Lab Report:
Bell's Inequality and Quantum Tomography

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Contents

1 Introduction

Over the past century, quantum mechanics has presented several counterintuitive phenomena that sharply depart from the established tenets of classical physics. Of all these peculiarities, *entanglement* stands out as one of the most intriguing and foundational aspects, placing quantum theory in direct conflict with our classical notions of locality and realism. In their seminal paper of 1935, Einstein, Podolsky, and Rosen (EPR) raised the question of whether quantum mechanics is a complete description of physical reality [2]. EPR's concerns led to the concept of *hidden variables* as possible explanations for the nonclassical correlations displayed by quantum systems.

It was not until 1964 that John Bell formulated a set of inequalities (now referred to as *Bell's inequalities*) to test if these hidden-variable theories could account for all observed quantum phenomena. Specifically, Bell's theorem shows that any local-realistic theory must obey these inequalities, whereas appropriately prepared quantum systems can violate them. Experimental violations of Bell's inequality therefore suggest that no local hidden-variable theory can fully capture the predictions of quantum mechanics.

Motivation and Aim of the Experiment

In this Advanced Laboratory Course experiment, we focus on generating and characterizing *polarization-entangled* photon pairs. The experiment is designed to address fundamental questions about quantum correlations and to explore key tools employed in modern quantum information science:

- **Generation of Entangled Photons**: Using a nonlinear optical process known as Spontaneous Parametric Down Conversion (SPDC), we create pairs of photons whose polarizations are entangled. By carefully adjusting the crystal orientation and compensators, we aim to produce one of the four *Bell states*, such as $|\phi^+\rangle = \frac{1}{\sqrt{2}}\big(|HH\rangle + |VV\rangle\big)$.

- **Measurement of Correlation Functions**: To confirm the presence of entanglement, we measure photon correlations in different polarization bases. In particular, we examine correlation functions for multiple angle settings of half-wave and quarter-wave plates. By rotating these wave plates, we can project our photonic qubits onto various polarization bases (e.g., horizontal/vertical, diagonal/antidiagonal, right/left circular). These correlation measurements are essential for characterizing entanglement and for performing quantum state tomography.

- **Violation of Bell's Inequality**: We combine the correlation measurements in carefully chosen settings to test the *Clauser-Horne-Shimony-Holt* (CHSH) form of Bell's inequality. Local hidden-variable theories demand that a certain combination of correlation values (the CHSH parameter S) does not exceed 2. Quantum mechanics, however, predicts S can reach values up to $2\sqrt{2}$. Our aim is to empirically demonstrate $S > 2$, thus ruling out local realism under the assumptions of the measurement.

- **Quantum State Tomography**: Beyond detecting the presence of entanglement, we further reconstruct the full density matrix of the generated two-photon state. Quantum tomography involves systematic measurements in a complete set of polarization bases (often taken to be X, Y, and Z for each qubit, leading to $3 \times 3 = 9$

total basis combinations). From these measurements, we retrieve the density matrix and can then quantify properties such as purity, fidelity (with respect to an ideal Bell state), and negativity (related to the Positive Partial Transpose criterion). These metrics help us evaluate the quality of our entangled source and quantify the degree of entanglement.

Structure of the Report

The sections that follow provide a detailed account of the physics background (*Qubits and Entanglement*, *Bell's Inequality*, and *Density Matrix and Quantum Tomography*). Subsequently, we describe the experimental apparatus used to generate and analyze polarization-entangled photons, including the specific steps required to observe the violation of Bell's inequality and to conduct a complete quantum state tomography. Finally, we present and discuss our measured results, comparing them with theoretical expectations. The experiment thereby serves as a practical demonstration of several core concepts in quantum information and quantum optics, connecting fundamental theory with cutting-edge applications such as quantum cryptography, teleportation, and quantum computing.

