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Mathematical Optimization Hands-On Tutorial

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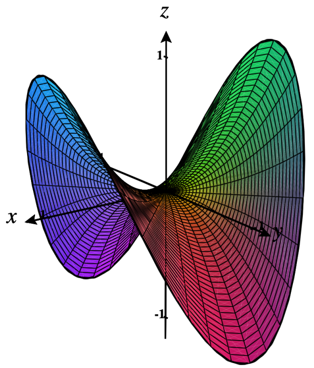
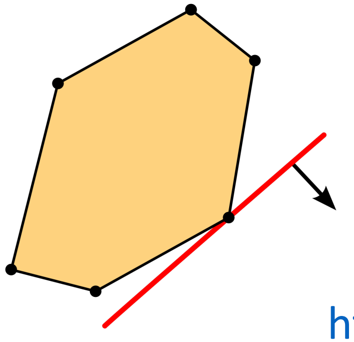
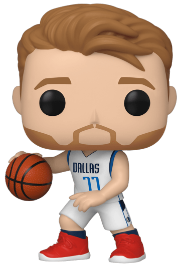


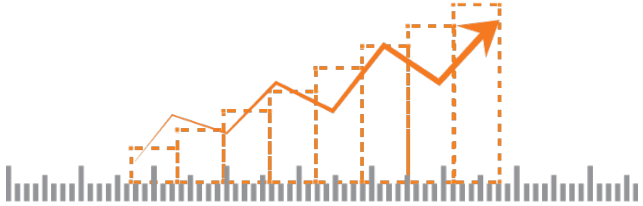


https://github.com/onatucsd/mathematical_opt_tutorial

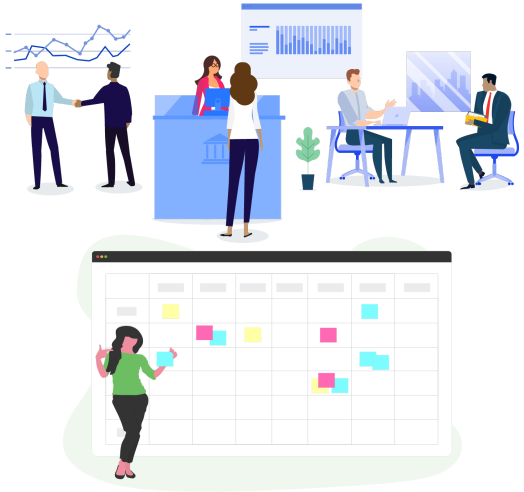
Mathematical Optimization

- “Optimization” comes from the same root as “optimal” which means “best”. When you “optimize” something, you are ”making it best”.
- “Best” can change:
  - If you are a basketball player, you want to:
    - **maximize** your shooting percentage
    - **minimize** your turnover ratio





- Mathematical optimization is a branch of applied mathematics which is useful in numerous fields:
  - Manufacturing
  - Transportation
  - Scheduling
  - Finance
  - Marketing



Mathematical Optimization

- Mathematical optimization (mathematical programming) finds the **best** choice among a set of **options** subject to a set of **constraints**.
- **Formulation in words:**

minimize	objective
by varying	variables
subject to	constraints

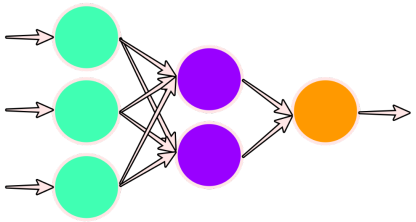
- **Objective function:** output you are trying to minimize (maximize)
- **Variables:** inputs you can control and influence system performance
- **Constraints:** equalities, or inequalities that restrict the variables.

