

# RESolve – a 1D Richards Equation Model in Matlab

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This report presents **RESolve**, a numerical method for the solution of the one-dimensional Richards equation in a saturated or unsaturated soil column. A sink term representing evapotranspiration is included. Different options for the boundary conditions (top and bottom) are available. The equation is solved numerically using MATLAB's **pdepe** solver. We present validation results confirming the correctness of our approach. We present an open-source test case, as well as a link to Github repository to promote the sustainable future use of **RESolve**.

## I. GOVERNING EQUATION AND BOUNDARY CONDITIONS

In this report we use the 1D Richards Equation to model the flow of moisture in a soil column. The equation is given here as:

$$\frac{\partial \theta(h)}{\partial t} = \frac{\partial}{\partial z} \left[ K(h) \left( \frac{\partial h}{\partial z} + 1 \right) \right] - \lambda(h, z, t), \quad z \in (0, L), \quad t > 0, \quad (1)$$

where

- $\theta(h)$  is the volumetric water content in notional units of  $\text{m}^3/\text{m}^3$
- $h$  is the pressure head in units of m. The pressure head is negative in unsaturated conditions;  $h = 0$  indicates saturation;
- $K(h)$  is the hydraulic conductivity, in units of  $\text{m} \cdot \text{s}^{-1}$
- $\lambda$  is a sink term representing evapotranspiration (ET), in units of  $\text{m} \cdot \text{s}^{-1}$
- $z$  is the vertical coordinate, increasing upward with  $z = 0$  at the bottom and  $z = L$  at the soil surface.

This sign convention follows the Hydrus system. The same convention is used in the standard paper on the solution of the Richards equation by Dogan and Motz [1]. The gravity term therefore appears with a positive sign (as  $K(h)$ ) in the flux expression. For the avoidance of doubt, the flux is:

$$f = K(h) \left( \frac{\partial h}{\partial z} + 1 \right). \quad (2)$$

### A. Soil Hydraulic Functions: van Genuchten model

To close the Richards equation, constitutive relationships are required for the soil the unsaturated hydraulic conductivity, among other things. We use the van Genuchten–Mualem

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(VG) model for this purpose. We first of all introduce the effective saturation:

$$S_e(h) = \begin{cases} \frac{\theta(h) - \theta_r}{\theta_s - \theta_r}, & h < 0, \\ 1, & h \geq 0, \end{cases} \quad (3)$$

where  $\theta_r$  and  $\theta_s$  are the residual and saturated water contents. Here,  $\theta_r$  and  $\theta_s$  are limiting values corresponding to a residual saturation under dry conditions, and a saturated condition. Hence,  $\theta_r \leq \theta \leq \theta_s$ . Correspondingly,  $0 \leq S_e \leq 1$ . For negative heads corresponding to unsaturated soils, the VG model links  $h$  to  $S_e$  via the relation:

$$S_e(h) = [1 + (\alpha|h|)^n]^{-m}, \quad (4)$$

Equation (4) describes the so-called **water-retention curve**. Here,  $\alpha$  is the air-entry pressure and is a parameter of the model. Also,

$$m = 1 - \frac{1}{n}, \quad \alpha > 0, \quad n > 1, \quad (5)$$

and  $n$  is a further model parameter. The volumetric water content then follows as

$$\theta(h) = \theta_r + S_e(\theta_s - \theta_r). \quad (6)$$

Finally, the permeability is given by:

$$K(h) = K_s S_e^\ell \left[ 1 - \left( 1 - S_e^{1/m} \right)^m \right]^2, \quad (7)$$

where  $K_s$  is the saturated hydraulic conductivity and  $\ell$  is the pore-connectivity parameter (commonly  $\ell = 0.5$ ).