By carrying out the tasks in the experiment manual, we gain a concrete understanding of:

- How to align and optimize the nonlinear crystal setup for consistent generation of entangled photons,
- The relevance of wave plate adjustments for projecting qubits onto different measurement bases,
- Strategies for collecting and analyzing coincidence counts that confirm quantum correlations,
- And finally, how to perform the data analysis leading to Bell inequality violation and complete state reconstruction.

These goals underscore the vital role of photonic qubits in testing the foundations of quantum mechanics, as well as their importance in emerging quantum technologies.

2 Theory

2.1 Qubits and Entanglement

A **qubit** is a two-level quantum system. For photonic qubits, we commonly identify the horizontal ($|H\rangle$) and vertical ($|V\rangle$) polarizations as computational basis states:

$$|0\rangle \equiv |H\rangle, \quad |1\rangle \equiv |V\rangle .$$

Any single-qubit polarization state can be written as a superposition

$$|\psi\rangle = a\,|H\rangle + b\,|V\rangle ,$$

with complex coefficients a and b satisfying $|a|^2 + |b|^2 = 1$. Geometrically, the state space of a single qubit can be visualized on the Bloch (or Poincaré) sphere.

Two-Qubit States. When two qubits are involved, the joint state lives in the tensor product of two 2-dimensional Hilbert spaces. A general two-qubit state can be expressed as

$$|\Psi\rangle = a_{HH}|HH\rangle + a_{HV}|HV\rangle + a_{VH}|VH\rangle + a_{VV}|VV\rangle .$$

If this state cannot be decomposed as a product of two single-qubit states, it is called *entangled*. One of the most famous families of entangled two-qubit states is the set of four *Bell states*:

$$\begin{aligned}
|\phi^+\rangle &= \frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle), \\
|\phi^-\rangle &= \frac{1}{\sqrt{2}}(|HH\rangle - |VV\rangle), \\
|\psi^+\rangle &= \frac{1}{\sqrt{2}}(|HV\rangle + |VH\rangle), \\
|\psi^-\rangle &= \frac{1}{\sqrt{2}}(|HV\rangle - |VH\rangle).
\end{aligned}$$

Entanglement and Local Realism. Entangled states exhibit correlations that cannot be explained by local hidden-variable theories. This fundamental nonlocality is at the heart of quantum mechanics and is what leads to the possibility of violating Bell's inequality in suitably designed experiments.

2.2 Bell's Inequality

Inspired by the Einstein-Podolsky-Rosen paradox, Bell formulated an inequality that any local-realistic theory must satisfy [?]. In the *Clauser-Horne-Shimony-Holt* (CHSH) form, it reads

$$S = |E(a,b) - E(a,b') + E(a',b) + E(a',b')| \leq 2,$$

where $E(a,b)$ represents the correlation coefficient for measurement settings a and b on two spatially separated qubits. Quantum mechanics predicts that *entangled* states can yield

$$S_{\text{QM}} \leq 2\sqrt{2},$$

thus violating the classical bound of 2. This violation is direct evidence that quantum correlations cannot be explained by local hidden variables.

Measurement Settings. In practice, to observe a violation of the Bell or CHSH inequality, we measure correlations in appropriately chosen linear (and sometimes circular) polarization bases. By adjusting wave plates and polarization beam splitters, we can measure along any basis. The "maximum" violation of $2\sqrt{2}$ is typically obtained for particular rotations separated by 45° or 22.5° increments in polarization.

2.3 Density Matrix and Quantum Tomography

While *pure states* can be represented by a state vector $|\psi\rangle$, real experimental states are often *mixed*, arising from statistical or environmental noise. In these cases, the density operator (or density matrix) ρ provides a complete description of the system:

$$\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i| ,$$

with probabilities p_i for the mixture of pure states $|\phi_i\rangle$.

Properties of the Density Matrix.

- ρ is *Hermitian*: $\rho = \rho^\dagger$.
- ρ is *positive semi-definite*: its eigenvalues are nonnegative.
- $\mathrm{Tr}(\rho) = 1$.

