M2 - Collective Risk Modelling

Discrete Convolution: $F_{X+Y}(x) = \sum_x F_Y(s-x) f_X(x)$, $f_{X+Y}(x) = \sum_x f_Y(s-x) f_X(x)$

Continuous Convolution: $F_{X+Y}(x) = \int_{-\infty}^{s} F_Y(s-x) f_X(x)\,dx$, $f_{X+Y}(x) = \int_{-\infty}^{s} f_Y(s-x) f_X(x)\,dx$

Collective Risk Model: $S = \sum_{i=1}^{N} Y_i$, $\mathbb{E}[S] = \mathbb{E}[N]\mathbb{E}[Y]$, $Var(S) = \mathbb{E}[N]\mathbb{E}[Y^2] + \mathbb{E}[Y]^2(Var(N) - \mathbb{E}[N])$, $M_N(\log(M_Y(t)))$. $F_S(x) = \sum_{n=0}^{\infty} \mathbb{P}(S \le x | N=n)\mathbb{P}(N=n)$

- Always discrete mass at 0, discrete/continuous/mixed for >0 depending on Y
- if Y is discrete: $f_S(s) = \sum_{n=0}^{\infty} \mathbb{P}(S=s|N=n)\mathbb{P}(N=n)$

Binomial Distribution: $N \sim Bin(v,p)$, $\mathbb{P}(N=k) = \binom{v}{k} p^k (1-p)^{v-k}$, $\binom{v}{k} = \frac{v!}{k!(v-k)!}$

Compound Binomial Distribution $S \sim CompBinom(v,p,G)$, also if $S_1 \perp \dots \perp S_n$: $\sum_{i=1}^{n} S_i \overset{d}{=} CompBinom(\sum_{i=1}^{n} v_i, p, G)$ *G is individual claim size distribution*

e.g. $S_{lc} = \sum_{i=1}^{N} Y_i \mathbb{1}_{\{Y_i > M\}} \overset{d}{=} CompBinom(v, p(1-G(M)), G_{lc})$, where $G_{lc} = \mathbb{P}(Y_1 \le y | Y_1 > M)$

Poisson Distribution: $N \sim Pois(\lambda v)$, $\mathbb{P}(N=n) = \frac{e^{-\lambda v}(\lambda v)^n}{n!}$ (as #trials $\to \infty$: binomial = poisson)

Compound Poisson Distribution $S \sim CompPois(\lambda v, G)$

Overdispersed: Variance is larger than mean (implemented in mixed poisson)

Mixed Poisson Distribution: Assume $\Lambda \sim H$ ($H(0)=0$, $\mathbb{E}[\Lambda]=\lambda$, $Var(\Lambda)>0$), $N|\Lambda \sim Pois(\Lambda v)$

$\mathbb{P}(N=n) = \int_0^{\infty} \frac{e^{-\lambda v}(\lambda v)^n}{n!} \cdot h(\lambda)\,d\lambda$, $\mathbb{E}[N] = \mathbb{E}[\Lambda]v = \lambda v$, $Var(N) = \lambda v + v^2 Var(\Lambda) > \mathbb{E}[N]$,

$M_N(t) = \mathbb{E}[\mathbb{E}[e^{tN}|\Lambda]] = \mathbb{E}[e^{\Lambda v(e^t-1)}] = M_\Lambda(v(e^t-1))$

Negative Binomial Distribution $(\Lambda = \lambda\Theta, \Theta \sim \Gamma(\gamma,\gamma))$: $M_\Theta = 1$, $\sigma^2_\Theta = \frac{1}{\gamma}$, $M_\Lambda = \lambda$, $\sigma^2_\Lambda = \lambda^2/\gamma$

$N|\Theta \overset{d}{=} NegBin(\lambda v, \gamma)$, proof with mgf of N, $p = \frac{\lambda v}{\lambda v + \gamma}$, #failures till γ^{th} success: $p_k = \binom{k+\gamma-1}{k} p^k (1-p)^\gamma$

$\Rightarrow$ Interpretation: $\mathbb{E}[N] = \lambda v$, $V(N) = \lambda v(1 + \frac{\lambda v}{\gamma}) > \mathbb{E}[\lambda v]$, $V_{co}(\frac{N}{v}) = \frac{\sigma}{\mu} = \sqrt{\frac{1}{\lambda v} + \frac{1}{\gamma}}$

