Subquandle Lattices

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Abstract

We provide a simple theorem which characterizes the inner automorphism group of a subquandle as a subquotient of the inner automorphism group of its ambient quandle. Additionally, we extend Saki and Kiani's [5] proof that infinite quandles may not have complemented sublattices by providing an example of an ind-finite quandle whose subquandle lattice is not complemented. Furthermore, we build upon Theorem 3.1 of Ehrman et al. [3] to provide a classification for subquandles whose set complements are also subquandles, which we call "strongly complemented." Finally, we provide a transitivity criterion for subquandle complementation.

1 Introduction

Quandles were first introduced as a complete classical knot invariant up to orientation reversal, by Joyce [1] in 1982. In recent years, quandles and racks have found numerous applications in knot theory and algebraic topology. Aside from these applications, their algebraic structure is interesting in its own right. Indeed, quandles capture the algebraic structure of conjugation in a group. Hence for any group, G, a quandle $\mathrm{Conj}(G)$ can be constructed, such that for all $g, h \in G$, $g \triangleright h = g^{-1}hg$.

The study of subuandles is particularly difficult, for subquandles lack constants such as an identity. Nonetheless, some progress has been made toward understanding them. For example, Saki and Kiani [5] showed that the lattice of subquandles is complemented in the finite case. Moreover, in Theorem 3.1 of [3], Ehrman et al. proved a quandle can be decomposed into its orbits under the action of the inner automorphism group. In this paper we relate complementedness to the orbit decomposition theorem by providing a complete classification of subquandles whose set complement in the ambient quandle is also a subquandle. We also use this to provide a simple criterion for complementedness.

2 Background and Definitions

Definition 2.1. [1] A *quandle* is a set Q with binary operations $\triangleright$ and $\triangleright^{-1}$ satisfying the following axioms for all $x, y, z \in Q$:

Q1. $x \triangleright x = x$.

Q2. $(x \triangleright y) \triangleright^{-1} y = x = (x \triangleright^{-1} y) \triangleright y$.

Q3. $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

In other words, $\triangleright$ is idempotent, right-invertible, and right-distributive. An algebraic structure $(Q, \triangleright)$ satisfying only Q2 and Q3 is called a *rack*.

Example 2.1. The Tait quandle $(\mathbf{T}_3, \triangleright)$ is a quandle with underlying set $\{1, 2, 3\}$ and operation

$\triangleright$	1	2	3
1	1	3	2
2	3	2	1
3	2	1	3

Example 2.2. The transpositions in the symmetric group S_n form a quandle under the operation of conjugation. In fact, the Tait quandle can be realized as the transpositions in S_3.

The following quandle is of particular relevance to us, as it will appear in several later examples.

Example 2.3. [7] Suppose A is an abelian group. Then, if we define $x \triangleright_{\text{dih}} y = 2y - x$ for any $x, y \in A$, $(A, \triangleright_{\text{dih}})$ forms a quandle. We call this quandle a *Takasaki quandle*, denoted $T(A)$. The operation $\triangleright_{\text{dih}}$ of $T(A)$ is called the *dihedral action*.

Note that $T(A)$ is always involutory, i.e., $(x \triangleright y) \triangleright y = x$ for all $x, y \in A$.

Definition 2.2. Given quandles $(Q_1, \triangleright_1), (Q_2, \triangleright_2)$, a *quandle homomorphism* is a map $h : Q_1 \to Q_2$ satisfying $h(x \triangleright_1 y) = h(x) \triangleright_2 h(y)$ for all $x, y \in Q_1$.

A *quandle isomorphism* ρ is a bijective quandle homomorphism, and ρ is a *quandle automorphism* if it maps $Q \to Q$. The quandle automorphisms of Q form a group $\text{Aut}(Q)$ under composition.

Definition 2.3. An *inner automorphism* on Q is an automorphism on Q of the form $S_y : x \to x \triangleright y$ for some $y \in Q$. Note that S_y is also called the *symmetry* at y. Denote the set of symmetries of Q as $S(Q)$.

The inner automorphisms of Q form a group $\text{Inn}(Q)$ under composition, which also happens to be a normal subgroup of $\text{Aut}(Q)$ [1].

Definition 2.4. The *orbit* of $s \in Q$ under is the subset of elements $t \in Q$ such that there exists some $\rho \in \mathrm{Inn}(Q)$ where $\rho(s) = t$. We denote the orbit of a particular element $x \in Q$ by $x \cdot \mathrm{Inn}(Q)$. The inner automorphism group acts on the quandle via right functional application, e.g., $x \cdot \rho = \rho(x)$.

Definition 2.5. A quandle Q is *connected* when there exists exactly one orbit in Q — for all x $\in Q$, the orbit of x is all of Q.

Definition 2.6. A subset Q' of a quandle $(Q, \rhd)$ is a *subquandle* of Q if it is a quandle closed under $\rhd$'s restriction to Q'. We write $Q' \preccurlyeq Q$ and $Q' \prec Q$ to denote that Q' is a subquandle and a proper subquandle of Q, respectively.

It is easy to see that orbits form a subquandle. Indeed, such subquandles are of prime importance for understanding the way in which a subquandle fits within the ambient quandle. Ehrman et. al. [3] proved their Orbit Decomposition Theorem, which shows how a quandle can be decomposed into its orbits. We shall briefly reiterate their definitions and results.

Consider a sequence of quandles $Q_1, \ldots, Q_n$. For each $i, j \in \mathbf{n}$, let $g_{ij} : \mathrm{Adconj}(Q_i) \to \mathrm{Aut}(Q_j)$ be a group homomorphism, and define the $n \times n$ matrix $M = (g_{ij})$. Ehrman et al. [3] defined the *semidisjoint union*, denoted $\#(Q_1, \ldots, Q_n, M)$, to be

$$\left(\coprod_{i=1}^{n} Q_i, \rhd \right)$$

where $\rhd$ is an operation on the semidisjoint union, defined $x \rhd y = x \cdot g_{ij}(|y|_{Q_i})$ if $x \in Q_i$ and $y \in Q_j$. If $\#(Q_1, \ldots, Q_n, M)$ happens to be a quandle, then we call the matrix M a *mesh*.

Ehrman et al. [3] showed that a mesh can be constructed such that a quandle Q can be expressed as a semidisjoint union of its orbits under its inner automorphism group. Hence the orbits of the inner automorphism group are of prime importance to quandle theory. It is also easy to see how this theorem extends to disjoint unions of orbits.