Quantum Tomography. To experimentally reconstruct ρ, we perform measurements in a complete set of bases, typically corresponding to the Pauli operators $(\sigma_x, \sigma_y, \sigma_z)$ for each qubit. For two qubits, one reconstructs the 4×4 matrix by combining measurement outcomes from nine basis settings (i.e., all combinations $XX, XY, \ldots, ZZ$). The measured coincidence counts allow one to extract expectation values $\langle \sigma_i \otimes \sigma_j \rangle$, which then determine the matrix elements of ρ.

In other words, we can write:

$$\rho = \frac{1}{4} \sum_{i,j=0}^{3} s_{ij}\, \sigma_i \otimes \sigma_j,$$

where σ_0 is the identity matrix and $\sigma_{1,2,3}$ are the usual Pauli matrices. The coefficients s_{ij} are obtained from the measured correlations and single-qubit observables.

Entanglement Witnesses and PPT Criterion. To identify entanglement, we can check if ρ has a nonpositive partial transpose (the *PPT criterion*) or if it violates an *entanglement witness* condition. For two qubits, the PPT criterion is both necessary and sufficient: a state is entangled if its partial transpose has a negative eigenvalue.

In the following sections of the report, we will describe how these theoretical ideas were put into practice to generate polarization-entangled photon pairs, measure their correlation functions, and perform quantum tomography to assess the entanglement fidelity and purity of the states we produced.

3 Experimental Setup

Experimental setup section.

4 Measurement and Results

As mentioned above, this experiment has 3 different sets of measurements:

- **Measurement of Correlation Functions**
- **Violation of Bell's Inequality**
- **Quantum State Tomography**

Thus, this section is also divided into 3 different sections. Each subsection is dedicated to one of the following.

4.1 Correlation Functions

In the first part of the experiment, the aim is to measure correlation functions for 2 different cases. In both cases, there are 2 different half-wave plates. One half-wave plate (HWP_B) is located in the pathway of the photon that goes to Bob, and the other one (HWP_A) is located in the pathway of the photon that goes to Alice. The angles of the half-wave plates are as follows:

- Case 1: $\alpha_{HWP_B} = 0°$, $\alpha_{HWP_A} = i°$ ($i \in \{0, 10, 20, 30, 40, 50, 60, 70, 80, 90\}$)
- Case 2: $\alpha_{HWP_B} = 22.5°$, $\alpha_{HWP_A} = i°$ ($i \in \{0, 10, 20, 30, 40, 50, 60, 70, 80, 90\}$)

For both cases, the angles for the half-wave plates were changed by rotating the half-wave plate. After each configuration, coincidence counts were recorded by the help of a C++ script. Counts for each state and case can be seen below:

α_{HWP_A}	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
C_{HH}	251	197	164	76	29	9	28	113	187	216
C_{HV}	6	18	67	133	196	195	173	111	45	8
C_{VH}	5	8	67	135	206	232	225	127	56	7
C_{VV}	205	199	165	73	22	10	36	88	162	234

Table 1: Coincidence counts for the first case where $\alpha_{HWP_B} = 0°$.

α_{HWP_A}	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
C_{HH}	101	161	167	139	108	42	14	25	48	111
C_{HV}	105	50	26	30	76	154	190	199	155	112
C_{VH}	86	45	18	42	91	167	186	180	126	82
C_{VV}	106	183	222	199	157	98	37	21	62	135

Table 2: Coincidence counts for the second case where $\alpha_{HWP_B} = 22.5°$.