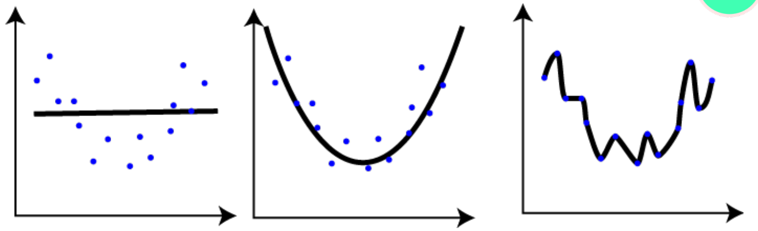
Mathematical Optimization Applications

- Portfolio optimization
  - **Objective:** minimize risk
  - **Variables:** amount of budget to allocate to each asset (e.g., stock, bond, etc.)
  - **Constraints:** total amount of budget available

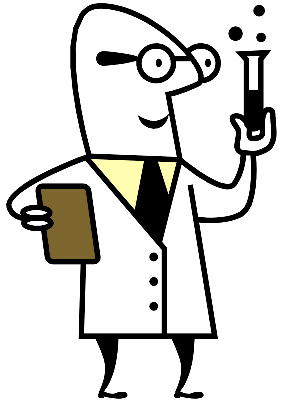
- Transportation problems
  - **Objective:** minimize transportation cost
  - **Variables:** routes to transport goods between warehouses and outlets
  - **Constraints:** outlets receive proper inventory

- Model fitting (machine learning)
  - **Objective:** minimize error in model predictions over a training set
  - **Variables:** parameters of the model
  - **Constraints:** model complexity





Mathematical Optimization Formulation



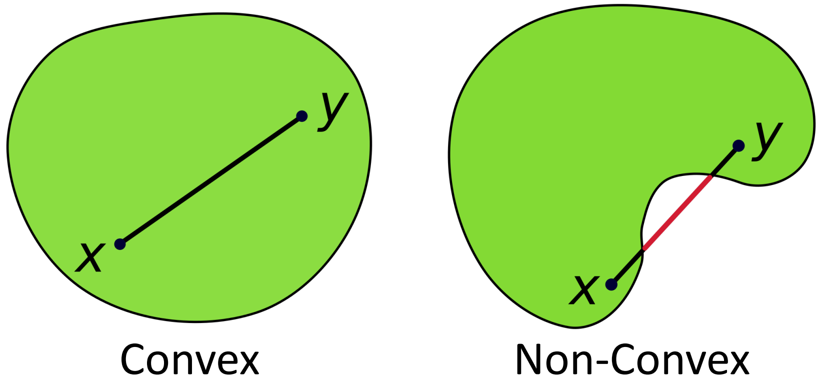
- **Standard form:**

$$
\begin{array}{lll}
\text{minimize} & f_0(x) & \\
\text{subject to} & f_i(x) \leq 0, & i = 1, \ldots, m \\
 & h_i(x) = 0, & i = 1, \ldots, p
\end{array}
$$

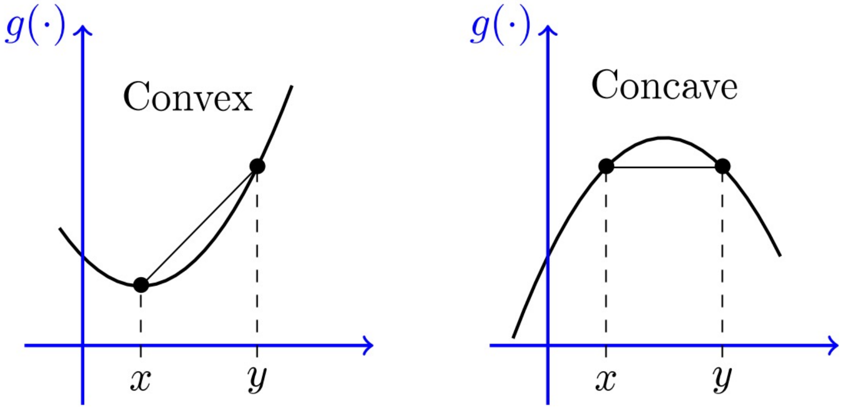
- $x \in \mathbb{R}^n$: decision variable
- $f_0 \colon \mathbb{R}^n \mapsto \mathbb{R}$: objective function
- $f_i \colon \mathbb{R}^n \mapsto \mathbb{R}$: inequality constraint function
- $h_i \colon \mathbb{R}^n \mapsto \mathbb{R}$: equality constraint function
- **Feasible set:** $D = \{x \in \mathbb{R}^n \mid f_i(x) \leq 0,\ i = 1, \ldots, m,\ h_i(x) = 0,\ i = 1, \ldots, p\}$
- **Feasibility problem:** Find $x \in D$

