Lab Report:

Bell's Inequality and Quantum Tomography

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1 Introduction

Over the past century, quantum mechanics has presented several counterintuitive phenomena that sharply depart from the established tenets of classical physics. Of all these peculiarities, *entanglement* stands out as one of the most intriguing and foundational aspects, placing quantum theory in direct conflict with our classical notions of locality and realism. In their seminal paper of 1935, Einstein, Podolsky, and Rosen (EPR) raised the question of whether quantum mechanics is a complete description of physical reality [4]. EPR's concerns led to the concept of *hidden variables* as possible explanations for the nonclassical correlations displayed by quantum systems.

It was not until 1964 that John Bell formulated a set of inequalities (now referred to as *Bell's inequalities*) to test if these hidden-variable theories could account for all observed quantum phenomena. Specifically, Bell's theorem shows that any local-realistic theory must obey these inequalities, whereas appropriately prepared quantum systems can violate them. Experimental violations of Bell's inequality therefore suggest that no local hidden-variable theory can fully capture the predictions of quantum mechanics.

Motivation and Aim of the Experiment

In this Advanced Laboratory Course experiment, we focus on generating and characterizing *polarization-entangled* photon pairs. The experiment is designed to address fundamental questions about quantum correlations and to explore key tools employed in modern quantum information science:

- **Generation of Entangled Photons**: Using a nonlinear optical process known as Spontaneous Parametric Down Conversion (SPDC), we create pairs of photons whose polarizations are entangled. By carefully adjusting the crystal orientation and compensators, we aim to produce one of the four *Bell states*, such as $|\phi^+\rangle = \frac{1}{\sqrt{2}}\big(|HH\rangle + |VV\rangle\big)$.

- **Measurement of Correlation Functions**: To confirm the presence of entanglement, we measure photon correlations in different polarization bases. In particular, we examine correlation functions for multiple angle settings of half-wave and quarter-wave plates. By rotating these wave plates, we can project our photonic qubits onto various polarization bases (e.g., horizontal/vertical, diagonal/antidiagonal, right/left circular). These correlation measurements are essential for characterizing entanglement and for performing quantum state tomography.

- **Violation of Bell's Inequality**: We combine the correlation measurements in carefully chosen settings to test the *Clauser-Horne-Shimony-Holt* (CHSH) form of Bell's inequality. Local hidden-variable theories demand that a certain combination of correlation values (the CHSH parameter S) does not exceed 2. Quantum mechanics, however, predicts S can reach values up to $2\sqrt{2}$. Our aim is to empirically demonstrate $S > 2$, thus ruling out local realism under the assumptions of the measurement.

- **Quantum State Tomography**: Beyond detecting the presence of entanglement, we further reconstruct the full density matrix of the generated two-photon state. Quantum tomography involves systematic measurements in a complete set of polarization bases (often taken to be X, Y, and Z for each qubit, leading to $3 \times 3 = 9$

total basis combinations). From these measurements, we retrieve the density matrix and can then quantify properties such as purity, fidelity (with respect to an ideal Bell state), and negativity (related to the Positive Partial Transpose criterion). These metrics help us evaluate the quality of our entangled source and quantify the degree of entanglement.

Structure of the Report

The sections that follow provide a detailed account of the physics background (*Qubits and Entanglement*, *Bell's Inequality*, and *Density Matrix and Quantum Tomography*). Subsequently, we describe the experimental apparatus used to generate and analyze polarization-entangled photons, including the specific steps required to observe the violation of Bell's inequality and to conduct a complete quantum state tomography. Finally, we present and discuss our measured results, comparing them with theoretical expectations. The experiment thereby serves as a practical demonstration of several core concepts in quantum information and quantum optics, connecting fundamental theory with cutting-edge applications such as quantum cryptography, teleportation, and quantum computing.

By carrying out the tasks in the experiment manual, we gain a concrete understanding of:

- How to align and optimize the nonlinear crystal setup for consistent generation of entangled photons,
- The relevance of wave plate adjustments for projecting qubits onto different measurement bases,
- Strategies for collecting and analyzing coincidence counts that confirm quantum correlations,
- And finally, how to perform the data analysis leading to Bell inequality violation and complete state reconstruction.

These goals underscore the vital role of photonic qubits in testing the foundations of quantum mechanics, as well as their importance in emerging quantum technologies.