In practice, it is better to solve the Richards Equation in the variable  $h$ . Therefore, Equation (1) is re-arranged. We have:

$$\frac{\partial \theta}{\partial t} = \frac{\partial \theta}{\partial h} \frac{\partial h}{\partial t}. \quad (8)$$

Here,  $\partial h / \partial \theta$  is computed analytically from Equation (6). In practice, we identify the capacity  $C(h)$ :

$$C(h) = (\theta_s - \theta_r) \frac{dS_e}{dh} + \theta \rho g (c_r + c_w), \quad (9)$$

and we solve the following PDE in place of Equation (1):

$$C(h) \frac{\partial h}{\partial t} = \frac{\partial}{\partial z} \left[ K(h) \left( \frac{\partial h}{\partial z} + 1 \right) \right] - \lambda(h, z, t), \quad z \in (0, L), \quad t > 0, \quad (10)$$

Here, following Mathis et al. [2],  $\rho g (c_r + c_w)$  is taken to be  $9.81 \times 10^{-7} \text{ m}^{-1}$ .

## B. Sink Term for Evapotranspiration

The sink term  $\lambda(h, z, t)$  accounts for water removal due to plant root uptake and/or soil surface evaporation. We follow the work by Mathias et al. [2] and use a Feddes-type reduction function, whereby

$$\lambda(z, t) = E_p(t) \phi_1(z) \phi_2(h) \quad (11)$$

Here,  $E_p(t)$  is the potential evaporation, which is obtained from meteorological conditions. The term  $\phi_1$  is the root function, which describes the distribution of roots in the soil column. We use:

$$\phi_1(z) = \begin{cases} \frac{a}{L_r} \left[ \frac{e^{-a} - e^{-az_M/L_r}}{(1+a)e^{-a} - 1} \right], & 0 \leq z_M \leq L_r, \\ 0, & z_M > L_r. \end{cases} \quad (12)$$

Here, we have momentarily used the Mathias convention for the vertical coordinate, with  $z_M = L - z$ , such that  $z_M = 0$  corresponds to the surface. The parameters  $a$  and  $L$  characterize the root distribution. These will be provided on a case-by-case basis. Finally,  $\phi_2$  is the plant stress function, described by Feddes et al. [3]:

$$\phi_2(h) = \begin{cases} 0, & h \geq h_a, \\ 1, & h_a > h > h_d, \\ 1 - \frac{h - h_d}{h_w - h_d}, & h_d \geq h \geq h_w, \\ 0, & h < h_w. \end{cases} \quad (13)$$

Notice that  $\phi_2(h)$  has a jump discontinuity at  $h_a$ . Again, the parameters  $h_a$ ,  $h_d$ , and  $h_w$  will be provided on a case-by-case basis, as these depend on the crop type.

### C. Two notes of Caution!

Care should be taken that parameters are selected such that:

$$\theta(h_w) > \theta_r. \quad (14)$$

Otherwise, the ET function will be active when the medium is effectively dry. Not only is this unphysical, but it will cause code instability and potentially, blowup. Additionally, it should be noticed that  $\phi_2$  is a jump discontinuity at  $h = h_a$ , which may not be desirable from the numerical point of view. For a plot of  $\phi_2(h)$ , see Figure 1.

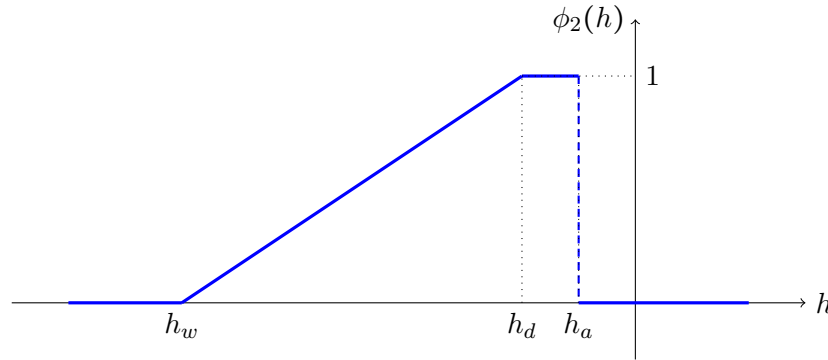


FIG. 1. Plot showing the plant stress function of Feddes et al. [3].

