700417 (a product of two numbers).

So the generalisation "For each n, $2^{2^n} + 1$ is a prime $(n \in \mathbf{N})$" is false.

Just this one example $2^{2^5} + 1$ is sufficient to disprove the generalisation. This is the counter example.

Thus, we have proved that the generalisation "For each $n, 2^{2^n} + 1$ is a prime $(n \in \mathbf{N})$" is not true in general.

Example 9 Every continuous function is differentiable.

Proof We consider some functions given by

(i) $f(x) = x^2$

(ii) $g(x) = e^x$

(iii) $h(x) = \sin x$

These functions are continuous for all values of x. If we check for their differentiability, we find that they are all differentiable for all the values of x. This makes us to believe that the generalisation "Every continuous function is differentiable" may be true. But if we check the differentiability of the function given by "$\phi(x) = |x|$" which is continuous, we find that it is not differentiable at $x = 0$. This means that the statement "Every continuous function is differentiable" is false, in general. Just this one function "$\phi(x) = |x|$" is sufficient to disprove the statement. Hence, "$\phi(x) = |x|$" is called a counter example to disprove "Every continuous function is differentiable".

Appendix 2

MATHEMATICAL MODELLING

A.2.1 Introduction

In class XI, we have learnt about mathematical modelling as an attempt to study some part (or form) of some real-life problems in mathematical terms, i.e., the conversion of a physical situation into mathematics using some suitable conditions. Roughly speaking mathematical modelling is an activity in which we make models to describe the behaviour of various phenomenal activities of our interest in many ways using words, drawings or sketches, computer programs, mathematical formulae etc.

In earlier classes, we have observed that solutions to many problems, involving applications of various mathematical concepts, involve mathematical modelling in one way or the other. Therefore, it is important to study mathematical modelling as a separate topic.

In this chapter, we shall further study mathematical modelling of some real-life problems using techniques/results from matrix, calculus and linear programming.

A.2.2 Why Mathematical Modelling?

Students are aware of the solution of word problems in arithmetic, algebra, trigonometry and linear programming etc. Sometimes we solve the problems without going into the physical insight of the situational problems. Situational problems need physical insight that is **introduction** of physical laws and some symbols to compare the mathematical results obtained with practical values. To solve many problems faced by us, we need a technique and this is what is known as *mathematical modelling.* Let us consider the following problems:

(i) To find the width of a river (particularly, when it is difficult to cross the river).

(ii) To find the optimal angle in case of shot-put (by considering the variables such as : the height of the thrower, resistance of the media, acceleration due to gravity etc.).

(iii) To find the height of a tower (particularly, when it is not possible to reach the top of the tower).

(iv) To find the temperature at the surface of the Sun.

(v) Why heart patients are not allowed to use lift? (without knowing the physiology of a human being).

(vi) To find the mass of the Earth.

(vii) Estimate the yield of pulses in India from the standing crops (a person is not allowed to cut all of it).

(viii) Find the volume of blood inside the body of a person (a person is not allowed to bleed completely).

(ix) Estimate the population of India in the year 2020 (a person is not allowed to wait till then).

All of these problems can be solved and infact have been solved with the help of Mathematics using mathematical modelling. In fact, you might have studied the methods for solving some of them in the present textbook itself. However, it will be instructive if you first try to solve them yourself and that too without the help of Mathematics, if possible, you will then appreciate the power of Mathematics and the need for mathematical modelling.

A.2.3 Principles of Mathematical Modelling

Mathematical modelling is a principled activity and so it has some principles behind it. These principles are almost philosophical in nature. Some of the basic principles of mathematical modelling are listed below in terms of instructions:

(i) Identify the need for the model. (for what we are looking for)

(ii) List the parameters/variables which are required for the model.

(iii) Identify the available relevent data. (what is given?)

(iv) Identify the circumstances that can be applied (assumptions)

(v) Identify the governing physical principles.

(vi) Identify

- (a) the equations that will be used.
- (b) the calculations that will be made.
- (c) the solution which will follow.

(vii) Identify tests that can check the

- (a) consistency of the model.
- (b) utility of the model.

(viii) Identify the parameter values that can improve the model.

The above principles of mathematical modelling lead to the following: steps for mathematical modelling.

Step 1: Identify the physical situation.

Step 2: Convert the physical situation into a mathematical model by introducing parameters / variables and using various known physical laws and symbols.

Step 3: Find the solution of the mathematical problem.

Step 4: Interpret the result in terms of the original problem and compare the result with observations or experiments.

Step 5: If the result is in good agreement, then accept the model. Otherwise modify the hypotheses / assumptions according to the physical situation and go to Step 2.

The above steps can also be viewed through the following diagram:

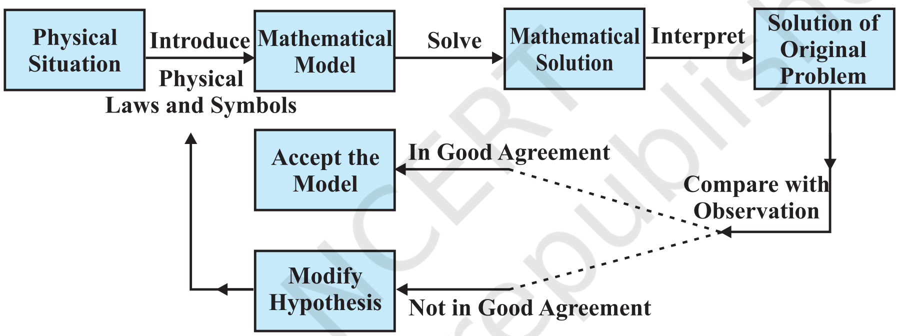


Fig A.2.1

Example 1 Find the height of a given tower using mathematical modelling.

Solution Step 1 Given physical situation is "to find the height of a given tower".

Step 2 Let AB be the given tower (Fig A.2.2). Let PQ be an observer measuring the height of the tower with his eye at P. Let $PQ = h$ and let height of tower be H. Let α be the angle of elevation from the eye of the observer to the top of the tower.

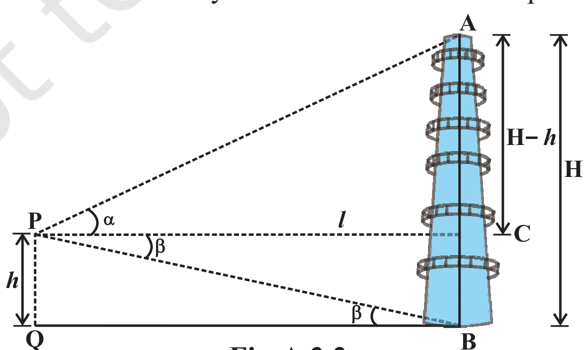


Fig A.2.2

Let $$l = \text{PC} = \text{QB}$$

Now $$\tan\alpha = \frac{\text{AC}}{\text{PC}} = \frac{\text{H}-h}{l}$$

or $$\text{H} = h + l\tan\alpha \qquad \text{... (1)}$$

Step 3 Note that the values of the parameters h, l and α (using sextant) are known to the observer and so (1) gives the solution of the problem.

Step 4 In case, if the foot of the tower is not accessible, i.e., when l is not known to the observer, let β be the angle of depression from P to the foot B of the tower. So from ΔPQB, we have

$$\tan\beta = \frac{\text{PQ}}{\text{QB}} = \frac{h}{l} \text{ or } l = h\cot\beta$$

Step 5 is not required in this situation as exact values of the parameters h, l, α and β are known.

Example 2 Let a business firm produces three types of products P_1, P_2 and P_3 that uses three types of raw materials R_1, R_2 and R_3. Let the firm has purchase orders from two clients F_1 and F_2. Considering the situation that the firm has a limited quantity of R_1, R_2 and R_3, respectively, prepare a model to determine the quantities of the raw material R_1, R_2 and R_3 required to meet the purchase orders.

Solution Step 1 The physical situation is well identified in the problem.

Step 2 Let A be a matrix that represents purchase orders from the two clients F_1 and F_2. Then, A is of the form

$$A = \begin{array}{c} \\ F_1 \\ F_2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} P_1 & P_2 & P_3 \end{array} \\ \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} \end{array}$$

Let B be the matrix that represents the amount of raw materials R_1, R_2 and R_3, required to manufacture each unit of the products P_1, P_2 and P_3. Then, B is of the form

$$B = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} R_1 & R_2 & R_3 \end{array} \\ \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} \end{array}$$

Step 3 Note that the product (which in this case is well defined) of matrices A and B is given by the following matrix

$$AB = \begin{array}{c} \\ F_1 \\ F_2 \end{array}\begin{array}{c} \begin{array}{ccc} R_1 & R_2 & R_3 \end{array} \\ \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} \end{array}$$

which in fact gives the desired quantities of the raw materials R_1, R_2 and R_3 to fulfill the purchase orders of the two clients F_1 and F_2.

Example 3 Interpret the model in Example 2, in case

$$A = \begin{bmatrix} 10 & 15 & 6 \\ 10 & 20 & 0 \end{bmatrix},\ B = \begin{bmatrix} 3 & 4 & 0 \\ 7 & 9 & 3 \\ 5 & 12 & 7 \end{bmatrix}$$

and the available raw materials are 330 units of R_1, 455 units of R_2 and 140 units of R_3.

Solution Note that

$$AB = \begin{bmatrix} 10 & 15 & 6 \\ 10 & 20 & 0 \end{bmatrix}\begin{bmatrix} 3 & 4 & 0 \\ 7 & 9 & 3 \\ 5 & 12 & 7 \end{bmatrix}$$

$$= \begin{array}{c} \\ F_1 \\ F_2 \end{array}\begin{array}{c} \begin{array}{ccc} R_1 & R_2 & R_3 \end{array} \\ \begin{bmatrix} 165 & 247 & 87 \\ 170 & 220 & 60 \end{bmatrix} \end{array}$$

This clearly shows that to meet the purchase order of F_1 and F_2, the raw material required is 335 units of R_1, 467 units of R_2 and 147 units of R_3 which is much more than the available raw material. Since the amount of raw material required to manufacture each unit of the three products is fixed, we can either ask for an increase in the available raw material or we may ask the clients to reduce their orders.

Remark If we replace A in Example 3 by A_1 given by

$$A_1 = \begin{bmatrix} 9 & 12 & 6 \\ 10 & 20 & 0 \end{bmatrix}$$

i.e., if the clients agree to reduce their purchase orders, then

$$A_1 B = \begin{bmatrix} 9 & 12 & 6 \\ 10 & 20 & 0 \end{bmatrix}\begin{bmatrix} 3 & 4 & 0 \\ 7 & 9 & 3 \\ 5 & 12 & 7 \end{bmatrix} = \begin{bmatrix} 141 & 216 & 78 \\ 170 & 220 & 60 \end{bmatrix}$$

This requires 311 units of R_1, 436 units of R_2 and 138 units of R_3 which are well below the available raw materials, i.e., 330 units of R_1, 455 units of R_2 and 140 units of R_3. Thus, if the revised purchase orders of the clients are given by A_1, then the firm can easily supply the purchase orders of the two clients.

Note One may further modify A so as to make full use of the available raw material.

Query Can we make a mathematical model with a given B and with fixed quantities of the available raw material that can help the firm owner to ask the clients to modify their orders in such a way that the firm makes the full use of its available raw material?

The answer to this query is given in the following example:

Example 4 Suppose P_1, P_2, P_3 and R_1, R_2, R_3 are as in Example 2. Let the firm has 330 units of R_1, 455 units of R_2 and 140 units of R_3 available with it and let the amount of raw materials R_1, R_2 and R_3 required to manufacture each unit of the three products is given by

$$B = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array} \begin{array}{c} \begin{array}{ccc} R_1 & R_2 & R_3 \end{array} \\ \begin{bmatrix} 3 & 4 & 0 \\ 7 & 9 & 3 \\ 5 & 12 & 7 \end{bmatrix} \end{array}$$

How many units of each product is to be made so as to utilise the full available raw material?

Solution Step 1 The situation is easily identifiable.

Step 2 Suppose the firm produces x units of P_1, y units of P_2 and z units of P_3. Since product P_1 requires 3 units of R_1, P_2 requires 7 units of R_1 and P_3 requires 5 units of R_1 (observe matrix B) and the total number of units, of R_1, available is 330, we have

$$3x + 7y + 5z = 330 \text{ (for raw material } R_1)$$

Similarly, we have

$$4x + 9y + 12z = 455 \text{ (for raw material } R_2)$$

and
$$3y + 7z = 140 \text{ (for raw material } R_3)$$

This system of equations can be expressed in matrix form as

$$\begin{bmatrix} 3 & 7 & 5 \\ 4 & 9 & 12 \\ 0 & 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 330 \\ 455 \\ 140 \end{bmatrix}$$

Step 3 Using elementary row operations, we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 35 \\ 5 \end{bmatrix}$$

This gives $x = 20$, $y = 35$ and $z = 5$. Thus, the firm can produce 20 units of P_1, 35 units of P_2 and 5 units of P_3 to make full use of its available raw material.

Remark One may observe that if the manufacturer decides to manufacture according to the available raw material and not according to the purchase orders of the two clients F_1 and F_2 (as in Example 3), he/she is unable to meet these purchase orders as F_1 demanded 6 units of P_3 where as the manufacturer can make only 5 units of P_3.

Example 5 A manufacturer of medicines is preparing a production plan of medicines M_1 and M_2. There are sufficient raw materials available to make 20000 bottles of M_1 and 40000 bottles of M_2, but there are only 45000 bottles into which either of the medicines can be put. Further, it takes 3 hours to prepare enough material to fill 1000 bottles of M_1, it takes 1 hour to prepare enough material to fill 1000 bottles of M_2 and there are 66 hours available for this operation. The profit is Rs 8 per bottle for M_1 and Rs 7 per bottle for M_2. How should the manufacturer schedule his/her production in order to maximise profit?

Solution Step 1 To find the number of bottles of M_1 and M_2 in order to maximise the profit under the given hypotheses.

Step 2 Let x be the number of bottles of type M_1 medicine and y be the number of bottles of type M_2 medicine. Since profit is Rs 8 per bottle for M_1 and Rs 7 per bottle for M_2, therefore the objective function (which is to be maximised) is given by

$$Z \equiv Z(x, y) = 8x + 7y$$

The objective function is to be maximised subject to the constraints (Refer Chapter 12 on Linear Programming)

$$\left.\begin{aligned} x &\le 20000 \\ y &\le 40000 \\ x + y &\le 45000 \\ 3x + y &\le 66000 \\ x \ge 0, y &\ge 0 \end{aligned}\right\} \quad \ldots (1)$$

Step 3 The shaded region OPQRST is the feasible region for the constraints (1) (Fig A.2.3). The co-ordinates of vertices O, P, Q, R, S and T are (0, 0), (20000, 0), (20000, 6000), (10500, 34500), (5000, 40000) and (0, 40000), respectively.

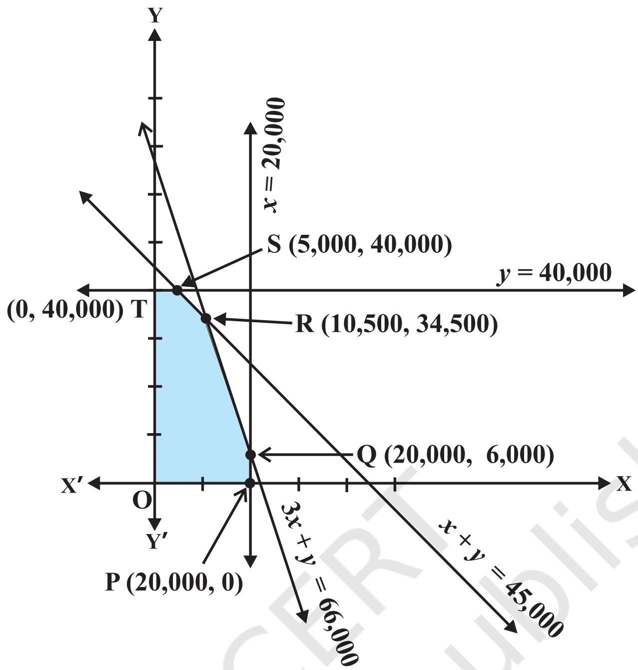


Fig A.2.3

Note that

Z at P (0, 0) = 0

Z at P (20000, 0) = 8 × 20000 = 160000

Z at Q (20000, 6000) = 8 × 20000 + 7 × 6000 = 202000

Z at R (10500, 34500) = 8 × 10500 + 7 × 34500 = 325500

Z at S = (5000, 40000) = 8 × 5000 + 7 × 40000 = 320000

Z at T = (0, 40000) = 7 × 40000 = 280000

Now observe that the profit is maximum at x = 10500 and y = 34500 and the maximum profit is ₹325500. Hence, the manufacturer should produce 10500 bottles of M_1 medicine and 34500 bottles of M_2 medicine in order to get maximum profit of ₹325500.

Example 6 Suppose a company plans to produce a new product that incur some costs (fixed and variable) and let the company plans to sell the product at a fixed price. Prepare a mathematical model to examine the profitability.

Solution Step 1 Situation is clearly identifiable.

Step 2 Formulation: We are given that the costs are of two types: fixed and variable. The fixed costs are independent of the number of units produced (e.g., rent and rates), while the variable costs increase with the number of units produced (e.g., material). Initially, we assume that the variable costs are directly proportional to the number of units produced — this should simplify our model. The company earn a certain amount of money by selling its products and wants to ensure that it is maximum. For convenience, we assume that all units produced are sold immediately.

The mathematical model

Let x = number of units produced and sold

C = total cost of production (in rupees)

I = income from sales (in rupees)

P = profit (in rupees)

Our assumptions above state that C consists of two parts:

(i) fixed cost = a (in rupees),

(ii) variable cost = b (rupees/unit produced).

Then
$$C = a + bx \qquad \text{... (1)}$$

Also, income I depends on selling price s (rupees/unit)

Thus
$$I = sx \qquad \text{... (2)}$$

The profit P is then the difference between income and costs. So

$$\begin{aligned} P &= I - C \\ &= sx - (a + bx) \\ &= (s - b)\, x - a \qquad \text{... (3)} \end{aligned}$$

We now have a mathematical model of the relationships (1) to (3) *between the variables* x, C, I, P, a, b, s. These variables may be classified as:

independent	x
dependent	C, I, P
parameters	a, b, s

The manufacturer, knowing x, a, b, s can determine P.

Step 3 From (3), we can observe that for the break even point (i.e., make neither profit nor loss), he must have P = 0, i.e., $x = \dfrac{a}{s-b}$ units.

Steps 4 and 5 In view of the break even point, one may conclude that if the company produces few units, i.e., less than $x = \dfrac{a}{s-b}$ units, then the company will suffer loss and if it produces large number of units, i.e., much more than $\frac{a}{s-b}$ units, then it can make huge profit. Further, if the break even point proves to be unrealistic, then another model could be tried or the assumptions regarding cash flow may be modified.

Remark From (3), we also have

$$\frac{dP}{dx} = s - b$$

This means that rate of change of P with respect to x depends on the quantity $s - b$, which is the difference of selling price and the variable cost of each product. Thus, in order to gain profit, this should be positive and to get large gains, we need to produce large quantity of the product and at the same time try to reduce the variable cost.

Example 7 Let a tank contains 1000 litres of brine which contains 250 g of salt per litre. Brine containing 200 g of salt per litre flows into the tank at the rate of 25 litres per minute and the mixture flows out at the same rate. Assume that the mixture is kept uniform all the time by stirring. What would be the amount of salt in the tank at any time t?

Solution Step 1 The situation is easily identifiable.

Step 2 Let $y = y(t)$ denote the amount of salt (in kg) in the tank at time t (in minutes) after the inflow, outflow starts. Further assume that y is a differentiable function.

When $t = 0$, i.e., before the inflow–outflow of the brine starts,

$$y = 250 \text{ g} \times 1000 = 250 \text{ kg}$$

Note that the change in y occurs due to the inflow, outflow of the mixture.

Now the inflow of brine brings salt into the tank at the rate of 5 kg per minute (as 25×200 g = 5 kg) and the outflow of brine takes salt out of the tank at the rate of $25\left(\frac{y}{1000}\right) = \frac{y}{40}$ kg per minute (as at time t, the salt in the tank is $\frac{y}{1000}$ kg).

Thus, the rate of change of salt with respect to t is given by

$$\frac{dy}{dt} = 5 - \frac{y}{40} \qquad \text{(Why?)}$$

or

$$\frac{dy}{dt} + \frac{1}{40}y = 5 \qquad \dots (1)$$

This gives a mathematical model for the given problem.

Step 3 Equation (1) is a linear equation and can be easily solved. The solution of (1) is given by

$$y e^{\frac{t}{40}} = 200 e^{\frac{t}{40}} + \mathrm{C} \text{ or } y(t) = 200 + \mathrm{C}\, e^{-\frac{t}{40}} \qquad \dots (2)$$

where, c is the constant of integration.

Note that when $t = 0$, $y = 250$. Therefore, $250 = 200 + \mathrm{C}$

or $\mathrm{C} = 50$

Then (2) reduces to

$$y = 200 + 50\, e^{-\frac{t}{40}} \qquad \dots (3)$$

or $$\frac{y-200}{50} = e^{-\frac{t}{40}}$$

or $$e^{\frac{t}{40}} = \frac{50}{y-200}$$

Therefore $$t = 40 \log_e \left(\frac{50}{y-200}\right) \qquad \dots (4)$$

Here, the equation (4) gives the time t at which the salt in tank is y kg.

Step 4 Since $e^{-\frac{t}{40}}$ is always positive, from (3), we conclude that $y > 200$ at all times Thus, the minimum amount of salt content in the tank is 200 kg.

Also, from (4), we conclude that $t > 0$ if and only if $0 < y - 200 < 50$ i.e., if and only if $200 < y < 250$ i.e., the amount of salt content in the tank after the start of inflow and outflow of the brine is between 200 kg and 250 kg.

Limitations of Mathematical Modelling

Till today many mathematical models have been developed and applied successfully to understand and get an insight into thousands of situations. Some of the subjects like mathematical physics, mathematical economics, operations research, bio-mathematics etc. are almost synonymous with mathematical modelling.

But there are still a large number of situations which are yet to be modelled. The reason behind this is that either the situation are found to be very complex or the mathematical models formed are mathematically intractable.

The development of the powerful computers and super computers has enabled us to mathematically model a large number of situations (even complex situations). Due to these fast and advanced computers, it has been possible to prepare more realistic models which can obtain better agreements with observations.

However, we do not have good guidelines for choosing various parameters / variables and also for estimating the values of these parameters / variables used in a mathematical model. Infact, we can prepare reasonably accurate models to fit any data by choosing five or six parameters / variables. We require a minimal number of parameters / variables to be able to estimate them accurately.

Mathematical modelling of large or complex situations has its own special problems. These type of situations usually occur in the study of world models of environment, oceanography, pollution control etc. Mathematical modellers from all disciplines — mathematics, computer science, physics, engineering, social sciences, etc., are involved in meeting these challenges with courage.

ANSWERS

EXERCISE 1.1

1. (i) Neither reflexive nor symmetric nor transitive.
   (ii) Neither reflexive nor symmetric but transitive.
   (iii) Reflexive and transitive but not symmetric.
   (iv) Reflexive, symmetric and transitive.
   (v) (a) Reflexive, symmetric and transitive.
       (b) Reflexive, symmetric and transitive.
       (c) Neither reflexive nor symmetric nor transitive.
       (d) Neither reflexive nor symmetric but transitive.
       (e) Neither reflexive nor symmetric nor transitive.
3. Neither reflexive nor symmetric nor transitive.
5. Neither reflexive nor symmetric nor transitive.
9. (i) {1, 5, 9}, (ii) {1}
12. T_1 is related to T_3.
13. The set of all triangles
14. The set of all lines $y = 2x + c$, $c \in \mathbf{R}$
15. B
16. C

EXERCISE 1.2

1. No
2. (i) Injective but not surjective
   (ii) Neither injective nor surjective
   (iii) Neither injective nor surjective
   (iv) Injective but not surjective
   (v) Injective but not surjective
7. (i) One-one and onto
   (ii) Neither one-one nor onto.
9. No
10. Yes
11. D
12. A

Miscellaneous Exercise on Chapter 1

3. No
4. $n!$
5. Yes
6. A
7. B

EXERCISE 2.1

1. $\frac{-\pi}{6}$ 2. $\frac{\pi}{6}$ 3. $\frac{\pi}{6}$ 4. $\frac{-\pi}{3}$

5. $\frac{2\pi}{3}$ 6. $-\frac{\pi}{4}$ 7. $\frac{\pi}{6}$ 8. $\frac{\pi}{6}$

9. $\frac{3\pi}{4}$ 10. $\frac{-\pi}{4}$ 11. $\frac{3\pi}{4}$ 12. $\frac{2\pi}{3}$

13. B 14. B

EXERCISE 2.2

3. $\frac{1}{2}\tan^{-1} x$ 4. $\frac{x}{2}$ 5. $\frac{\pi}{4} - x$

6. $\sin^{-1}\frac{x}{a}$ 7. $3\tan^{-1}\frac{x}{a}$ 8. $\frac{\pi}{4}$

9. $\frac{x+y}{1-xy}$ 10. $\frac{\pi}{3}$

11. $\frac{-\pi}{4}$ 12. $\frac{17}{6}$ 13. B 14. D

15. B

Miscellaneous Exercise on Chapter 2

1. $\frac{\pi}{6}$ 2. $\frac{\pi}{6}$ 11. $x = n\pi + \frac{\pi}{4}, n \in \mathbf{Z}$ 12. $x = \frac{1}{\sqrt{3}}$

13. D 14. C

EXERCISE 3.1

1. (i) 3×4 (ii) 12 (iii) $19, 35, -5, 12, \frac{5}{2}$
2. $1 \times 24, 2 \times 12, 3 \times 8, 4 \times 6, 6 \times 4, 8 \times 3, 12 \times 2, 24 \times 1; 1 \times 13, 13 \times 1$
3. $1 \times 18, 2 \times 9, 3 \times 6, 6 \times 3, 9 \times 2, 18 \times 1; 1 \times 5, 5 \times 1$
4. (i) $\begin{bmatrix} 2 & \frac{9}{2} \\ \frac{9}{2} & 8 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & \frac{1}{2} \\ 2 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} \frac{9}{2} & \frac{25}{2} \\ 8 & 18 \end{bmatrix}$

5. (i) $\begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{2} & 2 & \frac{3}{2} & 1 \\ 4 & \frac{7}{2} & 3 & \frac{5}{2} \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & -1 & -2 \\ 3 & 2 & 1 & 0 \\ 5 & 4 & 3 & 2 \end{bmatrix}$

6. (i) $x = 1, \quad y = 4, \quad z = 3$

(ii) $x = 4, \quad y = 2, \quad z = 0$ or $x = 2, \quad y = 4, \ z = 0$

(iii) $x = 2, \quad y = 4, \quad z = 3$

7. $a = 1, b = 2, c = 3, d = 4$

8. C **9.** B **10.** D

EXERCISE 3.2

1. (i) $A + B = \begin{bmatrix} 3 & 7 \\ 1 & 7 \end{bmatrix}$ (ii) $A - B = \begin{bmatrix} 1 & 1 \\ 5 & -3 \end{bmatrix}$

(iii) $3A - C = \begin{bmatrix} 8 & 7 \\ 6 & 2 \end{bmatrix}$ (iv) $AB = \begin{bmatrix} -6 & 26 \\ -1 & 19 \end{bmatrix}$ (v) $BA = \begin{bmatrix} 11 & 10 \\ 11 & 2 \end{bmatrix}$

2. (i) $\begin{bmatrix} 2a & 2b \\ 0 & 2a \end{bmatrix}$ (ii) $\begin{bmatrix} (a+b)^2 & (b+c)^2 \\ (a-c)^2 & (a-b)^2 \end{bmatrix}$

(iii) $\begin{bmatrix} 11 & 11 & 0 \\ 16 & 5 & 21 \\ 5 & 10 & 9 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

3. (i) $\begin{bmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{bmatrix}$ (iii) $\begin{bmatrix} -3 & -4 & 1 \\ 8 & 13 & 9 \end{bmatrix}$

(iv) $\begin{bmatrix} 14 & 0 & 42 \\ 18 & -1 & 56 \\ 22 & -2 & 70 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ -2 & 2 & 0 \end{bmatrix}$ (vi) $\begin{bmatrix} 14 & -6 \\ 4 & 5 \end{bmatrix}$

4. $A + B = \begin{bmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{bmatrix}$, $B - C = \begin{bmatrix} -1 & -2 & 0 \\ 4 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **6.** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

7. (i) $X = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}, Y = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ (ii) $X = \begin{bmatrix} \frac{2}{5} & \frac{-12}{5} \\ \frac{-11}{5} & 3 \end{bmatrix}, Y = \begin{bmatrix} \frac{2}{5} & \frac{13}{5} \\ \frac{14}{5} & -2 \end{bmatrix}$

8. $X = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$ **9.** $x = 3, y = 3$ **10.** $x = 3, y = 6, z = 9, t = 6$

11. $x = 3, y = -4$ **12.** $x = 2, y = 4, w = 3, z = 1$

15. $\begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$ **17.** $k = 1$

19. (a) ₹15000, ₹15000 (b) ₹5000, ₹25000

20. ₹20160 **21.** A **22.** B

EXERCISE 3.3

1. (i) $\begin{bmatrix} 5 & \frac{1}{2} & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} -1 & \sqrt{3} & 2 \\ 5 & 5 & 3 \\ 6 & 6 & -1 \end{bmatrix}$

4. $\begin{bmatrix} -4 & 5 \\ 1 & 6 \end{bmatrix}$ **9.** $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$

10. (i) $A = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 3 & \frac{1}{2} & \frac{-5}{2} \\ \frac{1}{2} & -2 & -2 \\ \frac{-5}{2} & -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & \frac{5}{2} & \frac{3}{2} \\ \frac{-5}{2} & 0 & 3 \\ \frac{-3}{2} & -3 & 0 \end{bmatrix}$ (iv) $A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$

11. A **12.** B

EXERCISE 3.4

1. D

Miscellaneous Exercise on Chapter 3

3. $x = \pm\frac{1}{\sqrt{2}}, y = \pm\frac{1}{\sqrt{6}}, z = \pm\frac{1}{\sqrt{3}}$

4. $x = -1$ **6.** $x = \pm 4\sqrt{3}$

7. (a) Total revenue in the market - I = ₹ 46000

Total revenue in the market - II = ₹ 53000

(b) ₹ 15000, ₹ 17000

8. $X = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$ **9.** C **10.** B **11.** C

EXERCISE 4.1

1. (i) 18 **2.** (i) 1, (ii) $x^3 - x^2 + 2$

5. (i) – 12, (ii) 46, (iii) 0, (iv) 5 **6.** 0

7. (i) $x = \pm\sqrt{3}$, (ii) $x = 2$ **8.** (B)

EXERCISE 4.2

1. (i) $\frac{15}{2}$, (ii) $\frac{47}{2}$, (iii) 15

3. (i) 0, 8, (ii) 0, 8 **4.** (i) $y = 2x$, (ii) $x - 3y = 0$ **5.** (D)

EXERCISE 4.3

1. (i) $M_{11} = 3, M_{12} = 0, M_{21} = -4, M_{22} = 2, A_{11} = 3, A_{12} = 0, A_{21} = 4, A_{22} = 2$

(ii) $M_{11} = d, M_{12} = b, M_{21} = c, M_{22} = a$

$A_{11} = d, A_{12} = -b, A_{21} = -c, A_{22} = a$

2. (i) $M_{11}= 1, M_{12}= 0, M_{13}= 0, M_{21} = 0, M_{22}= 1, M_{23}= 0, M_{31} = 0, M_{32}= 0, M_{33}= 1,$

$A_{11}= 1, A_{12}= 0, A_{13}= 0, A_{21}= 0, A_{22}= 1, A_{23}= 0, A_{31}= 0, A_{32}= 0, A_{33}= 1$

(ii) $M_{11}= 11, M_{12}= 6, M_{13}= 3, M_{21}= -4, M_{22}= 2, M_{23}= 1, M_{31}= -20, M_{32}= -13, M_{33}= 5$

$A_{11}= 11, A_{12}= -6, A_{13}= 3, A_{21}= 4, A_{22}= 2, A_{23}= -1, A_{31}= -20, A_{32}= 13, A_{33}= 5$

3. 7 **4.** $(x - y)(y - z)(z - x)$ **5.** (D)

EXERCISE 4.4

1. $\begin{matrix} 4 & -2 \\ -3 & 1 \end{matrix}$ **2.** $\begin{matrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{matrix}$ **5.** $\frac{1}{14}\begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$

6. $\frac{1}{13}\begin{matrix} 2 & -5 \\ 3 & -1 \end{matrix}$ **7.** $\frac{1}{10}\begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$ **8.** $\frac{-1}{3}\begin{matrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{matrix}$

9. $\frac{-1}{3}\begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$ **10.** $\begin{matrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{matrix}$ **11.** $\begin{matrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{matrix}$

13. $\frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ **14.** $a = -4, b = 1$ **15.** $A^{-1} = \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$

16. $\frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$ **17.** B **18.** B

EXERCISE 4.5

1. Consistent **2.** Consistent **3.** Inconsistent

4. Consistent **5.** Inconsistent **6.** Consistent

7. $x = 2, y = -3$ **8.** $x = \frac{-5}{11}, y = \frac{12}{11}$ **9.** $x = \frac{-6}{11}, y = \frac{-19}{11}$

10. $x = -1, y = 4$ **11.** $x = 1, y = \frac{1}{2}, z = \frac{-3}{2}$

12. $x = 2, y = -1, z = 1$ **13.** $x = 1, y = 2, z = -1$

14. $x = 2, y = 1, z = 3$

15. $\begin{matrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{matrix}$, $x = 1, y = 2, z = 3$

16. cost of onions per kg = ₹ 5
cost of wheat per kg = ₹ 8
cost of rice per kg = ₹ 8

Miscellaneous Exercise on Chapter 4

2. 1 **3.** $\begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

5. $-2(x^3 + y^3)$ **6.** xy **7.** $x = 2, y = 3, z = 5$

8. A **9.** D

EXERCISE 5.1

2. f is continuous at $x = 3$

3. (a), (b), (c) and (d) are all continuous functions

5. f is continuous at $x = 0$ and $x = 2$; Not continuous at $x = 1$

6. Discontinuous at $x = 2$ **7.** Discontinuous at $x = 3$

8. Discontinuous at $x = 0$
9. No point of discontinuity
10. No point of discontinuity
11. No point of discontinuity
12. f is discontinuous at $x = 1$
13. f is not continuous at $x = 1$
14. f is not continuous at $x = 1$ and $x = 3$
15. $x = 1$ is the only point of discontinuity
16. Continuous
17. $a = b + \dfrac{2}{3}$
18. For no value of λ, f is continuous at $x = 0$ but f is continuous at $x = 1$ for any value of λ.
20. f is continuous at $x = \pi$
21. (a), (b) and (c) are all continuous
22. Cosine function is continuous for all $x \in \mathbf{R}$; cosecant is continuous except for $x = n\pi$, $n \in \mathbf{Z}$; secant is continuous except for $x = (2n+1)\dfrac{\pi}{2}$, $n \in \mathbf{Z}$ and cotangent function is continuous except for $x = n\pi$, $n \in \mathbf{Z}$
23. There is no point of discontinuity.
24. Yes, f is continuous for all $x \in \mathbf{R}$
25. f is continuous for all $x \in \mathbf{R}$
26. $k = 6$
27. $k = \dfrac{3}{4}$
28. $k = \dfrac{-2}{\pi}$
29. $k = \dfrac{9}{5}$
30. $a = 2$, $b = 1$
34. There is no point of discontinuity.

EXERCISE 5.2

1. $2x \cos(x^2 + 5)$
2. $-\cos x \sin(\sin x)$
3. $a \cos(ax + b)$
4. $\dfrac{\sec(\tan\sqrt{x}).\tan(\tan\sqrt{x}).\sec^2\sqrt{x}}{2\sqrt{x}}$
5. $a \cos(ax + b) \sec(cx + d) + c \sin(ax + b) \tan(cx + d) \sec(cx + d)$
6. $10x^4 \sin x^5 \cos x^5 \cos x^3 - 3x^2 \sin x^3 \sin^2 x^5$
7. $\dfrac{-2\sqrt{2}\,x}{\sin x^2\sqrt{\sin 2x^2}}$
8. $-\dfrac{\sin\sqrt{x}}{2\sqrt{x}}$

EXERCISE 5.3

1. $\dfrac{\cos x - 2}{3}$
2. $\dfrac{2}{\cos y - 3}$
3. $-\dfrac{a}{2by + \sin y}$
4. $\dfrac{\sec^2 x - y}{x + 2y - 1}$
5. $-\dfrac{(2x+y)}{(x+2y)}$
6. $-\dfrac{(3x^2 + 2xy + y^2)}{(x^2 + 2xy + 3y^2)}$
7. $\dfrac{y \sin xy}{\sin 2y - x \sin xy}$
8. $\dfrac{\sin 2x}{\sin 2y}$
9. $\dfrac{2}{1+x^2}$
10. $\dfrac{3}{1+x^2}$
11. $\dfrac{2}{1+x^2}$
12. $\dfrac{-2}{1+x^2}$
13. $\dfrac{-2}{1+x^2}$
14. $\dfrac{2}{\sqrt{1-x^2}}$
15. $-\dfrac{2}{\sqrt{1-x^2}}$

EXERCISE 5.4

1. $\dfrac{e^x(\sin x - \cos x)}{\sin^2 x}, x \neq n\pi, n \in \mathbf{Z}$
2. $\dfrac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}, x \in (-1, 1)$
3. $3x^2 e^{x^3}$
4. $-\dfrac{e^{-x} \cos(\tan^{-1} e^{-x})}{1 + e^{-2x}}$
5. $-e^x \tan e^x,\ e^x \neq (2n+1)\dfrac{\pi}{2}, n \in \mathbf{N}$
6. $e^x + 2xe^{x^2} + 3x^2 e^{x^3} + 4x^3 e^{x^4} + 5x^4 e^{x^5}$
7. $\dfrac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}}, x > 0$
8. $\dfrac{1}{x \log x}, x > 1$
9. $-\dfrac{(x \sin x \cdot \log x + \cos x)}{x(\log x)^2}, x > 0$
10. $-\left(\dfrac{1}{x} + e^x\right) \sin(\log x + e^x), x > 0$

EXERCISE 5.5

1. $-\cos x \cos 2x \cos 3x\,[\tan x + 2\tan 2x + 3\tan 3x]$
2. $\dfrac{1}{2}\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\dfrac{1}{x-1} + \dfrac{1}{x-2} - \dfrac{1}{x-3} - \dfrac{1}{x-4} - \dfrac{1}{x-5}\right]$

3. $(\log x)^{\cos x}\left[\dfrac{\cos x}{x\log x}-\sin x\log(\log x)\right]$

4. $x^x(1+\log x)-2^{\sin x}\cos x\log 2$

5. $(x+3)(x+4)^2(x+5)^3(9x^2+70x+133)$

6. $\left(x+\dfrac{1}{x}\right)^x\left[\dfrac{x^2-1}{x^2+1}+\log\left(x+\dfrac{1}{x}\right)\right]+x^{1+\frac{1}{x}}\left(\dfrac{x+1-\log x}{x^2}\right)$

7. $(\log x)^{x-1}[1+\log x\,.\,\log(\log x)]+2x^{\log x-1}\,.\,\log x$

8. $(\sin x)^x(x\cot x+\log\sin x)+\dfrac{1}{2}\dfrac{1}{\sqrt{x-x^2}}$

9. $x^{\sin x}\left[\dfrac{\sin x}{x}+\cos x\log x\right]+(\sin x)^{\cos x}[\cos x\cot x-\sin x\log\sin x]$

10. $x^{x\cos x}[\cos x\,.\,(1+\log x)-x\sin x\log x]-\dfrac{4x}{(x^2-1)^2}$

11. $(x\cos x)^x[1-x\tan x+\log(x\cos x)]+(x\sin x)^{\frac{1}{x}}\left[\dfrac{x\cot x+1-\log(x\sin x)}{x^2}\right]$

12. $-\dfrac{yx^{y-1}+y^x\log y}{x^y\log x+xy^{x-1}}$

13. $\dfrac{y}{x}\left(\dfrac{y-x\log y}{x-y\log x}\right)$

14. $\dfrac{y\tan x+\log\cos y}{x\tan y+\log\cos x}$

15. $\dfrac{y(x-1)}{x(y+1)}$

16. $(1+x)(1+x^2)(1+x^4)(1+x^8)\left[\dfrac{1}{1+x}+\dfrac{2x}{1+x^2}+\dfrac{4x^3}{1+x^4}+\dfrac{8x^7}{1+x^8}\right];\ f'(1)=120$

17. $5x^4-20x^3+45x^2-52x+11$

EXERCISE 5.6

1. t^2

2. $\dfrac{b}{a}$

3. $-4\sin t$

4. $-\dfrac{1}{t^2}$

5. $\dfrac{\cos\theta-2\cos 2\theta}{2\sin 2\theta-\sin\theta}$

6. $-\cot\dfrac{\theta}{2}$

7. $-\cot 3t$

8. $\tan t$

9. $\dfrac{b}{a}\operatorname{cosec}\theta$

10. $\tan\theta$

EXERCISE 5.7

1. 2
2. $380\, x^{18}$
3. $-x \cos x - 2 \sin x$
4. $-\dfrac{1}{x^2}$
5. $x(5 + 6 \log x)$
6. $2e^x (5 \cos 5x - 12 \sin 5x)$
7. $9\, e^{6x} (3 \cos 3x - 4 \sin 3x)$
8. $-\dfrac{2x}{(1+x^2)^2}$
9. $-\dfrac{(1+\log x)}{(x \log x)^2}$
10. $-\dfrac{\sin(\log x) + \cos(\log x)}{x^2}$
12. $-\cot y \operatorname{cosec}^2 y$

Miscellaneous Exercise on Chapter 5

1. $27\,(3x^2 - 9x + 5)^8\,(2x - 3)$
2. $3\sin x \cos x\,(\sin x - 2\cos^4 x)$
3. $(5x)^{3\cos 2x}\left[\dfrac{3\cos 2x}{x} - 6\sin 2x \log 5x\right]$
4. $\dfrac{3}{2}\sqrt{\dfrac{x}{1-x^3}}$
5. $-\left[\dfrac{1}{\sqrt{4-x^2}\,\sqrt{2x+7}} + \dfrac{\cos^{-1}\dfrac{x}{2}}{(2x+7)^{\frac{3}{2}}}\right]$
6. $\dfrac{1}{2}$
7. $(\log x)^{\log x}\left[\dfrac{1}{x} + \dfrac{\log(\log x)}{x}\right],\ x > 1$
8. $(a \sin x - b \cos x) \sin (a \cos x + b \sin x)$
9. $(\sin x - \cos x)^{\sin x - \cos x} (\cos x + \sin x)\,(1 + \log(\sin x - \cos x)),\ \sin x > \cos x$
10. $x^x (1 + \log x) + a x^{a-1} + a^x \log a$
11. $x^{x^2-3}\left[\dfrac{x^2-3}{x} + 2x\log x\right] + (x-3)^{x^2}\left[\dfrac{x^2}{x-3} + 2x\log(x-3)\right]$
12. $\dfrac{6}{5}\cot\dfrac{t}{2}$
13. 0
17. $\dfrac{\sec^3 t}{at},\ 0 < t < \dfrac{\pi}{2}$

EXERCISE 6.1

1. (a) $6\pi\ \text{cm}^2/\text{cm}$ (b) $8\pi\ \text{cm}^2/\text{cm}$

2. $\frac{8}{3}\ \text{cm}^2/\text{s}$ **3.** $60\pi\ \text{cm}^2/\text{s}$ **4.** $900\ \text{cm}^3/\text{s}$

5. $80\pi\ \text{cm}^2/\text{s}$ **6.** $1.4\pi\ \text{cm/s}$

7. (a) -2 cm/min (b) $2\ \text{cm}^2/\text{min}$

8. $\frac{1}{\pi}$ cm/s **9.** $400\pi\ \text{cm}^3/\text{cm}$ **10.** $\frac{8}{3}$ cm/s

11. $(4, 11)$ and $\left(-4, \frac{-31}{3}\right)$ **12.** $2\pi\ \text{cm}^3/\text{s}$

13. $\frac{27}{8}\pi(2x+1)^2$ **14.** $\frac{1}{48\pi}$ cm/s **15.** ₹ 20.967

16. ₹ 208 **17.** B **18.** D

EXERCISE 6.2

4. (a) $\left(\frac{3}{4}, \infty\right)$ (b) $\left(-\infty, \frac{3}{4}\right)$

5. (a) $(-\infty, -2)$ and $(3, \infty)$ (b) $(-2, 3)$

6. (a) decreasing for $x < -1$ and increasing for $x > -1$

(b) decreasing for $x > -\frac{3}{2}$ and increasing for $x < -\frac{3}{2}$

(c) increasing for $-2 < x < -1$ and decreasing for $x < -2$ and $x > -1$

(d) increasing for $x < -\frac{9}{2}$ and decreasing for $x > -\frac{9}{2}$

(e) increasing in $(1, 3)$ and $(3, \infty)$, decreasing in $(-\infty, -1)$ and $(-1, 1)$.

8. $0 < x < 1$ and $x > 2$ **12.** A, B

13. D **14.** $a > -2$ **19.** D

EXERCISE 6.3

1. (i) Minimum Value = 3 (ii) Minimum Value = – 2
   (iii) Maximum Value = 10 (iv) Neither minimum nor maximum value
2. (i) Minimum Value = – 1; No maximum value
   (ii) Maximum Value = 3; No minimum value
   (iii) Minimum Value = 4; Maximum Value = 6
   (iv) Minimum Value = 2; Maximum Value = 4
   (v) Neither minimum nor Maximum Value
3. (i) local minimum at $x = 0$, local minimum value = 0
   (ii) local minimum at $x = 1$, local minimum value = – 2
   local maximum at $x = -1$, local maximum value = 2
   (iii) local maximum at $x = \frac{\pi}{4}$, local maximum value = $\sqrt{2}$
   (iv) local maximum at $x = \frac{3\pi}{4}$, local maximum value = $\sqrt{2}$
   local minimum at $x = \frac{7\pi}{4}$, local minimum value = $-\sqrt{2}$
   (v) local maximum at $x = 1$, local maximum value = 19
   local minimum at $x = 3$, local minimum value = 15
   (vi) local minimum at $x = 2$, local minimum value = 2
   (vii) local maximum at $x = 0$, local maximum value = $\frac{1}{2}$
   (viii) local maximum at $x = \frac{2}{3}$, local maximum value = $\frac{2\sqrt{3}}{9}$

5. (i) Absolute minimum value = – 8, absolute maximum value = 8

(ii) Absolute minimum value = – 1, absolute maximum value = $\sqrt{2}$

(iii) Absolute minimum value = – 10, absolute maximum value = 8

(iv) Absolute minimum value = 19, absolute maximum value = 3

6. Maximum profit = 113 unit.

7. Minima at $x = 2$, minimum value = – 39, Maxima at $x = 0$, maximum value = 25.

8. At $x = \frac{\pi}{4}$ and $\frac{5\pi}{4}$

9. Maximum value = $\sqrt{2}$

10. Maximum at $x = 3$, maximum value 89; maximum at $x = -2$, maximum value = 139

11. $a = 120$

12. Maximum at $x = 2\pi$, maximum value = 2π; Minimum at $x = 0$, minimum value = 0

13. 12, 12

14. 45, 15

15. 25, 10

16. 8, 8

17. 3 cm

18. $x = 5$ cm

21. radius = $\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm and height = $2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm

22. $\frac{112}{\pi+4}$ cm, $\frac{28\pi}{\pi+4}$ cm

27. A

28. D

29. C

Miscellaneous Exercise on Chapter 6

2. $b\sqrt{3}$ cm²/s

3. (i) $0 \le x \le \frac{\pi}{2}$ and $\frac{3\pi}{2} < x < 2\pi$ (ii) $\frac{\pi}{2} < x < \frac{3\pi}{2}$

4. (i) $x < -1$ and $x > 1$ (ii) $-1 < x < 1$

5. $\dfrac{3\sqrt{3}}{4}ab$ 6. Rs 1000

8. length $= \dfrac{20}{\pi + 4}$ m, breadth $= \dfrac{10}{\pi + 4}$ m

10. (i) local maxima at $x = \dfrac{2}{7}$ (ii) local minima at $x = 2$

(iii) point of inflection at $x = -1$

11. Absolute maximum $= \dfrac{5}{4}$, Absolute minimum $= 1$

14. $\dfrac{4\pi R^3}{3\sqrt{3}}$ 16. A

SUPPLEMENTARY MATERIAL

CHAPTER 5

Theorem 5 (To be on page 129 under the heading Theorem 5)

(i) Derivative of Exponential Function $f(x) = e^x$.

If $f(x) = e^x$, then

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x}$$

$$= e^x \cdot \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x}$$

$$= e^x \cdot 1 \quad [\text{since } \lim_{h \to 0} \frac{e^h - 1}{h} = 1]$$

Thus, $\frac{d}{dx}(e^x) = e^x$.

(ii) Derivative of logarithmic function $f(x) = \log_e x$.

If $f(x) = \log_e x$, then

$$f'(x) = \lim_{\Delta x \to 0} \frac{\log_e(x+\Delta x) - \log_e x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\log_e\left(1 + \frac{\Delta x}{x}\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{x} \frac{\log_e\left(1 + \frac{\Delta x}{x}\right)}{\frac{\Delta x}{x}}$$

$$= \frac{1}{x} \quad [\text{since } \lim_{h \to 0} \frac{\log_e(1+h)}{h} = 1]$$

Thus, $\frac{d}{dx}\log_e x = \frac{1}{x}$.