2 Theory

2.1 Qubits and Entanglement

A **qubit** is a two-level quantum system. For photonic qubits, we commonly identify the horizontal ($|H\rangle$) and vertical ($|V\rangle$) polarizations as computational basis states:

$$|0\rangle \equiv |H\rangle\,, \quad |1\rangle \equiv |V\rangle\,.$$

Any single-qubit polarization state can be written as a superposition

$$|\psi\rangle = a\,|H\rangle + b\,|V\rangle\,,$$

with complex coefficients a and b satisfying $|a|^2 + |b|^2 = 1$. Geometrically, the state space of a single qubit can be visualized on the Bloch (or Poincaré) sphere.

Two-Qubit States. When two qubits are involved, the joint state lives in the tensor product of two 2-dimensional Hilbert spaces. A general two-qubit state can be expressed as

$$|\Psi\rangle = a_{HH} |HH\rangle + a_{HV} |HV\rangle + a_{VH} |VH\rangle + a_{VV} |VV\rangle .$$

If this state cannot be decomposed as a product of two single-qubit states, it is called *entangled*. One of the most famous families of entangled two-qubit states is the set of four *Bell states*:

$$\begin{aligned}
|\phi^+\rangle &= \frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle),\\
|\phi^-\rangle &= \frac{1}{\sqrt{2}}(|HH\rangle - |VV\rangle),\\
|\psi^+\rangle &= \frac{1}{\sqrt{2}}(|HV\rangle + |VH\rangle),\\
|\psi^-\rangle &= \frac{1}{\sqrt{2}}(|HV\rangle - |VH\rangle).
\end{aligned}$$

Entanglement and Local Realism. Entangled states exhibit correlations that cannot be explained by local hidden-variable theories. This fundamental nonlocality is at the heart of quantum mechanics and is what leads to the possibility of violating Bell's inequality in suitably designed experiments.

2.2 Bell's Inequality

Albert Einstein, Boris Podolsky, and Nathan Rosen originally posed the question of whether quantum mechanics provides a complete description of reality [4], suggesting the possibility that yet-to-be-discovered *hidden variables* could explain the seemingly random outcomes of quantum experiments in a deterministic, local way. In 1964, John Bell offered a way to distinguish between the predictions of such local hidden-variable (LHV) theories and the predictions of quantum mechanics. Specifically, Bell derived an inequality that any LHV model must satisfy, whereas certain *entangled* quantum states can violate it [2].

CHSH Inequality. One particularly convenient form of Bell's inequality was introduced by Clauser, Horne, Shimony, and Holt (CHSH). In this framework, each of two distant observers (often called Alice and Bob) can choose between two measurement settings, labeled a and a' for Alice, and b and b' for Bob. Each measurement yields a binary outcome denoted by ± 1. We then define the correlation coefficient

$$E(a, b) \;=\; \langle A(a)\, B(b)\rangle,$$

where $A(a)$ and $B(b)$ are the measurement outcomes (each being ± 1) for the chosen settings a and b, respectively. These correlations must satisfy the CHSH version of Bell's inequality:

$$S \;=\; \Big|\, E(a, b) \;-\; E(a, b') \;+\; E(a', b) \;+\; E(a', b')\Big| \;\leq\; 2.$$

A local-realistic theory cannot surpass this bound. Quantum mechanics, however, predicts that certain entangled states allow for correlations such that

$$S_{\text{QM}} \;\leq\; 2\sqrt{2} \approx\; 2.828,$$

thus exceeding the classical limit of 2. Observing this *Bell violation* in an experiment is a strong indicator that the measured system cannot be described by any local hidden-variable model.

Correlation Functions and Measurement Settings. In practical optical experiments with photons, the observables often correspond to measuring the polarization along chosen axes. For instance, a single-qubit polarization measurement can be described by a projection onto linear polarizations (horizontal $|H\rangle$, vertical $|V\rangle$, or any rotation thereof) or onto circular polarizations (right-handed $|R\rangle$, left-handed $|L\rangle$). By adjusting wave plates and polarization beam splitters, we can implement the measurement settings a, a' (for Alice) and b, b' (for Bob). To maximize the possible violation (i.e., to achieve $S_{\text{QM}} = 2\sqrt{2}$ in an ideal scenario), one typically chooses measurement directions separated by $\pm 45°$ or $\pm 22.5°$.

