A CLASSIFYING INVARIANT OF KNOTS,
THE KNOT QUANDLE

David JOYCE

Department of Mathematics, Clark University, Worcester, MA 01610, USA

The two operations of conjugation in a group, $x \triangleright y = y^{-1}xy$ and $x \triangleright^{-1} y = yxy^{-1}$ satisfy certain identities. A set with two operations satisfying these identities is called a quandle. The Wirtinger presentation of the knot group involves only relations of the form $y^{-1}xy = z$ and so may be construed as presenting a quandle rather than a group. This quandle, called the knot quandle, is not only an invariant of the knot, but in fact a classifying invariant of the knot.

1. Introduction

In any group there are two operations of conjugation, $(x,y) \mapsto y^{-1}xy$ and $(x,y) \mapsto yxy^{-1}$. If we denote the first by $x \triangleright y$ and the second by $x \triangleright^{-1} y$, then these operations satisfy the following three identities.

Q1. $x \triangleright x = x$.
Q2. $(x \triangleright y) \triangleright^{-1} y = x = (x \triangleright^{-1} y) \triangleright y$.
Q3. $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

Definition 1.1. A *quandle* is a set equipped with two binary operations satisfying the identities Q1, Q2, and Q3.

Call the function $S(y)$ sending x to $x \triangleright y$ the *symmetry at y*. It will be more convenient for us to use the notation $xS(y)$ rather than $S(y)(x)$ when evaluating $S(y)$ at x. The axioms taken together say that the symmetry at each element of a quandle is an automorphism of the quandle which fixes that element.

Definition 1.2. For each group G, let Conj G be the quandle having the same underlying set as G along with the two operations of conjugation in G.

Theorem 4.2 below states that any equation in these two operations holding for all groups holds for all quandles. Thus, no more than these three axioms are necessary to capture the equational content of conjugation.

Particular classes of quandles have been studied before. The first study of self-distributivity is that of Burstin and Mayer [3]. They define 'distributive groups', or in modern terminology, distributive quasigroups, as quasigroups satisfying the two identities

$$(xy)z = (xz)(yz) \quad \text{and} \quad x(yz) = (xy)(xz).$$

It follows that a distributive quasigroup is idempotent, $xx = x$. Hence, quandle are a generalization of distributive quasigroups. Burstin and Mayer also defined an 'abelian distributive group' as one satisfying

$$(wx)(yz) = (wy)(xz).$$

This axiom also goes by the names 'entropy', 'mediality', 'surcommutativity', and 'symmetry'.

Definition 1.3. An *abelian quandle* is a quandle satisfying the identity

QAb. $(w \triangleright x) \triangleright (y \triangleright z) = (w \triangleright y) \triangleright (x \triangleright z).$

We will have occasion to study the largest abelian quotient of a given quandle. As an example of an abelian quandle consider the following. Let T be a nonsingular linear transformation on a vector space V. V becomes an abelian quandle with the operations

$$x \triangleright y = T(x - y) + y \quad \text{and} \quad x \triangleright^{-1} y = T^{-1}(x - y) + y.$$

A second class of quandles studied previously are those in which the symmetries $S(y)$ are all involutions, $S(y)^2$ is the identity. For this class, $x \triangleright y = x \triangleright^{-1} y$, which allows us to dispense with the second quandle operation. An equivalent condition is the identity.

QInv. $(x \triangleright y) \triangleright y = x.$

Definition 1.4. An *involutory quandle* is a quandle which satisfies QInv.

Alternatively, an involutory quandle may be defined as a set equipped with one binary operation which satisfies Q1, QInv and Q3. This is precisely the definition that Takasaki [11] uses for 'kei'.

Bruck [2] defined the *core* of a Moufang loop as the underlying set of the loop along with the operation $(x, y) \mapsto yx^{-1}y$. Moufang loop cores are involutory quandles. We do not need the generality afforded by Moufang loops, however, we will use the core of a group, denoted Core G for a group G.

Loos [5, 6] discovered that the intrinsic algebraic structure of a symmetric space is that of an involutory quandle. In the terminology of quandles, he defined a symmetric space as a differentiable manifold with a differentiable involutory

quandle structure in which each point is an isolated fixed point of the symmetry through it.

Related to involutory quandles are *n*-quandles.

Definition 1.5. For *n* an integer, let $x \triangleright^n y = xS(y)^n$. An *n-quandle* is a quandle satisfying the identity $x \triangleright^n y = x$. Hence, a 2-quandle is an involutory quandle.

It should be noted that quandles are seldom associative. In fact, the identity $(x \triangleright y) \triangleright z = x \triangleright (y \triangleright z)$ is equivalent to the identity $x \triangleright y = x$. To reduce the number of parentheses we use the notation $x \triangleright y \triangleright z$ for $(x \triangleright y) \triangleright z$.

2. Knot quandles

Consider a regular projection of a knot *K*, such as the trefoil knot in Fig. 1, and label the arcs $a, b, c, \ldots$, where by 'arc' is meant a segment from one underpass, over
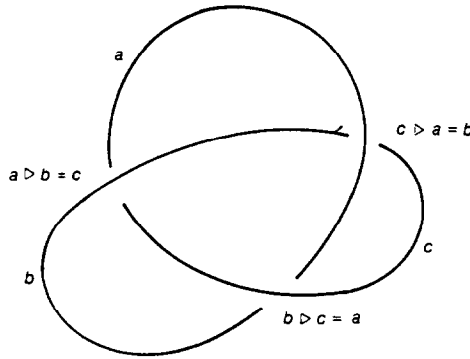

Fig. 1. The trefoil knot.

whatever overpasses there may be, to the next underpass. At each underpass, read a relation on the arcs, as '*a* under *b* gives *c*' $a \triangleright b = c$. Let $Q(K)$ be the quandle generated by the arcs with relations given by the underpasses. For instance,

$$Q(\text{trefoil}) = (a, b, c : a \triangleright b = c, b \triangleright c = a, c \triangleright a = b)$$

which is isomorphic to Core($Z/3Z$). The order of $Q(K)$ need not be equal to the number of arcs in the projection; it need not even be finite. A different regular projection of *K* will give the same $Q(K)$ up to isomorphism. Moreover, if *K* and *K'* are equivalent knots, then $Q(K)$ is isomorphic to $Q(K')$. Proofs and precise definitions will be supplied beginning in Section 12.

A similar construction gives the (non-involutory) quandle of a knot. An orientation of the knot is used to determine the relations. As expected, the knot quandle holds more information about the knot than the involutory knot quandle. It will be shown to be a complete invariant.

3. Representations and the general algebraic theory of quandles

There are various ways that groups may be used to represent quandles. First of all, Conj G, for G a group, is a quandle. Many quandles may be represented as subquandles of Conj G for appropriate G. Free quandles, for example, may be so represented as shown in Section 4. Secondly, homogeneous quandles may be represented as $H \setminus G$ for H a subgroup of G where an automorphism of G fixing H is needed to describe the quandle operations on $H \setminus G$ (see Theorem 7.1). Non-homogeneous quandles are representable as a union $H_1 \setminus G \cup H_2 \setminus G \cup \cdots$ where several automorphisms are used to describe the quandle operations. Finally, a quandle may be given as a set Q along with an action of a group G and a function $\varepsilon : Q \to G$ that describes the symmetries of Q. Such a construction is defined to be an augmented quandle in Section 9. We will be able to study some varieties of quandles by means of augmented quandles.

4. The equational theory of conjugation

In this section we show that the theory of quandles may be regarded as the theory of conjugation. Consider the two binary operations of conjugation,

$$(x, y) \mapsto y^{-1}xy = x \triangleright y \quad \text{and} \quad (x, y) \mapsto yxy^{-1} = x \triangleright^{-1} y,$$

on a group. We ask whether there are any equations involving only these two operations which hold uniformly for all groups other than those which hold in all quandles. To this end we show that free quandles may be faithfully represented as unions of conjugacy classes in free groups.

Theorem 4.1. *Let A be a set and F be the free group on A. Then the free quandle on A appears as the subquandle Q of* Conj F *consisting of the conjugates of the generators of F.*

Proof. Each element of Q is named as

$$a \triangleright^{e_1} b_1 \triangleright^{e_2} \cdots \triangleright^{e_n} b_n$$

where $a, b_1, \ldots, b_n \in A$ and $e_1, \ldots, e_n \in \{1, -1\}$. That is to say, the conjugates of a are of the form

$$b_n^{-e_n} \cdots b_1^{-e_1} a b_1^{e_1} \cdots b_n^{e_n}.$$

The equivalence on names is generated by two cases.