NOTES

MATHEMATICS

Textbook for Class XII

Part I

12079

विद्यया ऽमृतमश्नुते

एन सी ई आर टी
NCERT

राष्ट्रीय शैक्षिक अनुसंधान और प्रशिक्षण परिषद्
NATIONAL COUNCIL OF EDUCATIONAL RESEARCH AND TRAINING

Reprint 2024-25

12079 – MATHEMATICS
Textbook for Class XII

ISBN 81-7450-629-2

First Edition
November 2006 Agrahayana 1928

Reprinted
October 2007, December 2008, December 2009, January 2012, November 2012, November 2013, November 2014, December 2015, December 2016, December 2017, January 2019, August 2019, January 2021, July 2021 and November 2021

Revised Edition
October 2022 Kartika 1944

Reprinted
March 2024 Chaitra 1946

PD 275T SU

₹ 150.00

Printed on 80 GSM paper with NCERT watermark

Published at the Publication Division by the Secretary, National Council of Educational Research and Training, Sri Aurobindo Marg, New Delhi 110 016 and printed at Green World Publications (India) Pvt. Ltd., Mander Mode, Bamrauli, Allahabad (U.P.) 211 003

OFFICES OF THE PUBLICATION DIVISION, NCERT

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Cover
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Foreword

The National Curriculum Framework 2005, recommends that children's life at school must be linked to their life outside the school. This principle marks a departure from the legacy of bookish learning which continues to shape our system and causes a gap between the school, home and community. The syllabi and textbooks developed on the basis of NCF signify an attempt to implement this basic idea. They also attempt to discourage rote learning and the maintenance of sharp boundaries between different subject areas. We hope these measures will take us significantly further in the direction of a child-centred system of education outlined in the National Policy on Education (1986).

The success of this effort depends on the steps that school principals and teachers will take to encourage children to reflect on their own learning and to pursue imaginative activities and questions. We must recognise that, given space, time and freedom, children generate new knowledge by engaging with the information passed on to them by adults. Treating the prescribed textbook as the sole basis of examination is one of the key reasons why other resources and sites of learning are ignored. Inculcating creativity and initiative is possible if we perceive and treat children as participants in learning, not as receivers of a fixed body of knowledge.

These aims imply considerable change in school routines and mode of functioning. Flexibility in the daily time-table is as necessary as rigour in implementing the annual calendar so that the required number of teaching days are actually devoted to teaching. The methods used for teaching and evaluation will also determine how effective this textbook proves for making children's life at school a happy experience, rather than a source of stress or boredom. Syllabus designers have tried to address the problem of curricular burden by restructuring and reorienting knowledge at different stages with greater consideration for child psychology and the time available for teaching. The textbook attempts to enhance this endeavour by giving higher priority and space to opportunities for contemplation and wondering, discussion in small groups, and activities requiring hands-on experience.

NCERT appreciates the hard work done by the textbook development committee responsible for this book. We wish to thank the Chairperson of the advisory group in Science and Mathematics, Professor J.V. Narlikar and the Chief Advisor for this book, Professor P.K. Jain for guiding the work of this committee. Several teachers contributed to the development of this textbook; we are grateful to their principals for making this possible. We are indebted to the institutions and organisations which have generously permitted us to draw upon their resources, material and personnel. As an organisation committed to systemic reform and continuous improvement in the quality of its products, NCERT welcomes comments and suggestions which will enable us to undertake further revision and refinement.

Director
National Council of Educational
Research and Training

New Delhi
20 December 2005

Reprint 2024-25

Rationalisation of Content in the Textbook

In view of the COVID-19 pandemic, it is imperative to reduce content load on students. The National Education Policy 2020, also emphasises reducing the content load and providing opportunities for experiential learning with creative mindset. In this background, the NCERT has undertaken the exercise to rationalise the textbooks across all classes. Learning Outcomes already developed by the NCERT across classes have been taken into consideration in this exercise.

Contents of the textbooks have been rationalised in view of the following:

- Overlapping with similar content included in other subject areas in the same class
- Similar content included in the lower or higher class in the same subject
- Difficulty level
- Content, which is easily accessible to students without much interventions from teachers and can be learned by children through self-learning or peer-learning
- Content, which is irrelevant in the present context

This present edition, is a reformatted version after carrying out the changes given above.

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Preface

The National Council of Educational Research and Training (NCERT) had constituted 21 Focus Groups on Teaching of various subjects related to School Education, to review the National Curriculum Framework for School Education - 2000 (NCFSE - 2000) in face of new emerging challenges and transformations occurring in the fields of content and pedagogy under the contexts of National and International spectrum of school education. These Focus Groups made general and specific comments in their respective areas. Consequently, based on these reports of Focus Groups, National Curriculum Framework (NCF)-2005 was developed.

NCERT designed the new syllabi and constituted Textbook Development Teams for Classes XI and XII to prepare textbooks in mathematics under the new guidelines and new syllabi. The textbook for Class XI is already in use, which was brought in 2005.

The first draft of the present book (Class XII) was prepared by the team consisting of NCERT faculty, experts and practicing teachers. The draft was refined by the development team in different meetings. This draft of the book was exposed to a group of practicing teachers teaching mathematics at higher secondary stage in different parts of the country, in a review workshop organised by the NCERT at Delhi. The teachers made useful comments and suggestions which were incorporated in the draft textbook. The draft textbook was finalised by an editorial board constituted out of the development team. Finally, the Advisory Group in Science and Mathematics and the Monitoring Committee constituted by the HRD Ministry, Government of India have approved the draft of the textbook.

In the fitness of things, let us cite some of the essential features dominating the textbook. These characteristics have reflections in almost all the chapters. The existing textbook contain 13 main chapters and two appendices. Each Chapter contain the followings:

- Introduction: Highlighting the importance of the topic; connection with earlier studied topics; brief mention about the new concepts to be discussed in the chapter.
- Organisation of chapter into sections comprising one or more concepts/sub concepts.
- Motivating and introducing the concepts/sub concepts. Illustrations have been provided wherever possible.

- Proofs/problem solving involving deductive or inductive reasoning, multiplicity of approaches wherever possible have been inducted.
- Geometric viewing / visualisation of concepts have been emphasised whenever needed.
- Applications of mathematical concepts have also been integrated with allied subjects like science and social sciences.
- Adequate and variety of examples/exercises have been given in each section.
- For refocusing and strengthening the understanding and skill of problem solving and applicabilities, miscellaneous types of examples/exercises have been provided involving two or more sub concepts at a time at the end of the chapter. The scope of challenging problems to talented minority have been reflected conducive to the recommendation as reflected in NCF-2005.
- For more motivational purpose, brief historical background of topics have been provided at the end of the chapter and at the beginning of each chapter relevant quotation and photograph of eminent mathematician who have contributed significantly in the development of the topic undertaken, are also provided.
- Lastly, for direct recapitulation of main concepts, formulas and results, brief summary of the chapter has also been provided.

I am thankful to Professor Krishan Kumar, Director, NCERT who constituted the team and invited me to join this national endeavor for the improvement of mathematics education. He has provided us with an enlightened perspective and a very conducive environment. This made the task of preparing the book much more enjoyable and rewarding. I express my gratitude to Professor J.V. Narlikar, Chairperson of the Advisory Group in Science and Mathematics, for his specific suggestions and advice towards the improvement of the book from time to time. I, also, thank Prof. G. Ravindra, Joint Director, NCERT for his help from time to time.

I express my sincere thanks to Professor Hukum Singh, Chief Coordinator and Head DESM, Dr. V. P. Singh, Coordinator and Professor S. K. Singh Gautam who have been helping for the success of this project academically as well as administratively. Also, I would like to place on records my appreciation and thanks to all the members of the team and the teachers who have been associated with this noble cause in one or the other form.

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Acknowledgements

The Council gratefully acknowledges the valuable contributions of the following participants of the Textbook Review Workshop: Jagdish Saran, *Professor*, Deptt. of Statistics, University of Delhi; Quddus Khan, *Lecturer*, Shibli National P.G. College Azamgarh (U.P.); P.K. Tewari, *Assistant Commissioner* (Retd.), Kendriya Vidyalaya Sangathan; S.B. Tripathi, *Lecturer*, R.P.V.V. Surajmal Vihar, Delhi; O.N. Singh, *Reader*, RIE, Bhubaneswar, Orissa; Miss Saroj, *Lecturer*, Govt. Girls Senior Secondary School No.1, Roop Nagar, Delhi; P. Bhaskar Kumar, *PGT*, Jawahar Navodaya Vidyalaya, Lepakshi, Anantapur, (A.P.); Mrs. S. Kalpagam, *PGT*, K.V. NAL Campus, Bangalore; Rahul Sofat, *Lecturer*, Air Force Golden Jubilee Institute, Subroto Park, New Delhi; Vandita Kalra, *Lecturer,* Sarvodaya Kanya Vidyalaya, Vikaspuri, District Centre, New Delhi; Janardan Tripathi, *Lecturer,* Govt. R.H.S.S. Aizawl, Mizoram and Ms. Sushma Jaireth, *Reader,* DWS, NCERT, New Delhi.

The Council acknowledges the efforts of Deepak Kapoor, *Incharge*, Computer Station, Sajjad Haider Ansari, Rakesh Kumar and Nargis Islam, *D.T.P. Operators,* Monika Saxena,*Copy Editor* and Abhimanu Mohanty, *Proof Reader.*

The Contribution of APC-Office, administration of DESM and Publication Department is also duly acknowledged.

Contents

Part I

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Chapter 1

RELATIONS AND FUNCTIONS

❖ *There is no permanent place in the world for ugly mathematics ... . It may be very hard to define mathematical beauty but that is just as true of beauty of any kind, we may not know quite what we mean by a beautiful poem, but that does not prevent us from recognising one when we read it. — G. H. HARDY* ❖

1.1 Introduction

Lejeune Dirichlet (1805-1859)

Recall that the notion of relations and functions, domain, co-domain and range have been introduced in Class XI along with different types of specific real valued functions and their graphs. The concept of the term 'relation' in mathematics has been drawn from the meaning of relation in English language, according to which two objects or quantities are related if there is a recognisable connection or link between the two objects or quantities. Let A be the set of students of Class XII of a school and B be the set of students of Class XI of the same school. Then some of the examples of relations from A to B are

(i) $\{(a, b) \in \text{A} \times \text{B}: a \text{ is brother of } b\}$,

(ii) $\{(a, b) \in \text{A} \times \text{B}: a \text{ is sister of } b\}$,

(iii) $\{(a, b) \in \text{A} \times \text{B}: \text{age of } a \text{ is greater than age of } b\}$,

(iv) $\{(a, b) \in \text{A} \times \text{B}$: total marks obtained by a in the final examination is less than the total marks obtained by b in the final examination$\}$,

(v) $\{(a, b) \in \text{A} \times \text{B}: a \text{ lives in the same locality as } b\}$. However, abstracting from this, we define mathematically a relation R from A to B as an arbitrary subset of A × B.

If $(a, b) \in \text{R}$, we say that a is related to b under the relation R and we write as a R b. In general, $(a, b) \in \text{R}$, we do not bother whether there is a recognisable connection or link between a and b. As seen in Class XI, functions are special kind of relations.

In this chapter, we will study different types of relations and functions, composition of functions, invertible functions and binary operations.

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1.2 Types of Relations

In this section, we would like to study different types of relations. We know that a relation in a set A is a subset of A × A. Thus, the empty set ϕ and A × A are two extreme relations. For illustration, consider a relation R in the set A = {1, 2, 3, 4} given by R = $\{(a, b): a - b = 10\}$. This is the empty set, as no pair (a, b) satisfies the condition $a - b = 10$. Similarly, $R' = \{(a, b) : | a - b | \geq 0\}$ is the whole set A × A, as all pairs (a, b) in A × A satisfy $| a - b | \geq 0$. These two extreme examples lead us to the following definitions.

Definition 1 A relation R in a set A is called *empty relation*, if no element of A is related to any element of A, i.e., $R = \phi \subset A \times A$.

Definition 2 A relation R in a set A is called *universal relation*, if each element of A is related to every element of A, i.e., R = A × A.

Both the empty relation and the universal relation are some times called *trivial relations*.

Example 1 Let A be the set of all students of a boys school. Show that the relation R in A given by R = $\{(a, b) : a$ is sister of $b\}$ is the empty relation and $R' = \{(a, b)$: the difference between heights of a and b is less than 3 meters$\}$ is the universal relation.

Solution Since the school is boys school, no student of the school can be sister of any student of the school. Hence, $R = \phi$, showing that R is the empty relation. It is also obvious that the difference between heights of any two students of the school has to be less than 3 meters. This shows that $R' = A \times A$ is the universal relation.

Remark In Class XI, we have seen two ways of representing a relation, namely raster method and set builder method. However, a relation R in the set {1, 2, 3, 4} defined by R = $\{(a, b) : b = a + 1\}$ is also expressed as a R b if and only if $b = a + 1$ by many authors. We may also use this notation, as and when convenient.

If $(a, b) \in R$, we say that a is related to b and we denote it as a R b.

One of the most important relation, which plays a significant role in Mathematics, is an *equivalence relation*. To study equivalence relation, we first consider three types of relations, namely reflexive, symmetric and transitive.

Definition 3 A relation R in a set A is called

(i) *reflexive*, if $(a, a) \in R$, for every $a \in A$,

(ii) *symmetric*, if $(a_1, a_2) \in R$ implies that $(a_2, a_1) \in R$, for all $a_1, a_2 \in A$.

(iii) *transitive*, if $(a_1, a_2) \in R$ and $(a_2, a_3) \in R$ implies that $(a_1, a_3) \in R$, for all $a_1, a_2, a_3 \in A$.

Definition 4 A relation R in a set A is said to be an *equivalence relation* if R is reflexive, symmetric and transitive.

Example 2 Let T be the set of all triangles in a plane with R a relation in T given by R = $\{(T_1, T_2) : T_1$ is congruent to $T_2\}$. Show that R is an equivalence relation.

Solution R is reflexive, since every triangle is congruent to itself. Further, $(T_1, T_2) \in R \Rightarrow T_1$ is congruent to $T_2 \Rightarrow T_2$ is congruent to $T_1 \Rightarrow (T_2, T_1) \in R$. Hence, R is symmetric. Moreover, $(T_1, T_2), (T_2, T_3) \in R \Rightarrow T_1$ is congruent to T_2 and T_2 is congruent to $T_3 \Rightarrow T_1$ is congruent to $T_3 \Rightarrow (T_1, T_3) \in R$. Therefore, R is an equivalence relation.

Example 3 Let L be the set of all lines in a plane and R be the relation in L defined as R = $\{(L_1, L_2) : L_1$ is perpendicular to $L_2\}$. Show that R is symmetric but neither reflexive nor transitive.

Solution R is not reflexive, as a line L_1 can not be perpendicular to itself, i.e., $(L_1, L_1) \notin R$. R is symmetric as $(L_1, L_2) \in R$

$\Rightarrow$ L_1 is perpendicular to L_2

$\Rightarrow$ L_2 is perpendicular to L_1

$\Rightarrow$ $(L_2, L_1) \in R$.

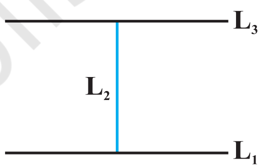


Fig 1.1

R is not transitive. Indeed, if L_1 is perpendicular to L_2 and L_2 is perpendicular to L_3, then L_1 can never be perpendicular to L_3. In fact, L_1 is parallel to L_3, i.e., $(L_1, L_2) \in R$, $(L_2, L_3) \in R$ but $(L_1, L_3) \notin R$.

Example 4 Show that the relation R in the set $\{1, 2, 3\}$ given by R = $\{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)\}$ is reflexive but neither symmetric nor transitive.

Solution R is reflexive, since (1, 1), (2, 2) and (3, 3) lie in R. Also, R is not symmetric, as $(1, 2) \in R$ but $(2, 1) \notin R$. Similarly, R is not transitive, as $(1, 2) \in R$ and $(2, 3) \in R$ but $(1, 3) \notin R$.

Example 5 Show that the relation R in the set **Z** of integers given by

$$R = \{(a, b) : 2 \text{ divides } a - b\}$$

is an equivalence relation.

Solution R is reflexive, as 2 divides $(a - a)$ for all $a \in \mathbf{Z}$. Further, if $(a, b) \in R$, then 2 divides $a - b$. Therefore, 2 divides $b - a$. Hence, $(b, a) \in R$, which shows that R is symmetric. Similarly, if $(a, b) \in R$ and $(b, c) \in R$, then $a - b$ and $b - c$ are divisible by 2. Now, $a - c = (a - b) + (b - c)$ is even (Why?). So, $(a - c)$ is divisible by 2. This shows that R is transitive. Thus, R is an equivalence relation in **Z**.

In Example 5, note that all even integers are related to zero, as $(0, \pm 2)$, $(0, \pm 4)$ etc., lie in R and no odd integer is related to 0, as $(0, \pm 1)$, $(0, \pm 3)$ etc., do not lie in R. Similarly, all odd integers are related to one and no even integer is related to one. Therefore, the set E of all even integers and the set O of all odd integers are subsets of **Z** satisfying following conditions:

(i) All elements of E are related to each other and all elements of O are related to each other.

(ii) No element of E is related to any element of O and vice-versa.

(iii) E and O are disjoint and $\mathbf{Z} = \mathrm{E} \cup \mathrm{O}$.

The subset E is called the *equivalence class containing zero* and is denoted by [0]. Similarly, O is the equivalence class containing 1 and is denoted by [1]. Note that $[0] \neq [1]$, $[0] = [2r]$ and $[1] = [2r + 1]$, $r \in \mathbf{Z}$. Infact, what we have seen above is true for an arbitrary equivalence relation R in a set X. Given an arbitrary equivalence relation R in an arbitrary set X, R divides X into mutually disjoint subsets A_i called partitions or subdivisions of X satisfying:

(i) all elements of A_i are related to each other, for all i.

(ii) no element of A_i is related to any element of A_j, $i \neq j$.

(iii) $\cup \mathrm{A}_j = \mathrm{X}$ and $\mathrm{A}_i \cap \mathrm{A}_j = \phi$, $i \neq j$.

The subsets A_i are called *equivalence classes*. The interesting part of the situation is that we can go reverse also. For example, consider a subdivision of the set **Z** given by three mutually disjoint subsets A_1, A_2 and A_3 whose union is **Z** with

$$\mathrm{A}_1 = \{x \in \mathbf{Z} : x \text{ is a multiple of } 3\} = \{..., -6, -3, 0, 3, 6, ...\}$$

$$\mathrm{A}_2 = \{x \in \mathbf{Z} : x - 1 \text{ is a multiple of } 3\} = \{..., -5, -2, 1, 4, 7, ...\}$$

$$\mathrm{A}_3 = \{x \in \mathbf{Z} : x - 2 \text{ is a multiple of } 3\} = \{..., -4, -1, 2, 5, 8, ...\}$$

Define a relation R in **Z** given by $\mathrm{R} = \{(a, b) : 3 \text{ divides } a - b\}$. Following the arguments similar to those used in Example 5, we can show that R is an equivalence relation. Also, A_1 coincides with the set of all integers in **Z** which are related to zero, A_2 coincides with the set of all integers which are related to 1 and A_3 coincides with the set of all integers in **Z** which are related to 2. Thus, $\mathrm{A}_1 = [0]$, $\mathrm{A}_2 = [1]$ and $\mathrm{A}_3 = [2]$. In fact, $\mathrm{A}_1 = [3r]$, $\mathrm{A}_2 = [3r + 1]$ and $\mathrm{A}_3 = [3r + 2]$, for all $r \in \mathbf{Z}$.

Example 6 Let R be the relation defined in the set $\mathrm{A} = \{1, 2, 3, 4, 5, 6, 7\}$ by $\mathrm{R} = \{(a, b) : \text{both } a \text{ and } b \text{ are either odd or even}\}$. Show that R is an equivalence relation. Further, show that all the elements of the subset $\{1, 3, 5, 7\}$ are related to each other and all the elements of the subset $\{2, 4, 6\}$ are related to each other, but no element of the subset $\{1, 3, 5, 7\}$ is related to any element of the subset $\{2, 4, 6\}$.

Solution Given any element a in A, both a and a must be either odd or even, so that $(a, a) \in$ R. Further, $(a, b) \in$ R $\Rightarrow$ both a and b must be either odd or even $\Rightarrow (b, a) \in$ R. Similarly, $(a, b) \in$ R and $(b, c) \in$ R $\Rightarrow$ all elements a, b, c, must be either even or odd simultaneously $\Rightarrow (a, c) \in$ R. Hence, R is an equivalence relation. Further, all the elements of $\{1, 3, 5, 7\}$ are related to each other, as all the elements of this subset are odd. Similarly, all the elements of the subset $\{2, 4, 6\}$ are related to each other, as all of them are even. Also, no element of the subset $\{1, 3, 5, 7\}$ can be related to any element of $\{2, 4, 6\}$, as elements of $\{1, 3, 5, 7\}$ are odd, while elements of $\{2, 4, 6\}$ are even.

EXERCISE 1.1

1. Determine whether each of the following relations are reflexive, symmetric and transitive:

   (i) Relation R in the set A = $\{1, 2, 3, ..., 13, 14\}$ defined as
   $$R = \{(x, y) : 3x - y = 0\}$$
   (ii) Relation R in the set **N** of natural numbers defined as
   $$R = \{(x, y) : y = x + 5 \text{ and } x < 4\}$$
   (iii) Relation R in the set A = $\{1, 2, 3, 4, 5, 6\}$ as
   $$R = \{(x, y) : y \text{ is divisible by } x\}$$
   (iv) Relation R in the set **Z** of all integers defined as
   $$R = \{(x, y) : x - y \text{ is an integer}\}$$
   (v) Relation R in the set A of human beings in a town at a particular time given by

   (a) R = $\{(x, y) : x$ and y work at the same place$\}$

   (b) R = $\{(x, y) : x$ and y live in the same locality$\}$

   (c) R = $\{(x, y) : x$ is exactly 7 cm taller than $y\}$

   (d) R = $\{(x, y) : x$ is wife of $y\}$

   (e) R = $\{(x, y) : x$ is father of $y\}$

2. Show that the relation R in the set **R** of real numbers, defined as R = $\{(a, b) : a \leq b^2\}$ is neither reflexive nor symmetric nor transitive.

3. Check whether the relation R defined in the set $\{1, 2, 3, 4, 5, 6\}$ as R = $\{(a, b) : b = a + 1\}$ is reflexive, symmetric or transitive.

4. Show that the relation R in **R** defined as R = $\{(a, b) : a \leq b\}$, is reflexive and transitive but not symmetric.

5. Check whether the relation R in **R** defined by R = $\{(a, b) : a \leq b^3\}$ is reflexive, symmetric or transitive.

6. Show that the relation R in the set $\{1, 2, 3\}$ given by $R = \{(1, 2), (2, 1)\}$ is symmetric but neither reflexive nor transitive.

7. Show that the relation R in the set A of all the books in a library of a college, given by $R = \{(x, y) : x$ and y have same number of pages$\}$ is an equivalence relation.

8. Show that the relation R in the set $A = \{1, 2, 3, 4, 5\}$ given by

$R = \{(a, b) : |a - b|$ is even$\}$, is an equivalence relation. Show that all the elements of $\{1, 3, 5\}$ are related to each other and all the elements of $\{2, 4\}$ are related to each other. But no element of $\{1, 3, 5\}$ is related to any element of $\{2, 4\}$.

9. Show that each of the relation R in the set $A = \{x \in \mathbf{Z} : 0 \leq x \leq 12\}$, given by

(i) $R = \{(a, b) : |a - b|$ is a multiple of $4\}$

(ii) $R = \{(a, b) : a = b\}$

is an equivalence relation. Find the set of all elements related to 1 in each case.

10. Give an example of a relation. Which is

(i) Symmetric but neither reflexive nor transitive.

(ii) Transitive but neither reflexive nor symmetric.

(iii) Reflexive and symmetric but not transitive.

(iv) Reflexive and transitive but not symmetric.

(v) Symmetric and transitive but not reflexive.

11. Show that the relation R in the set A of points in a plane given by $R = \{(P, Q)$: distance of the point P from the origin is same as the distance of the point Q from the origin$\}$, is an equivalence relation. Further, show that the set of all points related to a point $P \neq (0, 0)$ is the circle passing through P with origin as centre.

12. Show that the relation R defined in the set A of all triangles as $R = \{(T_1, T_2) : T_1$ is similar to $T_2\}$, is equivalence relation. Consider three right angle triangles T_1 with sides 3, 4, 5, T_2 with sides 5, 12, 13 and T_3 with sides 6, 8, 10. Which triangles among T_1, T_2 and T_3 are related?

13. Show that the relation R defined in the set A of all polygons as $R = \{(P_1, P_2) :$ P_1 and P_2 have same number of sides$\}$, is an equivalence relation. What is the set of all elements in A related to the right angle triangle T with sides 3, 4 and 5?

14. Let L be the set of all lines in XY plane and R be the relation in L defined as $R = \{(L_1, L_2) : L_1$ is parallel to $L_2\}$. Show that R is an equivalence relation. Find the set of all lines related to the line $y = 2x + 4$.

15. Let R be the relation in the set $\{1, 2, 3, 4\}$ given by R = $\{(1, 2), (2, 2), (1, 1), (4,4), (1, 3), (3, 3), (3, 2)\}$. Choose the correct answer.

(A) R is reflexive and symmetric but not transitive.

(B) R is reflexive and transitive but not symmetric.

(C) R is symmetric and transitive but not reflexive.

(D) R is an equivalence relation.

16. Let R be the relation in the set **N** given by R = $\{(a, b) : a = b - 2, b > 6\}$. Choose the correct answer.

(A) $(2, 4) \in$ R (B) $(3, 8) \in$ R (C) $(6, 8) \in$ R (D) $(8, 7) \in$ R

1.3 Types of Functions

The notion of a function along with some special functions like identity function, constant function, polynomial function, rational function, modulus function, signum function etc. along with their graphs have been given in Class XI.

Addition, subtraction, multiplication and division of two functions have also been studied. As the concept of function is of paramount importance in mathematics and among other disciplines as well, we would like to extend our study about function from where we finished earlier. In this section, we would like to study different types of functions.

Consider the functions f_1, f_2, f_3 and f_4 given by the following diagrams.

In Fig 1.2, we observe that the images of distinct elements of X_1 under the function f_1 are distinct, but the image of two distinct elements 1 and 2 of X_1 under f_2 is same, namely b. Further, there are some elements like e and f in X_2 which are not images of any element of X_1 under f_1, while all elements of X_3 are images of some elements of X_1 under f_3. The above observations lead to the following definitions:

Definition 5 A function $f : X \to Y$ is defined to be *one-one* (or *injective*), if the images of distinct elements of X under f are distinct, i.e., for every $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Otherwise, f is called *many-one*.

The function f_1 and f_4 in Fig 1.2 (i) and (iv) are one-one and the function f_2 and f_3 in Fig 1.2 (ii) and (iii) are many-one.

Definition 6 A function $f : X \to Y$ is said to be *onto* (or *surjective*), if every element of Y is the image of some element of X under f, i.e., for every $y \in Y$, there exists an element x in X such that $f(x) = y$.

The function f_3 and f_4 in Fig 1.2 (iii), (iv) are onto and the function f_1 in Fig 1.2 (i) is not onto as elements e, f in X_2 are not the image of any element in X_1 under f_1.

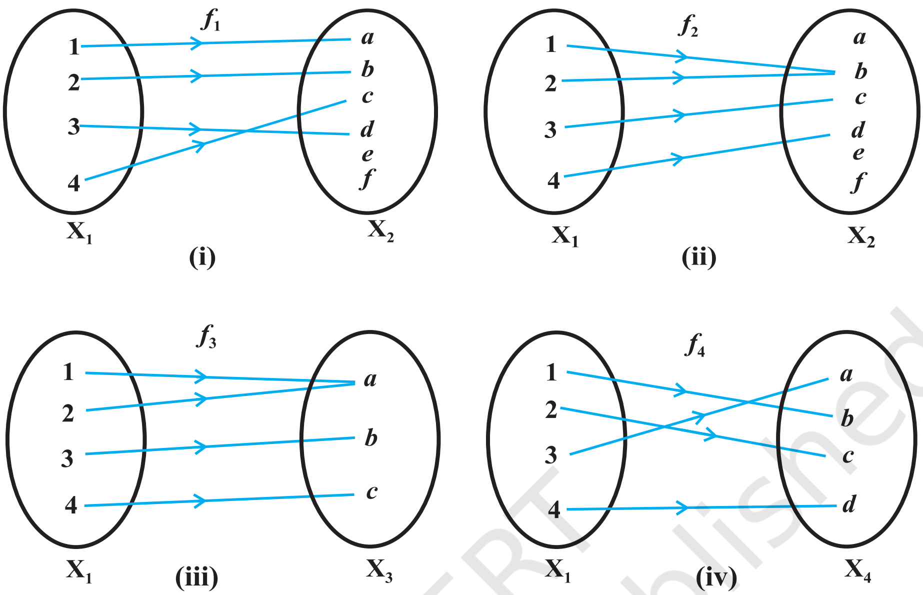


Fig 1.2 (i) to (iv)

Remark $f: X \to Y$ is onto if and only if Range of f = Y.

Definition 7 A function $f: X \to Y$ is said to be *one-one* and *onto* (or *bijective*), if f is both one-one and onto.

The function f_4 in Fig 1.2 (iv) is one-one and onto.

Example 7 Let A be the set of all 50 students of Class X in a school. Let $f: A \to \mathbf{N}$ be function defined by $f(x)$ = roll number of the student x. Show that f is one-one but not onto.

Solution No two different students of the class can have same roll number. Therefore, f must be one-one. We can assume without any loss of generality that roll numbers of students are from 1 to 50. This implies that 51 in **N** is not roll number of any student of the class, so that 51 can not be image of any element of X under f. Hence, f is not onto.

Example 8 Show that the function $f: \mathbf{N} \to \mathbf{N}$, given by $f(x) = 2x$, is one-one but not onto.

Solution The function f is one-one, for $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$. Further, f is not onto, as for $1 \in \mathbf{N}$, there does not exist any x in **N** such that $f(x) = 2x = 1$.

Example 9 Prove that the function $f : \mathbf{R} \to \mathbf{R}$, given by $f(x) = 2x$, is one-one and onto.

Solution f is one-one, as $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$. Also, given any real number y in R, there exists $\frac{y}{2}$ in R such that $f(\frac{y}{2}) = 2 \cdot (\frac{y}{2}) = y$. Hence, f is onto.

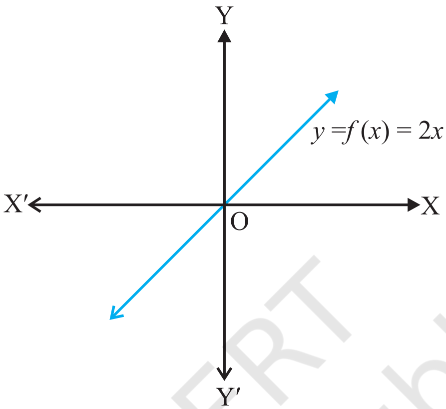


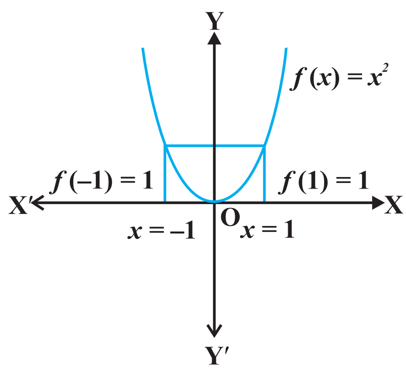
Fig 1.3

Example 10 Show that the function $f : \mathbf{N} \to \mathbf{N}$, given by $f(1) = f(2) = 1$ and $f(x) = x - 1$, for every $x > 2$, is onto but not one-one.

Solution f is not one-one, as $f(1) = f(2) = 1$. But f is onto, as given any $y \in \mathbf{N}$, $y \neq 1$, we can choose x as $y + 1$ such that $f(y + 1) = y + 1 - 1 = y$. Also for $1 \in \mathbf{N}$, we have $f(1) = 1$.

Example 11 Show that the function $f : \mathbf{R} \to \mathbf{R}$, defined as $f(x) = x^2$, is neither one-one nor onto.

Solution Since $f(-1) = 1 = f(1)$, f is not one-one. Also, the element -2 in the co-domain $\mathbf{R}$ is not image of any element x in the domain $\mathbf{R}$ (Why?). Therefore f is not onto.



The image of 1 and –1 under f is 1.

Fig 1.4

Example 12 Show that $f : \mathbf{N} \to \mathbf{N}$, given by

$$f(x) = \begin{cases} x + 1, \text{if } x \text{ is odd,} \\ x - 1, \text{if } x \text{ is even} \end{cases}$$

is both one-one and onto.

Solution Suppose $f(x_1) = f(x_2)$. Note that if x_1 is odd and x_2 is even, then we will have $x_1 + 1 = x_2 - 1$, i.e., $x_2 - x_1 = 2$ which is impossible. Similarly, the possibility of x_1 being even and x_2 being odd can also be ruled out, using the similar argument. Therefore, both x_1 and x_2 must be either odd or even. Suppose both x_1 and x_2 are odd. Then $f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 + 1 \Rightarrow x_1 = x_2$. Similarly, if both x_1 and x_2 are even, then also $f(x_1) = f(x_2) \Rightarrow x_1 - 1 = x_2 - 1 \Rightarrow x_1 = x_2$. Thus, f is one-one. Also, any odd number $2r + 1$ in the co-domain **N** is the image of $2r + 2$ in the domain **N** and any even number $2r$ in the co-domain **N** is the image of $2r - 1$ in the domain **N**. Thus, f is onto.

Example 13 Show that an onto function $f : \{1, 2, 3\} \to \{1, 2, 3\}$ is always one-one.

Solution Suppose f is not one-one. Then there exists two elements, say 1 and 2 in the domain whose image in the co-domain is same. Also, the image of 3 under f can be only one element. Therefore, the range set can have at the most two elements of the co-domain $\{1, 2, 3\}$, showing that f is not onto, a contradiction. Hence, f must be one-one.

Example 14 Show that a one-one function $f : \{1, 2, 3\} \to \{1, 2, 3\}$ must be onto.

Solution Since f is one-one, three elements of $\{1, 2, 3\}$ must be taken to 3 different elements of the co-domain $\{1, 2, 3\}$ under f. Hence, f has to be onto.

Remark The results mentioned in Examples 13 and 14 are also true for an arbitrary finite set X, i.e., a one-one function $f : X \to X$ is necessarily onto and an onto map $f : X \to X$ is necessarily one-one, for every finite set X. In contrast to this, Examples 8 and 10 show that for an infinite set, this may not be true. In fact, this is a characteristic difference between a finite and an infinite set.

EXERCISE 1.2

1. Show that the function $f : \mathbf{R_*} \to \mathbf{R_*}$ defined by $f(x) = \dfrac{1}{x}$ is one-one and onto, where $\mathbf{R_*}$ is the set of all non-zero real numbers. Is the result true, if the domain $\mathbf{R_*}$ is replaced by **N** with co-domain being same as $\mathbf{R_*}$?
2. Check the injectivity and surjectivity of the following functions:
   (i) $f : \mathbf{N} \to \mathbf{N}$ given by $f(x) = x^2$
   (ii) $f : \mathbf{Z} \to \mathbf{Z}$ given by $f(x) = x^2$
   (iii) $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^2$
   (iv) $f : \mathbf{N} \to \mathbf{N}$ given by $f(x) = x^3$
   (v) $f : \mathbf{Z} \to \mathbf{Z}$ given by $f(x) = x^3$
3. Prove that the Greatest Integer Function $f : \mathbf{R} \to \mathbf{R}$, given by $f(x) = [x]$, is neither one-one nor onto, where $[x]$ denotes the greatest integer less than or equal to x.

4. Show that the Modulus Function $f : \mathbf{R} \to \mathbf{R}$, given by $f(x) = |x|$, is neither one-one nor onto, where $|x|$ is x, if x is positive or 0 and $|x|$ is $-x$, if x is negative.

5. Show that the Signum Function $f : \mathbf{R} \to \mathbf{R}$, given by

$$f(x) = \begin{cases} 1, \text{ if } x > 0 \\ 0, \text{ if } x = 0 \\ \quad 1, \text{ if } x < 0 \end{cases}$$

is neither one-one nor onto.

6. Let A = {1, 2, 3}, B = {4, 5, 6, 7} and let f = {(1, 4), (2, 5), (3, 6)} be a function from A to B. Show that f is one-one.

7. In each of the following cases, state whether the function is one-one, onto or bijective. Justify your answer.

(i) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 3 - 4x$

(ii) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 1 + x^2$

8. Let A and B be sets. Show that $f : A \times B \to B \times A$ such that $f(a, b) = (b, a)$ is bijective function.

9. Let $f : \mathbf{N} \to \mathbf{N}$ be defined by $f(n) = \begin{cases} \dfrac{n+1}{2}, \text{ if } n \text{ is odd} \\ \dfrac{n}{2}, \text{ if } n \text{ is even} \end{cases}$ for all $n \in \mathbf{N}$.

State whether the function f is bijective. Justify your answer.

10. Let A = $\mathbf{R}$ – {3} and B = $\mathbf{R}$ – {1}. Consider the function $f : A \to B$ defined by $f(x) = \left(\dfrac{x-2}{x-3}\right)$. Is f one-one and onto? Justify your answer.

11. Let $f : \mathbf{R} \to \mathbf{R}$ be defined as $f(x) = x^4$. Choose the correct answer.

(A) f is one-one onto (B) f is many-one onto

(C) f is one-one but not onto (D) f is neither one-one nor onto.

12. Let $f : \mathbf{R} \to \mathbf{R}$ be defined as $f(x) = 3x$. Choose the correct answer.

(A) f is one-one onto (B) f is many-one onto

(C) f is one-one but not onto (D) f is neither one-one nor onto.

1.4 Composition of Functions and Invertible Function

Definition 8 Let $f: \mathrm{A} \rightarrow \mathrm{B}$ and $g: \mathrm{B} \rightarrow \mathrm{C}$ be two functions. Then the composition of f and g, denoted by gof, is defined as the function $gof: \mathrm{A} \rightarrow \mathrm{C}$ given by

$$gof(x) = g(f(x)), \ \forall \ x \in \mathrm{A}.$$

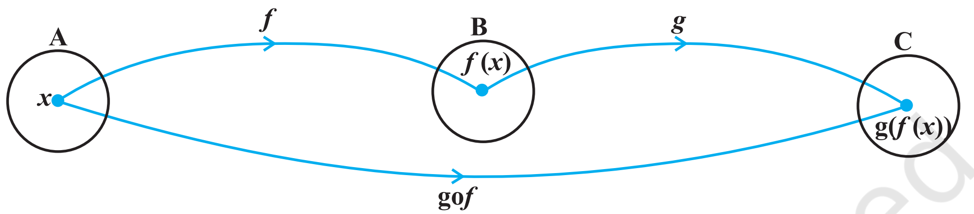


Fig 1.5

Example 15 Let $f: \{2, 3, 4, 5\} \rightarrow \{3, 4, 5, 9\}$ and $g: \{3, 4, 5, 9\} \rightarrow \{7, 11, 15\}$ be functions defined as $f(2) = 3$, $f(3) = 4$, $f(4) = f(5) = 5$ and $g(3) = g(4) = 7$ and $g(5) = g(9) = 11$. Find gof.

Solution We have $gof(2) = g(f(2)) = g(3) = 7$, $gof(3) = g(f(3)) = g(4) = 7$, $gof(4) = g(f(4)) = g(5) = 11$ and $gof(5) = g(5) = 11$.

Example 16 Find gof and fog, if $f: \mathbf{R} \rightarrow \mathbf{R}$ and $g: \mathbf{R} \rightarrow \mathbf{R}$ are given by $f(x) = \cos x$ and $g(x) = 3x^2$. Show that $gof \neq fog$.

Solution We have $gof(x) = g(f(x)) = g(\cos x) = 3 (\cos x)^2 = 3 \cos^2 x$. Similarly, $fog(x) = f(g(x)) = f(3x^2) = \cos (3x^2)$. Note that $3\cos^2 x \neq \cos 3x^2$, for $x = 0$. Hence, $gof \neq fog$.

Definition 9 A function $f: \mathrm{X} \rightarrow \mathrm{Y}$ is defined to be *invertible*, if there exists a function $g: \mathrm{Y} \rightarrow \mathrm{X}$ such that $gof = \mathrm{I}_{\mathrm{X}}$ and $fog = \mathrm{I}_{\mathrm{Y}}$. The function g is called the *inverse of* f and is denoted by f^{-1}.

Thus, if f is invertible, then f must be one-one and onto and conversely, if f is one-one and onto, then f must be invertible. This fact significantly helps for proving a function f to be invertible by showing that f is one-one and onto, specially when the actual inverse of f is not to be determined.

Example 17 Let $f: \mathbf{N} \rightarrow \mathrm{Y}$ be a function defined as $f(x) = 4x + 3$, where, $\mathrm{Y} = \{y \in \mathbf{N}: y = 4x + 3 \text{ for some } x \in \mathbf{N}\}$. Show that f is invertible. Find the inverse.

Solution Consider an arbitrary element y of Y. By the definition of Y, $y = 4x + 3$, for some x in the domain $\mathbf{N}$. This shows that $x = \dfrac{(y-3)}{4}$. Define $g: \mathrm{Y} \rightarrow \mathbf{N}$ by

$g(y) = \dfrac{(y-3)}{4}$. Now, $gof(x) = g(f(x)) = g(4x + 3) = \dfrac{(4x+3-3)}{4} = x$ and

$fog(y) = f(g(y)) = f\left(\dfrac{(y-3)}{4}\right) = \dfrac{4(y-3)}{4} + 3 = y - 3 + 3 = y$. This shows that $gof = I_N$ and $fog = I_Y$, which implies that f is invertible and g is the inverse of f.

Miscellaneous Examples

Example 18 If R_1 and R_2 are equivalence relations in a set A, show that $R_1 \cap R_2$ is also an equivalence relation.

Solution Since R_1 and R_2 are equivalence relations, $(a, a) \in R_1$, and $(a, a) \in R_2$ $\forall\, a \in A$. This implies that $(a, a) \in R_1 \cap R_2$, $\forall\, a$, showing $R_1 \cap R_2$ is reflexive. Further, $(a, b) \in R_1 \cap R_2 \Rightarrow (a, b) \in R_1$ and $(a, b) \in R_2 \Rightarrow (b, a) \in R_1$ and $(b, a) \in R_2 \Rightarrow (b, a) \in R_1 \cap R_2$, hence, $R_1 \cap R_2$ is symmetric. Similarly, $(a, b) \in R_1 \cap R_2$ and $(b, c) \in R_1 \cap R_2 \Rightarrow (a, c) \in R_1$ and $(a, c) \in R_2 \Rightarrow (a, c) \in R_1 \cap R_2$. This shows that $R_1 \cap R_2$ is transitive. Thus, $R_1 \cap R_2$ is an equivalence relation.

Example 19 Let R be a relation on the set A of ordered pairs of positive integers defined by (x, y) R (u, v) if and only if $xv = yu$. Show that R is an equivalence relation.

Solution Clearly, (x, y) R (x, y), $\forall\, (x, y) \in A$, since $xy = yx$. This shows that R is reflexive. Further, (x, y) R $(u, v) \Rightarrow xv = yu \Rightarrow uy = vx$ and hence (u, v) R (x, y). This shows that R is symmetric. Similarly, (x, y) R (u, v) and (u, v) R $(a, b) \Rightarrow xv = yu$ and $ub = va \Rightarrow xv\dfrac{a}{u} = yu\dfrac{a}{u} \Rightarrow xv\dfrac{b}{v} = yu\dfrac{a}{u} \Rightarrow xb = ya$ and hence (x, y) R (a, b). Thus, R is transitive. Thus, R is an equivalence relation.

Example 20 Let X = {1, 2, 3, 4, 5, 6, 7, 8, 9}. Let R_1 be a relation in X given by $R_1 = \{(x, y) : x - y$ is divisible by $3\}$ and R_2 be another relation on X given by $R_2 = \{(x, y): \{x, y\} \subset \{1, 4, 7\}\}$ or $\{x, y\} \subset \{2, 5, 8\}$ or $\{x, y\} \subset \{3, 6, 9\}\}$. Show that $R_1 = R_2$.

Solution Note that the characteristic of sets {1, 4, 7}, {2, 5, 8} and {3, 6, 9} is that difference between any two elements of these sets is a multiple of 3. Therefore, $(x, y) \in R_1 \Rightarrow x - y$ is a multiple of $3 \Rightarrow \{x, y\} \subset \{1, 4, 7\}$ or $\{x, y\} \subset \{2, 5, 8\}$ or $\{x, y\} \subset \{3, 6, 9\} \Rightarrow (x, y) \in R_2$. Hence, $R_1 \subset R_2$. Similarly, $\{x, y\} \in R_2 \Rightarrow \{x, y\}$

$\subset \{1, 4, 7\}$ or $\{x, y\} \subset \{2, 5, 8\}$ or $\{x, y\} \subset \{3, 6, 9\} \Rightarrow x - y$ is divisible by $3 \Rightarrow \{x, y\} \in R_1$. This shows that $R_2 \subset R_1$. Hence, $R_1 = R_2$.

Example 21 Let $f: X \to Y$ be a function. Define a relation R in X given by $R = \{(a, b): f(a) = f(b)\}$. Examine whether R is an equivalence relation or not.

Solution For every $a \in X$, $(a, a) \in R$, since $f(a) = f(a)$, showing that R is reflexive. Similarly, $(a, b) \in R \Rightarrow f(a) = f(b) \Rightarrow f(b) = f(a) \Rightarrow (b, a) \in R$. Therefore, R is symmetric. Further, $(a, b) \in R$ and $(b, c) \in R \Rightarrow f(a) = f(b)$ and $f(b) = f(c) \Rightarrow f(a) = f(c) \Rightarrow (a, c) \in R$, which implies that R is transitive. Hence, R is an equivalence relation.

Example 22 Find the number of all one-one functions from set $A = \{1, 2, 3\}$ to itself.

Solution One-one function from $\{1, 2, 3\}$ to itself is simply a permutation on three symbols 1, 2, 3. Therefore, total number of one-one maps from $\{1, 2, 3\}$ to itself is same as total number of permutations on three symbols 1, 2, 3 which is $3! = 6$.

Example 23 Let $A = \{1, 2, 3\}$. Then show that the number of relations containing (1, 2) and (2, 3) which are reflexive and transitive but not symmetric is three.

Solution The smallest relation R_1 containing (1, 2) and (2, 3) which is reflexive and transitive but not symmetric is $\{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$. Now, if we add the pair (2, 1) to R_1 to get R_2, then the relation R_2 will be reflexive, transitive but not symmetric. Similarly, we can obtain R_3 by adding (3, 2) to R_1 to get the desired relation. However, we can not add two pairs (2, 1), (3, 2) or single pair (3, 1) to R_1 at a time, as by doing so, we will be forced to add the remaining pair in order to maintain transitivity and in the process, the relation will become symmetric also which is not required. Thus, the total number of desired relations is three.

Example 24 Show that the number of equivalence relation in the set $\{1, 2, 3\}$ containing (1, 2) and (2, 1) is two.

Solution The smallest equivalence relation R_1 containing (1, 2) and (2, 1) is $\{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$. Now we are left with only 4 pairs namely (2, 3), (3, 2), (1, 3) and (3, 1). If we add any one, say (2, 3) to R_1, then for symmetry we must add (3, 2) also and now for transitivity we are forced to add (1, 3) and (3, 1). Thus, the only equivalence relation bigger than R_1 is the universal relation. This shows that the total number of equivalence relations containing (1, 2) and (2, 1) is two.

Example 25 Consider the identity function $I_{\mathbf{N}} : \mathbf{N} \to \mathbf{N}$ defined as $I_{\mathbf{N}}(x) = x \ \forall x \in \mathbf{N}$. Show that although $I_{\mathbf{N}}$ is onto but $I_{\mathbf{N}} + I_{\mathbf{N}} : \mathbf{N} \to \mathbf{N}$ defined as

$$(I_{\mathbf{N}} + I_{\mathbf{N}})(x) = I_{\mathbf{N}}(x) + I_{\mathbf{N}}(x) = x + x = 2x \text{ is not onto.}$$

Solution Clearly I_N is onto. But $I_N + I_N$ is not onto, as we can find an element 3 in the co-domain **N** such that there does not exist any x in the domain **N** with $(I_N + I_N)(x) = 2x = 3$.

Example 26 Consider a function $f : \left[0, \frac{\pi}{2}\right] \to \mathbf{R}$ given by $f(x) = \sin x$ and $g : \left[0, \frac{\pi}{2}\right] \to \mathbf{R}$ given by $g(x) = \cos x$. Show that f and g are one-one, but $f + g$ is not one-one.

Solution Since for any two distinct elements x_1 and x_2 in $\left[0, \frac{\pi}{2}\right]$, $\sin x_1 \neq \sin x_2$ and $\cos x_1 \neq \cos x_2$, both f and g must be one-one. But $(f + g)(0) = \sin 0 + \cos 0 = 1$ and $(f+g)\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 1$. Therefore, $f + g$ is not one-one.

