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Solution to Tutorial 4

COMP 1210

Question 1

Given $X = \{a, b, c\}$.

(a) Now, given the relation $R_1 = \{(a, a), (a, b), (b, b), (b, a)\}$

R_1 is not reflexive not for all $x \in X$, $(x, x) \in R_1$. That is $c \in X$, but $(c, c) \notin R_1$.

R_1 is symmetric since $(a, b) \in R_1$ $\Longrightarrow (b, a) \in R_1$. So, both (a, b) and (b, a) are in R_1.

R_1 is transitive for (a, b) and (b, a) $\in R_1$ then $(a, a) \in R_1$. Also, for (b, a) and (a, b) $\in R_1$ then $(b, b) \in R_1$. We also have (a, a) and $(a, b) \in R_1$ implies that $(a, b) \in R_1$. Finally, for (b, a) and $(a, a) \in R_1 \Longrightarrow (b, a) \in R_1$. So, for (x, y) and $(y, z) \in R_1$ then we have $(x, z) \in R_1$.

Now, since R_1 is not reflexive it is not an equivalence relation.

(b) Given $R_2 = \{(a,a), (b,b), (c,c)\}$

Now, R_2 is reflexive since $(i, i) \in R_2 \ \forall\, i \in X$. That is, for $a, b, c \in X$, (a,a), (b,b) and $(c,c) \in R_2$.

R_2 is symmetric since the hypothesis (argument) ~~conclusion~~ does not hold (false) so, the ~~conclusion~~ is true. That is, there is no if condition

R_2 is transitive since the compound hypothesis does not hold true, so the argument ~~conclusion~~ is true. That is, there is no if condition.

So, R_2 is an equivalence relation.

(2) Question 2

Given statement xRy iff x is a multiple of y or y is a multiple of x, since we have the or ~~[illegible]~~ connective then for the argument to be valid only one clause must be satisfied.

Reflexive

Now, is xRx $\forall x$? Is x a multiple of x $\forall x$?

Well, $xRx \iff x = 1(x)$, where $1 \in \mathbb{Z}$. So, x is a multiple of x $\forall x \in \mathbb{N}$. Take for example, $2R2 \iff 2 = 1(2)$, where $1 \in \mathbb{Z}$.

Therefore, R is reflexive.

Symmetric

We consider whether xRy implies yRx.

Now, xRy means that either x is a multiple of y or y is a multiple of x. —— (1)

yRx means that either y is a multiple of x or x is a multiple of y. —— (2)

Statement (2) follows from statement (1), they are saying the same thing.

That is, $xRy \iff x = y(k)$, for $k \in \mathbb{Z}$ or $y = x(k)$, $k \in \mathbb{Z}$.

<u>Example</u>

$2R6 \implies 6 = 2(3)$, where $3 \in \mathbb{Z}$.

R is therefore symmetric.

<u>Transitive</u>

Now, if xRy and $yRz \implies xRz$.

Now, we have $xRy \implies x = y(k)$, for $k \in \mathbb{Z}$

Also, $yRz \implies y = z(k)$, $k \in \mathbb{Z}$.

Now, $xRz \implies x = z(k)$, $k \in \mathbb{Z}$.

{Note: You could have called each integer a different letter]

But for the case xRz both numbers might not have a common factor at all times.

For example, $2R6$ since 6 is a multiple of 2, and $6R3$ since 6 is a multiple of 3. So, $2R6$ and $6R3$. However, $2 \not{R} 3$, since neither is 2 a multiple of 3 nor is 3 a multiple of 2. Therefore, R is not transitive.

So, R is not an equivalence relation.

Question 3

(3) Given $x \sim y \iff x - y = 4k$, we have:

Reflexive

For all $x \in \mathbb{Z}$, $x \sim x \iff x - x = 0 = 4(0)$, where $0 \in \mathbb{Z}$. So, $\sim$ is reflexive.

Symmetric

Let $x, y \in \mathbb{Z}$. Then $x \sim y \iff x - y = 4k$, for $k \in \mathbb{Z}$.

Now, $y - x = -(x-y) = 4(-k)$ and $-k \in \mathbb{Z}$,

$\implies y \sim x$.

So, $\sim$ is symmetric.

