1. Aggregate claims from a risk have a compound Poisson distribution with Poisson parameter 50, and individual claim sizes X's are distributed according to the following cumulative distribution function

$$F(x) = 1 - 0.4e^{-0.01x} - 0.6e^{-0.02x}, \qquad x \geq 0.$$

   (a) Show that the individual claim sizes follow a mixed distribution in the form of

$$X = \alpha X_1 + \beta X_2$$

   where both X_1 and X_2 are independent to each other. Determine the distribution of both X_1 and X_2 and their associated parameter(s) as well as parameters α and β. (3 marks)

$$\begin{aligned} f(x) &= 0.004e^{-0.01x} + 0.012e^{-0.02x} \\ X &= 0.4X_1 + 0.6X_2, \\ X_1 &\sim \text{Exp}(0.01), \quad X_2 \sim \text{Exp}(0.02) \end{aligned}$$

   (b) Calculate the value of $P(S \leq 4500)$ using a normal approximation. (4 marks)

$$\begin{aligned} m_1 &= \frac{0.4}{0.01} + \frac{0.6}{0.02} = 70 \\ m_2 &= 2\left(\frac{0.4}{0.01^2} + \frac{0.6}{0.02^2}\right) = 11000 \\ m_3 &= 6\left(\frac{0.4}{0.01^3} + \frac{0.6}{0.02^3}\right) = 2850000 \\ E(S) &= 50(70) = 3500 \\ \text{Var}(S) &= 50(11000) = 550000 \\ P(S \leq 4500) &= P(Z \leq \frac{4500 - 3500}{\sqrt{550000}}) \\ &= P(Z \leq 1.3484) = 0.911 \end{aligned}$$

   (c) Suppose that we use a compound negative binomial model to study the aggregate claims instead, where the mean of claims count $E(N)$ under the negative binomial distribution is the same as the mean of claims count under the Poisson distribution.

Without performing any numerical calculations, explain whether the corresponding approximate value of $P(S \leq 4500)$ using a normal approximation is larger than, equal to, or smaller than your answer in part (b). (3 marks)

Both models have identical $E(S) = E(N)E(X) = 3500$.

The negative binomial model features over-dispersion where $\text{Var}(N) > E(N)$, whereas the Poisson model has equi-dispersion $\text{Var}(N) = E(N)$. Hence, the negative binomial model has a larger $\text{Var}(S) = E(N)\text{Var}(X) + \text{Var}(N)\left(E(X)\right)^2$ than that of the Poisson model.

Therefore we are effectively finding $P(Z <$ smaller positive magnitude$)$ and we obtain a smaller corresponding $P(S \leq 4500)$.

2. Consider the Archimedean generator

$$\psi_{\alpha,\beta}(t) = \left(t^{-\alpha} - 1\right)^{\beta}, \qquad \text{for } \alpha > 0,\ \beta \geq 1.$$

satisfying the definition of Archimedian copula in which

$$C_{\alpha,\beta}(u,v) = \psi_{\alpha,\beta}^{-1}\left(\psi_{\alpha,\beta}(u) + \psi_{\alpha,\beta}(v)\right),$$

where we can easily see that $\psi_{\alpha,\beta}(1) = 0$, $\psi'_{\alpha,\beta}(t) < 0$ and $\psi''_{\alpha,\beta}(t) > 0$

(a) Without determining the inverse of the generator function, derive an expression for the copula $C_{\alpha,\beta}(u,v)$ in terms of u and v. (2 marks)

From

$$\psi_{\alpha,\beta}\left(C_{\alpha,\beta}(u,v)\right) = \psi_{\alpha,\beta}(u) + \psi_{\alpha,\beta}(v),$$

we obtain

$$\left(C_{\alpha,\beta}(u,v)^{-\alpha} - 1\right)^{\beta} = \left(u^{-\alpha} - 1\right)^{\beta} + \left(v^{-\alpha} - 1\right)^{\beta}$$

and hence

$$C_{\alpha,\beta}(u,v) = \left\{\left[\left(u^{-\alpha} - 1\right)^{\beta} + \left(v^{-\alpha} - 1\right)^{\beta}\right]^{1/\beta} + 1\right\}^{-1/\alpha}.$$

(b) Show that the copula

$$C(u,v) = \frac{uv}{u + v - uv}$$

belongs to the family of copulas $C_{\alpha,\beta}(u,v)$ defined by the generator $\psi_{\alpha,\beta}(t)$. Show your steps clearly. (3 marks)

Notice how the exponent powers of u and v in $C(u,v)$ are 1, so we try $\beta = 1$ and $\alpha = 1$ and obtain

$$\begin{aligned}
C_{1,1}(u,v) &= \left\{\left[\left(u^{-1}-1\right)^1 + \left(v^{-1}-1\right)^1\right]^{1/1} + 1\right\}^{-1/1} \\
&= \frac{1}{1/u + 1/v - 1} \\
&= \frac{uv}{v + u - uv}.
\end{aligned}$$

This is the same copula as the implicit one derived in the lecture.