### D. Boundary Conditions

Equation (1) is solved with a variety of different boundary conditions, depending on the context.

**Bottom Boundary conditions:** We look at two possibilities here:

1. Free Drainage – this is a Neuman boundary condition, such that:

$$f = K(h), \quad z = 0. \quad (15)$$

Since  $f = K(h) (\partial_z h + 1)$ , this amounts to  $\partial_z h = 0$  at  $z = 0$ .

2. Saturation boundary condition – this is a Dirichlet condition, such that

$$h = h_{ref}, \quad z = 0. \quad (16)$$

Here,  $h_{ref}$  is a constant reference level. To describe a saturated soil at  $z = 0$ , we take  $h_{ref} = 0$ .

Care should be taken here to define ‘numerical saturation’ slightly differently from  $h = 0$  as Equation (10) may have trouble handling the extreme cases  $\theta = \theta_s$  (corresponding to  $h = 0$ ) and  $\theta = \theta_r$  (corresponding to  $h = -\infty$ ). Consequently, we introduce a ‘numerical saturation level’  $h_{ref}$ , such that:

$$\theta(h_{ref}) = 0.999\theta_s. \quad (17)$$

Here,  $\theta(h_{ref})$  is evaluated using Equation (6)

**Top Boundary conditions:** At the soil surface  $z = L$ , the standard approach is to use the rainfall rate as a flux boundary condition:

$$f = q_{rain}(t), \quad z = L, \quad (18)$$

where  $q_{rain}(t)$  is the imposed rainfall rate, in units of  $\text{m} \cdot \text{s}^{-1}$ . In practice, this is obtained from meteorological data. The meteorological data is often in  $\text{mm} \cdot \text{day}^{-1}$  and hence, care should be taken to input a rainfall flux in SI units into the PDE solver.

A more detailed and realistic approach takes account of the ponding phenomenon. Thus, we calculate a maximum reference flux:

$$f_{max} = K(h_{ref}) \left( \frac{h_{ref} - h(L - \Delta z)}{\Delta z} + 1 \right). \quad (19)$$

Here, finite-difference methodology is used: a uniform grid of spacing  $\Delta z$  is assumed, such that  $L - \Delta z$  is the gridpoint just below the surface, and

$$\frac{h_{ref} - h(L - \Delta z)}{\Delta z} \quad (20)$$

is an approximation to a reference gradient  $(\partial h / \partial z)_{ref}$ , at which point the surface flux corresponds to the surface flux at saturation. For the avoidance of doubt,  $h_{ref}$  is calculated from Equation (17). With this limiting flux identified, we are in a position to impose **ponding logic** at  $z = L$ :

$$f = \begin{cases} q_{rain}(t), & \text{if } q_{rain}(t) < f_{max}, \\ q_{rain}(t), & \text{if } q_{rain}(t) \geq f_{max}, \end{cases}, \text{ at } z = L. \quad (21)$$

## E. Initial Conditions

The initial pressure head distribution is specified as

$$h(z, 0) = h_0(z), \quad (22)$$

where  $h_0(z)$  may be uniform or vary with depth. In practice, we use  $h_0(z) = -z + z_{wt}$ , where  $z_{wt}$  is the initial level of the water-table. This follows the initial condition in Mathias et al. [2], but with the Hydrus convention whereby  $z = L$  is the top of the domain. This can also be seen to be an equilibrium solution of the Richards Equation,  $\partial_z h + 1 = 0$ , albeit one that satisfies particular boundary conditions.