Miscellaneous Exercise on Chapter 1

1. Show that the function $f : \mathbf{R} \to \{x \in \mathbf{R} : -1 < x < 1\}$ defined by $f(x) = \frac{x}{1+|x|}$, $x \in \mathbf{R}$ is one one and onto function.
2. Show that the function $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^3$ is injective.
3. Given a non empty set X, consider P(X) which is the set of all subsets of X. Define the relation R in P(X) as follows:

   For subsets A, B in P(X), ARB if and only if $A \subset B$. Is R an equivalence relation on P(X)? Justify your answer.
4. Find the number of all onto functions from the set $\{1, 2, 3, ..., n\}$ to itself.
5. Let $A = \{-1, 0, 1, 2\}$, $B = \{-4, -2, 0, 2\}$ and $f, g : A \to B$ be functions defined by $f(x) = x^2 - x$, $x \in A$ and $g(x) = 2\left|x - \frac{1}{2}\right| - 1$, $x \in A$. Are f and g equal? Justify your answer. (Hint: One may note that two functions $f : A \to B$ and $g : A \to B$ such that $f(a) = g(a)$ $\forall a \in A$, are called equal functions).

6. Let A = {1, 2, 3}. Then number of relations containing (1, 2) and (1, 3) which are reflexive and symmetric but not transitive is

(A) 1 (B) 2 (C) 3 (D) 4

7. Let A = {1, 2, 3}. Then number of equivalence relations containing (1, 2) is

(A) 1 (B) 2 (C) 3 (D) 4

Summary

In this chapter, we studied different types of relations and equivalence relation, composition of functions, invertible functions and binary operations. The main features of this chapter are as follows:

- *Empty relation* is the relation R in X given by R = $\phi \subset$ X × X.
- *Universal relation* is the relation R in X given by R = X × X.
- *Reflexive relation* R in X is a relation with $(a, a) \in$ R $\forall\, a \in$ X.
- *Symmetric relation* R in X is a relation satisfying $(a, b) \in$ R implies $(b, a) \in$ R.
- *Transitive relation* R in X is a relation satisfying $(a, b) \in$ R and $(b, c) \in$ R implies that $(a, c) \in$ R.
- *Equivalence relation* R in X is a relation which is reflexive, symmetric and transitive.
- *Equivalence class* $[a]$ containing $a \in$ X for an equivalence relation R in X is the subset of X containing all elements b related to a.
- A function f : X $\rightarrow$ Y is *one-one* (or *injective*) if
  $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \;\forall\; x_1, x_2 \in$ X.
- A function f : X $\rightarrow$ Y is *onto* (or *surjective*) if given any $y \in$ Y, $\exists\, x \in$ X such that $f(x) = y$.
- A function f : X $\rightarrow$ Y is *one-one and onto* (or *bijective*), if f is both one-one and onto.
- Given a finite set X, a function f : X $\rightarrow$ X is one-one (respectively onto) if and only if f is onto (respectively one-one). This is the characteristic property of a finite set. This is not true for infinite set

Historical Note

The concept of function has evolved over a long period of time starting from R. Descartes (1596-1650), who used the word 'function' in his manuscript "*Geometrie*" in 1637 to mean some positive integral power x^n of a variable x while studying geometrical curves like hyperbola, parabola and ellipse. James Gregory (1636-1675) in his work " *Vera Circuli et Hyperbolae Quadratura*" (1667) considered function as a quantity obtained from other quantities by successive use of algebraic operations or by any other operations. Later G. W. Leibnitz (1646-1716) in his manuscript "*Methodus tangentium inversa, seu de functionibus*" written in 1673 used the word 'function' to mean a quantity varying from point to point on a curve such as the coordinates of a point on the curve, the slope of the curve, the tangent and the normal to the curve at a point. However, in his manuscript "*Historia*" (1714), Leibnitz used the word 'function' to mean quantities that depend on a variable. He was the first to use the phrase 'function of x'. John Bernoulli (1667-1748) used the notation ϕx for the first time in 1718 to indicate a function of x. But the general adoption of symbols like f, F, ϕ, ψ ... to represent functions was made by Leonhard Euler (1707-1783) in 1734 in the first part of his manuscript "*Analysis Infinitorium*". Later on, Joeph Louis Lagrange (1736-1813) published his manuscripts "*Theorie des functions analytiques*" in 1793, where he discussed about analytic function and used the notion $f(x)$, F(x), $\phi(x)$ etc. for different function of x. Subsequently, Lejeunne Dirichlet (1805-1859) gave the definition of function which was being used till the set theoretic definition of function presently used, was given after set theory was developed by Georg Cantor (1845-1918). The set theoretic definition of function known to us presently is simply an abstraction of the definition given by Dirichlet in a rigorous manner.

Chapter 2

INVERSE TRIGONOMETRIC FUNCTIONS

❖ *Mathematics, in general, is fundamentally the science of self-evident things. — FELIX KLEIN* ❖

2.1 Introduction

Aryabhata (476-550 A. D.)

In Chapter 1, we have studied that the inverse of a function f, denoted by f^{-1}, exists if f is one-one and onto. There are many functions which are not one-one, onto or both and hence we can not talk of their inverses. In Class XI, we studied that trigonometric functions are not one-one and onto over their natural domains and ranges and hence their inverses do not exist. In this chapter, we shall study about the restrictions on domains and ranges of trigonometric functions which ensure the existence of their inverses and observe their behaviour through graphical representations. Besides, some elementary properties will also be discussed.

The inverse trigonometric functions play an important role in calculus for they serve to define many integrals. The concepts of inverse trigonometric functions is also used in science and engineering.

2.2 Basic Concepts

In Class XI, we have studied trigonometric functions, which are defined as follows:

sine function, i.e., sine : $\mathbf{R} \to [-1, 1]$

cosine function, i.e., cos : $\mathbf{R} \to [-1, 1]$

tangent function, i.e., tan : $\mathbf{R} - \{x : x = (2n+1)\frac{\pi}{2}, n \in \mathbf{Z}\} \to \mathbf{R}$

cotangent function, i.e., cot : $\mathbf{R} - \{x : x = n\pi, n \in \mathbf{Z}\} \to \mathbf{R}$

secant function, i.e., sec : $\mathbf{R} - \{x : x = (2n+1)\frac{\pi}{2}, n \in \mathbf{Z}\} \to \mathbf{R} - (-1, 1)$

cosecant function, i.e., cosec : $\mathbf{R} - \{x : x = n\pi, n \in \mathbf{Z}\} \to \mathbf{R} - (-1, 1)$

We have also learnt in Chapter 1 that if $f : X \to Y$ such that $f(x) = y$ is one-one and onto, then we can define a unique function $g : Y \to X$ such that $g(y) = x$, where $x \in X$ and $y = f(x)$, $y \in Y$. Here, the domain of g = range of f and the range of g = domain of f. The function g is called the inverse of f and is denoted by f^{-1}. Further, g is also one-one and onto and inverse of g is f. Thus, $g^{-1} = (f^{-1})^{-1} = f$. We also have

$$(f^{-1} \text{ o } f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x$$

and $$(f \text{ o } f^{-1})(y) = f(f^{-1}(y)) = f(x) = y$$

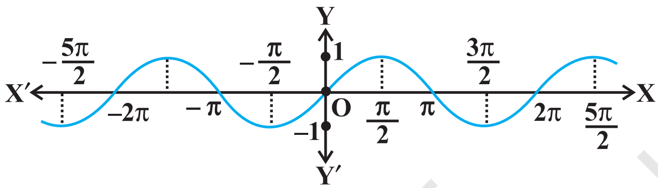
Since the domain of sine function is the set of all real numbers and range is the closed interval [–1, 1]. If we restrict its domain to $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, then it becomes one-one and onto with range [– 1, 1]. Actually, sine function restricted to any of the intervals $\left[\frac{-3\pi}{2}, \frac{\pi}{2}\right]$, $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ etc., is one-one and its range is [–1, 1]. We can, therefore, define the inverse of sine function in each of these intervals. We denote the inverse of sine function by $\sin^{-1}$ (arc sine function). Thus, $\sin^{-1}$ is a function whose domain is [– 1, 1] and range could be any of the intervals $\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right]$, $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ or $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, and so on. Corresponding to each such interval, we get a *branch* of the function $\sin^{-1}$. The branch with range $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is called the *principal value branch, whereas other intervals as range give different branches of* $\sin^{-1}$. When we refer to the function $\sin^{-1}$, we take it as the function whose domain is [–1, 1] and range is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. We write $\sin^{-1} : [-1, 1] \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

From the definition of the inverse functions, it follows that $\sin(\sin^{-1} x) = x$ if $-1 \le x \le 1$ and $\sin^{-1}(\sin x) = x$ if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. In other words, if $y = \sin^{-1} x$, then $\sin y = x$.

Remarks

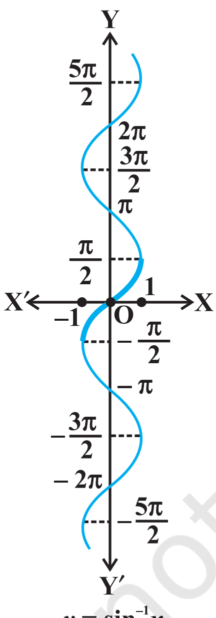
(i) We know from Chapter 1, that if $y = f(x)$ is an invertible function, then $x = f^{-1}(y)$. Thus, the graph of $\sin^{-1}$ function can be obtained from the graph of original function by interchanging x and y axes, i.e., if (a, b) is a point on the graph of sine function, then (b, a) becomes the corresponding point on the graph of inverse of sine function. Thus, the graph of the function $y = \sin^{-1} x$ can be obtained from the graph of $y = \sin x$ by interchanging x and y axes. The graphs of $y = \sin x$ and $y = \sin^{-1} x$ are as given in Fig 2.1 (i), (ii), (iii). The dark portion of the graph of $y = \sin^{-1} x$ represent the principal value branch.

(ii) It can be shown that the graph of an inverse function can be obtained from the corresponding graph of original function as a mirror image (i.e., reflection) along the line $y = x$. This can be visualised by looking the graphs of $y = \sin x$ and $y = \sin^{-1} x$ as given in the same axes (Fig 2.1 (iii)).



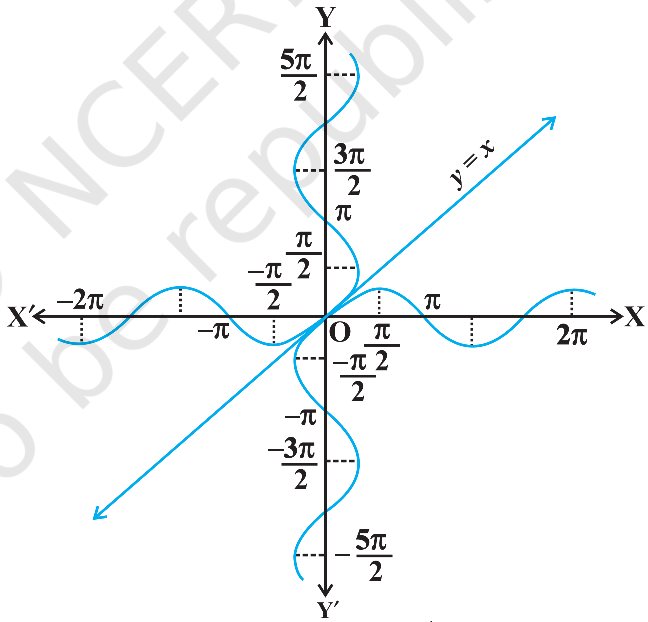
$y = \sin x$

Fig 2.1 (i)



$y = \sin^{-1} x$

Fig 2.1 (ii)



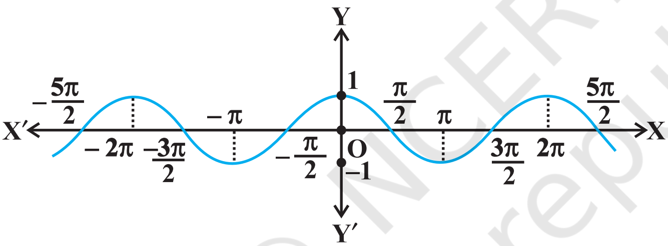
$y = \sin x$ and $y = \sin^{-1} x$

Fig 2.1 (iii)

Like sine function, the cosine function is a function whose domain is the set of all real numbers and range is the set [–1, 1]. If we restrict the domain of cosine function to [0, π], then it becomes one-one and onto with range [–1, 1]. Actually, cosine function restricted to any of the intervals $[-\pi, 0]$, $[0,\pi]$, $[\pi, 2\pi]$ etc., is bijective with range as $[-1, 1]$. We can, therefore, define the inverse of cosine function in each of these intervals. We denote the inverse of the cosine function by $\cos^{-1}$ (arc cosine function). Thus, $\cos^{-1}$ is a function whose domain is $[-1, 1]$ and range could be any of the intervals $[-\pi, 0]$, $[0, \pi]$, $[\pi, 2\pi]$ etc. Corresponding to each such interval, we get a branch of the function $\cos^{-1}$. The branch with range $[0, \pi]$ is called the *principal value branch* of the function $\cos^{-1}$. We write

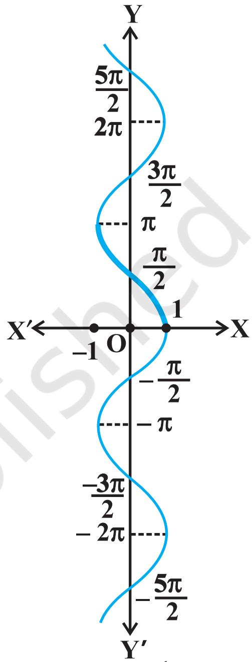
$$\cos^{-1} : [-1, 1] \to [0, \pi].$$

The graph of the function given by $y = \cos^{-1} x$ can be drawn in the same way as discussed about the graph of $y = \sin^{-1} x$. The graphs of $y = \cos x$ and $y = \cos^{-1} x$ are given in Fig 2.2 (i) and (ii).



$y = \cos x$

Fig 2.2 (i)



$y = \cos^{-1} x$

Fig 2.2 (ii)

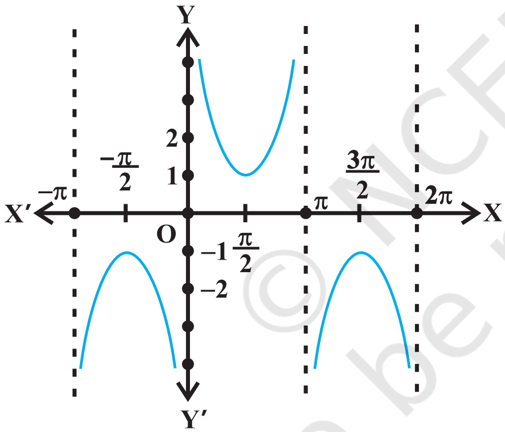
Let us now discuss $\text{cosec}^{-1}x$ and $\sec^{-1}x$ as follows:

Since, $\text{cosec}\, x = \dfrac{1}{\sin x}$, the domain of the cosec function is the set $\{x : x \in \mathbf{R}$ and $x \neq n\pi,\ n \in \mathbf{Z}\}$ and the range is the set $\{y : y \in \mathbf{R},\ y \geq 1 \text{ or } y \leq -1\}$ i.e., the set $\mathbf{R} - (-1, 1)$. It means that $y = \text{cosec}\, x$ assumes all real values except $-1 < y < 1$ and is not defined for integral multiple of π. If we restrict the domain of cosec function to $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] - \{0\}$, then it is one to one and onto with its range as the set $\mathbf{R} - (-1, 1)$. Actually, cosec function restricted to any of the intervals $\left[\dfrac{-3\pi}{2}, \dfrac{-\pi}{2}\right] - \{-\pi\}$, $\left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right] - \{0\}$, $\left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right] - \{\pi\}$ etc., is bijective and its range is the set of all real numbers $\mathbf{R} - (-1, 1)$.

Thus $\operatorname{cosec}^{-1}$ can be defined as a function whose domain is **R** – (–1, 1) and range could be any of the intervals $\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right] - \{-\pi\}$, $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$, $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right] - \{\pi\}$ etc. The function corresponding to the range $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ is called the *principal value branch* of $\operatorname{cosec}^{-1}$. We thus have principal branch as

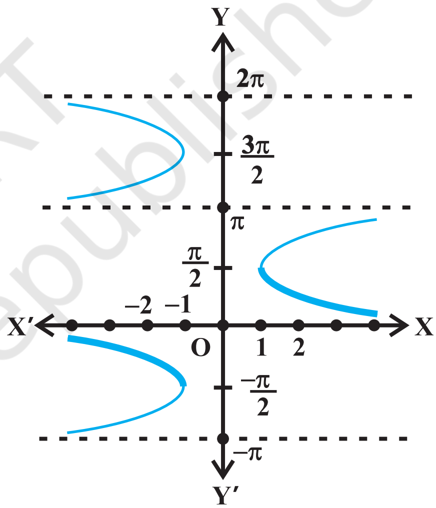
$$\operatorname{cosec}^{-1} : \mathbf{R} - (-1, 1) \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$$

The graphs of $y = \operatorname{cosec} x$ and $y = \operatorname{cosec}^{-1} x$ are given in Fig 2.3 (i), (ii).



$y = \operatorname{cosec} x$

Fig 2.3 (i)



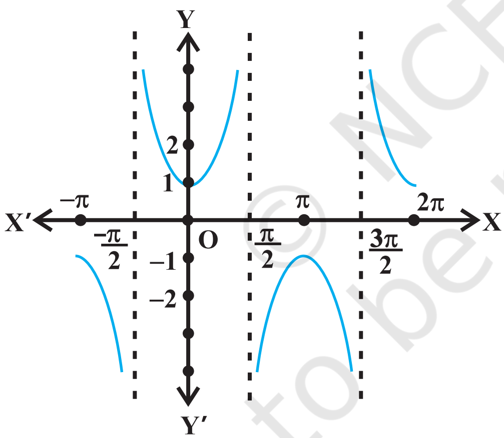
$y = \operatorname{cosec}^{-1} x$

Fig 2.3 (ii)

Also, since $\sec x = \frac{1}{\cos x}$, the domain of $y = \sec x$ is the set $\mathbf{R} - \{x : x = (2n + 1)\frac{\pi}{2}$, $n \in \mathbf{Z}\}$ and range is the set **R** – (–1, 1). It means that sec (secant function) assumes all real values except $-1 < y < 1$ and is not defined for odd multiples of $\frac{\pi}{2}$. If we restrict the domain of secant function to $[0, \pi] - \{\frac{\pi}{2}\}$, then it is one-one and onto with its range as the set **R** – (–1, 1). Actually, secant function restricted to any of the intervals $[-\pi, 0]-\{\frac{-\pi}{2}\}$, $[0, \pi]-\left\{\frac{\pi}{2}\right\}$, $[\pi, 2\pi]-\{\frac{3\pi}{2}\}$ etc., is bijective and its range is $\mathbf{R}-\{-1, 1\}$. Thus $\sec^{-1}$ can be defined as a function whose domain is $\mathbf{R}-(-1, 1)$ and range could be any of the intervals $[-\pi, 0]-\{\frac{-\pi}{2}\}$, $[0, \pi]-\{\frac{\pi}{2}\}$, $[\pi, 2\pi]-\{\frac{3\pi}{2}\}$ etc. Corresponding to each of these intervals, we get different branches of the function $\sec^{-1}$. The branch with range $[0, \pi]-\{\frac{\pi}{2}\}$ is called the *principal value branch* of the function $\sec^{-1}$. We thus have

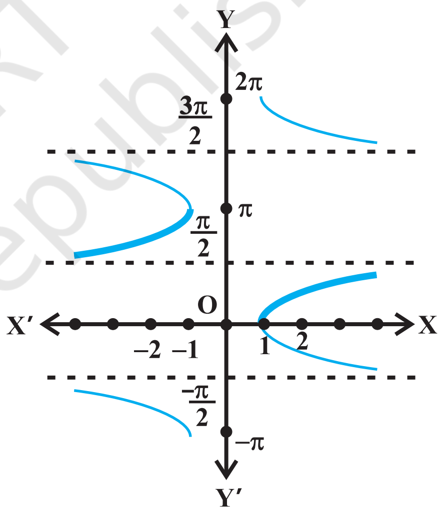
$$\sec^{-1} : \mathbf{R}-(-1,1) \rightarrow [0, \pi]-\{\frac{\pi}{2}\}$$

The graphs of the functions $y = \sec x$ and $y = \sec^{-1} x$ are given in Fig 2.4 (i), (ii).



$y = \sec x$

Fig 2.4 (i)



$y = \sec^{-1}x$

Fig 2.4 (ii)

Finally, we now discuss $\tan^{-1}$ and $\cot^{-1}$

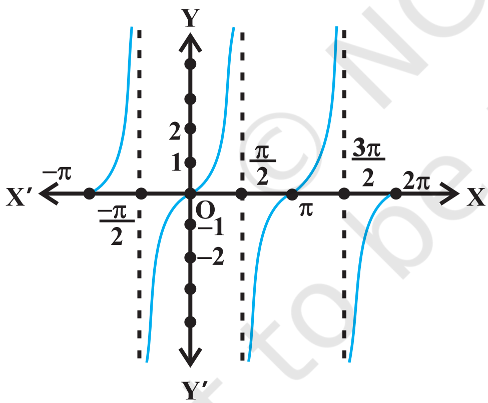
We know that the domain of the tan function (tangent function) is the set $\{x : x \in \mathbf{R} \text{ and } x \neq (2n+1)\frac{\pi}{2}, n \in \mathbf{Z}\}$ and the range is **R**. It means that tan function is not defined for odd multiples of $\frac{\pi}{2}$. If we restrict the domain of tangent function to

$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, then it is one-one and onto with its range as **R**. Actually, tangent function restricted to any of the intervals $\left(\frac{-3\pi}{2}, \frac{-\pi}{2}\right)$, $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ etc., is bijective and its range is **R**. Thus $\tan^{-1}$ can be defined as a function whose domain is **R** and range could be any of the intervals $\left(\frac{-3\pi}{2}, \frac{-\pi}{2}\right)$, $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and so on. These intervals give different branches of the function $\tan^{-1}$. The branch with range $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ is called the *principal value branch* of the function $\tan^{-1}$.

We thus have

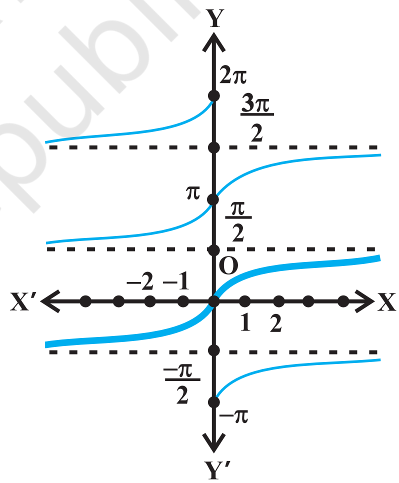
$$\tan^{-1} : \mathbf{R} \to \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

The graphs of the function $y = \tan x$ and $y = \tan^{-1}x$ are given in Fig 2.5 (i), (ii).



$y = \tan x$

Fig 2.5 (i)



$y = \tan^{-1}x$

Fig 2.5 (ii)

We know that domain of the cot function (cotangent function) is the set $\{x : x \in \mathbf{R} \text{ and } x \neq n\pi, n \in \mathbf{Z}\}$ and range is **R**. It means that cotangent function is not defined for integral multiples of π. If we restrict the domain of cotangent function to (0, π), then it is bijective with and its range as **R**. In fact, cotangent function restricted to any of the intervals (–π, 0), (0, π), (π, 2π) etc., is bijective and its range is **R**. Thus $\cot^{-1}$ can be defined as a function whose domain is the **R** and range as any of the intervals $(-\pi, 0)$, $(0, \pi)$, $(\pi, 2\pi)$ etc. These intervals give different branches of the function $\cot^{-1}$. The function with range $(0, \pi)$ is called the *principal value branch* of the function $\cot^{-1}$. We thus have

$$\cot^{-1} : \mathbf{R} \to (0, \pi)$$

The graphs of $y = \cot x$ and $y = \cot^{-1}x$ are given in Fig 2.6 (i), (ii).

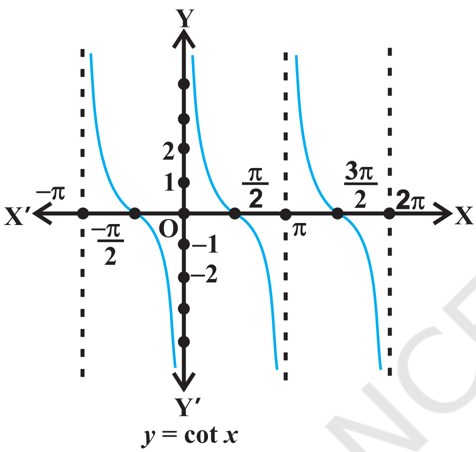


Fig 2.6 (i)

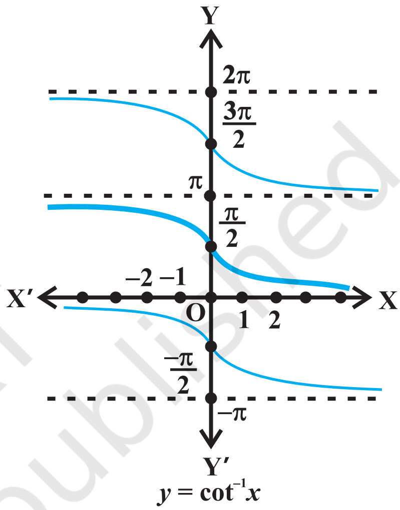


Fig 2.6 (ii)

The following table gives the inverse trigonometric function (principal value branches) along with their domains and ranges.

$\sin^{-1}$	:	$[-1, 1]$	$\to$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}$	:	$[-1, 1]$	$\to$	$[0, \pi]$
cosec^{-1}	:	$\mathbf{R} - (-1,1)$	$\to$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$\sec^{-1}$	:	$\mathbf{R} - (-1, 1)$	$\to$	$[0, \pi] - \{\frac{\pi}{2}\}$
$\tan^{-1}$	:	$\mathbf{R}$	$\to$	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1}$	:	$\mathbf{R}$	$\to$	$(0, \pi)$

Note

1. $\sin^{-1}x$ should not be confused with $(\sin x)^{-1}$. In fact $(\sin x)^{-1} = \dfrac{1}{\sin x}$ and similarly for other trigonometric functions.
2. Whenever no branch of an inverse trigonometric functions is mentioned, we mean the principal value branch of that function.
3. The value of an inverse trigonometric functions which lies in the range of principal branch is called the *principal value* of that inverse trigonometric functions.

We now consider some examples:

Example 1 Find the principal value of $\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$.

Solution Let $\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right) = y$. Then, $\sin y = \dfrac{1}{\sqrt{2}}$.

We know that the range of the principal value branch of $\sin^{-1}$ is $\left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$ and $\sin\left(\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$. Therefore, principal value of $\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$ is $\dfrac{\pi}{4}$

Example 2 Find the principal value of $\cot^{-1}\left(\dfrac{-1}{\sqrt{3}}\right)$

Solution Let $\cot^{-1}\left(\dfrac{-1}{\sqrt{3}}\right) = y$. Then,

$$\cot y = \frac{-1}{\sqrt{3}} = -\cot\left(\frac{\pi}{3}\right) = \cot\left(\pi - \frac{\pi}{3}\right) = \cot\left(\frac{2\pi}{3}\right)$$

We know that the range of principal value branch of $\cot^{-1}$ is $(0, \pi)$ and $\cot\left(\dfrac{2\pi}{3}\right) = \dfrac{-1}{\sqrt{3}}$. Hence, principal value of $\cot^{-1}\left(\dfrac{-1}{\sqrt{3}}\right)$ is $\dfrac{2\pi}{3}$

EXERCISE 2.1

Find the principal values of the following:

1. $\sin^{-1}\left(-\dfrac{1}{2}\right)$ **2.** $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$ **3.** $\operatorname{cosec}^{-1}(2)$

4. $\tan^{-1}(-\sqrt{3})$ **5.** $\cos^{-1}\left(-\dfrac{1}{2}\right)$ **6.** $\tan^{-1}(-1)$

7. $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ **8.** $\cot^{-1}(\sqrt{3})$ **9.** $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

10. $\text{cosec}^{-1}(-\sqrt{2})$

Find the values of the following:

11. $\tan^{-1}(1) + \cos^{-1} \; -\frac{1}{2} \; + \sin^{-1} \; -\frac{1}{2}$ **12.** $\cos^{-1} \; \frac{1}{2} \; + 2\sin^{-1} \; \frac{1}{2}$

13. If $\sin^{-1} x = y$, then

(A) $0 \le y \le \pi$ (B) $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(C) $0 < y < \pi$ (D) $-\frac{\pi}{2} < y < \frac{\pi}{2}$

14. $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$ is equal to

(A) π (B) $-\frac{\pi}{3}$ (C) $\frac{\pi}{3}$ (D) $\frac{2\pi}{3}$

2.3 Properties of Inverse Trigonometric Functions

In this section, we shall prove some important properties of inverse trigonometric functions. It may be mentioned here that these results are valid within the principal value branches of the corresponding inverse trigonometric functions and wherever they are defined. Some results may not be valid for all values of the domains of inverse trigonometric functions. In fact, they will be valid only for some values of x for which inverse trigonometric functions are defined. We will not go into the details of these values of x in the domain as this discussion goes beyond the scope of this textbook.

Let us recall that if $y = \sin^{-1}x$, then $x = \sin y$ and if $x = \sin y$, then $y = \sin^{-1}x$. This is equivalent to

$$\sin(\sin^{-1} x) = x,\ x \in [-1, 1] \text{ and } \sin^{-1}(\sin x) = x,\ x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

For suitable values of domain similar results follow for remaining trigonometric functions.

We now consider some examples.

Example 3 Show that

(i) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\sin^{-1}x, \; -\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$

(ii) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1}x, \; \frac{1}{\sqrt{2}} \leq x \leq 1$

Solution

(i) Let $x = \sin\theta$. Then $\sin^{-1}x = \theta$. We have

$$\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$$

$$= \sin^{-1}(2\sin\theta\cos\theta) = \sin^{-1}(\sin 2\theta) = 2\theta$$

$$= 2\sin^{-1}x$$

(ii) Take $x = \cos\theta$, then proceeding as above, we get, $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1}x$

Example 4 Express $\tan^{-1}\frac{\cos x}{1-\sin x}, \; \frac{-3\pi}{2} < x < \frac{\pi}{2}$ in the simplest form.

Solution We write

$$\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) = \tan^{-1}\left[\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}\right]$$

$$= \tan^{-1}\left[\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}\right]$$

$$= \tan^{-1}\left[\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right] = \tan^{-1}\left[\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right] = \frac{\pi}{4}+\frac{x}{2}$$

Example 5 Write $\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right)$, $x > 1$ in the simplest form.

Solution Let $x = \sec\theta$, then $\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$

Therefore, $\cot^{-1}\frac{1}{\sqrt{x^2-1}} = \cot^{-1}(\cot\theta) = \theta = \sec^{-1} x$, which is the simplest form.

EXERCISE 2.2

Prove the following:

1. $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$, $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

2. $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$, $x \in \left[\frac{1}{2}, 1\right]$

Write the following functions in the simplest form:

3. $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$, $x \neq 0$

4. $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$, $0 < x < \pi$

5. $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$, $\frac{-\pi}{4} < x < \frac{3\pi}{4}$

6. $\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}$, $|x| < a$

7. $\tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right)$, $a > 0$; $\frac{-a}{\sqrt{3}} < x < \frac{a}{\sqrt{3}}$

Find the values of each of the following:

8. $\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$

9. $\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right]$, $|x| < 1$, $y > 0$ and $xy < 1$

Find the values of each of the expressions in Exercises 16 to 18.

10. $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

11. $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

12. $\tan\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$

13. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ is equal to

(A) $\frac{7\pi}{6}$ (B) $\frac{5\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{6}$

14. $\sin\left(\frac{\pi}{3}-\sin^{-1}(-\frac{1}{2})\right)$ is equal to

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) 1

15. $\tan^{-1}\sqrt{3}-\cot^{-1}(-\sqrt{3})$ is equal to

(A) π (B) $-\frac{\pi}{2}$ (C) 0 (D) $2\sqrt{3}$

Miscellaneous Examples

Example 6 Find the value of $\sin^{-1}(\sin\frac{3\pi}{5})$

Solution We know that $\sin^{-1}(\sin x)=x$. Therefore, $\sin^{-1}(\sin\frac{3\pi}{5})=\frac{3\pi}{5}$

But $\frac{3\pi}{5}\notin\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, which is the principal branch of $\sin^{-1} x$

However $\sin(\frac{3\pi}{5})=\sin(\pi-\frac{3\pi}{5})=\sin\frac{2\pi}{5}$ and $\frac{2\pi}{5}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Therefore $\sin^{-1}(\sin\frac{3\pi}{5})=\sin^{-1}(\sin\frac{2\pi}{5})=\frac{2\pi}{5}$

Miscellaneous Exercise on Chapter 2

Find the value of the following:

1. $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$

2. $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$

Prove that

3. $2\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{24}{7}$

4. $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{36}$

5. $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$

6. $\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$

7. $\tan^{-1}\frac{63}{16} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$

Prove that

8. $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\frac{1-x}{1+x}$, $x \in [0, 1]$

9. $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}$, $x \in \left(0, \frac{\pi}{4}\right)$

10. $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x$, $-\frac{1}{\sqrt{2}} \le x \le 1$ [Hint: Put $x = \cos 2\theta$]

Solve the following equations:

11. $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

12. $\tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1}x, (x > 0)$

13. $\sin(\tan^{-1}x)$, $|x| < 1$ is equal to

(A) $\frac{x}{\sqrt{1-x^2}}$ (B) $\frac{1}{\sqrt{1-x^2}}$ (C) $\frac{1}{\sqrt{1+x^2}}$ (D) $\frac{x}{\sqrt{1+x^2}}$

14. $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$, then x is equal to

(A) $0, \frac{1}{2}$ (B) $1, \frac{1}{2}$ (C) 0 (D) $\frac{1}{2}$

Summary

- The domains and ranges (principal value branches) of inverse trigonometric functions are given in the following table:

Functions	**Domain**	**Range (Principal Value Branches)**
$y = \sin^{-1} x$	$[-1, 1]$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$y = \operatorname{cosec}^{-1} x$	$\mathbf{R} - (-1,1)$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$y = \sec^{-1} x$	$\mathbf{R} - (-1, 1)$	$[0, \pi] - \{\frac{\pi}{2}\}$
$y = \tan^{-1} x$	$\mathbf{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \cot^{-1} x$	$\mathbf{R}$	$(0, \pi)$

- $\sin^{-1}x$ should not be confused with $(\sin x)^{-1}$. In fact $(\sin x)^{-1} = \frac{1}{\sin x}$ and similarly for other trigonometric functions.
- The value of an inverse trigonometric functions which lies in its principal value branch is called the *principal value* of that inverse trigonometric functions.

For suitable values of domain, we have

- $y = \sin^{-1} x \Rightarrow x = \sin y$
- $x = \sin y \Rightarrow y = \sin^{-1} x$
- $\sin (\sin^{-1} x) = x$
- $\sin^{-1} (\sin x) = x$

Historical Note

The study of trigonometry was first started in India. The ancient Indian Mathematicians, Aryabhata (476A.D.), Brahmagupta (598 A.D.), Bhaskara I (600 A.D.) and Bhaskara II (1114 A.D.) got important results of trigonometry. All this knowledge went from India to Arabia and then from there to Europe. The Greeks had also started the study of trigonometry but their approach was so clumsy that when the Indian approach became known, it was immediately adopted throughout the world.

In India, the predecessor of the modern trigonometric functions, known as the sine of an angle, and the introduction of the sine function represents one of the main contribution of the *siddhantas* (Sanskrit astronomical works) to mathematics.

Bhaskara I (about 600 A.D.) gave formulae to find the values of sine functions for angles more than 90°. A sixteenth century Malayalam work *Yuktibhasa* contains a proof for the expansion of sin (A + B). Exact expression for sines or cosines of 18°, 36°, 54°, 72°, etc., were given by Bhaskara II.

The symbols $\sin^{-1} x$, $\cos^{-1} x$, etc., for arc sin x, arc cos x, etc., were suggested by the astronomer Sir John F.W. Hersehel (1813) The name of Thales (about 600 B.C.) is invariably associated with height and distance problems. He is credited with the determination of the height of a great pyramid in Egypt by measuring shadows of the pyramid and an auxiliary staff (or gnomon) of known height, and comparing the ratios:

$$\frac{\mathrm{H}}{\mathrm{S}} = \frac{h}{s} = \tan \text{(sun's altitude)}$$

Thales is also said to have calculated the distance of a ship at sea through the proportionality of sides of similar triangles. Problems on height and distance using the similarity property are also found in ancient Indian works.

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Chapter 3

MATRICES

❖ *The essence of Mathematics lies in its freedom. — CANTOR* ❖

3.1 Introduction

The knowledge of matrices is necessary in various branches of mathematics. Matrices are one of the most powerful tools in mathematics. This mathematical tool simplifies our work to a great extent when compared with other straight forward methods. The evolution of concept of matrices is the result of an attempt to obtain compact and simple methods of solving system of linear equations. Matrices are not only used as a representation of the coefficients in system of linear equations, but utility of matrices far exceeds that use. Matrix notation and operations are used in electronic spreadsheet programs for personal computer, which in turn is used in different areas of business and science like budgeting, sales projection, cost estimation, analysing the results of an experiment etc. Also, many physical operations such as magnification, rotation and reflection through a plane can be represented mathematically by matrices. Matrices are also used in cryptography. This mathematical tool is not only used in certain branches of sciences, but also in genetics, economics, sociology, modern psychology and industrial management.

In this chapter, we shall find it interesting to become acquainted with the fundamentals of matrix and matrix algebra.

3.2 Matrix

Suppose we wish to express the information that Radha has 15 notebooks. We may express it as [15] with the understanding that the number inside [ ] is the number of notebooks that Radha has. Now, if we have to express that Radha has 15 notebooks and 6 pens. We may express it as [15 6] with the understanding that first number inside [ ] is the number of notebooks while the other one is the number of pens possessed by Radha. Let us now suppose that we wish to express the information of possession of notebooks and pens by Radha and her two friends Fauzia and Simran which is as follows:

Radha	has	15	notebooks	and	6 pens,
Fauzia	has	10	notebooks	and	2 pens,
Simran	has	13	notebooks	and	5 pens.

Now this could be arranged in the tabular form as follows:

	Notebooks	**Pens**
Radha	15	6
Fauzia	10	2
Simran	13	5

and this can be expressed as

$$\begin{array}{cl} \begin{bmatrix} 15 & 6 \\ 10 & 2 \\ 13 & 5 \end{bmatrix} & \begin{array}{l} \leftarrow \text{First row} \\ \leftarrow \text{Second row} \\ \leftarrow \text{Third row} \end{array} \\ \begin{array}{cc} \uparrow & \uparrow \\ \text{First} & \text{Second} \\ \text{Column} & \text{Column} \end{array} & \end{array}$$

or

	Radha	**Fauzia**	**Simran**
Notebooks	15	10	13
Pens	6	2	5

which can be expressed as:

$$\begin{array}{cl} \begin{bmatrix} 15 & 10 & 13 \\ 6 & 2 & 5 \end{bmatrix} & \begin{array}{l} \leftarrow \text{First row} \\ \leftarrow \text{Second row} \end{array} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{First} & \text{Second} & \text{Third} \\ \text{Column} & \text{Column} & \text{Column} \end{array} & \end{array}$$

In the first arrangement the entries in the first column represent the number of note books possessed by Radha, Fauzia and Simran, respectively and the entries in the second column represent the number of pens possessed by Radha, Fauzia and Simran, respectively. Similarly, in the second arrangement, the entries in the first row represent the number of notebooks possessed by Radha, Fauzia and Simran, respectively. The entries in the second row represent the number of pens possessed by Radha, Fauzia and Simran, respectively. An arrangement or display of the above kind is called a *matrix*. Formally, we define matrix as:

Definition 1 A *matrix* is an ordered rectangular array of numbers or functions. The numbers or functions are called the elements or the entries of the matrix.

We denote matrices by capital letters. The following are some examples of matrices:

$$A = \begin{bmatrix} -2 & 5 \\ 0 & \sqrt{5} \\ 3 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 2+i & 3 & -\frac{1}{2} \\ 3.5 & -1 & 2 \\ \sqrt{3} & 5 & \frac{5}{7} \end{bmatrix}, \quad C = \begin{bmatrix} 1+x & x^3 & 3 \\ \cos x & \sin x + 2 & \tan x \end{bmatrix}$$

In the above examples, the horizontal lines of elements are said to constitute, ***rows*** of the matrix and the vertical lines of elements are said to constitute, ***columns*** of the matrix. Thus A has 3 rows and 2 columns, B has 3 rows and 3 columns while C has 2 rows and 3 columns.

3.2.1 *Order of a matrix*

A matrix having m rows and n columns is called a matrix of *order* $m \times n$ or simply $m \times n$ matrix (read as an m by n matrix). So referring to the above examples of matrices, we have A as 3×2 matrix, B as 3×3 matrix and C as 2×3 matrix. We observe that A has $3 \times 2 = 6$ elements, B and C have 9 and 6 elements, respectively.

In general, an $m \times n$ matrix has the following rectangular array:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

or $\quad A = [a_{ij}]_{m \times n}, \; 1 \le i \le m, \; 1 \le j \le n \quad i, j \in \mathbf{N}$

Thus the i^{th} row consists of the elements $a_{i1}, a_{i2}, a_{i3}, \ldots, a_{in}$, while the j^{th} column consists of the elements $a_{1j}, a_{2j}, a_{3j}, \ldots, a_{mj}$,

In general a_{ij}, is an element lying in the i^{th} row and j^{th} column. We can also call it as the $(i, j)^{\text{th}}$ element of A. The number of elements in an $m \times n$ matrix will be equal to mn.

Note In this chapter

1. We shall follow the notation, namely $A = [a_{ij}]_{m \times n}$ to indicate that A is a matrix of order $m \times n$.
2. We shall consider only those matrices whose elements are real numbers or functions taking real values.

We can also represent any point (x, y) in a plane by a matrix (column or row) as $\begin{bmatrix} x \\ y \end{bmatrix}$ (or $[x, y]$). For example point P(0, 1) as a matrix representation may be given as

$$P = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ or } [0\ 1].$$

Observe that in this way we can also express the vertices of a closed rectilinear figure in the form of a matrix. For example, consider a quadrilateral ABCD with vertices A (1, 0), B (3, 2), C (1, 3), D (–1, 2).

Now, quadrilateral ABCD in the matrix form, can be represented as

$$X = \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 3 & 1 & -1 \\ 0 & 2 & 3 & 2 \end{bmatrix}_{2 \times 4} \end{array} \quad \text{or} \quad Y = \begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 1 & 3 \\ -1 & 2 \end{bmatrix}_{4 \times 2}$$

Thus, matrices can be used as representation of vertices of geometrical figures in a plane.

Now, let us consider some examples.

Example 1 Consider the following information regarding the number of men and women workers in three factories I, II and III

	Men workers	Women workers
I	30	25
II	25	31
III	27	26

Represent the above information in the form of a 3 × 2 matrix. What does the entry in the third row and second column represent?

Solution The information is represented in the form of a 3 × 2 matrix as follows:

$$A = \begin{bmatrix} 30 & 25 \\ 25 & 31 \\ 27 & 26 \end{bmatrix}$$

The entry in the third row and second column represents the number of women workers in factory III.

Example 2 If a matrix has 8 elements, what are the possible orders it can have?

Solution We know that if a matrix is of order $m \times n$, it has mn elements. Thus, to find all possible orders of a matrix with 8 elements, we will find all ordered pairs of natural numbers, whose product is 8.

Thus, all possible ordered pairs are (1, 8), (8, 1), (4, 2), (2, 4)

Hence, possible orders are 1×8, 8×1, 4×2, 2×4

Example 3 Construct a 3 × 2 matrix whose elements are given by $a_{ij} = \frac{1}{2}|i - 3j|$.

Solution In general a 3 × 2 matrix is given by $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$.

Now $\quad a_{ij} = \frac{1}{2}|i - 3j|$, $i = 1, 2, 3$ and $j = 1, 2$.

Therefore $\quad a_{11} = \frac{1}{2}|1 - 3 \times 1| = 1 \qquad a_{12} = \frac{1}{2}|1 - 3 \times 2| = \frac{5}{2}$

$$a_{21} = \frac{1}{2}|2 - 3 \times 1| = \frac{1}{2} \qquad a_{22} = \frac{1}{2}|2 - 3 \times 2| = 2$$

$$a_{31} = \frac{1}{2}|3 - 3 \times 1| = 0 \qquad a_{32} = \frac{1}{2}|3 - 3 \times 2| = \frac{3}{2}$$

Hence the required matrix is given by $A = \begin{bmatrix} 1 & \frac{5}{2} \\ \frac{1}{2} & 2 \\ 0 & \frac{3}{2} \end{bmatrix}$.

3.3 Types of Matrices

In this section, we shall discuss different types of matrices.

(i) **Column matrix**

A matrix is said to be a *column matrix* if it has only one column.

For example, $A = \begin{bmatrix} 0 \\ \sqrt{3} \\ -1 \\ 1/2 \end{bmatrix}$ is a column matrix of order 4 × 1.

In general, $A = [a_{ij}]_{m \times 1}$ is a column matrix of order $m \times 1$.

(ii) **Row matrix**

A matrix is said to be a *row matrix* if it has only one row.

For example, $B = \begin{bmatrix} -\frac{1}{2} & \sqrt{5} & 2 & 3 \end{bmatrix}_{1\times 4}$ is a row matrix.

In general, $B = [b_{ij}]_{1 \times n}$ is a row matrix of order $1 \times n$.

(iii) **Square matrix**

A matrix in which the number of rows are equal to the number of columns, is said to be a *square matrix*. Thus an $m \times n$ matrix is said to be a square matrix if $m = n$ and is known as a square matrix of order 'n'.

For example $A = \begin{bmatrix} 3 & -1 & 0 \\ \frac{3}{2} & 3\sqrt{2} & 1 \\ 4 & 3 & -1 \end{bmatrix}$ is a square matrix of order 3.

In general, $A = [a_{ij}]_{m \times m}$ is a square matrix of order m.

Note If $A = [a_{ij}]$ is a square matrix of order n, then elements (entries) $a_{11}, a_{22}, ..., a_{nn}$ are said to constitute the *diagonal*, of the matrix A. Thus, if $A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & 4 & -1 \\ 3 & 5 & 6 \end{bmatrix}$.

Then the elements of the diagonal of A are 1, 4, 6.

(iv) **Diagonal matrix**

A square matrix $B = [b_{ij}]_{m \times m}$ is said to be a *diagonal matrix* if all its non diagonal elements are zero, that is a matrix $B = [b_{ij}]_{m \times m}$ is said to be a diagonal matrix if $b_{ij} = 0$, when $i \neq j$.

For example, $A = [4]$, $B = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$, $C = \begin{bmatrix} -1.1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, are diagonal matrices of order 1, 2, 3, respectively.

(v) **Scalar matrix**

A diagonal matrix is said to be a *scalar matrix* if its diagonal elements are equal, that is, a square matrix $B = [b_{ij}]_{n \times n}$ is said to be a scalar matrix if

$$b_{ij} = 0, \quad \text{when } i \neq j$$
$$b_{ij} = k, \quad \text{when } i = j, \text{ for some constant } k.$$

For example

$A = [3]$, $\quad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\quad C = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$

are scalar matrices of order 1, 2 and 3, respectively.

(vi) **Identity matrix**

A square matrix in which elements in the diagonal are all 1 and rest are all zero is called an *identity matrix*. In other words, the square matrix $A = [a_{ij}]_{n \times n}$ is an identity matrix, if $a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

We denote the identity matrix of order n by I_n. When order is clear from the context, we simply write it as I.

For example [1], $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices of order 1, 2 and 3, respectively.

Observe that a scalar matrix is an identity matrix when $k = 1$. But every identity matrix is clearly a scalar matrix.

(vii) **Zero matrix**

A matrix is said to be *zero matrix* or *null matrix* if all its elements are zero.

For example, [0], $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, [0, 0] are all zero matrices. We denote zero matrix by O. Its order will be clear from the context.

3.3.1 *Equality of matrices*

Definition 2 Two matrices A = $[a_{ij}]$ and B = $[b_{ij}]$ are said to be equal if

(i) they are of the same order

(ii) each element of A is equal to the corresponding element of B, that is $a_{ij} = b_{ij}$ for all i and j.

For example, $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ are equal matrices but $\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ are not equal matrices. Symbolically, if two matrices A and B are equal, we write A = B.

If $\begin{bmatrix} x & y \\ z & a \\ b & c \end{bmatrix} = \begin{bmatrix} -1.5 & 0 \\ 2 & \sqrt{6} \\ 3 & 2 \end{bmatrix}$, then $x = -1.5$, $y = 0$, $z = 2$, $a = \sqrt{6}$, $b = 3$, $c = 2$

Example 4 If $\begin{bmatrix} x+3 & z+4 & 2y-7 \\ -6 & a-1 & 0 \\ b-3 & -21 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3y-2 \\ -6 & -3 & 2c+2 \\ 2b+4 & -21 & 0 \end{bmatrix}$

Find the values of a, b, c, x, y and z.

Solution As the given matrices are equal, therefore, their corresponding elements must be equal. Comparing the corresponding elements, we get

$$x + 3 = 0, \qquad z + 4 = 6, \qquad 2y - 7 = 3y - 2$$
$$a - 1 = -3, \qquad 0 = 2c + 2 \qquad b - 3 = 2b + 4,$$

Simplifying, we get

$$a = -2,\ b = -7,\ c = -1,\ x = -3,\ y = -5,\ z = 2$$

Example 5 Find the values of a, b, c, and d from the following equation:

$$\begin{bmatrix} 2a+b & a-2b \\ 5c-d & 4c+3d \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}$$

Solution By equality of two matrices, equating the corresponding elements, we get

$$2a + b = 4 \qquad 5c - d = 11$$
$$a - 2b = -3 \qquad 4c + 3d = 24$$

Solving these equations, we get

$$a = 1, b = 2, c = 3 \text{ and } d = 4$$

EXERCISE 3.1

1. In the matrix $A = \begin{bmatrix} 2 & 5 & 19 & -7 \\ 35 & -2 & \frac{5}{2} & 12 \\ \sqrt{3} & 1 & -5 & 17 \end{bmatrix}$, write:

(i) The order of the matrix, (ii) The number of elements,

(iii) Write the elements $a_{13}, a_{21}, a_{33}, a_{24}, a_{23}$.