(1) If $a = b_1$, then $a \rhd^{e_1} b_1 \rhd^{e_2} \cdots \rhd^{e_n} b_n$ names the same element as

$$a \rhd^{e_2} b_2 \rhd^{e_3} \cdots \rhd^{e_n} b_n.$$

(2) If $b_i = b_{i+1}$ and $e_i + e_{i+1} = 0$, then $a \rhd^{e_1} b_1 \rhd^{e_2} \cdots \rhd^{e_n} b_n$ names the same element as

$$a \rhd^{e_1} \cdots \rhd^{e_{i-1}} b_{i-1} \rhd^{e_{i+2}} b_{i+2} \rhd \cdots \rhd^{e_n} b_n.$$

Now let each a in A be assigned to an element $f(a)$ in a quandle P. If f extends to Q, then we must have

$$f(a \rhd^{e_1} b_1 \rhd^{e_2} \cdots \rhd^{e_n} b_n) = f(a) \rhd^{e_1} f(b_1) \rhd^{e_2} \cdots \rhd^{e_n} f(b_n).$$

We must only show that this extension is well defined. But this follows directly from that fact that in P the analogues of (1) and (2) hold for

$$f(a) \rhd^{e_1} f(b_1) \rhd^{e_2} \cdots \rhd^{e_n} f(b_n). \qquad \square$$

Theorem 4.2. *Any equation holding in* Conj G *for all groups* G *holds in all quandles.*

Proof. Let E be an equation holding in Conj G for all groups G. In particular E holds in Conj F for free groups F, hence, E holds in free quandles. Whence, E holds in all quandles. $\square$

Although the equations true for conjugation in groups are also true in quandles, there are many statements true for conjugation in groups but false for quandles in general. For instance, the implication

$$p \rhd q = p \quad \text{implies} \quad q \rhd p = q$$

holds in Conj G for all groups G but is false for the quandle Cs(4) described in Section 6.

5. Automorphism groups of quandles

Let Q be a quandle. We define three automorphism groups for Q. First, there is the group consisting of all automorphisms, the *full automorphism group of* Q, Aut Q. Second, there is the subgroup of Aut Q generated by all the symmetries of Q, called the *inner automorphism group of* Q, Inn Q. Third, there is the subgroup of Inn Q generated by automorphisms of the form $S(x)S(y)^{-1}$ for x, y in Q, called the *transvection group of* Q, Trans Q. Inn Q is a normal subgroup of Aut Q, and Trans Q is normal in both Inn Q and Aut Q. The quotient group Inn Q/Trans Q is a cyclic group. The elements of Trans Q are the automorphisms of the form $S(x_1)^{e_1} \cdots S(x_n)^{e_n}$ such that $e_1 + \cdots + e_n = 0$.

To illustrate these groups let Q be $\mathbb{R}^2$ with $x \triangleright y = 2y - x$ considered as a quandle in the category of topological spaces. Then Aut Q consists of the affine transformations of $\mathbb{R}^2$. Inn Q includes the point symmetries and the translations. Trans Q includes only translations.

6. Representation of quandles as conjugacy classes

Two elements x and y of a quandle Q are said to be *behaviorally equivalent* if $S(x) = S(y)$, that is, $z \triangleright x = z \triangleright y$ for all z in Q. Behavioral equivalence is a congruence relation, $\equiv_b$, on the quandle, and $Q/\equiv_b$ is isomorphic to the image $S(Q)$ as a subquandle of Conj Inn Q. The elements of Q are behaviorally distinct iff S is an injection, in which case Q is isomorphic to a union of conjugacy classes in Inn Q.

Even if the elements of a quandle are not all behaviorally distinct, the quandle may be isomorphic to a union of conjugacy classes of some group. There is a universal group in which to attempt to represent a quandle as a subset closed under conjugation. As noted previously, every group G may be considered to be a quandle, Conj G, with conjugation as the quandle operation. Adjointly, every quandle Q gives rise to a group, Adconj Q, generated by the elements of Q modulo the relations of conjugation. Precisely, Adconj Q has the presentation

$$(\bar{x}, \text{ for } x \in Q: \overline{x \triangleright y} = \bar{y}^{-1} \overline{xy}, \text{ for } x, y \in Q)$$

The function $\eta : Q \to$ Conj Adconj Q sending x to $\bar{x}$ is a quandle homomorphism whose image is a union of conjugacy classes of Adconj Q. Also, η has the universal property that for any quandle homomorphism $h : Q \to$ Conj G, G a group, there exists a unique group homomorphism $H :$ Adconj $Q \to G$ such that $h = H \circ \eta$. Thus, if any $h : Q \to$ Conj G is monic, then η is monic.

But η need not be injective in general. Consider the involutory quandle Cs(4) of order 3 determined by the equations $a \triangleright b = c$, $c \triangleright b = a$, $b \triangleright a = b \triangleright c = b$. Since $b \triangleright a = b$, $\bar{a}$ commutes with $\bar{b}$. But $a \triangleright b = c$, so $\bar{b}^{-1} \bar{a} \bar{b} = \bar{c}$. Therefore, $\bar{a} = \bar{c}$, and η is not injective.

Later, when we consider the quandle associated to a knot, the non-injectivity of η will be important. For example, the quandles associated to the square and granny knots are distinct, but the Adconj groups of these quandles (which are the knot groups) are isomorphic, and for each, η is not injective.

7. Representation of quandles as coset classes

Let s be an automorphism on a group G. We may define quandle operations on G by $x \triangleright y = s(xy^{-1})y$ and $x \triangleright^{-1} y = s^{-1}(xy^{-1})y$. Denote the resulting quandle by $(G; s)$. Let H be a subgroup of G whose elements are fixed by s. Then $H \setminus G$ inherits this quandle structure.

$$Hx \triangleright Hy = Hs(xy^{-1})y$$

This quandle $(H \setminus G; s)$ is a quotient of $(G; s)$. G acts on the right on $(H \setminus G; s)$ by $(Hx)y = H(xy)$, and the action is by quandle automorphisms. Since G acts transitively on $H \setminus G$, it follows that $(H \setminus G; s)$ is a *homogeneous* quandle, that is, there is a quandle automorphism sending any element to any other element of the quandle.

We are mainly interested in the case when s is an inner automorphism of G, $s(x) = z^{-1}xz$ for some fixed element z of G. Then $x \triangleright y = z^{-1}xy^{-1}zy$. When H contains z, the operation of $(H \setminus G; z) = (H \setminus G; s)$ is

$$Hx \triangleright Hy = Hxy^{-1}zy.$$

Theorem 7.1. *Every homogeneous quandle is representable as $(H \setminus G; z)$.*

Proof. Let Q be a homogeneous quandle and $G = \text{Aut } Q$. Fix $p \in Q$. Let $z = S(p)$, symmetry at p, and let s be conjugation by z in G. Let $e : (G; s) \to Q$ be evaluation at p, defined by $e(x) = px$. Then e is a quandle homomorphism. Indeed,

$$e(x \triangleright y) = e(z^{-1}xy^{-1}zy) = pxy^{-1}zy = (pxy^{-1} \triangleright p)y$$

$$= pxy^{-1}y \triangleright py = e(x) \triangleright e(y).$$

Since Q is homogeneous, e is surjective. Let H be the stability subgroup of p, $H = \{x \in G \mid px = p\}$. Then e factors through $(H \setminus G; z)$ since $p = pH$. Moreover, $(H \setminus G; z) \to Q$ is injective, for if $pHx = pHy$, then $pxy^{-1} = p$, $xy^{-1} \in H$, and so $Hx = Hy$. Thus Q is isomorphic to $(H \setminus G; z)$. $\square$

Some adjustments are needed to represent nonhomogeneous quandles. Given a group G, elements $z_1, z_2, \ldots$ of G, and subgroups $H_1, H_2, \ldots$ of G such that for each index i, H_i is contained in the centralizer of z_i, we form a quandle $(H_1, H_2, \ldots \setminus G; z_1, z_2, \ldots)$ as the disjoint union of $H_1 \setminus G$, $H_2 \setminus G$, ... with the quandle operation

$$H_i x \triangleright H_j y = H_i xy^{-1}z_j y.$$

Theorem 7.2. *Every quandle is representable as*

$$(H_1, H_2, \ldots \setminus G; z_1, z_2, \ldots).$$

Proof. Let Q be a quandle and $G = \text{Aut } Q$. Let $Q_1, Q_2, \ldots$ be the orbits of the action of G on Q. For each index i choose $p_i \in Q_i$, let $z_i = S(p_i)$, and let H_i be the stability subgroup of p_i. Then for each i, H_i is contained in the centralizer of z_i, and so we have a quandle $P = (H_1, H_2, \ldots \setminus G; z_1, z_2, \ldots)$ as described above. Define $e : P \to Q$ by $H_i x \mapsto p_i x$. As in the proof of the previous proposition e may be shown to be an isomorphism. $\square$

8. Algebraic connectivity

We say that a quandle Q is *algebraically connected* (or just *connected* when there will be no confusion with topological connectivity) if the inner automorphism group Inn Q acts transitively on Q. In other words, Q is connected iff for each pair a, b in Q there are $a_1, a_2, \ldots, a_n$ in Q and $e_1, e_2, \ldots, e_n$ in $\{1, -1\}$ such that

$$a \rhd^{e_1} a_1 \rhd^{e_2} \cdots \rhd^{e_n} a_n = b.$$

Let Q be a quandle and q a point of Q. The *q-fibre* of a map $g: Q \to Q''$ is the subquandle $Q' = \{p \in Q \mid g(p) = g(q)\}$ of Q. Suppose that Q'' is a quotient of Q, that is, Q'' is given by a congruence on Q. In general the q-fibre does not determine Q''; just consider quandles whose operation is first projection.

