Solution to Tutorial 6

Comp1210

Question 1

(i) For all $a, b \in S$,

$$a \otimes b = a + b + ab$$
$$= b + a + ba$$
$$= b \otimes a.$$

Therefore, $\otimes$ is commutative.

(ii) Let $a, b, c \in S$. Then,

$$a \otimes (b \otimes c) = a \otimes (b + c + bc)$$
$$= a + b + c + bc + a(b + c + bc)$$
$$= a + b + c + bc + ab + ac + abc.$$

Also,

$$(a \otimes b) \otimes c = (a + b + ab) \otimes c$$
$$= a + b + ab + c + (a + b + ab)c$$
$$= a + b + ab + c + ac + bc + abc.$$

Since $a \otimes (b \otimes c) = (a \otimes b) \otimes c$, then $\otimes$ is associative.

(iii) Suppose $e \in S$ is the identity with respect to $\otimes$, $\forall\, x \in S$,
$x \otimes e = x$ and also
$x \otimes e = x + e + xe$. Equating for $x \otimes e$ we have $x = x + e + xe$.
$\therefore\ x - x = e + xe$. So, $0 = e(1+x)$.
$\therefore\ e = 0$ or $1 + x = 0$. So, $e = 0$ or $x = -1$;
but $1 + x \neq 0$ since $x \neq -1$. Therefore, $e = 0$ is the identity.

(iv) Suppose $\bar{x} \in S$ represent the inverse of $x \in S$. Then $x \otimes \bar{x} = e = 0$ and also $x \otimes \bar{x} = x + \bar{x} + x\bar{x}$. Equating we have $0 = x + \bar{x} + x\bar{x}$.
$\therefore\ -x = \bar{x}(1+x)$. So, $\bar{x} = \dfrac{-x}{1+x}$,
where $1 + x \neq 0 \Rightarrow x \neq -1$.
So, every element have inverses except $x = -1$.

(v) Now, $5 \otimes x = 5 + x + 5x = 4$.
So, $6x = 4 - 5$. So, $6x = -1$.
Therefore, $x = -1/6$.

Question 2

(i) $\forall\ (m,n), (p,q) \in \mathbb{R} \times \mathbb{R}$,

$$\begin{aligned}(m,n) * (p,q) &= (mq + np, nq)\\ &= (np + mq, nq)\\ &= (pn + qm, qn)\\ &= (p,q) * (m,n).\end{aligned}$$

Therefore, $*$ is commutative.

Let $(m,n), (p,q), (r,s) \in \mathbb{R} \times \mathbb{R}$.
Now, $(m,n) * [(p,q) * (r,s)] =$
$(m,n) * (ps + qr, qs) =$
$(mqs + n(ps + qr), nqs) =$
$(mqs + nps + nqr, nqs)$.

Also, $[(m,n) * (p,q)] * (r,s) =$

$(mq + np, nq) * (r,s) =$

$((mq+np)s + nqr, nqs) =$

$(mqs + nps + nqr, nqs)$.

Since $(m,n) * [(p,q) * (r,s)] =$

$[(m,n) * (p,q)] * (r,s)$ then

$*$ is associative.

(ii) Suppose $(e_1, e_2) \in \mathbb{R} \times \mathbb{R}$ represent the identity with respect to $*$, $\forall$ $(x,y) \in \mathbb{R} \times \mathbb{R}$,

$(x,y) * (e_1, e_2) = (x,y)$. Also,

$(x,y) * (e_1, e_2) = (xe_2 + ye_1, ye_2)$.

Equating we have

$(x,y) = (xe_2 + ye_1, ye_2)$.

Now, since the corresponding elements in the ordered pairs are equal, we have

$x = xe_2 + ye_1$ and $y = ye_2$.

So, from 2^{nd} equation, $\frac{y}{y} = e_2 = 1$.

$\therefore$ from eqn(1), $x = x(1) + ye_1$.

So, $x = x + ye_1$. $\therefore$ $x - x = ye_1$.

So, $0 = ye_1$. Therefore, $e_1 = \frac{0}{y} = 0$.

Therefore, $(e_1, e_2) = (0, 1)$.

(iii) Suppose $(i_1, i_2) \in \mathbb{R} \times \mathbb{R}$ represent the inverse of $(a, b) \in \mathbb{R} \times \mathbb{R}$. Then,

$(a,b) * (i_1, i_2) = (e_1, e_2) = (0, 1)$. Also,

$(a,b) * (i_1, i_2) = (ai_2 + bi_1, bi_2)$.

Equating we have

$(0,1) = (ai_2 + bi_1, bi_2)$. Therefore,

$0 = ai_2 + bi_1 \text{ —— } (1)$ and $1 = bi_2 \text{ —— } (2)$.

$\frac{1}{b} = i_2$, $b \neq 0$ and from (1), So, from (2),

$0 = a\left(\frac{1}{b}\right) + bi_1$. So, $-\frac{a}{b} = bi_1$.