Transitive

Let $x, y, z \in \mathbb{Z}$. Now,

$x \sim y$ and $y \sim z \implies$
$x - y = 4k$ and $y - z = 4l$, where $k, l \in \mathbb{Z}$. Now,

$$x - z = (x-y) + (y-z) = 4k + 4l = 4(k+l),$$

where $k + l \in \mathbb{Z}$, $\implies x \sim z$.

Hence, $x \sim z$, so $\sim$ is transitive.

Thus, $\sim$ is an equivalence relation.

The equivalence classes of $x=2$ and $x=4$ are :

$$[2] = \{y \in \mathbb{Z} : yR2\}$$
$$= \{y \in \mathbb{Z} : y-2 = 4k, k \in \mathbb{Z}\}$$
$$= \{y \in \mathbb{Z} : y = 4k+2, k \in \mathbb{Z}\}$$
$$= \{\ldots, \underset{k=-2}{-6}, \underset{k=-1}{-2}, \underset{k=0}{2}, \underset{k=1}{6}, \underset{k=2}{10}, \ldots\}$$

where $-2 \leq k \leq 2$

Similarly,

$$[4] = \{y \in \mathbb{Z} : yR4\}$$
$$= \{y \in \mathbb{Z} : y-4 = 4k, k \in \mathbb{Z}\}$$
$$= \{y \in \mathbb{Z} : y = 4k+4, k \in \mathbb{Z}\}$$
$$= \{\ldots, \underset{k=-2}{-4}, \underset{k=-1}{0}, \underset{k=0}{4}, \underset{k=1}{8}, \ldots\},$$

for $-2 \leq k \leq 1$

Question 4

Given $x \equiv y \bmod 5$ which means that $5 \mid x-y$ (which is to say 5 divides $x-y$, so $x-y$ is a multiple of 5).

We have the following:

Reflexive

$\forall\ x \in \mathbb{Z}$, $x \equiv x \bmod 5 \Rightarrow$

$x - x = 0 = 5(0)$, $0 \in \mathbb{Z}$.

Thus, $x \equiv x \bmod 5 \Rightarrow x - x = 5(k)$, $k \in \mathbb{Z}$.

Therefore, the relation is reflexive.

Symmetric

Let $x, y \in \mathbb{Z}$, now, $x \equiv y \bmod 5 \Rightarrow$ $x - y = 5k$, $k \in \mathbb{Z}$. We also have that $y - x = -(x-y) = -5k = 5(-k)$, $-k \in \mathbb{Z}$, $\Rightarrow y \equiv x \bmod 5$. Then the relation is symmetric.

Transitive

Let $x, y, z \in \mathbb{Z}$. Suppose $x \equiv y \bmod 5$ and $y \equiv z \bmod 5 \Rightarrow$ $x - y = 5k$ and $y - z = 5l$; $k, l \in \mathbb{Z}$.

Now,

$$x - z = x - y + y - z = (x-y) + (y-z)$$
$= 5k + 5l = 5(k+l)$, where

$k+l \in \mathbb{Z}$, $\Rightarrow x \equiv z \bmod 5$.

So, the relation is transitive.

Therefore "$\equiv \bmod 5$" is an equivalence relation.

Equivalence Classes of $x = 2, 4$

$$[2] = \{y \in \mathbb{Z} : y \equiv 2 \bmod 5\}$$
$$= \{y \in \mathbb{Z} : y - 2 = 5k,\ k \in \mathbb{Z}\}$$
$$= \{y \in \mathbb{Z} : y = 5k + 2,\ k \in \mathbb{Z}\}$$
$$= \{\dots, \underset{k=-2}{-8}, \underset{k=-1}{-3}, \underset{k=0}{2}, \underset{k=1}{7}, \underset{k=2}{12}, \dots\},$$

where $-2 \le k \le 2$

Similarly,

$$[4] = \{y \in \mathbb{Z} : y \equiv 4 \bmod 5\}$$
$$= \{y \in \mathbb{Z} : y - 4 = 5k,\ k \in \mathbb{Z}\}$$
$$= \{y \in \mathbb{Z} : y = 5k + 4,\ k \in \mathbb{Z}\}$$
$$= \{\dots, -11, -6, -1, 4, 9, 14, \dots\},$$

where $-3 \le k \le 2$

(5) Question 5

Given $(a,b) R (c,d) \iff a+d = b+c$ we have the following:

Reflexive

For all $(a,b) \in \mathbb{Z} \times \mathbb{Z}$, $(a,b) R (a,b) \Longrightarrow a+b = b+a$ which is true. Therefore, R is reflexive.