Convex Optimization Terminology

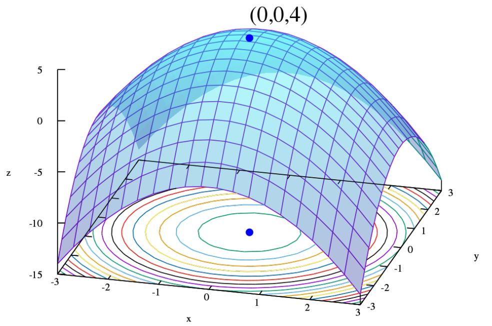
Convex set: A set C is convex if the line segment between any two points in C is contained in C, i.e., $\forall x, y \in C, \forall \alpha \in [0,1]$ -> $\alpha x + (1-\alpha) y \in C$.
Examples: The empty set $\emptyset$, hyperplanes, lines, etc.



Convex function: A function g: $\mathbb{R}^n \mapsto \mathbb{R}$ is convex if
1) Domain of the function is convex.
2) Line segment between any two points on the graph of the function lies above the graph between two points.



Convex Optimization



- **Standard form:**

minimize	$f_0(x)$
subject to	$f_i(x) \leq 0, \quad i = 1, \dots, m$
	$h_i(x) = 0, \quad i = 1, \dots, p$

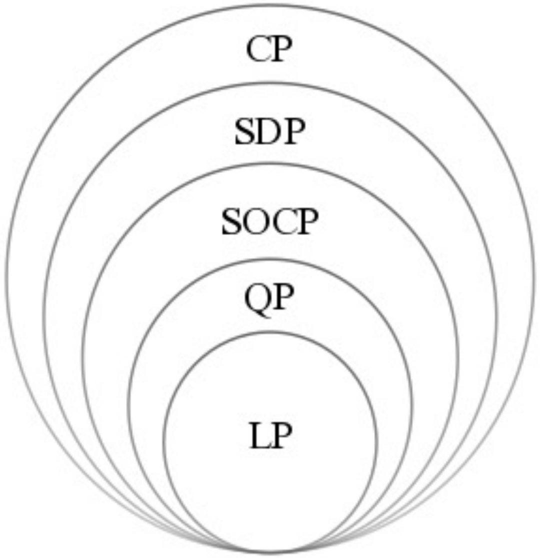
- $x \in \mathbb{R}^n$: decision variable
- $f_0 : \mathbb{R}^n \mapsto \mathbb{R}$ is a convex function.
- $f_i : \mathbb{R}^n \mapsto \mathbb{R}$: are convex functions.
- $h_i : \mathbb{R}^n \mapsto \mathbb{R}$: are affine functions (i.e., $h_i(x) = a_i{}^T x + b_i$).
- We minimize a convex objective function f_0 over a convex set C.

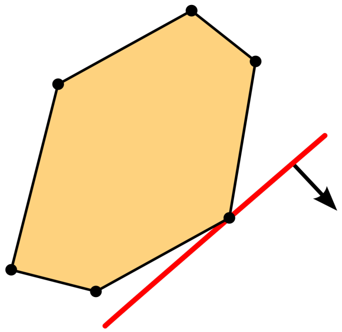
"...in fact, the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity."
- R. Tyrrell Rockafellar, in *SIAM Review*, 1993