$\Rightarrow$ Additional uncertainty not diversifiable $V_{co}(\frac{N}{v}) \to \gamma^{-\frac{1}{2}} > 0$ as $v \to \infty$

Compound Negative Binomial Distribution: $S \sim CompNB(\lambda v, \gamma, G)$. $N \sim NegBinom(\lambda v, \gamma)$

Aggregation Property: let $S_i \sim CompPoi(\lambda_i v_i, G_i) \Rightarrow S = \sum_{i=1}^{n} S_i$, $v = \sum v_i$, $\lambda = \sum \lambda_i$, $G = \sum \frac{\lambda_i v_i}{\lambda v} G_i$

e.g. $S_i = y_i N_i$ $(v_i = 1)$, y_i is degen. dist. ($G(y_i)=1$, $G(y)=0$, $y_i \ne y$), $S \sim \sum^{m} S_i \sim CompPoi(\lambda v, G(y))$, $v = \sum^{m} v_i = m$, $\lambda = \sum^{m} \frac{v_i}{v}\lambda_i = \frac{\sum^m \lambda_i}{m}$, $G(y_j) = \sum^{m} \frac{\lambda_i v_i}{\lambda v} G_i(y_j) = \frac{\lambda_j}{\sum^m \lambda_i}$

Disjoint Decomposition Theorem: $Y_i = Y_i \mathbb{1}_{\{Y_i \in A_1\}} + \dots + Y_i \mathbb{1}_{\{Y_i \in A_n\}}$, in each partition A_k: $v_k = v$, $\lambda_k = \lambda p^{(k)}$

$\Rightarrow$ meaning the volume remains constant in each partition, but the claims frequencies λ_k change proportionally to the probabilities of falling in each partition A_k (thinning of pois process)

$\Rightarrow S_k = \sum_{i=1}^{N} Y_i \mathbb{1}_{\{(Y_i, I_i) \in A_k\}} \sim CompPoi(\lambda_k v_k, G_k)$, $\lambda_k v_k = \lambda v p^{(k)} > 0$, $G_k(y) = \mathbb{P}(Y_1 \le y | (Y_1, I_1) \in A_k)$

Sparse Vector Algorithm: $S \sim CompPoi(\lambda, g(y_i) = \pi_i)$, $i = 1, \dots, m$. $S = y_1 N_1 + \dots + y_m N_m$, $N_1 \perp \dots \perp N_m$, $N_i \sim Poi(\lambda_i = \lambda \pi_i)$, $p_i(x) = \mathbb{P}(iN_i = x) = \mathbb{P}(N_i = \frac{x_i}{i})$

Large Claim Seperation: let M be claim threshold. $S_{sc} = \sum_{i=1}^{n} Y_i \mathbb{1}_{\{Y_i \le M\}}$, $S_{lc} = \sum_{i=1}^{n} Y_i \mathbb{1}_{\{Y_i > M\}}$, $S = S_{lc} + S_{sc}$, $S_{sc} \sim CompPoi(\lambda_{sc} v = \lambda G(M) v, G_{sc}(y) = \mathbb{P}(Y_1 \le y | Y_1 \le M))$, and $S_{lc} \sim CompPoi(\lambda_{lc} v = \lambda(1-G(M))v, G_{lc}(y) = \mathbb{P}(Y_1 \le y | Y_1 > M))$, proof using mgf

e.g. $M_{S_{lc}}(r) = \mathbb{E}[e^{r\sum_{i=1}^{N} Y_i \mathbb{1}_{\{Y_i > M\}}}] = \mathbb{E}[\prod_{i=1}^{N} e^{r Y_i \mathbb{1}_{\{Y_i > M\}}}] = \mathbb{E}[\prod_{i=1}^{N} (e^{r Y_i} \mathbb{1}_{\{Y_i > M\}} + \mathbb{1}_{\{Y_i < M\}})]$
$= \mathbb{E}[e^{r Y_i | Y_i > M}][1 - G(M)] + G(M) = M_N(\log(M_{Y|Y>M}(r)(1-G(M)) + G(M)))$