3 The Inner Automorphism Group of The Subquandle

In this section, we show that the inner automorphism group of a subquandle is a subquotient of the inner automorphism group of the ambient quandle.

Theorem 3.1. Let Q be a quandle and $Q' \preccurlyeq Q$. Then there exists some subgroup $S_{Q'} \leq \mathrm{Inn}(Q)$ and some normal subgroup $K_{Q'} \trianglelefteq S_{Q'}$ such that $\mathrm{Inn}(Q') \cong S_{Q'}/K_{Q'}$.

Proof. Define $S_{Q'} := \langle \{S_q : q \in Q'\} \rangle \leq \mathrm{Inn}(Q)$, where each $S_q \in \mathrm{Inn}(Q)$. Consider $\tau : S_{Q'} \to \mathrm{Inn}(Q')$ defined by $\tau(f) = f|_{Q'}$.

Claim 3.1. τ is well-defined.

Proof. For $z \in Q$ and $\epsilon \in \{-1, 1\}$, we write S_z^ϵ to mean the symmetry $(x \mapsto x \triangleright^\epsilon z) \in \mathrm{Inn}(Q)$. From properties of bijections, we have the following:

Fact 1. Let $n \in \mathbb{N}$, and let $Q' \subseteq Q$ be sets. For each $1 \leq i \leq n$, let $f_i : Q \to Q$ satisfy $f_i[Q'] \subseteq Q'$. Then,

$$(f_n \circ f_{n-1} \circ \cdots \circ f_1)|_{Q'} = (f_n|_{Q'}) \circ (f_{n-1}|_{Q'}) \cdots \circ (f_1|_{Q'}).$$

Note that each restriction $f_i|_{Q'}$ is well-defined by the assumption $f_i[Q'] \subseteq Q'$.

Fact 2. Let $Q' \subseteq Q$ be sets, and suppose $f : Q \to Q$ is a bijection such that $f[Q'] = Q'$. Then $(f|_{Q'})^{-1} = f^{-1}|_{Q'}$.

The first fact allows us to freely move restrictions to Q' in and out of compositions of symmetries of Q. The second fact allows us to freely move inverses of symmetries of Q in and out of restrictions to Q'. Now, to show τ maps every element of $S_{Q'}$ into $\mathrm{Inn}(Q')$, suppose $f \in S_{Q'}$ has a decomposition $f = S_{x_n}^{\epsilon_n} \circ S_{x_{n-1}}^{\epsilon_{n-1}} \cdots \circ S_{x_1}^{\epsilon_1}$ for some $n \in \mathbb{N}$ and each $x_i \in Q'$, $\epsilon_i \in \{-1, 1\}$. Then,

$$\begin{aligned}
\tau(f) &= (S_{x_n}^{\epsilon_n} \circ S_{x_{n-1}}^{\epsilon_{n-1}} \circ \cdots \circ S_{x_1}^{\epsilon_1})|_{Q'} \\
&= (S_{x_n}^{\epsilon_n}|_{Q'}) \circ \cdots \circ (S_{x_1}^{\epsilon_1}|_{Q'}) \\
&= (S_{x_n}|_{Q'})^{\epsilon_n} \circ \cdots \circ (S_{x_1}|_{Q'})^{\epsilon_1} \in \mathrm{Inn}(Q').
\end{aligned}$$

Hence, $\tau[S_{Q'}] \subseteq \mathrm{Inn}(Q')$. Finally, it is clear that if $f, g \in S_{Q'}$ are such that $f = g$, then $f|_{Q'} = g|_{Q'}$. $\square$

Claim 3.2. τ is surjective.

Proof. Fix $\alpha \in \mathrm{Inn}(Q')$, and suppose $\alpha = T_{x_n}^{\epsilon_n} \circ \cdots \circ T_{x_1}^{\epsilon_1}$, where each $T_{x_i}^{\epsilon_i} \in \mathrm{Inn}(Q')$ is defined via $T_{x_i}^{\epsilon_i}(x) := x \triangleright^{\epsilon_i} x_i$. Note that $S_{x_n}^{\epsilon_n} \circ \cdots \circ S_1^{\epsilon_1}$ is an element of $\mathrm{Inn}(Q)$. Moreover,

$$\begin{aligned}
\tau(S_{x_n}^{\epsilon_n} \circ \cdots \circ S_1^{\epsilon_1}) &= (S_{x_n}^{\epsilon_n} \circ \cdots \circ S_{x_1}^{\epsilon_1})|_{Q'} \\
&= S_{x_n}^{\epsilon_n}|_{Q'} \circ \cdots \circ S_{x_1}^{\epsilon_1}|_{Q'} \\
&= (S_{x_n}|_{Q'})^{\epsilon_n} \circ \cdots \circ (S_{x_1}|_{Q'})^{\epsilon_1} \\
&= T_{x_n}^{\epsilon_n} \circ \cdots \circ T_{x_1}^{\epsilon_1} = \alpha.
\end{aligned}$$

$\square$

Claim 3.3. τ is a group homomorphism.

Proof. Let $f, g \in S_{Q'}$ be arbitrary, and suppose $f = S_{x_n}^{e_n} \circ \cdots \circ S_{x_1}^{e_1}$ and $g = S_{y_m}^{f_m} \circ \cdots \circ S_{y_1}^{f_1}$, such that $m, n \in \mathbb{N}$, and each $x_i, y_j \in Q'$, $e_i, f_j \in \{-1, 1\}$. Since each symmetry S_k fixes Q', f and g must also fix Q', as they are compositions of symmetries and inverse symmetries. Hence, $\tau(f \circ g) = (f \circ g)|_{Q'} = f|_{Q'} \circ g|_{Q'} = \tau(f) \circ \tau(g)$. $\square$

Therefore, τ is a surjective group homomorphism. By the First Isomorphism Theorem of groups, $S_{Q'}/\ker(\tau) \cong \mathrm{Inn}(Q')$, as desired. $\square$

When referring to the restriction map from the subgroup $S_{Q'}$ to $\mathrm{Inn}(Q')$, we will simply reference it as "the map τ" when the specific map can be inferred from context.

In order to illustrate this result, we prove a special (although admittedly rare) case in which the inner automorphism group of the ambient quandle is isomorphic to the subquandle in a non-trivial way.