## II. NUMERICAL SOLUTION USING PDEPE

Our solver **RESolve** does not directly solve the Richards Equation (10). Instead, we use MATLAB's **pdepe**, which can solve PDEs in the general form

$$c(x, t, u, u_x) u_t = x^{-m_D} \frac{\partial}{\partial x} (x^{m_D} f(x, t, u, u_x)) + s(x, t, u, u_x), \quad (23)$$

where  $m_D$  specifies the dimensionality ( $m_D = 0$  for the present 1D Cartesian geometry). To convert the generic form (23) to suit the Richards Equation, we identify:

$$x \equiv z, \quad u \equiv h, \quad m_D = 0. \quad (24)$$

We also identify:

$$\begin{aligned} c(z, t, u, u_z) &= (\theta_s - \theta_r) \frac{dS_e}{dh} + \theta \rho g (c_r + c_w), \\ f(z, t, u, u_z) &= K(u) (u_z - 1), \\ s(z, t, u, u_z) &= -\lambda(u, z, t). \end{aligned}$$

### A. Implementation of Boundary Conditions in PDEPE

PDEPE expects boundary conditions in the form

$$p(z, t, u) + q(z, t) f(z, t, u, u_z) = 0.$$

We go through the various options now:

1. **Bottom, free drainage:** In the case of free drainage, we impose  $f = K(h)$ , hence  $p + qf(h) = 0$ , hence  $q = 1$  and  $p = -K(h)$ .
2. **Bottom, saturated:** In the case of saturation, we impose  $h = h_{ref}$ , hence  $q = 0$  and  $p = u - h_{ref}$ .
3. **Top, rainfall flux:** The rainfall flux condition is

$$f = q_{rain}(t), \quad z = L. \quad (25a)$$

Thus,

$$p = -q_{rain}(t), \quad q = 1. \quad (25b)$$

To implement the ponding boundary condition, only a small modification to Equation (25) is required.

## B. Spatial and Temporal Discretization

A uniform spatial mesh is used:

$$z_i = i\Delta z, \quad i = 0, \dots, N.$$

The time integration uses MATLAB's built-in ODE solver (`ode15s`), controlled by user-specified `tspan`.

## C. MATLAB Structure

The model is written in a single Matlab function `richards_pdepe.m`, which includes a main function setting up the spatial grid, time vector, soil hydraulic functions, rainfall input, and calls to `pdepe`. The main function contains subfunctions:

- a PDE function defining  $c$ ,  $f$ , and  $s$ ,
- a boundary condition function,
- an initial condition function.

Additional functions in separate files are included to give effect to the Van Genuchten model, to handle rainfall and evapotranspiration inputs, and also, to perform postprocessing, as required.

## III. VALIDATION

We validate our solver with respect to the results of Mathias (SM) et al. [2]. For the root distribution function and the plant-stress function, we use the parameters given in that paper and summarized here again in Table I. Furthermore, the meteorological data in that paper were kindly provided by SM upon request. Consequently, the daily rainfall data and the monthly potential evapotranspiration levels are available for the present validation purposes, and were obtained from the Theale gauging station in the Kennet valley, Berkshire, UK. The time series are from January 1961 to December 1997. We further use a soil column

TABLE I. Parameters Used in the Root Distribution and Plant Stress Functions

Parameter	Value	Description
$L_r$	1 m	Root zone depth
$a$	2 m	Shape parameter of the root distribution function
$h_a$	-0.05 m	Pressure head for onset of aeration stress
$h_d$	-4 m	Pressure head for reduced uptake due to dryness
$h_w$	-150 m	Wilting point pressure head

depth  $L = 3$  m and a constant initial condition  $h_0(z) = -0.1$  m. We run the simulations from January 1st 1985, with a view to comparing `RESolve` and the model of SM within the window from January 1st 1988 to December 31st 1995. Results for the sandy soil (VG parameters in Table II) are shown in Figure 2. Here, we are plotting

$$\Theta(t) = \int_0^L \theta(t, z) dz. \quad (26)$$

Results for the clay soil (VG parameters in Table II also) are shown in Figure 3. It is important to consider this case as well, since here, the ponding condition is triggered during high-rainfall events. In both figures, visual inspection with the reference paper shows that