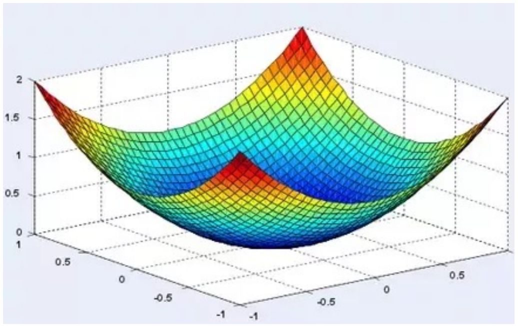
minimize	$f_0(x)$
subject to	$x \in C$

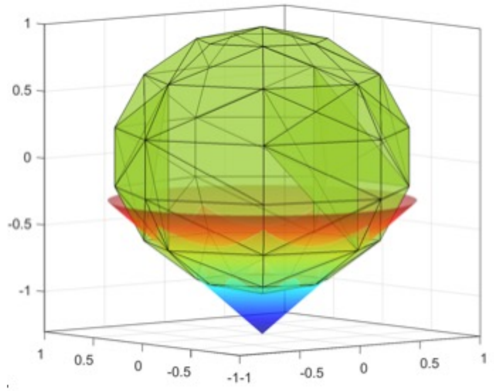
Convex Optimization Problems

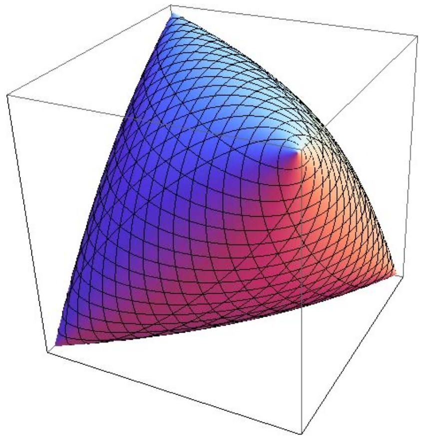
- **Linear Programming (LP)**
  - affine objective and constraints.
- Quadratic Programming (QP)
  - (convex) quadratic objective, and affine constraints.
- Second Order Cone Programming (SOCP)
- Semidefinite Programming (SDP)
- Cone Programming (CP)











Linear Programming (LP)

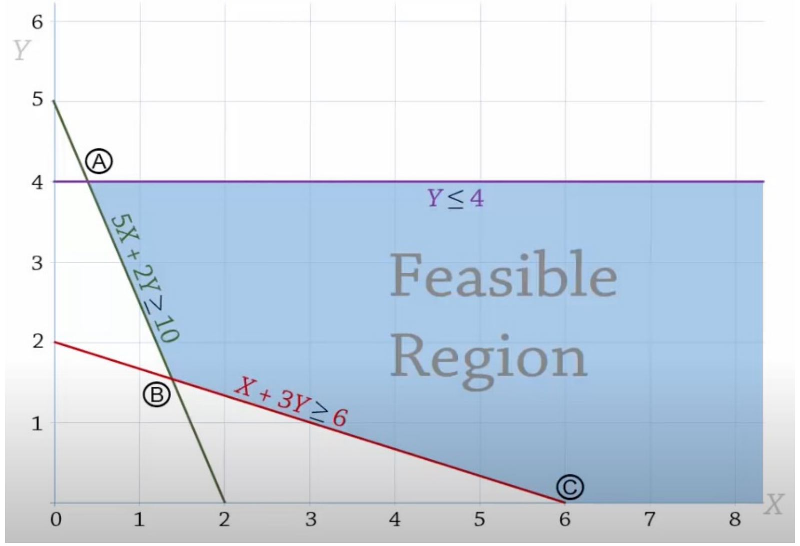
- Linear programming is developed during **World War II** to plan expenditures and returns in order to reduce costs to the army.
- **Linear programming (LP)** refers to an optimization problem in which (1) the decision variables are continuous, (2) the objective function is linear, and (3) each constraint is a linear inequality or equality.
- LP in standard form:

$$
\begin{array}{ll}
\text{minimize} & c_1 x_1 + \ldots + c_n x_n \\
\text{subject to} & a_{11} x_1 + \ldots + a_{1n} x_n = b_1 \\
& \quad \vdots \quad + \ldots + \quad \vdots \quad = \vdots \\
& a_{m1} x_1 + \ldots + a_{mn} x_n = b_m \\
& x_i \geq 0, \ \forall i
\end{array}
\quad \Longleftrightarrow \quad
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b \\
& x \geq 0
\end{array}
$$

LP Solution Methods

- The Graphical Solution Approach
  - Applicable if and only if the LP contains **two decision variables**.
  - Search only the **corner points** of the feasible region.
    - Based on the fundamental theorem of LP.