2. If a matrix has 24 elements, what are the possible orders it can have? What, if it has 13 elements?

3. If a matrix has 18 elements, what are the possible orders it can have? What, if it has 5 elements?

4. Construct a 2×2 matrix, $A = [a_{ij}]$, whose elements are given by:

(i) $a_{ij} = \frac{(i+j)^2}{2}$ (ii) $a_{ij} = \frac{i}{j}$ (iii) $a_{ij} = \frac{(i+2j)^2}{2}$

5. Construct a 3×4 matrix, whose elements are given by:

(i) $a_{ij} = \frac{1}{2}|-3i + j|$ (ii) $a_{ij} = 2i - j$

6. Find the values of x, y and z from the following equations:

(i) $\begin{bmatrix} 4 & 3 \\ x & 5 \end{bmatrix} = \begin{bmatrix} y & z \\ 1 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} x+y & 2 \\ 5+z & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$ (iii) $\begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$

7. Find the value of a, b, c and d from the equation:

$$\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$$

8. $A = [a_{ij}]_{m \times n}$ is a square matrix, if

(A) $m < n$ (B) $m > n$ (C) $m = n$ (D) None of these

9. Which of the given values of x and y make the following pair of matrices equal

$$\begin{bmatrix} 3x+7 & 5 \\ y+1 & 2-3x \end{bmatrix}, \begin{bmatrix} 0 & y-2 \\ 8 & 4 \end{bmatrix}$$

(A) $x = \frac{-1}{3}, \quad y = 7$ (B) Not possible to find

(C) $y = 7, \quad x = \frac{-2}{3}$ (D) $x = \frac{-1}{3}, \quad y = \frac{-2}{3}$

10. The number of all possible matrices of order 3 × 3 with each entry 0 or 1 is:

(A) 27 (B) 18 (C) 81 (D) 512

3.4 Operations on Matrices

In this section, we shall introduce certain operations on matrices, namely, addition of matrices, multiplication of a matrix by a scalar, difference and multiplication of matrices.

3.4.1 *Addition of matrices*

Suppose Fatima has two factories at places A and B. Each factory produces sport shoes for boys and girls in three different price categories labelled 1, 2 and 3. The quantities produced by each factory are represented as matrices given below:

$$\begin{array}{cc} & \text{Factory at A} \\ & \begin{array}{cc} \text{Boys} & \text{Girls} \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 80 & 60 \\ 75 & 65 \\ 90 & 85 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & \text{Factory at B} \\ & \begin{array}{cc} \text{Boys} & \text{Girls} \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 90 & 50 \\ 70 & 55 \\ 75 & 75 \end{bmatrix} \end{array}$$

Suppose Fatima wants to know the total production of sport shoes in each price category. Then the total production

In category 1 : for boys (80 + 90), for girls (60 + 50)

In category 2 : for boys (75 + 70), for girls (65 + 55)

In category 3 : for boys (90 + 75), for girls (85 + 75)

This can be represented in the matrix form as $\begin{bmatrix} 80+90 & 60+50 \\ 75+70 & 65+55 \\ 90+75 & 85+75 \end{bmatrix}$.

This new matrix is the **sum** of the above two matrices. We observe that the sum of two matrices is a matrix obtained by adding the corresponding elements of the given matrices. Furthermore, the two matrices have to be of the same order.

Thus, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ is a 2 × 3 matrix and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$ is another 2×3 matrix. Then, we define $A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}$.

In general, if $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of the same order, say $m \times n$. Then, the sum of the two matrices A and B is *defined* as a matrix $C = [c_{ij}]_{m \times n}$, where $c_{ij} = a_{ij} + b_{ij}$, for all possible values of i and j.

Example 6 Given $A = \begin{bmatrix} \sqrt{3} & 1 & -1 \\ 2 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & \sqrt{5} & 1 \\ -2 & 3 & \frac{1}{2} \end{bmatrix}$, find A + B

Since A, B are of the same order 2 × 3. Therefore, addition of A and B is defined and is given by

$$A + B = \begin{bmatrix} 2+\sqrt{3} & 1+\sqrt{5} & 1-1 \\ 2-2 & 3+3 & 0+\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2+\sqrt{3} & 1+\sqrt{5} & 0 \\ 0 & 6 & \frac{1}{2} \end{bmatrix}$$

Note

1. We emphasise that if A and B are not of the same order, then A + B is not defined. For example if $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}$, then A + B is not defined.
2. We may observe that addition of matrices is an example of binary operation on the set of matrices of the same order.

3.4.2 *Multiplication of a matrix by a scalar*

Now suppose that Fatima has doubled the production at a factory A in all categories (refer to 3.4.1).

Previously quantities (in standard units) produced by factory A were

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cc} \text{Boys} & \text{Girls} \end{array} \\ \begin{bmatrix} 80 & 60 \\ 75 & 65 \\ 90 & 85 \end{bmatrix} \end{array}$$

Revised quantities produced by factory A are as given below:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cc} \text{Boys} & \text{Girls} \end{array} \\ \begin{bmatrix} 2\times 80 & 2\times 60 \\ 2\times 75 & 2\times 65 \\ 2\times 90 & 2\times 85 \end{bmatrix} \end{array}$$

This can be represented in the matrix form as $\begin{bmatrix} 160 & 120 \\ 150 & 130 \\ 180 & 170 \end{bmatrix}$. We observe that the new matrix is obtained by multiplying each element of the previous matrix by 2.

In general, we may define *multiplication of a matrix* by a scalar as follows: if $A = [a_{ij}]_{m \times n}$ is a matrix and k is a scalar, then kA is another matrix which is obtained by multiplying each element of A by the scalar k.

In other words, $kA = k[a_{ij}]_{m \times n} = [k(a_{ij})]_{m \times n}$, that is, $(i, j)^{\text{th}}$ element of kA is ka_{ij} for all possible values of i and j.

For example, if $A = \begin{bmatrix} 3 & 1 & 1.5 \\ \sqrt{5} & 7 & -3 \\ 2 & 0 & 5 \end{bmatrix}$, then

$$3A = 3\begin{bmatrix} 3 & 1 & 1.5 \\ \sqrt{5} & 7 & -3 \\ 2 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 3 & 4.5 \\ 3\sqrt{5} & 21 & -9 \\ 6 & 0 & 15 \end{bmatrix}$$

Negative of a matrix The negative of a matrix is denoted by – A. We define – A = (– 1) A.

For example, let $A = \begin{bmatrix} 3 & 1 \\ -5 & x \end{bmatrix}$, then – A is given by

$$-A = (-1)A = (-1)\begin{bmatrix} 3 & 1 \\ -5 & x \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 5 & -x \end{bmatrix}$$

Difference of matrices If $A = [a_{ij}]$, $B = [b_{ij}]$ are two matrices of the same order, say $m \times n$, then difference A – B is defined as a matrix $D = [d_{ij}]$, where $d_{ij} = a_{ij} - b_{ij}$, for all value of i and j. In other words, D = A – B = A + (–1) B, that is sum of the matrix A and the matrix – B.

Example 7 If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$, then find 2A – B.

Solution We have

$$2A - B = 2\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \end{bmatrix} + \begin{bmatrix} -3 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2-3 & 4+1 & 6-3 \\ 4+1 & 6+0 & 2-2 \end{bmatrix} = \begin{bmatrix} -1 & 5 & 3 \\ 5 & 6 & 0 \end{bmatrix}$$

3.4.3 *Properties of matrix addition*

The addition of matrices satisfy the following properties:

(i) **Commutative Law** If $A = [a_{ij}]$, $B = [b_{ij}]$ are matrices of the same order, say $m \times n$, then A + B = B + A.

Now $A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$

$= [b_{ij} + a_{ij}]$ (addition of numbers is commutative)

$= ([b_{ij}] + [a_{ij}]) = B + A$

(ii) **Associative Law** For any three matrices $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$ of the same order, say $m \times n$, (A + B) + C = A + (B + C).

Now $(A + B) + C = ([a_{ij}] + [b_{ij}]) + [c_{ij}]$

$= [a_{ij} + b_{ij}] + [c_{ij}] = [(a_{ij} + b_{ij}) + c_{ij}]$

$= [a_{ij} + (b_{ij} + c_{ij})]$ (Why?)

$= [a_{ij}] + [(b_{ij} + c_{ij})] = [a_{ij}] + ([b_{ij}] + [c_{ij}]) = A + (B + C)$

(iii) **Existence of additive identity** Let A = $[a_{ij}]$ be an $m \times n$ matrix and O be an $m \times n$ zero matrix, then A + O = O + A = A. In other words, O is the additive identity for matrix addition.

(iv) **The existence of additive inverse** Let A = $[a_{ij}]_{m \times n}$ be any matrix, then we have another matrix as – A = $[- a_{ij}]_{m \times n}$ such that A + (– A) = (– A) + A= O. So – A is the additive inverse of A or negative of A.

3.4.4 *Properties of scalar multiplication of a matrix*

If A = $[a_{ij}]$ and B = $[b_{ij}]$ be two matrices of the same order, say $m \times n$, and k and l are scalars, then

(i) $k(\text{A} + \text{B}) = k\,\text{A} + k\text{B}$, (ii) $(k + l)\text{A} = k\,\text{A} + l\,\text{A}$

(ii) $k\,(\text{A} + \text{B}) = k\,([a_{ij}] + [b_{ij}])$

$$= k\,[a_{ij} + b_{ij}] = [k\,(a_{ij} + b_{ij})] = [(k\,a_{ij}) + (k\,b_{ij})]$$

$$= [k\,a_{ij}] + [k\,b_{ij}] = k\,[a_{ij}] + k\,[b_{ij}] = k\text{A} + k\text{B}$$

(iii) $(k + l)\,\text{A} = (k + l)\,[a_{ij}]$

$$= [(k + l)\,a_{ij}] + [k\,a_{ij}] + [l\,a_{ij}] = k\,[a_{ij}] + l\,[a_{ij}] = k\,\text{A} + l\,\text{A}$$

Example 8 If $\text{A} = \begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$ and $\text{B} = \begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix}$, then find the matrix X, such that 2A + 3X = 5B.

Solution We have 2A + 3X = 5B

or $\quad 2\text{A} + 3\text{X} - 2\text{A} = 5\text{B} - 2\text{A}$

or $\quad 2\text{A} - 2\text{A} + 3\text{X} = 5\text{B} - 2\text{A}$ (Matrix addition is commutative)

or $\quad \text{O} + 3\text{X} = 5\text{B} - 2\text{A}$ (– 2A is the additive inverse of 2A)

or $\quad 3\text{X} = 5\text{B} - 2\text{A}$ (O is the additive identity)

or $\quad \text{X} = \frac{1}{3}(5\text{B} - 2\text{A})$

or $$\text{X} = \frac{1}{3}\left(5\begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix} - 2\begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}\right) = \frac{1}{3}\left(\begin{bmatrix} 10 & -10 \\ 20 & 10 \\ -25 & 5 \end{bmatrix} + \begin{bmatrix} -16 & 0 \\ -8 & 4 \\ -6 & -12 \end{bmatrix}\right)$$

$$= \frac{1}{3}\begin{bmatrix} 10-16 & -10+0 \\ 20-8 & 10+4 \\ -25-6 & 5-12 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -6 & -10 \\ 12 & 14 \\ -31 & -7 \end{bmatrix} = \begin{bmatrix} -2 & \frac{-10}{3} \\ 4 & \frac{14}{3} \\ \frac{-31}{3} & \frac{-7}{3} \end{bmatrix}$$

Example 9 Find X and Y, if $X+Y=\begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ and $X-Y=\begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$.

Solution We have $(X+Y)+(X-Y)=\begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}+\begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$.

or $$(X + X) + (Y - Y) = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} \Rightarrow 2X=\begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix}$$

or $$X = \frac{1}{2}\begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$$

Also $$(X + Y) - (X - Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$$

or $$(X - X) + (Y + Y) = \begin{bmatrix} 5-3 & 2-6 \\ 0 & 9+1 \end{bmatrix} \Rightarrow 2Y = \begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix}$$

or $$Y = \frac{1}{2}\begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$$

Example 10 Find the values of x and y from the following equation:

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

Solution We have

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x & 10 \\ 14 & 2y-6 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

or $\quad \begin{bmatrix} 2x+3 & 10-4 \\ 14+1 & 2y-6+2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x+3 & 6 \\ 15 & 2y-4 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$

or $\quad 2x + 3 = 7 \quad$ and $\quad 2y - 4 = 14 \quad$ (Why?)

or $\quad 2x = 7 - 3 \quad$ and $\quad 2y = 18$

or $\quad x = \frac{4}{2} \quad$ and $\quad y = \frac{18}{2}$

i.e. $\quad x = 2 \quad$ and $\quad y = 9.$

Example 11 Two farmers Ramkishan and Gurcharan Singh cultivates only three varieties of rice namely Basmati, Permal and Naura. The sale (in Rupees) of these varieties of rice by both the farmers in the month of September and October are given by the following matrices A and B.

September Sales (in Rupees)

$$\begin{array}{cc} & \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \text{A} = & \begin{bmatrix} 10{,}000 & 20{,}000 & 30{,}000 \\ 50{,}000 & 30{,}000 & 10{,}000 \end{bmatrix} \begin{array}{l} \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array} \end{array}$$

October Sales (in Rupees)

$$\begin{array}{cc} & \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \text{B} = & \begin{bmatrix} 5000 & 10{,}000 & 6000 \\ 20{,}000 & 10{,}000 & 10{,}000 \end{bmatrix} \begin{array}{l} \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array} \end{array}$$

(i) Find the combined sales in September and October for each farmer in each variety.

(ii) Find the decrease in sales from September to October.

(iii) If both farmers receive 2% profit on gross sales, compute the profit for each farmer and for each variety sold in October.

Solution

(i) Combined sales in September and October for each farmer in each variety is given by

$$\begin{array}{cc} & \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \text{A} + \text{B} = & \begin{bmatrix} 15{,}000 & 30{,}000 & 36{,}000 \\ 70{,}000 & 40{,}000 & 20{,}000 \end{bmatrix} \begin{array}{l} \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array} \end{array}$$

(ii) Change in sales from September to October is given by

$$A - B = \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \left[\begin{array}{ccc} 5000 & 10,000 & 24,000 \\ 30,000 & 20,000 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

(iii) $2\% \text{ of } B = \frac{2}{100} \times B = 0.02 \times B$

$$= 0.02 \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \left[\begin{array}{ccc} 5000 & 10,000 & 6000 \\ 20,000 & 10,000 & 10,000 \end{array}\right] \end{array} \begin{array}{l} \\ \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

$$= \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \left[\begin{array}{ccc} 100 & 200 & 120 \\ 400 & 200 & 200 \end{array}\right] \end{array} \begin{array}{l} \\ \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

Thus, in October Ramkishan receives ₹ 100, ₹ 200 and ₹ 120 as profit in the sale of each variety of rice, respectively, and Grucharan Singh receives profit of ₹ 400, ₹ 200 and ₹ 200 in the sale of each variety of rice, respectively.

3.4.5 *Multiplication of matrices*

Suppose Meera and Nadeem are two friends. Meera wants to buy 2 pens and 5 story books, while Nadeem needs 8 pens and 10 story books. They both go to a shop to enquire about the rates which are quoted as follows:

Pen – ₹ 5 each, story book – ₹ 50 each.

How much money does each need to spend? Clearly, Meera needs ₹ $(5 \times 2 + 50 \times 5)$ that is ₹ 260, while Nadeem needs $(8 \times 5 + 50 \times 10)$ ₹, that is ₹ 540. In terms of matrix representation, we can write the above information as follows:

Requirements **Prices per piece (in Rupees)** **Money needed (in Rupees)**

$$\begin{bmatrix} 2 & 5 \\ 8 & 10 \end{bmatrix} \qquad \begin{bmatrix} 5 \\ 50 \end{bmatrix} \qquad \begin{bmatrix} 5 \times 2 + 5 \times 50 \\ 8 \times 5 + 10 \times 50 \end{bmatrix} = \begin{bmatrix} 260 \\ 540 \end{bmatrix}$$

Suppose that they enquire about the rates from another shop, quoted as follows:

pen – ₹ 4 each, story book – ₹ 40 each.

Now, the money required by Meera and Nadeem to make purchases will be respectively ₹ $(4 \times 2 + 40 \times 5) =$ ₹ 208 and ₹ $(8 \times 4 + 10 \times 40) =$ ₹ 432

Again, the above information can be represented as follows:

Requirements **Prices per piece (in Rupees)** **Money needed (in Rupees)**

$$\begin{bmatrix} 2 & 5 \\ 8 & 10 \end{bmatrix} \qquad \begin{bmatrix} 4 \\ 40 \end{bmatrix} \qquad \begin{bmatrix} 4\times 2+40\times 5 \\ 8\times 4+10\times 40 \end{bmatrix} = \begin{bmatrix} 208 \\ 432 \end{bmatrix}$$

Now, the information in both the cases can be combined and expressed in terms of matrices as follows:

Requirements **Prices per piece (in Rupees)** **Money needed (in Rupees)**

$$\begin{bmatrix} 2 & 5 \\ 8 & 10 \end{bmatrix} \qquad \begin{bmatrix} 5 & 4 \\ 50 & 40 \end{bmatrix} \qquad \begin{bmatrix} 5\times 2+5\times 50 & 4\times 2+40\times 5 \\ 8\times 5+10\times 50 & 8\times 4+10\times 40 \end{bmatrix}$$

$$= \begin{bmatrix} 260 & 208 \\ 540 & 432 \end{bmatrix}$$

The above is an example of multiplication of matrices. We observe that, for multiplication of two matrices A and B, the number of columns in A should be equal to the number of rows in B. Furthermore for getting the elements of the product matrix, we take rows of A and columns of B, multiply them element-wise and take the sum. Formally, we define multiplication of matrices as follows:

The *product* of two matrices A and B is *defined* if the number of columns of A is equal to the number of rows of B. Let A = $[a_{ij}]$ be an $m \times n$ matrix and B = $[b_{jk}]$ be an $n \times p$ matrix. Then the product of the matrices A and B is the matrix C of order $m \times p$. To get the $(i, k)^{\text{th}}$ element c_{ik} of the matrix C, we take the i^{th} row of A and k^{th} column of B, multiply them elementwise and take the sum of all these products. In other words, if A = $[a_{ij}]_{m \times n}$, B = $[b_{jk}]_{n \times p}$, then the i^{th} row of A is $[a_{i1}\ a_{i2}\ \ldots\ a_{in}]$ and the k^{th} column of

B is $\begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{nk} \end{bmatrix}$, then $c_{ik} = a_{i1} b_{1k} + a_{i2} b_{2k} + a_{i3} b_{3k} + \ldots + a_{in} b_{nk} = \sum_{j=1}^{n} a_{ij} b_{jk}$.

The matrix C = $[c_{ik}]_{m \times p}$ is the product of A and B.

For example, if $C = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix}$, then the product CD is defined and is given by $CD = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix}$. This is a 2 × 2 matrix in which each entry is the sum of the products across some row of C with the corresponding entries down some column of D. These four computations are

Entry in first row first column $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} (1)(2)+(-1)(-1)+(2)(5) & ? \\ ? & ? \end{bmatrix}$

Entry in first row second column $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} 13 & (1)(7)+(-1)(1)+2(-4) \\ ? & ? \end{bmatrix}$

Entry in second row first column $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} 13 & -2 \\ 0(2)+3(-1)+4(5) & ? \end{bmatrix}$

Entry in second row second column $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} 13 & -2 \\ 17 & 0(7)+3(1)+4(-4) \end{bmatrix}$

Thus $CD = \begin{bmatrix} 13 & -2 \\ 17 & -13 \end{bmatrix}$

Example 12 Find AB, if $A = \begin{bmatrix} 6 & 9 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 6 & 0 \\ 7 & 9 & 8 \end{bmatrix}$.

Solution The matrix A has 2 columns which is equal to the number of rows of B. Hence AB is defined. Now

$$AB = \begin{bmatrix} 6(2)+9(7) & 6(6)+9(9) & 6(0)+9(8) \\ 2(2)+3(7) & 2(6)+3(9) & 2(0)+3(8) \end{bmatrix}$$

$$= \begin{bmatrix} 12+63 & 36+81 & 0+72 \\ 4+21 & 12+27 & 0+24 \end{bmatrix} = \begin{bmatrix} 75 & 117 & 72 \\ 25 & 39 & 24 \end{bmatrix}$$

Remark If AB is defined, then BA need not be defined. In the above example, AB is defined but BA is not defined because B has 3 column while A has only 2 (and not 3) rows. If A, B are, respectively $m \times n$, $k \times l$ matrices, then both AB and BA are defined **if and only if** $n = k$ and $l = m$. In particular, if both A and B are square matrices of the same order, then both AB and BA are defined.

Non-commutativity of multiplication of matrices

Now, we shall see by an example that even if AB and BA are both defined, it is not necessary that AB = BA.

Example 13 If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$, then find AB, BA. Show that AB $\neq$ BA.

Solution Since A is a 2×3 matrix and B is 3×2 matrix. Hence AB and BA are both defined and are matrices of order 2×2 and 3×3, respectively. Note that

$$AB = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2-8+6 & 3-10+3 \\ -8+8+10 & -12+10+5 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$$

and $$BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 2-12 & -4+6 & 6+15 \\ 4-20 & -8+10 & 12+25 \\ 2-4 & -4+2 & 6+5 \end{bmatrix} = \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$$

Clearly AB $\neq$ BA

In the above example both AB and BA are of different order and so AB $\neq$ BA. But one may think that perhaps AB and BA could be the same if they were of the same order. But it is not so, here we give an example to show that even if AB and BA are of same order they may not be same.

Example 14 If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $AB = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

and $$BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$ Clearly AB $\neq$ BA.

Thus matrix multiplication is not commutative.

Note This does not mean that AB ≠ BA for every pair of matrices A, B for which AB and BA, are defined. For instance,

If $A=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$, $B=\begin{bmatrix}3 & 0\\ 0 & 4\end{bmatrix}$, then $AB = BA = \begin{bmatrix}3 & 0\\ 0 & 8\end{bmatrix}$

Observe that multiplication of diagonal matrices of same order will be commutative.

Zero matrix as the product of two non zero matrices

We know that, for real numbers a, b if $ab = 0$, then either $a = 0$ or $b = 0$. This need not be true for matrices, we will observe this through an example.

Example 15 Find AB, if $A=\begin{bmatrix}0 & -1\\ 0 & 2\end{bmatrix}$ and $B=\begin{bmatrix}3 & 5\\ 0 & 0\end{bmatrix}$.

Solution We have $AB=\begin{bmatrix}0 & -1\\ 0 & 2\end{bmatrix}\begin{bmatrix}3 & 5\\ 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$.

Thus, if the product of two matrices is a zero matrix, it is not necessary that one of the matrices is a zero matrix.

3.4.6 *Properties of multiplication of matrices*

The multiplication of matrices possesses the following properties, which we state without proof.

1. **The associative law** For any three matrices A, B and C. We have (AB) C = A (BC), whenever both sides of the equality are defined.
2. **The distributive law** For three matrices A, B and C.
   (i) A (B+C) = AB + AC
   (ii) (A+B) C = AC + BC, whenever both sides of equality are defined.
3. **The existence of multiplicative identity** For every square matrix A, there exist an identity matrix of same order such that IA = AI = A.

Now, we shall verify these properties by examples.

Example 16 If $A=\begin{bmatrix}1 & 1 & -1\\ 2 & 0 & 3\\ 3 & -1 & 2\end{bmatrix}$, $B=\begin{bmatrix}1 & 3\\ 0 & 2\\ -1 & 4\end{bmatrix}$ and $C=\begin{bmatrix}1 & 2 & 3 & -4\\ 2 & 0 & -2 & 1\end{bmatrix}$, find A(BC), (AB)C and show that (AB)C = A(BC).

Solution We have $AB = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 1+0+1 & 3+2-4 \\ 2+0-3 & 6+0+12 \\ 3+0-2 & 9-2+8 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 18 \\ 1 & 15 \end{bmatrix}$

$$(AB)\ (C) = \begin{bmatrix} 2 & 1 \\ -1 & 18 \\ 1 & 15 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 2+2 & 4+0 & 6-2 & -8+1 \\ -1+36 & -2+0 & -3-36 & 4+18 \\ 1+30 & 2+0 & 3-30 & -4+15 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 & 4 & -7 \\ 35 & -2 & -39 & 22 \\ 31 & 2 & -27 & 11 \end{bmatrix}$$

Now $\qquad BC = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1+6 & 2+0 & 3-6 & -4+3 \\ 0+4 & 0+0 & 0-4 & 0+2 \\ -1+8 & -2+0 & -3-8 & 4+4 \end{bmatrix}$

$$= \begin{bmatrix} 7 & 2 & -3 & -1 \\ 4 & 0 & -4 & 2 \\ 7 & -2 & -11 & 8 \end{bmatrix}$$

Therefore $\qquad A(BC) = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 7 & 2 & -3 & -1 \\ 4 & 0 & -4 & 2 \\ 7 & -2 & -11 & 8 \end{bmatrix}$

$$= \begin{bmatrix} 7+4-7 & 2+0+2 & -3-4+11 & -1+2-8 \\ 14+0+21 & 4+0-6 & -6+0-33 & -2+0+24 \\ 21-4+14 & 6+0-4 & -9+4-22 & -3-2+16 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 & 4 & -7 \\ 35 & -2 & -39 & 22 \\ 31 & 2 & -27 & 11 \end{bmatrix}.$$ Clearly, (AB) C = A (BC)

Example 17 If $A=\begin{bmatrix}0&6&7\\-6&0&8\\7&-8&0\end{bmatrix}, B=\begin{bmatrix}0&1&1\\1&0&2\\1&2&0\end{bmatrix}, C=\begin{bmatrix}2\\-2\\3\end{bmatrix}$

Calculate AC, BC and (A + B)C. Also, verify that (A + B)C = AC + BC

Solution Now, $A+B=\begin{bmatrix}0&7&8\\-5&0&10\\8&-6&0\end{bmatrix}$

So $$(A+B)\,C=\begin{bmatrix}0&7&8\\-5&0&10\\8&-6&0\end{bmatrix}\begin{bmatrix}2\\-2\\3\end{bmatrix}=\begin{bmatrix}0-14+24\\-10+0+30\\16+12+0\end{bmatrix}=\begin{bmatrix}10\\20\\28\end{bmatrix}$$

Further $$AC=\begin{bmatrix}0&6&7\\-6&0&8\\7&-8&0\end{bmatrix}\begin{bmatrix}2\\-2\\3\end{bmatrix}=\begin{bmatrix}0-12+21\\-12+0+24\\14+16+0\end{bmatrix}=\begin{bmatrix}9\\12\\30\end{bmatrix}$$

and $$BC=\begin{bmatrix}0&1&1\\1&0&2\\1&2&0\end{bmatrix}\begin{bmatrix}2\\-2\\3\end{bmatrix}=\begin{bmatrix}0-2+3\\2+0+6\\2-4+0\end{bmatrix}=\begin{bmatrix}1\\8\\-2\end{bmatrix}$$

So $$AC+BC=\begin{bmatrix}9\\12\\30\end{bmatrix}+\begin{bmatrix}1\\8\\-2\end{bmatrix}=\begin{bmatrix}10\\20\\28\end{bmatrix}$$

Clearly, $(A+B)\,C = AC + BC$

Example 18 If $A=\begin{bmatrix}1&2&3\\3&-2&1\\4&2&1\end{bmatrix}$, then show that $A^3 - 23A - 40\,I = O$

Solution We have $A^2=A.A=\begin{bmatrix}1&2&3\\3&-2&1\\4&2&1\end{bmatrix}\begin{bmatrix}1&2&3\\3&-2&1\\4&2&1\end{bmatrix}=\begin{bmatrix}19&4&8\\1&12&8\\14&6&15\end{bmatrix}$

So $$A^3 = A\,A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix} = \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix}$$

Now

$$A^3 - 23A - 40I = \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} - 23\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} - 40\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} + \begin{bmatrix} -23 & -46 & -69 \\ -69 & 46 & -23 \\ -92 & -46 & -23 \end{bmatrix} + \begin{bmatrix} -40 & 0 & 0 \\ 0 & -40 & 0 \\ 0 & 0 & -40 \end{bmatrix}$$

$$= \begin{bmatrix} 63-23-40 & 46-46+0 & 69-69+0 \\ 69-69+0 & -6+46-40 & 23-23+0 \\ 92-92+0 & 46-46+0 & 63-23-40 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Example 19 In a legislative assembly election, a political group hired a public relations firm to promote its candidate in three ways: telephone, house calls, and letters. The cost per contact (in paise) is given in matrix A as

$$A = \begin{array}{c} \textit{Cost per contact} \\ \begin{bmatrix} 40 \\ 100 \\ 50 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Telephone} \\ \text{Housecall} \\ \text{Letter} \end{array}$$

The number of contacts of each type made in two cities X and Y is given by

$$B = \begin{array}{c} \begin{array}{ccc} \text{Telephone} & \text{Housecall} & \text{Letter} \end{array} \\ \begin{bmatrix} 1000 & 500 & 5000 \\ 3000 & 1000 & 10,000 \end{bmatrix} \end{array} \begin{array}{l} \\ \to X \\ \to Y \end{array}$$. Find the total amount spent by the group in the two cities X and Y.

Solution We have

$$\mathrm{BA} = \begin{bmatrix} 40{,}000 + 50{,}000 + 250{,}000 \\ 120{,}000 + 100{,}000 + 500{,}000 \end{bmatrix} \begin{matrix} \to \mathrm{X} \\ \to \mathrm{Y} \end{matrix}$$

$$= \begin{bmatrix} 340{,}000 \\ 720{,}000 \end{bmatrix} \begin{matrix} \to \mathrm{X} \\ \to \mathrm{Y} \end{matrix}$$

So the total amount spent by the group in the two cities is 340,000 paise and 720,000 paise, i.e., ₹ 3400 and ₹ 7200, respectively.

EXERCISE 3.2

1. Let $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}, C = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$

Find each of the following:

(i) A + B (ii) A – B (iii) 3A – C

(iv) AB (v) BA

2. Compute the following:

(i) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ (ii) $\begin{bmatrix} a^2+b^2 & b^2+c^2 \\ a^2+c^2 & a^2+b^2 \end{bmatrix} + \begin{bmatrix} 2ab & 2bc \\ -2ac & -2ab \end{bmatrix}$

(iii) $\begin{bmatrix} -1 & 4 & -6 \\ 8 & 5 & 16 \\ 2 & 8 & 5 \end{bmatrix} + \begin{bmatrix} 12 & 7 & 6 \\ 8 & 0 & 5 \\ 3 & 2 & 4 \end{bmatrix}$ (iv) $\begin{bmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{bmatrix} + \begin{bmatrix} \sin^2 x & \cos^2 x \\ \cos^2 x & \sin^2 x \end{bmatrix}$

3. Compute the indicated products.

(i) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ (ii) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [2 \quad 3 \quad 4]$ (iii) $\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{bmatrix}$ (v) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}$

(vi) $\begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}$

4. If $A=\begin{bmatrix}1 & 2 & -3\\ 5 & 0 & 2\\ 1 & -1 & 1\end{bmatrix}$, $B=\begin{bmatrix}3 & -1 & 2\\ 4 & 2 & 5\\ 2 & 0 & 3\end{bmatrix}$ and $C=\begin{bmatrix}4 & 1 & 2\\ 0 & 3 & 2\\ 1 & -2 & 3\end{bmatrix}$, then compute (A+B) and (B – C). Also, verify that A + (B – C) = (A + B) – C.

5. If $A=\begin{bmatrix}\frac{2}{3} & 1 & \frac{5}{3}\\ \frac{1}{3} & \frac{2}{3} & \frac{4}{3}\\ \frac{7}{3} & 2 & \frac{2}{3}\end{bmatrix}$ and $B=\begin{bmatrix}\frac{2}{5} & \frac{3}{5} & 1\\ \frac{1}{5} & \frac{2}{5} & \frac{4}{5}\\ \frac{7}{5} & \frac{6}{5} & \frac{2}{5}\end{bmatrix}$, then compute 3A – 5B.

6. Simplify $\cos\theta\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}+\sin\theta\begin{bmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{bmatrix}$

7. Find X and Y, if

(i) $X+Y=\begin{bmatrix}7 & 0\\ 2 & 5\end{bmatrix}$ and $X-Y=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$

(ii) $2X+3Y=\begin{bmatrix}2 & 3\\ 4 & 0\end{bmatrix}$ and $3X+2Y=\begin{bmatrix}2 & -2\\ -1 & 5\end{bmatrix}$

8. Find X, if $Y=\begin{bmatrix}3 & 2\\ 1 & 4\end{bmatrix}$ and $2X+Y=\begin{bmatrix}1 & 0\\ -3 & 2\end{bmatrix}$

9. Find x and y, if $2\begin{bmatrix}1 & 3\\ 0 & x\end{bmatrix}+\begin{bmatrix}y & 0\\ 1 & 2\end{bmatrix}=\begin{bmatrix}5 & 6\\ 1 & 8\end{bmatrix}$

10. Solve the equation for x, y, z and t, if $2\begin{bmatrix}x & z\\ y & t\end{bmatrix}+3\begin{bmatrix}1 & -1\\ 0 & 2\end{bmatrix}=3\begin{bmatrix}3 & 5\\ 4 & 6\end{bmatrix}$

11. If $x\begin{bmatrix}2\\ 3\end{bmatrix}+y\begin{bmatrix}-1\\ 1\end{bmatrix}=\begin{bmatrix}10\\ 5\end{bmatrix}$, find the values of x and y.

12. Given $3\begin{bmatrix}x & y\\ z & w\end{bmatrix}=\begin{bmatrix}x & 6\\ -1 & 2w\end{bmatrix}+\begin{bmatrix}4 & x+y\\ z+w & 3\end{bmatrix}$, find the values of x, y, z and w.

13. If $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$, show that F($x$) F($y$) = F($x + y$).

14. Show that

(i) $\begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \neq \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

15. Find $A^2 - 5A + 6I$, if $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

16. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$, prove that $A^3 - 6A^2 + 7A + 2I = 0$

17. If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find k so that $A^2 = kA - 2I$

18. If $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ and I is the identity matrix of order 2, show that

$$I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

19. A trust fund has ₹ 30,000 that must be invested in two different types of bonds. The first bond pays 5% interest per year, and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide ₹ 30,000 among the two types of bonds. If the trust fund must obtain an annual total interest of:

(a) ₹ 1800 (b) ₹ 2000

20. The bookshop of a particular school has 10 dozen chemistry books, 8 dozen physics books, 10 dozen economics books. Their selling prices are ₹ 80, ₹ 60 and ₹ 40 each respectively. Find the total amount the bookshop will receive from selling all the books using matrix algebra.

Assume X, Y, Z, W and P are matrices of order $2 \times n$, $3 \times k$, $2 \times p$, $n \times 3$ and $p \times k$, respectively. Choose the correct answer in Exercises 21 and 22.

21. The restriction on n, k and p so that PY + WY will be defined are:

(A) $k = 3, p = n$ (B) k is arbitrary, $p = 2$

(C) p is arbitrary, $k = 3$ (D) $k = 2, p = 3$

22. If $n = p$, then the order of the matrix $7X - 5Z$ is:

(A) $p \times 2$ (B) $2 \times n$ (C) $n \times 3$ (D) $p \times n$

3.5. Transpose of a Matrix

In this section, we shall learn about transpose of a matrix and special types of matrices such as symmetric and skew symmetric matrices.

Definition 3 If $A = [a_{ij}]$ be an $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of A is called the *transpose* of A. Transpose of the matrix A is denoted by A′ or (A^T). In other words, if $A = [a_{ij}]_{m \times n}$, then $A' = [a_{ji}]_{n \times m}$. For example,

$$\text{if } A = \begin{bmatrix} 3 & 5 \\ \sqrt{3} & 1 \\ 0 & \frac{-1}{5} \end{bmatrix}_{3 \times 2}, \text{ then } A' = \begin{bmatrix} 3 & \sqrt{3} & 0 \\ 5 & 1 & \frac{-1}{5} \end{bmatrix}_{2 \times 3}$$

3.5.1 Properties of transpose of the matrices

We now state the following properties of transpose of matrices without proof. These may be verified by taking suitable examples.

For any matrices A and B of suitable orders, we have

(i) $(A')' = A$, (ii) $(kA)' = kA'$ (where k is any constant)

(iii) $(A + B)' = A' + B'$ (iv) $(A B)' = B' A'$

Example 20 If $A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$, verify that

(i) $(A')' = A$, (ii) $(A + B)' = A' + B'$,

(iii) $(kB)' = kB'$, where k is any constant.

Solution

(i) We have

$$A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 3 & 4 \\ \sqrt{3} & 2 \\ 2 & 0 \end{bmatrix} \Rightarrow (A')' = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix} = A$$

Thus $(A')' = A$

(ii) We have

$$A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix},\ B = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix} \Rightarrow A + B = \begin{bmatrix} 5 & \sqrt{3}-1 & 4 \\ 5 & 4 & 4 \end{bmatrix}$$

Therefore $$(A + B)' = \begin{bmatrix} 5 & 5 \\ \sqrt{3}-1 & 4 \\ 4 & 4 \end{bmatrix}$$

Now $$A' = \begin{bmatrix} 3 & 4 \\ \sqrt{3} & 2 \\ 2 & 0 \end{bmatrix},\ B' = \begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 4 \end{bmatrix},$$

So $$A' + B' = \begin{bmatrix} 5 & 5 \\ \sqrt{3}-1 & 4 \\ 4 & 4 \end{bmatrix}$$

Thus $(A + B)' = A' + B'$

(iii) We have

$$kB = k\begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2k & -k & 2k \\ k & 2k & 4k \end{bmatrix}$$

Then $$(kB)' = \begin{bmatrix} 2k & k \\ -k & 2k \\ 2k & 4k \end{bmatrix} = k\begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 4 \end{bmatrix} = kB'$$

Thus $(kB)' = kB'$

Example 21 If $A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & -6 \end{bmatrix}$, verify that $(AB)' = B'A'$.

Solution We have

$$A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & -6 \end{bmatrix}$$

then $$AB = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 3 & -6 \end{bmatrix} = \begin{bmatrix} -2 & -6 & 12 \\ 4 & 12 & -24 \\ 5 & 15 & -30 \end{bmatrix}$$

Now $$A' = [-2\ 4\ 5],\ B' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} \begin{bmatrix} -2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix} = (AB)'$$

Clearly $(AB)' = B'A'$

3.6 Symmetric and Skew Symmetric Matrices

Definition 4 A square matrix $A = [a_{ij}]$ is said to be *symmetric* if $A' = A$, that is, $[a_{ij}] = [a_{ji}]$ for all possible values of i and j.

For example $A = \begin{bmatrix} \sqrt{3} & 2 & 3 \\ 2 & -1.5 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ is a symmetric matrix as $A' = A$

Definition 5 A square matrix $A = [a_{ij}]$ is said to be *skew symmetric* matrix if $A' = -A$, that is $a_{ji} = -a_{ij}$ for all possible values of i and j. Now, if we put $i = j$, we have $a_{ii} = -a_{ii}$. Therefore $2a_{ii} = 0$ or $a_{ii} = 0$ for all i's.

This means that all the diagonal elements of a skew symmetric matrix are zero.

For example, the matrix $B = \begin{bmatrix} 0 & e & f \\ -e & 0 & g \\ -f & -g & 0 \end{bmatrix}$ is a skew symmetric matrix as $B' = -B$

Now, we are going to prove some results of symmetric and skew-symmetric matrices.

Theorem 1 For any square matrix A with real number entries, $A + A'$ is a symmetric matrix and $A - A'$ is a skew symmetric matrix.

Proof Let $B = A + A'$, then

$$\begin{aligned} B' &= (A + A')' \\ &= A' + (A')' \text{ (as } (A + B)' = A' + B') \\ &= A' + A \text{ (as } (A')' = A) \\ &= A + A' \text{ (as } A + B = B + A) \\ &= B \end{aligned}$$

Therefore $\quad B = A + A'$ is a symmetric matrix

Now let $\quad C = A - A'$

$$\begin{aligned} C' &= (A - A')' = A' - (A')' \qquad \text{(Why?)} \\ &= A' - A \qquad \text{(Why?)} \\ &= -(A - A') = -C \end{aligned}$$

Therefore $\quad C = A - A'$ is a skew symmetric matrix.

Theorem 2 Any square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix.

Proof Let A be a square matrix, then we can write

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

From the Theorem 1, we know that $(A + A')$ is a symmetric matrix and $(A - A')$ is a skew symmetric matrix. Since for any matrix A, $(kA)' = kA'$, it follows that $\frac{1}{2}(A + A')$

is symmetric matrix and $\frac{1}{2}(A - A')$ is skew symmetric matrix. Thus, any square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix.

Example 22 Express the matrix $B = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

Solution Here

$$B' = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}$$

Let $$P = \frac{1}{2}(B + B') = \frac{1}{2}\begin{bmatrix} 4 & -3 & -3 \\ -3 & 6 & 2 \\ -3 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix},$$

Now $$P' = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} = P$$

Thus $P = \frac{1}{2}(B + B')$ is a symmetric matrix.

Also, let $$Q = \frac{1}{2}(B - B') = \frac{1}{2}\begin{bmatrix} 0 & -1 & -5 \\ 1 & 0 & 6 \\ 5 & -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix}$$

Then $$Q' = \begin{bmatrix} 0 & \frac{1}{2} & \frac{5}{3} \\ \frac{-1}{2} & 0 & -3 \\ \frac{-5}{2} & 3 & 0 \end{bmatrix} = -Q$$

Thus $Q = \frac{1}{2}(B - B')$ is a skew symmetric matrix.

Now $$P + Q = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} + \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = B$$

Thus, B is represented as the sum of a symmetric and a skew symmetric matrix.

EXERCISE 3.3

1. Find the transpose of each of the following matrices:

(i) $\begin{bmatrix} 5 \\ \frac{1}{2} \\ -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} -1 & 5 & 6 \\ \sqrt{3} & 5 & 6 \\ 2 & 3 & -1 \end{bmatrix}$

2. If $A = \begin{bmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$, then verify that

(i) $(A + B)' = A' + B'$, (ii) $(A - B)' = A' - B'$

3. If $A' = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, then verify that

(i) $(A + B)' = A' + B'$ (ii) $(A - B)' = A' - B'$

4. If $A' = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$, then find $(A + 2B)'$

5. For the matrices A and B, verify that $(AB)' = B'A'$, where

(i) $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 5 & 7 \end{bmatrix}$

6. If (i) $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then verify that $A'A = I$

(ii) If $A = \begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix}$, then verify that $A'A = I$

7. (i) Show that the matrix $A = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 3 \end{bmatrix}$ is a symmetric matrix.

(ii) Show that the matrix $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ is a skew symmetric matrix.

8. For the matrix $A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$, verify that

(i) $(A + A')$ is a symmetric matrix

(ii) $(A - A')$ is a skew symmetric matrix

9. Find $\frac{1}{2}(A + A')$ and $\frac{1}{2}(A - A')$, when $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$

10. Express the following matrices as the sum of a symmetric and a skew symmetric matrix:

(i) $\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix}$

Choose the correct answer in the Exercises 11 and 12.

11. If A, B are symmetric matrices of same order, then AB – BA is a

(A) Skew symmetric matrix (B) Symmetric matrix

(C) Zero matrix (D) Identity matrix

12. If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, and $A + A' = I$, then the value of α is

(A) $\dfrac{\pi}{6}$ (B) $\dfrac{\pi}{3}$

(C) π (D) $\dfrac{3\pi}{2}$

3.7 Invertible Matrices

Definition 6 If A is a square matrix of order *m*, and if there exists another square matrix B of the same order *m*, such that AB = BA = I, then B is called the *inverse* matrix of A and it is denoted by A^{-1}. In that case A is said to be invertible.

For example, let $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ be two matrices.

Now
$$AB = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 4-3 & -6+6 \\ 2-2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Also $BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$. Thus B is the inverse of A, in other words $B = A^{-1}$ and A is inverse of B, i.e., $A = B^{-1}$

Note

1. A rectangular matrix does not possess inverse matrix, since for products BA and AB to be defined and to be equal, it is necessary that matrices A and B should be square matrices of the same order.
2. If B is the inverse of A, then A is also the inverse of B.

Theorem 3 (Uniqueness of inverse) Inverse of a square matrix, if it exists, is unique.

Proof Let A = $[a_{ij}]$ be a square matrix of order m. If possible, let B and C be two inverses of A. We shall show that B = C.

Since B is the inverse of A

$$AB = BA = I \qquad \text{... (1)}$$

Since C is also the inverse of A

$$AC = CA = I \qquad \text{... (2)}$$

Thus $\qquad B = BI = B\,(AC) = (BA)\,C = IC = C$

Theorem 4 If A and B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$.

Proof From the definition of inverse of a matrix, we have

$$(AB)\,(AB)^{-1} = 1$$

or $\qquad A^{-1}(AB)\,(AB)^{-1} = A^{-1}I \qquad$ (Pre multiplying both sides by A^{-1})

or $\qquad (A^{-1}A)\,B\,(AB)^{-1} = A^{-1} \qquad$ (Since $A^{-1}\,I = A^{-1}$)

or $\qquad IB\,(AB)^{-1} = A^{-1}$

or $\qquad B\,(AB)^{-1} = A^{-1}$

or $\qquad B^{-1}\,B\,(AB)^{-1} = B^{-1}\,A^{-1}$

or $\qquad I\,(AB)^{-1} = B^{-1}\,A^{-1}$

Hence $\qquad (AB)^{-1} = B^{-1}\,A^{-1}$

EXERCISE 3.4

1. Matrices A and B will be inverse of each other only if

(A) AB = BA (B) AB = BA = 0

(C) AB = 0, BA = I (D) AB = BA = I

Miscellaneous Examples

Example 23 If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then prove that $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$, $n \in \mathbf{N}$.

Solution We shall prove the result by using principle of mathematical induction.

We have $\quad P(n)$: If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$, $n \in \mathbf{N}$

$$P(1):\ A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \text{ so } A^1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Therefore, the result is true for $n = 1$.

Let the result be true for $n = k$. So

$$P(k):\ A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \text{ then } A^k = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$$

Now, we prove that the result holds for $n = k+1$

Now $$A^{k+1} = A \cdot A^k = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos k\theta - \sin\theta\sin k\theta & \cos\theta\sin k\theta + \sin\theta\cos k\theta \\ -\sin\theta\cos k\theta + \cos\theta\sin k\theta & -\sin\theta\sin k\theta + \cos\theta\cos k\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta + k\theta) & \sin(\theta + k\theta) \\ -\sin(\theta + k\theta) & \cos(\theta + k\theta) \end{bmatrix} = \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

Therefore, the result is true for $n = k + 1$. Thus by principle of mathematical induction, we have $A^n = \begin{bmatrix} \cos n\,\theta & \sin n\,\theta \\ -\sin n\,\theta & \cos n\,\theta \end{bmatrix}$, holds for all natural numbers.

Example 24 If A and B are symmetric matrices of the same order, then show that AB is symmetric if and only if A and B commute, that is AB = BA.

Solution Since A and B are both symmetric matrices, therefore A′ = A and B′ = B.

Let AB be symmetric, then (AB)′ = AB

But (AB)′ = B′A′= BA (Why?)

Therefore BA = AB

Conversely, if AB = BA, then we shall show that AB is symmetric.

Now (AB)′ = B′A′

= B A (as A and B are symmetric)

= AB

Hence AB is symmetric.

Example 25 Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}, C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$. Find a matrix D such that CD – AB = O.

Solution Since A, B, C are all square matrices of order 2, and CD – AB is well defined, D must be a square matrix of order 2.

Let $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then CD – AB = 0 gives

$$\begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix} = O$$

or $$\begin{bmatrix} 2a+5c & 2b+5d \\ 3a+8c & 3b+8d \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 43 & 22 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

or $$\begin{bmatrix} 2a+5c-3 & 2b+5d \\ 3a+8c-43 & 3b+8d-22 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

By equality of matrices, we get

$$2a + 5c - 3 = 0 \quad \text{... (1)}$$

$$3a + 8c - 43 = 0 \quad \text{... (2)}$$

$$2b + 5d = 0 \quad \text{... (3)}$$

and $$3b + 8d - 22 = 0 \quad \text{... (4)}$$

Solving (1) and (2), we get $a = -191$, $c = 77$. Solving (3) and (4), we get $b = -110$, $d = 44$.

Therefore $$D = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix}$$

Miscellaneous Exercise on Chapter 3

1. If A and B are symmetric matrices, prove that AB – BA is a skew symmetric matrix.

2. Show that the matrix B′AB is symmetric or skew symmetric according as A is symmetric or skew symmetric.

3. Find the values of x, y, z if the matrix $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfy the equation $A'A = I$.

4. For what values of x : $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = O$?

5. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$.

6. Find x, if $\begin{bmatrix} x & -5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = O$

7. A manufacturer produces three products x, y, z which he sells in two markets. Annual sales are indicated below:

Market	**Products**		
I	10,000	2,000	18,000
II	6,000	20,000	8,000

(a) If unit sale prices of x, y and z are ₹ 2.50, ₹ 1.50 and ₹ 1.00, respectively, find the total revenue in each market with the help of matrix algebra.