- **Example Settings for Maximum Violation**: For an entangled state such as the singlet state $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|HV\rangle - |VH\rangle)$, one can choose measurement bases in the equatorial plane of the Bloch sphere at angles differing by 45°. This arrangement leads to the largest predicted quantum correlations and the maximal violation of the CHSH inequality.

In an actual experiment, one records the coincidence counts (simultaneous photon detection events) in the relevant detectors. From these, one computes empirical estimates of $E(a, b)$, $E(a, b')$, $E(a', b)$, and $E(a', b')$. Substituting these estimates into the CHSH parameter S reveals whether or not a Bell violation is observed.

2.3 Density Matrix and Quantum Tomography

Despite being derived largely for *pure states*, the theoretical arguments surrounding entanglement and Bell's inequalities are readily extended to *mixed states* through the density matrix formalism. Real-world experiments inevitably suffer from noise and imperfections, meaning that the generated states may not be perfectly pure. Hence, the density matrix ρ offers the most general description of the quantum state.

Definition and Properties. A density matrix (or density operator) ρ for a quantum state is written as:

$$\rho \;=\; \sum_i p_i \; |\phi_i\rangle \, \langle\phi_i| \, ,$$

where each $|\phi_i\rangle$ is a pure state and p_i is the classical probability associated with that pure state, satisfying $\sum_i p_i = 1$. The density matrix must obey:

- **Hermiticity**: $\rho = \rho^\dagger$.
- **Positive semi-definiteness**: $\rho \geq 0$, implying all eigenvalues are nonnegative.
- **Normalization**: $\text{Tr}(\rho) = 1$.

For *pure states*, ρ can be expressed as $\rho = |\psi\rangle \, \langle\psi|$, and it additionally satisfies $\rho^2 = \rho$, giving $\text{Tr}(\rho^2) = 1$. Mixed states have $\text{Tr}(\rho^2) < 1$.

Quantum State Tomography. Tomography is the procedure by which the density matrix is experimentally reconstructed from a set of measurement outcomes. For a single qubit, one typically measures in at least three orthogonal bases (commonly the eigenbases of the Pauli matrices: X, Y, Z). For a two-qubit system, a complete tomography usually requires measurements of *both* qubits in all combinations of the X, Y, Z bases, giving nine distinct settings in total (e.g., $XX, XY, XZ, YX, \ldots, ZZ$).

From each setting, one extracts coincidence counts C_{ij} that correspond to projecting onto the basis states $|i\rangle \otimes |j\rangle$ (where i, j can be H, V, or any other basis label). Normalizing these counts yields probabilities, which in turn give the expectation values of the tensor products of Pauli matrices,

$$\langle \sigma_i \otimes \sigma_j \rangle .$$

Using the identity:

$$\rho \;=\; \frac{1}{4} \sum_{i,j=0}^{3} s_{ij}\, \sigma_i \otimes \sigma_j,$$

where σ_0 is the 2×2 identity matrix and $\sigma_{1,2,3} \equiv (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli operators, we can solve for the coefficients s_{ij} by matching them to the measured correlation values.

Entanglement Criteria: PPT and Entanglement Witnesses. Once the density matrix is reconstructed, determining whether the state is entangled is achieved by any of several criteria:

- **Positive Partial Transpose (PPT) Criterion:** For a bipartite system in a state ρ, define the partial transpose (with respect to one subsystem, say Alice) by transposing only the indices pertaining to that subsystem. If ρ is separable, this partial transpose remains a valid density matrix (i.e. it remains positive semi-definite). Conversely, if the partial transpose has at least one negative eigenvalue, then ρ is an *entangled* state. For two qubits, this PPT criterion is both necessary and sufficient.

- **Entanglement Witnesses:** An operator W is called an entanglement witness if it is constructed so that $\mathrm{Tr}(W\,\rho_{\mathrm{sep}}) \geq 0$ for *all* separable states ρ_{sep}, while $\mathrm{Tr}(W\,\rho_{\mathrm{ent}}) < 0$ for *at least one* entangled state ρ_{ent}. Thus, if the measured state yields a negative expectation value for W, it is guaranteed to be entangled. In practice, these witnesses can be optimized for specific target states (e.g. certain Bell states) and can offer a relatively simple experimental test of entanglement.