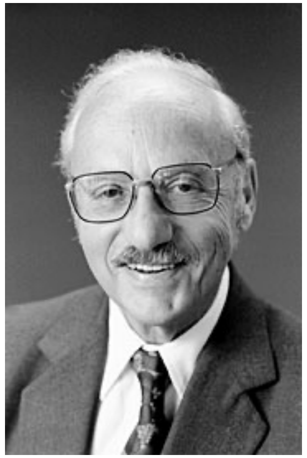
minimize	$z = 5x + 7y$
subject to	$x + 3y \geq 6$
	$5x + 2y \geq 10$
	$y \leq 4$
	$x, y \geq 0$



Corner points:
A) $x = 1, y = 4 \mapsto z = 33$
B) $\mathbf{x = 1.38, y = 1.54 \mapsto z = 17.7}$
C) $x = 6, y = 0 \mapsto z = 30$
Optimal Solution:
$x^* = 1.38, y^* = 1.54$
Optimal Value:
$z^* = 17.7$

LP Solution Methods

- The Simplex Algorithm
  - **George Dantzig** in 1949 developed an efficient procedure for solving LP.
  - **Most widely** used method in commercial computer packages.
  - Move from one **extreme point** to the other (better or equally good).
  - When it finds an **optimum**, it identifies this fact and stops.
  - Uses very simple mathematics which are easy to implement.



- Solvers
  - **Solver:** A **software** package that includes one or more **algorithms** (e.g., Simplex) to find solutions to one or more **classes of problem** (e.g., LP).
  - GUROBI, MOSEK, CPLEX, Excel Solver, etc.

LP Example 1: Manufacturing

- Imagine you are an operations manager of a factory. You need to decide a production plan for the next week. The factory produces **three** products (P, Q, and R) using **four** machines (A, B, C, and D). Each of the four machines performs a unique process. There is one machine of each type, and each machine is available for **2400 minutes** per week. Storage from one week to the next is not permitted. The operating expenses associated with the plant are **$6000 per week**, regardless of how many components and products are made. The $6000 includes all expenses except for material costs. Your goal is to **determine the production quantity** of each product to **maximize the total profit** given the **available resources**.

LP Example: Manufacturing

Machine Data – Unit Processing Time (min)

Machine	Product P	Product Q	Product R
A	20	10	10
B	12	28	16
C	15	6	16
D	10	15	0

Product Data

Item	Product P	Product Q	Product R
Revenue per unit	$90	$100	$70
Material cost per unit	$45	$40	$20
Maximum allowable sales	100	40	60

LP Example: Manufacturing

- Step 1: Define the **decision variables**
  - Company policy is to **determine the production quantity of each product:**
    - p: number of units of P to be produced
    - q: number of units of Q to be produced
    - r: number of units of R to be produced
- Step 2: Choose an **objective function**
  - Company wants to **maximize the total profit:**
    - maximize $z = 45p + 60q + 50r - 6000$
- Step 3: Identify the **constraints**
  - Machine availabilities:
    - $20p + 10q + 10r \leq 2400$
    - $12p + 28q + 16r \leq 2400$
    - $15p + 6q + 16r \leq 2400$
    - $10p + 15q + 0r \leq 2400$
  - Maximum allowable sales constraints:
    - $p \leq 100$, $q \leq 40$, $r \leq 60$
  - Non-negativity constraints:
    - $p \geq 0$, $q \geq 0$, $r \geq 0$

maximize $\quad z = 45p + 60q + 50r$

subject to

$$20p + 10q + 10r \leq 2400$$
$$12p + 28q + 16r \leq 2400$$
$$15p + 6q + 16r \leq 2400$$
$$10p + 15q + 0r \leq 2400$$
$$0 \leq p \leq 100$$
$$0 \leq q \leq 40$$
$$0 \leq r \leq 60$$