M3 - Individual Claim Size Modelling

Zero Inflated Severity Model $(X = IB)$: $\mathbb{P}(I=1) = q$, $\mathbb{P}(I=0) = 1-q$, $F_X(x) = 1 - q + q F_B(x)$, $M_X(t) = 1 - q + q M_B(t)$, $\mathbb{E}[X] = q\mathbb{E}[B]$, $V(X) = q(1-q)\mathbb{E}[B]^2 + qV(B)$, $\mathbb{E}[X|I] = I\mathbb{E}[B]$, $V(X|I) = I^2 V(B)$

case $B = b$: $\mathbb{E}[X] = bq$, $V(X) = b^2 Var(I) = b^2 q(1-q)$

Likelihood (with left truncation/right censoring): (t_j, x_j, δ_j), t_j = left truncation point, x_j = claim value, δ_j = indicator if limit has been reached, likelihood:
$L(\theta; \vec{x}) = \prod_j \left[\frac{f(x_j;\theta)}{1-F(t_j;\theta)}\right]^{1-\delta_j} \prod_j \left[\frac{1-F(x_j;\theta)}{1-F(t_j;\theta)}\right]^{\delta_j}$

Akaike Information Criteria: $AIC = -2\ell + 2d$, ℓ = max log-likelihood, d = #parameters estimated

Bayesian Information Criteria: $BIC = -2\ell + \log(n) d$, and n = #parameters

Deductibles: $Y = \max(0, X-d) = 0$ for $X \le d$ or $X-d$ for $X > d$

Limit: $Y = \min(X, M) = X$ for $X \le M$ or M for $X > M$

useful formula: $\mathbb{E}[(X \wedge M)^k] = \int_0^M x^k f_X(x)\,dx + M^k(1 - F_X(M)) = \sum_{i: x_i \le M} x_i^k \mathbb{P}(X = x_i) + M^k(1 - F_X(M))$.

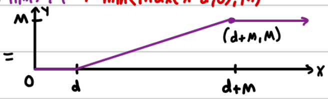
$\mathbb{E}[X \wedge M] = \int_0^M 1 - F_X(x)\,dx = \sum_{i: 0 \le x_i \le M} (1 - F_X(x_i))(x_{i+1} - x_i)$. $(X-c)_+ = X - (X \wedge c)$.

$\mathbb{E}[(X-d)_+] = \mathbb{E}[X] - \mathbb{E}[X \wedge d] = \int_d^\infty 1 - F_X(x)\,dx = \sum_{i: x_i \ge d} (1 - F_X(x_i))(x_{i+1} - x_i)$

Stop Loss Premium: $P_d = \mathbb{E}[(X-d)_+] = \mathbb{P}(X > d) e(d)$, $e(d) = \mathbb{E}[X - d | X > d]$. e.g. $X \sim exp(\beta)$, $P_d = (1/\beta)e^{-\beta d}$

if claim has both deductible d and max. limit M: $Y = \min(\max(X-d, 0), M)$:

$$= \begin{cases} 0, & X \le d \\ X - d, & d < X < d+M \\ M, & X \ge d+M \end{cases} = (X \wedge (d+M)) - (X \wedge d) =$$



Reinsurance:

- Proportional: insurer $= Y = \alpha X$, reinsurer $= Z = (1-\alpha)X$. $M_Y = \alpha M_X$, $\sigma_Y^2 = \alpha^2 \sigma_X^2$, $S_Y = S_X$
- Non-Proportional: excess of loss: $Y = \min(X, d)$, $Z = \max(X-d, 0)$, reinsurer could apply limit $M \Rightarrow Y = \min(X,d) + \max(X-M-d, 0)$, $Z = \min(\max(X-d,0), M)$. stop loss: let $S = \sum^N X$ $Y = \min(S,d)$, $Z = \max(S-d, 0)$ $(P_d \equiv (S-d)_+)$, reinsurer could apply limit $M \Rightarrow$ $Y = \min(S,d) + \max(S-d-M, 0)$, $Z = \min(\max(S-d,0), M)$

M4 - Approximations for Compound Distributions

Panjer Distributions $(a,b,0)$: $\mathbb{P}(N=k) = (a + \frac{b}{k})\mathbb{P}(N=k-1)$, or $\frac{p_k}{p_{k-1}} = (a + \frac{b}{k})$, $k = 1, 2, \dots$

	a	b	$\mathbb{P}(N=0)$
$Pois(\lambda)$	0	λ	$e^{-\lambda}$
$NegBin(\gamma,p)$	p	$(\gamma-1)p$	$(1-p)^\gamma$
$Bin(v,p)$	$-p/(1-p)$	$(v+1)p/(1-p)$	$(1-p)^v$

