Solution to Tutorial #2

COMP 1210

Question 1

Given that $1+2+3+4+\cdots+n = \frac{n(n+1)}{2}$ $\forall\, n \in \mathbb{N}$, we have:

Let $p(n) = 1+2+3+4+\cdots+n = \frac{n(n+1)}{2}$ $\forall\, n \geq 1$.

Now, for the base case we let $n=1$.

Now, $1+2+3+4+\cdots+n = \sum_{j=1}^{n} j$

$$= \frac{n(n+1)}{2}.$$

Now, left hand side: when $n=1$

$$\sum_{j=1}^{n} j = \sum_{j=1}^{1} j = 1$$ and the

right hand side: when $n=1$

$$\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

Since L.H.S = R.H.S. = 1

then true for $n=1$. So, $p(1)$ is true.

When $n=k$, we assume that

$$p(k) = 1+2+3+4+\cdots+k = \sum_{j=1}^{k} j$$

$$= \frac{k(k+1)}{2} \text{ is true for } k \geq 1$$

When $n=k+1$, we want to show that $n=k$ being true implies that $n=k+1$ is also true; that is, $p(k) \Longrightarrow p(k+1)$ is true.

So, when $n=k+1$, we have

$$1+2+3+4+\cdots+k+k+1 =$$

$$\sum_{j=1}^{k+1} j = \sum_{j=1}^{k} j + k+1$$

$$= \frac{k(k+1)}{2} + k+1 =$$

$$\frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

So, true for $n=k+1$; that is $p(k+1)$ is true.

Hence, true for all $n \geq 1$ or $n \in \mathbb{Z}^+$.

Question 2

Given that $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$.

$\forall\ n \geq 1$, we have:

let $p(n) = 1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1)$

$$= \frac{n(n+1)(n+2)}{3} \quad \forall\ n \in \mathbb{Z}^+.$$

So,

$$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1) = \sum_{j=1}^{n} j(j+1)$$

$$= \frac{n(n+1)(n+2)}{3} \quad \forall\ n \geq 1.$$

When $n=1$, left hand side:

$$\sum_{j=1}^{1} j(j+1) = 1(1+1) = 1\cdot 2 \quad \text{and}$$

right hand side: $\dfrac{1(1+1)(1+2)}{3} =$

$$\frac{1(2)(\cancel{3})}{\cancel{3}} = 1\cdot 2.$$

Since $L.H.S = 1\cdot 2 = R.H.S$, then true for $p(1)$.

When $n=k$, we assume that

$$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + k(k+1) =$$

$$\sum_{j=1}^{k} j(j+1) = \frac{k(k+1)(k+2)}{3} \quad \text{is}$$

true for all $k \in \mathbb{Z}^+$.

When $n = k+1$, we want to show that $n = k$ being true implies that $n = k+1$ is also true.

Now, we have that

$$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + k(k+1) + (k+1)(k+2)$$

$$= \sum_{j=1}^{k+1} j(j+1) = \sum_{j=1}^{k} j(j+1) + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} =$$

$$\frac{(k+1)(k+2)(k+3)}{3}.$$

So, true for $p(k+1)$. Hence, true for all $n \geq 1$.

Question 3

Given that $1+3+5+\cdots+(2n-1) = n^2$

for all $n \in \mathbb{Z}^+$, we have:

Let $p(n) = 1+3+5+\cdots+(2n-1) = n^2$
$\forall\ n \in \mathbb{Z}^+$.

Now, we have that

$$1+3+5+\cdots+(2n-1) = \sum_{i=1}^{n}(2i-1) = n^2$$

$\forall\ n \geq 1$.

When $n=1$, Left hand side:

$$\sum_{i=1}^{n}(2i-1) = \sum_{i=1}^{1} 2i-1 = 2(1)-1$$
$$= 1$$

For the right hand side:

$$n^2 = 1^2 = 1$$

Since L.H.S. $=1=$ R.H.S. then true for $p(1)$.

When $n=k$ we assume that

$$1+3+5+\cdots+(2k-1) = \sum_{i=1}^{k}(2i-1) = k^2$$

is true for $k \geq 1$.

When $n=k+1$ we want to show that $p(k)$ being true implies that $p(k+1)$ is also true.

That is,

$$1+3+5+\cdots+(2k-1)+(2k+1)=$$

$$\sum_{i=1}^{k+1}(2i-1)=\sum_{i=1}^{k}(2i-1)+(2k+1)=$$

$$=k^2+2k+1=(k+1)^2.$$

Therefore true for $p(k+1)$.
Hence, true for all $n \in \mathbb{Z}^+$.