Theorem 3.2. Suppose that quandles Q_1 and Q_2 are isomorphic. Let α: $Q_1 \to Q_2$ be a quandle isomorphism. Then α induces a group homomorphism $\overline{\alpha} : \mathrm{Adconj}(Q_1) \to \mathrm{Aut}(Q_2)$, defined by

$$\overline{\alpha}(|x_1|^{e_1} \dots |x_n|^{e_n}) = S_{x_n}^{e_n} \dots S_{x_1}^{e_1}.$$

Define $\overline{\alpha^{-1}}$ similarly. Denote the canonical homomorphism for Q_i by c_i [1]. Define the matrix of group homomorphisms

$$M = \begin{pmatrix} c_1 & \overline{\alpha} \\ \overline{\alpha^{-1}} & c_2 \end{pmatrix}.$$

Then $\#(Q_1, Q_2, M)$ is a quandle and

$$\mathrm{Inn}(\#(Q_1, Q_2, M)) \cong \mathrm{Inn}(Q_1) \cong \mathrm{Inn}(Q_2).$$

Proof. First, we must show that $\overline{\alpha}$ and $\overline{\alpha^{-1}}$ are well defined. Suppose that $x \rhd_1 y = z \in Q_1$ for any $x, y \in Q_1$. Recall from Joyce (section 6) [1] that $S_{a \rhd b} = S_b^{-1} S_a S_b$ for any a, b in any quandle Q. Then it follows that

$$\begin{aligned}
\overline{\alpha}(z) &= \overline{\alpha}(|x \rhd_1 y|_{Q_1}) \\
&= S_{\alpha(x \rhd_1 y)} \\
&= S_{\alpha(x) \rhd_2 \alpha(y)} \\
&= S_{\alpha(y)}^{-1} S_{\alpha(x)} S_{\alpha(y)} \\
&= \overline{\alpha}(|y|_{Q_1}^{-1}) \overline{\alpha}(|x|_{Q_1}) \overline{\alpha}(|y|_{Q_1}) \\
&= \overline{\alpha}(|y|_{Q_1}^{-1} |x|_{Q_1} |y|_{Q_1}).
\end{aligned}$$

Hence $\overline{\alpha}$ is well defined. By symmetry, this also applies to $\overline{\alpha^{-1}}$.

Now to show that Q is a quandle, we proceed to show that M is a mesh by use of Theorem 3.2 of [3]. Because there are only two quandles, the second condition listed is vacuously true, hence we only need to verify that the first condition holds.

The first condition collapses into the following:

1. $\left(x \cdot \overline{\alpha^{-1}}(|y|_{Q_2})\right) \rhd_1 z = (x \rhd_1 z) \cdot \overline{\alpha^{-1}}\left(\left|y \cdot \overline{\alpha}(|z|_{Q_1})\right|_{Q_2}\right)$ for all $x, z \in Q_1$ and $y \in Q_2$

2. $\left(x\cdot\overline{\alpha}\big(|y|_{Q_1}\big)\right)\rhd_2 z = (x\rhd_2 z)\overline{\alpha}\left(\left|y\cdot\overline{\alpha^{-1}}(|z|_{Q_2})\right|_{Q_1}\right)$ for all $x,z\in Q_2$ and $y\in Q_1$.

By symmetry, it suffices to prove only the first condition.

Let $x,z\in Q_1$ and $y\in Q_2$. For the left hand side we have

$$\begin{aligned}
LHS:\left(\overline{\alpha^{-1}}\big(|y|_{Q_2}\big)\right)\rhd_1 z &= (x\rhd_1 z)\cdot\overline{\alpha^{-1}}\left(\left|y\cdot S_{\alpha(z)}\right|_{Q_2}\right)\\
&= (x\rhd_1 z)\cdot\overline{\alpha^{-1}}\big(|y\rhd_2\alpha(z)|_{Q_2}\big)\\
&= (x\rhd_1 z)\cdot S_{\alpha^{-1}(y\rhd_2\alpha(z))}\\
&= (x\rhd_1 z)\rhd_1\big(\alpha^{-1}(y\rhd_2\alpha(z))\big)\\
&= (x\rhd_1 z)\rhd_1(\alpha^{-1}(y)\rhd_2\alpha^{-1}(\alpha(z)))\\
&= (x\rhd z)\rhd_1(\alpha^{-1}(y)\rhd_1 z)
\end{aligned}$$

and on the right hand side we have

$$\begin{aligned}
RHS:(x\rhd_1 z)\cdot\overline{\alpha^{-1}}\left(\left|y\cdot\overline{\alpha}(|z|_{Q_1})\right|_{Q_2}\right) &= (x\cdot S_{\alpha^{-1}(y)})\rhd_1 z\\
&= (x\rhd_1 z)\rhd_1(\alpha^{-1}(y)\rhd_1 z).
\end{aligned}$$

Since the LHS and the RHS coincide, the first condition is satisfied. This proves that the first condition of Theorem 3.2 [3] is satisfied. Hence M is a mesh, and $\#(Q_1,Q_2,M)$ is a quandle.

Let S_{Q_1} denote the subgroup of $\mathrm{Inn}(\#(Q_1,Q_2,M))$ generated by the symmetries at elements in Q_1, and similarly for Q_2. We show that $S_{Q_1}=S_{Q_2}$. First, consider any $S_x\in Q_1$ for some $x\in Q_1$ (note that we are showing that the generating sets are in fact equal). Let z be arbitrary in $\#(Q_1,Q_2,M)$. There are two cases. Either $z\in Q_1$ or $z\in Q_2$.

Case 1. Suppose $z\in Q_1$. Then $z\rhd\alpha(x)=z\cdot\overline{\alpha^{-1}}(|\alpha(x)|_{Q_2})=z\cdot S_{\alpha^{-1}(\alpha(x))}=z\rhd_1 x=z\rhd x$.

Case 2. Suppose $z\in Q_2$. Then $z\rhd x=z\cdot S_{\alpha(x)}=z\rhd_2\alpha(x)=z\rhd\alpha(x)$.

In either case, $z\rhd\alpha(x)=z\rhd x$. Because z was arbitrary in $\#(Q_1,Q_2,M)$, x and $\alpha(x)$ act the same on $\#(Q_1,Q_2,M)$. Hence, $S_x=S_{\alpha(x)}$. But $\alpha(x)\in Q_2$. As a result element in the generating set for S_{Q_1} is also in the generating set for S_{Q_2}. We can apply the same argument backwards; it follows the generating sets are equal. In short, $S_{Q_1}=S_{Q_2}$. Moreover, it follows by definition of the inner automorphism group, and from the fact that Q_1 and Q_2 contain all of the elements of $\#(Q_1,Q_2,M)$, that $S_{Q_1}=\mathrm{Inn}(Q_1)$.