Symmetric

Let $(a,b), (c,d) \in \mathbb{Z} \times \mathbb{Z}$. Now, $(a,b) R (c,d) \Longrightarrow a+d = b+c \Longrightarrow d+a = c+b \Longrightarrow c+b = d+a \Longrightarrow (c,d) R (a,b)$. So R is symmetric.

Transitive

Let $(a,b), (c,d), (e,f) \in \mathbb{Z} \times \mathbb{Z}$.
Now, $(a,b) R (c,d)$ and $(c,d) R (e,f)$
$\Longrightarrow \underbrace{a+d = b+c}_{(1)}$ and $\underbrace{c+f = d+e}_{(2)}$

From (1), $d-c = b-a$ and from (2), $d-c = f-e$. Equating for $d-c$ we obtain $b-a = f-e \Longrightarrow b+e = a+f$
$\Longrightarrow a+f = b+e \Longrightarrow (a,b) R (e,f)$

So, R is transitive. Hence R is an equivalence relation.

The equivalence class of (x,y) is given by

$$[(x,y)] = \{(a,b) \in \mathbb{Z}\times\mathbb{Z} : (a,b)R(x,y)\}$$
$$= \{(a,b) \in \mathbb{Z}\times\mathbb{Z} : a+y = b+x\}$$
$$= \{(a,b) \in \mathbb{Z}\times\mathbb{Z} : y-x = b-a\}$$

Question 6

Given $(a,b) \in R$ iff $\exists\, m \in \mathbb{N}$ such $b = ma$, we have the following:

Reflexive

$\forall\, a \in \mathbb{N}$, $(a,a) \in R$ or aRa
$\Rightarrow a = a(1)$, where $1 \in \mathbb{N}$. So, $(a,a) \in R$ for all $a \in \mathbb{N}$ and so R is reflexive.

Symmetric

Let $a,b \in \mathbb{N}$. Now, $(a,b) \in R$ implies that $b = ma$, for $m \in \mathbb{N}$. Now, $b = ma \Rightarrow a = \frac{1}{m} b$; $\frac{1}{m} \notin \mathbb{N}$,

(6)

Since if m is an integer, in general, $\frac{1}{m}$ is not an integer.

So, $a = \frac{1}{m} b \not\Rightarrow (b,a) \in R$.

For example, $(3,6) \in R$ since $6 = 3(2)$, $2 \in N$, but $(6,3) \notin R$ since $3 = 6(1/2)$, $\frac{1}{2} \notin N$.

Hence, R is not symmetric.

Transitive

Let $a, b, c \in N$. Now, $(a,b) \in R$ and $(b,c) \in R \Rightarrow b = ma$ and $c = nb$ for $m, n \in N$.

From 2^{nd} equation $b = \frac{c}{n}$. Substituting for b in first equation gives

$\frac{c}{n} = ma \Rightarrow c = mna \Rightarrow$

$(a,c) \in R$, where $mn \in N$.

Therefore, R is transitive.

Note: You can think of specific examples, for instance

$(6,12) \in R$ and $(12,24) \in R \Rightarrow (6,24) \in R$

Now, R is not an equivalence relation.

Question 7

(i) Given $x R y \Leftrightarrow x - y = pk$, for $k \in \mathbb{Z}$.

Reflexive

For all $x \in \mathbb{Z}$, $x R x \Rightarrow$
$x - x = 0 = p(0)$, where $0 \in \mathbb{Z}$.
Therefore, R is reflexive.

Symmetric

Let $x, y \in \mathbb{Z}$. Now, $x R y \Rightarrow$
$x - y = pk$, $k \in \mathbb{Z}$. Also, we have
that $y - x = -(x-y) = -pk =$
$p(-k)$, for $-k \in \mathbb{Z}$. So,
$y - x = p(-k) \Rightarrow y R x$.
So, R is symmetric.

