Problem Set #2

Consider a bipartite system existing in state $|\psi\rangle = \lambda_1|00\rangle + \lambda_2|11\rangle$, where $\lambda_1 = (1 + i) / \sqrt{3}$ and $\lambda_2 = 1 / \sqrt{3}$. Calculate the following quantities given in questions 1-4 if each subsystem has a Hilbert space spanned by $\{|0\rangle,|1\rangle\}$:

Q1) Quantum coherence present in the joint system in terms of (i) *relative entropy of coherence* and (ii) *l_1 norm of coherence*.
Q2) Total correlations shared (i) in the original state $|\psi\rangle$ and (ii) in the modified state ρ_d = *diagonal*($|\psi\rangle\langle\psi|$) in terms of *mutual information*.
Q3) Quantum entanglement shared in the joint state in terms of *entanglement entropy* (*entanglement of formation*).
Q4) Quantum entanglement shared in the joint state in terms of *entanglement negativity*.

Calculate the following quantities given in questions 5-8 if the bipartite system under consideration exists in state $|\psi\rangle$ with a probability of 3/5, in state $|01\rangle$ with a probability of 1/5, and in state $|10\rangle$ with a probability of 1/5.

Q5) Quantum coherence present in the joint system in terms of (i) *relative entropy of coherence* and (ii) *l_1 norm of coherence*.
Q6) Total correlations shared (i) in the original state ρ and (ii) in the modified state ρ_d = *diagonal*(ρ) in terms of *mutual information*.
Q7) Quantum entanglement shared in the joint state in terms of *entanglement of formation*.
Q8) Quantum entanglement shared in the joint state in terms *of entanglement negativity*.

A thermal state is defined as

$$\rho_\beta = \exp(-\beta H) / Z = \sum_j \exp(-\beta E_j) / Z\, |E_j\rangle\langle E_j|$$

where H is the Hamiltonian of the system, $\{E_j, |E_j\rangle\}$ are eigenvalues of eigenvectors of this Hamiltonian, $\beta = 1/(k_B T)$ is the inverse temperature of the environment, and Z is the partition function that equals to

$$Z = \mathrm{tr}(\exp(-\beta H)) = \sum_j \exp(-\beta E_j).$$

Q9) Show that $S(\rho||\rho_\beta) \geq S(\rho_\beta||\rho_\beta)$ implies $\beta\, \Delta U \geq \Delta S$, where $U = \langle H\rangle$ is the expected value of energy, $\Delta U = U(\rho) - U(\rho_\beta)$, $\Delta S = S(\rho) - S(\rho_\beta)$. Here, define entropy $S(\bullet)$ and relative entropy $S(\bullet||\bullet)$ using the natural logarithm.
Q10) Consider an isolated composite system that consists of two two-level systems and initially exist in the following state

$$\rho_{12}(t_0) = \rho_\beta \otimes \rho_{\beta'} + \lambda\, (|01\rangle\langle 10|+|10\rangle\langle 01|)$$

where $H^{(1)} = E_1\, |0\rangle\langle 0|_1 + E_2\, |1\rangle\langle 1|_1$ and $H^{(2)} = \varepsilon_1\, |0\rangle\langle 0|_2 + \varepsilon_2\, |1\rangle\langle 1|_2$. Assume that the joint state evolves to $\rho_{12}(t_1)$ by a heat-exchange unitary interaction between the subsystems. Show that $\beta\, Q_1 + \beta'\, Q_2 \geq \Delta I$ where Q_j is the amount of heat that flows into the jth subsystem and I denotes the mutual information between subsystems. Note that the entropy is invariant under unitary evolutions.