Question 4

Given $2^n \geq n^2$ for $n = 4, 5, \ldots,$

we let $p(n) = 2^n \geq n^2$ for all $n \geq 4$.

Now, when $n=4$, we have that

$2^4 \geq 4^2 \Rightarrow 16 = 16.$

So, since $2^4 = 16$ and $4^2 = 16$

then true for $n=4$.

When $n=k$, we assume that

$2^k \geq k^2 \quad \forall\, k \geq 4.$

When $n=k+1$ we want to show that $p(k)$ being true implies that $p(k+1)$ is also true.

That is, $2^{k+1} = 2^k \cdot 2^1 \geq 2 \cdot k^2$ ④

So, $2 \cdot 2^k \geq 2k^2$

$= k^2 + k^2 + 2k - 2k + 1 - 1$

$= (k^2 + 2k + 1) + k^2 - 2k - 1$

$= (k+1)^2 + k^2 - 2k - 1$

$\geq (k+1)^2$,

since $k^2 > 2k - 1 = -(2k+1)$, $k \geq 4$.

Therefore, true for $n = k+1$;

Hence, true for all $n \geq 4$.

OR we could use $2k^2 = (k+1)^2 + (k-1) - 2$

So, $2^{k+1} = 2 \cdot 2^k \geq 2 \cdot k^2 \geq (k+1)^2$

Question 5

Given that $2n+1 \leq 2^n$, where $n = 3, 4, \ldots$

We let $p(n) = 2n+1 \leq 2^n$ or we can say let $p(n): 2n+1 \leq 2^n$ $\forall n \geq 3$.

When $n = 3$, we have

Left hand side: $2(3)+1 = 6+1 = 7$

and right hand side: $2^3 = 8$

So, $2(3)+1 \leq 2^3 \Rightarrow 7 \leq 8$

So, true for $p(3)$.

When $n=k$, we assume that

$2k+1 \le 2^k$ for $k=3,4,\dots$

When $n=k+1$ we want to show that $p(k)$ being true implies that $p(k+1)$ is also true.

So, we have

$$2(k+1)+1 = 2k+1+2$$

Now, $(2k+1)+2 \le 2^k + 2^k - 2^k + 2$

$$= 2^k \cdot 2 - 2^k + 2$$

$$= 2^{k+1} - (2^k - 2)$$

$$\le 2^{k+1}$$

$$\boxed{\text{So, } 2(k+1)+1 \le 2^{k+1}}$$

Therefore, true for $n=k+1$ and hence true for all $n \geq 3$.

Question 6

(5)

Given $5^{2n}-1$ is a multiple of 8, $\forall\, n \in \mathbb{N}$, we let
$p(n)$: $5^{2n}-1$ is a multiple of 8 $\forall\, n \in \mathbb{N}$.

For the base case we let $n=1$, so, we have
$$5^{2(1)}-1 = 5^2-1 = 25-1 = 24 = 8(3),$$
which is divisible by 8; so it is a multiple of 8. Therefore, true for $n=1$.

When $n=k$, we assume that $5^{2k}-1 = 8q$, $q \in \mathbb{Z}$, is divisible by 8.

When $n=k+1$, we want to show that $p(k) \Rightarrow p(k+1)$.

That is, $5^{2(k+1)}-1 = 5^{2k+2}-1$
$$= 5^2 \cdot 5^{2k} - 1 = 25 \cdot 5^{2k} - 1$$
$$= 25(5^{2k}-1)+24$$
$$= 25(8q)+24 = 8(25q)+8(3)$$

$= 8(259 + 3)$ which is divisible by 8, since it is a multiple of 8.

Question 7

Given

$E = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$, $A = \{1, 2, 3, 4, 5, 7\}$