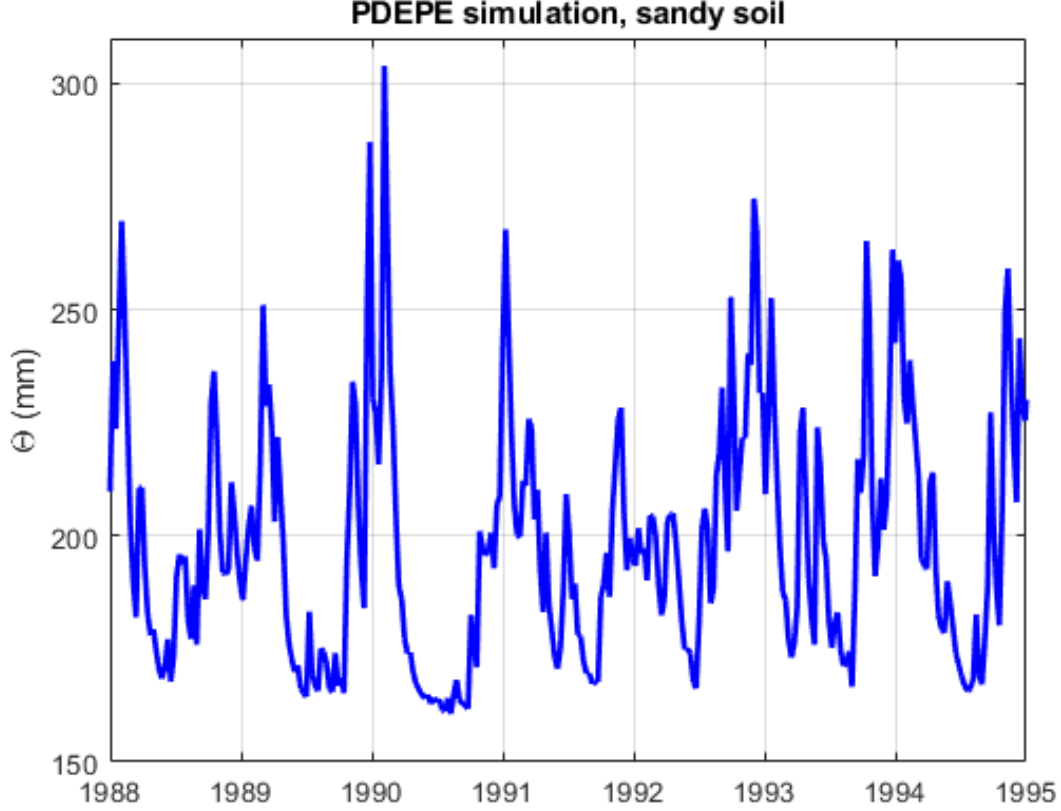


FIG. 2. Time series plot showing  $\Theta(t)$  as computed using `RESolve` for a sandy soil. Results may be compared with Figure 2 in Reference [2].

the agreement between `RESolve` and the numerical method of Mathias et al. [2] is good, confirming the suitability of `pdepde` for the purpose of solving the 1D Richards Equation.