Theorem 8.1. *Let Q be an algebraically connected quandle and q a point of Q. Then every quotient of Q is determined by its q-fibre. Consequently, every congruence on Q is determined by any one of its congruence classes.*

Proof. Let Q'' be a quotient of Q with q-fibre Q'. Let $a, b \in Q$. By the connectivity of Q there is an inner automorphism x such that $ax = b$. Since homomorphisms respect inner automorphisms, it follows that $g(a) = g(b)$ iff $g(ax) = g(bx)$. Hence, $g(a) = g(b)$ iff $\exists x \in$ Inn Q such that $ax = b$ and $bx \in Q'$. Thus, Q' determines Q''. □

9. Augmented quandles

An augmented quandle is intended to be a quandle with a group acting on it by quandle automorphisms such that the group contains 'coherent' representatives of the symmetries of the quandle. Since the symmetries and the action determine the quandle operations, it is redundant to require the original quandle operations.

Definition 9.1. An *augmented quandle* (Q, G) consists of a set Q, a group G equipped with a right action on the set Q, and a function $\varepsilon: Q \to G$ called the *augmentation map* which satisfy AQ1 and AQ2.

AQ1. $q\varepsilon(q) = q \ \forall q \in Q$.

AQ2. $\varepsilon(qx) = x^{-1}\varepsilon(q)x \ \forall q \in Q, \ \forall x \in G$.

Given an augmented quandle (Q, G), we can define quandle operations on Q by

$$p \rhd q = p\varepsilon(q)$$
$$p \rhd^{-1} q = p\varepsilon(q)^{-1}.$$

Then Q is a quandle, the action of G on Q is by quandle automorphisms, and the augmentation map is a quandle homomorphism $\varepsilon: Q \to \mathrm{Conj}\, G$.

Definition 9.2. A *morphism of augmented quandles* from (P, H) to (Q, G) consists of a group homomorphism $g: G \to H$ and a function $f: Q \to P$ such that the diagram

$$
\begin{array}{ccccc}
Q \times G & \longrightarrow & Q & \overset{\varepsilon}{\longrightarrow} & G \\
{\scriptstyle f \times g}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow \\
P \times H & \longrightarrow & P & \overset{\varepsilon}{\longrightarrow} & H
\end{array}
$$

commutes. It follows that f is a quandle homomorphism.

Examples. Fix a quandle Q. Two examples of augmented quandles with underlying quandle Q are $(Q, \mathrm{Aut}\, Q)$ and $(Q, \mathrm{Inn}\, Q)$. The augmentation in each case is the function that has been denoted S. In the category of augmentations of Q, $(Q, \mathrm{Aut}\, Q)$ is the terminator. That is, for each augmentation (Q, G) there is a unique group homomorphism $f: G \to \mathrm{Aut}\, Q$ such that the diagram

$$
\begin{array}{ccccc}
Q \times G & \longrightarrow & Q & \overset{\varepsilon}{\longrightarrow} & G \\
{\scriptstyle 1 \times f}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle f}\downarrow \\
Q \times \mathrm{Aut}\, Q & \longrightarrow & Q & \overset{S}{\longrightarrow} & \mathrm{Aut}\, Q
\end{array}
$$

commutes. The map f is readily defined from the action $Q \times G \to Q$.

A third augmentation of Q is $(Q, \mathrm{Adconj}\, Q)$, (see Section 6). the augmentation is $\eta: Q \to \mathrm{Adconj}\, Q$, while the group action is defined as

$$
p(\bar{q}_1^{e_1} \cdots \bar{q}_n^{e_n}) = p \rhd^{e_1} q_1 \rhd^{e_2} \cdots \rhd^{e_n} q_n.
$$

where $p \in Q$ and $\bar{q}_1^{e_1} \cdots \bar{q}_n^{e_n}$ is an arbitrary element of $\mathrm{Adconj}\, Q$, $q_i \in Q$, $e_i \in \{-1, 1\}$ for $i = 1, \ldots, n$. This is a well-defined group action. Axiom AQ1 clearly holds, and since $\eta(Q)$ generates $\mathrm{Adconj}\, Q$, AQ2, follows from the fact that $\eta: Q \to \mathrm{Conj}\, \mathrm{Adconj}\, Q$ is a quandle homomorphism as noted in Section 6. This example supplies the coterminator in the category of augmentations of Q.

We now consider constructions in the category AQ of augmented quandles. Products, equalizers, and limits in general are constructed coordinatewise. For instance $(Q, G) \times (P, H) = (Q \times P, G \times H)$ with coordinatewise action and augmentation map. Colimits are less obvious. Let U be the forgetful functor from AQ to the category of groups. $U(Q, G) = G$. The functor U has a left adjoint T and a right adjoint V. That U has a left adjoint is automatic and uninteresting; $T(G) = (\emptyset, G)$. On the other hand the existence of a right adjoint is unexpected.

$V(G) = (\text{Conj } G, G)$ where G acts on $\text{Conj } G$ by conjugation, and the function $\varepsilon : \text{Conj } G \to G$ is the identity.

The existence of a right adjoint for U simplifies the construction of colimits in AQ. For if (Q, G) is the colimit, $\text{colim}(Q_i, G_i)$, then G is the colimit, $\text{colim } G_i$ in the category of groups. Unfortunately, the forgetful functor from AQ to the category of quandles has no right adjoint. We use another construction of augmented quandles to complete the description of their colimits.

Let (Q, G) be an augmented quandle and $f : G \to H$ be a group homomorphism. Then $Q \times H$ is a right H-set with action $(q, x)y = (q, xy)$. Define an H-set congruence on $Q \times H$ by $(q, y) \equiv (p, z)$ iff $yz^{-1} = f(x)$ and $p = qx$ for some x in G. Let $q \otimes y$ denote the congruence class of (q, y), and let $Q \otimes_G H$ denote the set of congruence classes. It follows that

$$(q \otimes y)z = q \otimes yz \quad \text{and} \quad q \otimes f(x) = qx \otimes 1.$$

Define $\varepsilon : Q \otimes H \to H$ by $\varepsilon(q \otimes y) = y^{-1}(f \circ \varepsilon)(q)y$. Then ε is well defined, and $(Q \otimes H, H)$ is an augmented quandle. Also, there is a function $i : Q \to Q \otimes H$, $i(q) = q \otimes 1$, which along with f gives a morphism $(i, f) : (Q, G) \to (Q \otimes H, H)$ of augmented quandles.

Lemma 9.3. *Let (Q, G) be an augmented quandle and $f : G \to H$ be a group homomorphism. Then*

$$
\begin{array}{ccc}
(\emptyset, G) & \longrightarrow & (Q, G) \\
\downarrow{\scriptstyle (1, f)} & & \downarrow{\scriptstyle (i, f)} \\
(\emptyset, H) & \longrightarrow & (Q \otimes_G H, H)
\end{array}
$$

is a pushout square. $\square$

Now consider an arbitrary colimit $(Q, G) = \text{colim}(Q_j, G_j)$ in the category AQ. As noted above G is the colimit, $\text{colim } G_j$, in the category of groups. By the preceding proposition, $(Q_j, G_j) \to (Q, G)$ factors uniquely through $(Q_j, G_j) \to (Q_j \otimes_{G_j} G, G)$ for each j. Consequently, $(Q, G) \cong \text{colim}(Q_j \otimes_{G_j} G, G)$. This reduces the construction of colimits to the case when there is a single augmentation group for all the quandles $P_j = Q_j \otimes_{G_j} G$, and all the maps $(P_j, G) \to (P_k, G)$ are of the form $(f, 1)$.

In this case let $P = \text{colim } P_j$ in the category of sets. Then P has a unique right G-action consistent with the G-actions on the P_j, and there is a unique function $\varepsilon : P \to G$ consistent with the augmentations $\varepsilon : P_j \to G$. With this action and with $\varepsilon : P \to G$, (P, G) is an augmented quandle. Moreover, (P, G) is isomorphic to $\text{colim}(P_j, G)$. We summarize this result.