Now we show that $S_{Q_1}\cong\mathrm{Inn}(Q_1)$. We do this by showing the restriction map τ is monic. Suppose that there is $\sigma\in S_{Q_1}$ such that $\tau(\sigma)=\sigma|_{Q_1}=1_{Q_1}$. By definition of the inner automorphism group, there exist $x_1,\ldots,x_n\in\#(Q_1,Q_2,M)$ and

$\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$ such that $\sigma : y \to y \rhd^{\epsilon_1} x_1 \rhd^{\epsilon_2} \ldots \rhd^{\epsilon_n} x_n$. Let $x \in \#(Q_1, Q_2, M)$ be arbitrary. Then

$$\begin{aligned} x \cdot \sigma &= x \rhd^{\epsilon_1} x_1 \ldots \rhd^{\epsilon_n} x_n \\ &= x \rhd_2^{\epsilon_1} \alpha(x_1) \ldots \rhd_2^{\epsilon_n} \alpha(x_n) \\ &= \alpha(\alpha^{-1}(x) \rhd_2^{\epsilon_1} \alpha(x_1)) \ldots \rhd_2^{\epsilon_n} \alpha(x_n) \\ &= \alpha\big(\alpha^{-1}(x) \rhd_1^{\epsilon_1} x_1 \ldots \rhd_1^{\epsilon_n} x_n\big) \\ &= \alpha(\alpha^{-1}(x)) \\ &= x, \end{aligned}$$

By definition, σ must act trivially on the elements of Q_1 since it is in $\ker \tau$. Since x was arbitrary, σ acts trivially on all elements of $\#(Q_1, Q_2, M)$. Since, σ was arbitrary in $\ker \tau$, it follows that all elements of $\ker \tau$ are the identity. We have shown τ to injective, and τ is, by the previous theorems, already known to be surjective and a homomorphism, hence τ is an isomorphism. Hence $\mathrm{Inn}(\#(Q_1, Q_2, M)) \cong S_{Q_1} \cong \mathrm{Inn}(Q_1) \cong \mathrm{Inn}(Q_2)$. $\square$

4 Subquandle Lattices

As shown by Saki and Kiani in [5], the set of subquandles of any quandle Q under inclusion forms a lattice. We denote this lattice as $\mathcal{L}(Q)$. Given two subquandles $Q_1, Q_2 \preccurlyeq Q$, their *meet* is $Q_1 \wedge Q_2 = Q_1 \cap Q_2$ and their *join* is $Q_1 \vee Q_2 = \langle\langle Q_1 \cup Q_2 \rangle\rangle$. We will also write $A \subseteq Q$ write $\langle\langle A \rangle\rangle$ to denote the subquandle of Q generated by A. Note that $Q_1 \wedge Q_2$ is the first subquandle below both Q_1 and Q_2 in $\mathcal{L}(Q)$, and $Q_1 \vee Q_2$ is the first subquandle above both Q_2 in $\mathcal{L}(Q)$. Another perspective is that $Q_1 \wedge Q_2$ is the largest subquandle of in Q contained in Q_1 and Q_2, while $Q_1 \vee Q_2$ is the smallest subquandle containing Q_1 and Q_2.

It is also worth noting that $Q_1 \vee Q_2$ is not necessarily $Q_1 \cup Q_2$, as $Q_1 \cup Q_2$ may not be a subquandle of Q. For example, if Q is the Tait quandle on $\{1, 2, 3\}$, $\{1\} \preccurlyeq Q$ and $\{2\} \preccurlyeq Q$ but $\{1, 2\} \not\preccurlyeq Q$.

Definition 4.1. The subquandle lattice $\mathcal{L}(Q)$ of a quandle Q is *complemented* if for any $Q_1 \preccurlyeq Q$, there exists some $Q_2 \preccurlyeq Q$ such that $Q_1 \vee Q_2 = Q$ and $Q_1 \wedge Q_2 = \emptyset$.

Definition 4.2. We say $M \prec Q$ is *maximal* if there is no $N \prec Q$ such that $M \prec N$, i.e., there is no element above M in $\mathcal{L}(Q)$. Observe that any maximal subquandle M of a quandle Q must be complemented by $\{x\}$ for any $x \in Q \setminus M$.

The definitions for complemented and maximal subracks are analagous, except racks may not necessarily have singleton subracks. Saki and Kiani [5] proved subrack lattices are necessarily complemented for finite racks but not necessarily complemented for infinite racks:

Theorem 4.1. [5] Let R be a finite rack. Then the following hold:

- The intersection of the maximal subracks of R is empty.
- Suppose that $R_1, R_2 \preccurlyeq R$. Then R_1 is complemented in $R_1 \vee R_2$, and its complement is contained within R_2.

Corollary 4.1. [5] If R is a finite rack, then $\mathcal{L}(R)$ is complemented.

Let the quandle operation $\triangleright^*$ be given by $x \triangleright^* y = 2x - y$.

Theorem 4.2. [5] The subquandle lattice of $(\mathbb{Q}, \triangleright^*)$ is not complemented. In particular, $\{0\} \preccurlyeq \mathbb{Q}$ has no complement.

In Saki and Kiani's proof, the complementation of finite subrack lattices $\mathcal{L}(R)$ relies the containment of all subracks of a R in some maximal subrack of R [5]. In particular, the "top layer" of $\mathcal{L}(R)$ is composed of a set of pairwise-disjoint maximal subracks under which every other proper subrack of R lies. However, in the infinite case, maximal subracks may not even exist.

Proposition 4.1. $(\mathbb{Q}, \triangleright^*)$ has no maximal subquandles.