and $B = \{2, 3, 7, 8, 9, 12, 15\}$

(i) $n(A) = 6$ (ii) $A \cup B = \{1, 2, 3, 4, 5, 7, 8, 9, 12, 15\}$

So, $n(A \cup B) = 10$

(iii) $A \cap B = \{2, 3, 7\}$

Now, $(A \cap B)^c = \{1, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$

So, $n(A \cap B)^c = 17$

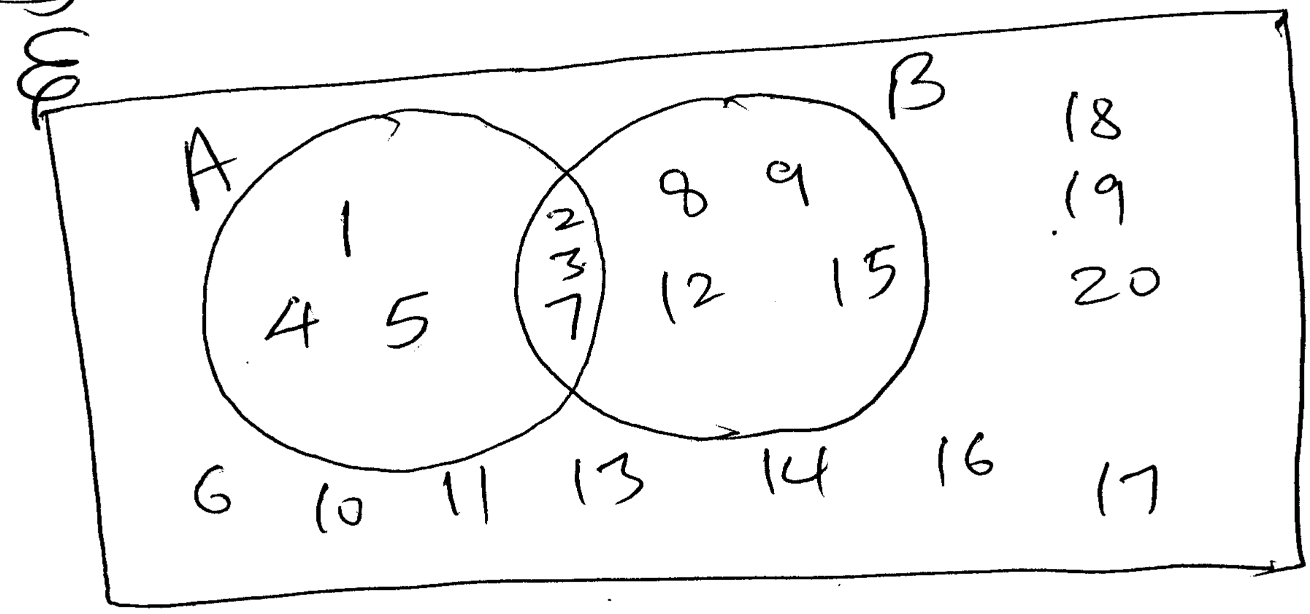
(iv) $(A \cup B)^c = \{6, 10, 11, 13, 14, 16, 17, 18, 19, 20\}$

(v) $A^c = \{6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$

(vi) $B^c = \{1, 4, 5, 6, 10, 11, 13, 14, 16, 17, 18, 19, 20\}$

$n(B^c) = 13.$

(vii) The Venn diagram is ~~given~~ shown below:



Question 8

$n(\xi) = 30$

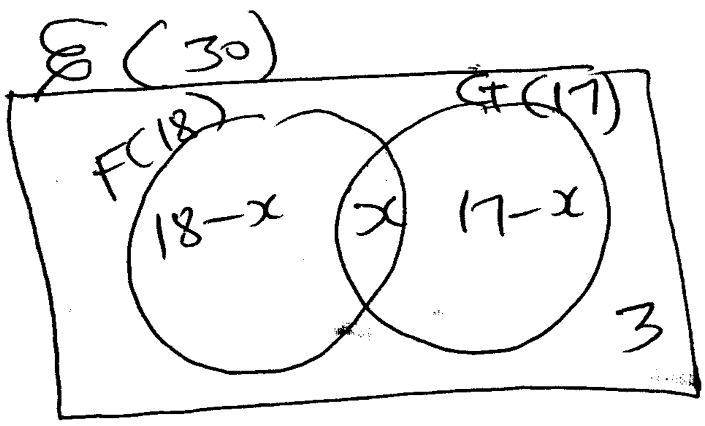
$n(F) = 18$

$n(G) = 17$

$n(F \cup G)' = 3$

Let F represent French and G German.

The Venn diagram is shown below:



Let x represent the number of pupils that take both French and German

So, we have $n(F \cup G) = n(F) + n(G) - n(F \cap G)$

So, $30 = 18 + 17 - n(F \cap G) + 3$

So, $30 - 3 = 35 - n(F \cap G)$

$\therefore n(F \cap G) = 35 - 27 = 8$

So, the number of pupils that take both subjects is 8

Alternatively,

$18 - \cancel{x} + \cancel{x} + 17 - x + 3 = 30$

$\therefore 35 - x + 3 = 30$

$38 - 30 = x$

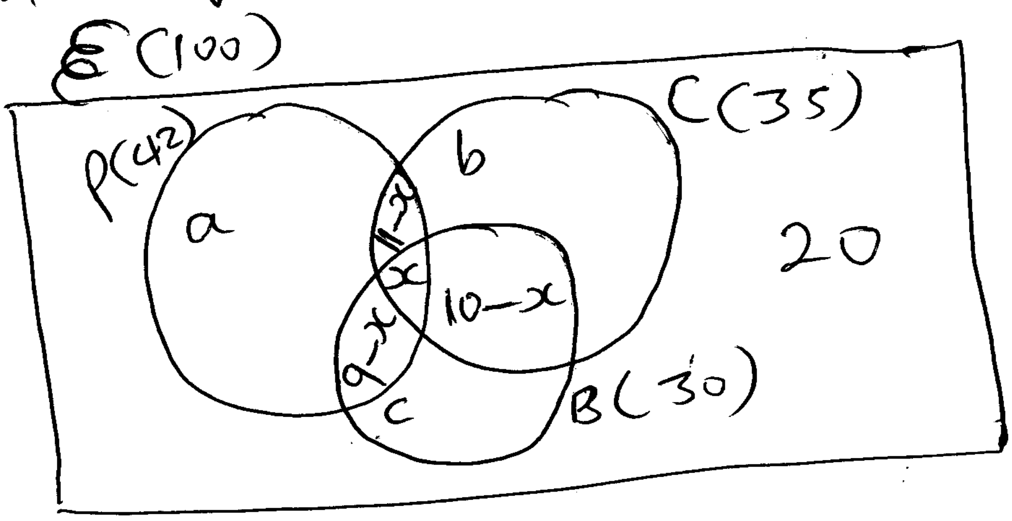
$\therefore x = 8$

Question 9

Let P represent Physics, C represent Chemistry and B represent Botany.

$n(\mathcal{E}) = 100$, $n(P) = 42$, $n(C) = 35$, $n(B) = 30$, $n(P \cup C \cup B)' = 20$, $n(B \cap P) = 9$, $n(B \cap C) = 10$ and $n(P \cap C) = 11$.

The Venn diagram is given below:

(i) 

Let x represent the number of students that take all three subjects.

(ii) Now, using the set notation, we have $n(P \cup B \cup C) =$

$n(P) + n(B) + n(C) - n(P \cap B) - n(P \cap C) - n(B \cap C) + n(P \cap B \cap C)$

Therefore,

$(100-20) = 42+30+35-9-11-10$
$+ n(P\cap B\cap C)$

So, $80 = 107-30+n(P\cap B\cap C)$

$\therefore\ 80-77 = n(P\cap B\cap C)$

So, $n(P\cap B\cap C) = 3$

Therefore, the number of students that take all three subjects is 3.

Alternatively, using algebra, we proceed as follows:

$a+11-\cancel{x}+\cancel{x}+9-x = 42$

$\therefore\ a+20-x = 42$

So, $a = x+22$

Also, $b+11-\cancel{x}+\cancel{x}+10-x = 35$

So, $b+21-x = 35$

$\therefore\ b = x+14$

and $c+9-\cancel{x}+\cancel{x}+10-x = 30$

$\therefore\ c+19-x = 30$

$\therefore\ c = x+11$

Hence, we have

$42+x+14+10-\cancel{x}+\cancel{x}+11+20 = 100$

$\therefore\ 97+x = 100 \implies x = 3.$

(iii) The number of students that take Physics only is $a = 3 + 22$

$\therefore \quad a = 25$

(iv) The number of students that take Botany and chemistry only is

$10 - x = 10 - 3 = 7.$

Question 10

(i) Given $(A \cap B)' = A' \cup B'$.

Suppose $x \in (A \cap B)' \iff x \notin (A \cap B)$

$\iff x \notin A$ or $x \notin B$

$\iff x \in A'$ or $x \in B'$

$\iff x \in (A' \cup B')$

Therefore, $(A \cap B)' = A' \cup B'$

(ii) Given $A' - B' = B - A$.

Suppose $x \in A' - B' \iff x \in A'$ and $x \notin B'$

$\iff x \notin A$ and $x \in B$

$\iff x \in B$ and $x \notin A$

$\iff x \in (B - A)$.

So, $A' - B' = B - A$.

(iii)

Given $A-(B\cap C)=(A-B)\cup(A-C)$.

Suppose

$$x\in A-(B\cap C) \Longleftrightarrow x\in A \text{ and } x\notin(B\cap C)$$
$$\Longleftrightarrow x\in A \text{ and } x\notin B \text{ or } x\notin C$$
$$\Longleftrightarrow x\in A \text{ and } x\notin B \text{ or } x\in A \text{ and } x\notin C$$
$$\Longleftrightarrow x\in(A-B) \text{ or } x\in(A-C$$
$$\Longleftrightarrow x\in(A-B)\cup(A-C)$$

Therefore

$$A-(B\cap C)=(A-B)\cup(A-C).$$