$\therefore$ $i_1 = -\frac{a}{b^2}$, $b^2 \neq 0$. So, $(i_1, i_2) = \left(\frac{-a}{b^2}, \frac{1}{b}\right)$.

Question 3

(i) $\forall$ $(a,b), (c,d) \in S$. Then

$$(a,b) \boxplus (c,d) = (a+c, bd)$$
$$= (c+a, db)$$
$$= (c,d) \boxplus (a,b).$$

So, $\boxplus$ is commutative.

Let $(a,b), (c,d), (e,f) \in S$. Now,

$(a,b) \boxplus [(c,d) \boxplus (e,f)] =$

$(a,b) \boxplus (c+e, df) = (a+c+e, bdf)$.

Also, $[(a,b) \boxplus (c,d)] \boxplus (e,f) =$

$(a+c, bd) \boxplus (e,f) =$

$(a+c+e, bdf)$. Since

$(a,b) \boxplus [(c,d) \boxplus (e,f)] =$

$[(a,b) \boxplus (c,d)] \boxplus (e,f)$ then $\boxplus$ is associative.

(ii) Suppose $(e_1, e_2) \in S$ represent the identity with respect to $\boxplus$, then $\forall$ $(x, y) \in S$,

$(x, y) \boxplus (e_1, e_2) = (x, y)$. Also,
$(x, y) \boxplus (e_1, e_2) = (x + e_1, y e_2)$.
Equating we have $(x, y) = (x + e_1, y e_2)$.
Therefore, $x = x + e_1$ and $y = y e_2$.
From 2nd equation $e_2 = \frac{y}{y} = 1$ and from 1st equation $x - x = e_1$. So, $e_1 = 0$.
Thus, $(e_1, e_2) = (0, 1)$.

(iii) Suppose $(\bar{x}_1, \bar{x}_2) \in S$ represent the inverse of $(x, y) \in S$. Then,
$(x, y) \boxplus (\bar{x}_1, \bar{x}_2) = (e_1, e_2) = (0, 1)$.
Also, $(x, y) \boxplus (\bar{x}_1, \bar{x}_2) = (x + \bar{x}_1, y \bar{x}_2)$.
Equating we have $(0, 1) = (x + \bar{x}_1, y \bar{x}_2)$.
So, $0 = x + \bar{x}_1$ and $1 = y \bar{x}_2$.

Therefore, $\bar{x}_1 = -x$ and $\bar{x}_2 = \frac{1}{y}$, where $y \neq 0$.
Hence, the inverse is
$(\bar{x}_1, \bar{x}_2) = (-x, \frac{1}{y})$.

Question 4

(i) $\forall\, a,b \in \mathbb{R}^+$, $a \heartsuit b = \dfrac{ab}{a+b+1}$
$= \dfrac{ba}{b+a+1} = b \heartsuit a$.

So, $\heartsuit$ is commutative.
Let $a, b, c \in \mathbb{R}^+$. Now,

$a \heartsuit (b \heartsuit c) = a \heartsuit \left(\dfrac{bc}{b+c+1}\right) =$

$$\frac{a\left(\frac{bc}{b+c+1}\right)}{a + \frac{bc}{b+c+1} + 1} = \frac{\frac{abc}{b+c+1}}{\frac{ab+ac+a+bc+b+c+1}{b+c+1}}$$

$$= \frac{abc}{b+\cancel{c}+1} \times \frac{b+\cancel{c}+1}{ab+ac+a+bc+b+c+1}$$

$$= \frac{abc}{ab+ac+a+bc+b+c+1}$$

Also, $(a \heartsuit b) \heartsuit c = \left(\frac{ab}{a+b+1}\right) \heartsuit c =$

$$\frac{\left(\frac{ab}{a+b+1}\right)c}{\frac{ab}{a+b+1}+c+1} =$$

$$\frac{\frac{abc}{a+b+1}}{\frac{ab+ac+bc+c+a+b+1}{a+b+1}} =$$

$$\frac{abc}{a+\cancel{b}+1} \times \frac{a+\cancel{b}+1}{ab+ac+bc+c+a+b+1} =$$

$$\frac{abc}{ab+ac+bc+c+a+b+1}$$

Since $a \heartsuit (b \heartsuit c) = (a \heartsuit b) \heartsuit c$
then $\heartsuit$ is associative.

(ii) Suppose $e \in \mathbb{R}^+$ represent the identity with respect to $\heartsuit$, then for all $x \in \mathbb{R}^+$,

$x \heartsuit e = x$. Also,

$$x \heartsuit e = \frac{xe}{x+e+1}.$$ Equating for $x \heartsuit e$, we have

$$x = \frac{xe}{x+e+1}.$$ So,

$$x(x+e+1) = xe$$

$$\therefore x^2 + xe + x = xe.$$

Now, $xe - xe = x^2 + x$.

So, $0 = x^2 + x$. Now, since e vanishes from the equation then there is no identity with respect to $\heartsuit$ on $\mathbb{R}^+$