Transitive

Let $x, y, z \in \mathbb{Z}$. Now, $x R y$ and
$y R z \Rightarrow x - y = pk$ and $y - z = pl$,
for $k, l \in \mathbb{Z}$. Adding both equations
we have $x - z = pk + pl = p(k+l)$
$\Rightarrow x R z$, for $k + l \in \mathbb{Z}$.
Therefore, R is transitive.
Thus, R is an equivalence relation.

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(ii) The equivalence class of $x \in \mathbb{Z}$ is given by

$$[x] = \{y \in \mathbb{Z} : yRx\}$$
$$= \{y \in \mathbb{Z} : y - x = pk, k \in \mathbb{Z}\}$$
$$= \{y \in \mathbb{Z} : y = x + pk, k \in \mathbb{Z}\}$$

Question 8

(i) Given $(m, n) \in \nabla \iff m - n = 2k$,

We have the following;

Reflexive

$\forall\ m \in \mathbb{Z}$, $(m, m) \in \nabla \implies m - m = 0$
$= 2(0),\ 0 \in \mathbb{Z}$.

So, ∇ is reflexive.

Symmetric

Let $m, n \in \mathbb{Z}$, now $(m, n) \in \nabla \implies$
$m - n = 2k$, $k \in \mathbb{Z}$. Now, we have
that $n - m = -(m - n) = -2k = 2(-k)$
$\implies (n, m) \in \nabla$, for $-k \in \mathbb{Z}$.
So, ∇ is symmetric.

Transitive

Let $m, n, r \in \mathbb{Z}$. Now,

$(m, n) \in \nabla$ and $(n, r) \in \nabla \implies$

$m - n = 2k$ and $n - r = 2l$, for $k, l \in \mathbb{Z}$. Adding both equations we have

$m - r = 2k + 2l = 2(k + l) \implies$

$(m, r) \in \nabla$, for $k + l \in \mathbb{Z}$.

Therefore, ∇ is transitive.

Thus ∇ is an equivalence relation.

(ii) The equivalence class of $x \in \mathbb{Z}$ is given by

$$[x] = \{y \in \mathbb{Z} : (y, x) \in \nabla\}$$
$$= \{y \in \mathbb{Z} : y - x = 2k, k \in \mathbb{Z}\}$$
$$= \{y \in \mathbb{Z} : y = x + 2k, k \in \mathbb{Z}\}$$

(iii) The distinct equivalence classes of ∇ are given by

$$[x] = \{y \in \mathbb{Z} : y = x + 2k, k \in \mathbb{Z}\}$$
$$= \{y \in \mathbb{Z} : y = 2n + 2k; k \in \mathbb{Z}, x = 2n\}$$
$$= \{y \in \mathbb{Z} : y = 2(n + k); k \in \mathbb{Z}, x \text{ even}\}$$
$$= \{\ldots, -2, 0, 2, 4, \ldots\}$$

8 So, for even x, we have

$$[x] = \{\cdots, -4, -2, 0, 2, 4, \cdots\}$$

for $k \in \mathbb{Z}$ and $n \in \mathbb{Z}$. That is, $x = 2n$.

Similarly,

$$\begin{aligned}[x] &= \{y \in \mathbb{Z} : y = x + 2k,\ k \in \mathbb{Z}\} \\ &= \{y \in \mathbb{Z} : y = 2n+1+2k;\ k \in \mathbb{Z},\ x = 2n+1\} \\ &= \{y \in \mathbb{Z} : y = 2(n+k)+1;\ k \in \mathbb{Z},\ \text{odd } x\} \\ &= \{\cdots, -3, -1, 1, 3, 5, \cdots\}.\end{aligned}$$

Therefore, for odd x; that is, $x = 2n+1$, we have

$$[x] = \{\cdots, -3, -1, 1, 3, 5, \cdots\}.$$

for $k, n \in \mathbb{Z}$.

Note that when $x = 2n$ and $x = 2n+1$,

$$[2n] \cap [2n+1] = \{\ \} \text{ or } \phi \text{ and}$$

$$[2n] \cup [2n+1] = \mathbb{Z}.$$