TABLE II. van Genuchten Parameters for Sandy and Clay Soils

Parameter	Symbol	Sandy Soil	Clay Soil
Residual water content	$\theta_r$	0.0515	0.0961
Saturated water content	$\theta_s$	0.3769	0.4616
VG parameter	$\alpha$	3.321	2.711
VG parameter	$n$	2.503	1.149
Saturated hydraulic conductivity ( $\text{mm} \cdot \text{day}^{-1}$ )	$K_s$	3220	108.5
Pore-connectivity parameter	$\ell$	-0.8653	-5.153

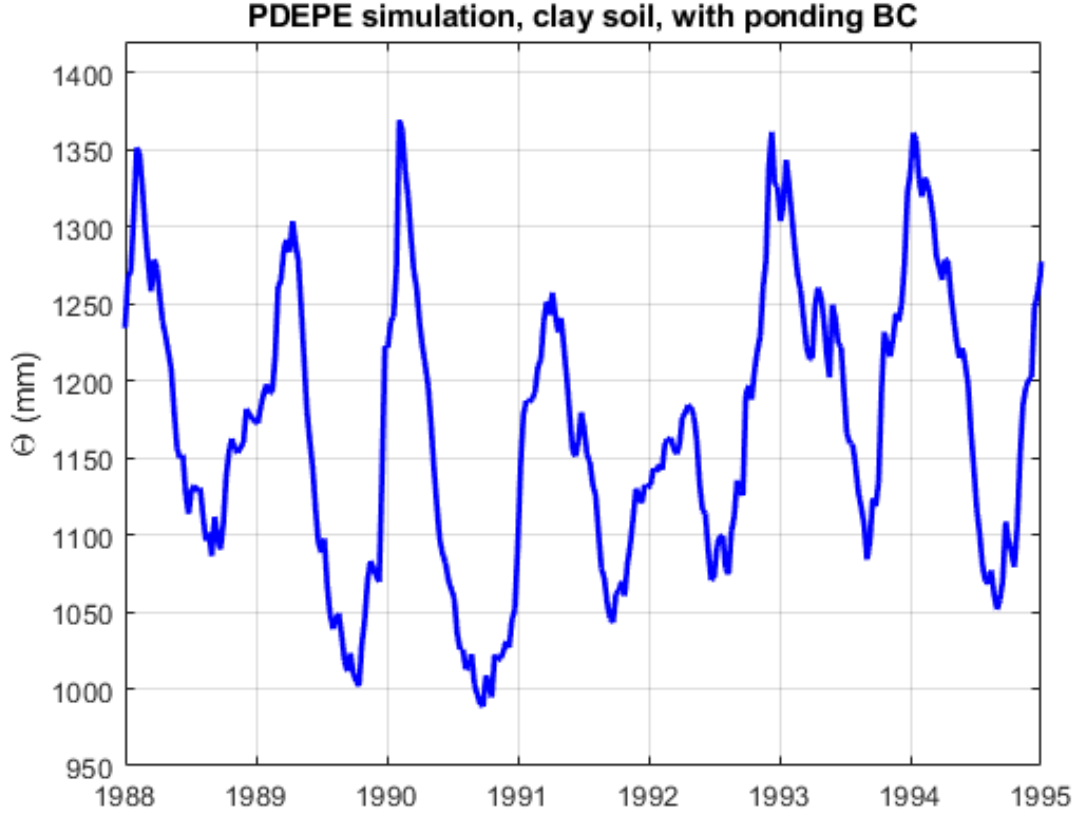


FIG. 3. Time series plot showing  $\Theta(t)$  as computed using **RESolve** for a clay soil. Results may be compared with Figure 3 in Reference [2].

#### IV. OPEN-SOURCE TEST CASE

We present an open-source test case which may be used to validate other codes or familiarize a user with **RESolve**. The test case involves a two-year simulation with soil data appropriate for Johnstown Castle, Co. Wexford, Ireland, and provided in Tables III–IV. The VG parameters are obtained from the Irish Soil Map, by depth-averaging. Hence, the presented model is an effective single-layer model. We solve the equations using a soil column  $L = 1.8\text{ m}$  with a saturated bottom boundary condition. We use the initial condition

$$h_0(z) = -z - 0.5\text{ m}. \quad (27)$$

We simulate for four years. The first year is used for burn-in, so that after one year of simulation time, the model effectively forgets the initial condition, which may not be physical. We use synthetic meteorological data, with

$$q_{rainfall}(t) = q_{rf,max} \left\{ \frac{1}{2} [\cos(2\pi t/T_{year}) + 1] \right\}, \quad (28a)$$

$$EP(t) = EP_{max} \left\{ \frac{1}{2} [\cos(2\pi t/T_{year}) + 1] \right\}. \quad (28b)$$