Panjer Distribution $(a,b,1)$: $\frac{p_k}{p_{k-1}} = a + \frac{b}{k}$, $k = 2, 3, \dots$, p_0 can be any $p_0 \in [0,1]$

- Zero-truncated dist.: $p_k^T = 0$ for $k=0$ or $\frac{p_k}{1-p_0}$, $k = 1, 2, \dots$. p_k from $(a,b,0)$
- Zero-modified dist.: $p_k^M = p_0^M$ for $k=0$ or $\frac{1-p_0^M}{1-p_0} p_k$, $k = 1, 2, \dots$

Panjer Recursion: $S = \sum^{N} Y_i$, N is a (a,b) dist., Y_i's ≥ 0 i.i.d with pmf $g_j = \mathbb{P}(Y=j)$

$f_S(s) = \frac{1}{1 - a g_0} \sum_{j=1}^{s} (a + b \cdot \frac{j}{s}) g_j f_S(s-j)$, $s = 1, 2, \dots$ $f_S(0) = \mathbb{P}(N=0)$ for $g_0 = 0$ or $M_N(\log(g_0))$ if $g_0 > 0$. e.g. comp. pois. $f_S(s) = \frac{\lambda}{s} \sum_{j=1}^{s} j g_j f_S(s-j)$, $f_S(0) = e^{\lambda(g_0 - 1)}$

Normal Approx.: $\mathbb{P}(S \le s) = \Phi\left(\frac{s - \mathbb{E}[S]}{sd(S)}\right)$

$S \overset{d}{=} CompPois(\lambda v, G)$, $\frac{S - \lambda v \mathbb{E}[Y]}{\sqrt{\lambda v \mathbb{E}[Y^2]}} \overset{d}{\approx} N(0,1)$ as $v \to \infty$, Only holds for finite $\mathbb{E}[Y^2]$, so only good for small claims

Translated Gamma/LN Approximation: $X = k + Z$, $Z \sim LN(\mu, \sigma^2) \sim \Gamma(\gamma, c)$

match expected value, variance, and skew.

e.g. fit translated gamma to $CompPois(\lambda v, G(y)) \overset{d}{=} S$, need to solve:

$\mathbb{E}[S] = \lambda v \mathbb{E}[Y] = k + \gamma/c$, $Var(S) = \lambda v \mathbb{E}[Y^2] = \gamma/c^2$, $S_S = \frac{\mathbb{E}[Y^3]}{(\lambda v)^{1/2}\mathbb{E}[Y^2]^{3/2}} = \frac{2}{\sqrt{\gamma}}$

$\gamma = \left(\frac{2(\lambda v)^{1/2}\mathbb{E}[Y^2]^{3/2}}{\mathbb{E}[Y^3]}\right)^2$, $c = \left(\frac{\gamma}{\lambda v \mathbb{E}[Y^2]}\right)^{1/2}$, $k = \lambda v \mathbb{E}[Y] - \frac{\gamma}{c}$ (also $\mathbb{E}[S^3] = \lambda\mathbb{E}[Y^3]$)

e.g. translated LN: $\mathbb{E} = k + e^{\mu + \frac{\sigma^2}{2}}$, $Var = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}$, $Skew = (e^{\sigma^2} + 2)\sqrt{e^{\sigma^2} - 1}$

M6 - Extreme Value Theory

Measures of Tail Weight: **Mean Excess Function:** if linear increase $\Rightarrow$ heavy tail. **Existence of Moments:** Fewer finite moments $\Rightarrow$ heavier tails. **Limiting Density/Survival Ratio:** $\lim_{x\to\infty}\frac{S_A(x)}{S_B(x)} = \lim_{x\to\infty}\frac{f_A(x)}{f_B(x)}$ if ∞ A has heavier tail, if 0 A has lighter tail, if c tails are comparable. **Hazard Rate Function:** $h(x) = \frac{f(x)}{1-F(x)}$, if increasing in $x \Rightarrow$ light-tailed, if decreasing in x $\Rightarrow$ heavy tailed, if constant $\Rightarrow$ exponential tail.

