Solution to Tutorial #3

COMP 1210

Question 1

(a) Given $f(x) = 4x-16$, we want to find which real x is "good" that is, will make $f(x)$ real.

Now, any $x \in \mathbb{R}$ ~~is real~~ will make $f(x)$ real.

Thus, the "natural domain" is $\mathbb{R}$.

Now, the range of $f(x)$ is $(-\infty, \infty)$ or $\mathbb{R}$.

(b) Given $f(x) = \sqrt{3x-6}$, we set $3x-6 \geq 0$ to find the domain of f.

So, $x \in$ Domain f is $3x-6 \geq 0$; i.e, $x \geq 2$. So, Domain $f = \{x \in \mathbb{R} : x \geq 2\}$

$= [2, \infty)$.

The Range of f is $\sqrt{3x-6} \geq 0$; i.e,
Range $f = \{y \in \mathbb{R} : y \geq 0\}$

(c) Given $f(x) = \sqrt{2-x}$, we set $2-x \geq 0 \Rightarrow 2 \geq x$ which is the same as $x \leq 2$. So,

Domain $f = \{x \in \mathbb{R} : x \leq 2\}$

The Range of $f = \{y \in \mathbb{R} : y \geq 0\}$

(d) Given $f(x) = \frac{2x}{(x-1)(3x+9)}$, we set $(x-1)(3x+9) = 0$ to find the restriction on the domain; that is, $x = 1$ or $x = -3$. So, the domain of f is all real numbers of x except $x = 1$ or -3. That is,

Domain $f = \{x \in \mathbb{R} : x \neq 1, -3\}$

Now, the Range of $f = \{x : f(x) \in \mathbb{R}\}$
$= \mathbb{R}$..

(e) Given $f(x) = \frac{1}{\sqrt{x-3}}$, we need $x-3 > 0$; i.e, $x > 3$. So,

Domain $f = \{x \in \mathbb{R} : x > 3\}$

Range $f(x) = \{y \in \mathbb{R} : y = \frac{1}{\sqrt{x-3}} : x > 3\}$
$= \{y \in \mathbb{R} : y > 0\}$

Now, $y = \frac{1}{\sqrt{x-3}} \Longrightarrow y^2 = \frac{1}{x-3} > 0$

or $x-3 = \frac{1}{y^2}$. Therefore,

$x = \frac{1}{y^2} + 3 > 3$.

Thus, $\frac{1}{y^2} > 0$. So, $y^2 > 0$; take

$y > 0$.

Question 2

(a) Given $f(x) = x - 3$

Injective (1-1)

Suppose $f(x) = f(y)$. Then $x - 3 = y - 3$.
Therefore, $x = y$. Hence f is 1-1.

Surjective (onto)

Suppose $y \in \mathbb{R}$ (codomain) then
$y + 3 \in \mathbb{R}$ (domain).
Now, $f(y+3) = y + 3 - 3 = y$.
Thus, f is onto.

Rough work

$y \in \mathbb{R}$ if f is onto $\exists\, x^* \in \mathbb{R}$ domain such that $f(x^*) = x^* - 3 = y$. i.e, $x^* = y + 3 \in \mathbb{R}$. That is, x^* is in domain

(b) Given $f(x) = 2x + 3$.

Injective

Suppose $f(x) = f(y)$. Then $2x + 3 = 2y + 3$

Thus, $2x = 2y \Rightarrow x = y$.

Therefore, f is one-to-one.

Surjective

Suppose $y \in \mathbb{R}$ then $\frac{y-3}{2} \in \mathbb{R}$ domain

and $f\left(\frac{y-3}{2}\right) = 2\left(\frac{y-3}{2}\right) + 3 = y$.

Therefore, f is onto.

Question 3

(a) Now, given $f(x): \mathbb{Z} \rightarrow \mathbb{Z}$ and $f(x) = x - 3$.

One-to-one

Suppose $f(x) = f(y)$. Then $x - 3 = y - 3$. Therefore $x = y$. Hence f is 1-1.

Onto

Suppose $y \in \mathbb{Z}$ (codomain)
Then $y + 3 \in \mathbb{Z}$ (domain) and $f(y+3) = y$. Thus, f is onto.

(b) Given $f(x) = 2x + 3$.

One-to-one

Suppose $f(x) = f(y)$ then $2x + 3 = 2y + 3$. Thus $2x = 2y$ or $x = y$. Therefore, f is one-to-one.