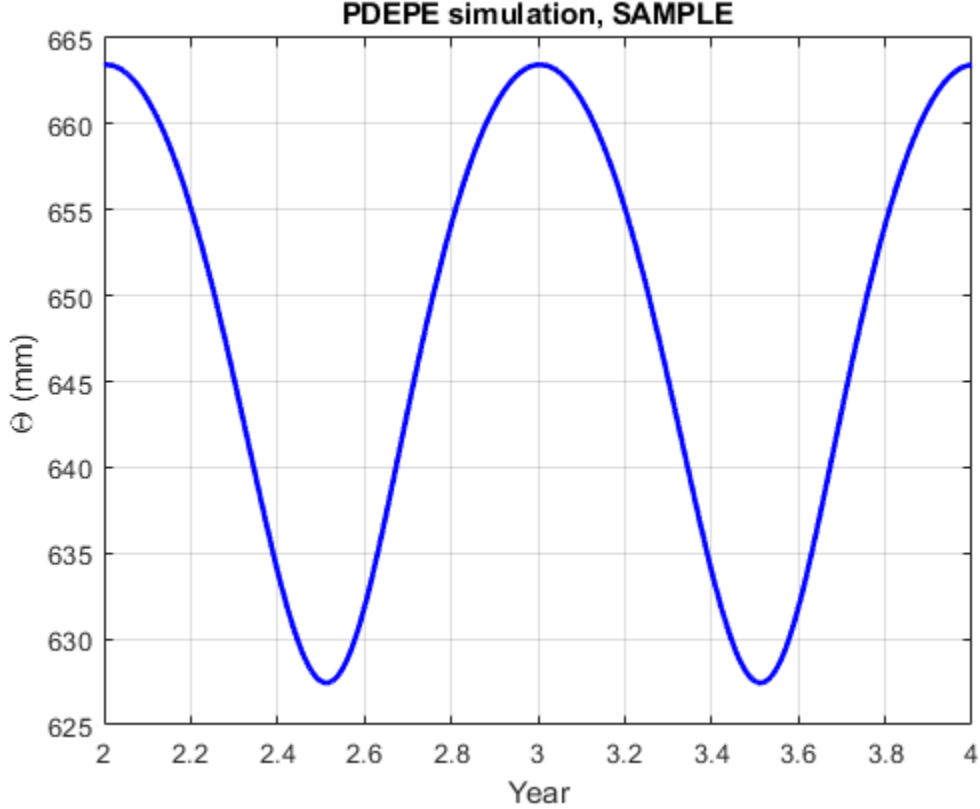


FIG. 4. Simulated time series of  $\Theta(t)$  for Johnstown-castle-type soil

As always,  $t$  is measured in seconds, and  $T_{year} = 365 \times 24 \times 3600$ . We also use  $q_{rf,max} = 20 \text{ mm} \cdot \text{day}^{-1}$  and  $EP_{max} = 10 \text{ mm} \cdot \text{day}^{-1}$ . Sample results are  $\Theta(t)$  are shown in Figure 4. We also show the space-time behaviour of  $h(t, z)$  and  $\theta(t, z)$  in Figure 5, showing the clear seasonality trends in the moisture levels in the soil.