Theorem 9.4. *A colimit $\text{colim}(Q_j, G_j)$ in AQ is isomorphic to $\text{colim}(Q_j \otimes_{G_j} G, G)$*

where $G = \text{colim } G_j$ in the category of groups. It is also isomorphic to (P, G) where $P = \text{colim } Q_j \otimes_{G_j} G$ in the category of sets. $\square$

10. Some quotients and some free quandles

Some quotients of quandles are described by normal subgroups of an augmentation group. In particular, the largest abelian quotient and the largest involutory quotient are so described. Let (Q, G) be an augmented quandle and N be a normal subgroup of G. Let G/N be the quotient group with elements denoted as xN for x in G. Let Q/N be the quandle $Q \otimes_G (G/N)$. Then the elements of Q/N correspond to equivalence classes $qN = \{qn \in Q \mid n \in N\}$ for q in Q. The action $Q/N \times G/N \to Q/N$ is given by $(qN)(xN) = (qx)N$, and the augmentation $\varepsilon : Q/N \to G/N$ is given by $\varepsilon(qN) = \varepsilon(q)N$.

Let us first consider the abelianization of a quandle, that is, its largest abelian quotient. Let (Q, G) be an augmented quandle. In order that Q be abelian, we require:

QAb. $(p \triangleright q) \triangleright (r \triangleright s) = (p \triangleright r) \triangleright (q \triangleright s)$.

Equivalently, $\varepsilon(q)\varepsilon(r\varepsilon(s)) = \varepsilon(r)\varepsilon(q\varepsilon(s))$. That is, every element of the form

$$\varepsilon(q)\varepsilon(s)^{-1}\varepsilon(r)\varepsilon(q)^{-1}\varepsilon(s)\varepsilon(r)^{-1} \tag{$*$}$$

be equal to 1. Let N be the normal subgroup of G generated by the elements of the form $(*)$. Then the quotient $(Q/N, G/N)$ of (Q, G) is assured to be abelian. It is evident that $(Q/N, G/N)$ has the universal property that each map $(Q, G) \to (P, H)$ factors uniquely through $(Q/N, G/N)$ whenever P is an abelian quandle. Moreover, if $\varepsilon(Q)$ generates G, then Q/N is the abelianization of Q.

Theorem 10.1. *Let A be a set and G be the group generated by A modulo the relations $ab^{-1}c = cb^{-1}a$ for conjugates a, b, c of the generators of G. Then the free abelian quandle on A appears as the conjugates of the generators of G as a subquandle of* $\text{Conj } G$. $\square$

What has been done above for abelian quandles can be done for many other varieties of quandles. The method works for any variety determined by equations of the form

$$p \triangleright^{e_1} \phi_1 \triangleright^{e_2} \cdots \triangleright^{e_n} \phi_n = p \triangleright^{f_1} \psi_1 \triangleright^{f_2} \cdots \triangleright^{f_m} \psi_m$$

where the ϕ_1 and ψ_j are expressions not involving p. For example, the identity for n-quandles, $p \triangleright^n q = p$ is of this form.

Theorem 10.2. *Let (Q, G) be an augmented quandle such that $\varepsilon(Q)$ generates G, and*

let n be a positive integer. Let N be the normal subgroup of G generated by elements of the form $\varepsilon(q)^n$. Then Q/N is the largest quotient of Q which is an n-quandle.   □

Corollary 10.3. *Let A be a set and G be the group presented as*

$$G = (a \text{ for } a \in A : a^n = 1 \text{ for } a \in A).$$

Then the free n-quandle on A consists of the conjugates of the generators of G.   □

Specializing to the case $n = 2$, we discover a description of free involutory quandles.

Corollary 10.4. *The free involutory quandle on two elements is isomorphic to Core Z with generators 0 and 1.*

Proof. Let $A = \{a, b\}$ and $G = (a, b : a^2 = b^2 = 1)$. The quandle of conjugates of a and b in G is $Q = \{ak^k \mid k \in Z\}$. Verification that $ax^n \rhd ax^m = ax^{2n-m}$ shows that $f : Q \to \text{Core } Z$, $f(ax^n) = n$, is an isomorphism of quandles.   □

Finally, we present the free abelian involutory quandles.

Theorem 10.5. *The free abelian involutory quandle on $n + 1$ generators appears as*

$$P = \{(k_1, \dots, k_n) \in Z^n \mid \text{at most one } k_i \text{ is odd}\}$$

as a subquandle of Core Z^n. The generators are

$$e_0 = (0, 0, \dots, 0), \quad e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1).$$

Proof. Following the method used above, we let G be the group presented as

$$G = (a_0, \dots, a_n : a_i^2 = 1, a_i a_j a_k = a_k a_j a_i, \text{ all } i, j, k),$$

and let Q include the conjugates of the generators of G. Then Q is the free abelian involutory quandle on $A = \{a_0, \dots, a_n\}$. Since P is an abelian involutory quandle, there is a unique map $h : Q \to P$ such that $h(a_i) = e_i$, all i. We will show h is an isomorphism. Let $t_i = a_0 a_i$ for $i = 0, \dots, n$. Then $t_j t_k = t_k t_j$. The conjugates of a_i are of the form

$$a_i \rhd a_{j_1} \rhd \cdots \rhd a_{j_r} = a_i \rhd a_{j_1} a_{j_2} \cdots a_{j_r}.$$

and r may be taken to be even since $a_i \rhd a_i = a_i$. Since

$$a_i \rhd a_{j_1} \rhd \cdots \rhd a_{j_r} = a_i \rhd t_{j_1}^{-1} t_{j_2} \cdots t_{j_{r+1}}^{-1} t_{j_r},$$

we find Q consists of the elements of the form

$$a_i \rhd t_1^{k_1} \cdots t_n^{k_n} \quad \text{with } k_i \in Z, \ i = 1, \dots, n.$$

We now have that

$$h(a_i \triangleright t_1^{k_1} \cdots t_n^{k_n}) = e_i + (2k_1, \ldots, 2k_n)$$

which clearly displays the bijectivity of h. $\square$

Alternatively, we may describe the free abelian involutory quandle on $n + 1$ generators as $\{(k_0, \ldots, k_n) \in Z^{n+1} \mid$ exactly one k_i is odd$\}$ as a subquandle of Core Z^{n+1}.

11. Involutory quandles and geodesics

The fact that symmetric spaces are involutory quandles and that their structure is determined by distance along geodesics suggests that involutory quandles is general be determined by some kind of geodesic. Consider, for example, the *integral line quandle*, $L = \text{Core } Z$. Interpret L as the integral points on a line. Then for m, n in L, $m \triangleright n = 2n - m$ is found by moving along the line from m through n the same distance beyond n as m is beyond n. The suggestion may be formalized as follows.

Definition 11.1. An *involutory quandle with geodesics* is a set Q of points along with a collection of functions, called geodesics, $g : L \to Q$, where L is the integral line quandle Core Z, satisfying three axioms.

QG1. *Every pair of points lies in the image of some geodesic.*

QG2. *Whenever a pair of points x, y lies in the image of two geodesics, $f(m) = x$, $f(n) = y$, $f(m') = x$, $g(n') = y$, it is the case that $f(m \triangleright n) = g(m' \triangleright n')$. We denote this point $f(m \triangleright n)$ as $x \triangleright y$.*

QG3. *A geodesic reflected through a point is a geodesic; precisely, if x is a point and f a geodesic, then there exists a geodesic g such that for all m, n in L, there exist p, q in L such that $f(m) \triangleright x = g(p)$, $f(n) \triangleright x = g(q)$, and $f(m \triangleright n) \triangleright x = g(p \triangleright q)$. See Fig. 2.*
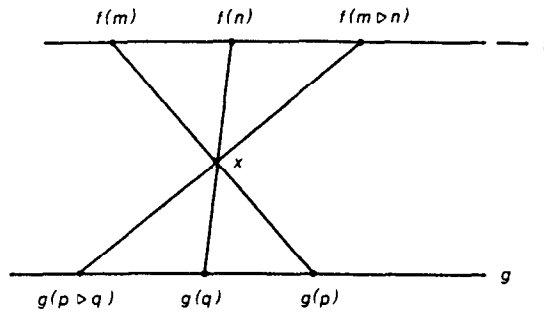

Fig. 2. QG3.

It is easily seen that an involutory quandle with geodesics is an involutory quandle. The operation $\rhd$ is as defined in QG2. Fig. 3 displays a typical involutory quandle with geodesics.
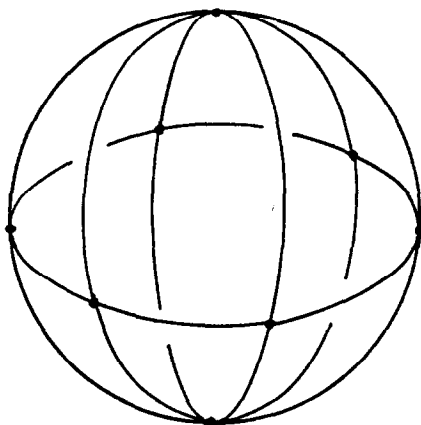

Fig. 3. An involutory quandle with geodesics.