Proof. Fix an arbitrary $M \prec \mathbb{Q}$. We use the fact that maximal subquandles must be complemented by every singleton in their set-theoretic complement and show that this cannot hold for M. By Theorem 5.2 of [5], $\{0\}$ has no complement in $\mathcal{L}(\mathbb{Q})$, so $\{0\}$ is not a complement of M. Thus, if $0 \notin M$, M cannot be maximal. Now suppose $0 \in M$. By Lemma 5.1 in [5], $M \leq \mathbb{Q}$. Since $\mathbb{Q}$ has no maximal subgroups, it follows that there is some $q \in \mathbb{Q} \setminus M$ such that $\langle M \cup \{q\} \rangle$ is a proper subgroup of $\mathbb{Q}$. Hence, $\langle\langle M \cup \{q\} \rangle\rangle \subseteq \langle M \cup \{q\} \rangle \subset Q$, so $\{q\}$ is not a complement of M in $\mathcal{L}(Q)$. Therefore, M cannot be maximal. $\square$

4.1 Ind-finite Quandles

Although $\mathbb{Q}$ is an infinite quandle with a non-complemented sublattice, $\mathbb{Q}$ does not have any clear relationship to finite quandles, which do have complemented sublattices. In this section, we capture one type of relationship an infinite quandle can have with finite quandles by examining ind-finite quandles. We also provide an example which shows that the subquandle lattices of ind-finite quandles are not necessarily complemented.

Definition 4.3. A set $(A, \leq)$ is *directed* if for all $a, b \in A$, there exists some $c \in A$ such that $a \leq c$ and $b \leq c$. In other words, every pair of elements in A has an upper bound.

Definition 4.4. We say that a nonempty quandle Q is *ind-finite* if there is a family of finite subquandles $\{Q_i\}_{i \in \mathcal{I}}$ indexed by a directed set $\mathcal{I}$ such that each $Q_i \preccurlyeq Q$, $|Q_i| < \infty$, $Q_i \prec Q_{i+1}$, and $Q = \bigcup_{i \in \mathcal{I}} Q_i$.

There is an equivalent formulation of ind-finiteness which will soon be useful.

Lemma 4.1. Suppose Q is a nonempty quandle. Then, Q is ind-finite if and only if every finitely generated subquandle of Q is finite.

Proof. Suppose that $Q = \bigcup_{i \in \mathcal{I}} Q_i$ is ind-finite, and fix an arbitrary finitely generated subquandle Q' of Q. Let $\{g_1, \ldots, g_n\}$ be a finite generating set for Q'. Then, for each $1 \le j \le n$, since $g_j \in Q$, there exists some $i(j) \in \mathcal{I}$ such that $g_j \in Q_{i(j)}$. Clearly,

$$Q' = \langle\langle \{g_1, g_2, \ldots, g_n\} \rangle\rangle \subseteq \langle\langle Q_{i(1)}, \ldots, Q_{i(n)} \rangle\rangle = \langle\langle \cup_j Q_{i(j)} \rangle\rangle.$$

But, since $\preccurlyeq$ is a total order on $\{Q_i\}_{i \in \mathcal{I}}$, $\bigcup_j Q_{i(j)} = Q_m$, where $m = \max_{1 \le j \le n} i(j)$. Thus, $\langle\langle Q_{i(1)}, \ldots, Q_{i(n)} \rangle\rangle = \langle\langle Q_m \rangle\rangle = Q_m$. Since Q_m is finite by assumption and $Q' \subseteq Q_m$, Q' must also be finite.

Now suppose that every finitely generated subquandle of Q is finite. Let $x \in Q$ be arbitrary, and define $Q_0 = \langle\langle x \rangle\rangle$ and $\mathcal{I} = |Q|$. For each $i \in \mathcal{I}$, while $\bigcup_{k \le i} Q_k \ne Q$, let $Q_{i+1} = \langle\langle Q_i, y \rangle\rangle$ for some $y \in Q \setminus \bigcup_{k \le i} Q_k$. Since each Q_i is finitely generated subquandle of Q by construction, each Q_i must also be finite. Moreover, $\bigcup_{i \in |Q|} Q_i = Q$, since every element of Q can be reached by this construction, and $Q_i \prec Q_{i+1}$ for each $i \in \mathcal{I}$. By definition, Q is ind-finite. $\square$

Proposition 4.2. $(\mathbb{Q}, \triangleright^*)$ is not ind-finite.

Proof. Fix any $x, y \in \mathbb{Q}$ such that $x < y$. By Theorem 3.3 in [6],

$$\langle\langle x, y \rangle\rangle = \{r_0 x + r_1 y\},$$

where $r_0, r_1 \in \mathbb{Z}$, exactly one of r_0, r_1 is odd, and $r_0 + r_1 = 1$. In particular, $y + \mathbb{N}k \subseteq \langle\langle x, y \rangle\rangle$, where $k := y - x$. Hence, $\langle\langle x, y \rangle\rangle$ is finitely generated and also infinite, so by the previous lemma, $\mathbb{Q}$ cannot be ind-finite. $\square$

Proposition 4.3. $(\mathbb{Q}/\mathbb{Z}, \triangleright^*)$ is ind-finite.

Proof. Let $P = \{2, 3, 5, 7, \ldots\}$ be the sequence of positive primes, and let the sequence P_c be the concatenation of all of the finite consecutive subsequences of P, namely

$$P_c = \{2, 2, 3, 2, 3, 5, 2, 3, 5, 7, \ldots\}.$$

Define another sequence P_Π via $P_\Pi(i) = \prod_{k \le i} P_c(i)$, so

$$P_\Pi = \{2, 4, 12, 24, 72, 360, 720, 2160, \ldots\}.$$

Let D be the sequence formed by the positive non-unit divisors of entries in P_Π, namely $D(i) = \{k : k \in \mathbb{Z}^{>1},\ k \mid P_\Pi(i)\}$. It is clear from prime factorization that every positive integer greater than 1 eventually appears as a divisor of an entry of P_Π, so it must eventually appear as an element of some $D(i)$.

Let R be the sequence of cosets (in $\mathbb{Q}/\mathbb{Z}$) of reciprocals elements of D, namely

$R(i) = \{\frac{1}{d} + \mathbb{Z} : d \in D(i)\}$. Note that each $\langle\langle R(i)\rangle\rangle$ must be finite: for any $d \in D(i)$, $d \mid P_{\Pi}(i) := k$, so $\langle\langle R(i)\rangle\rangle \subseteq \{-(k-1)/k + \mathbb{Z}, \ldots, \mathbb{Z}, \ldots, (k-1)/k + \mathbb{Z}\}$, a set of cardinality $2k-1$. Hence, the $\langle\langle R(i)\rangle\rangle$ form an ascending chain of finite subquandles of $\mathbb{Q}/\mathbb{Z}$.