(b) If the unit costs of the above three commodities are ₹ 2.00, ₹ 1.00 and 50 paise respectively. Find the gross profit.

8. Find the matrix X so that $X\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}=\begin{bmatrix}-7 & -8 & -9\\ 2 & 4 & 6\end{bmatrix}$

Choose the correct answer in the following questions:

9. If $A=\begin{bmatrix}\alpha & \beta\\ \gamma & -\alpha\end{bmatrix}$ is such that $A^2 = I$, then

(A) $1+\alpha^2+\beta\gamma = 0$ (B) $1-\alpha^2+\beta\gamma = 0$

(C) $1-\alpha^2-\beta\gamma = 0$ (D) $1+\alpha^2-\beta\gamma = 0$

10. If the matrix A is both symmetric and skew symmetric, then

(A) A is a diagonal matrix (B) A is a zero matrix

(C) A is a square matrix (D) None of these

11. If A is square matrix such that $A^2 = A$, then $(I + A)^3 - 7A$ is equal to

(A) A (B) I – A (C) I (D) 3A

Summary

- A matrix is an ordered rectangular array of numbers or functions.
- A matrix having m rows and n columns is called a matrix of order $m \times n$.
- $[a_{ij}]_{m \times 1}$ is a column matrix.
- $[a_{ij}]_{1 \times n}$ is a row matrix.
- An $m \times n$ matrix is a square matrix if $m = n$.
- $A = [a_{ij}]_{m \times m}$ is a diagonal matrix if $a_{ij} = 0$, when $i \neq j$.
- $A = [a_{ij}]_{n \times n}$ is a scalar matrix if $a_{ij} = 0$, when $i \neq j$, $a_{ij} = k$, (k is some constant), when $i = j$.
- $A = [a_{ij}]_{n \times n}$ is an identity matrix, if $a_{ij} = 1$, when $i = j$, $a_{ij} = 0$, when $i \neq j$.
- A zero matrix has all its elements as zero.
- $A = [a_{ij}] = [b_{ij}] = B$ if (i) A and B are of same order, (ii) $a_{ij} = b_{ij}$ for all possible values of i and j.

- $k\text{A} = k[a_{ij}]_{m \times n} = [k(a_{ij})]_{m \times n}$
- $-\text{A} = (-1)\text{A}$
- $\text{A} - \text{B} = \text{A} + (-1)\,\text{B}$
- $\text{A} + \text{B} = \text{B} + \text{A}$
- $(\text{A} + \text{B}) + \text{C} = \text{A} + (\text{B} + \text{C})$, where A, B and C are of same order.
- $k(\text{A} + \text{B}) = k\text{A} + k\text{B}$, where A and B are of same order, k is constant.
- $(k + l)\,\text{A} = k\text{A} + l\text{A}$, where k and l are constant.
- If $\text{A} = [a_{ij}]_{m \times n}$ and $\text{B} = [b_{jk}]_{n \times p}$, then $\text{AB} = \text{C} = [c_{ik}]_{m \times p}$, where $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$
- (i) $\text{A(BC)} = \text{(AB)C}$, (ii) $\text{A(B + C)} = \text{AB} + \text{AC}$, (iii) $\text{(A + B)C} = \text{AC} + \text{BC}$
- If $\text{A} = [a_{ij}]_{m \times n}$, then A' or $\text{A}^{\text{T}} = [a_{ji}]_{n \times m}$
- (i) $(\text{A}')' = \text{A}$, (ii) $(k\text{A})' = k\text{A}'$, (iii) $(\text{A} + \text{B})' = \text{A}' + \text{B}'$, (iv) $(\text{AB})' = \text{B}'\text{A}'$
- A is a symmetric matrix if $\text{A}' = \text{A}$.
- A is a skew symmetric matrix if $\text{A}' = -\text{A}$.
- Any square matrix can be represented as the sum of a symmetric and a skew symmetric matrix.
- If A and B are two square matrices such that $\text{AB} = \text{BA} = \text{I}$, then B is the inverse matrix of A and is denoted by A^{-1} and A is the inverse of B.
- Inverse of a square matrix, if it exists, is unique.

NOTES

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Chapter 4

DETERMINANTS

❖ *All Mathematical truths are relative and conditional. — C.P. STEINMETZ* ❖

4.1 Introduction

P.S. Laplace
(1749-1827)

In the previous chapter, we have studied about matrices and algebra of matrices. We have also learnt that a system of algebraic equations can be expressed in the form of matrices. This means, a system of linear equations like

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

can be represented as $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. Now, this system of equations has a unique solution or not, is determined by the number $a_1 b_2 - a_2 b_1$. (Recall that if $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$ or, $a_1 b_2 - a_2 b_1 \neq 0$, then the system of linear equations has a unique solution). The number $a_1 b_2 - a_2 b_1$ which determines uniqueness of solution is associated with the matrix $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ and is called the determinant of A or det A. Determinants have wide applications in Engineering, Science, Economics, Social Science, etc.

In this chapter, we shall study determinants up to order three only with real entries. Also, we will study various properties of determinants, minors, cofactors and applications of determinants in finding the area of a triangle, adjoint and inverse of a square matrix, consistency and inconsistency of system of linear equations and solution of linear equations in two or three variables using inverse of a matrix.

4.2 Determinant

To every square matrix $A = [a_{ij}]$ of order n, we can associate a number (real or complex) called determinant of the square matrix A, where $a_{ij} = (i, j)^{th}$ element of A.

This may be thought of as a function which associates each square matrix with a unique number (real or complex). If M is the set of square matrices, K is the set of numbers (real or complex) and $f : M \to K$ is defined by $f(A) = k$, where $A \in M$ and $k \in K$, then $f(A)$ is called the determinant of A. It is also denoted by $|A|$ or det A or Δ.

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then determinant of A is written as } |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det(A)$$

Remarks

(i) For matrix A, $|A|$ is read as determinant of A and not modulus of A.

(ii) Only square matrices have determinants.

4.2.1 *Determinant of a matrix of order one*

Let $A = [a]$ be the matrix of order 1, then determinant of A is defined to be equal to a

4.2.2 *Determinant of a matrix of order two*

Let $$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$ be a matrix of order 2 × 2,

then the determinant of A is defined as:

$$\det(A) = |A| = \Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Example 1 Evaluate $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$.

Solution We have $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = 2(2) - 4(-1) = 4 + 4 = 8.$

Example 2 Evaluate $\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix}$

Solution We have

$$\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix} = x(x) - (x+1)(x-1) = x^2 - (x^2 - 1) = x^2 - x^2 + 1 = 1$$

4.2.3 *Determinant of a matrix of order 3 × 3*

Determinant of a matrix of order three can be determined by expressing it in terms of second order determinants. This is known as expansion of a determinant along a row (or a column). There are six ways of expanding a determinant of order

3 corresponding to each of three rows (R_1, R_2 and R_3) and three columns (C_1, C_2 and C_3) giving the same value as shown below.

Consider the determinant of square matrix $A = [a_{ij}]_{3 \times 3}$

i.e.,
$$|A| = \begin{vmatrix} \boldsymbol{a_{11}} & \boldsymbol{a_{12}} & \boldsymbol{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expansion along first Row (R_1)

Step 1 Multiply first element a_{11} of R_1 by $(-1)^{(1+1)}$ $[(-1)^{\text{sum of suffixes in } a_{11}}]$ and with the second order determinant obtained by deleting the elements of first row (R_1) and first column (C_1) of | A | as a_{11} lies in R_1 and C_1,

i.e.,
$$(-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Step 2 Multiply 2nd element a_{12} of R_1 by $(-1)^{1+2}$ $[(-1)^{\text{sum of suffixes in } a_{12}}]$ and the second order determinant obtained by deleting elements of first row (R_1) and 2nd column (C_2) of | A | as a_{12} lies in R_1 and C_2,

i.e.,
$$(-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

Step 3 Multiply third element a_{13} of R_1 by $(-1)^{1+3}$ $[(-1)^{\text{sum of suffixes in } a_{13}}]$ and the second order determinant obtained by deleting elements of first row (R_1) and third column (C_3) of | A | as a_{13} lies in R_1 and C_3,

i.e.,
$$(-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Step 4 Now the expansion of determinant of A, that is, | A | written as sum of all three terms obtained in steps 1, 2 and 3 above is given by

$$\det A = |A| = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$
$$+ (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

or
$$|A| = a_{11} (a_{22} a_{33} - a_{32} a_{23}) - a_{12} (a_{21} a_{33} - a_{31} a_{23})$$
$$+ a_{13} (a_{21} a_{32} - a_{31} a_{22})$$

$$= a_{11}\, a_{22}\, a_{33} - a_{11}\, a_{32}\, a_{23} - a_{12}\, a_{21}\, a_{33} + a_{12}\, a_{31}\, a_{23} + a_{13}\, a_{21}\, a_{32} - a_{13}\, a_{31}\, a_{22} \qquad \ldots (1)$$

Note We shall apply all four steps together.

Expansion along second row (R_2)

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \boldsymbol{a_{21}} & \boldsymbol{a_{22}} & \boldsymbol{a_{23}} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding along R_2, we get

$$|A| = (-1)^{2+1} a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{2+2} a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{2+3} a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= -a_{21}\,(a_{12}\, a_{33} - a_{32}\, a_{13}) + a_{22}\,(a_{11}\, a_{33} - a_{31}\, a_{13}) - a_{23}\,(a_{11}\, a_{32} - a_{31}\, a_{12})$$

$$|A| = -a_{21}\, a_{12}\, a_{33} + a_{21}\, a_{32}\, a_{13} + a_{22}\, a_{11}\, a_{33} - a_{22}\, a_{31}\, a_{13} - a_{23}\, a_{11}\, a_{32} + a_{23}\, a_{31}\, a_{12}$$

$$= a_{11}\, a_{22}\, a_{33} - a_{11}\, a_{23}\, a_{32} - a_{12}\, a_{21}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{13}\, a_{21}\, a_{32} - a_{13}\, a_{31}\, a_{22} \qquad \ldots (2)$$

Expansion along first Column (C_1)

$$|A| = \begin{vmatrix} \boldsymbol{a_{11}} & a_{12} & a_{13} \\ \boldsymbol{a_{21}} & a_{22} & a_{23} \\ \boldsymbol{a_{31}} & a_{32} & a_{33} \end{vmatrix}$$

By expanding along C_1, we get

$$|A| = a_{11}\,(-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}\,(-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\,(-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11}\,(a_{22}\, a_{33} - a_{23}\, a_{32}) - a_{21}\,(a_{12}\, a_{33} - a_{13}\, a_{32}) + a_{31}\,(a_{12}\, a_{23} - a_{13}\, a_{22})$$

$$\begin{aligned} |A| &= a_{11}\, a_{22}\, a_{33} - a_{11}\, a_{23}\, a_{32} - a_{21}\, a_{12}\, a_{33} + a_{21}\, a_{13}\, a_{32} + a_{31}\, a_{12}\, a_{23} \\ &\quad - a_{31}\, a_{13}\, a_{22} \\ &= a_{11}\, a_{22}\, a_{33} - a_{11}\, a_{23}\, a_{32} - a_{12}\, a_{21}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{13}\, a_{21}\, a_{32} \\ &\quad - a_{13}\, a_{31}\, a_{22} \qquad \ldots (3) \end{aligned}$$

Clearly, values of $|A|$ in (1), (2) and (3) are equal. It is left as an exercise to the reader to verify that the values of $|A|$ by expanding along R_3, C_2 and C_3 are equal to the value of $|A|$ obtained in (1), (2) or (3).

Hence, expanding a determinant along any row or column gives same value.

Remarks

(i) For easier calculations, we shall expand the determinant along that row or column which contains maximum number of zeros.

(ii) While expanding, instead of multiplying by $(-1)^{i+j}$, we can multiply by +1 or –1 according as $(i + j)$ is even or odd.

(iii) Let $A = \begin{bmatrix} 2 & 2 \\ 4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$. Then, it is easy to verify that A = 2B. Also $|A| = 0 - 8 = -8$ and $|B| = 0 - 2 = -2$.

Observe that, $|A| = 4(-2) = 2^2|B|$ or $|A| = 2^n|B|$, where $n = 2$ is the order of square matrices A and B.

In general, if $A = kB$ where A and B are square matrices of order n, then $|A| = k^n |B|$, where $n = 1, 2, 3$

Example 3 Evaluate the determinant $\Delta = \begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix}$.

Solution Note that in the third column, two entries are zero. So expanding along third column (C_3), we get

$$\Delta = 4\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}$$
$$= 4(-1 - 12) - 0 + 0 = -52$$

Example 4 Evaluate $\Delta = \begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix}$.

Solution Expanding along R_1, we get

$$\Delta = 0\begin{vmatrix} 0 & \sin\beta \\ -\sin\beta & 0 \end{vmatrix} - \sin\alpha\begin{vmatrix} -\sin\alpha & \sin\beta \\ \cos\alpha & 0 \end{vmatrix} - \cos\alpha\begin{vmatrix} -\sin\alpha & 0 \\ \cos\alpha & -\sin\beta \end{vmatrix}$$

$$= 0 - \sin\alpha\,(0 - \sin\beta\cos\alpha) - \cos\alpha\,(\sin\alpha\sin\beta - 0)$$
$$= \sin\alpha\sin\beta\cos\alpha - \cos\alpha\sin\alpha\sin\beta = 0$$

Example 5 Find values of x for which $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$.

Solution We have $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$

i.e. $\quad 3 - x^2 = 3 - 8$

i.e. $\quad x^2 = 8$

Hence $\quad x = \pm 2\sqrt{2}$

EXERCISE 4.1

Evaluate the determinants in Exercises 1 and 2.

1. $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$

2. (i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ (ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

3. If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, then show that $|2A| = 4|A|$

4. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, then show that $|3A| = 27|A|$

5. Evaluate the determinants

(i) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$ (ii) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

(iii) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$ (iv) $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

6. If $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$, find $|A|$

7. Find values of x, if

(i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$ (ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

8. If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, then x is equal to

(A) 6 (B) ± 6 (C) -6 (D) 0

4.3 Area of a Triangle

In earlier classes, we have studied that the area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3), is given by the expression $\frac{1}{2}[x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)]$. Now this expression can be written in the form of a determinant as

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \text{... (1)}$$

Remarks

(i) Since area is a positive quantity, we always take the absolute value of the determinant in (1).

(ii) If area is given, use both positive and negative values of the determinant for calculation.

(iii) The area of the triangle formed by three collinear points is zero.

Example 6 Find the area of the triangle whose vertices are (3, 8), (– 4, 2) and (5, 1).

Solution The area of triangle is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} 3 & 8 & 1 \\ -4 & 2 & 1 \\ 5 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[3(2-1)-8(-4-5)+1(-4-10)\right]$$

$$= \frac{1}{2}\left(3+72-14\right) = \frac{61}{2}$$

Example 7 Find the equation of the line joining A(1, 3) and B (0, 0) using determinants and find k if D(k, 0) is a point such that area of triangle ABD is 3sq units.

Solution Let P (x, y) be any point on AB. Then, area of triangle ABP is zero (Why?). So

$$\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 3 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

This gives $\quad \frac{1}{2}(y-3x) = 0$ or $y = 3x$,

which is the equation of required line AB.

Also, since the area of the triangle ABD is 3 sq. units, we have

$$\frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ k & 0 & 1 \end{vmatrix} = \pm 3$$

This gives, $\frac{-3k}{2} = \pm 3$, i.e., $k = \mp\, 2$.

EXERCISE 4.2

1. Find area of the triangle with vertices at the point given in each of the following :
   (i) (1, 0), (6, 0), (4, 3) (ii) (2, 7), (1, 1), (10, 8)
   (iii) (–2, –3), (3, 2), (–1, –8)
2. Show that points
   A (a, $b + c$), B (b, $c + a$), C (c, $a + b$) are collinear.
3. Find values of k if area of triangle is 4 sq. units and vertices are
   (i) (k, 0), (4, 0), (0, 2) (ii) (–2, 0), (0, 4), (0, k)
4. (i) Find equation of line joining (1, 2) and (3, 6) using determinants.
   (ii) Find equation of line joining (3, 1) and (9, 3) using determinants.
5. If area of triangle is 35 sq units with vertices (2, – 6), (5, 4) and (k, 4). Then k is
   (A) 12 (B) –2 (C) –12, –2 (D) 12, –2

4.4 Minors and Cofactors

In this section, we will learn to write the expansion of a determinant in compact form using minors and cofactors.

Definition 1 Minor of an element a_{ij} of a determinant is the determinant obtained by deleting its ith row and jth column in which element a_{ij} lies. Minor of an element a_{ij} is denoted by M_{ij}.

Remark Minor of an element of a determinant of order $n(n \geq 2)$ is a determinant of order $n - 1$.

Example 8 Find the minor of element 6 in the determinant $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

Solution Since 6 lies in the second row and third column, its minor M_{23} is given by

$$M_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = 8 - 14 = -6 \text{ (obtained by deleting } R_2 \text{ and } C_3 \text{ in } \Delta).$$

Definition 2 Cofactor of an element a_{ij}, denoted by A_{ij} is defined by

$$A_{ij} = (-1)^{i+j} M_{ij}, \text{ where } M_{ij} \text{ is minor of } a_{ij}.$$

Example 9 Find minors and cofactors of all the elements of the determinant $\begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix}$

Solution Minor of the element a_{ij} is M_{ij}

Here $a_{11} = 1$. So M_{11} = Minor of $a_{11} = 3$

M_{12} = Minor of the element $a_{12} = 4$

M_{21} = Minor of the element $a_{21} = -2$

M_{22} = Minor of the element $a_{22} = 1$

Now, cofactor of a_{ij} is A_{ij}. So

$A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (3) = 3$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (4) = -4$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (-2) = 2$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (1) = 1$

Example 10 Find minors and cofactors of the elements a_{11}, a_{21} in the determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Solution By definition of minors and cofactors, we have

$$\text{Minor of } a_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}\, a_{33} - a_{23}\, a_{32}$$

$$\text{Cofactor of } a_{11} = A_{11} = (-1)^{1+1} \; M_{11} = a_{22}\, a_{33} - a_{23}\, a_{32}$$

$$\text{Minor of } a_{21} = M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12}\, a_{33} - a_{13}\, a_{32}$$

$$\text{Cofactor of } a_{21} = A_{21} = (-1)^{2+1} \; M_{21} = (-1)\,(a_{12}\, a_{33} - a_{13}\, a_{32}) = -\, a_{12}\, a_{33} + a_{13}\, a_{32}$$

Remark Expanding the determinant Δ, in Example 21, along R_1, we have

$$\Delta = (-1)^{1+1}\, a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2}\, a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} \; a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$= a_{11}\, A_{11} + a_{12}\, A_{12} + a_{13}\, A_{13}$, where A_{ij} is cofactor of a_{ij}

= sum of product of elements of R_1 with their corresponding cofactors

Similarly, Δ can be calculated by other five ways of expansion that is along R_2, R_3, C_1, C_2 and C_3.

Hence Δ = sum of the product of elements of any row (or column) with their corresponding cofactors.

Note If elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero. For example,

$$\Delta = a_{11}\, A_{21} + a_{12}\, A_{22} + a_{13}\, A_{23}$$

$$= a_{11}\; (-1)^{1+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}\; (-1)^{1+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\; (-1)^{1+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \text{ (since } R_1 \text{ and } R_2 \text{ are identical)}$$

Similarly, we can try for other rows and columns.

Example 11 Find minors and cofactors of the elements of the determinant

$\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$ and verify that $a_{11}\,A_{31} + a_{12}\,A_{32} + a_{13}\,A_{33} = 0$

Solution We have $M_{11} = \begin{vmatrix} 0 & 4 \\ 5 & -7 \end{vmatrix} = 0 - 20 = -20$; $A_{11} = (-1)^{1+1}(-20) = -20$

$$M_{12} = \begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} = -42 - 4 = -46; \qquad A_{12} = (-1)^{1+2}(-46) = 46$$

$$M_{13} = \begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix} = 30 - 0 = 30; \qquad A_{13} = (-1)^{1+3}(30) = 30$$

$$M_{21} = \begin{vmatrix} -3 & 5 \\ 5 & -7 \end{vmatrix} = 21 - 25 = -4; \qquad A_{21} = (-1)^{2+1}(-4) = 4$$

$$M_{22} = \begin{vmatrix} 2 & 5 \\ 1 & -7 \end{vmatrix} = -14 - 5 = -19; \qquad A_{22} = (-1)^{2+2}(-19) = -19$$

$$M_{23} = \begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix} = 10 + 3 = 13; \qquad A_{23} = (-1)^{2+3}(13) = -13$$

$$M_{31} = \begin{vmatrix} -3 & 5 \\ 0 & 4 \end{vmatrix} = -12 - 0 = -12; \qquad A_{31} = (-1)^{3+1}(-12) = -12$$

$$M_{32} = \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} = 8 - 30 = -22; \qquad A_{32} = (-1)^{3+2}(-22) = 22$$

and $M_{33} = \begin{vmatrix} 2 & -3 \\ 6 & 0 \end{vmatrix} = 0 + 18 = 18; \qquad A_{33} = (-1)^{3+3}(18) = 18$

Now $a_{11} = 2,\ a_{12} = -3,\ a_{13} = 5;\ A_{31} = -12,\ A_{32} = 22,\ A_{33} = 18$

So $a_{11}\,A_{31} + a_{12}\,A_{32} + a_{13}\,A_{33}$

$= 2\,(-12) + (-3)\,(22) + 5\,(18) = -24 - 66 + 90 = 0$

EXERCISE 4.3

Write Minors and Cofactors of the elements of following determinants:

1. (i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

2. (i) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

3. Using Cofactors of elements of second row, evaluate $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$.

4. Using Cofactors of elements of third column, evaluate $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$.

5. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and A_{ij} is Cofactors of a_{ij}, then value of Δ is given by

(A) $a_{11} A_{31} + a_{12} A_{32} + a_{13} A_{33}$ (B) $a_{11} A_{11} + a_{12} A_{21} + a_{13} A_{31}$

(C) $a_{21} A_{11} + a_{22} A_{12} + a_{23} A_{13}$ (D) $a_{11} A_{11} + a_{21} A_{21} + a_{31} A_{31}$

4.5 Adjoint and Inverse of a Matrix

In the previous chapter, we have studied inverse of a matrix. In this section, we shall discuss the condition for existence of inverse of a matrix.

To find inverse of a matrix A, i.e., A^{-1} we shall first define adjoint of a matrix.

4.5.1 *Adjoint of a matrix*

Definition 3 The adjoint of a square matrix $A = [a_{ij}]_{n \times n}$ is defined as the transpose of the matrix $[A_{ij}]_{n \times n}$, where A_{ij} is the cofactor of the element a_{ij}. Adjoint of the matrix A is denoted by *adj* A.

Let $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then $$adj\ \mathrm{A} = \text{Transpose of} \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} & \mathrm{A}_{13} \\ \mathrm{A}_{21} & \mathrm{A}_{22} & \mathrm{A}_{23} \\ \mathrm{A}_{31} & \mathrm{A}_{32} & \mathrm{A}_{33} \end{bmatrix} = \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{21} & \mathrm{A}_{31} \\ \mathrm{A}_{12} & \mathrm{A}_{22} & \mathrm{A}_{32} \\ \mathrm{A}_{13} & \mathrm{A}_{23} & \mathrm{A}_{33} \end{bmatrix}$$

Example 12 Find *adj* A for A $= \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

Solution We have $\mathrm{A}_{11} = 4$, $\mathrm{A}_{12} = -1$, $\mathrm{A}_{21} = -3$, $\mathrm{A}_{22} = 2$

Hence $$adj\ \mathrm{A} = \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{21} \\ \mathrm{A}_{12} & \mathrm{A}_{22} \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

Remark For a square matrix of order 2, given by

$$\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The *adj* A can also be obtained by interchanging a_{11} and a_{22} and by changing signs of a_{12} and a_{21}, i.e.,

$$adj\ \mathrm{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Change sign Interchange

We state the following theorem without proof.

Theorem 1 If A be any given square matrix of order *n*, then

$$\mathrm{A}(adj\ \mathrm{A}) = (adj\ \mathrm{A})\,\mathrm{A} = |\mathrm{A}|\,\mathrm{I},$$

where I is the identity matrix of order *n*

Verification

Let $$\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{ then } adj\ \mathrm{A} = \begin{bmatrix} \mathrm{A}_{11} & \mathrm{A}_{21} & \mathrm{A}_{31} \\ \mathrm{A}_{12} & \mathrm{A}_{22} & \mathrm{A}_{32} \\ \mathrm{A}_{13} & \mathrm{A}_{23} & \mathrm{A}_{33} \end{bmatrix}$$

Since sum of product of elements of a row (or a column) with corresponding cofactors is equal to |A| and otherwise zero, we have

$$\mathrm{A}\,(adj\,\mathrm{A}) = \begin{bmatrix} |\mathrm{A}| & 0 & 0 \\ 0 & |\mathrm{A}| & 0 \\ 0 & 0 & |\mathrm{A}| \end{bmatrix} = |\mathrm{A}| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |\mathrm{A}|\ \mathrm{I}$$

Similarly, we can show $(adj\,\mathrm{A})\,\mathrm{A} = |\mathrm{A}|\ \mathrm{I}$

Hence $\mathrm{A}\,(adj\,\mathrm{A}) = (adj\,\mathrm{A})\,\mathrm{A} = |\mathrm{A}|\ \mathrm{I}$

Definition 4 A square matrix A is said to be singular if $|\mathrm{A}| = 0$.

For example, the determinant of matrix $\mathrm{A} = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$ is zero

Hence A is a singular matrix.

Definition 5 A square matrix A is said to be non-singular if $|\mathrm{A}| \neq 0$

Let $\mathrm{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Then $|\mathrm{A}| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0.$

Hence A is a nonsingular matrix

We state the following theorems without proof.

Theorem 2 If A and B are nonsingular matrices of the same order, then AB and BA are also nonsingular matrices of the same order.

Theorem 3 The determinant of the product of matrices is equal to product of their respective determinants, that is, $|\mathrm{AB}| = |\mathrm{A}|\ |\mathrm{B}|$, where A and B are square matrices of the same order

Remark We know that $(adj\,\mathrm{A})\,\mathrm{A} = |\mathrm{A}|\ \mathrm{I} = \begin{bmatrix} |\mathrm{A}| & 0 & 0 \\ 0 & |\mathrm{A}| & 0 \\ 0 & 0 & |\mathrm{A}| \end{bmatrix}, |\mathrm{A}| \neq 0$

Writing determinants of matrices on both sides, we have

$$|(adj\,\mathrm{A})\,\mathrm{A}| = \begin{vmatrix} |\mathrm{A}| & 0 & 0 \\ 0 & |\mathrm{A}| & 0 \\ 0 & 0 & |\mathrm{A}| \end{vmatrix}$$

i.e. $$|(adj\ A)|\ |A| = |A|^3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad \text{(Why?)}$$

i.e. $$|(adj\ A)|\ |A| = |A|^3\ (1)$$

i.e. $$|(adj\ A)| = |A|^2$$

In general, if A is *a* square matrix of order n, then $|adj(A)| = |A|^{n-1}$.

Theorem 4 A square matrix A is invertible if and only if A is nonsingular matrix.

Proof Let A be invertible matrix of order n and I be the identity matrix of order n. Then, there exists a square matrix B of order n such that AB = BA = I

Now $\quad$ AB = I. So $|AB| = |I|$ or $|A|\ |B| = 1$ (since $|I| = 1, |AB| = |A||B|$)

This gives $\quad |A| \neq 0$. Hence A is nonsingular.

Conversely, let A be nonsingular. Then $|A| \neq 0$

Now $$A\ (adj\ A) = (adj\ A)\ A = |A|\ I \quad \text{(Theorem 1)}$$

or $$A\left(\frac{1}{|A|}adj\ A\right) = \left(\frac{1}{|A|}adj\ A\right)A = I$$

or $$AB = BA = I, \text{ where } B = \frac{1}{|A|}adj\ A$$

Thus $$A \text{ is invertible and } A^{-1} = \frac{1}{|A|}adj\ A$$

Example 13 If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, then verify that A *adj* A = | A | I. Also find A^{-1}.

Solution We have $|A| = 1(16 - 9) - 3(4 - 3) + 3(3 - 4) = 1 \neq 0$

Now $A_{11} = 7$, $A_{12} = -1$, $A_{13} = -1$, $A_{21} = -3$, $A_{22} = 1$, $A_{23} = 0$, $A_{31} = -3$, $A_{32} = 0$, $A_{33} = 1$

Therefore $$adj\ A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Now $$A\,(adj\ A) = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7-3-3 & -3+3+0 & -3+0+3 \\ 7-4-3 & -3+4+0 & -3+0+3 \\ 7-3-4 & -3+3+0 & -3+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\,.\,I$$

Also $$A^{-1} = \frac{1}{|A|}\,adj\ A = \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Example 14 If $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, then verify that $(AB)^{-1} = B^{-1}A^{-1}$.

Solution We have $AB = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 5 & -14 \end{bmatrix}$

Since, $|AB| = -11 \neq 0$, $(AB)^{-1}$ exists and is given by

$$(AB)^{-1} = \frac{1}{|AB|}\,adj\,(AB) = -\frac{1}{11}\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$$

Further, $|A| = -11 \neq 0$ and $|B| = 1 \neq 0$. Therefore, A^{-1} and B^{-1} both exist and are given by

$$A^{-1} = -\frac{1}{11}\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}, B^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

Therefore $$B^{-1}A^{-1} = -\frac{1}{11}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix} = -\frac{1}{11}\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$$

Hence $(AB)^{-1} = B^{-1}A^{-1}$

Example 15 Show that the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^2 - 4A + I = O$, where I is 2×2 identity matrix and O is 2×2 zero matrix. Using this equation, find A^{-1}.

Solution We have $A^2 = A.A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

Hence $\quad A^2 - 4A + I = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

Now $\quad A^2 - 4A + I = O$

Therefore $\quad A A - 4A = - I$

or $\quad A\ A (A^{-1}) - 4\ A A^{-1} = - I\ A^{-1}$ (Post multiplying by A^{-1} because $|A| \neq 0$)

or $\quad A (A A^{-1}) - 4I = - A^{-1}$

or $\quad AI - 4I = - A^{-1}$

or $\quad A^{-1} = 4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Hence $\quad A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

EXERCISE 4.4

Find adjoint of each of the matrices in Exercises 1 and 2.

1. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **2.** $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

Verify A (*adj* A) = (*adj* A) A = |A| I in Exercises 3 and 4

3. $\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$ **4.** $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

Find the inverse of each of the matrices (if it exists) given in Exercises 5 to 11.

5. $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$ **6.** $\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$ **7.** $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$ **9.** $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$ **10.** $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$

12. Let $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$. Verify that $(AB)^{-1} = B^{-1} A^{-1}$.

13. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$. Hence find A^{-1}.

14. For the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the numbers a and b such that $A^2 + aA + bI = O$.

15. For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

Show that $A^3 - 6A^2 + 5A + 11\,I = O$. Hence, find A^{-1}.

16. If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Verify that $A^3 - 6A^2 + 9A - 4I = O$ and hence find A^{-1}

17. Let A be a nonsingular square matrix of order 3 × 3. Then $|adj\ A|$ is equal to

(A) $|A|$ (B) $|A|^2$ (C) $|A|^3$ (D) $3|A|$

18. If A is an invertible matrix of order 2, then det (A^{-1}) is equal to

(A) det (A) (B) $\dfrac{1}{\det(A)}$ (C) 1 (D) 0

4.6 Applications of Determinants and Matrices

In this section, we shall discuss application of determinants and matrices for solving the system of linear equations in two or three variables and for checking the consistency of the system of linear equations.

Consistent system A system of equations is said to be *consistent* if its solution (one or more) exists.

Inconsistent system A system of equations is said to be *inconsistent* if its solution does not exist.

Note In this chapter, we restrict ourselves to the system of linear equations having unique solutions only.

4.6.1 *Solution of system of linear equations using inverse of a matrix*

Let us express the system of linear equations as matrix equations and solve them using inverse of the coefficient matrix.

Consider the system of equations

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

Let $$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Then, the system of equations can be written as, AX = B, i.e.,

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Case I If A is a nonsingular matrix, then its inverse exists. Now

$$AX = B$$

or $$A^{-1}(AX) = A^{-1} B \quad \text{(premultiplying by } A^{-1}\text{)}$$

or $$(A^{-1}A)X = A^{-1} B \quad \text{(by associative property)}$$

or $$I X = A^{-1} B$$

or $$X = A^{-1} B$$

This matrix equation provides unique solution for the given system of equations as inverse of a matrix is unique. This method of solving system of equations is known as Matrix Method.

Case II If A is a singular matrix, then $|A| = 0$.

In this case, we calculate (*adj* A) B.

If (*adj* A) B ≠ O, (O being zero matrix), then solution does not exist and the system of equations is called inconsistent.

If (*adj* A) B = O, then system may be either consistent or inconsistent according as the system have either infinitely many solutions or no solution.

Example 16 Solve the system of equations

$$2x + 5y = 1$$
$$3x + 2y = 7$$

Solution The system of equations can be written in the form AX = B, where

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

Now, $|A| = -11 \neq 0$, Hence, A is nonsingular matrix and so has a unique solution.

Note that $$A^{-1} = -\frac{1}{11}\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$$

Therefore $$X = A^{-1}B = -\frac{1}{11}\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

i.e. $$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{11}\begin{bmatrix} -33 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Hence $$x = 3, y = -1$$

Example 17 Solve the following system of equations by matrix method.

$$3x - 2y + 3z = 8$$
$$2x + y - z = 1$$
$$4x - 3y + 2z = 4$$

Solution The system of equations can be written in the form AX = B, where

$$A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

We see that

$$|A| = 3(2-3) + 2(4+4) + 3(-6-4) = -17 \neq 0$$

Hence, A is nonsingular and so its inverse exists. Now

$A_{11} = -1$, $A_{12} = -8$, $A_{13} = -10$

$A_{21} = -5$, $A_{22} = -6$, $A_{23} = 1$

$A_{31} = -1$, $A_{32} = 9$, $A_{33} = 7$

Therefore $$A^{-1} = -\frac{1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

So $$X = A^{-1}B = -\frac{1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

i.e. $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{17}\begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence $x = 1, y = 2$ and $z = 3$.

Example 18 The sum of three numbers is 6. If we multiply third number by 3 and add second number to it, we get 11. By adding first and third numbers, we get double of the second number. Represent it algebraically and find the numbers using matrix method.

Solution Let first, second and third numbers be denoted by x, y and z, respectively. Then, according to given conditions, we have

$$x + y + z = 6$$
$$y + 3z = 11$$
$$x + z = 2y \text{ or } x - 2y + z = 0$$

This system can be written as A X = B, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & 2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

Here $|A| = 1(1+6) - (0-3) + (0-1) = 9 \neq 0$. Now we find *adj* A

$A_{11} = 1(1 + 6) = 7$, $A_{12} = -(0 - 3) = 3$, $A_{13} = -1$

$A_{21} = -(1 + 2) = -3$, $A_{22} = 0$, $A_{23} = -(-2 - 1) = 3$

$A_{31} = (3 - 1) = 2$, $A_{32} = -(3 - 0) = -3$, $A_{33} = (1 - 0) = 1$

Hence $$adj\ A = \begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$$

Thus $$A^{-1} = \frac{1}{|A|}\, adj\,(A) = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$$

Since $$X = A^{-1} B$$

$$X = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 42-33+0 \\ 18+\ 0+0 \\ -6+33+0 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 \\ 18 \\ 27 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Thus $$x = 1,\ y = 2,\ z = 3$$

EXERCISE 4.5

Examine the consistency of the system of equations in Exercises 1 to 6.

1. $x + 2y = 2$
$2x + 3y = 3$

2. $2x - y = 5$
$x + y = 4$

3. $x + 3y = 5$
$2x + 6y = 8$

4. $x + y + z = 1$
$2x + 3y + 2z = 2$
$ax + ay + 2az = 4$

5. $3x - y - 2z = 2$
$2y - z = -1$
$3x - 5y = 3$

6. $5x - y + 4z = 5$
$2x + 3y + 5z = 2$
$5x - 2y + 6z = -1$

Solve system of linear equations, using matrix method, in Exercises 7 to 14.

7. $5x + 2y = 4$
$7x + 3y = 5$

8. $2x - y = -2$
$3x + 4y = 3$

9. $4x - 3y = 3$
$3x - 5y = 7$

10. $5x + 2y = 3$
$3x + 2y = 5$

11. $2x + y + z = 1$
$x - 2y - z = \frac{3}{2}$
$3y - 5z = 9$

12. $x - y + z = 4$
$2x + y - 3z = 0$
$x + y + z = 2$

13. $2x + 3y + 3z = 5$
$x - 2y + z = -4$
$3x - y - 2z = 3$

14. $x - y + 2z = 7$
$3x + 4y - 5z = -5$
$2x - y + 3z = 12$

15. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}. Using A^{-1} solve the system of equations

$$\begin{aligned} 2x - 3y + 5z &= 11 \\ 3x + 2y - 4z &= -5 \\ x + y - 2z &= -3 \end{aligned}$$

16. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is ₹60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is ₹90. The cost of 6 kg onion 2 kg wheat and 3 kg rice is ₹70. Find cost of each item per kg by matrix method.

Miscellaneous Examples

Example 19 Use product $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 9 & 2 & 3 \\ 6 & 1 & 2 \end{bmatrix}$ to solve the system of equations

$$\begin{aligned} x - y + 2z &= 1 \\ 2y - 3z &= 1 \\ 3x - 2y + 4z &= 2 \end{aligned}$$

Solution Consider the product $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

$$= \begin{bmatrix} -2- 9+12 & 0-2+2 & 1+3-4 \\ 0+18-18 & 0+4-3 & 0-6+6 \\ -6-18+24 & 0-4+4 & 3+6-8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

Now, given system of equations can be written, in matrix form, as follows

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

or
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 9 & 2 & 3 \\ 6 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} -2+0+2 \\ 9+2-6 \\ 6+1-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$$

Hence $x = 0, y = 5$ and $z = 3$

Miscellaneous Exercises on Chapter 4

1. Prove that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of θ.

2. Evaluate $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$.

3. If $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, find $(AB)^{-1}$

4. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$. Verify that

(i) $[adj\ A]^{-1} = adj\ (A^{-1})$ (ii) $(A^{-1})^{-1} = A$

5. Evaluate $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

6. Evaluate $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$

Using properties of determinants in Exercises 11 to 15, prove that:

7. Solve the system of equations

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4$$

$$\frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1$$

$$\frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2$$

Choose the correct answer in Exercise 17 to 19.

8. If x, y, z are nonzero real numbers, then the inverse of matrix $A=\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ is

(A) $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$ (B) $xyz\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

(C) $\frac{1}{xyz}\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ (D) $\frac{1}{xyz}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

9. Let $A=\begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$, where $0 \le \theta \le 2\pi$. Then

(A) $\text{Det}(A) = 0$ (B) $\text{Det}(A) \in (2, \infty)$

(C) $\text{Det}(A) \in (2, 4)$ (D) $\text{Det}(A) \in [2, 4]$

Summary

- Determinant of a matrix $A = [a_{11}]_{1\times 1}$ is given by $|a_{11}| = a_{11}$
- Determinant of a matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is given by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\, a_{22} - a_{12}\, a_{21}$$

- Determinant of a matrix $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is given by (expanding along R_1)

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

For any square matrix A, the |A| satisfy following properties.

- Area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

- Minor of an element a_{ij} of the determinant of matrix A is the determinant obtained by deleting i^{th} row and j^{th} column and denoted by M_{ij}.
- Cofactor of a_{ij} of given by $A_{ij} = (-1)^{i+j}\, M_{ij}$
- Value of determinant of a matrix A is obtained by sum of product of elements of a row (or a column) with corresponding cofactors. For example,

$$|A| = a_{11}\, A_{11} + a_{12}\, A_{12} + a_{13}\, A_{13}.$$

- If elements of one row (or column) are multiplied with cofactors of elements of any other row (or column), then their sum is zero. For example, $a_{11}\, A_{21} + a_{12}\, A_{22} + a_{13}\, A_{23} = 0$

- If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $adj\ A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$, where A_{ij} is cofactor of a_{ij}
- $A\,(adj\ A) = (adj\ A)\,A = |A|\,I$, where A is square matrix of order n.
- A square matrix A is said to be singular or non-singular according as $|A| = 0$ or $|A| \neq 0$.
- If AB = BA = I, where B is square matrix, then B is called inverse of A. Also $A^{-1} = B$ or $B^{-1} = A$ and hence $(A^{-1})^{-1} = A$.
- A square matrix A has inverse if and only if A is non-singular.
- $A^{-1} = \dfrac{1}{|A|}(adj\ A)$
- If $a_1 x + b_1 y + c_1 z = d_1$
  $a_2 x + b_2 y + c_2 z = d_2$
  $a_3 x + b_3 y + c_3 z = d_3$,
  then these equations can be written as A X = B, where
  $$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$
- Unique solution of equation AX = B is given by $X = A^{-1} B$, where $|A| \neq 0$.
- A system of equation is consistent or inconsistent according as its solution exists or not.
- For a square matrix A in matrix equation AX = B
  (i) $|A| \neq 0$, there exists unique solution
  (ii) $|A| = 0$ and $(adj\ A)\,B \neq 0$, then there exists no solution
  (iii) $|A| = 0$ and $(adj\ A)\,B = 0$, then system may or may not be consistent.

Historical Note

The Chinese method of representing the coefficients of the unknowns of several linear equations by using rods on a calculating board naturally led to the discovery of simple method of elimination. The arrangement of rods was precisely that of the numbers in a determinant. The Chinese, therefore, early developed the idea of subtracting columns and rows as in simplification of a determinant *Mikami, China*, *pp* 30, 93.

Seki Kowa, the greatest of the Japanese Mathematicians of seventeenth century in his work '*Kai Fukudai no Ho*' in 1683 showed that he had the idea of determinants and of their expansion. But he used this device only in eliminating a quantity from two equations and not directly in the solution of a set of simultaneous linear equations. T. Hayashi, "*The Fakudoi and Determinants in Japanese Mathematics*," in the proc. of the Tokyo Math. Soc., V.

Vendermonde was the first to recognise determinants as independent functions. He may be called the formal founder. Laplace (1772), gave general method of expanding a determinant in terms of its complementary minors. In 1773 Lagrange treated determinants of the second and third orders and used them for purpose other than the solution of equations. In 1801, Gauss used determinants in his theory of numbers.

The next great contributor was Jacques - Philippe - Marie Binet, (1812) who stated the theorem relating to the product of two matrices of m-columns and n-rows, which for the special case of $m = n$ reduces to the multiplication theorem.

Also on the same day, Cauchy (1812) presented one on the same subject. He used the word 'determinant' in its present sense. He gave the proof of multiplication theorem more satisfactory than Binet's.

The greatest contributor to the theory was Carl Gustav Jacob Jacobi, after this the word determinant received its final acceptance.

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Chapter 5

CONTINUITY AND DIFFERENTIABILITY

❖ *The whole of science is nothing more than a refinement of everyday thinking." — ALBERT EINSTEIN* ❖

5.1 Introduction

This chapter is essentially a continuation of our study of differentiation of functions in Class XI. We had learnt to differentiate certain functions like polynomial functions and trigonometric functions. In this chapter, we introduce the very important concepts of continuity, differentiability and relations between them. We will also learn differentiation of inverse trigonometric functions. Further, we introduce a new class of functions called exponential and logarithmic functions. These functions lead to powerful techniques of differentiation. We illustrate certain geometrically obvious conditions through differential calculus. In the process, we will learn some fundamental theorems in this area.

Sir Issac Newton (1642-1727)

5.2 Continuity

We start the section with two informal examples to get a feel of continuity. Consider the function

$$f(x) = \begin{cases} 1, \text{ if } x \le 0 \\ 2, \text{ if } x > 0 \end{cases}$$

This function is of course defined at every point of the real line. Graph of this function is given in the Fig 5.1. One can deduce from the graph that the value of the function at *nearby* points on x-axis remain *close* to each other except at $x = 0$. At the points near and to the left of 0, i.e., at points like $-0.1, -0.01, -0.001$, the value of the function is 1. At the points near and to the right of 0, i.e., at points like 0.1, 0.01,

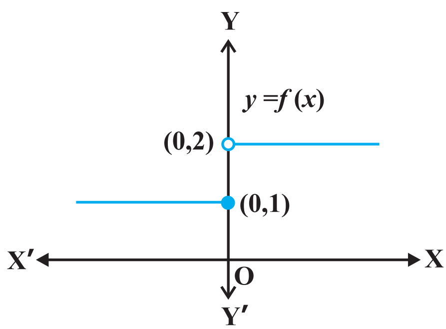

Fig 5.1

0.001, the value of the function is 2. Using the language of left and right hand limits, we may say that the left (respectively right) hand limit of f at 0 is 1 (respectively 2). In particular the left and right hand limits do not coincide. We also observe that the value of the function at $x = 0$ concides with the left hand limit. Note that when we try to draw the graph, we cannot draw it in one stroke, i.e., without lifting pen from the plane of the paper, we can not draw the graph of this function. In fact, we need to lift the pen when we come to 0 from left. This is one instance of function being not continuous at $x = 0$.

Now, consider the function defined as

$$f(x) = \begin{cases} 1, \text{ if } x \neq 0 \\ 2, \text{ if } x = 0 \end{cases}$$

This function is also defined at every point. Left and the right hand limits at $x = 0$ are both equal to 1. But the value of the function at $x = 0$ equals 2 which does not coincide with the common value of the left and right hand limits. Again, we note that we cannot draw the graph of the function without lifting the pen. This is yet another instance of a function being not continuous at $x = 0$.

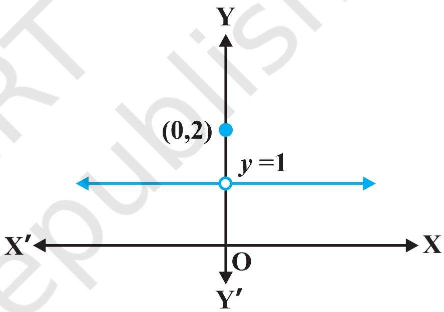

Fig 5.2

Naively, we may say that a function is continuous at a fixed point if we can draw the graph of the function *around* that point without lifting the pen from the plane of the paper.

Mathematically, it may be phrased precisely as follows:

Definition 1 Suppose f is a real function on a subset of the real numbers and let c be a point in the domain of f. Then f is continuous at c if

$$\lim_{x \to c} f(x) = f(c)$$

More elaborately, if the left hand limit, right hand limit and the value of the function at $x = c$ exist and equal to each other, then f is said to be continuous at $x = c$. Recall that if the right hand and left hand limits at $x = c$ coincide, then we say that the common value is the limit of the function at $x = c$. Hence we may also rephrase the definition of continuity as follows: *a function is continuous at $x = c$ if the function is defined at $x = c$ and if the value of the function at $x = c$ equals the limit of the function at $x = c$.* If f is not continuous at c, we say f is *discontinuous* at c and c is called a *point of discontinuity* of f.

Example 1 Check the continuity of the function f given by $f(x) = 2x + 3$ at $x = 1$.

Solution First note that the function is defined at the given point $x = 1$ and its value is 5. Then find the limit of the function at $x = 1$. Clearly

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} (2x + 3) = 2(1) + 3 = 5$$

Thus $$\lim_{x \to 1} f(x) = 5 = f(1)$$

Hence, f is continuous at $x = 1$.

Example 2 Examine whether the function f given by $f(x) = x^2$ is continuous at $x = 0$.

Solution First note that the function is defined at the given point $x = 0$ and its value is 0. Then find the limit of the function at $x = 0$. Clearly

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 = 0^2 = 0$$

Thus $$\lim_{x \to 0} f(x) = 0 = f(0)$$

Hence, f is continuous at $x = 0$.

Example 3 Discuss the continuity of the function f given by $f(x) = | x |$ at $x = 0$.

Solution By definition

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Clearly the function is defined at 0 and $f(0) = 0$. Left hand limit of f at 0 is

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x) = 0$$

Similarly, the right hand limit of f at 0 is

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0$$

Thus, the left hand limit, right hand limit and the value of the function coincide at $x = 0$. Hence, f is continuous at $x = 0$.

Example 4 Show that the function f given by

$$f(x) = \begin{cases} x^3 + 3, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$$

is not continuous at $x = 0$.

Solution The function is defined at $x = 0$ and its value at $x = 0$ is 1. When $x \neq 0$, the function is given by a polynomial. Hence,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} (x^3 + 3) = 0^3 + 3 = 3$$

Since the limit of f at $x = 0$ does not coincide with $f(0)$, the function is not continuous at $x = 0$. It may be noted that $x = 0$ is the only point of discontinuity for this function.

Example 5 Check the points where the constant function $f(x) = k$ is continuous.

Solution The function is defined at all real numbers and by definition, its value at any real number equals k. Let c be any real number. Then

$$\lim_{x\to c} f(x) = \lim_{x\to c} k = k$$

Since $f(c) = k = \lim_{x\to c} f(x)$ for any real number c, the function f is continuous at every real number.

Example 6 Prove that the identity function on real numbers given by $f(x) = x$ is continuous at every real number.

Solution The function is clearly defined at every point and $f(c) = c$ for every real number c. Also,

$$\lim_{x\to c} f(x) = \lim_{x\to c} x = c$$

Thus, $\lim_{x\to c} f(x) = c = f(c)$ and hence the function is continuous at every real number.

Having defined continuity of a function at a given point, now we make a natural extension of this definition to discuss continuity of a function.