Proposition 11.2. *Every involutory quandle is representable as an involutory quandle with geodesics.*

Proof. Recall Corollary 10.4 which states that L is the free involutory quandle on two points. Let Q be the given quandle. For each pair of points x, y in Q there is a unique quandle map $f: L \to Q$ such that $f(0) = x$ and $f(1) = y$. Take all such maps as geodesics. Clearly, QG1 holds. For points x, y, if f is a geodesic such that $f(m) = x$ and $f(n) = y$, then $f(m \rhd n) = x \rhd y$, hence, QG2 holds. Finally, given a geodesic f and a point x, the geodesic g required for QG3 is that such that $g(0) = f(0) \rhd x$ and $g(1) = f(1) \rhd x$. $\quad\square$

12. The fundamental quandle of a pair of spaces

Let P be the category of pointed pairs of topological spaces. An object of P is a pair of spaces (K, X), K a subspace of X, along with a distinguished point $*$ in $X - K$ called the basepoint. A map $f: (K, X) \to (L, Y)$ is given by a continuous map $f: X \to Y$ such that $f^{-1}(L) = K$ and $f(*) = *$. Let $X \wedge I$ be the quotient $X \times I / \{*\} \times I$ of the space $X \times I$, $I = [0, 1]$. Two maps $f, g: (K, X) \to (L, Y)$ in P are said to be *homotopic*, written $f \sim g$, if there is a map $h: (K \times I, X \wedge I) \to (L, Y)$ in P such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for all x in X. This concept of homotopy in P yields a quotient category $[P]$ with the same objects as P where maps are homotopy classes of maps in P.

One object of $[P]$ is the circle $S = (\emptyset, S^1)$ where S^1 is the unit circle in the complex plane with basepoint at 1. S is a cogroup in $[P]$, that is, S has a group structure in $[P]^{op}$. The homotopy classes of maps from S to any object (K, X) in P is a group $\pi(K, X)$ which is just the fundamental group of $X - K$, $\pi(K, X) = \pi_1(X - K)$. This group gives only partial information about the way that K is situated in S. For instance, when K is a knot in 3-space X, $\pi(K, X)$ is the knot group, and although the knot group distinguishes many knots, it fails to distinguish between the square knot and the granny knot. We will replace the pair $S = (\emptyset, S^1)$ by a pair where the subspace forms an integral part of the whole. In doing so, we will not have a cogroup, but only a coquandle.

Since we will be dealing with the cogroup S in some detail, let us describe its structure in detail. We need a comultiplication $\mu : S \to S \vee S$, a coinversion $\sigma : S \to S$, and a coidentity $S \to (\emptyset, 1)$. There is only one map $S \to (\emptyset, 1)$, so it is the coidentity. The coinversion σ is defined by $\sigma(z) = z^{-1}$. Represent $S \vee S$, as $S \times \{1, 2\}$ with the points $(1, 1)$ and $(1, 2)$ identified. Then the comultiplication μ is defined by

$$\mu(e^{it}) = \begin{cases} (e^{2it}, 1) & \text{for } 0 \le t \le \pi, \\ (e^{2it}, 2) & \text{for } \pi \le t \le 2\pi. \end{cases}$$

Let N be the object $(0, Y)$ in P where Y is the subspace of the complex plane consisting of the union of the closed unit disk $\{z \in \mathbb{C} \mid |z| \le 1\}$ and the 'rope' $\{z \in \mathbb{C} \mid z$ real and $1 \le z \le 5\}$, where 0 denotes $\{0 \in \mathbb{C}\}$, and the basepoint $*$ of X is 5. We will show that N is a coquandle, but not directly. Instead, we will show that S and N together form a co-augmented-quandle. This entails the construction of two maps in P, $a : N \to N \vee S$ and $d : S \to N$, so that in $[P]^{op}$ the two axioms AQ1 and AQ2 (Section 9) are satisfied. Once this is done, S and N will represent a contravarient functor from $[P]$ to the category of augmented quandles which will extend the fundamental group functor.

Let $d : S \to N$ wrap the circle around the disk of N by way of the rope of N.

$$d(e^{it}) = \begin{cases} 5 - 8t/\pi & \text{for } 0 \le t \le \pi/2, \\ e^{2i(t - \pi/2)} & \text{for } \pi/2 \le t \le 3\pi/2, \\ 8t/\pi - 11 & \text{for } 3\pi/2 \le t \le 2\pi. \end{cases}$$

The map d may be interpreted as the boundary of N. Let the map $a : N \to N \vee S$ place the disk of N onto the disk of $N \vee S$, then stretch the rope of N along the rope of $N \vee S$ to the basepoint and around the circle of $N \vee S$.

$$a(z) = \begin{cases} z \in N & \text{if } |z| \le 1, \\ 2z - 1 \in N & \text{if } 1 \le z \le 3, \\ e^{i(z - 3)\pi} \in S & \text{if } 3 \le z \le 5. \end{cases}$$

In order to show that a gives a group action, we must show that the diagram

$$
\begin{array}{ccc}
N & \xrightarrow{\ a\ } & N \vee S \\
a \downarrow & & \downarrow a \vee 1 \\
N \vee S & \xrightarrow{\ 1 \vee \mu\ } & N \vee S \vee S
\end{array}
$$

commutes up to homotopy. Both maps $f = a(a \vee 1)$ and $g = a(1 \vee \mu)$ place the disk of N onto the disk of $N \vee S \vee S$ then stretch the rope of N along the rope of $N \vee S \vee S$ and around each circle of $N \vee S \vee S$. Restricted to the disk of N, f equals g. They only differ with regard to the rate that they stretch the rope, hence they are homotopic.

In order to show AQ1 holds we must show that the diagram

$$
\begin{array}{ccc}
N \xrightarrow{\ a\ } N \vee S \xrightarrow{\ 1 \vee d\ } N \vee N \\
 \searrow_{1} \downarrow \delta \\
 N
\end{array}
$$

commutes up to homotopy. Represent $N \vee N$ as $N \times \{1, 2\}$ with basepoints identified. Then the map $\delta : N \vee N \to N$ is given by $\delta(z, n) = z$. Let $f = a(1 \vee d)\delta$. Then $f : N \to N$ is formulated as

$$
f(z) = \begin{cases}
z & \text{if } |z| \leq 1, \\
2(z - 1) + 1 & \text{if } 1 \leq z \leq 3, \\
5 - 8(z - 3) & \text{if } 3 \leq z \leq 3.5, \\
\exp(2i(z - 3.5)\pi) & \text{if } 3.5 \leq z \leq 4.5, \\
8(z - 4.5) + L & \text{if } 4.5 \leq z \leq 5.
\end{cases}
$$

Verbally described, f places the disk of N onto itself, then stretches the rope along and back itself, around the disk, and back to the basepoint. In the category P a homotopy is not allowed to pass any point of $X - A$ through A; in particular, the rope of N may not pass through the origin. The required homotopy, $f \sim 1$, may be made by rotating the disk counterclockwise one revolution while contracting the string to its initial position.

In order to show Axiom AQ2 holds, we must verify that the diagram

$$
\begin{array}{ccc}
S \xrightarrow{\ \mu\ } S \vee S \xrightarrow{\ \mu \vee 1\ } S \vee S \vee S \\
d \downarrow \downarrow \begin{pmatrix} \sigma & 0 \\ 0 & d \\ 1 & 0 \end{pmatrix} \\
N \xrightarrow{ a } N \vee S
\end{array}
$$

commutes up to homotopy. Both of the compositions describe a loop in $N \vee S$ starting at the basepoint, clockwise around the circle of $N \vee S$, around the boundary of N in $N \vee S$, and counterclockwise around the circle of $N \vee S$ back to the basepoint. The two loops only differ by their rate, and hence are homotopic.

Thus, S and N together represent a contravariant functor AQ from $[P]$ to the category of augmented quandles. Let (K, X) be an object of P. A *noose about K* is a map $v : N \to (K, X)$ in P. If α is a loop in $X - K$ and v is a noose about K, let $v\alpha = a(v \vee \alpha)$ and $\varepsilon(v) = dv$. We call $\varepsilon(v)$ the boundary loop of v. If β is another loop in $X - K$, we have the homotopies

$$(v\alpha)\beta \sim \beta(\alpha\beta) \quad \text{and} \quad \varepsilon(v\alpha) \sim \alpha^{-1}\varepsilon(v)\alpha.$$

If μ and v are nooses about K, let $\mu \rhd v = \mu\varepsilon(v)$. Then the homotopy classes of nooses in X about K form a quandle, $Q(K, X)$. We name $Q(K, X)$, *the fundamental quandle* of (K, X), and we name $AQ(K, X)$ *the fundamental augmented quandle* of (K, X). $AQ(K, X)$ is the fundamental quandle augmented by the fundamental group of $X - K$.