4.1.1 Calculation of Correlation Functions

To calculate the correlation values, first we need to calculate relative frequencies for each state. Relative frequencies are calculated by dividing the coincidence counts by the total number of counts:

$$f_{ij} = \frac{C_{ij}}{\sum_{i,j} C_{ij}} \quad (i, j \in \{H, V\}) \tag{1}$$

After calculating the relative frequencies, we can calculate the correlation values using the following formula with the relative frequencies from Table 1 and Table 2:

$$K_{ij}^{ex} = f_{HH} - f_{HV} - f_{VH} + f_{VV} \tag{2}$$

with $i, j \in \{H, V\}$. Correlation values for both cases can be seen below:

α_{HWP_A}	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
Case 1	0.953	0.877	0.421	-0.279	-0.775	-0.915	-0.723	-0.084	0.551	0.936
Case 2	0.040	0.567	0.797	0.648	0.227	-0.393	-0.761	-0.784	-0.437	0.118

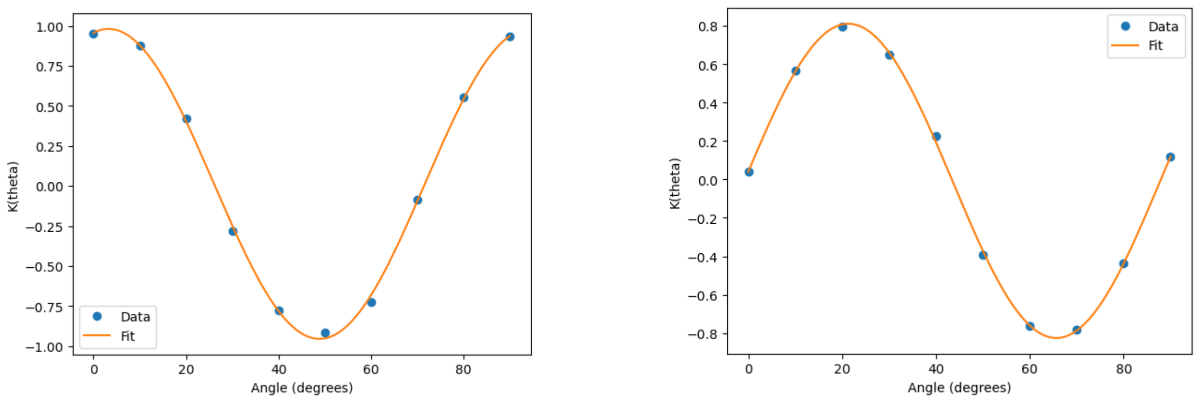
Table 3: Calculated correlation values.

4.1.2 Visibility

The visibility can be used to parameterize the contrast of measured graphs [1]. It is defined as the ratio of the difference between the maximum and minimum value of the correlation function to the sum of the maximum and minimum values. Since a correlation function can be bounded by -1 and 1, the visibility of this function would lead to $\frac{2}{0}$ in a perfect experimental setup. This is why we need to fit the correlation function to a function to calculate the visibility. Since the correlation function is a sinusoidal function, we can fit it to a sinusoidal function. The fit function is given as:

$$f(\theta) = A \cdot \sin(B \cdot \theta + C) + D \tag{3}$$

The fits for both cases can be seen below:



(a) Fit for the first case.

(b) Fit for the second case.

Figure 1: Comparison of fits for the first and second cases.

Discussion on the necessity of two correlation functions for entanglement detection.
Calculation of visibility using the provided fit function.

4.2 Violation of Bell's Inequality

Experimental settings for Alice and Bob's angles.
Calculation of correlation functions and error propagation.
Interpretation of results demonstrating Bell inequality violation.

4.3 Quantum State Tomography

Measurement procedure for the four Bell states ($|\phi^+\rangle$, $|\phi^-\rangle$, $|\psi^+\rangle$, $|\psi^-\rangle$).
Reconstruction of density matrices from experimental data.
Calculation of fidelity, purity, and eigenvalues.
Proof of entanglement using PPT criterion and entanglement witnesses.

5 Discussion

Interpret the results and discuss their implications. Mention any limitations and possible future work.

6 Conclusion

Conclusion section. add references from the bibliography [?]

References

[1] Lab course: Bell's inequality and quantum tomography. *Max Planck Institute of Quantum Optics*, 2020.

[2] A. Einstein, B. Podolsky, and N. Rosen. Can quantum-mechanical description of physical reality be considered complete? *Physical review*, 47(10):777, 1935.