Onto

Now, integers ($\mathbb{Z}$) are in the codomain but there are no corresponding $x \in \mathbb{Z}$ (domain) at some instances such that $f(x) = y$. For example, $4 \in \mathbb{Z}$ (codomain) but there is no $x \in \mathbb{Z}$ (domain) such that $f(x) = 4$.

Since $f(x) = 4 \implies 2x + 3 = 4$.

That is, $x = \frac{1}{2} \notin \mathbb{Z}$. So, f is not onto.

Question 4

(a) Given $A = \{a, b, c\}$ and

$B = \{1, 2, 3\}$.

Now, we have

$$A \times B = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3), (c,1), (c,2), (c,3)\}$$

Now, the functions from A to B are given by

$\{(a,1), (b,1), (c,1)\} = f_1$

$\{(a,2), (b,2), (c,2)\} = f_2$

$\{(a,3), (b,3), (c,3)\} = f_3$

$\{(a,1), (b,1), (c,2)\} = f_4$

$\{(a,1), (b,2), (c,1)\} = f_5$

$\{(a,2), (b,1), (c,1)\} = f_6$

$\{(a,1), (b,1), (c,3)\} = f_7$

$f_8 = \{(a,1), (b,3), (c,1)\}$

$f_9 = \{(a,3), (b,1), (c,1)\}$

$f_{10} = \{(a,3), (b,3), (c,2)\}$

$f_{11} = \{(a,3), (b,2), (c,3)\}$

$f_{12} = \{(a,2), (b,3), (c,3)\}$

$f_{13} = \{(a,3), (b,3), (c,1)\}$

$f_{14} = \{(a,3), (b,1), (c,3)\}$

$f_{15} = \{(a,1), (b,3), (c,3)\}$

$f_{16} = \{(a,2), (b,2), (c,1)\}$

$f_{17} = \{(a,2), (b,1), (c,2)\}$

$f_{18} = \{(a,1), (b,2), (c,2)\}$

$f_{19} = \{(a,2), (b,2), (c,3)\}$

$f_{20} = \{(a,2), (b,3), (c,2)\}$

$f_{21} = \{(a,3), (b,2), (c,2)\}$

$f_{22} = \{(a,1), (b,2), (c,3)\}$

$f_{23} = \{(a,1), (b,3), (c,2)\}$

$f_{24} = \{(a,3), (b,2), (c,1)$

$f_{25} = \{(a,2), (b,1), (c,3)\}$

$f_{26} = \{(a,2), (b,3), (c,1)\}$

$f_{27} = \{(a,3), (b,1), (c,2)\}$

Now, functions f_{22} to f_{27} are both injective and surjective.

The others, that is, functions f_1 to f_{26} are neither injective nor surjective.

⑤

(b) Given $X = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$ and $f = \{(x, x^2) : x \in X\}$.

Now, f as a set of ordered pairs is given by

$$f = \{(-5, 25), (-4, 16), (-3, 9), (-2, 4), (-1, 1), (0, 0), (5, 25), (4, 16), (3, 9), (2, 4), (1, 1)\}$$

Injective

Now, f is not injective since $f(-3) = 9 = f(3)$ and $-3 \neq 3$.

Surjective

Now, f is not surjective since $-4 \in X$ but -4 is not a second element of any ordered pair. Also, any of the other numbers could be used as an example for which the surjective property does not hold.

Question 5

Given $f : A \to B$ given by $f(x) = \frac{1}{2-x}$ and $g : B \to B$ be given by $g(x) = \frac{1}{x+2}$.

Now, note that $x \in A$. Then $x < 0$ and $f(x) = \frac{1}{2-x} > 0 \in B$.

Thus, $f(x) \in B$, $\forall x \in A$ and $g \circ f(x)$ is defined for all $x \in A$.

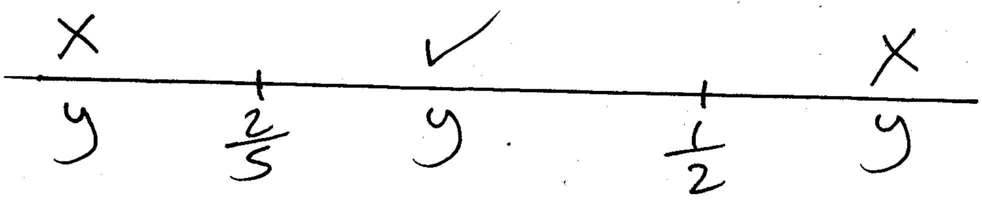
$$\text{Now, } g \circ f(x) = g(f(x)) = \frac{1}{f(x)+2}$$

$$= \frac{1}{\frac{1}{2-x} + 2} = \frac{1}{\frac{5-2x}{2-x}}$$

$$= \frac{2-x}{5-2x} = \frac{x-2}{2x-5}.$$

Now, suppose y is in range of $g \circ f$.