As N is a coquandle, a presentation of its structure is in order. The structure is given by two maps, $q, q' : N \to N \vee N$, q used to represent $\rhd$ and q' to represent $\rhd^{-1}$. The map q is to be homotopic to $a(1 \vee d)$. Let $N \vee N$ be $N \times \{1, 2\}$ with basepoints identified. Then such a map is

$$q(z) = \begin{cases} (z, 1) & \text{if } |z| \le 1, \\ (4(z-1)+1, 1) & \text{if } 1 \le z \le 2, \\ (5-4(z-2), 2) & \text{if } 2 \le z \le 3, \\ (\exp(2\pi i(z-3)), 2) & \text{if } 3 \le z \le 4, \\ 4(z-4)+1, 2) & \text{if } 4 \le z \le 5. \end{cases}$$

The map q places the disk of N onto the first disk of $N \vee N$ and stretches the rope of N along the first rope of $N \vee N$ and around the boundary of the second N. The map q' is defined similarly except that $(\exp(-2\pi i(z-3)), 2)$ is used in the case $3 \le z \le 4$.

In forthcoming proofs we will have occasion to compose nooses with paths as well as loops. If v is a noose in X about K with basepoint $*$, and α is a path in $X - K$ from $*$ to $*'$, then let $v\alpha$ denote the composition. $v\alpha$ is a noose in X about K with basepoint $*'$. Let Disk v denote the noose with basepoint $v(1)$, $(\text{Disk } v : N \to (K, X)$ is constantly $v(1)$ on the rope of N). Let Rope v denote the path from $v(1)$ to $*$ along the image of v. Then $v \sim (\text{Disk } v)(\text{Rope } v)$.

Theorem 12.1. *Let* $(0, D)$ *be the object in P where D is the closed unit disk in the plane* $\mathbb{C}$, *0 is the origin, and* $*$ *is 1. Any element of* $Q(0, D)$ *is uniquely representable as a noose* $v : N \to (0, D)$,

$$v(z) = \begin{cases} re^{in\theta} & \text{if } |z| \le 1, \ z = re^{i\theta}, \\ 1 & \text{if } 1 \le z \le 5 \end{cases}$$

for an integer n. n is the winding number of the boundary of the element around 0. The quandle operation on $Q(0, D)$ satisfies $x \triangleright y = x$.   □

13. The Seifert–Van Kampen theorem

Recall the Seifert–Van Kampen theorem for the fundamental group. Let X be an arcwise connected topological space with a basepoint, and let $\{U_i\}$ be a covering of X by arcwise connected open sets closed under pairwise intersections such that each open set U_i contains the basepoint. Then

$$\pi_1(X) = \text{colim } \pi_1(U_i).$$

We will prove an analogous theorem for the fundamental quandle of a pair of spaces.

Theorem 13.1. *Let (K, X) be an object in P. Let $\{U_i\}$ be a covering of X closed under pairwise intersection. Assume for each index i that U_i is a neighborhood of $U_i \cap K$ and that $U_i - K$ is arcwise connected and contains the basepoint of X. Then*

$$AQ(K, X) = \text{colim } AQ(U_i \cap K, U_i).$$

Proof. Let $AQ(K, X) = (Q, G)$. Then by the Seifert–Van Kampen theorem, $G = \pi_1(X - K) = \text{colim } \pi_1(U_i - K)$. According to remarks in Section 10, the $\text{colim } AQ(U_i \cap K, U_i)$ is then of the form (L, G). By the universal property of (L, G) there is a unique $f: L \to Q$ determined by the maps $AQ(U_i \cap K, U_i) \to AQ(K, X) = (Q, G)$. We will show f is an isomorphism.

Surjectivity of f. By Theorem 9.4, it suffices to show that every noose γ about K is homotopic to some $\alpha\beta$ where α is a noose in some $(U_i \cap K, U_i)$, and β is a loop in $X - K$. Given γ, an arbitrary noose about K, $\gamma(0)$ lies in some U_i. Since $\gamma^{-1}(U_i)$ is a neighborhood of 0 in N, by an appropriate homotopy we may adjust γ so that we may assume $\gamma(z) \in U_i$ for $|z| \leq 1$. Choose a path δ in $U_i - K$ from $\gamma(1)$ to $*$. Let α be the noose (Disk γ)δ in $(U_i \cap K, U_i)$ and β be the loop δ^{-1}(Rope γ) in $X - K$. Then $\alpha\beta = (\text{Disk } \gamma)\delta\delta^{-1}(\text{Rope } \gamma) \sim (\text{Disk } \gamma)(\text{Rope } \gamma) \sim \gamma$ as required.

Injectivity of f. It suffices to show that if $\alpha\beta \sim \alpha'\beta'$ where α is a noose in $(U_i \cap K, U_i)$, β and β' are loops in $X - K$, and α' is a noose in $(U_{i'} \cap K, U_{i'})$, then as elements of L, $\alpha\beta = \alpha'\beta'$. Let H effect the homotopy $\alpha\beta \sim \alpha'\beta'$;

$$H: N \times I \to X, \qquad H(z, 0) = (\alpha\beta)(z), \qquad H(z, 1) = (\alpha'\beta')(z).$$

The inverse images of the open sets U_i under $H(0, t)$ cover the unit interval I. Hence, we may divide I into subintervals $0 = t_0 \leq t_1 \leq \cdots \leq t_n = 1$, so that for each $j = 1, \dots, n$ there is an index $i(j)$ for which $H(0, t) \in U_{i(j)}$ when $t_{j-1} \leq t \leq t_j$. There is an $r > 0$ so that for $|z| \leq r$ and $t_{j-1} \leq t \leq t_j$ we have $H(z, t) \in U_{i(j)}$. One r suffices for all j. By appropriate adjustments of α, α', and H we may assume $r = 1$.

Fig. 4 illustrates the remainder of the proof. For $j = 0, 1, \ldots, n$ define the nooses $\gamma_j(z) = H(z, t_j)$ in (K, X). Also choose paths δ_j from $\gamma_j(1)$ to $*$ in $U_{i(j-1)} \cap U_{i(j)} - K$ for $j = 1, 2, \ldots, n-1$, and set $\delta_0 = \text{Rope } \alpha$ and $\delta_n = \text{Rope } \alpha'$.

For $j = 0, \ldots, n$ define α_j to be the noose $(\text{Disk } \gamma_j)\delta_j$ in $(U_{i(j-1)} \cap U_{i(j)} \cap K,$ $U_{i(j-1)} \cap U_{i(j)})$ and β_j to be the loop $\delta_j^{-1}(\text{Rope } \gamma_j)$ in $X - K$. Then $\alpha = \alpha_0$ and $\alpha' = \alpha_n$. As nooses in $(U_i \cap K, U_i)$ and $(U_i' \cap K, U_i')$, $\alpha\beta \sim \alpha_0\beta_0$ and $\alpha'\beta' \sim \alpha_n\beta_n$, respectively. In order to show $\alpha\beta$ equals $\alpha'\beta'$ in L, we will show for $j = 1, \ldots, n$ that $\alpha_{j-1}\beta_{j-1}$ equals $\alpha_j\beta_j$.
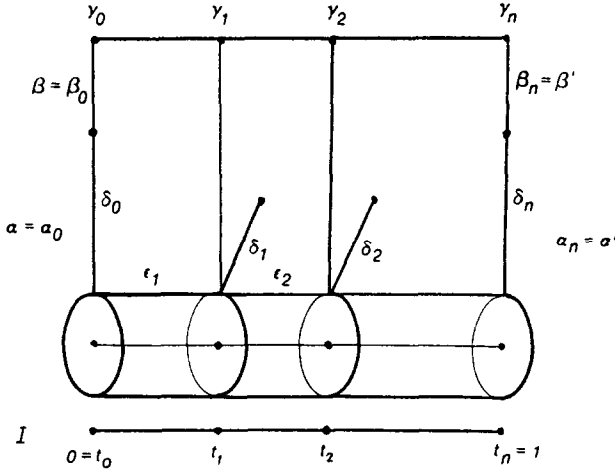

Fig. 4.