To verify that the union of this chain is indeed $\mathbb{Q}/\mathbb{Z}$, choose arbitrary $\frac{m}{n} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$. Then, by previous reasoning, there exists some $i \in \mathbb{N}$ such that $n \in D(i)$. Thus, $\frac{1}{n} + \mathbb{Z} \in R(i)$, which implies $\frac{m}{n} + \mathbb{Z} \in \langle\langle R(i)\rangle\rangle$. Hence, since $\frac{m}{n} + \mathbb{Z}$ was arbitrary,

$$\mathbb{Q}/\mathbb{Z} = \bigcup_{i \in \mathbb{N}} \langle\langle R(i)\rangle\rangle.$$

By definition, $\mathbb{Q}/\mathbb{Z}$ is ind-finite. $\square$

Proposition 4.4. $(\mathbb{Q}/\mathbb{Z}, \triangleright^*)$ does not have a complemented subquandle lattice and has no maximal subquandles.

Proof. Note that $\mathbb{Q}/\mathbb{Z} = \{[\frac{m}{n}] : \frac{m}{n} \in \mathbb{Q}\}$, where $[\frac{m}{n}] = \frac{m}{n} + \mathbb{Z}$. We are assuming $\mathbb{Q}/\mathbb{Z}$ is equipped with the operation $[x] \triangleright^* [y] = [2x - y]$, where $x, y \in \mathbb{Q}$. Thus, in the proofs of Lemma 5.1 and Theorem 5.2 in [5], we may replace the elements $x \in \mathbb{Q}$ with their corresponding cosets $[x] \in \mathbb{Q}/\mathbb{Z}$ to obtain the following analogous results for $\mathbb{Q}/\mathbb{Z}$:

1. The nonempty subquandles of $\mathbb{Q}/\mathbb{Z}$ are exactly the cosets of the subgroups of $\mathbb{Q}/\mathbb{Z}$.
2. $\mathcal{L}((\mathbb{Q}/\mathbb{Z}, \triangleright^*))$ is not complemented. In particular, $\{\mathbb{Z}\}$ has no complement.

To show $\mathbb{Q}/\mathbb{Z}$ has no maximal subquandles, fix an arbitrary $M \prec \mathbb{Q}/\mathbb{Z}$ — we show that M cannot be maximal. As in Proposition 4.2, if $0 \notin M$, M cannot be maximal. Additionally, if $0 \notin M$, note that $\mathbb{Q}/\mathbb{Z}$ has no maximal subgroups, so the same argument from Proposition 4.1 shows that M cannot be maximal. $\square$

4.2 Profinite Quandles

Profinite quandles are the dual to ind-finite quandles — in particular, ind-finite quandles are the direct limit of a direct system of finite quandles, while profinite quandles are the inverse limit of an inverse system of finite quandles.

As seen with examples infinite and ind-finite quandles, they do not necessarily have complemeneted lattices. This motivates the following open question:

Question 4.1. Are the subquandle lattices of profinite quandles complemented?

5 Strongly Complemented Subquandles

In this section, we investigate a special class of complemented subquandles: subquandles whose set-theoretic complement is also a subquandle.

Definition 5.1. Let Q be a quandle and $Q' \preccurlyeq Q$. Then Q' is *strongly complemented* in Q if $Q \setminus Q' \preccurlyeq Q$.

Interestingly enough, strongly complemented subquandles of a quandle Q can also be described in terms of actions of $\mathrm{Inn}(Q)$ — this description will be made explicit by the following theorem. Before proceeding, we describe the two different actions of $\mathrm{Inn}(Q)$ which will be used throughout the rest of this paper. We shall make extensive use of both of these actions, often together, so it is essential to have an efficient way of distinguishing between them.

The first action is the usual right action $\cdot : Q \times \mathrm{Inn}(Q) \to Q$ given by $q \cdot \sigma = \sigma(q)$. As described previously, the action of $\mathrm{Inn}(Q)$ creates orbits of individual elements of $q \in Q$, which we denote by $q \cdot Q$. Hence, for $Q' \preccurlyeq Q$, we use the notation $Q' \cdot \mathrm{Inn}(Q)$ to refer to the union of the orbits of elements of Q', i.e., $Q' \cdot \mathrm{Inn}(Q) = \bigcup_{q \in Q'} q \cdot \mathrm{Inn}(Q)$.

The second action is the right action $\cdot : \mathcal{L}(Q) \times \mathrm{Inn}(Q) \to \mathcal{L}(Q)$ given by $Q' \cdot \sigma = \sigma[Q']$ for Q' a subquandle of Q. Here, $\sigma[Q']$ denotes the image of Q' under σ. Thus, we use the notation $[Q'] \cdot \mathrm{Inn}(Q)$ to refer to the set of images of Q' under elements of $\mathrm{Inn}(Q)$, i.e., $[Q'] \cdot \mathrm{Inn}(Q) = \{\sigma[Q'] : q \in Q'\}$.

With these two actions defined, we proceed to our theorem classifying strongly complemented subquandles:

Theorem 5.1. Let Q be a quandle, and let $Q' \preccurlyeq Q$. Denote the subquandle lattice of Q by $\mathcal{L}(Q)$. The following are equivalent:

1. $Q \setminus Q' \preccurlyeq Q$,
2. Q' is a union of orbits under the action of $\mathrm{Inn}(Q)$ on Q,
3. Q' is a fixed point of the action of $\mathrm{Inn}(Q)$ on $\mathcal{L}(Q)$,
4. $Q = \#(Q', Q \setminus Q', M)$ for a mesh M.

Proof.

(2) $\Leftrightarrow$ (3): First, suppose that $Q' = Q' \cdot \mathrm{Inn}(Q)$. So, fix orbits $Q_1, \ldots, Q_n$ such that $Q' = Q_1 \cup \cdots \cup Q_n$. Now, choose any $x \in Q'$ and any $\sigma \in \mathrm{Inn}(Q)$. Then, $x \in Q_i$ for some $1 \leq i \leq n$, which implies that $x = x_i \cdot \gamma$ for some $x_i \in Q_i$ and for some $\gamma \in \mathrm{Inn}(Q)$. It follows that $x\sigma = (x_i \cdot \gamma) \cdot \sigma = x_i \cdot (\gamma\sigma)$. Thus, since $\gamma\sigma \in \mathrm{Inn}(Q)$, $x \cdot \sigma \in x_i \cdot \mathrm{Inn}(Q)$. But since $x_i \cdot \mathrm{Inn}(Q) \subseteq Q'$, we have $x \cdot \sigma \in Q'$. Since $x \in Q'$ was arbitrary, $\sigma[Q'] \subseteq Q'$, which implies $\sigma[Q'] = Q'$ because σ is injective. And, because $\sigma \in \mathrm{Inn}(Q)$ was arbitrary, $[Q'] \cdot \mathrm{Inn}(Q) = \{\sigma[Q'] : q \in Q'\} = \{Q'\}$, as desired.