TABLE III. Parameters Used in the Root Distribution and Plant Stress Functions (Johnstown Castle)

Parameter	Value	Description
$L_r$	0.25 m	Root zone depth
$a$	1.55 m	Shape parameter of the root distribution function
$h_a$	-0.25 m	Pressure head for onset of aeration stress
$h_d$	-3 m	Pressure head for reduced uptake due to dryness
$h_w$	-10 m	Wilting point pressure head

## V. CONCLUSION

Summarizing, we have introduced **RESolve**, a 1D Richards-equation mathematical model, to describe the moisture level in a saturated or unsaturated soil. The model is implemented

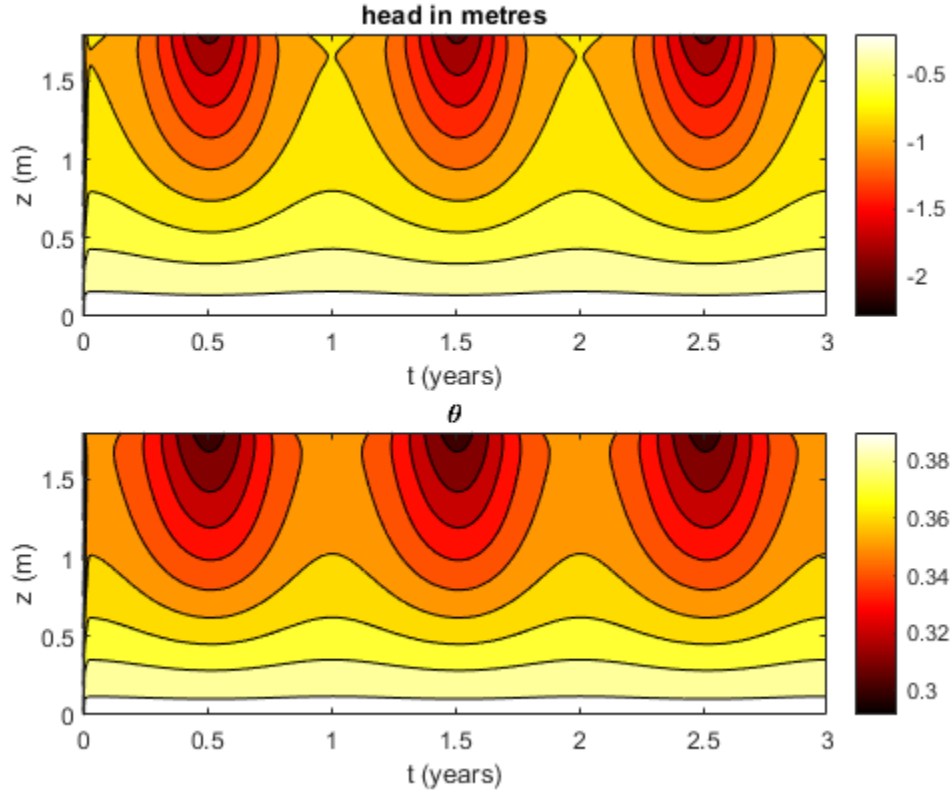


FIG. 5. Space-time plot of  $h(t, z)$  and  $\theta(t, z)$  for Johnstown-castle-type soil

TABLE IV. van Genuchten Parameters for Sandy and Clay Soils (Johnstown Castle)

Parameter	Symbol	Value
Residual water content	$\theta_r$	0.077
Saturated water content	$\theta_s$	0.396
VG parameter	$\alpha$	0.894
VG parameter	$n$	1.424
Saturated hydraulic conductivity ( $\text{m} \cdot \text{s}^{-1}$ )	$K_s$	$0.195/(24 \times 3600)$
Pore-connectivity parameter	$\ell$	0.5

by using the `pdepe` function in Matlab, meaning that the nitty-gritty of discretizing the PDE and handling the boundary conditions is left to an effective ‘black box’, enabling the user to focus on the model parametrization. A choice of different physically-relevant boundary conditions is available to the user. We have carefully validated our solver against the published results of Mathias *et al.* [2], for various soil types. We have presented a SAMPLE case, which we make freely available for validation and testing on the Github

repository onaraighl/RESolve.

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