State Characterization: Fidelity and Purity. With a reconstructed ρ at hand, we can also quantify the "closeness" to an ideal target state $|\psi\rangle\langle\psi|$ by computing the *fidelity*:

$$F(\rho, |\psi\rangle) \;=\; \langle\psi|\, \rho\, |\psi\rangle .$$

If ρ is pure, F reaches 1 if and only if $\rho = |\psi\rangle\langle\psi|$. More generally, the fidelity is between 0 and 1 and provides a convenient measure of how well the experimentally produced state matches the desired theoretical one.

Moreover, the *purity* of the state can be quantified via

$$\mathcal{P} \;=\; \mathrm{Tr}(\rho^2).$$

A perfectly pure state has $\mathcal{P} = 1$, whereas a completely mixed (maximally disordered) state in a d-dimensional system has $\mathcal{P} = \frac{1}{d}$.

In summary, Bell's inequality (particularly its CHSH form) and quantum tomography provide complementary ways to investigate and quantify entanglement in photonic qubits. By carefully choosing measurement bases, recording correlation functions, and reconstructing the density matrix, one can conclusively demonstrate the nonclassical, nonlocal nature of quantum mechanics.

3 Experimental Setup

Experimental setup section.

4 Measurement and Results

As mentioned above, this experiment has 3 different sets of measurements:

- **Measurement of Correlation Functions**
- **Violation of Bell's Inequality**
- **Quantum State Tomography**

Thus, this section is also divided into 3 different sections. Each subsection is dedicated to one of the following.

4.1 Correlation Functions

In the first part of the experiment, the aim is to measure correlation functions for 2 different cases. In both cases, there are 2 different half-wave plates. One half-wave plate (HWP_B) is located in the pathway of the photon that goes to Bob, and the other one (HWP_A) is located in the pathway of the photon that goes to Alice. The angles of the half-wave plates are as follows:

- Case 1: $\alpha_{HWP_B} = 0°$, $\alpha_{HWP_A} = i°$ ($i \in \{0, 10, 20, 30, 40, 50, 60, 70, 80, 90\}$)
- Case 2: $\alpha_{HWP_B} = 22.5°$, $\alpha_{HWP_A} = i°$ ($i \in \{0, 10, 20, 30, 40, 50, 60, 70, 80, 90\}$)

For both cases, the angles for the half-wave plates were changed by rotating the half-wave plate. After each configuration, coincidence counts were recorded by the help of a C++ script. Counts for each state and case can be seen below:

α_{HWP_A}	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
C_{HH}	251	197	164	76	29	9	28	113	187	216
C_{HV}	6	18	67	133	196	195	173	111	45	8
C_{VH}	5	8	67	135	206	232	225	127	56	7
C_{VV}	205	199	165	73	22	10	36	88	162	234

Table 1: Coincidence counts for the first case where $\alpha_{HWP_B} = 0°$.

α_{HWP_A}	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
C_{HH}	101	161	167	139	108	42	14	25	48	111
C_{HV}	105	50	26	30	76	154	190	199	155	112
C_{VH}	86	45	18	42	91	167	186	180	126	82
C_{VV}	106	183	222	199	157	98	37	21	62	135

Table 2: Coincidence counts for the second case where $\alpha_{HWP_B} = 22.5°$.

4.1.1 Calculation of Correlation Functions

To calculate the correlation values, first we need to calculate relative frequencies for each state. Relative frequencies are calculated by dividing the coincidence counts by the total number of counts:

$$f_{ij} = \frac{C_{ij}}{\sum_{i,j} C_{ij}} \quad (i, j \in \{H, V\}) \tag{1}$$

After calculating the relative frequencies, we can calculate the correlation values using the following formula with the relative frequencies from Table 1 and Table 2:

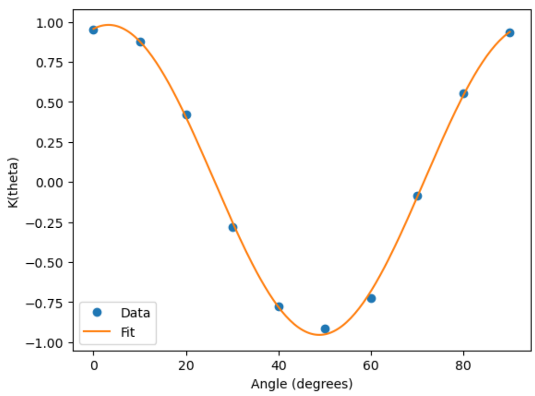
$$K_{ij}^{ex} = f_{HH} - f_{HV} - f_{VH} + f_{VV} \tag{2}$$

with $i, j \in \{H, V\}$. Correlation values for both cases can be seen below:

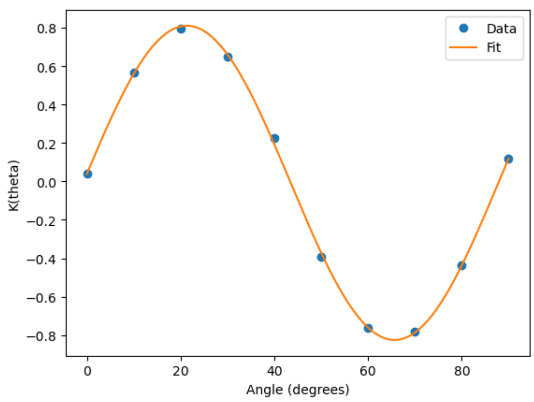
α_{HWP_A}	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
Case 1	0.953	0.877	0.421	-0.279	-0.775	-0.915	-0.723	-0.084	0.551	0.936
Case 2	0.040	0.567	0.797	0.648	0.227	-0.393	-0.761	-0.784	-0.437	0.118

Table 3: Calculated correlation values.

The relation between the correlation values and the angles can be visualized by plotting the correlation values against the angles. Since the correlation values are sinusoidal functions, we can fit the correlation values to a sinusoidal function. The fits for both cases can be seen below:



(a) Case 1.



(b) Case 2.

Figure 1: Correlation values for both cases. $f(\theta) = A \cdot \sin(B \cdot \theta + C) + D$ is used as the fit function.

4.1.2 Visibility

The visibility can be used to parameterize the contrast of measured graphs [1]. It is defined as the ratio of the difference between the maximum and minimum value of the correlation function to the sum of the maximum and minimum values. Since a correlation function can be bounded by -1 and 1, the visibility of this function would lead to $\frac{2}{0}$ in a perfect experimental setup. To deduce the visibility of the correlation function, we can use a fit function that maps a flat correlation function to 0 and a sinusoidal correlation function to 1. This can be easily achieved by using the following formula:

$$\mathcal{V} = \frac{\max_\theta \tilde{f}(\theta) - \min_\theta \tilde{f}(\theta)}{2}$$

This formula will result in a value between 0 and 1, where 0 indicates a flat correlation function and 1 indicates a correlation function varrying between -1 and 1. By using this formula, we can calculate the visibility of the correlation functions for both cases. *Case 1* has a visibility of 0.934, while *Case 2* has a visibility of 0.791.

4.1.3 Necessity of Two Correlation Functions

In this experiment, (as any other Bell test experiment) two correlation functions are needed to detect entanglement. Entanglement is a quantum correlation and to be able to detect it, we need to detect correlations (measuring correlation functions) in at least two different bases. Just measuring the correlation in one basis is not enough to detect entanglement, since correlations in one basis can be explained by classical correlations and local hidden variables[2, 3].

4.2 Violation of Bell's Inequality

In the second part of the experiment, the aim is to demonstrate the violation of CHSH inequality which is a generalization of Bell's theorem[3]. As in the first part of the experiment here we also have one half-wave plate for each party. Unlike the first part, we only have 4 sets of angles for the half-wave plates. The angles are as follows:

Alice		Bob	
α	α'	β	β'
$22.5°$	$0°$	$11.25°$	$-11.25°$

Table 4: Measurement settings for Alice and Bob.

Experimental settings for Alice and Bob's angles.
Calculation of correlation functions and error propagation.
Interpretation of results demonstrating Bell inequality violation.

4.3 Quantum State Tomography

Measurement procedure for the four Bell states ($|\phi^+\rangle$, $|\phi^-\rangle$, $|\psi^+\rangle$, $|\psi^-\rangle$).
Reconstruction of density matrices from experimental data.
Calculation of fidelity, purity, and eigenvalues.
Proof of entanglement using PPT criterion and entanglement witnesses.

5 Discussion

Interpret the results and discuss their implications. Mention any limitations and possible future work.

6 Conclusion

Conclusion section. add references from the bibliography [4]

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