Block Maxima Notation: $X_{n,n} = \min(X_1,\dots,X_n)$, $X_{1,n} = \max(X_1,\dots,X_n)$ $\quad X_{n,n} \le X_{n-1,n} \le \dots \le X_{1,n}$

Limiting Argument: (if $X_i \sim \exp(\lambda)$), $\mathbb{P}\left(X_{1,n} \le \frac{x+\log(n)}{\lambda}\right) \approx \Lambda(x) \Rightarrow \mathbb{P}(X_{1,n} \le y) \approx \Lambda(\lambda y - \log n)$ — Gumbel cdf

GEV: $H_{\gamma;\mu,\sigma}(x) = \begin{cases} e^{-\left(1+\gamma\cdot\frac{x-\mu}{\sigma}\right)^{-1/\gamma}}, & \gamma \ne 0 \\ e^{-e^{-\frac{x-\mu}{\sigma}}}, & \gamma = 0 \end{cases}$

$\gamma<0$: upper bounded weibull (small tail) $x < \mu - \sigma/\gamma$

$\gamma=0$: Gumbel (middle size tail) $x \in \mathbb{R}$

$\gamma>0$: Fréchet (heavy tail) $x > \mu - \sigma/\gamma$

- e.g. distributions: $\gamma<0$: beta, uniform, triangular; $\gamma=0$: chi-squared, exponential, gamma, log normal, normal, weibull; $\gamma>0$: Burr, F, log-gamma, Pareto, t
- **Steps:** Find a_n and b_n: $\lim_{n\to\infty}\mathbb{P}\left(\frac{X_{1,n}-b_n}{a_n} \le x\right) = \lim_{n\to\infty}\left(F(a_n x+b_n)^n\right)$ to match GEV form range of x: $x \ge \lim_{n\to\infty}\left(-\frac{b_n}{a_n}\right)$, $n(1-F(a_n x+b_n)) = \tau(x)$, where $H(x) = e^{-\tau(x)}$ to find a_n/b_n
- **For Frechet ($\gamma>0$):** limiting distribution of normalised maximum has $\gamma>0$ if: $\bar F(x) = 1-F(x) = x^{-\frac{1}{\gamma}}L(x)$, where $\lim_{x\to\infty}\frac{L(tx)}{L(x)} = 1, t>0$ (also $\mathbb{E}[X^k] = \infty$ for $k > \frac{1}{\gamma}$)
- **For Gumbel ($\gamma=0$):** limiting distribution of normalised maximum has $\gamma=0$ if: $\mathbb{E}[X^k] < \infty$, $\forall k>0$
- **For Weibull ($\gamma<0$):** limiting distribution of normalised maximum has $\gamma<0$ if: it has a finite endpoint ($x_F < \infty$, where $x_F = \sup\{x\in\mathbb{R} : F(x) < 1\}$)

GPD Asymptotic Tail Behaviour: if pricing EoL, retention limit is the t-year quantile: $u_t = F^{-1}(1-\frac{1}{t})$. **Threshold Exceedance:** $F_u(x) = \frac{F(x+u)-F(u)}{1-F(u)} = \mathbb{P}(Y-u \le x \mid Y>u)$

GPD: $G_{\gamma,\sigma}(x) = \begin{cases} 1-\left(1+\gamma\cdot\frac{x}{\sigma}\right)^{-1/\gamma}, & \gamma\ne 0 \\ 1-e^{-\frac{x}{\sigma}}, & \gamma = 0 \end{cases}$

$\gamma<0$: upper bound (light tail) $x\in(0,\sigma/|\gamma|)$

$\gamma=0$: exponential (base case) $x \ge 0$

$\gamma>0$: Pareto (heavy tail) $x \ge 0$

GPD Central Moment (in excess of u): $\mathbb{E}[X] = \frac{\sigma}{1-\gamma}, \gamma<1$, $Var(X) = \frac{\sigma^2}{(1-\gamma)^2(1-2\gamma)}, \gamma<\frac{1}{2}$,
$\hookrightarrow$ $\mathbb{E}[X^k] < \infty$ if $\gamma < \frac{1}{k}$, $\mathbb{E}[X]$ is the stop loss premium (since X is the excess over u)

Choice of u: We need to estimate tail $\bar F(u+x) = \bar F(u)\bar F_u(x)$, for a fixed large u and $x \ge 0$. $\mathbb{P}(\text{exceeding } u) = \widehat{\bar F(u)} = \frac{n_u}{n}$, then: $\widehat{\bar F_u(x)} = \bar G_{\hat\gamma(u),\hat\sigma(u)}(x)$. Limitations: Larger u is better approximation, but it reduces amount of data to estimate $\bar F(u), \gamma, \sigma$.