Suppose that $[Q'] \cdot \mathrm{Inn}(Q) = \{Q'\}$. Choose any $x \in Q'$. Because $\sigma(x) \in Q'$ for any $\sigma \in \mathrm{Inn}(Q)$, the orbit $x \cdot \mathrm{Inn}(Q)$ is contained entirely within Q'. Since

$x \in Q'$ was arbitrary, $Q \cdot \mathrm{Inn}(Q) \subseteq Q'$. Moreover, $Q' \subseteq Q \cdot \mathrm{Inn}(Q)$, as each $x \in x \cdot \mathrm{Inn}(Q)$. Thus, we have expressed Q' as a union of orbits of the action of $\mathrm{Inn}(Q)$ on Q, as desired.

Before proceeding, we establish a useful lemma:

Lemma 5.1. If Q' is not a fixed point under the action of $\mathrm{Inn}(Q)$ on $\mathcal{L}(Q)$, then there exists some $x \in Q \setminus Q'$ such that the symmetry at x acting on Q' does not fix Q'. In other words, there exists some $y \in Q'$ such that $y \triangleright x \notin Q'$.

Proof. Since $[Q'] \cdot \mathrm{Inn}(Q) \neq \{Q'\}$, by definition, there exists some $\sigma \in \mathrm{Inn}(Q)$ such that $\sigma[Q'] \neq Q'$. Now, assume for the sake of contradiction that all of the symmetries of elements in Q fix Q'. Then the compositions and inverses of any number of symmetries must also fix Q'; all elements of $\mathrm{Inn}(Q)$ must fix Q', a contradiction. Hence, there must exist some x whose symmetry does not fix Q'. Accordingly, there exists some $y \in Q'$ such that $y \triangleright x \notin Q'$. Clearly x cannot be in Q' as Q' is a subquandle and is closed under $\triangleright$. Thus, $x \in Q \setminus Q'$, as desired. □

(1) $\Leftrightarrow$ (2): If Q' is a union of orbits, so is $Q \setminus Q'$, so Q' is strongly complemented. For the converse, assume for the sake of contradiction that $Q' \preccurlyeq Q$ is strongly complemented but not a union of orbits. Because Q' is a union of orbits, Q' is a fixed point under the action of $\mathrm{Inn}(Q)$ on $\mathcal{L}(Q)$. Hence, by the previous lemma, there exists some $q \in Q'$ such that there exists some $y \in Q \setminus Q'$ such that $q \triangleright y \notin Q'$, so $q \triangleright y \in Q \setminus Q'$. Because $Q \setminus Q'$ is a subquandle, it is closed under $\triangleright^{-1}$, so $q = (q \triangleright y) \triangleright^{-1} y \in Q \setminus Q'$. Therefore, $q \in Q' \cap (Q \setminus Q')$. But Q' and $Q \setminus Q'$ are disjoint, so this is a contradiction.

(1) $\Leftrightarrow$ (4): If Q' is strongly complemented, Q' is a union of orbits. Because Theorem 3.1 from [3] also holds for semidisjoint unions of unions of orbits, we obtain $Q = \#(Q', Q \setminus Q', M)$ for some mesh M. The converse of this is true; it follows directly from definition of the semidisjoint union.

□

Corollary 5.1. The only strongly complemented subquandles of a connected quandle Q are Q and $\emptyset$.

5.1 Chains of Strongly Complemented Subquandles

Here, we provide a transitivity criterion for subquandle complementation. We begin with the following lemma.

Lemma 5.2. Let $Q'' \preccurlyeq Q' \preccurlyeq Q$ such that Q'' is strongly complemented within Q', and Q' is strongly complemented within Q. Then, for any $\gamma \in \mathrm{Inn}(Q)$, $Q''\gamma$ is strongly complemented within Q'.

Proof. Since $Q'' \preccurlyeq Q'$ is strongly complemented in Q', it follows that $Q' \setminus Q'' \preccurlyeq Q'$. Since $\gamma \in \mathrm{Inn}(Q)$ and since the images of subquandles under inner automorphisms are subquandles of Q, $(Q' \setminus Q'')\gamma \preccurlyeq Q$. Because Q' is strongly complemented within Q, the previous theorem implies that $[Q'] \cdot \mathrm{Inn}(Q) = \{Q'\}$. Since $Q' \setminus Q'' \subset Q'$ and $\gamma : Q' \to Q'$, $(Q' \setminus Q'')\gamma \subset Q'$. But, since we showed $(Q' \setminus Q'')\gamma$ is a quandle, $(Q' \setminus Q'')\gamma \preccurlyeq Q'$. Also, since γ is a bijection, $(Q' \setminus Q'')\gamma = Q'\gamma \setminus Q''\gamma$. Because $[Q'] \cdot \mathrm{Inn}(Q) = \{Q'\}$, $Q'\gamma = Q'$. Substituting, we have $(Q' \setminus Q'')\gamma = Q' \setminus Q''\gamma \preccurlyeq Q'$. By the definition of strong complementedness, we have shown $Q''\gamma$ is strongly complemented within Q'. □

Lemma 5.3. Let $\sigma \in \mathrm{Inn}(Q)$. Then there exists some $\sigma' \in \langle \{S_x : x \in Q \setminus Q'\} \rangle \leq \mathrm{Inn}(Q)$ such that $Q''\sigma = Q''\sigma'$.

Proof. Since $\sigma \in \mathrm{Inn}(Q)$, $\sigma = S_{x_1}^{\epsilon_1} \dots S_{x_n}^{\epsilon_n}$ for each $x_i \in Q$, $\epsilon_i \in \{-1, 1\}$. To prove our lemma, we will show that by removing all instances of symmetries at elements in Q' which appear in the representation of σ, we obtain another element of $\mathrm{Inn}(Q)$ which acts the same as σ upon Q'' in $\mathcal{L}(Q)$.