Then $y = \frac{x-2}{2x-5}$; that is, $x = \frac{5y-2}{2y-1} < 0$.



Thus, $\frac{2}{5} < y < \frac{1}{2}$.

Hence, the range of $g \circ f \in (2/5, 1/2)$.

Question 6

(a) Given $f(x) = \sqrt{x-1}$, $g(x) = \sqrt{x-2}$

Now, $f \circ g(x) = f(g(x)) = \sqrt{g(x) - 1} =$ $\sqrt{\sqrt{x-2} - 1}$. We note that

$f \circ g(x)$ is defined for x such that $\sqrt{x-2} - 1 \geq 0$; that is, $\sqrt{x-2} \geq 1$.

That is, $\{x : x \geq 3\}$.

(b) Given $f(x) = \frac{1}{x-1}$ and $g(x) = \frac{x-1}{x+1}$.

Now, $f \circ g(x) = f(g(x)) = \frac{1}{g(x) - 1} =$

$$\frac{1}{\frac{x-1}{x+1} - 1} = \frac{x+1}{-2} = -\frac{1}{2}(x+1).$$

Question 7

(a) Given $f(x) = -x-4$.

Suppose $f(x) = f(y)$. Then $-x-4 = -y-4$; that is, $x = y$ and so f is one-to-one.

Now, let $y = -x-4 \Rightarrow x = -y-4$.
Now, let $x^* = x = -y-4$. Substituting, we have $f(x^*) = -(-y-4)-4 = y+\cancel{4}-\cancel{4}$
$= y$.

$\therefore f(x^*) = y$. Notice if $y \in \mathbb{R}$ then $x = -y-4 \in \mathbb{R}$. So, $\forall y \in \mathbb{R}, \exists x^* = -y-4 \in \mathbb{R}$ such that $f(x^*) = y$.
Hence f is onto.

Now, to find the inverse we let $y = f(x)$; i.e, $y = -x-4$.
Now, interchanging variables:
$x = -y-4$. Solving for y:
$y = -x-4$. Therefore, $f^{-1}(x) = -x-4$.

(b) Given $f(x) = (x+1)^2$, $x \geq -1$.

Suppose $x, y \geq -1$ and $f(x) = f(y)$.
Then $(x+1)^2 = (y+1)^2$ or
$x^2 + 2x + 1 = y^2 + 2y + 1 \Longrightarrow$
$x^2 - y^2 + 2(x-y) = 0 \Longrightarrow$
$(x-y)(x+y) + 2(x-y) = 0 \Longrightarrow$
$(x-y)(x+y+2) = 0.$
Now, either $\overset{\text{Case 1}}{(x-y)=0}$ or $\overset{\text{Case 2}}{x+y+2=0}$
Since $x, y \geq -1$ we have $x+y+2=0$
$\iff x=-1=y$. Also, $(x-y)=0 \iff$
$x=y$ (Both cases imply that $x=y$)
Thus, $(x-y)(x+y+2)=0 \iff x=y$.
Hence, $f(x)$ is one-to-one.

We let $y = (x+1)^2$ and we have
that $-\sqrt{y} = x+1$ or $\sqrt{y} = x+1 \overset{\text{i.e.}}{\Longrightarrow}$
$x = \sqrt{y} - 1$ or $x = -\sqrt{y} - 1$.
Since we must ensure that $x \in$ domain;
that is, $x \geq -1$.
So, we choose $x^* = \sqrt{y} - 1$ and so

$f(x^*) = (\sqrt{y} \cancel{-1} \cancel{+1})^2 = (\sqrt{y})^2 = y.$

So, $\forall y \in$ codomain, $\exists\, x^* = \sqrt{y} - 1 \geq -1$
$\in$ domain

such that $f(x^*) = y$.

Therefore, f is onto.