Definition 2 A real function f is said to be continuous if it is continuous at every point in the domain of f.

This definition requires a bit of elaboration. Suppose f is a function defined on a closed interval $[a, b]$, then for f to be continuous, it needs to be continuous at every point in $[a, b]$ including the end points a and b. Continuity of f at a means

$$\lim_{x\to a^+} f(x) = f(a)$$

and continuity of f at b means

$$\lim_{x\to b^-} f(x) = f(b)$$

Observe that $\lim_{x\to a^-} f(x)$ *and* $\lim_{x\to b^+} f(x)$ *do not make sense. As a consequence of this definition, if f is defined only at one point, it is continuous there, i.e., if the domain of f is a singleton, f is a continuous function.*

Example 7 Is the function defined by $f(x) = |x|$, a continuous function?

Solution We may rewrite f as

$$f(x) = \begin{cases} -x, \text{ if } x < 0 \\ x, \quad \text{if } x \geq 0 \end{cases}$$

By Example 3, we know that f is continuous at $x = 0$.

Let c be a real number such that $c < 0$. Then $f(c) = -c$. Also

$$\lim_{x \to c} f(x) = \lim_{x \to c} (-x) = -c \qquad \text{(Why?)}$$

Since $\lim_{x \to c} f(x) = f(c)$, f is continuous at all negative real numbers.

Now, let c be a real number such that $c > 0$. Then $f(c) = c$. Also

$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c \qquad \text{(Why?)}$$

Since $\lim_{x \to c} f(x) = f(c)$, f is continuous at all positive real numbers. Hence, f is continuous at all points.

Example 8 Discuss the continuity of the function f given by $f(x) = x^3 + x^2 - 1$.

Solution Clearly f is defined at every real number c and its value at c is $c^3 + c^2 - 1$. We also know that

$$\lim_{x \to c} f(x) = \lim_{x \to c} (x^3 + x^2 - 1) = c^3 + c^2 - 1$$

Thus $\lim_{x \to c} f(x) = f(c)$, and hence f is continuous at every real number. This means f is a continuous function.

Example 9 Discuss the continuity of the function f defined by $f(x) = \dfrac{1}{x}$, $x \neq 0$.

Solution Fix any non zero real number c, we have

$$\lim_{x \to c} f(x) = \lim_{x \to c} \frac{1}{x} = \frac{1}{c}$$

Also, since for $c \neq 0$, $f(c) = \dfrac{1}{c}$, we have $\lim_{x \to c} f(x) = f(c)$ and hence, f is continuous at every point in the domain of f. Thus f is a continuous function.

We take this opportunity to explain the concept of *infinity*. This we do by analysing the function $f(x) = \frac{1}{x}$ near $x = 0$. To carry out this analysis we follow the usual trick of finding the value of the function at real numbers *close* to 0. Essentially we are trying to find the right hand limit of f at 0. We tabulate this in the following (Table 5.1).

Table 5.1

x	1	0.3	0.2	$0.1 = 10^{-1}$	$0.01 = 10^{-2}$	$0.001 = 10^{-3}$	10^{-n}
$f(x)$	1	3.333...	5	10	$100 = 10^2$	$1000 = 10^3$	10^n

We observe that as x gets closer to 0 from the right, the value of $f(x)$ shoots up higher. This may be rephrased as: the value of $f(x)$ may be made larger than any given number by choosing a positive real number *very close* to 0. In symbols, we write

$$\lim_{x \to 0^+} f(x) = +\infty$$

(to be read as: the right hand limit of $f(x)$ at 0 is plus infinity). We wish to emphasise that $+\infty$ is NOT a real number and hence the right hand limit of f at 0 does not exist (as a real number).

Similarly, the left hand limit of f at 0 may be found. The following table is self explanatory.

Table 5.2

x	-1	-0.3	-0.2	-10^{-1}	-10^{-2}	-10^{-3}	-10^{-n}
$f(x)$	-1	$-3.333...$	-5	-10	-10^2	-10^3	-10^n

From the Table 5.2, we deduce that the value of $f(x)$ may be made smaller than any given number by choosing a negative real number *very close* to 0. In symbols, we write

$$\lim_{x \to 0^-} f(x) = -\infty$$

(to be read as: the left hand limit of $f(x)$ at 0 is minus infinity). Again, we wish to emphasise that $-\infty$ is NOT a real number and hence the left hand limit of f at 0 does not exist (as a real number). The graph of the reciprocal function given in Fig 5.3 is a geometric representation of the above mentioned facts.

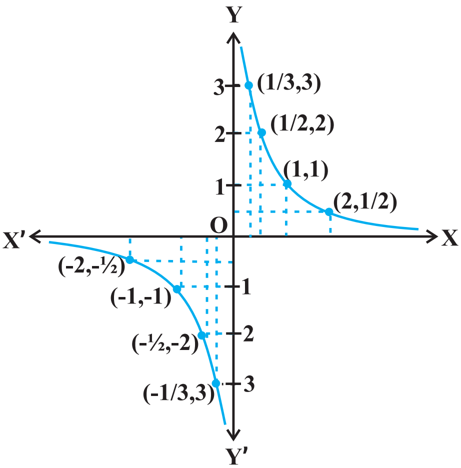

Fig 5.3

Example 10 Discuss the continuity of the function f defined by

$$f(x) = \begin{cases} x+2, \text{ if } x \le 1 \\ x-2, \text{ if } x > 1 \end{cases}$$

Solution The function f is defined at all points of the real line.

Case 1 If $c < 1$, then $f(c) = c + 2$. Therefore, $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c}(x+2) = c+2$

Thus, f is continuous at all real numbers less than 1.

Case 2 If $c > 1$, then $f(c) = c - 2$. Therefore,

$$\lim_{x\to c} f(x) = \lim_{x\to c} (x-2) = c-2 = f(c)$$

Thus, f is continuous at all points $x > 1$.

Case 3 If $c = 1$, then the left hand limit of f at $x = 1$ is

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x+2) = 1+2 = 3$$

The right hand limit of f at $x = 1$ is

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-2) = 1-2 = -1$$

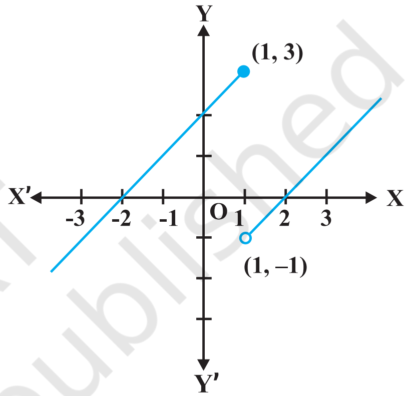

Fig 5.4

Since the left and right hand limits of f at $x = 1$ do not coincide, f is not continuous at $x = 1$. Hence $x = 1$ is the only point of discontinuity of f. The graph of the function is given in Fig 5.4.

Example 11 Find all the points of discontinuity of the function f defined by

$$f(x) = \begin{cases} x+2, \text{ if } x < 1 \\ 0, \quad \text{if } x = 1 \\ x-2, \text{ if } x > 1 \end{cases}$$

Solution As in the previous example we find that f is continuous at all real numbers $x \neq 1$. The left hand limit of f at $x = 1$ is

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x+2) = 1+2 = 3$$

The right hand limit of f at $x = 1$ is

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-2) = 1-2 = -1$$

Since, the left and right hand limits of f at $x = 1$ do not coincide, f is not continuous at $x = 1$. Hence $x = 1$ is the only point of discontinuity of f. The graph of the function is given in the Fig 5.5.

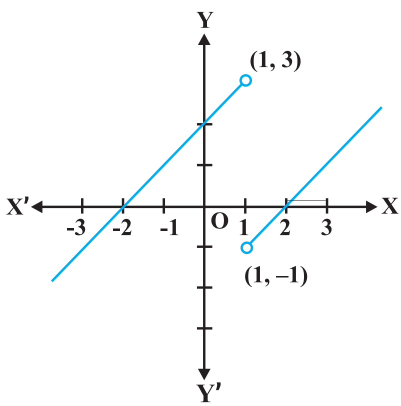

Fig 5.5

Example 12 Discuss the continuity of the function defined by

$$f(x) = \begin{cases} x+2, \text{ if } x<0 \\ -x+2, \text{ if } x>0 \end{cases}$$

Solution Observe that the function is defined at all real numbers except at 0. Domain of definition of this function is

$D_1 \cup D_2$ where $D_1 = \{x \in \mathbf{R} : x < 0\}$ and
$D_2 = \{x \in \mathbf{R} : x > 0\}$

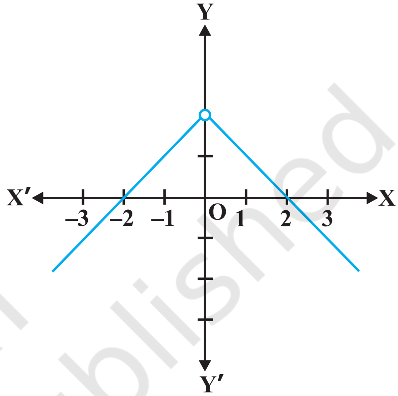

Fig 5.6

Case 1 If $c \in D_1$, then $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} (x + 2)$ $= c + 2 = f(c)$ and hence f is continuous in D_1.

Case 2 If $c \in D_2$, then $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} (-x + 2)$ $= -c + 2 = f(c)$ and hence f is continuous in D_2.

Since f is continuous at all points in the domain of f, we deduce that f is continuous. Graph of this function is given in the Fig 5.6. Note that to graph this function we need to lift the pen from the plane of the paper, but we need to do that only for those points where the function is not defined.

Example 13 Discuss the continuity of the function f given by

$$f(x) = \begin{cases} x, & \text{if } x \geq 0 \\ x^2, & \text{if } x < 0 \end{cases}$$

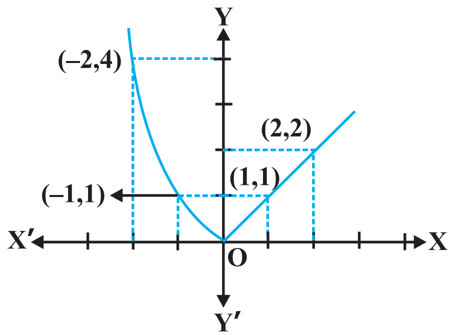

Fig 5.7

Solution Clearly the function is defined at every real number. Graph of the function is given in Fig 5.7. By inspection, it seems prudent to partition the domain of definition of f into three disjoint subsets of the real line.

Let $D_1 = \{x \in \mathbf{R} : x < 0\}$, $D_2 = \{0\}$ and
$D_3 = \{x \in \mathbf{R} : x > 0\}$

Case 1 At any point in D_1, we have $f(x) = x^2$ and it is easy to see that it is continuous there (see Example 2).

Case 2 At any point in D_3, we have $f(x) = x$ and it is easy to see that it is continuous there (see Example 6).

Case 3 Now we analyse the function at $x = 0$. The value of the function at 0 is $f(0) = 0$. The left hand limit of f at 0 is

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 = 0^2 = 0$$

The right hand limit of f at 0 is

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0$$

Thus $\lim_{x \to 0} f(x) = 0 = f(0)$ and hence f is continuous at 0. This means that f is continuous at every point in its domain and hence, f is a continuous function.

Example 14 Show that every polynomial function is continuous.

Solution Recall that a function p is a polynomial function if it is defined by $p(x) = a_0 + a_1 x + ... + a_n x^n$ for some natural number n, $a_n \neq 0$ and $a_i \in \mathbf{R}$. Clearly this function is defined for every real number. For a fixed real number c, we have

$$\lim_{x \to c} p(x) = p(c)$$

By definition, p is continuous at c. Since c is any real number, p is continuous at every real number and hence p is a continuous function.

Example 15 Find all the points of discontinuity of the greatest integer function defined by $f(x) = [x]$, where $[x]$ denotes the greatest integer less than or equal to x.

Solution First observe that f is defined for all real numbers. Graph of the function is given in Fig 5.8. From the graph it looks like that f is discontinuous at every integral point. Below we explore, if this is true.

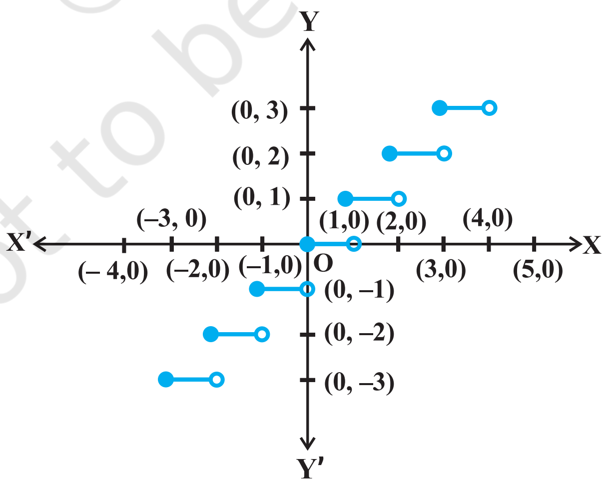

Fig 5.8

Case 1 Let c be a real number which is not equal to any integer. It is evident from the graph that for all real numbers *close* to c the value of the function is equal to $[c]$; i.e., $\lim_{x\to c} f(x) = \lim_{x\to c} [x] = [c]$. Also $f(c) = [c]$ and hence the function is continuous at all real numbers not equal to integers.

Case 2 Let c be an integer. Then we can find a sufficiently small real number $r > 0$ such that $[c - r] = c - 1$ whereas $[c + r] = c$.

This, in terms of limits mean that

$$\lim_{x\to c^-} f(x) = c - 1, \quad \lim_{x\to c^+} f(x) = c$$

Since these limits cannot be equal to each other for any c, the function is discontinuous at every integral point.

5.2.1 Algebra of continuous functions

In the previous class, after having understood the concept of limits, we learnt some algebra of limits. Analogously, now we will study some algebra of continuous functions. Since continuity of a function at a point is entirely dictated by the limit of the function at that point, it is reasonable to expect results analogous to the case of limits.

Theorem 1 Suppose f and g be two real functions continuous at a real number c. Then

(1) $f + g$ is continuous at $x = c$.

(2) $f - g$ is continuous at $x = c$.

(3) $f \,.\, g$ is continuous at $x = c$.

(4) $\left(\dfrac{f}{g}\right)$ is continuous at $x = c$, (provided $g(c) \neq 0$).

Proof We are investigating continuity of $(f + g)$ at $x = c$. Clearly it is defined at $x = c$. We have

$$\begin{aligned}
\lim_{x\to c}(f+g)(x) &= \lim_{x\to c}[f(x)+g(x)] && \text{(by definition of } f+g\text{)}\\
&= \lim_{x\to c} f(x) + \lim_{x\to c} g(x) && \text{(by the theorem on limits)}\\
&= f(c) + g(c) && \text{(as } f \text{ and } g \text{ are continuous)}\\
&= (f+g)(c) && \text{(by definition of } f+g\text{)}
\end{aligned}$$

Hence, $f + g$ is continuous at $x = c$.

Proofs for the remaining parts are similar and left as an exercise to the reader.

Remarks

(i) As a special case of (3) above, if f is a constant function, i.e., $f(x) = \lambda$ for some real number λ, then the function $(\lambda \,.\, g)$ defined by $(\lambda \,.\, g)(x) = \lambda \,.\, g(x)$ is also continuous. In particular if $\lambda = -1$, the continuity of f implies continuity of $-f$.

(ii) As a special case of (4) above, if f is the constant function $f(x) = \lambda$, then the function $\frac{\lambda}{g}$ defined by $\frac{\lambda}{g}(x) = \frac{\lambda}{g(x)}$ is also continuous wherever $g(x) \neq 0$. In particular, the continuity of g implies continuity of $\frac{1}{g}$.

The above theorem can be exploited to generate many continuous functions. They also aid in deciding if certain functions are continuous or not. The following examples illustrate this:

Example 16 Prove that every rational function is continuous.

Solution Recall that every rational function f is given by

$$f(x) = \frac{p(x)}{q(x)}, \quad q(x) \neq 0$$

where p and q are polynomial functions. The domain of f is all real numbers except points at which q is zero. Since polynomial functions are continuous (Example 14), f is continuous by (4) of Theorem 1.

Example 17 Discuss the continuity of sine function.

Solution To see this we use the following facts

$$\lim_{x \to 0} \sin x = 0$$

We have not proved it, but is intuitively clear from the graph of $\sin x$ near 0.

Now, observe that $f(x) = \sin x$ is defined for every real number. Let c be a real number. Put $x = c + h$. If $x \to c$ we know that $h \to 0$. Therefore

$$\begin{aligned}
\lim_{x \to c} f(x) &= \lim_{x \to c} \sin x \\
&= \lim_{h \to 0} \sin(c + h) \\
&= \lim_{h \to 0} [\sin c \cos h + \cos c \sin h] \\
&= \lim_{h \to 0} [\sin c \cos h] + \lim_{h \to 0} [\cos c \sin h] \\
&= \sin c + 0 = \sin c = f(c)
\end{aligned}$$

Thus $\lim_{x \to c} f(x) = f(c)$ and hence f is a continuous function.

Remark A similar proof may be given for the continuity of cosine function.

Example 18 Prove that the function defined by $f(x) = \tan x$ is a continuous function.

Solution The function $f(x) = \tan x = \dfrac{\sin x}{\cos x}$. This is defined for all real numbers such that $\cos x \neq 0$, i.e., $x \neq (2n+1)\dfrac{\pi}{2}$. We have just proved that both sine and cosine functions are continuous. Thus tan x being a quotient of two continuous functions is continuous wherever it is defined.

An interesting fact is the behaviour of continuous functions with respect to composition of functions. Recall that if f and g are two real functions, then

$$(f \text{ o } g)(x) = f(g(x))$$

is defined whenever the range of g is a subset of domain of f. The following theorem (stated without proof) captures the continuity of composite functions.

Theorem 2 Suppose f and g are real valued functions such that (f o g) is defined at c. If g is continuous at c and if f is continuous at $g(c)$, then (f o g) is continuous at c.

The following examples illustrate this theorem.

Example 19 Show that the function defined by $f(x) = \sin(x^2)$ is a continuous function.

Solution Observe that the function is defined for every real number. The function f may be thought of as a composition g o h of the two functions g and h, where $g(x) = \sin x$ and $h(x) = x^2$. Since both g and h are continuous functions, by Theorem 2, it can be deduced that f is a continuous function.

Example 20 Show that the function f defined by

$$f(x) = |1 - x + |x||,$$

where x is any real number, is a continuous function.

Solution Define g by $g(x) = 1 - x + |x|$ and h by $h(x) = |x|$ for all real x. Then

$$\begin{aligned}(h \text{ o } g)(x) &= h(g(x)) \\ &= h(1 - x + |x|) \\ &= |1 - x + |x|| = f(x)\end{aligned}$$

In Example 7, we have seen that h is a continuous function. Hence g being a sum of a polynomial function and the modulus function is continuous. But then f being a composite of two continuous functions is continuous.

EXERCISE 5.1

1. Prove that the function $f(x) = 5x - 3$ is continuous at $x = 0$, at $x = -3$ and at $x = 5$.
2. Examine the continuity of the function $f(x) = 2x^2 - 1$ at $x = 3$.
3. Examine the following functions for continuity.

   (a) $f(x) = x - 5$ (b) $f(x) = \dfrac{1}{x-5}, x \neq 5$

   (c) $f(x) = \dfrac{x^2 - 25}{x+5}, x \neq -5$ (d) $f(x) = |x - 5|$
4. Prove that the function $f(x) = x^n$ is continuous at $x = n$, where n is a positive integer.
5. Is the function f defined by

$$f(x) = \begin{cases} x, & \text{if } x \leq 1 \\ 5, & \text{if } x > 1 \end{cases}$$

   continuous at $x = 0$? At $x = 1$? At $x = 2$?

Find all points of discontinuity of f, where f is defined by

6. $f(x) = \begin{cases} 2x+3, & \text{if } x \leq 2 \\ 2x-3, & \text{if } x > 2 \end{cases}$

7. $f(x) = \begin{cases} |x|+3, & \text{if } x \leq -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x+2, & \text{if } x \geq 3 \end{cases}$

8. $f(x) = \begin{cases} \dfrac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

9. $f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$

10. $f(x) = \begin{cases} x+1, & \text{if } x \geq 1 \\ x^2+1, & \text{if } x < 1 \end{cases}$

11. $f(x) = \begin{cases} x^3-3, & \text{if } x \leq 2 \\ x^2+1, & \text{if } x > 2 \end{cases}$

12. $f(x) = \begin{cases} x^{10}-1, & \text{if } x \leq 1 \\ x^2, & \text{if } x > 1 \end{cases}$

13. Is the function defined by

$$f(x) = \begin{cases} x+5, & \text{if } x \leq 1 \\ x-5, & \text{if } x > 1 \end{cases}$$

   a continuous function?

Discuss the continuity of the function *f*, where *f* is defined by

14. $f(x)=\begin{cases}3, & \text{if } 0\le x\le 1\\ 4, & \text{if } 1<x<3\\ 5, & \text{if } 3\le x\le 10\end{cases}$

15. $f(x)=\begin{cases}2x, & \text{if } x<0\\ 0, & \text{if } 0\le x\le 1\\ 4x, & \text{if } x>1\end{cases}$

16. $f(x)=\begin{cases}-2, & \text{if } x\le -1\\ 2x, & \text{if } -1<x\le 1\\ 2, & \text{if } x>1\end{cases}$

17. Find the relationship between *a* and *b* so that the function *f* defined by

$$f(x)=\begin{cases}ax+1, & \text{if } x\le 3\\ bx+3, & \text{if } x>3\end{cases}$$

is continuous at $x = 3$.

18. For what value of λ is the function defined by

$$f(x)=\begin{cases}\lambda(x^2-2x), & \text{if } x\le 0\\ 4x+1, & \text{if } x>0\end{cases}$$

continuous at $x = 0$? What about continuity at $x = 1$?

19. Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral points. Here $[x]$ denotes the greatest integer less than or equal to x.

20. Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at $x = \pi$?

21. Discuss the continuity of the following functions:

(a) $f(x) = \sin x + \cos x$ (b) $f(x) = \sin x - \cos x$

(c) $f(x) = \sin x \, . \cos x$

22. Discuss the continuity of the cosine, cosecant, secant and cotangent functions.

23. Find all points of discontinuity of *f*, where

$$f(x)=\begin{cases}\dfrac{\sin x}{x}, & \text{if } x<0\\ x+1, & \text{if } x\ge 0\end{cases}$$

24. Determine if *f* defined by

$$f(x)=\begin{cases}x^2\sin\dfrac{1}{x}, & \text{if } x\ne 0\\ 0, & \text{if } x=0\end{cases}$$

is a continuous function?

25. Examine the continuity of f, where f is defined by

$$f(x)=\begin{cases}\sin x-\cos x, & \text{if } x\neq 0\\ -1, & \text{if } x=0\end{cases}$$

Find the values of k so that the function f is continuous at the indicated point in Exercises 26 to 29.

26. $f(x)=\begin{cases}\dfrac{k\cos x}{\pi-2x}, & \text{if } x\neq\dfrac{\pi}{2}\\ 3, & \text{if } x=\dfrac{\pi}{2}\end{cases}$ at $x=\dfrac{\pi}{2}$

27. $f(x)=\begin{cases}kx^2, & \text{if } x\leq 2\\ 3, & \text{if } x>2\end{cases}$ at $x = 2$

28. $f(x)=\begin{cases}kx+1, & \text{if } x\leq \pi\\ \cos x, & \text{if } x>\pi\end{cases}$ at $x = \pi$

29. $f(x)=\begin{cases}kx+1, & \text{if } x\leq 5\\ 3x-5, & \text{if } x>5\end{cases}$ at $x = 5$

30. Find the values of a and b such that the function defined by

$$f(x)=\begin{cases}5, & \text{if } x\leq 2\\ ax+b, & \text{if } 2<x<10\\ 21, & \text{if } x\geq 10\end{cases}$$

is a continuous function.

31. Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.

32. Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

33. Examine that $\sin|x|$ is a continuous function.

34. Find all the points of discontinuity of f defined by $f(x) = |x| - |x + 1|$.

5.3. Differentiability

Recall the following facts from previous class. We had defined the derivative of a real function as follows:

Suppose f is a real function and c is a point in its domain. The derivative of f at c is defined by

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

provided this limit exists. Derivative of f at c is denoted by $f'(c)$ or $\frac{d}{dx}(f(x))|_c$. The function defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

wherever the limit exists is defined to be the derivative of f. The derivative of f is denoted by $f'(x)$ or $\frac{d}{dx}(f(x))$ or if $y = f(x)$ by $\frac{dy}{dx}$ or y'. The process of finding derivative of a function is called differentiation. We also use the phrase *differentiate* $f(x)$ *with respect to* x to mean *find* $f'(x)$.

The following rules were established as a part of algebra of derivatives:

(1) $(u \pm v)' = u' \pm v'$

(2) $(uv)' = u'v + uv'$ (Leibnitz or product rule)

(3) $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$, wherever $v \neq 0$ (Quotient rule).

The following table gives a list of derivatives of certain standard functions:

Table 5.3

$f(x)$	x^n	$\sin x$	$\cos x$	$\tan x$
$f'(x)$	nx^{n-1}	$\cos x$	$-\sin x$	$\sec^2 x$

Whenever we defined derivative, we had put a caution *provided the limit exists*. Now the natural question is; what if it doesn't? The question is quite pertinent and so is its answer. If $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ does not exist, we say that f is not differentiable at c. In other words, we say that a function f is differentiable at a point c in its domain if both $\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$ and $\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$ are finite and equal. A function is said to be differentiable in an interval $[a, b]$ if it is differentiable at every point of $[a, b]$. As in case of continuity, at the end points a and b, we take the right hand limit and left hand limit, which are nothing but left hand derivative and right hand derivative of the function at a and b respectively. Similarly, a function is said to be differentiable in an interval (a, b) if it is differentiable at every point of (a, b).

Theorem 3 If a function f is differentiable at a point c, then it is also continuous at that point.

Proof Since f is differentiable at c, we have

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

But for $x \neq c$, we have

$$f(x) - f(c) = \frac{f(x) - f(c)}{x - c} \cdot (x - c)$$

Therefore $$\lim_{x \to c} [f(x) - f(c)] = \lim_{x \to c} \left[\frac{f(x) - f(c)}{x - c} \cdot (x - c)\right]$$

or $$\lim_{x \to c} [f(x)] - \lim_{x \to c} [f(c)] = \lim_{x \to c} \left[\frac{f(x) - f(c)}{x - c}\right] \cdot \lim_{x \to c} [(x - c)]$$

$$= f'(c) \cdot 0 = 0$$

or $$\lim_{x \to c} f(x) = f(c)$$

Hence f is continuous at $x = c$.

Corollary 1 Every differentiable function is continuous.

We remark that the converse of the above statement is not true. Indeed we have seen that the function defined by $f(x) = |x|$ is a continuous function. Consider the left hand limit

$$\lim_{h \to 0^-} \frac{f(0 + h) - f(0)}{h} = \frac{-h}{h} = -1$$

The right hand limit

$$\lim_{h \to 0^+} \frac{f(0 + h) - f(0)}{h} = \frac{h}{h} = 1$$

Since the above left and right hand limits at 0 are not equal, $\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}$ does not exist and hence f is not differentiable at 0. Thus f is not a differentiable function.

5.3.1 *Derivatives of composite functions*

To study derivative of composite functions, we start with an illustrative example. Say, we want to find the derivative of f, where

$$f(x) = (2x + 1)^3$$

One way is to expand $(2x + 1)^3$ using binomial theorem and find the derivative as a polynomial function as illustrated below.

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left[(2x+1)^3\right]$$

$$= \frac{d}{dx}(8x^3 + 12x^2 + 6x + 1)$$

$$= 24x^2 + 24x + 6$$

$$= 6\,(2x + 1)^2$$

Now, observe that $\qquad f(x) = (h \text{ o } g)\,(x)$

where $g(x) = 2x + 1$ and $h(x) = x^3$. Put $t = g(x) = 2x + 1$. Then $f(x) = h(t) = t^3$. Thus

$$\frac{df}{dx} = 6\,(2x + 1)^2 = 3(2x + 1)^2 \,.\, 2 = 3t^2 \,.\, 2 = \frac{dh}{dt}\cdot\frac{dt}{dx}$$

The advantage with such observation is that it simplifies the calculation in finding the derivative of, say, $(2x + 1)^{100}$. We may formalise this observation in the following theorem called the chain rule.

Theorem 4 (Chain Rule) Let f be a real valued function which is a composite of two functions u and v; i.e., $f = v \text{ o } u$. Suppose $t = u\,(x)$ and if both $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist, we have

$$\frac{df}{dx} = \frac{dv}{dt}\cdot\frac{dt}{dx}$$

We skip the proof of this theorem. Chain rule may be extended as follows. Suppose f is a real valued function which is a composite of three functions u, v and w; i.e.,

$f = (w \text{ o } u) \text{ o } v$. If $t = v\,(x)$ and $s = u\,(t)$, then

$$\frac{df}{dx} = \frac{d(w\text{o}u)}{dt}\cdot\frac{dt}{dx} = \frac{dw}{ds}\cdot\frac{ds}{dt}\cdot\frac{dt}{dx}$$

provided all the derivatives in the statement exist. Reader is invited to formulate chain rule for composite of more functions.

Example 21 Find the derivative of the function given by $f(x) = \sin\,(x^2)$.

Solution Observe that the given function is a composite of two functions. Indeed, if $t = u(x) = x^2$ and $v(t) = \sin t$, then

$$f(x) = (v \text{ o } u)\,(x) = v(u(x)) = v(x^2) = \sin x^2$$

Put $t = u(x) = x^2$. Observe that $\frac{dv}{dt} = \cos t$ and $\frac{dt}{dx} = 2x$ exist. Hence, by chain rule

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos t \cdot 2x$$

It is normal practice to express the final result only in terms of x. Thus

$$\frac{df}{dx} = \cos t \cdot 2x = 2x \cos x^2$$

EXERCISE 5.2

Differentiate the functions with respect to x in Exercises 1 to 8.

1. $\sin (x^2 + 5)$ **2.** $\cos (\sin x)$ **3.** $\sin (ax + b)$

4. $\sec (\tan (\sqrt{x}))$ **5.** $\frac{\sin (ax+b)}{\cos (cx+d)}$ **6.** $\cos x^3 . \sin^2 (x^5)$

7. $2\sqrt{\cot(x^2)}$ **8.** $\cos(\sqrt{x})$

9. Prove that the function f given by

$$f(x) = |x - 1|, x \in \mathbf{R}$$

is not differentiable at $x = 1$.

10. Prove that the greatest integer function defined by

$$f(x) = [x], 0 < x < 3$$

is not differentiable at $x = 1$ and $x = 2$.

5.3.2 *Derivatives of implicit functions*

Until now we have been differentiating various functions given in the form $y = f(x)$. But it is not necessary that functions are always expressed in this form. For example, consider one of the following relationships between x and y:

$$x - y - \pi = 0$$

$$x + \sin xy - y = 0$$

In the first case, we can *solve for y* and rewrite the relationship as $y = x - \pi$. In the second case, it does not seem that there is an easy way to *solve for y*. Nevertheless, there is no doubt about the dependence of y on x in either of the cases. When a relationship between x and y is expressed in a way that it is easy to *solve for y* and write $y = f(x)$, we say that y is given as an *explicit function* of x. In the latter case it is implicit that y is a function of x and we say that the relationship of the second type, above, gives function *implicitly*. In this subsection, we learn to differentiate implicit functions.

Example 22 Find $\frac{dy}{dx}$ if $x - y = \pi$.

Solution One way is to solve for y and rewrite the above as

$$y = x - \pi$$

But then
$$\frac{dy}{dx} = 1$$

Alternatively, *directly* differentiating the relationship w.r.t., x, we have

$$\frac{d}{dx}(x-y) = \frac{d\pi}{dx}$$

Recall that $\frac{d\pi}{dx}$ means to differentiate the constant function taking value π everywhere w.r.t., x. Thus

$$\frac{d}{dx}(x) - \frac{d}{dx}(y) = 0$$

which implies that

$$\frac{dy}{dx} = \frac{dx}{dx} = 1$$

Example 23 Find $\frac{dy}{dx}$, if $y + \sin y = \cos x$.

Solution We differentiate the relationship directly with respect to x, i.e.,

$$\frac{dy}{dx} + \frac{d}{dx}(\sin y) = \frac{d}{dx}(\cos x)$$

which implies using chain rule

$$\frac{dy}{dx} + \cos y \cdot \frac{dy}{dx} = -\sin x$$

This gives
$$\frac{dy}{dx} = -\frac{\sin x}{1+\cos y}$$

where
$$y \neq (2n + 1)\,\pi$$

5.3.3 *Derivatives of inverse trigonometric functions*

We remark that inverse trigonometric functions are continuous functions, but we will not prove this. Now we use chain rule to find derivatives of these functions.

Example 24 Find the derivative of f given by $f(x) = \sin^{-1} x$ assuming it exists.

Solution Let $y = \sin^{-1} x$. Then, $x = \sin y$.

Differentiating both sides w.r.t. x, we get

$$1 = \cos y \frac{dy}{dx}$$

which implies that
$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1} x)}$$

Observe that this is defined only for $\cos y \neq 0$, i.e., $\sin^{-1} x \neq -\frac{\pi}{2}, \frac{\pi}{2}$, i.e., $x \neq -1, 1$, i.e., $x \in (-1, 1)$.

To make this result a bit more attractive, we carry out the following manipulation. Recall that for $x \in (-1, 1)$, $\sin(\sin^{-1} x) = x$ and hence

$$\cos^2 y = 1 - (\sin y)^2 = 1 - (\sin(\sin^{-1} x))^2 = 1 - x^2$$

Also, since $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\cos y$ is positive and hence $\cos y = \sqrt{1-x^2}$

Thus, for $x \in (-1, 1)$,

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

$f(x)$	$\sin^{-1} x$	$\cos^{-1} x$	$\tan^{-1} x$
$f^1(x)$	$1/\sqrt{1-x^2}$	$-1/\sqrt{1-x^2}$	$1/1+x^2$
Domain off	(-1, 1)	(-1, 1)	R

EXERCISE 5.3

Find $\frac{dy}{dx}$ in the following:

1. $2x + 3y = \sin x$ **2.** $2x + 3y = \sin y$ **3.** $ax + by^2 = \cos y$

4. $xy + y^2 = \tan x + y$ **5.** $x^2 + xy + y^2 = 100$ **6.** $x^3 + x^2y + xy^2 + y^3 = 81$

7. $\sin^2 y + \cos xy = \kappa$ **8.** $\sin^2 x + \cos^2 y = 1$ **9.** $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

10. $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

11. $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$

12. $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$

13. $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right), -1 < x < 1$

14. $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right), -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

15. $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}}$

5.4 Exponential and Logarithmic Functions

Till now we have learnt some aspects of different classes of functions like polynomial functions, rational functions and trigonometric functions. In this section, we shall learn about a new class of (related) functions called exponential functions and logarithmic functions. It needs to be emphasized that many statements made in this section are motivational and precise proofs of these are well beyond the scope of this text.

The Fig 5.9 gives a sketch of $y = f_1(x) = x$, $y = f_2(x) = x^2$, $y = f_3(x) = x^3$ and $y = f_4(x) = x^4$. Observe that the curves get steeper as the power of x increases. Steeper the curve, faster is the rate of growth. What this means is that for a fixed increment in the value of x (> 1), the increment in the value of $y = f_n(x)$ increases as n increases for $n = 1, 2, 3, 4$. It is conceivable that such a statement is true for all positive values of n, where $f_n(x) = x^n$. Essentially, this means that the graph of $y = f_n(x)$ *leans* more towards the y-axis as n increases. For example, consider $f_{10}(x) = x^{10}$ and $f_{15}(x) = x^{15}$. If x increases from 1 to 2, f_{10} increases from 1 to 2^{10} whereas f_{15} increases from 1 to 2^{15}. Thus, for the same increment in x, f_{15} grow faster than f_{10}.

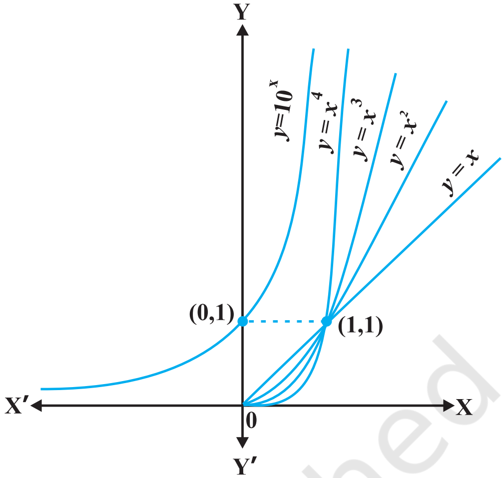

Fig 5.9

Upshot of the above discussion is that the growth of polynomial functions is dependent on the degree of the polynomial function – higher the degree, greater is the growth. The next natural question is: Is there a function which grows faster than any polynomial function. The answer is in affirmative and an example of such a function is

$$y = f(x) = 10^x.$$

Our claim is that this function f grows faster than $f_n(x) = x^n$ for any positive integer n. For example, we can prove that 10^x grows faster than $f_{100}(x) = x^{100}$. For large values of x like $x = 10^3$, note that $f_{100}(x) = (10^3)^{100} = 10^{300}$ whereas $f(10^3) = 10^{10^3} = 10^{1000}$. Clearly $f(x)$ is much greater than $f_{100}(x)$. It is not difficult to prove that for all $x > 10^3$, $f(x) > f_{100}(x)$. But we will not attempt to give a proof of this here. Similarly, by choosing large values of x, one can verify that $f(x)$ grows faster than $f_n(x)$ for any positive integer n.

Definition 3 The exponential function with positive base $b > 1$ is the function

$$y = f(x) = b^x$$

The graph of $y = 10^x$ is given in the Fig 5.9.

It is advised that the reader plots this graph for particular values of b like 2, 3 and 4. Following are some of the salient features of the exponential functions:

(1) Domain of the exponential function is **R**, the set of all real numbers.

(2) Range of the exponential function is the set of all positive real numbers.

(3) The point (0, 1) is always on the graph of the exponential function (this is a restatement of the fact that $b^0 = 1$ for any real $b > 1$).

(4) Exponential function is ever increasing; i.e., as we move from left to right, the graph rises above.

(5) For very large negative values of x, the exponential function is very close to 0. In other words, in the second quadrant, the graph approaches x-axis (but never meets it).

Exponential function with base 10 is called the *common exponential function*. In the Appendix A.1.4 of Class XI, it was observed that the sum of the series

$$1+\frac{1}{1!}+\frac{1}{2!}+\dots$$

is a number between 2 and 3 and is denoted by e. Using this e as the base we obtain an extremely important exponential function $y = e^x$.

This is called *natural exponential function*.

It would be interesting to know if the inverse of the exponential function exists and has *nice* interpretation. This search motivates the following definition.

Definition 4 Let $b > 1$ be a real number. Then we say logarithm of a to base b is x if $b^x = a$.

Logarithm of a to base b is denoted by $\log_b a$. Thus $\log_b a = x$ if $b^x = a$. Let us work with a few explicit examples to get a feel for this. We know $2^3 = 8$. In terms of logarithms, we may rewrite this as $\log_2 8 = 3$. Similarly, $10^4 = 10000$ is equivalent to saying $\log_{10} 10000 = 4$. Also, $625 = 5^4 = 25^2$ is equivalent to saying $\log_5 625 = 4$ or $\log_{25} 625 = 2$.

On a slightly more mature note, fixing a base $b > 1$, we may look at logarithm as a function from positive real numbers to all real numbers. This function, called the *logarithmic function*, is defined by

$$\log_b : \mathbf{R}^+ \to \mathbf{R}$$
$$x \to \log_b x = y \quad \text{if } b^y = x$$

As before if the base $b = 10$, we say it is *common logarithms* and if $b = e$, then we say it is *natural logarithms*. Often natural logarithm is denoted by *ln*. *In this chapter, log x denotes the logarithm function to base e*, i.e., *ln x* will be written as simply $\log x$. The Fig 5.10 gives the plots of logarithm function to base 2, e and 10.

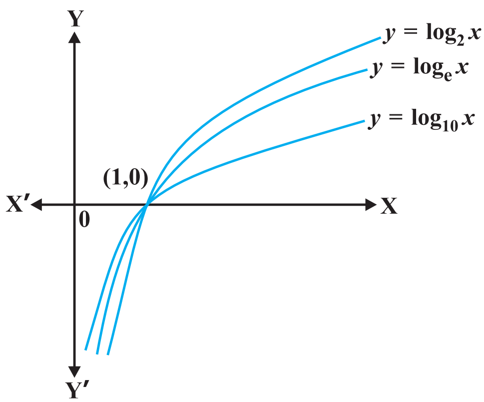

Fig 5.10

Some of the important observations about the logarithm function to any base $b > 1$ are listed below:

(1) We cannot make a meaningful definition of logarithm of non-positive numbers and hence the domain of log function is $\mathbf{R}^+$.

(2) The range of log function is the set of all real numbers.

(3) The point (1, 0) is always on the graph of the log function.

(4) The log function is ever increasing, i.e., as we move from left to right the graph rises above.

(5) For x very near to zero, the value of log x can be made lesser than any given real number. In other words in the fourth quadrant the graph approaches y-axis (but never meets it).

(6) Fig 5.11 gives the plot of $y = e^x$ and $y = ln\, x$. It is of interest to observe that the two curves are the mirror images of each other reflected in the line $y = x$.

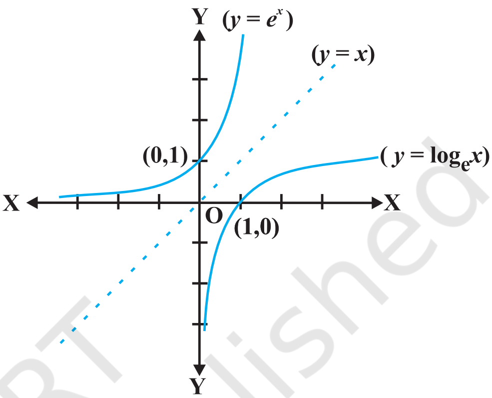

Fig 5.11

Two properties of 'log' functions are proved below:

(1) There is a standard change of base rule to obtain $\log_a p$ in terms of $\log_b p$. Let $\log_a p = \alpha$, $\log_b p = \beta$ and $\log_b a = \gamma$. This means $a^\alpha = p$, $b^\beta = p$ and $b^\gamma = a$.

Substituting the third equation in the first one, we have

$$(b^\gamma)^\alpha = b^{\gamma\alpha} = p$$

Using this in the second equation, we get

$$b^\beta = p = b^{\gamma\alpha}$$

which implies $\quad \beta = \alpha\gamma$ or $\alpha = \dfrac{\beta}{\gamma}$. But then

$$\mathbf{\log_a p = \frac{\log_b p}{\log_b a}}$$

(2) Another interesting property of the log function is its effect on products. Let $\log_b pq = \alpha$. Then $b^\alpha = pq$. If $\log_b p = \beta$ and $\log_b q = \gamma$, then $b^\beta = p$ and $b^\gamma = q$. But then $b^\alpha = pq = b^\beta b^\gamma = b^{\beta + \gamma}$

which implies $\alpha = \beta + \gamma$, i.e.,

$$\mathbf{\log_b pq = \log_b p + \log_b q}$$

A particularly interesting and important consequence of this is when $p = q$. In this case the above may be rewritten as

$$\log_b p^2 = \log_b p + \log_b p = 2 \log p$$

An easy generalisation of this (left as an exercise!) is

$$\log_b p^n = n \log p$$

for any positive integer n. In fact this is true for any real number n, but we will not attempt to prove this. On the similar lines the reader is invited to verify

$$\log_b \frac{x}{y} = \log_b x - \log_b y$$

Example 25 Is it true that $x = e^{\log x}$ for all real x?

Solution First, observe that the domain of log function is set of all positive real numbers. So the above equation is not true for non-positive real numbers. Now, let $y = e^{\log x}$. If $y > 0$, we may take logarithm which gives us $\log y = \log (e^{\log x}) = \log x \,.\, \log e = \log x$. Thus $y = x$. Hence $x = e^{\log x}$ is true only for positive values of x.

One of the striking properties of the natural exponential function in differential calculus is that it doesn't change during the process of differentiation. This is captured in the following theorem whose proof we skip.

Theorem 5*

(1) The derivative of e^x w.r.t., x is e^x; i.e., $\frac{d}{dx}(e^x) = e^x$.

(2) The derivative of $\log x$ w.r.t., x is $\frac{1}{x}$; i.e., $\frac{d}{dx}(\log x) = \frac{1}{x}$.

Example 26 Differentiate the following w.r.t. x:

(i) e^{-x} (ii) $\sin(\log x), x > 0$ (iii) $\cos^{-1}(e^x)$ (iv) $e^{\cos x}$

Solution

(i) Let $y = e^{-x}$. Using chain rule, we have

$$\frac{dy}{dx} = e^{-x} \cdot \frac{d}{dx}(-x) = -e^{-x}$$

(ii) Let $y = \sin(\log x)$. Using chain rule, we have

$$\frac{dy}{dx} = \cos(\log x) \cdot \frac{d}{dx}(\log x) = \frac{\cos(\log x)}{x}$$

* *Please see supplementary material on Page 222.*

(iii) Let $y = \cos^{-1}(e^x)$. Using chain rule, we have

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(e^x)^2}} \cdot \frac{d}{dx}(e^x) = \frac{-e^x}{\sqrt{1-e^{2x}}}$$

(iv) Let $y = e^{\cos x}$. Using chain rule, we have

$$\frac{dy}{dx} = e^{\cos x} \cdot (-\sin x) = -(\sin x)\, e^{\cos x}$$

EXERCISE 5.4

Differentiate the following w.r.t. x:

1. $\dfrac{e^x}{\sin x}$ **2.** $e^{\sin^{-1} x}$ **3.** e^{x^3}

4. $\sin(\tan^{-1} e^{-x})$ **5.** $\log(\cos e^x)$ **6.** $e^x + e^{x^2} + ... + e^{x^5}$

7. $\sqrt{e^{\sqrt{x}}},\ x > 0$ **8.** $\log(\log x),\ x > 1$ **9.** $\dfrac{\cos x}{\log x},\ x > 0$

10. $\cos(\log x + e^x),\ x > 0$

5.5. Logarithmic Differentiation

In this section, we will learn to differentiate certain special class of functions given in the form

$$y = f(x) = [u(x)]^{v(x)}$$

By taking logarithm (to base e) the above may be rewritten as

$$\log y = v(x) \log [u(x)]$$

Using chain rule we may differentiate this to get

$$\frac{1}{y} \cdot \frac{dy}{dx} = v(x) \cdot \frac{1}{u(x)} \cdot u'(x) + v'(x) \cdot \log [u(x)]$$

which implies that

$$\frac{dy}{dx} = y\left[\frac{v(x)}{u(x)} \cdot u'(x) + v'(x) \cdot \log[u(x)]\right]$$

The main point to be noted in this method is that $f(x)$ and $u(x)$ must always be positive as otherwise their logarithms are not defined. This process of differentiation is known as *logarithms differentiation* and is illustrated by the following examples:

Example 27 Differentiate $\sqrt{\dfrac{(x-3)(x^2+4)}{3x^2+4x+5}}$ w.r.t. x.

Solution Let $y = \sqrt{\dfrac{(x-3)(x^2+4)}{(3x^2+4x+5)}}$

Taking logarithm on both sides, we have

$$\log y = \frac{1}{2}[\log(x-3) + \log(x^2+4) - \log(3x^2+4x+5)]$$

Now, differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{(x-3)} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5}\right]$$

or $$\frac{dy}{dx} = \frac{y}{2}\left[\frac{1}{(x-3)} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5}\right]$$

$$= \frac{1}{2}\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}\left[\frac{1}{(x-3)} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5}\right]$$

Example 28 Differentiate a^x w.r.t. x, where a is a positive constant.

Solution Let $y = a^x$. Then

$$\log y = x \log a$$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{y}\frac{dy}{dx} = \log a$$

or $$\frac{dy}{dx} = y \log a$$

Thus $$\frac{d}{dx}(a^x) = a^x \log a$$

Alternatively $$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x\log a}) = e^{x\log a}\frac{d}{dx}(x\log a)$$

$$= e^{x\log a} \cdot \log a = a^x \log a.$$

Example 29 Differentiate $x^{\sin x}$, $x > 0$ w.r.t. x.

Solution Let $y = x^{\sin x}$. Taking logarithm on both sides, we have

$$\log y = \sin x \log x$$

Therefore
$$\frac{1}{y} \cdot \frac{dy}{dx} = \sin x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(\sin x)$$

or
$$\frac{1}{y}\frac{dy}{dx} = (\sin x)\frac{1}{x} + \log x \cos x$$

or
$$\frac{dy}{dx} = y\left[\frac{\sin x}{x} + \cos x \log x\right]$$

$$= x^{\sin x}\left[\frac{\sin x}{x} + \cos x \log x\right]$$

$$= x^{\sin x - 1} \cdot \sin x + x^{\sin x} \cdot \cos x \log x$$

Example 30 Find $\frac{dy}{dx}$, if $y^x + x^y + x^x = a^b$.

Solution Given that $y^x + x^y + x^x = a^b$.