Fix j between 1 and n. We must show $\alpha_{j-1} = \alpha_j\beta_j\beta_{j-1}^{-1}$. Let ε_j be the path from $\gamma_{j-1}(1)$ to $\gamma_j(1)$ in $U_{i(j-1)} - K$ given by $\varepsilon_j(t) = H(1, t_{j-1} + t(t_j - t_{j-1}))$. The portion of the homotopy H on $[1, 5] \times [t_{j-1}, t_j]$ yields a homotopy in $X - K$ of ε_j to $(\text{Rope } \gamma_{j-1})(\text{Rope } \gamma_j)^{-1}$. Thus

$$\beta_{j-1}\beta_j^{-1} = \delta_{j-1}^{-1}(\text{Rope } \gamma_{j-1})(\text{Rope } \gamma_j)^{-1}\delta_j = \delta_{j-1}^{-1}\varepsilon_j\delta_j$$

as elements of $\pi_1(X - K)$. The noose $\alpha_j\delta_{j-1}^{-1}\varepsilon_j\delta_j$ lies entirely in $U_{i(j-1)}$ as does α_{j-1}. Moreover, the restriction of H to $\{|z| \leq 1\} \times [t_{j-1}, t_j]$ has an image in $U_{i(j-1)}$ and yields a homotopy of α_{j-1} to $\alpha_j\delta_{j-1}^{-1}\varepsilon_j\delta_j$ in $(U_{i(j-1)} \cap K, U_{i(j-1)})$. Thus, $\alpha_{j-1} = \alpha_j\delta_{j-1}^{-1}\varepsilon_j\delta_j = \alpha_j\beta_{j-1}\beta_j^{-1}$ as elements of L. $\square$

Corollary 13.2. *Let (K, X) be an object in P. Let U and V be open subsets of X such that $U - K$, $V - K$, and $(U \cap V) - K$ are arcwise connected and contain the basepoint of X. Then $AQ(K, X)$ is the pushout*

$$
\begin{array}{ccc}
AQ(U \cap V \cap K, U \cap V) & \longrightarrow & AQ(V \cap K, V) \\
\downarrow & & \downarrow \\
AQ(U \cap K, U) & \longrightarrow & AQ(K, X). \quad \square
\end{array}
$$

14. Knot quandles

Recall that a *knot* is a subspace K of the 3-sphere $X = S^3$ which is homeomorphic
to a circle. A *link* is a subspace K of X homeomorphic to a disjoint union of circles.
Two knots or links K and K' are *equivalent* if there is a homeomorphism h of X such
that $h(K) = K'$, that is, in the category P, (K, X) is homeomorphic to (K', X). We will
deal with oriented spaces; we assume X and K are endowed with orientations. (If K
is a link, then we assume each component circle is oriented.) An *oriented
equivalence* of K and K' is an orientation preserving homeomorphism h of X such
that $h(K) = K'$ and h preserves the orientation of each component of K. An
equivalence class of oriented knots or links (under oriented equivalence) is called an
oriented knot type or *oriented link type,* respectively. We assume all knots, links,
and equivalences are oriented and henceforth omit the adjective 'oriented'.

The fundamental group of $X - K$, $\pi_1(K, X)$, is called the *knot group* (or *link
group*). This definition assumes either that $X - K$ has a designated basepoint or else
that the knot group is only defined up to noncanonical isomorphism; we assume a
basepoint.

Recall that a *tame knot* is a knot equivalent to a closed polygonal curve in X.
Some of the results below are restricted to tame knots and tame links.

Associated to a knot (K, X) we have the fundamental quandle $Q(K, X)$. An
element of $Q(K, X)$ is represented by a noose v about K. The boundary loop $\varepsilon(v)$
may or may not link with K. In order to decide when a loop in $X - K$ links once with
K, it suffices to choose a generator of $H_1(X - K) \cong Z$. Then loops homologous to
that generator have linking number 1 with K. Since we assume K and X have
orientations, such a generator may be naturally chosen (say, by the right-hand rule).

Let f be the composition

$$Q(K, X) \xrightarrow{\ \varepsilon\ } \pi_1(X - K) \to H_1(X - K),$$

and let $\bar{Q} = \bar{Q}(K, X) = f^{-1}(\text{generator})$. Then $\bar{Q}$ consists of the nooses linking once
with K. $\bar{Q}$ is an invariant of the knot type of K; if $(K, X) \cong (K', X)$, then $\bar{Q}(K, X) \cong
\bar{Q}(K', X)$. The boundaries of the nooses in $\bar{Q}$ are called *meridians* of K. $\pi_1(X - K)$
acts on $\bar{Q}(K, X)$ as well as (K, X). Call $\bar{Q}$ the *knot quandle* of the knot (K, X).

15. A presentation of the knot quandle

Let (K, X) be a tame knot, such as the figure-8 knot shown in Fig. 5. Recall
Wirtinger's presentation for the knot group. Project the knot onto a suitably chosen
plane so that the image contains no triple points and only finitely many, n, double
points. Such a projection is called a *regular projection*. The n 'underpoints' (one for
each double point) divide the knot into n arcs, an arc going from one underpass,
over whatever overpasses there may be, to another underpass. Label the arcs
$a_1, a_2, \ldots, a_n$, placing the labels each to the right of the knot (using the orientation of
(K, X)). For each arc a_i pass a loop x_i in $X - K$ under the arc a_i from right to left.
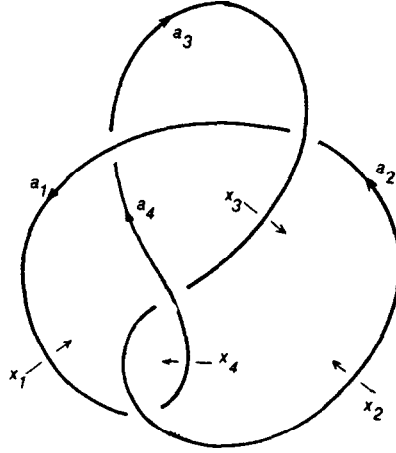
Fig. 5. The figure-8 knot.

The loops $x_1, x_2, \ldots, x_n$ generate the knot group. Each underpass yields one relation among the loops. For instance, the intersection of the figure-8 knot where a_1 passes under a_2 to become a_4 yields the relation $x_2^{-1} x_1 x_2 = x_4$. Together these n generators and n relations give a presentation of the knot. For the figure-8 knot we have the presentation

$$G(K) = (x_1, x_2, x_3, x_4 : x_3^{-1} x_1 x_3 = x_2, x_4 x_2 x_4^{-1} = x_3,$$

$$x_1^{-1} x_3 x_1 = x_4, x_2 x_4 x_2^{-1} = x_1).$$

Since each relation states that one generator is a conjugate of another, we may give a presentation of a quandle just by using quandle notation. For the figure-8 knot we then have

$$Q(K) = (x_1, x_2, x_3, x_4 : x_1 \triangleright x_3 = x_2, x_2 \triangleright^{-1} x_4 = x_3,$$

$$x_3 \triangleright x_1 = x_4, x_4 \triangleright^{-1} x_2 = x_1).$$

Note that we have described $Q(K)$ so that Adconj $Q(K) = G(K)$. We may arrive at the same presentation of $Q(K)$ more simply. Take a regular projection of the knot K. Label the arcs putting the labels always on one side of the knot. For each intersection derive a relation of one of the two forms as illustrated in Fig. 6. Then the n relations on the n generators present the quandle $Q(K)$.
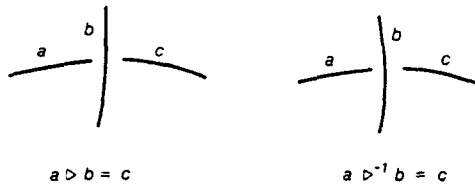

$$a \triangleright b = c \qquad\qquad a \triangleright^{-1} b = c$$

Fig. 6. Intersections.

Theorem 15.1. *The quandle $Q(K)$ of a tame knot K as presented above is an invariant of the knot type of K.*

Proof. We give a direct combinatorial demonstration. There are three basic deformations, called Reidemeister moves of regular projections of knots which do not change the knot type. These three moves account for equivalences among tame knots in the following sense. If two tame knots are equivalent, then for any regular projections of the two knots there is a finite sequence of Reidemeister moves transforming one projection into the other, see [1, 8]. In order to show the invariance of $Q(K)$ it suffices to show invariance under these three moves.