Optimal Solution: $p^* = 81.82$, $q^* = 16.36$, $r^* = 60$, $z^* = 7664$

LP Example: Scheduling

- Dunder Mifflin Paper Company, Inc. requires different numbers of full-time employees on different days of the week. Each full-time employee must **work five** consecutive days and then receive **two days off**. Formulate an LP to **minimize** the **number of full-time employees**.

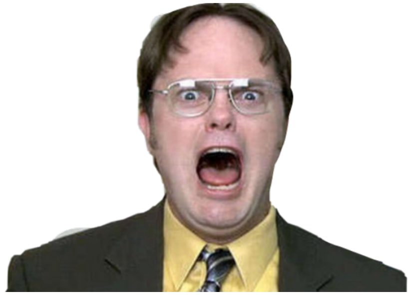
Day of the Week	Number of Employees Required
1 (Monday)	17
2 (Tuesday)	13
3 (Wednesday)	15
4 (Thursday)	19
5 (Friday)	14
6 (Saturday)	16
7 (Sunday)	11




LP Example: Scheduling

DUNDER MIFFLIN INC.
PAPER COMPANY

- Step 1: Define the **decision variables**
  - Decide **how many employees** we must hire:
    - y_i: number of employees who work on day i, i=1,2,…,7.
    - x_i: number of employees whose five consecutive days of work begin on day i, i=1,2,…,7.
      - They work on days i, i+1, i+2, i+3, and i+4.
      - e.g., x_1: number of employees beginning to work on Monday.
- Step 2: Choose an **objective function**
  - Minimize the number of hired employees:
    - minimize z = $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$
- Step 3: Identify the **constraints**
  - Number of employees required for each day:
    - $x_1 + x_4 + x_5 + x_6 + x_7 \geq 17$ (Monday)
    - $x_1 + x_2 + x_5 + x_6 + x_7 \geq 13$ (Tuesday)
    - $x_1 + x_2 + x_3 + x_6 + x_7 \geq 15$ (Wednesday)
    - $x_1 + x_2 + x_3 + x_4 + x_7 \geq 19$ (Thursday)
    - $x_1 + x_2 + x_3 + x_4 + x_5 \geq 14$ (Friday)
    - $x_2 + x_3 + x_4 + x_5 + x_6 \geq 16$ (Saturday)
    - $x_3 + x_4 + x_5 + x_6 + x_7 \geq 11$ (Sunday)
  - Non-negativity of the decision variables:
    - $x_i \geq 0$ ∀i, i=1,2,..7 (non-negativity)



minimize $z = \sum_{i=1}^{7} x_i$

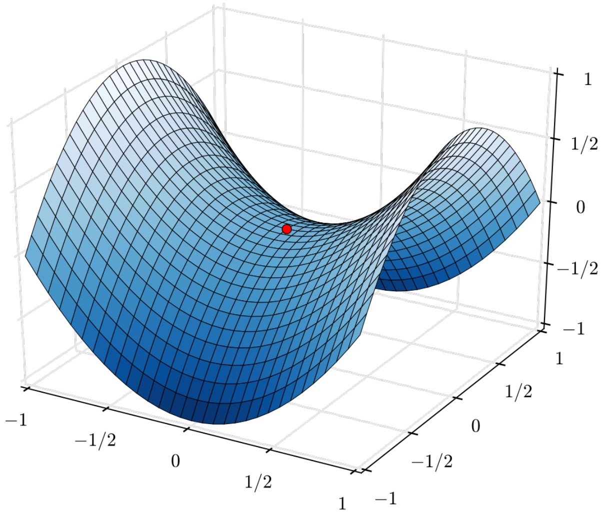
subject to

$$x_1 + x_4 + x_5 + x_6 + x_7 \geq 17$$
$$x_1 + x_2 + x_5 + x_6 + x_7 \geq 13$$
$$x_1 + x_2 + x_3 + x_6 + x_7 \geq 15$$
$$x_1 + x_2 + x_3 + x_4 + x_7 \geq 19$$
$$x_1 + x_2 + x_3 + x_4 + x_5 \geq 14$$
$$x_2 + x_3 + x_4 + x_5 + x_6 \geq 16$$
$$x_3 + x_4 + x_5 + x_6 + x_7 \geq 11$$
$$x_i \geq 0 \quad \forall i, i=1,2,..7$$