We proceed by way of minimal counterexample. Suppose we have one $x_i \in Q'$ such that $S_{x_i}^{\epsilon_i}$ appears somewhere in the string for σ, and such that the inner automorphism σ' formed by simply removing instances of $S_{x_i}^{\epsilon_i}$ from the string is such that $Q''\sigma \neq Q''\sigma'$. Let $\gamma = S_{x_1}^{\epsilon_1} \dots S_{x_{i-1}}^{\epsilon_{i-1}}$ and $\gamma' = S_{x_{i+1}}^{\epsilon_{i+1}} \dots S_{x_n}^{\epsilon_n}$. By associativity, we have $\sigma = \gamma S_{x_i}^{\epsilon_i} \gamma'$. By the associativity of group action, we have $Q''\sigma = ((Q''\gamma)S_{x_i}^{\epsilon_i})\gamma'$. Moreover, by the previous lemma, $Q''\gamma$ is strongly complemented within Q'. Hence, $Q''\gamma$ is a fixed point of the action of $\mathrm{Inn}(Q')$ on $\mathcal{L}(Q')$, as follows by the Equivalence Theorem. Since both x_i and Q'' are contained within Q', S_{x_i} and its inverse ought to act the same upon elements of Q', as it does in ita image under the restriction map τ. From this it follows that $(Q''\gamma)S_{x_i}^{\epsilon_i} = Q''\gamma$. Now, substituting, we have the identity

$$Q''\sigma = Q''(\gamma S_{x_i}^{\epsilon_i} \gamma') = ((Q''\gamma)S_{x_i}^{\epsilon_i})\gamma' = (Q''\gamma)\gamma' = Q''(\gamma\gamma') = Q''\sigma'.$$

But this contradicts our construction of $Q''\sigma \neq Q''\sigma'$. Hence, there can be no counterexamples. We can always construct some σ' in $\langle S(Q \setminus Q') \rangle$ such that $Q''\sigma = Q''\sigma'$; we can do this by simply removing instances of symmetries which are not at elements in Q'. □

Lemma 5.4. Suppose $Q_2 \preccurlyeq Q_1 \preccurlyeq Q$ and that Q_2 is strongly complemented within Q. Then, Q_2 is strongly complemented in Q_1.

Proof. Since Q_2 is strongly complemented in Q, it is a union of orbits as follows by our equivalence theorem. Moreover, $S_{Q_1} = \langle S(Q_1) \rangle \leq \mathrm{Inn}(Q)$, where $S(Q_1)$ denotes the image of Q_1 under the function which maps elements to their symmetries. By the previous lemma it follows that Q_2 is a union of orbits under the action of S_{Q_1}. Restricting under the homomorphism τ as described in the first section we note that $\mathrm{Inn}(Q_1)$ will act the same upon Q_1 as S_{Q_1}. Hence Q_1 is a union of orbits under the action of $\mathrm{Inn}(Q_1)$. Hence Q_2 is strongly complemented in Q_1. □

Theorem 5.2. Let $Q'' \preccurlyeq Q' \preccurlyeq Q$ such that Q'' is strongly complemented within Q', while Q' is strongly complemented within Q. Then Q'' is complemented within Q.

Proof. By the previous lemma, it follows that Q'' is strongly complemented within $Q'' \cdot \text{Inn}(Q)$. Since $Q'' \cdot \text{Inn}(Q)$ is a union of orbits by construction, it follows that $Q \setminus Q'' \cdot \text{Inn}(Q) \preccurlyeq Q$. Obviously, $Q'' \cap Q \setminus Q'' \cdot \text{Inn}(Q)$, for $Q'' \preccurlyeq Q'' \cdot \text{Inn}(Q)$. In order to show that $Q'' \cdot \text{Inn}(Q)$ is a complement for Q'' in Q, it remains to be shown that $Q'' \vee Q \cdot \text{Inn}(Q) = Q$. It suffices to show that $Q \cdot \text{Inn}(Q) \subseteq Q'' \vee Q \cdot \text{Inn}(Q)$.

Now, chose any $x \in Q'' \cdot \text{Inn}(Q)$. Then there is some $\sigma \in \text{Inn}(Q)$ such that $x \in Q''\sigma$. By Lemma 5.5, there exists some $\sigma' \in < S_q : q \in Q \setminus Q' >$ such that $\sigma' Q'' = \sigma Q''$. Hence there exists $q_1, \ldots, q_n \in Q \setminus Q'$ and $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$ such that $\sigma' = S_{q_1}^{\epsilon_1} \ldots S_{q_n}^{\epsilon_n}$. Hence by definition of the action of $\text{Inn}(Q)$, $Q''\sigma' = \{q \triangleright^{\epsilon_1} q_1 \ldots \triangleright^{\epsilon_n} q_n : q \in Q''\}$. Hence $x = q \triangleright^{\epsilon_1} q_1 \ldots \triangleright^{\epsilon_n} q_n$, and by definition of the join, $x \in Q'' \vee Q \setminus Q'' \cdot \text{Inn}(Q)$. But x was arbitrary in $Q'' \cdot \text{Inn}(Q)$, so it follows that $Q'' \cdot \text{Inn}(Q) \subset Q'' \vee Q \setminus Q'' \cdot \text{Inn}(Q)$. From this it follows that the join condition is satisfied. Hence $Q \setminus Q'' \cdot \text{Inn}(Q)$ is a subquandle for Q'' in Q.

□

It should be noted that this theorem is interesting only for subquandles of infinite quandles of orbit decomposition of at least two. Not only does this theorem show that Q'' is complemented when it lies two links down in a chain of strongly complemented subquandles of Q, but it also provides the complement for Q'', namely the set complement of the union of the orbits of its elements under the action of the inner automorphism group of Q. Moreover, an even more interesting consequence of this theorem is that, since the complement for Q'' is a union of orbits, we know by our equivalence theorem it follows that the complement for Q'' is strongly complemented.

An interesting question would be whether complementedness is carried by further links in the chain of strongly complemented subquandles. Suppose we have a chain of strongly complemented subquandles, $Q_3 \preccurlyeq Q_2 \preccurlyeq Q_1 \preccurlyeq Q$. A natural attempt would be to consider $Q \setminus Q_3 \cdot \text{Inn}(Q)$, though the join condition is not nearly as straightforward to show, if it holds at all. Hopefully future research will be able to either manage to prove that complementedness is transitive across chains of strongly complemented of arbitrary length, or else find a counterexample. Either result would significantly add to the allure of this result.

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