M5 - Copulas

Pearson's Corr. Coef.: $\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$. **Kendall's Tau:** $\tau(Z_i,Z_j) = \mathbb{P}((Z_i-Z_i')(Z_j-Z_j')>0) - \mathbb{P}((Z_i-Z_i')(Z_j-Z_j')<0)$, $\tau(Z_i,Z_j) = 4\mathbb{E}[F(Z_i,Z_j)]-1$. **Spearman's Rho:** $Z_i\sim F_i$, $Z_j \sim F_j$ $r(Z_i,Z_j) = \rho(F_i(Z_i),F_j(Z_j))$. **Sklar's Theorem:** $\exists F(x_1,\dots,x_n) = C(F_1(x),\dots,F_n(x))$

Copula Conditions: non-decreasing, right continuous. ① $\lim_{u_k\to 0} C(u_1,u_2) = 0, k=1,2$ ② $\lim_{u_k\to 1} C(u_1,u_2) = u_k, k=1,2$ ③ $C(v_1,v_2) - C(u_1,v_2) - C(v_1,u_2) + C(u_1,u_2) \ge 0$, $\forall u_1 \le v_1, \forall u_2\le v_2$

Invariance Property: if $\vec X$ has copula C, and $T_1,\dots,T_n$ are strictly increasing $\Rightarrow$ $(T_1(X_1),\dots,T_n(X_n)) \sim$ copula C, so copula holds under log, inflation, etc.

Gaussian Copula: $C(u_1,\dots,u_n) = \Phi_\Sigma(\Phi^{-1}(u_1),\dots,\Phi^{-1}(u_n))$ or $C(u_1,u_n) = \Phi_\rho(\Phi^{-1}(u_1),\Phi^{-1}(u_2))$

Fréchet Bounds: all copulas satisfy $L_F(u_1,\dots,u_n) \le C(u_1,\dots,u_n) \le U_F(u_1,\dots,u_n)$, where $L_F = \max(0,\sum_{k=1}^n u_k - (n-1))$, $U_F = \min(u_1,\dots,u_n)$

Comonotonicity Copula: $C(u,v) = \min(u,v)$, where the R.V's are perf. pos. dependent $\quad \lambda_u = \lambda_L = 1$

Countermonotonicity Copula: $C(u,v) = \max(u+v-1,0)$, where R.V's are perf. neg. dependent $\quad \lambda_u=\lambda_L=0$

Survival Copulas: $\bar F(x_1,x_2) = \mathbb{P}(X_1>x,X_2>x) = \bar C(\bar F_1(x_1),\bar F_2(x_2))$, where $\bar C(1-u,1-v) = 1-u-v+C(u,v)$, since $\bar F(x_1,x_2) = 1-F_1(x)-F_2(x)+F(x_1,x_2)$

Coef. of Lower Tail Dependence: $\lambda_L = \lim_{u\to 0^+}\mathbb{P}(X_1\le F_{X_1}^{-1}(u) \mid X_2 \le F_{X_2}^{-1}(u)) = \lim_{u\to 0^+}\frac{C(u,u)}{u} \in [0,1]$

Coef. of Upper Tail Dependence: $\lambda_u = \lim_{u\to 1^-}\mathbb{P}(X_1\ge F_{X_1}^{-1}(u) \mid X_2 \ge F_{X_2}^{-1}(u)) = \lim_{u\to 1^-}\frac{\bar C(1-u,1-u)}{1-u} = \lim_{u\to 0^+}\frac{\bar C(u,u)}{u} \in [0,1]$ ($\lambda = 0$ (no dependence), $\lambda = 1$ (full dependence))

Archimedian Copulas: if $C(u_1,\dots,u_n) = \psi^{-1}(\psi(u_1)+\dots+\psi(u_n))$, ψ = "generator", $\psi(1) = 0$, ψ is strictly decreasing ($\psi'(u)<0, \forall u$), ψ is convex ($\psi''(u) \ge 0, \forall u$).
Kendalls tau relation: $\tau = 1+4\int_0^1 \frac{\psi(t)}{\psi'(t)}dt$