(c) Assume that we can extend the parameter range of $C_{\alpha,\beta}(u,v)$ to include $\alpha = 0$. Identify $C_{0,1}(u,v)$, a well known copula. (3 marks)

$$\begin{aligned}
C_{0,1}(u,v) &= \lim_{\alpha\to 0}\left(u^{-\alpha} + v^{-\alpha} - 1\right)^{-1/\alpha} \\
&= \lim_{\alpha\to 0}\left(\frac{(uv)^\alpha}{v^\alpha + u^\alpha - (uv)^\alpha}\right)^{1/\alpha} \\
&= \lim_{\alpha\to 0}\frac{uv}{\left(v^\alpha + u^\alpha - (uv)^\alpha\right)^{-1/\alpha}} \\
&= \frac{uv}{1} = uv,
\end{aligned}$$

that is, we obtain the independence copula.

3. Suppose that the random variable X has the following probability density function

$$f_X(x) = \alpha e^{-\alpha x}, \quad x > 0$$

for parameter $\alpha > 0$.

(a) Determine the probability density function of $Y = e^X$. (2 marks)

$$\begin{aligned}
P(X \le x) &= \int_0^x \alpha e^{-\alpha z} dz = 1 - e^{-\alpha x} \\
P(Y \le y) &= P(e^X \le y) \\
&= P(X \le \log y) = 1 - y^{-\alpha}, \quad y > 1 \\
f_Y(y) &= \alpha y^{-\alpha-1}, \quad y > 1
\end{aligned}$$

(b) By considering the hazard rate function of both X and Y, compare their tail weight. (3 marks)

$$\begin{aligned} h(x) &= \frac{f(x)}{S(x)} \\ &= \frac{\alpha e^{-\alpha x}}{1-(1-e^{-\alpha x})} = \alpha \\ h(y) &= \frac{f(y)}{S(y)} \\ &= \frac{\alpha y^{-\alpha-1}}{y^{-\alpha}} = \frac{\alpha}{y} \end{aligned}$$

Since $h(x)$ is constant and $h(y)$ is a decreasing function of y, Y has a heavier tail weight than X.

(c) Keeping the value of α fixed, how does the survival function $S(y) = P(Y > y)$ behave as $y \to \infty$? Discuss. (1 mark)

We have $S(y) = 1 - (1 - y^{-\alpha}) = y^{-\alpha}$, so the survival function decays to zero according to the magnitude of α.

(d) Determine the derivative of $S(y)$ with respect to α. What can you conclude from the result? (2 marks)

$$\begin{aligned} \frac{dS(y)}{d\alpha} &= y^{-\alpha} \log \frac{1}{y} \\ &= -\log y \times y^{-\alpha} < 0 \end{aligned}$$

So the larger the value of α, the smaller the value of $S(y)$, that is, the survival function decays faster (i.e., having a lighter tail, other things being equal).

(e) Define random variable $Z = Y - 1$. Show that Z belongs to the Generalised Pareto distribution (GPD) and identify the associated parameters. (2 marks)

$$\begin{aligned} P(Z \leq z) &= P(Y \leq z + 1) \\ &= 1 - (z+1)^{-\alpha} = 1 - \left(1 + \gamma \frac{z}{\sigma}\right)^{-\frac{1}{\gamma}} = G_{\gamma,\sigma}(z) \end{aligned}$$

where we have $\sigma = \gamma = \frac{1}{\alpha}$.

(f) By considering the survival function $S(y)$, comment on whether Y is of Generalised Extreme Value (GEV) distribution of Fréchet type by examining the definition of tail index in

$$S(y) = y^{-\frac{1}{\gamma}} L(y)$$

for slowly varying function $L(y)$. (1 mark)

Since we can write $S(y) = y^{-\alpha}$, it follows immediately that the tail index is given by $-\frac{1}{\gamma} = -\alpha$, that is, Y is of Fréchet type with shape parameter $\gamma = \frac{1}{\alpha} > 0$.

END OF TEST