Putting $u = y^x$, $v = x^y$ and $w = x^x$, we get $u + v + w = a^b$

Therefore
$$\frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} = 0 \qquad \text{... (1)}$$

Now, $u = y^x$. Taking logarithm on both sides, we have

$$\log u = x \log y$$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{u} \cdot \frac{du}{dx} = x\frac{d}{dx}(\log y) + \log y \frac{d}{dx}(x)$$

$$= x\frac{1}{y} \cdot \frac{dy}{dx} + \log y \cdot 1$$

So
$$\frac{du}{dx} = u\left(\frac{x}{y}\frac{dy}{dx} + \log y\right) = y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right] \qquad \text{... (2)}$$

Also $v = x^y$

Taking logarithm on both sides, we have

$$\log v = y \log x$$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{v}\cdot\frac{dv}{dx} = y\frac{d}{dx}(\log x) + \log x\frac{dy}{dx}$$

$$= y\cdot\frac{1}{x} + \log x\cdot\frac{dy}{dx}$$

So $$\frac{dv}{dx} = v\left[\frac{y}{x} + \log x\frac{dy}{dx}\right]$$

$$= x^y\left[\frac{y}{x} + \log x\frac{dy}{dx}\right] \quad \text{... (3)}$$

Again $$w = x^x$$

Taking logarithm on both sides, we have

$$\log w = x \log x.$$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{w}\cdot\frac{dw}{dx} = x\frac{d}{dx}(\log x) + \log x\cdot\frac{d}{dx}(x)$$

$$= x\cdot\frac{1}{x} + \log x\cdot 1$$

i.e. $$\frac{dw}{dx} = w\,(1 + \log x)$$

$$= x^x\,(1 + \log x) \quad \text{... (4)}$$

From (1), (2), (3), (4), we have

$$y^x\left(\frac{x}{y}\frac{dy}{dx} + \log y\right) + x^y\left(\frac{y}{x} + \log x\frac{dy}{dx}\right) + x^x\,(1 + \log x) = 0$$

or $$(x\,.\,y^{x-1} + x^y\,.\,\log x)\,\frac{dy}{dx} = -\,x^x\,(1 + \log x) - y\,.\,x^{y-1} - y^x \log y$$

Therefore $$\frac{dy}{dx} = \frac{-[y^x\log y + y\,.\,x^{y-1} + x^x(1+\log x)]}{x\,.\,y^{x-1} + x^y\log x}$$

EXERCISE 5.5

Differentiate the functions given in Exercises 1 to 11 w.r.t. x.

1. $\cos x \cdot \cos 2x \cdot \cos 3x$

2. $\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

3. $(\log x)^{\cos x}$

4. $x^x - 2^{\sin x}$

5. $(x+3)^2 \cdot (x+4)^3 \cdot (x+5)^4$

6. $\left(x+\dfrac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

7. $(\log x)^x + x^{\log x}$

8. $(\sin x)^x + \sin^{-1}\sqrt{x}$

9. $x^{\sin x} + (\sin x)^{\cos x}$

10. $x^{x\cos x} + \dfrac{x^2+1}{x^2-1}$

11. $(x\cos x)^x + (x\sin x)^{\frac{1}{x}}$

Find $\dfrac{dy}{dx}$ of the functions given in Exercises 12 to 15.

12. $x^y + y^x = 1$

13. $y^x = x^y$

14. $(\cos x)^y = (\cos y)^x$

15. $xy = e^{(x-y)}$

16. Find the derivative of the function given by $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$ and hence find $f'(1)$.

17. Differentiate $(x^2 - 5x + 8)(x^3 + 7x + 9)$ in three ways mentioned below:
    (i) by using product rule
    (ii) by expanding the product to obtain a single polynomial.
    (iii) by logarithmic differentiation.
    Do they all give the same answer?

18. If u, v and w are functions of x, then show that

$$\frac{d}{dx}(u.\,v.\,w) = \frac{du}{dx}\,v.\,w + u\,.\,\frac{dv}{dx}\,.\,w + u\,.\,v\,\frac{dw}{dx}$$

in two ways - first by repeated application of product rule, second by logarithmic differentiation.

5.6 Derivatives of Functions in Parametric Forms

Sometimes the relation between two variables is neither explicit nor implicit, but some link of a third variable with each of the two variables, separately, establishes a relation between the first two variables. In such a situation, we say that the relation between them is expressed via a third variable. The third variable is called the parameter. More precisely, a relation expressed between two variables x and y in the form $x = f(t)$, $y = g(t)$ is said to be parametric form with t as a parameter.

In order to find derivative of function in such form, we have by chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

or $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \left(\text{whenever } \frac{dx}{dt} \neq 0\right)$$

Thus $$\frac{dy}{dx} = \frac{g'(t)}{f'(t)} \left(\text{as } \frac{dy}{dt} = g'(t) \text{ and } \frac{dx}{dt} = f'(t)\right) \text{ [provided } f'(t) \neq 0]$$

Example 31 Find $\frac{dy}{dx}$, if $x = a \cos \theta$, $y = a \sin \theta$.

Solution Given that

$$x = a \cos \theta,\ y = a \sin \theta$$

Therefore $$\frac{dx}{d\theta} = -a \sin \theta,\ \frac{dy}{d\theta} = a \cos \theta$$

Hence $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a \cos \theta}{-a \sin \theta} = -\cot \theta$$

Example 32 Find $\frac{dy}{dx}$, if $x = at^2$, $y = 2at$.

Solution Given that $x = at^2$, $y = 2at$

So $$\frac{dx}{dt} = 2at \quad \text{and} \quad \frac{dy}{dt} = 2a$$

Therefore $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

Example 33 Find $\frac{dy}{dx}$, if $x = a\,(\theta + \sin\theta)$, $y = a\,(1 - \cos\theta)$.

Solution We have $\frac{dx}{d\theta} = a(1 + \cos\theta)$, $\frac{dy}{d\theta} = a\,(\sin\theta)$

Therefore
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1+\cos\theta)} = \tan\frac{\theta}{2}$$

Note It may be noted here that $\frac{dy}{dx}$ is expressed in terms of parameter only without directly involving the main variables x and y.

Example 34 Find $\frac{dy}{dx}$, if $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Solution Let $x = a\cos^3\theta$, $y = a\sin^3\theta$. Then

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = (a\cos^3\theta)^{\frac{2}{3}} + (a\sin^3\theta)^{\frac{2}{3}}$$
$$= a^{\frac{2}{3}}(\cos^2\theta + (\sin^2\theta) = a^{\frac{2}{3}}$$

Hence, $x = a\cos^3\theta$, $y = a\sin^3\theta$ is parametric equation of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Now $\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$ and $\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

Therefore
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta = -\sqrt[3]{\frac{y}{x}}$$

EXERCISE 5.6

If x and y are connected parametrically by the equations given in Exercises 1 to 10, without eliminating the parameter, Find $\frac{dy}{dx}$.

1. $x = 2at^2$, $y = at^4$
2. $x = a \cos \theta$, $y = b \cos \theta$
3. $x = \sin t$, $y = \cos 2t$
4. $x = 4t$, $y = \frac{4}{t}$
5. $x = \cos \theta - \cos 2\theta$, $y = \sin \theta - \sin 2\theta$
6. $x = a(\theta - \sin \theta)$, $y = a(1 + \cos \theta)$
7. $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}$, $y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$
8. $x = a\left(\cos t + \log \tan \frac{t}{2}\right)$ $y = a \sin t$
9. $x = a \sec \theta$, $y = b \tan \theta$
10. $x = a(\cos \theta + \theta \sin \theta)$, $y = a(\sin \theta - \theta \cos \theta)$
11. If $x = \sqrt{a^{\sin^{-1} t}}$, $y = \sqrt{a^{\cos^{-1} t}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$

5.7 Second Order Derivative

Let $\quad y = f(x)$. Then

$$\frac{dy}{dx} = f'(x) \qquad \text{... (1)}$$

If $f'(x)$ is differentiable, we may differentiate (1) again w.r.t. x. Then, the left hand side becomes $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ which is called the *second order derivative* of y w.r.t. x and is denoted by $\frac{d^2y}{dx^2}$. The second order derivative of $f(x)$ is denoted by $f''(x)$. It is also denoted by $D^2 y$ or y'' or y_2 if $y = f(x)$. We remark that higher order derivatives may be defined similarly.

Example 35 Find $\frac{d^2y}{dx^2}$, if $y = x^3 + \tan x$.

Solution Given that $y = x^3 + \tan x$. Then

$$\frac{dy}{dx} = 3x^2 + \sec^2 x$$

Therefore
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(3x^2 + \sec^2 x\right)$$
$$= 6x + 2 \sec x \,.\, \sec x \tan x = 6x + 2 \sec^2 x \tan x$$

Example 36 If $y = \text{A} \sin x + \text{B} \cos x$, then prove that $\frac{d^2y}{dx^2} + y = 0$.

Solution We have

$$\frac{dy}{dx} = \text{A} \cos x - \text{B} \sin x$$

and
$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\text{A} \cos x - \text{B} \sin x)$$
$$= -\text{A} \sin x - \text{B} \cos x = -y$$

Hence
$$\frac{d^2y}{dx^2} + y = 0$$

Example 37 If $y = 3e^{2x} + 2e^{3x}$, prove that $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$.

Solution Given that $y = 3e^{2x} + 2e^{3x}$. Then

$$\frac{dy}{dx} = 6e^{2x} + 6e^{3x} = 6\,(e^{2x} + e^{3x})$$

Therefore
$$\frac{d^2y}{dx^2} = 12e^{2x} + 18e^{3x} = 6\,(2e^{2x} + 3e^{3x})$$

Hence
$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 6\,(2e^{2x} + 3e^{3x})$$
$$- 30\,(e^{2x} + e^{3x}) + 6\,(3e^{2x} + 2e^{3x}) = 0$$

Example 38 If $y = \sin^{-1} x$, show that $(1 - x^2) \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 0$.

Solution We have $y = \sin^{-1}x$. Then

$$\frac{dy}{dx} = \frac{1}{\sqrt{(1-x^2)}}$$

or
$$\sqrt{(1-x^2)}\,\frac{dy}{dx} = 1$$

So
$$\frac{d}{dx}\left(\sqrt{(1-x^2)}\,.\,\frac{dy}{dx}\right) = 0$$

or
$$\sqrt{(1-x^2)}\cdot\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot\frac{d}{dx}\left(\sqrt{(1-x^2)}\right) = 0$$

or
$$\sqrt{(1-x^2)}\cdot\frac{d^2y}{dx^2} - \frac{dy}{dx}\cdot\frac{2x}{2\sqrt{1-x^2}} = 0$$

Hence
$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$$

Alternatively, Given that $y = \sin^{-1}x$, we have

$$y_1 = \frac{1}{\sqrt{1-x^2}},\ \text{i.e., } \left(1-x^2\right)y_1^2 = 1$$

So
$$(1-x^2)\,.\,2y_1y_2 + y_1^2(0-2x) = 0$$

Hence
$$(1 - x^2)\,y_2 - xy_1 = 0$$

EXERCISE 5.7

Find the second order derivatives of the functions given in Exercises 1 to 10.

1. $x^2 + 3x + 2$ **2.** x^{20} **3.** $x\,.\cos x$

4. $\log x$ **5.** $x^3 \log x$ **6.** $e^x \sin 5x$

7. $e^{6x} \cos 3x$ **8.** $\tan^{-1} x$ **9.** $\log(\log x)$

10. $\sin(\log x)$

11. If $y = 5\cos x - 3\sin x$, prove that $\dfrac{d^2y}{dx^2} + y = 0$

12. If $y = \cos^{-1} x$, Find $\dfrac{d^2y}{dx^2}$ in terms of y alone.

13. If $y = 3 \cos (\log x) + 4 \sin (\log x)$, show that $x^2 y_2 + xy_1 + y = 0$

14. If $y = Ae^{mx} + Be^{nx}$, show that $\dfrac{d^2y}{dx^2} - (m+n)\dfrac{dy}{dx} + mny = 0$

15. If $y = 500e^{7x} + 600e^{-7x}$, show that $\dfrac{d^2y}{dx^2} = 49y$

16. If $e^y (x + 1) = 1$, show that $\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2$

17. If $y = (\tan^{-1} x)^2$, show that $(x^2 + 1)^2 y_2 + 2x (x^2 + 1) y_1 = 2$

Miscellaneous Examples

Example 39 Differentiate w.r.t. x, the following function:

(i) $\sqrt{3x+2} + \dfrac{1}{\sqrt{2x^2+4}}$ (ii) $\log_7 (\log x)$

Solution

(i) Let $y = \sqrt{3x+2} + \dfrac{1}{\sqrt{2x^2+4}} = (3x+2)^{\frac{1}{2}} + (2x^2+4)^{-\frac{1}{2}}$

Note that this function is defined at all real numbers $x > -\dfrac{2}{3}$. Therefore

$$\frac{dy}{dx} = \frac{1}{2}(3x+2)^{\frac{1}{2}-1} \cdot \frac{d}{dx}(3x+2) + \left(-\frac{1}{2}\right)(2x^2+4)^{-\frac{1}{2}-1} \cdot \frac{d}{dx}(2x^2+4)$$

$$= \frac{1}{2}(3x+2)^{-\frac{1}{2}} \cdot (3) - \left(\frac{1}{2}\right)(2x^2+4)^{-\frac{3}{2}} \cdot 4x$$

$$= \frac{3}{2\sqrt{3x+2}} - \frac{2x}{(2x^2+4)^{\frac{3}{2}}}$$

This is defined for all real numbers $x > -\dfrac{2}{3}$.

(ii) Let $y = \log_7 (\log x) = \dfrac{\log(\log x)}{\log 7}$ (by change of base formula).

The function is defined for all real numbers $x > 1$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\log 7}\frac{d}{dx}(\log(\log x))\\ &= \frac{1}{\log 7}\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)\\ &= \frac{1}{x\log 7\log x}\end{aligned}$$

Example 40 Differentiate the following w.r.t. x.

(i) $\cos^{-1}(\sin x)$ (ii) $\tan^{-1}\left(\dfrac{\sin x}{1+\cos x}\right)$ (iii) $\sin^{-1}\left(\dfrac{2^{x+1}}{1+4^x}\right)$

Solution

(i) Let $f(x) = \cos^{-1}(\sin x)$. Observe that this function is defined for all real numbers. We may rewrite this function as

$$\begin{aligned}f(x) &= \cos^{-1}(\sin x)\\ &= \cos^{-1}\left[\cos\left(\frac{\pi}{2}-x\right)\right]\\ &= \frac{\pi}{2}-x\end{aligned}$$

Thus $f'(x) = -1$.

(ii) Let $f(x) = \tan^{-1}\left(\dfrac{\sin x}{1+\cos x}\right)$. Observe that this function is defined for all real numbers, where $\cos x \neq -1$; i.e., at all odd multiples of π. We may rewrite this function as

$$\begin{aligned}f(x) &= \tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)\\ &= \tan^{-1}\left[\frac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{2\cos^2\frac{x}{2}}\right]\end{aligned}$$

$$= \tan^{-1}\left[\tan\left(\frac{x}{2}\right)\right] = \frac{x}{2}$$

Observe that we could cancel $\cos\left(\frac{x}{2}\right)$ in both numerator and denominator as it is not equal to zero. Thus $f'(x) = \frac{1}{2}$.

(iii) Let $f(x) = \sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$. To find the domain of this function we need to find all x such that $-1 \le \frac{2^{x+1}}{1+4^x} \le 1$. Since the quantity in the middle is always positive, we need to find all x such that $\frac{2^{x+1}}{1+4^x} \le 1$, i.e., all x such that $2^{x+1} \le 1 + 4^x$. We may rewrite this as $2 \le \frac{1}{2^x} + 2^x$ which is true for all x. Hence the function is defined at every real number. By putting $2^x = \tan\theta$, this function may be rewritten as

$$f(x) = \sin^{-1}\left[\frac{2^{x+1}}{1+4^x}\right]$$

$$= \sin^{-1}\left[\frac{2^x \cdot 2}{1+\left(2^x\right)^2}\right]$$

$$= \sin^{-1}\left[\frac{2\tan\theta}{1+\tan^2\theta}\right]$$

$$= \sin^{-1}[\sin 2\theta]$$

$$= 2\theta = 2\tan^{-1}(2^x)$$

Thus $$f'(x) = 2\cdot\frac{1}{1+\left(2^x\right)^2}\cdot\frac{d}{dx}(2^x)$$

$$= \frac{2}{1+4^x}\cdot(2^x)\log 2$$

$$= \frac{2^{x+1}\log 2}{1+4^x}$$

Example 41 Find $f'(x)$ if $f(x) = (\sin x)^{\sin x}$ for all $0 < x < \pi$.

Solution The function $y = (\sin x)^{\sin x}$ is defined for all positive real numbers. Taking logarithms, we have

$$\log y = \log (\sin x)^{\sin x} = \sin x \log (\sin x)$$

Then
$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(\sin x \log (\sin x))$$

$$= \cos x \log (\sin x) + \sin x \,.\, \frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)$$

$$= \cos x \log (\sin x) + \cos x$$

$$= (1 + \log (\sin x)) \cos x$$

Thus
$$\frac{dy}{dx} = y((1 + \log (\sin x)) \cos x) = (1 + \log (\sin x)) (\sin x)^{\sin x} \cos x$$

Example 42 For a positive constant a find $\frac{dy}{dx}$, where

$$y = a^{t+\frac{1}{t}}, \text{ and } x = \left(t+\frac{1}{t}\right)^{a}$$

Solution Observe that both y and x are defined for all real $t \neq 0$. Clearly

$$\frac{dy}{dt} = \frac{d}{dt}\left(a^{t+\frac{1}{t}}\right) = a^{t+\frac{1}{t}}\frac{d}{dt}\left(t+\frac{1}{t}\right)\cdot\log a$$

$$= a^{t+\frac{1}{t}}\left(1-\frac{1}{t^2}\right)\log a$$

Similarly
$$\frac{dx}{dt} = a\left[t+\frac{1}{t}\right]^{a-1}\cdot\frac{d}{dt}\left(t+\frac{1}{t}\right)$$

$$= a\left[t+\frac{1}{t}\right]^{a-1}\cdot\left(1-\frac{1}{t^2}\right)$$

$\frac{dx}{dt} \neq 0$ only if $t \neq \pm 1$. Thus for $t \neq \pm 1$,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a^{t+\frac{1}{t}}\left(1-\frac{1}{t^2}\right)\log a}{a\left[t+\frac{1}{t}\right]^{a-1}\cdot\left(1-\frac{1}{t^2}\right)}$$

$$= \frac{a^{t+\frac{1}{t}}\log a}{a\left(t+\frac{1}{t}\right)^{a-1}}$$

Example 43 Differentiate $\sin^2 x$ w.r.t. $e^{\cos x}$.

Solution Let $u(x) = \sin^2 x$ and $v(x) = e^{\cos x}$. We want to find $\frac{du}{dv} = \frac{du/dx}{dv/dx}$. Clearly

$$\frac{du}{dx} = 2\sin x\cos x \text{ and } \frac{dv}{dx} = e^{\cos x}(-\sin x) = -(\sin x)\, e^{\cos x}$$

Thus
$$\frac{du}{dv} = \frac{2\sin x\cos x}{-\sin x\, e^{\cos x}} = -\frac{2\cos x}{e^{\cos x}}$$

Miscellaneous Exercise on Chapter 5

Differentiate w.r.t. x the function in Exercises 1 to 11.

1. $(3x^2 - 9x + 5)^9$

2. $\sin^3 x + \cos^6 x$

3. $(5x)^{3\cos 2x}$

4. $\sin^{-1}(x\sqrt{x}),\ 0 \le x \le 1$

5. $\dfrac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}},\ -2 < x < 2$

6. $\cot^{-1}\left[\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right],\ 0 < x < \dfrac{\pi}{2}$

7. $(\log x)^{\log x},\ x > 1$

8. $\cos(a\cos x + b\sin x)$, for some constant a and b.

9. $(\sin x - \cos x)^{(\sin x - \cos x)},\ \dfrac{\pi}{4} < x < \dfrac{3\pi}{4}$

10. $x^x + x^a + a^x + a^a$, for some fixed $a > 0$ and $x > 0$

11. $x^{x^2-3}+(x-3)^{x^2}$, for $x > 3$

12. Find $\dfrac{dy}{dx}$, if $y = 12\,(1 - \cos t)$, $x = 10\,(t - \sin t)$, $-\dfrac{\pi}{2}<t<\dfrac{\pi}{2}$

13. Find $\dfrac{dy}{dx}$, if $y = \sin^{-1} x + \sin^{-1}\sqrt{1-x^2}$, $0 < x < 1$

14. If $x\sqrt{1+y}+y\sqrt{1+x}=0$, for , $-1 < x < 1$, prove that

$$\frac{dy}{dx}=-\frac{1}{(1+x)^2}$$

15. If $(x - a)^2 + (y - b)^2 = c^2$, for some $c > 0$, prove that

$$\frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}$$

is a constant independent of a and b.

16. If $\cos y = x \cos (a + y)$, with $\cos a \neq \pm 1$, prove that $\dfrac{dy}{dx}=\dfrac{\cos^2(a+y)}{\sin a}$.

17. If $x = a\,(\cos t + t \sin t)$ and $y = a\,(\sin t - t \cos t)$, find $\dfrac{d^2y}{dx^2}$.

18. If $f(x) = |\,x\,|^3$, show that $f''(x)$ exists for all real x and find it.

19. Using the fact that $\sin(A + B) = \sin A \cos B + \cos A \sin B$ and the differentiation, obtain the sum formula for cosines.

20. Does there exist a function which is continuous everywhere but not differentiable at exactly two points? Justify your answer.

21. If $y=\begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, prove that $\dfrac{dy}{dx}=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

22. If $y = e^{a\cos^{-1}x}$, $-1 \le x \le 1$, show that $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}-a^2y=0$.

Summary

- A real valued function is ***continuous*** at a point in its domain if the limit of the function at that point equals the value of the function at that point. A function is continuous if it is continuous on the whole of its domain.
- Sum, difference, product and quotient of continuous functions are continuous. i.e., if f and g are continuous functions, then

  $(f \pm g)(x) = f(x) \pm g(x)$ is continuous.

  $(f \,.\, g)(x) = f(x) \,.\, g(x)$ is continuous.

  $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ (wherever $g(x) \neq 0$) is continuous.
- Every differentiable function is continuous, but the converse is not true.
- Chain rule is rule to differentiate composites of functions. If $f = v \circ u$, $t = u(x)$ and if both $\dfrac{dt}{dx}$ and $\dfrac{dv}{dt}$ exist then

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$$

- Following are some of the standard derivatives (in appropriate domains):

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(e^x\right) = e^x \qquad \frac{d}{dx}\left(\log x\right) = \frac{1}{x}$$

- Logarithmic differentiation is a powerful technique to differentiate functions of the form $f(x) = [u(x)]^{v(x)}$. Here both $f(x)$ and $u(x)$ need to be positive for this technique to make sense.

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Chapter 6

APPLICATION OF DERIVATIVES

❖ *With the Calculus as a key, Mathematics can be successfully applied to the explanation of the course of Nature." — WHITEHEAD* ❖

6.1 Introduction

In Chapter 5, we have learnt how to find derivative of composite functions, inverse trigonometric functions, implicit functions, exponential functions and logarithmic functions. In this chapter, we will study applications of the derivative in various disciplines, e.g., in engineering, science, social science, and many other fields. For instance, we will learn how the derivative can be used (i) to determine rate of change of quantities, (ii) to find the equations of tangent and normal to a curve at a point, (iii) to find turning points on the graph of a function which in turn will help us to locate points at which largest or smallest value (locally) of a function occurs. We will also use derivative to find intervals on which a function is increasing or decreasing. Finally, we use the derivative to find approximate value of certain quantities.

6.2 Rate of Change of Quantities

Recall that by the derivative $\frac{ds}{dt}$, we mean the rate of change of distance s with respect to the time t. In a similar fashion, whenever one quantity y varies with another quantity x, satisfying some rule $y = f(x)$, then $\frac{dy}{dx}$ (or $f'(x)$) represents the rate of change of y with respect to x and $\left.\frac{dy}{dx}\right]_{x=x_0}$ (or $f'(x_0)$) represents the rate of change of y with respect to x at $x = x_0$.

Further, if two variables x and y are varying with respect to another variable t, i.e., if $x = f(t)$ and $y = g(t)$, then by Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}, \text{ if } \frac{dx}{dt} \neq 0$$

Thus, the rate of change of y with respect to x can be calculated using the rate of change of y and that of x both with respect to t.

Let us consider some examples.

Example 1 Find the rate of change of the area of a circle per second with respect to its radius r when $r = 5$ cm.

Solution The area A of a circle with radius r is given by $A = \pi r^2$. Therefore, the rate of change of the area A with respect to its radius r is given by $\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$.

When $r = 5$ cm, $\frac{dA}{dr} = 10\pi$. Thus, the area of the circle is changing at the rate of 10π cm^2/s.

Example 2 The volume of a cube is increasing at a rate of 9 cubic centimetres per second. How fast is the surface area increasing when the length of an edge is 10 centimetres ?

Solution Let x be the length of a side, V be the volume and S be the surface area of the cube. Then, $V = x^3$ and $S = 6x^2$, where x is a function of time t.

Now $$\frac{dV}{dt} = 9\text{cm}^3/\text{s (Given)}$$

Therefore $$9 = \frac{dV}{dt} = \frac{d}{dt}(x^3) = \frac{d}{dx}(x^3)\cdot\frac{dx}{dt} \quad \text{(By Chain Rule)}$$

$$= 3x^2 \cdot \frac{dx}{dt}$$

or $$\frac{dx}{dt} = \frac{3}{x^2} \qquad \text{... (1)}$$

Now $$\frac{dS}{dt} = \frac{d}{dt}(6x^2) = \frac{d}{dx}(6x^2)\cdot\frac{dx}{dt} \quad \text{(By Chain Rule)}$$

$$= 12x \cdot \left(\frac{3}{x^2}\right) = \frac{36}{x} \quad \text{(Using (1))}$$

Hence, when $x = 10$ cm, $\frac{dS}{dt} = 3.6\ \text{cm}^2/\text{s}$

Example 3 A stone is dropped into a quiet lake and waves move in circles at a speed of 4cm per second. At the instant, when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing?

Solution The area A of a circle with radius r is given by $A = \pi r^2$. Therefore, the rate of change of area A with respect to time t is

$$\frac{dA}{dt} = \frac{d}{dt}(\pi r^2) = \frac{d}{dr}(\pi r^2)\cdot\frac{dr}{dt} = 2\pi r\frac{dr}{dt} \qquad \text{(By Chain Rule)}$$

It is given that $\qquad \dfrac{dr}{dt} = 4\text{cm/s}$

Therefore, when $r = 10$ cm, $\qquad \dfrac{dA}{dt} = 2\pi(10)(4) = 80\pi$

Thus, the enclosed area is increasing at the rate of 80π cm^2/s, when $r = 10$ cm.

Note $\dfrac{dy}{dx}$ is positive if y increases as x increases and is negative if y decreases as x increases.

Example 4 The length x of a rectangle is decreasing at the rate of 3 cm/minute and the width y is increasing at the rate of 2cm/minute. When x =10cm and y = 6cm, find the rates of change of (a) the perimeter and (b) the area of the rectangle.

Solution Since the length x is decreasing and the width y is increasing with respect to time, we have

$$\frac{dx}{dt} = -3 \text{ cm/min} \qquad \text{and} \qquad \frac{dy}{dt} = 2 \text{ cm/min}$$

(a) The perimeter P of a rectangle is given by

$$P = 2(x + y)$$

Therefore $\qquad \dfrac{dP}{dt} = 2\left(\dfrac{dx}{dt} + \dfrac{dy}{dt}\right) = 2(-3+2) = -2 \text{ cm/min}$

(b) The area A of the rectangle is given by

$$A = x \cdot y$$

Therefore $\qquad \dfrac{dA}{dt} = \dfrac{dx}{dt}\cdot y + x\cdot\dfrac{dy}{dt}$

$= -3(6) + 10(2) \qquad$ (as x = 10 cm and y = 6 cm)

$= 2$ cm^2/min

Example 5 The total cost C(x) in Rupees, associated with the production of x units of an item is given by

$$C(x) = 0.005\, x^3 - 0.02\, x^2 + 30x + 5000$$

Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output.

Solution Since marginal cost is the rate of change of total cost with respect to the output, we have

Marginal cost (MC) $= \dfrac{dC}{dx} = 0.005(3x^2) - 0.02(2x) + 30$

When $x = 3$, MC $= 0.015(3^2) - 0.04(3) + 30$

$$= 0.135 - 0.12 + 30 = 30.015$$

Hence, the required marginal cost is ₹ 30.02 (nearly).

Example 6 The total revenue in Rupees received from the sale of x units of a product is given by R(x) = $3x^2 + 36x + 5$. Find the marginal revenue, when $x = 5$, where by marginal revenue we mean the rate of change of total revenue with respect to the number of items sold at an instant.

Solution Since marginal revenue is the rate of change of total revenue with respect to the number of units sold, we have

Marginal Revenue (MR) $= \dfrac{dR}{dx} = 6x + 36$

When $x = 5$, MR $= 6(5) + 36 = 66$

Hence, the required marginal revenue is ₹ 66.

EXERCISE 6.1

1. Find the rate of change of the area of a circle with respect to its radius r when
   (a) $r = 3$ cm (b) $r = 4$ cm
2. The volume of a cube is increasing at the rate of 8 cm³/s. How fast is the surface area increasing when the length of an edge is 12 cm?
3. The radius of a circle is increasing uniformly at the rate of 3 cm/s. Find the rate at which the area of the circle is increasing when the radius is 10 cm.
4. An edge of a variable cube is increasing at the rate of 3 cm/s. How fast is the volume of the cube increasing when the edge is 10 cm long?
5. A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/s. At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing?

6. The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference?

7. The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When $x = 8$cm and $y = 6$cm, find the rates of change of (a) the perimeter, and (b) the area of the rectangle.

8. A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.

9. A balloon, which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the later is 10 cm.

10. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall ?

11. A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y-coordinate is changing 8 times as fast as the x-coordinate.

12. The radius of an air bubble is increasing at the rate of $\frac{1}{2}$ cm/s. At what rate is the volume of the bubble increasing when the radius is 1 cm?

13. A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x+1)$. Find the rate of change of its volume with respect to x.

14. Sand is pouring from a pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

15. The total cost C (x) in Rupees associated with the production of x units of an item is given by

$$C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000.$$

Find the marginal cost when 17 units are produced.

16. The total revenue in Rupees received from the sale of x units of a product is given by

$$R(x) = 13x^2 + 26x + 15.$$

Find the marginal revenue when $x = 7$.

Choose the correct answer for questions 17 and 18.

17. The rate of change of the area of a circle with respect to its radius r at $r = 6$ cm is

(A) 10π (B) 12π (C) 8π (D) 11π

18. The total revenue in Rupees received from the sale of x units of a product is given by

$R(x) = 3x^2 + 36x + 5$. The marginal revenue, when $x = 15$ is

(A) 116 (B) 96 (C) 90 (D) 126

6.3 Increasing and Decreasing Functions

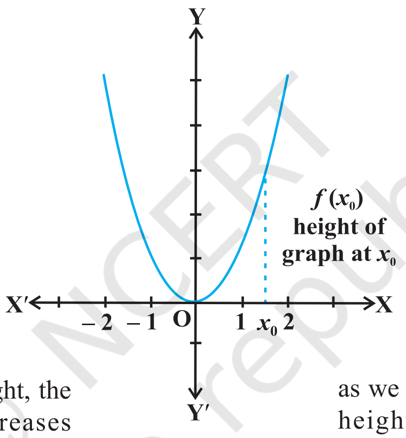
In this section, we will use differentiation to find out whether a function is increasing or decreasing or none.

Consider the function f given by $f(x) = x^2$, $x \in \mathbf{R}$. The graph of this function is a parabola as given in Fig 6.1.

Values left to origin

x	$f(x) = x^2$
-2	4
$-\frac{3}{2}$	$\frac{9}{4}$
-1	1
$-\frac{1}{2}$	$\frac{1}{4}$
0	0

as we move from left to right, the height of the graph decreases



Values right to origin

x	$f(x) = x^2$
0	0
$\frac{1}{2}$	$\frac{1}{4}$
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

as we move from left to right, the height of the graph increases

Fig 6.1

First consider the graph (Fig 6.1) to the right of the origin. Observe that as we move from left to right along the graph, the height of the graph continuously increases. For this reason, the function is said to be increasing for the real numbers $x > 0$.

Now consider the graph to the left of the origin and observe here that as we move from left to right along the graph, the height of the graph continuously decreases. Consequently, the function is said to be decreasing for the real numbers $x < 0$.

We shall now give the following analytical definitions for a function which is increasing or decreasing on an interval.

Definition 1 Let I be an interval contained in the domain of a real valued function f. Then f is said to be

(i) increasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in$ I.

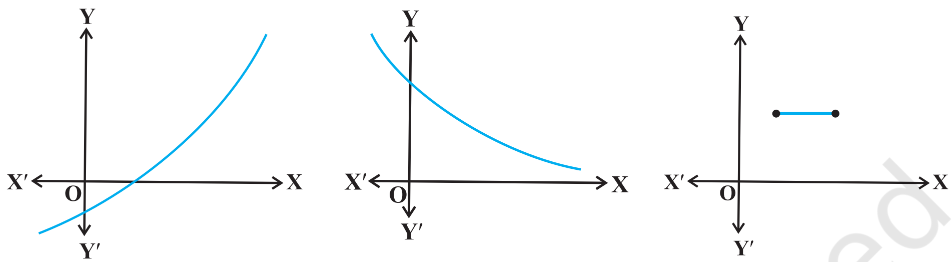
(ii) decreasing on I, if x_1, x_2 in I $\Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in$ I.

(iii) constant on I, if $f(x) = c$ for all $x \in$ I, where c is a constant.

(iv) decreasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in$ I.

(v) strictly decreasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in$ I.

For graphical representation of such functions see Fig 6.2.



Strictly Increasing function
(i)

Strictly Decreasing function
(ii)

Neither Increasing nor Decreasing function
(iii)

Fig 6.2

We shall now define when a function is increasing or decreasing at a point.

Definition 2 Let x_0 be a point in the domain of definition of a real valued function f. Then f is said to be increasing, decreasing at x_0 if there exists an open interval I containing x_0 such that f is increasing, decreasing, respectively, in I.

Let us clarify this definition for the case of increasing function.

Example 7 Show that the function given by $f(x) = 7x - 3$ is increasing on **R**.

Solution Let x_1 and x_2 be any two numbers in **R**. Then

$$x_1 < x_2 \Rightarrow 7x_1 < 7x_2 \Rightarrow 7x_1 - 3 < 7x_2 - 3 \Rightarrow f(x_1) < f(x_2)$$

Thus, by Definition 1, it follows that f is strictly increasing on **R**.

We shall now give the first derivative test for increasing and decreasing functions. The proof of this test requires the Mean Value Theorem studied in Chapter 5.

Theorem 1 Let f be continuous on $[a, b]$ and differentiable on the open interval (a,b). Then

(a) f is increasing in $[a,b]$ if $f'(x) > 0$ for each $x \in (a, b)$

(b) f is decreasing in $[a,b]$ if $f'(x) < 0$ for each $x \in (a, b)$

(c) f is a constant function in $[a,b]$ if $f'(x) = 0$ for each $x \in (a, b)$

Proof (a) Let $x_1, x_2 \in [a, b]$ be such that $x_1 < x_2$.

Then, by Mean Value Theorem (Theorem 8 in Chapter 5), there exists a point c between x_1 and x_2 such that

$$f(x_2) - f(x_1) = f'(c)\ (x_2 - x_1)$$

i.e. $$f(x_2) - f(x_1) > 0 \qquad \text{(as } f'(c) > 0 \text{ (given))}$$

i.e. $$f(x_2) > f(x_1)$$

Thus, we have

$$x_1 < x_2 \quad f(x_1) \quad f(x_2), \text{ for all } x_1, x_2 \quad [a,b]$$

Hence, f is an increasing function in $[a,b]$.

The proofs of part (b) and (c) are similar. It is left as an exercise to the reader.

Remarks

There is a more generalised theorem, which states that if *f*¢(x) > 0 for x in an interval excluding the end points and f is continuous in the interval, then f is increasing. Similarly, if *f*¢(x) < 0 for x in an interval excluding the end points and f is continuous in the interval, then f is decreasing.

Example 8 Show that the function f given by

$$f(x) = x^3 - 3x^2 + 4x,\ x \in \mathbf{R}$$

is increasing on **R**.

Solution Note that

$$\begin{aligned} f'(x) &= 3x^2 - 6x + 4 \\ &= 3(x^2 - 2x + 1) + 1 \\ &= 3(x - 1)^2 + 1 > 0, \text{ in every interval of } \mathbf{R} \end{aligned}$$

Therefore, the function f is increasing on **R**.

Example 9 Prove that the function given by $f(x) = \cos x$ is

(a) decreasing in $(0, \pi)$

(b) increasing in $(\pi, 2\pi)$, and

(c) neither increasing nor decreasing in $(0, 2\pi)$.

Solution Note that $f'(x) = -\sin x$

(a) Since for each $x \in (0, \pi)$, $\sin x > 0$, we have $f'(x) < 0$ and so f is decreasing in $(0, \pi)$.

(b) Since for each $x \in (\pi, 2\pi)$, $\sin x < 0$, we have $f'(x) > 0$ and so f is increasing in $(\pi, 2\pi)$.

(c) Clearly by (a) and (b) above, f is neither increasing nor decreasing in $(0, 2\pi)$.

Example 10 Find the intervals in which the function f given by $f(x) = x^2 - 4x + 6$ is

(a) increasing (b) decreasing

Solution We have

$$f(x) = x^2 - 4x + 6$$

or

$$f'(x) = 2x - 4$$

Fig 6.3

Therefore, $f'(x) = 0$ gives $x = 2$. Now the point $x = 2$ divides the real line into two disjoint intervals namely, $(-\infty, 2)$ and $(2, \infty)$ (Fig 6.3). In the interval $(-\infty, 2)$, $f'(x) = 2x - 4 < 0$.

Therefore, f is decreasing in this interval. Also, in the interval $(2, \infty)$, $f'(x) > 0$ and so the function f is increasing in this interval.

Example 11 Find the intervals in which the function f given by $f(x) = 4x^3 - 6x^2 - 72x + 30$ is (a) increasing (b) decreasing.

Solution We have

$$f(x) = 4x^3 - 6x^2 - 72x + 30$$

or

$$\begin{aligned} f'(x) &= 12x^2 - 12x - 72 \\ &= 12(x^2 - x - 6) \\ &= 12(x - 3)(x + 2) \end{aligned}$$

Therefore, $f'(x) = 0$ gives $x = -2, 3$. The points $x = -2$ and $x = 3$ divides the real line into three disjoint intervals, namely, $(-\infty, -2)$, $(-2, 3)$ and $(3, \infty)$.

Fig 6.4

In the intervals $(-\infty, -2)$ and $(3, \infty)$, $f'(x)$ is positive while in the interval $(-2, 3)$, $f'(x)$ is negative. Consequently, the function f is increasing in the intervals $(-\infty, -2)$ and $(3, \infty)$ while the function is decreasing in the interval $(-2, 3)$. However, f is neither increasing nor decreasing in **R**.

Interval	Sign of $f'(x)$	Nature of function f
$(-\infty, -2)$	$(-)\ (-) > 0$	f is increasing
$(-2, 3)$	$(-)\ (+) < 0$	f is decreasing
$(3, \infty)$	$(+)\ (+) > 0$	f is increasing

Example 12 Find intervals in which the function given by $f(x) = \sin 3x$, $x \in \left[0, \frac{\pi}{2}\right]$ is (a) increasing (b) decreasing.

Solution We have

$$f(x) = \sin 3x$$

or

$$f'(x) = 3\cos 3x$$

Therefore, $f'(x) = 0$ gives $\cos 3x = 0$ which in turn gives $3x = \frac{\pi}{2}, \frac{3\pi}{2}$ (as $x \in \left[0, \frac{\pi}{2}\right]$ implies $3x \in \left[0, \frac{3\pi}{2}\right]$). So $x = \frac{\pi}{6}$ and $\frac{\pi}{2}$. The point $x = \frac{\pi}{6}$ divides the interval $\left[0, \frac{\pi}{2}\right]$ into two disjoint intervals $\left[0, \frac{\pi}{6}\right)$ and $\left(\frac{\pi}{6}, \frac{\pi}{2}\right]$.

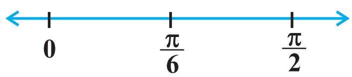


Fig 6.5

Now, $f'(x) > 0$ for all $x \in \left[0, \frac{\pi}{6}\right)$ as $0 \le x < \frac{\pi}{6} \Rightarrow 0 \le 3x < \frac{\pi}{2}$ and $f'(x) < 0$ for all $x \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right)$ as $\frac{\pi}{6} < x < \frac{\pi}{2} \Rightarrow \frac{\pi}{2} < 3x < \frac{3\pi}{2}$.

Therefore, f is increasing in $\left[0, \frac{\pi}{6}\right)$ and decreasing in $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$.

Also, the given function is continuous at $x = 0$ and $x = \frac{\pi}{6}$. Therefore, by Theorem 1, f is increasing on $\left[0, \frac{\pi}{6}\right]$ and decreasing on $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$.

Example 13 Find the intervals in which the function f given by

$$f(x) = \sin x + \cos x,\ 0 \le x \le 2\pi$$

is increasing or decreasing.

Solution We have

$$f(x) = \sin x + \cos x,$$

or

$$f'(x) = \cos x - \sin x$$

Now $f'(x) = 0$ gives $\sin x = \cos x$ which gives that $x = \frac{\pi}{4}, \frac{5\pi}{4}$ as $0 \le x \le 2\pi$

The points $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ divide the interval $[0, 2\pi]$ into three disjoint intervals, namely, $\left[0, \frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right]$.

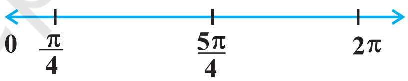


Fig 6.6

Note that $\quad f'(x) > 0$ if $x \in \left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$

or $\quad f$ is increasing in the intervals $\left[0, \frac{\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right]$

Also $\quad f'(x) < 0$ if $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

or $\quad f$ is decreasing in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

Interval	Sign of $f'(x)$	Nature of function
$\left[0, \frac{\pi}{4}\right)$	> 0	f is increasing
$\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$	< 0	f is decreasing
$\left(\frac{5\pi}{4}, 2\pi\right]$	> 0	f is increasing

EXERCISE 6.2

1. Show that the function given by $f(x) = 3x + 17$ is increasing on **R**.
2. Show that the function given by $f(x) = e^{2x}$ is increasing on **R**.
3. Show that the function given by $f(x) = \sin x$ is

   (a) increasing in $\left(0, \frac{\pi}{2}\right)$ (b) decreasing in $\left(\frac{\pi}{2}, \pi\right)$

   (c) neither increasing nor decreasing in $(0, \pi)$
4. Find the intervals in which the function f given by $f(x) = 2x^2 - 3x$ is

   (a) increasing (b) decreasing
5. Find the intervals in which the function f given by $f(x) = 2x^3 - 3x^2 - 36x + 7$ is

   (a) increasing (b) decreasing
6. Find the intervals in which the following functions are strictly increasing or decreasing:

   (a) $x^2 + 2x - 5$ (b) $10 - 6x - 2x^2$

   (c) $-2x^3 - 9x^2 - 12x + 1$ (d) $6 - 9x - x^2$

   (e) $(x + 1)^3 (x - 3)^3$
7. Show that $y = \log(1+x) - \frac{2x}{2+x}$, $x > -1$, is an increasing function of x throughout its domain.
8. Find the values of x for which $y = [x(x - 2)]^2$ is an increasing function.
9. Prove that $y = \frac{4\sin\theta}{(2+\cos\theta)} - \theta$ is an increasing function of θ in $\left[0, \frac{\pi}{2}\right]$.

10. Prove that the logarithmic function is increasing on $(0, \infty)$.

11. Prove that the function f given by $f(x) = x^2 - x + 1$ is neither strictly increasing nor decreasing on $(-1, 1)$.

12. Which of the following functions are decreasing on $0, \dfrac{\pi}{2}$?

(A) $\cos x$ (B) $\cos 2x$ (C) $\cos 3x$ (D) $\tan x$

13. On which of the following intervals is the function f given by $f(x) = x^{100} + \sin x - 1$ decreasing ?

(A) (0,1) (B) $\dfrac{\pi}{2}, \pi$ (C) $0, \dfrac{\pi}{2}$ (D) None of these

14. For what values of a the function f given by $f(x) = x^2 + ax + 1$ is increasing on [1, 2]?

15. Let I be any interval disjoint from $[-1, 1]$. Prove that the function f given by $f(x) = x + \dfrac{1}{x}$ is increasing on I.

16. Prove that the function f given by $f(x) = \log \sin x$ is increasing on $\left(0, \dfrac{\pi}{2}\right)$ and decreasing on $\left(\dfrac{\pi}{2}, \pi\right)$.

17. Prove that the function f given by $f(x) = \log |\cos x|$ is decreasing on $\left(0, \dfrac{\pi}{2}\right)$ and increasing on $\left(\dfrac{3\pi}{2}, 2\pi\right)$.

18. Prove that the function given by $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing in **R**.

19. The interval in which $y = x^2 e^{-x}$ is increasing is

(A) $(-\infty, \infty)$ (B) $(-2, 0)$ (C) $(2, \infty)$ (D) $(0, 2)$

6.4 Maxima and Minima

In this section, we will use the concept of derivatives to calculate the maximum or minimum values of various functions. In fact, we will find the 'turning points' of the graph of a function and thus find points at which the graph reaches its highest (or lowest) *locally*. The knowledge of such points is very useful in sketching the graph of a given function. Further, we will also find the absolute maximum and absolute minimum of a function that are necessary for the solution of many applied problems.

Let us consider the following problems that arise in day to day life.

(i) The profit from a grove of orange trees is given by $P(x) = ax + bx^2$, where a,b are constants and x is the number of orange trees per acre. How many trees per acre will maximise the profit?

(ii) A ball, thrown into the air from a building 60 metres high, travels along a path given by $h(x) = 60 + x - \frac{x^2}{60}$, where x is the horizontal distance from the building and $h(x)$ is the height of the ball . What is the maximum height the ball will reach?

(iii) An Apache helicopter of enemy is flying along the path given by the curve $f(x) = x^2 + 7$. A soldier, placed at the point (1, 2), wants to shoot the helicopter when it is nearest to him. What is the nearest distance?

In each of the above problem, there is something common, i.e., we wish to find out the maximum or minimum values of the given functions. In order to tackle such problems, we first formally define maximum or minimum values of a function, points of local maxima and minima and test for determining such points.

Definition 3 Let f be a function defined on an interval I. Then

(a) f is said to have a *maximum value* in I, if there exists a point c in I such that $f(c) > f(x)$, for all $x \in$ I.

The number $f(c)$ is called the maximum value of f in I and the point c is called a *point of maximum value* of f in I.

(b) f is said to have a minimum value in I, if there exists a point c in I such that $f(c) < f(x)$, for all $x \in$ I.

The number $f(c)$, in this case, is called the minimum value of f in I and the point c, in this case, is called a *point of minimum value* of f in I.

(c) f is said to have an extreme value in I if there exists a point c in I such that $f(c)$ is either a maximum value or a minimum value of f in I.

The number $f(c)$, in this case, is called an *extreme value* of f in I and the point c is called an *extreme point*.

Remark In Fig 6.7(a), (b) and (c), we have exhibited that graphs of certain particular functions help us to find maximum value and minimum value at a point. Infact, through graphs, we can even find maximum/minimum value of a function at a point at which it is not even differentiable (Example 15).

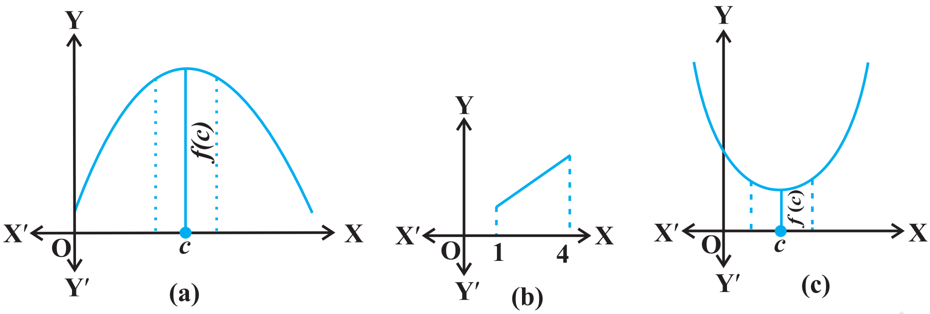


Fig 6.7

Example 14 Find the maximum and the minimum values, if any, of the function f given by

$$f(x) = x^2, x \in \mathbf{R}.$$

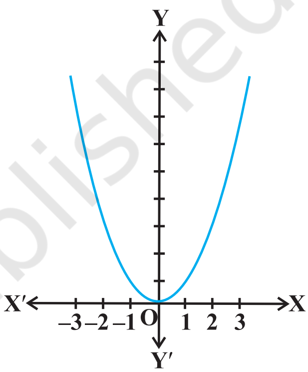


Fig 6.8

Solution From the graph of the given function (Fig 6.8), we have $f(x) = 0$ if $x = 0$. Also

$$f(x) \geq 0, \text{ for all } x \in \mathbf{R}.$$

Therefore, the minimum value of f is 0 and the point of minimum value of f is $x = 0$. Further, it may be observed from the graph of the function that f has no maximum value and hence no point of maximum value of f in **R**.

> **Note** If we restrict the domain of f to $[-2, 1]$ only, then f will have maximum value $(-2)^2 = 4$ at $x = -2$.

Example 15 Find the maximum and minimum values of f, if any, of the function given by $f(x) = |x|, x \in \mathbf{R}$.