To find the inverse, we let
$y = (x+1)^2$ and so, $x = (y+1)^2$ when we interchange variables. Now, solving for y we obtain
$y = \sqrt{x} - 1$. Hence, $f^{-1}(x) = \sqrt{x} - 1$.

Question 8

Given $A = \{x : x \geq 2\}$ and
$B = \{x : x \geq -4\}$ and
$f : A \rightarrow B$.

Now, suppose $f(x) = f(y) \Rightarrow$
$x^2 - 4x = y^2 - 4y \Rightarrow x^2 - y^2 - 4(x-y) = 0$
$\Rightarrow (x-y)(x+y) - 4(x-y) = 0 \Rightarrow$
$(x-y)(x+y-4) = 0$.
Either $x - y = 0 \Rightarrow x = y$ OR
$x + y - 4 = 0 \Leftrightarrow x = y = 2$. Therefore,
$(x-y)(x+y-4) = 0 \Leftrightarrow x = y$.
That is, f is one-to-one.

Now, let $y = x^2 - 4x$ and we obtain
$x^2 - 4x - y = 0$.
Using the quadratic formula:
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where $a = 1$,
$b = -4$ and $c = -y$, we have

$$x = \frac{4 \pm \sqrt{16+4y}}{2} = 2 \pm \frac{2\sqrt{4+y}}{2}$$
$$= 2 \pm \sqrt{4+y}$$

Now, suppose $y \in B$, then $\exists x = 2 \pm \sqrt{4+y} \in A$ such that $f(x) = y$.

We choose $x^* = 2 + \sqrt{4+y}$.

So, $\forall \overset{\in \text{codomain}}{y \geq -4}$, $\exists x^* = \overset{\in \text{domain}}{2 + \sqrt{4+y}} \geq 2$ such that $f(x^*) = y$.

Hence, f is onto.

Now, since we chose $x = 2 + \sqrt{4+y}$, we can deduce $f^{-1}(x)$.

Now it follows that $y = 2 + \sqrt{4+x}$ and so,

$f^{-1}(x) = 2 + \sqrt{4+x}$ since $x \geq 2$ seeing that $x \in A$.

⑨

Question 9

We have that $f: A \to B$ and $g: B \to C$

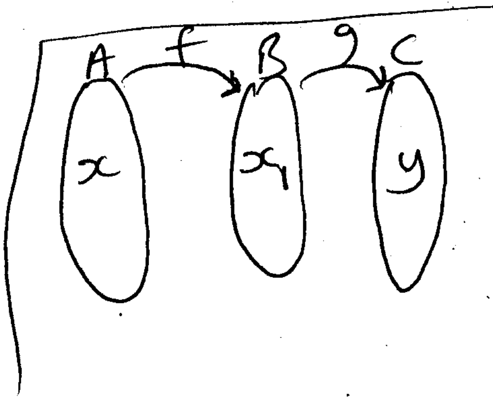
(a) Given $f: A \to B$ and $g: B \to C$.

Suppose that f and g are injective and suppose $(g \circ f)(x) = (g \circ f)(y)$ for $x, y \in A$. Then $g[f(x)] = g[f(y)]$, but g injective implies that $f(x) = f(y)$. Also, f injective implies that $x = y$. Thus, $g \circ f(x)$ is injective.

(b) Given $f: A \to B$ and $g: B \to C$.

Let $y \in C$. Now, g surjective implies that $\exists x_1 \in B \overset{\text{such that}}{\ni} g(x_1) = y$. But $x_1 \in B$ and f surjective implies that $\exists\, x \in A \ni f(x) = x_1$; that is, $g(f(x)) = g(x_1) = y$.

Thus, $g \circ f$ is surjective.



Question 10

Given that $f: A \to B$ and

$C_1, C_2 \subseteq A$ and $D \subseteq B$

(a) Let $y \in f(C_1 \cap C_2) \implies$

$\exists x \in C_1 \cap C_2$ such that $f(x) = y$.

Now, $x \in C_1$ and $x \in C_2$.

Thus, $y = f(x) \in f(C_1)$ and $y = f(x) \in f(C_2)$

Thus, $y \in f(C_1) \cap f(C_2)$.

Therefore, $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$.

(b) Let $y \in f[f^{-1}(D)]$. Then

$\exists x \in f^{-1}(D)$ such that $f(x) = y$.

But $x \in f^{-1}(D) \implies y = f(x) \in D$.

Therefore, $f[f^{-1}(D)] \subseteq D$.