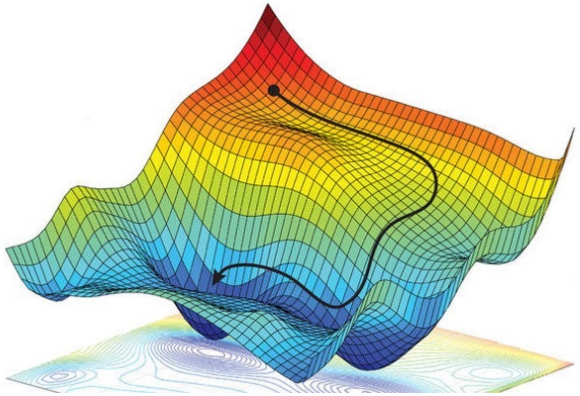
Optimal Solution: $x_1^* = 6.3$, $x_2^* = 3.3$, $x_3* = 2$, $x_4^* = 7.3$, $x_5^* = 0$, $x_6* = 3.3$, $x_7* = 0$, z* = 22.3

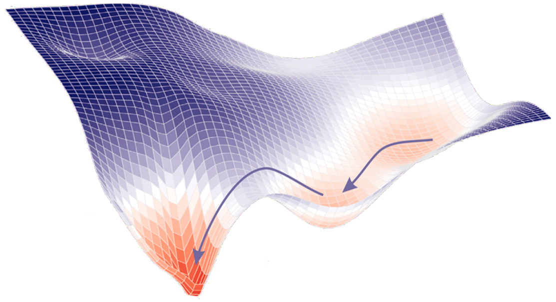
Non-convex Optimization

- **Non-convex optimization problem:** any problem where the objective or any of the constraints are non-convex.
- Non-convex problems are difficult to solve:
  - Saddle points
  - Many local minima
  - NP-hard



- Non-convex Optimization Problems
  - **Integer programming (IP)**
  - Linear-fractional programming (LFP)
  - Geometric programming (GP)





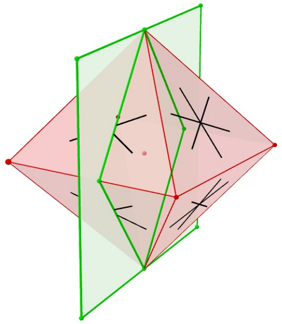
Integer Programming (IP)

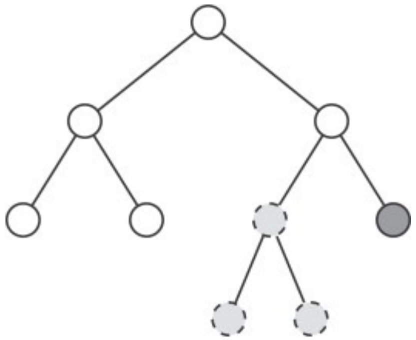
- In previous LP examples, we assume that decision variables can only take continuous values:
  - number of units to be produced during a week
  - number of employees required
- However, mostly this assumption is NOT realistic. We need these variables to take INTEGER values:
  - **Pure Integer Programming (IP):** All variables are required to be integer.
  - **Mixed Integer Programming (MIP):** If some variables are restricted to be integer and some are not.
  - **Binary (0-1) Integer Programming (BIP):** The case where the integer variables are restricted to be 0 or 1.