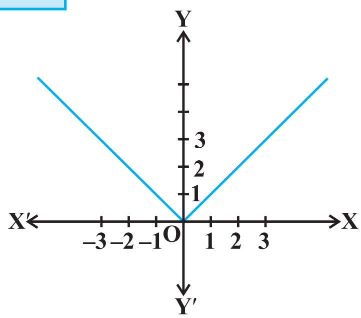


Fig 6.9

Solution From the graph of the given function (Fig 6.9) , note that

$$f(x) \geq 0, \text{ for all } x \in \mathbf{R} \text{ and } f(x) = 0 \text{ if } x = 0.$$

Therefore, the function f has a minimum value 0 and the point of minimum value of f is $x = 0$. Also, the graph clearly shows that f has no maximum value in **R** and hence no point of maximum value in **R**.

> **Note**
>
> (i) If we restrict the domain of f to $[-2, 1]$ only, then f will have maximum value $|-2| = 2$.

(ii) One may note that the function f in Example 27 is not differentiable at $x = 0$.

Example 16 Find the maximum and the minimum values, if any, of the function given by

$$f(x) = x, x \in (0, 1).$$

Solution The given function is an increasing (strictly) function in the given interval (0, 1). From the graph (Fig 6.10) of the function f, it seems that, it should have the minimum value at a point closest to 0 on its right and the maximum value at a point closest to 1 on its left. Are such points available? Of course, not. It is not possible to locate such points. Infact, if a point x_0 is closest to 0, then we find $\frac{x_0}{2} < x_0$ for all $x_0 \in (0,1)$. Also, if x_1 is closest to 1, then $\frac{x_1+1}{2} > x_1$ for all $x_1 \in (0,1)$.

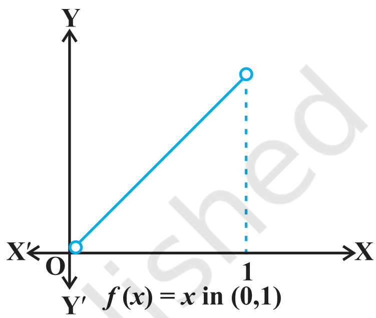


Fig 6.10

Therefore, the given function has neither the maximum value nor the minimum value in the interval (0,1).

Remark The reader may observe that in Example 16, if we include the points 0 and 1 in the domain of f, i.e., if we extend the domain of f to [0,1], then the function f has minimum value 0 at $x = 0$ and maximum value 1 at $x = 1$. Infact, we have the following results (The proof of these results are beyond the scope of the present text)

Every monotonic function assumes its maximum/minimum value at the end points of the domain of definition of the function.

A more general result is

Every continuous function on a closed interval has a maximum and a minimum value.

Note By a monotonic function f in an interval I, we mean that f is either increasing in I or decreasing in I.

Maximum and minimum values of a function defined on a closed interval will be discussed later in this section.

Let us now examine the graph of a function as shown in Fig 6.11. Observe that at points A, B, C and D on the graph, the function changes its nature from decreasing to increasing or vice-versa. These points may be called *turning points* of the given function. Further, observe that at turning points, the graph has either a little hill or a little valley. Roughly speaking, the function has minimum value in some neighbourhood (interval) of each of the points A and C which are at the bottom of their respective

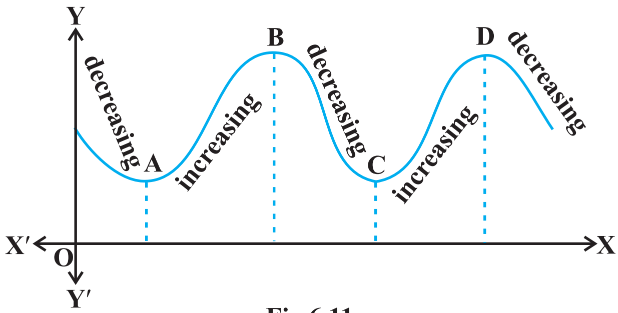


Fig 6.11

valleys. Similarly, the function has maximum value in some neighbourhood of points B and D which are at the top of their respective hills. For this reason, the points A and C may be regarded as points of *local minimum value* (or *relative minimum value*) and points B and D may be regarded as points of *local maximum value* (or *relative maximum value*) for the function. The *local maximum value* and *local minimum value* of the function are referred to as *local maxima* and *local minima*, respectively, of the function.

We now formally give the following definition

Definition 4 Let f be a real valued function and let c be an interior point in the domain of f. Then

(a) c is called a point of *local maxima* if there is an $h > 0$ such that

$$f(c) \geq f(x), \text{ for all } x \text{ in } (c - h, c + h), x \neq c$$

The value $f(c)$ is called the *local maximum value* of f.

(b) c is called a point of *local minima* if there is an $h > 0$ such that

$$f(c) \leq f(x), \text{ for all } x \text{ in } (c - h, c + h)$$

The value $f(c)$ is called the *local minimum value* of f.

Geometrically, the above definition states that if $x = c$ is a point of local maxima of f, then the graph of f around c will be as shown in Fig 6.12(a). Note that the function f is increasing (i.e., $f'(x) > 0$) in the interval $(c - h, c)$ and decreasing (i.e., $f'(x) < 0$) in the interval $(c, c + h)$.

This suggests that $f'(c)$ must be zero.

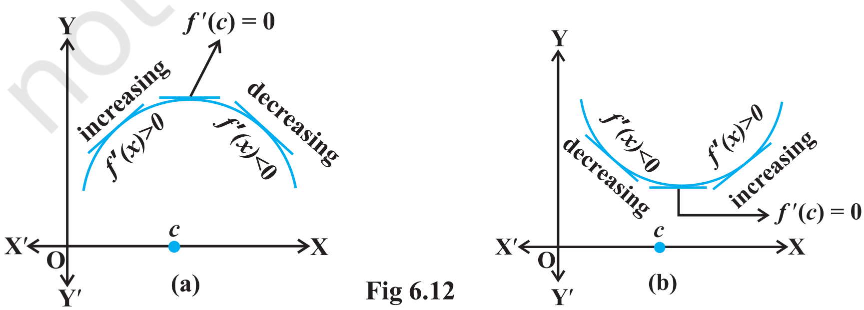


Fig 6.12

Similarly, if c is a point of local minima of f, then the graph of f around c will be as shown in Fig 6.14(b). Here f is decreasing (i.e., $f'(x) < 0$) in the interval $(c - h, c)$ and increasing (i.e., $f'(x) > 0$) in the interval $(c, c + h)$. This again suggest that $f'(c)$ must be zero.

The above discussion lead us to the following theorem (without proof).

Theorem 2 Let f be a function defined on an open interval I. Suppose $c \in$ I be any point. If f has a local maxima or a local minima at $x = c$, then either $f'(c) = 0$ or f is not differentiable at c.

Remark The converse of above theorem need not be true, that is, a point at which the derivative vanishes need not be a point of local maxima or local minima. For example, if $f(x) = x^3$, then $f'(x) = 3x^2$ and so $f'(0) = 0$. But 0 is neither a point of local maxima nor a point of local minima (Fig 6.13).

Note A point c in the domain of a function f at which either $f'(c) = 0$ or f is not differentiable is called a *critical point* of f. Note that if f is continuous at c and $f'(c) = 0$, then there exists an $h > 0$ such that f is differentiable in the interval $(c - h, c + h)$.

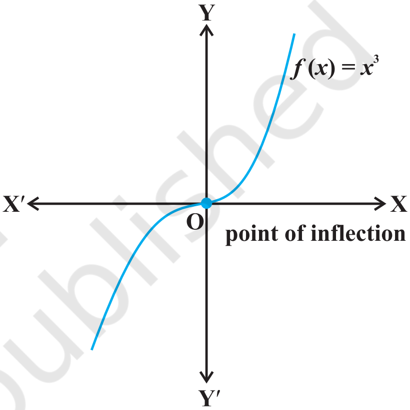


Fig 6.13

We shall now give a working rule for finding points of local maxima or points of local minima using only the first order derivatives.

Theorem 3 (First Derivative Test) Let f be a function defined on an open interval I. Let f be continuous at a critical point c in I. Then

(i) If $f'(x)$ changes sign from positive to negative as x increases through c, i.e., if $f'(x) > 0$ at every point sufficiently close to and to the left of c, and $f'(x) < 0$ at every point sufficiently close to and to the right of c, then c is a point of *local maxima*.

(ii) If $f'(x)$ changes sign from negative to positive as x increases through c, i.e., if $f'(x) < 0$ at every point sufficiently close to and to the left of c, and $f'(x) > 0$ at every point sufficiently close to and to the right of c, then c is a point of *local minima*.

(iii) If $f'(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Infact, such a point is called *point of inflection* (Fig 6.13).

Note If c is a point of local maxima of f, then $f(c)$ is a local maximum value of f. Similarly, if c is a point of local minima of f, then $f(c)$ is a local minimum value of f.

Figures 6.13 and 6.14, geometrically explain Theorem 3.

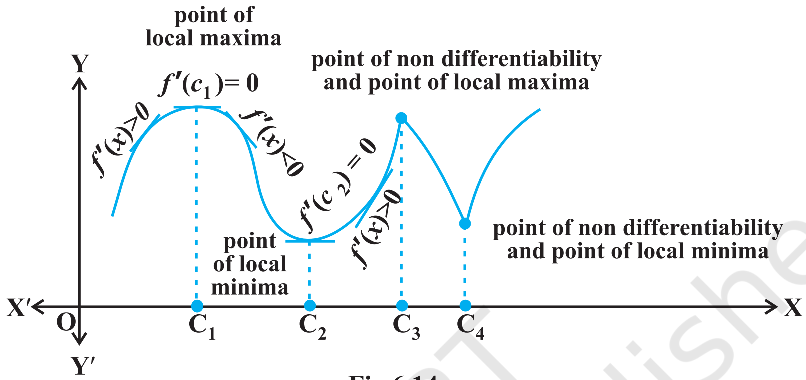


Fig 6.14

Example 17 Find all points of local maxima and local minima of the function f given by

$$f(x) = x^3 - 3x + 3.$$

Solution We have

$$f(x) = x^3 - 3x + 3$$

or $\quad f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$

or $\quad f'(x) = 0$ at $x = 1$ and $x = -1$

Thus, $x = \pm 1$ are the only critical points which could possibly be the points of local maxima and/or local minima of f. Let us first examine the point $x = 1$.

Note that for values close to 1 and to the right of 1, $f'(x) > 0$ and for values close to 1 and to the left of 1, $f'(x) < 0$. Therefore, by first derivative test, $x = 1$ is a point of local minima and local minimum value is $f(1) = 1$. In the case of $x = -1$, note that $f'(x) > 0$, for values close to and to the left of -1 and $f'(x) < 0$, for values close to and to the right of -1. Therefore, by first derivative test, $x = -1$ is a point of local maxima and local maximum value is $f(-1) = 5$.

Values of x		Sign of $f'(x) = 3(x-1)(x+1)$
Close to 1	to the right (say 1.1 etc.)	>0
	to the left (say 0.9 etc.)	<0
Close to -1	to the right (say -0.9 etc.)	<0
	to the left (say -1.1 etc.)	>0

Example 18 Find all the points of local maxima and local minima of the function f given by

$$f(x) = 2x^3 - 6x^2 + 6x + 5.$$

Solution We have

$$f(x) = 2x^3 - 6x^2 + 6x + 5$$

or $$f'(x) = 6x^2 - 12x + 6 = 6(x-1)^2$$

or $$f'(x) = 0 \quad \text{at} \quad x = 1$$

Thus, $x = 1$ is the only critical point of f. We shall now examine this point for local maxima and/or local minima of f. Observe that $f'(x) \geq 0$, for all $x \in \mathbf{R}$ and in particular $f'(x) > 0$, for values close to 1 and to the left and to the right of 1. Therefore, by first derivative test, the point $x = 1$ is neither a point of local maxima nor a point of local minima. Hence $x = 1$ is a point of inflexion.

Remark One may note that since $f'(x)$, in Example 30, never changes its sign on **R**, graph of f has no turning points and hence no point of local maxima or local minima.

We shall now give another test to examine local maxima and local minima of a given function. This test is often easier to apply than the first derivative test.

Theorem 4 (Second Derivative Test) Let f be a function defined on an interval I and $c \in$ I. Let f be twice differentiable at c. Then

(i) $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$

The value $f(c)$ is local maximum value of f.

(ii) $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$

In this case, $f(c)$ is local minimum value of f.

(iii) The test fails if $f'(c) = 0$ and $f''(c) = 0$.

In this case, we go back to the first derivative test and find whether c is a point of local maxima, local minima or a point of inflexion.

Note As f is twice differentiable at c, we mean second order derivative of f exists at c.

Example 19 Find local minimum value of the function f given by $f(x) = 3 + |x|$, $x \in \mathbf{R}$.

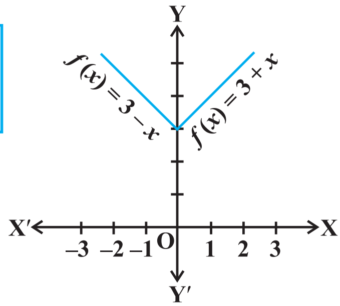


Fig 6.15

Solution Note that the given function is not differentiable at $x = 0$. So, second derivative test fails. Let us try first derivative test. Note that 0 is a critical point of f. Now to the left of 0, $f(x) = 3 - x$ and so $f'(x) = -1 < 0$. Also to the right of 0, $f(x) = 3 + x$ and so $f'(x) = 1 > 0$. Therefore, by first derivative test, $x = 0$ is a point of local minima of f and local minimum value of f is $f(0) = 3$.

Example 20 Find local maximum and local minimum values of the function f given by

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 12$$

Solution We have

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 12$$

or $\quad f'(x) = 12x^3 + 12x^2 - 24x = 12x(x-1)(x+2)$

or $\quad f'(x) = 0$ at $x = 0$, $x = 1$ and $x = -2$.

Now $\quad f''(x) = 36x^2 + 24x - 24 = 12(3x^2 + 2x - 2)$

or
$$\begin{cases} f''(0) &= -24 < 0 \\ f''(1) &= 36 > 0 \\ f''(-2) &= 72 > 0 \end{cases}$$

Therefore, by second derivative test, $x = 0$ is a point of local maxima and local maximum value of f at $x = 0$ is $f(0) = 12$ while $x = 1$ and $x = -2$ are the points of local minima and local minimum values of f at $x = -1$ and -2 are $f(1) = 7$ and $f(-2) = -20$, respectively.

Example 21 Find all the points of local maxima and local minima of the function f given by

$$f(x) = 2x^3 - 6x^2 + 6x + 5.$$

Solution We have

$$f(x) = 2x^3 - 6x^2 + 6x + 5$$

or
$$\begin{cases} f'(x) = 6x^2 - 12x + 6 = 6(x-1)^2 \\ f''(x) = 12(x-1) \end{cases}$$

Now $f'(x) = 0$ gives $x = 1$. Also $f''(1) = 0$. Therefore, the second derivative test fails in this case. So, we shall go back to the first derivative test.

We have already seen (Example 18) that, using first derivative test, $x = 1$ is neither a point of local maxima nor a point of local minima and so it is a point of inflexion.

Example 22 Find two positive numbers whose sum is 15 and the sum of whose squares is minimum.

Solution Let one of the numbers be x. Then the other number is $(15 - x)$. Let $S(x)$ denote the sum of the squares of these numbers. Then

$$S(x) = x^2 + (15 - x)^2 = 2x^2 - 30x + 225$$

or $$\begin{cases} S'(x) = 4x - 30 \\ S''(x) = 4 \end{cases}$$

Now $S'(x) = 0$ gives $x = \frac{15}{2}$. Also $S''\left(\frac{15}{2}\right) = 4 > 0$. Therefore, by second derivative test, $x = \frac{15}{2}$ is the point of local minima of S. Hence the sum of squares of numbers is minimum when the numbers are $\frac{15}{2}$ and $15 - \frac{15}{2} = \frac{15}{2}$.

Remark Proceeding as in Example 34 one may prove that the two positive numbers, whose sum is k and the sum of whose squares is minimum, are $\frac{k}{2}$ and $\frac{k}{2}$.

Example 23 Find the shortest distance of the point $(0, c)$ from the parabola $y = x^2$, where $\frac{1}{2} \le c \le 5$.

Solution Let (h, k) be any point on the parabola $y = x^2$. Let D be the required distance between (h, k) and $(0, c)$. Then

$$D = \sqrt{(h-0)^2 + (k-c)^2} = \sqrt{h^2 + (k-c)^2} \qquad \text{... (1)}$$

Since (h, k) lies on the parabola $y = x^2$, we have $k = h^2$. So (1) gives

$$D \equiv D(k) = \sqrt{k + (k-c)^2}$$

or $$D'(k) = \frac{1 + 2(k-c)}{2\sqrt{k + (k-c)^2}}$$

Now $$D'(k) = 0 \text{ gives } k = \frac{2c-1}{2}$$

Observe that when $k < \frac{2c-1}{2}$, then $2(k-c) + 1 < 0$, i.e., $D'(k) < 0$. Also when $k > \frac{2c-1}{2}$, then $D'(k) > 0$. So, by first derivative test, D (k) is minimum at $k = \frac{2c-1}{2}$.

Hence, the required shortest distance is given by

$$D\left(\frac{2c-1}{2}\right)=\sqrt{\frac{2c-1}{2}+\left(\frac{2c-1}{2}-c\right)^2}=\frac{\sqrt{4c-1}}{2}$$

Note The reader may note that in Example 35, we have used first derivative test instead of the second derivative test as the former is easy and short.

Example 24 Let AP and BQ be two vertical poles at points A and B, respectively. If AP = 16 m, BQ = 22 m and AB = 20 m, then find the distance of a point R on AB from the point A such that $RP^2 + RQ^2$ is minimum.

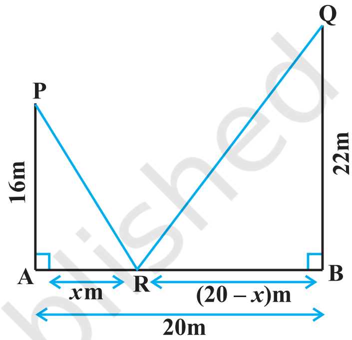


Fig 6.16

Solution Let R be a point on AB such that AR = x m. Then RB = $(20 - x)$ m (as AB = 20 m). From Fig 6.16, we have

$$RP^2 = AR^2 + AP^2$$

and $$RQ^2 = RB^2 + BQ^2$$

Therefore
$$\begin{aligned} RP^2 + RQ^2 &= AR^2 + AP^2 + RB^2 + BQ^2 \\ &= x^2 + (16)^2 + (20 - x)^2 + (22)^2 \\ &= 2x^2 - 40x + 1140 \end{aligned}$$

Let $$S \equiv S(x) = RP^2 + RQ^2 = 2x^2 - 40x + 1140.$$

Therefore $$S'(x) = 4x - 40.$$

Now $S'(x) = 0$ gives $x = 10$. Also $S''(x) = 4 > 0$, for all x and so $S''(10) > 0$. Therefore, by second derivative test, $x = 10$ is the point of local minima of S. Thus, the distance of R from A on AB is AR = x =10 m.

Example 25 If length of three sides of a trapezium other than base are equal to 10cm, then find the area of the trapezium when it is maximum.

Solution The required trapezium is as given in Fig 6.17. Draw perpendiculars DP and

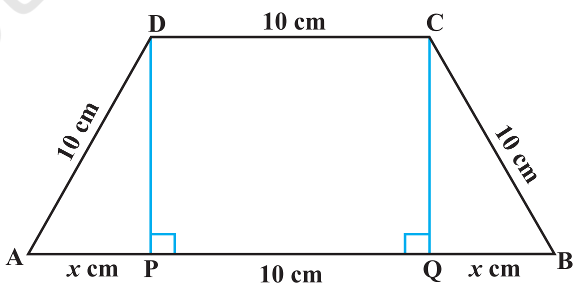


Fig 6.17

CQ on AB. Let AP = x cm. Note that ΔAPD ~ ΔBQC. Therefore, QB = x cm. Also, by Pythagoras theorem, DP = QC = $\sqrt{100-x^2}$. Let A be the area of the trapezium. Then

$$A \equiv A(x) = \frac{1}{2}\text{(sum of parallel sides) (height)}$$

$$= \frac{1}{2}(2x+10+10)\left(\sqrt{100-x^2}\right)$$

$$= (x+10)\left(\sqrt{100-x^2}\right)$$

or $$A'(x) = (x+10)\frac{(-2x)}{2\sqrt{100-x^2}} + \left(\sqrt{100-x^2}\right)$$

$$= \frac{-2x^2-10x+100}{\sqrt{100-x^2}}$$

Now $A'(x) = 0$ gives $2x^2 + 10x - 100 = 0$, i.e., $x = 5$ and $x = -10$.
Since x represents distance, it can not be negative.
So, $x = 5$. Now

$$A''(x) = \frac{\sqrt{100-x^2}\,(-4x-10) - (-2x^2-10x+100)\dfrac{(-2x)}{2\sqrt{100-x^2}}}{100-x^2}$$

$$= \frac{2x^3-300x-1000}{(100-x^2)^{\frac{3}{2}}} \text{ (on simplification)}$$

or $$A''(5) = \frac{2(5)^3-300(5)-1000}{(100-(5)^2)^{\frac{3}{2}}} = \frac{-2250}{75\sqrt{75}} = \frac{-30}{\sqrt{75}} < 0$$

Thus, area of trapezium is maximum at $x = 5$ and the area is given by

$$A(5) = (5+10)\sqrt{100-(5)^2} = 15\sqrt{75} = 75\sqrt{3}\text{ cm}^2$$

Example 26 Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

Solution Let OC = r be the radius of the cone and OA = h be its height. Let a cylinder with radius OE = x inscribed in the given cone (Fig 6.18). The height QE of the cylinder is given by

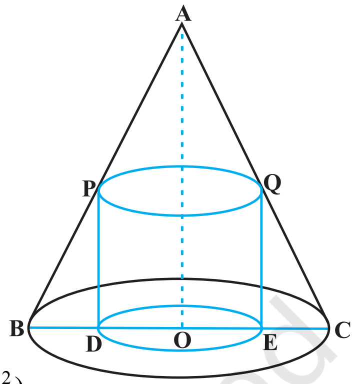


Fig 6.18

$$\frac{QE}{OA} = \frac{EC}{OC} \quad \text{(since } \Delta QEC \sim \Delta AOC\text{)}$$

or $$\frac{QE}{h} = \frac{r-x}{r}$$

or $$QE = \frac{h(r-x)}{r}$$

Let S be the curved surface area of the given cylinder. Then

$$S \equiv S(x) = \frac{2\pi x h(r-x)}{r} = \frac{2\pi h}{r}(rx - x^2)$$

or $$\begin{cases} S'(x) = \dfrac{2\pi h}{r}(r-2x) \\ S''(x) = \dfrac{-4\pi h}{r} \end{cases}$$

Now $S'(x) = 0$ gives $x = \frac{r}{2}$. Since $S''(x) < 0$ for all x, $S''\left(\frac{r}{2}\right) < 0$. So $x = \frac{r}{2}$ is a point of maxima of S. Hence, the radius of the cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

6.4.1 Maximum and Minimum Values of a Function in a Closed Interval

Let us consider a function f given by

$$f(x) = x + 2, x \in (0, 1)$$

Observe that the function is continuous on (0, 1) and neither has a maximum value nor has a minimum value. Further, we may note that the function even has neither a local maximum value nor a local minimum value.

However, if we extend the domain of f to the closed interval [0, 1], then f still may not have a local maximum (minimum) values but it certainly does have maximum value $3 = f(1)$ and minimum value $2 = f(0)$. The maximum value 3 of f at $x = 1$ is called *absolute maximum* value (*global maximum* or *greatest value*) of f on the interval [0, 1]. Similarly, the minimum value 2 of f at $x = 0$ is called the *absolute minimum* value (*global minimum* or *least value*) of f on [0, 1].

Consider the graph given in Fig 6.19 of a continuous function defined on a closed interval $[a, d]$. Observe that the function f has a local minima at $x = b$ and local

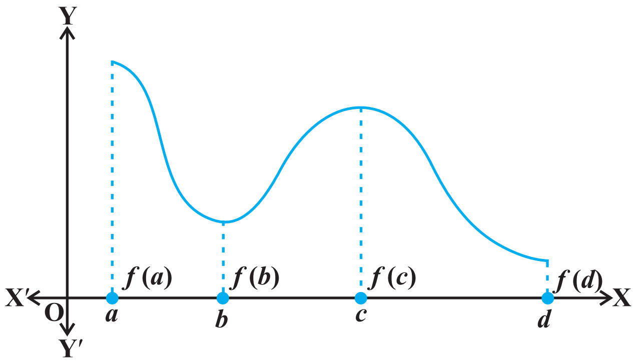


Fig 6.19

minimum value is $f(b)$. The function also has a local maxima at $x = c$ and local maximum value is $f(c)$.

Also from the graph, it is evident that f has absolute maximum value $f(a)$ and absolute minimum value $f(d)$. Further note that the absolute maximum (minimum) value of f is different from local maximum (minimum) value of f.

We will now state two results (without proof) regarding absolute maximum and absolute minimum values of a function on a closed interval I.

Theorem 5 Let f be a continuous function on an interval I = $[a, b]$. Then f has the absolute maximum value and f attains it at least once in I. Also, f has the absolute minimum value and attains it at least once in I.

Theorem 6 Let f be a differentiable function on a closed interval I and let c be any interior point of I. Then

(i) $f'(c) = 0$ if f attains its absolute maximum value at c.

(ii) $f'(c) = 0$ if f attains its absolute minimum value at c.

In view of the above results, we have the following working rule for finding absolute maximum and/or absolute minimum values of a function in a given closed interval $[a, b]$.

Working Rule

Step 1: Find all critical points of f in the interval, i.e., find points x where either $f'(x) = 0$ or f is not differentiable.

Step 2: Take the end points of the interval.

Step 3: At all these points (listed in Step 1 and 2), calculate the values of f.

Step 4: Identify the maximum and minimum values of f out of the values calculated in Step 3. This maximum value will be the absolute maximum (greatest) value of f and the minimum value will be the absolute minimum (least) value of f.

Example 27 Find the absolute maximum and minimum values of a function f given by

$$f(x) = 2x^3 - 15x^2 + 36x + 1 \text{ on the interval } [1, 5].$$

Solution We have

$$f(x) = 2x^3 - 15x^2 + 36x + 1$$

or
$$f'(x) = 6x^2 - 30x + 36 = 6(x-3)(x-2)$$

Note that $f'(x) = 0$ gives $x = 2$ and $x = 3$.

We shall now evaluate the value of f at these points and at the end points of the interval [1, 5], i.e., at $x = 1$, $x = 2$, $x = 3$ and at $x = 5$. So

$$f(1) = 2(1^3) - 15(1^2) + 36(1) + 1 = 24$$
$$f(2) = 2(2^3) - 15(2^2) + 36(2) + 1 = 29$$
$$f(3) = 2(3^3) - 15(3^2) + 36(3) + 1 = 28$$
$$f(5) = 2(5^3) - 15(5^2) + 36(5) + 1 = 56$$

Thus, we conclude that absolute maximum value of f on [1, 5] is 56, occurring at $x = 5$, and absolute minimum value of f on [1, 5] is 24 which occurs at $x = 1$.

Example 28 Find absolute maximum and minimum values of a function f given by

$$f(x) = 12x^{\frac{4}{3}} - 6x^{\frac{1}{3}}, \; x \in [-1, 1]$$

Solution We have

$$f(x) = 12x^{\frac{4}{3}} - 6x^{\frac{1}{3}}$$

or
$$f'(x) = 16x^{\frac{1}{3}} - \frac{2}{x^{\frac{2}{3}}} = \frac{2(8x-1)}{x^{\frac{2}{3}}}$$

Thus, $f'(x) = 0$ gives $x = \frac{1}{8}$. Further note that $f'(x)$ is not defined at $x = 0$. So the critical points are $x = 0$ and $x = \frac{1}{8}$. Now evaluating the value of f at critical points $x = 0, \frac{1}{8}$ and at end points of the interval $x = -1$ and $x = 1$, we have

$$f(-1) = 12(-1)^{\frac{4}{3}} - 6(-1)^{\frac{1}{3}} = 18$$
$$f(0) = 12(0) - 6(0) = 0$$

$$f\left(\frac{1}{8}\right) = 12\left(\frac{1}{8}\right)^{\frac{4}{3}} - 6\left(\frac{1}{8}\right)^{\frac{1}{3}} = \frac{-9}{4}$$

$$f(1) = 12(1)^{\frac{4}{3}} - 6(1)^{\frac{1}{3}} = 6$$

Hence, we conclude that absolute maximum value of f is 18 that occurs at $x = -1$ and absolute minimum value of f is $\frac{-9}{4}$ that occurs at $x = \frac{1}{8}$.

Example 29 An Apache helicopter of enemy is flying along the curve given by $y = x^2 + 7$. A soldier, placed at (3, 7), wants to shoot down the helicopter when it is nearest to him. Find the nearest distance.

Solution For each value of x, the helicopter's position is at point $(x, x^2 + 7)$. Therefore, the distance between the helicopter and the soldier placed at (3,7) is

$$\sqrt{(x-3)^2 + (x^2+7-7)^2}\text{, i.e., }\sqrt{(x-3)^2 + x^4}\,.$$

Let $\qquad f(x) = (x - 3)^2 + x^4$

or $\qquad f'(x) = 2(x - 3) + 4x^3 = 2(x - 1)(2x^2 + 2x + 3)$

Thus, $f'(x) = 0$ gives $x = 1$ or $2x^2 + 2x + 3 = 0$ for which there are no real roots. Also, there are no end points of the interval to be added to the set for which f' is zero, i.e., there is only one point, namely, $x = 1$. The value of f at this point is given by $f(1) = (1 - 3)^2 + (1)^4 = 5$. Thus, the distance between the solider and the helicopter is $\sqrt{f(1)} = \sqrt{5}$.

Note that $\sqrt{5}$ is either a maximum value or a minimum value. Since

$$\sqrt{f(0)} = \sqrt{(0-3)^2 + (0)^4} = 3 > \sqrt{5}\,,$$

it follows that $\sqrt{5}$ is the minimum value of $\sqrt{f(x)}$. Hence, $\sqrt{5}$ is the minimum distance between the soldier and the helicopter.

EXERCISE 6.3

1. Find the maximum and minimum values, if any, of the following functions given by

   (i) $f(x) = (2x - 1)^2 + 3$ (ii) $f(x) = 9x^2 + 12x + 2$

   (iii) $f(x) = -(x - 1)^2 + 10$ (iv) $g(x) = x^3 + 1$

2. Find the maximum and minimum values, if any, of the following functions given by

(i) $f(x) = |x + 2| - 1$ (ii) $g(x) = -|x + 1| + 3$

(iii) $h(x) = \sin(2x) + 5$ (iv) $f(x) = |\sin 4x + 3|$

(v) $h(x) = x + 1, x \in (-1, 1)$

3. Find the local maxima and local minima, if any, of the following functions. Find also the local maximum and the local minimum values, as the case may be:

(i) $f(x) = x^2$ (ii) $g(x) = x^3 - 3x$

(iii) $h(x) = \sin x + \cos x, 0 < x < \dfrac{\pi}{2}$

(iv) $f(x) = \sin x - \cos x, \ 0 < x < 2\pi$

(v) $f(x) = x^3 - 6x^2 + 9x + 15$ (vi) $g(x) = \dfrac{x}{2} + \dfrac{2}{x}, \quad x > 0$

(vii) $g(x) = \dfrac{1}{x^2 + 2}$ (viii) $f(x) = x\sqrt{1-x}, \ 0 < x < 1$

4. Prove that the following functions do not have maxima or minima:

(i) $f(x) = e^x$ (ii) $g(x) = \log x$

(iii) $h(x) = x^3 + x^2 + x + 1$

5. Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals:

(i) $f(x) = x^3, x \in [-2, 2]$ (ii) $f(x) = \sin x + \cos x, x \in [0, \pi]$

(iii) $f(x) = 4x - \dfrac{1}{2}x^2, \ x \in \left[-2, \dfrac{9}{2}\right]$ (iv) $f(x) = (x-1)^2 + 3, \ x \in [-3, 1]$

6. Find the maximum profit that a company can make, if the profit function is given by

$$p(x) = 41 - 72x - 18x^2$$

7. Find both the maximum value and the minimum value of

$3x^4 - 8x^3 + 12x^2 - 48x + 25$ on the interval $[0, 3]$.

8. At what points in the interval $[0, 2\pi]$, does the function $\sin 2x$ attain its maximum value?

9. What is the maximum value of the function $\sin x + \cos x$?

10. Find the maximum value of $2x^3 - 24x + 107$ in the interval $[1, 3]$. Find the maximum value of the same function in $[-3, -1]$.

11. It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value, on the interval [0, 2]. Find the value of a.

12. Find the maximum and minimum values of $x + \sin 2x$ on $[0, 2\pi]$.

13. Find two numbers whose sum is 24 and whose product is as large as possible.

14. Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum.

15. Find two positive numbers x and y such that their sum is 35 and the product $x^2 y^5$ is a maximum.

16. Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.

17. A square piece of tin of side 18 cm is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible.

18. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum ?

19. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

20. Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.

21. Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimetres, find the dimensions of the can which has the minimum surface area?

22. A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?

23. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

24. Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ time the radius of the base.

25. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

26. Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$.

Choose the correct answer in Questions 27 and 29.

27. The point on the curve $x^2 = 2y$ which is nearest to the point (0, 5) is

(A) $(2\sqrt{2},4)$ (B) $(2\sqrt{2},0)$ (C) (0, 0) (D) (2, 2)

28. For all real values of x, the minimum value of $\dfrac{1-x+x^2}{1+x+x^2}$ is

(A) 0 (B) 1 (C) 3 (D) $\dfrac{1}{3}$

29. The maximum value of $[x(x-1)+1]^{\frac{1}{3}}$, $0 \le x \le 1$ is

(A) $\left(\dfrac{1}{3}\right)^{\frac{1}{3}}$ (B) $\dfrac{1}{2}$ (C) 1 (D) 0

Miscellaneous Examples

Example 30 A car starts from a point P at time $t = 0$ seconds and stops at point Q. The distance x, in metres, covered by it, in t seconds is given by

$$x = t^2\left(2 - \frac{t}{3}\right)$$

Find the time taken by it to reach Q and also find distance between P and Q.

Solution Let v be the velocity of the car at t seconds.

Now $$x = t^2\left(2 - \frac{t}{3}\right)$$

Therefore $$v = \frac{dx}{dt} = 4t - t^2 = t(4 - t)$$

Thus, $v = 0$ gives $t = 0$ and/or $t = 4$.

Now $v = 0$ at P as well as at Q and at P, $t = 0$. So, at Q, $t = 4$. Thus, the car will reach the point Q after 4 seconds. Also the distance travelled in 4 seconds is given by

$$x]_{t=4} = 4^2\left(2 - \frac{4}{3}\right) = 16\left(\frac{2}{3}\right) = \frac{32}{3}\text{ m}$$

Example 31 A water tank has the shape of an inverted right circular cone with its axis vertical and vertex lowermost. Its semi-vertical angle is $\tan^{-1}(0.5)$. Water is poured into it at a constant rate of 5 cubic metre per hour. Find the rate at which the level of the water is rising at the instant when the depth of water in the tank is 4 m.

Solution Let r, h and α be as in Fig 6.20. Then $\tan\alpha = \frac{r}{h}$.

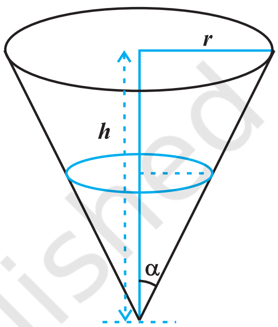


Fig 6.20

So $$\alpha = \tan^{-1}\left(\frac{r}{h}\right).$$

But $$\alpha = \tan^{-1}(0.5) \quad \text{(given)}$$

or $$\frac{r}{h} = 0.5$$

or $$r = \frac{h}{2}$$

Let V be the volume of the cone. Then

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}$$

Therefore $$\frac{dV}{dt} = \frac{d}{dh}\left(\frac{\pi h^3}{12}\right)\cdot\frac{dh}{dt} \quad \text{(by Chain Rule)}$$

$$= \frac{\pi}{4}h^2\frac{dh}{dt}$$

Now rate of change of volume, i.e., $\frac{dV}{dt} = 5\ \text{m}^3/\text{h}$ and $h = 4$ m.

Therefore $$5 = \frac{\pi}{4}(4)^2\cdot\frac{dh}{dt}$$

or $$\frac{dh}{dt} = \frac{5}{4\pi} = \frac{35}{88}\ \text{m/h} \quad \left(\pi = \frac{22}{7}\right)$$

Thus, the rate of change of water level is $\frac{35}{88}$ m/h.

Example 32 A man of height 2 metres walks at a uniform speed of 5 km/h away from a lamp post which is 6 metres high. Find the rate at which the length of his shadow increases.

Solution In Fig 6.21, Let AB be the lamp-post, the lamp being at the position B and let MN be the man at a particular time t and let AM = l metres. Then, MS is the shadow of the man. Let MS = s metres.

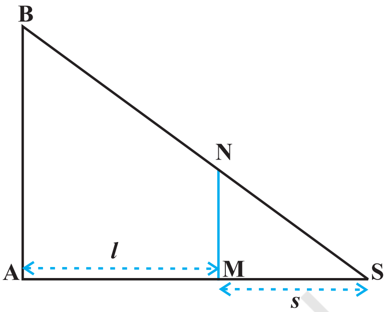


Fig 6.21

Note that $\Delta MSN \sim \Delta ASB$

or $$\frac{MS}{AS} = \frac{MN}{AB}$$

or AS = 3s (as MN = 2 and AB = 6 (given))

Thus AM = 3s – s = 2s. But AM = l

So l = 2s

Therefore $$\frac{dl}{dt} = 2\frac{ds}{dt}$$

Since $\frac{dl}{dt} = 5$ km/h. Hence, the length of the shadow increases at the rate $\frac{5}{2}$ km/h.

Example 33 Find intervals in which the function given by

$$f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$$

is (a) increasing (b) decreasing.

Solution We have

$$f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$$

Therefore $$f'(x) = \frac{3}{10}(4x^3) - \frac{4}{5}(3x^2) - 3(2x) + \frac{36}{5}$$

$$= \frac{6}{5}(x-1)(x+2)(x-3) \qquad \text{(on simplification)}$$

Now $f'(x) = 0$ gives $x = 1, x = -2$, or $x = 3$. The points $x = 1, -2$, and 3 divide the real line into four disjoint intervals namely, $(-\infty, -2)$, $(-2, 1)$, $(1, 3)$ and $(3, \infty)$ (Fig 6.22).

Fig 6.22

Consider the interval $(-\infty, -2)$, i.e., when $-\infty < x < -2$.

In this case, we have $x - 1 < 0$, $x + 2 < 0$ and $x - 3 < 0$.

(In particular, observe that for $x = -3$, $f'(x) = (x - 1)(x + 2)(x - 3) = (-4)(-1)(-6) < 0$)

Therefore, $\quad f'(x) < 0$ when $-\infty < x < -2$.

Thus, the function f is decreasing in $(-\infty, -2)$.

Consider the interval $(-2, 1)$, i.e., when $-2 < x < 1$.

In this case, we have $x - 1 < 0$, $x + 2 > 0$ and $x - 3 < 0$

(In particular, observe that for $x = 0$, $f'(x) = (x - 1)(x + 2)(x - 3) = (-1)(2)(-3) = 6 > 0$)

So $\quad f'(x) > 0$ when $-2 < x < 1$.

Thus, $\quad f$ is increasing in $(-2, 1)$.

Now consider the interval $(1, 3)$, i.e., when $1 < x < 3$. In this case, we have $x - 1 > 0$, $x + 2 > 0$ and $x - 3 < 0$.

So, $\quad f'(x) < 0$ when $1 < x < 3$.

Thus, $\quad f$ is decreasing in $(1, 3)$.

Finally, consider the interval $(3, \infty)$, i.e., when $x > 3$. In this case, we have $x - 1 > 0$, $x + 2 > 0$ and $x - 3 > 0$. So $f'(x) > 0$ when $x > 3$.

Thus, f is increasing in the interval $(3, \infty)$.

Example 34 Show that the function f given by

$$f(x) = \tan^{-1}(\sin x + \cos x),\ x > 0$$

is always an increasing function in $\left(0, \frac{\pi}{4}\right)$.

Solution We have

$$f(x) = \tan^{-1}(\sin x + \cos x),\ x > 0$$

Therefore

$$f'(x) = \frac{1}{1 + (\sin x + \cos x)^2}(\cos x - \sin x)$$

$$= \frac{\cos x - \sin x}{2 + \sin 2x} \qquad \text{(on simplification)}$$

Note that $2 + \sin 2x > 0$ for all x in $\left[0, \frac{\pi}{4}\right]$.

Therefore $\quad f'(x) > 0$ if $\cos x - \sin x > 0$

or $\quad f'(x) > 0$ if $\cos x > \sin x$ or $\cot x > 1$

Now $\quad \cot x > 1$ if $\tan x < 1$, i.e., if $0 < x < \frac{\pi}{4}$

Thus $\quad f'(x) > 0$ in $\left(0, \frac{\pi}{4}\right)$

Hence f is increasing function in $\left(0, \frac{\pi}{4}\right)$.

Example 35 A circular disc of radius 3 cm is being heated. Due to expansion, its radius increases at the rate of 0.05 cm/s. Find the rate at which its area is increasing when radius is 3.2 cm.

Solution Let r be the radius of the given disc and A be its area. Then

$$\mathrm{A} = \pi r^2$$

or
$$\frac{d\mathrm{A}}{dt} = 2\pi r \frac{dr}{dt} \qquad \text{(by Chain Rule)}$$

Now approximate rate of increase of radius $= dr = \frac{dr}{dt}\Delta t = 0.05$ cm/s.

Therefore, the approximate rate of increase in area is given by

$$d\mathrm{A} = \frac{d\mathrm{A}}{dt}(\Delta t) = 2\pi r\left(\frac{dr}{dt}\Delta t\right)$$

$$= 2\pi\,(3.2)\,(0.05) = 0.320\pi \text{ cm}^2\text{/s} \quad (r = 3.2 \text{ cm})$$

Example 36 An open topped box is to be constructed by removing equal squares from each corner of a 3 metre by 8 metre rectangular sheet of aluminium and folding up the sides. Find the volume of the largest such box.

Solution Let x metre be the length of a side of the removed squares. Then, the height of the box is x, length is $8 - 2x$ and breadth is $3 - 2x$ (Fig 6.23). If V(x) is the volume of the box, then

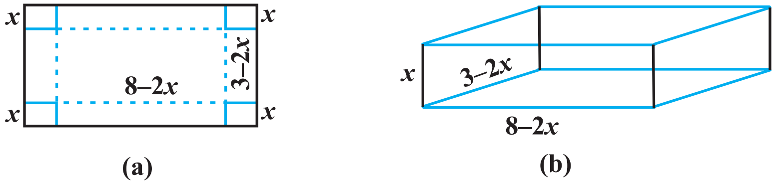


Fig 6.23

$$\begin{aligned} \mathrm{V}(x) &= x(3-2x)(8-2x) \\ &= 4x^3 - 22x^2 + 24x \end{aligned}$$

Therefore $\begin{cases} \mathrm{V}'(x) = 12x^2 - 44x + 24 = 4(x-3)(3x-2) \\ \mathrm{V}''(x) = 24x - 44 \end{cases}$

Now $\mathrm{V}'(x) = 0$ gives $x = 3, \frac{2}{3}$. But $x \neq 3$ (Why?)

Thus, we have $x = \frac{2}{3}$. Now $\mathrm{V}''\left(\frac{2}{3}\right) = 24\left(\frac{2}{3}\right) - 44 = -28 < 0$.

Therefore, $x = \frac{2}{3}$ is the point of maxima, i.e., if we remove a square of side $\frac{2}{3}$ metre from each corner of the sheet and make a box from the remaining sheet, then the volume of the box such obtained will be the largest and it is given by

$$\begin{aligned} \mathrm{V}\left(\frac{2}{3}\right) &= 4\left(\frac{2}{3}\right)^3 - 22\left(\frac{2}{3}\right)^2 + 24\left(\frac{2}{3}\right) \\ &= \frac{200}{27}\ \mathrm{m}^3 \end{aligned}$$

Example 37 Manufacturer can sell x items at a price of rupees $\left(5 - \frac{x}{100}\right)$ each. The cost price of x items is Rs $\left(\frac{x}{5} + 500\right)$. Find the number of items he should sell to earn maximum profit.

Solution Let $S(x)$ be the selling price of x items and let $C(x)$ be the cost price of x items. Then, we have

$$S(x) = \left(5 - \frac{x}{100}\right)x = 5x - \frac{x^2}{100}$$

and
$$C(x) = \frac{x}{5} + 500$$

Thus, the profit function $P(x)$ is given by

$$P(x) = S(x) - C(x) = 5x - \frac{x^2}{100} - \frac{x}{5} - 500$$

i.e.
$$P(x) = \frac{24}{5}x - \frac{x^2}{100} - 500$$

or
$$P'(x) = \frac{24}{5} - \frac{x}{50}$$

Now $P'(x) = 0$ gives $x = 240$. Also $P''(x) = \frac{-1}{50}$. So $P''(240) = \frac{-1}{50} < 0$

Thus, $x = 240$ is a point of maxima. Hence, the manufacturer can earn maximum profit, if he sells 240 items.

Miscellaneous Exercise on Chapter 6

1. Show that the function given by $f(x) = \frac{\log x}{x}$ has maximum at $x = e$.
2. The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base ?
3. Find the intervals in which the function f given by
$$f(x) = \frac{4\sin x - 2x - x\cos x}{2 + \cos x}$$
is (i) increasing (ii) decreasing.
4. Find the intervals in which the function f given by $f(x) = x^3 + \frac{1}{x^3}$, $x \neq 0$ is

   (i) increasing (ii) decreasing.

5. Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with its vertex at one end of the major axis.

6. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank costs Rs 70 per sq metres for the base and Rs 45 per square metre for sides. What is the cost of least expensive tank?

7. The sum of the perimeter of a circle and square is k, where k is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.

8. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

9. A point on the hypotenuse of a triangle is at distance a and b from the sides of the triangle.

   Show that the minimum length of the hypotenuse is $\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

10. Find the points at which the function f given by $f(x) = (x-2)^4 (x+1)^3$ has
    (i) local maxima (ii) local minima
    (iii) point of inflexion

11. Find the absolute maximum and minimum values of the function f given by

$$f(x) = \cos^2 x + \sin x,\ x \in [0, \pi]$$

12. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

13. Let f be a function defined on $[a, b]$ such that $f'(x) > 0$, for all $x \in (a, b)$. Then prove that f is an increasing function on (a, b).

14. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

15. Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi vertical angle α is one-third that of the cone and the greatest volume of cylinder is $\frac{4}{27}\pi h^3 \tan^2 \alpha$.

16. A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic metre per hour. Then the depth of the wheat is increasing at the rate of

(A) 1 m/h (B) 0.1 m/h

(C) 1.1 m/h (D) 0.5 m/h

Summary

- If a quantity *y* varies with another quantity *x*, satisfying some rule $y = f(x)$, then $\frac{dy}{dx}$ (or $f'(x)$) represents the rate of change of *y* with respect to *x* and $\left.\frac{dy}{dx}\right]_{x=x_0}$ (or $f'(x_0)$) represents the rate of change of *y* with respect to *x* at $x = x_0$.
- If two variables *x* and *y* are varying with respect to another variable *t*, i.e., if $x = f(t)$ and $y = g(t)$, then by Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}, \text{ if } \frac{dx}{dt} \neq 0.$$

- A function f is said to be
  (a) increasing on an interval (*a*, *b*) if
  $x_1 < x_2$ in $(a, b) \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in (a, b)$.
  Alternatively, if $f'(x) \geq 0$ for each *x* in (*a*, *b*)
  (b) decreasing on (*a*,*b*) if
  $x_1 < x_2$ in $(a, b) \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in (a, b)$.
  (c) constant in (*a*, *b*), if $f(x) = c$ for all $x \in (a, b)$, where *c* is a constant.
- A point *c* in the domain of a function *f* at which either $f'(c) = 0$ or *f* is not differentiable is called a *critical point* of *f*.
- ***First Derivative Test*** Let *f* be a function defined on an open interval I. Let *f* be continuous at a critical point *c* in I. Then
  (i) If $f'(x)$ changes sign from positive to negative as *x* increases through c, i.e., if $f'(x) > 0$ at every point sufficiently close to and to the left of *c*, and $f'(x) < 0$ at every point sufficiently close to and to the right of *c*, then *c* is a point of *local maxima*.

(ii) If $f'(x)$ changes sign from negative to positive as x increases through c, i.e., if $f'(x) < 0$ at every point sufficiently close to and to the left of c, and $f'(x) > 0$ at every point sufficiently close to and to the right of c, then c is a point of *local minima*.

(iii) If $f'(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Infact, such a point is called *point of inflexion*.

- ***Second Derivative Test*** Let f be a function defined on an interval I and $c \in$ I. Let f be twice differentiable at c. Then

  (i) $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$

  The values $f(c)$ is local maximum value of f.

  (ii) $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$

  In this case, $f(c)$ is local minimum value of f.

  (iii) The test fails if $f'(c) = 0$ and $f''(c) = 0$.

  In this case, we go back to the first derivative test and find whether c is a point of maxima, minima or a point of inflexion.

- Working rule for finding absolute maxima and/or absolute minima

  Step 1: Find all critical points of f in the interval, i.e., find points x where either $f'(x) = 0$ or f is not differentiable.

  Step 2: Take the end points of the interval.

  Step 3: At all these points (listed in Step 1 and 2), calculate the values of f.

  Step 4: Identify the maximum and minimum values of f out of the values calculated in Step 3. This maximum value will be the absolute maximum value of f and the minimum value will be the absolute minimum value of f.