Clayton Copula: $C(u_1,u_2) = (u_1^{-\theta}+u_2^{-\theta}-1)^{-\frac{1}{\theta}}$, $\theta\in(0,\infty)$. $\psi(t) = \frac{1}{\theta}(t^{-\theta}-1)$, $\psi^{-1}(s) = (1+\theta s)^{-\frac{1}{\theta}}$,
$\tau = \frac{\theta}{2+\theta} \Leftrightarrow \theta = \frac{2\tau}{1-\tau}$, $\lambda_L = \lim_{u\to 0^+}\frac{(2u^{-\theta}-1)^{-1/\theta}}{u} = \lim_{u\to 0^+}\frac{(2-u^\theta)^{-1/\theta}u}{u} = 2^{-\frac{1}{\theta}}$, $\lambda_u = \lim_{u\to 1^-}\frac{1-2u+(2u^{-\theta}-1)^{-1/\theta}}{1-u} = 0$
$\Rightarrow$ Copula is asymmetric with (pos.) lower tail dependence, $\lambda_L = 0$ ($\theta\to 0$), $\lambda_L = 1$ ($\theta\to\infty$)

Frank Copula: $C(u_1,u_2) = -\frac{1}{\theta}\log\left(1+\frac{(e^{-\theta u_1}-1)(e^{-\theta u_2}-1)}{e^{-\theta}-1}\right)$, $\theta\in\mathbb{R}\setminus\{0\}$. $\psi(t) = -\log\left(\frac{e^{-\theta t}-1}{e^{-\theta}-1}\right)$, $\psi^{-1}(s) = -\frac{1}{\theta}\log(1+e^{-s}(e^{-\theta}-1))$, $\tau = 1-\frac{4}{\theta}+\frac{4}{\theta^2}\int_0^\theta\frac{t}{e^t-1}dt$, $\lambda_u=\lambda_L=0 \Rightarrow$ Copula is symmetric

Gumbel Copula: $C(u_1,u_2) = e^{\left[-((-\log u_1)^\theta+(-\log u_2)^\theta)^{1/\theta}\right]}$, $\theta\in[1,\infty)$. $\psi(t) = (-\log t)^\theta$, $\psi^{-1}(s) = e^{-t^{1/\theta}}$,
$\tau = \frac{\theta-1}{\theta} \Leftrightarrow \theta = \frac{1}{1-\tau}$, $\lambda_L = 0$, $\lambda_u = 2-2^{\frac{1}{\theta}}$

Distributions/Formulas

Gamma $\Gamma(\alpha,\beta)$: $g(x) = \frac{\beta^\alpha x^{\alpha-1}e^{-\beta x}}{\Gamma(\alpha)}$, $\mathbb{E} = \frac{\alpha}{\beta}$, $V = \frac{\alpha}{\beta^2}$, $\varsigma = \frac{2}{\sqrt\alpha}$ (pos. skew), $M_X(t) = \left(\frac{\beta}{\beta-t}\right)^\alpha (t<\beta)$, $\mathbb{E}[X^k] = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)\beta^k}$

Inverse Gaussian $IG(\alpha,\beta)$: $g(x) = \frac{\alpha x^{-3/2}}{\sqrt{2\pi\beta}}e^{-\frac{(\alpha-\beta x)^2}{2\beta x}}$ $(x>0)$, $\mathbb{E} = \frac{\alpha}{\beta}$, $V = \frac{\alpha}{\beta^2}$, $\varsigma = \frac{3}{\sqrt\alpha}$, $M_X(t) = e^{\alpha\left(1-\sqrt{1-(2t/\beta)}\right)}$ $(t<\beta/2)$

Weibull (τ,c): $g(x) = (c\tau)(cx)^{\tau-1}e^{-(cx)^\tau}$ $(x>0)$, $G(x) = 1-e^{-(cx)^\tau}$, $\mathbb{E} = \frac{\Gamma(1+1/\tau)}{c}$, $V = \frac{\Gamma(1+2/\tau)}{c^2}-\mu^2$, $\varsigma = \left[\frac{\Gamma(1+3/\tau)}{c^2}-3\mu\sigma^2-\mu^3\right]/\sigma^3$, $\mathbb{E}[X^k] = \frac{\Gamma(1+k/\tau)}{c^k}$, if $Z\sim\exp(1) \Rightarrow Z^{1/\tau}/c \sim W(\tau,c)$

$X\sim LN(\mu,\sigma^2)$: $\log(X)\sim N(\mu,\sigma^2)$, $\mathbb{E} = e^{\mu+\frac{\sigma^2}{2}}$, $V = e^{2\mu+\sigma^2}(e^{\sigma^2}-1)$, $\varsigma = (e^{\sigma^2}+2)(e^{\sigma^2}-1)$