IP Solution Approaches

- **Enumerate** all the possible solutions and then choose the best one.
  - Only suitable for very small-sized problems.
  - What about larger problems?
    - e.g., IP model with 100 binary variables have $2^{100} = (1.268)10^{30}$ possible solutions!

- **Branch & Bound Algorithm:** This algorithm consists of a systematic enumeration of all candidate solutions, where large subsets of fruitless candidates are discarded **en masse**, by using upper and lower estimated bounds of the quantity being optimized.
- **Implicit Enumeration**
- **Cutting Plane Algorithm**
- **Lagrangian Relaxation**





MIP Example: Diet Problem

- After a never-ending quarantine period, you realized that you gained 10 pounds. You immediately decided to make a better diet plan. In this diet, you are only allowed to consume apples, oranges, and milk. Apples and oranges should be consumed in integer quantities whereas there is no such restriction for milk. Each apple costs **$1.5**, each orange costs **$2**, and each gallon of milk costs **$3**. Each day, you must ingest at least **500 calories**, **6 gr of protein** and **10 gr of sugar**. The nutritional content per unit of each food is given in the table. Also, **number of apples consumed should be greater than the half of the number of oranges consumed**. Formulate a mixed integer programming model that can be used to satisfy your daily nutritional requirements at minimum cost.

Food	Calorie	Protein	Sugar
Apple	70	0.5	2
Orange	90	0.2	3
Milk	400	4	2.5

MIP Example: Diet Problem

- Decision variables:
  - x_1: number of apples to be consumed
  - x_2: number of oranges to be consumed
  - x_3: gallons of milk to be consumed

minimize $\quad z = 1.5x_1 + 2x_2 + 3x_3$ (minimize total cost)

subject to $\quad 70x_1 + 90x_2 + 400x_3 \geq 500$ (calorie constraint)

$0.5x_1 + 0.2x_2 + 4x_3 \geq 6$ (protein constraint)

$2x_1 + 3x_2 + 2.5x_3 \geq 10$ (sugar constraint)

$x_1 - 0.5x_2 \geq 0.1$ (amount constraint)

$x_1, x_2, x_3 \geq 0$ (non-negativity)

x_1, x_2 integer (integrality restrictions)

Optimal Solution: x_1* = 2, x_2* = 1, $x_3* = 1.2$, z* = 8.6

BIP Example: Investment

- CEO of Hawthorne Wipes, Inc. Pierce Hawthorne decided to make an investment in order to use his father's $24,000 heritage. He has four investment opportunities shown in the table:

Investment	Required Amount	Present Value
Shirley's Sandwiches	$9,000	$13,000
Greendale Community College	$7,000	$11,000
Señor Chang's security school	$7,000	$10,000
Abed's new movie	$3,000	$5,000

- Pierce also has additional restrictions:
  - He can only make two investments.
  - If he invests in Greendale, he also must invest in Abed's movie.
  - If he invests in Shirley's Sandwiches, he cannot invest in Chang's school.
- Into which investments should Pierce place the money to **maximize the total present value**?

BIP Example: Investment

- Decision variables:
  - x_i: binary variable (0 or 1) indicating investment i is made or not (i = 1,2,3,4)
    - e.g., x_1= 1 -> investment is made in Shirley's Sandwiches.

maximize	z = $13x_1 + 11x_2 + 10x_3 + 5x_4$ (maximize total present value)
subject to	$9x_1 + 7x_2 + 7x_3 + 3x_4 \leq 24$ (total heritage constraint)
	$x_1 + x_2 + x_3 + x_4 = 2$ (total number of investment constraint)
	$x_2 - x_4 \leq 0$ (1st dependency constraint)
	$x_1 + x_3 \leq 1$ (2nd dependency constraint)
	$x_i \in \{0,1\} \quad \forall i, i = 1,2,3,4$ (binary variable constraints)

Optimal Solution: $x_1^* = 1$, $x_2^* = 0$, $x_{3*} = 0$, $x_4^* = 1$, z* = 18

Thank You

Any questions?

Feel free to reach out to me for any further questions:

E-mail: ogungor@ucsd.edu