Log-gamma ($\log X\sim\Gamma(\alpha,\beta)$): $g(x) = \frac{c^\gamma}{\Gamma(\gamma)}(\log(x))^{\gamma-1}x^{-(c+1)}$ $(x>0)$, $\mathbb{E} = \left(\frac{c}{c-1}\right)^\gamma (c>1)$, $V = \left(\frac{c}{c-2}\right)^\gamma-\mu^2 (c>2)$, $\varsigma = \left[\left(\frac{c}{c-3}\right)^\gamma-3\mu\sigma^2-\mu^3\right]/\sigma^3$, $\mathbb{E}[X^k] = \left(\frac{c}{c-k}\right)^\gamma (c>k)$

Pareto(θ,α): $g(x) = \frac{\alpha}{\theta}\left(\frac{x}{\theta}\right)^{-(\alpha+1)}$ $(x\ge\theta)$, $G(x) = 1-\left(\frac{\theta}{x}\right)^\alpha$, $\mathbb{E} = \theta\cdot\frac{\alpha}{\alpha-1} (\alpha>1)$, $V = \theta^2\cdot\frac{\alpha}{(\alpha-1)^2(\alpha-2)} (\alpha>2)$, $\varsigma = \frac{2(1+\alpha)}{\alpha-3}\cdot\left(\frac{\alpha-2}{\alpha}\right)^{1/2} (\alpha>3)$. Let $Z = Y-\theta$: $F_Z(z) = 1-\left(\frac{\theta}{\theta+z}\right)^\alpha$, $z\ge 0$

Definition of e^a: $\lim_{n\to\infty}\left(\left(1+\frac{a}{n}\right)^n\right) = e^a$ or $\lim_{n\to 0}\left((1-na)^{-\frac{1}{n}}\right) = e^a$

MGF: $M_X(t) = \mathbb{E}[e^{Xt}]$ **PGF:** $P_X(t) = \mathbb{E}[t^X]$ **CGF:** $k_X(t) = \log M_X(t)$, $\mathbb{E}[X] = k_X'(0)$, $Var(X) = k''(0)$

if $Y = G(X)$: $F_Y(x) = F_X(G^{-1}(y))$ for $G\uparrow$ or $1-F_X(G^{-1}(y))$ for $G\downarrow$. $f_Y(y) = f_X(G^{-1}(x))\left|\frac{d}{dy}G^{-1}(y)\right|$

Poisson(λ): $f_X(x) = \frac{\lambda^x e^{-\lambda}}{x!}$, $\mathbb{E} = V = \lambda$, $\varsigma = \frac{1}{\sqrt\lambda}$, $M_X(t) = e^{\lambda(e^t-1)}$

NegBinom(r,p): $f_X(x) = \binom{x+r-1}{x}(1-p)^x p^r$, $\mathbb{E} = \frac{r(1-p)}{p}$, $V = \frac{r(1-p)}{p^2}$, $M_X(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r$ $t<-\log(1-p)$, $\varsigma = \frac{2-p}{\sqrt{(1-p)r}}$, "# failures k before observing r successes, with prob. p"

Binom(n,p): $f_X(x) = \binom{n}{x}p^x(1-p)^{n-x}$, $\mathbb{E} = np$, $V = np(1-p)$, $\varsigma = \frac{1-2p}{\sqrt{np(1-p)}}$, $M_X(t) = (1-p+pe^t)^n$

CompPois $\mathbb{E}, V, \varsigma$: $N\sim Pois(\lambda)$, $m_k = \mathbb{E}[X^k]$, $\mathbb{E}[S] = \lambda m_1$, $V(S) = \lambda m_2$, $\varsigma_S = \mathbb{E}\left[\left(\frac{S-\mu_S}{\sigma_S}\right)^3\right] = \frac{\lambda m_3}{(\lambda m_2)^{3/2}}$

Taylor Series: $e^{-x} = \sum_{n=0}^\infty\frac{(-x)^n}{n!} = 1-x+\dots$, $\log(1-x) = \sum_{n=1}^\infty(-1)^{n+1}\frac{x^n}{n} = x-\frac{x^2}{2}+\dots \quad |x|<1$

Integration by Parts: $\int u\,dv = uv - \int v\,du$