The first move Ω_1 removes or adds a kink. For Ω_1 we have two cases depending on which side of the arc is labeled. Fig. 7 indicates invariance under Ω_1 since in a quandle the identities $x \triangleright x = x$ and $x \triangleright^{-1} x = x$ are satisfied. The move Ω_2 slides one arc under another. It requires that $x \triangleright y \triangleright^{-1} y = x$ and $x \triangleright^{-1} y \triangleright y = x$ as shown in Fig. 8.
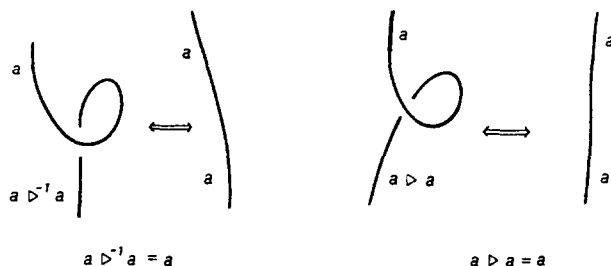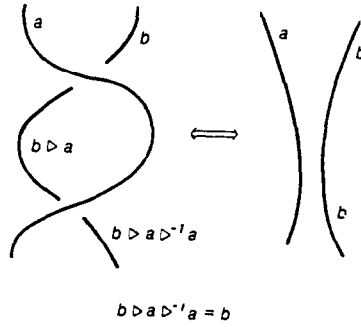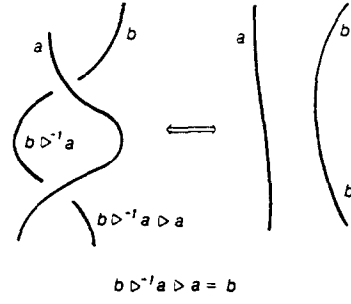

Fig. 7. Equations for Ω_1.

The move Ω_3 slides an arc under an intersection. This last move yields four requirements depending on the labeling.

Fig. 9 illustrates one of the requirements, which turns out to be Q3. The other three requirements are also satisfied by quandles. Since the requirements for the invariance of the knot quandle under the Reidemeister moves are all satisfied, it follows that the quandle is an invariant of the knot type.   □

Theorem 15.2. *The quandle $Q(K)$ of a tame knot K as presented in this section is isomorphic to the knot quandle $\bar{Q}(K)$ defined in Section 14.*

Proof. Assume that the regular projection of the knot to the place is projection from the basepoint ∗. Label the arcs in order $a_1, a_2, \ldots, a_n$. For $i = 1, \ldots, n$ let b_i be a path down from ∗ directly to the center of the arc a_i. Let γ_i be the loop which travels from ∗ down b_i along a_i and a_{i+1} then up b_{i+1} back to ∗. Let U_i be a small toroidal neighborhood of γ_i. Then $U_{i-1} \cap U_i$ is a neighborhood of b_i. Let $V = X - K$. Then $X = V \cup U_1 \cup \cdots \cup U_n$. According to the Seifert–Van Kampen theorem for quandles, in order to construct $\bar{Q}(K)$, we need to know only the augmented quandles of

$$b \triangleright^{-1} a \triangleright a = b$$

$$b \triangleright a \triangleright^{-1} a = b$$

Fig. 8. Equations for Ω_2.
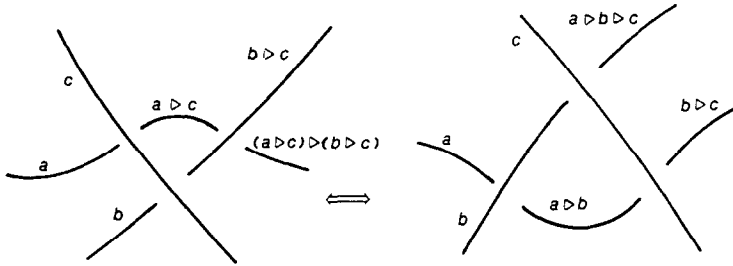
Fig. 9. $(a \triangleright c) \triangleright (b \triangleright c) = (a \triangleright b) \triangleright c$.

$(U_i \cap K, U_i)$, $(U_i \cap U_{i+1} \cap K, U_i \cap U_{i+1})$, $(\emptyset, V)$, $(\emptyset, U_i \cap V)$, and $(\emptyset, U_i \cap U_{i+1} \cap V)$, and the maps between them.

Let v_i be a noose in $U_{i-1} \cap U_i$ linking once about a_i. Then $\pi_1(U_{i-1} \cap U_i - K) = (x_i)$ is the free group on one element $x_i = \partial(v_i)$. Hence, by Theorem 12.1,

$$AQ(U_{i-1} \cap U_i \cap K, U_{i-1} \cap U_i) = (\mathrm{Conj}(x_i), (x_i)).$$

Restricting to the nooses linking once with K, we have

$$\overline{AQ}(U_{i-1} \cap U_i \cap K, U_{i-1} \cap U_i) = (\{x_i\}, (x_i)).$$

Let $G = \pi_1(V) = \pi_1(X - K)$ and $G_i = \pi_1(U_i - k)$. Then $\overline{AQ}(U_i \cap K, U_i) = (\{x_i\} \otimes_{(x_i)} G_i, G_i)$, since

$$AQ(U_i \cap K, U_i) = (\mathrm{Conj}(x_i) \otimes_{(x_i)} G_i, G_i).$$

As described in Section 9, in order to find $\overline{AQ}(K, X)$ we may first tensor the various $\overline{AQ}$ with $G = \pi_1(X - K)$. Then according to Theorems 9.4 and 13.1, upon taking the colimit of the various $\overline{AQ} \otimes G$, we will have $\overline{AQ}(K, X)$. Both $\overline{AQ}(U_{i-1} \cap U_i \cap K, U_{i-1} \cap U_i)$ and $\overline{AQ}(U_i \cap K, U_i)$ become $(\{x_i\} \otimes_{(x_i)} G, G)$ when tensored with G. Thus, $\bar{Q}$ is generated by $x_1, \ldots, x_n$ modulo the relations induced by tensoring with G. These relations are determined by the action of the generators of G on $\bar{Q}$ and the relations among the generators of G. It is exactly these relations which were used in the definition of $Q(K)$. Thus, $\bar{Q}(K) \cong Q(K)$. $\square$

We have already noted and used the fact that $\mathrm{Adconj}\, Q(K)$ is the knot group $G(K)$. We now know $\mathrm{Adconj}\, \bar{Q}(K) \cong G$. In particular $\varepsilon(\bar{Q})$ generates G.

Corollary 15.3. *Let K be a tame knot, G its knot group, and Q its know quandle. Then $Q \times G \to Q$ is a transitive group action. Consequently, Q is algebraically connected.*

Proof. In order to show G acts transitively on Q it suffices to show that for generators a, b of Q there is an x in G such that $ax = b$. But by passing under sufficiently many arcs $b_1, b_2, \ldots, b_k$ of the regular projections, the arc a becomes the arc b. Hence, $b = a\varepsilon(b_1)^{\pm 1} \cdots \varepsilon(b_k)^{\pm 1}$. $\square$

The theorems of this section hold not only for knots, but also for links when careful attention is paid to the orientation of the components of the links. An exception is the preceding corollary which holds only for knots. The quandle of a link is algebraically connected iff the link is a knot.

16. A representation of the knot quandle

Recall that a *meridian* about a knot K is a loop in the complement of K that links once with K and bounds a disk intersecting K at one point. Equivalently, a meridian is a boundary of a noose in the knot quandle. Let U be a regular neighborhood of K in X, that is, U is the image of $S^1 \times (\text{disk})$ embedded in X with $K = \mathrm{image}(S^1 \times \{0\})$. The boundary ∂U of U is a torus. Connect U to the basepoint $*$ by a path γ in $X - U$. Then the inclusion $U \cup \gamma \subsetneq X - K$ induces a homomorphism from $\pi_1(U \cup \gamma) \cong Z \oplus Z$ to $G = \pi_1(X - K)$. This is a monomorphism unless K is a trivial knot. The image of this map is called a *peripheral subgroup* of the knot group G. Each peripheral subgroup P contains exactly one meridian. Another distinguished element of P is the *longitude l. l* is a generator of the subgroup of P consisting of loops which are not linked with K.

Theorem 16.1. *Let K be a tame knot with group G and quandle Q. Let $v \in Q$ and $G_v = \{x \in G \mid vx = v\}$. Then G_v is a peripheral subgroup of G.*

Proof. Let U be a regular neighborhood of K containing the disk of the noose v. Connect U to $*$ by Rope v. Then the loops in $U \cup$ Rope v form a peripheral subgroup P of G. We show $P = G_v$. Without loss of generality we may assume $*$ lies in U and Rope $v = \{*\}$. Let $U = f(S^1 \times \text{disk})$, $\partial U = f(S^1 \times S^1)$, $* = f(1, 1)$. We may also assume $\varepsilon(v) = f(S^1 \times 1)$. P is generated by the meridian $\varepsilon(v)$ and the longitude $l = f(1 \times S^1)$. vl is homotopic to v (slide the disk of vl around the solid torus U by one revolution). Also, $vm \sim v$, hence, $P \subseteq G_v$. Let β be a loop in $X - K$ such that $v\beta \sim v$. The homotopy H of $v\beta$ to v may be chosen so that the disk portion of the homotopy lies inside U. Let γ be the loop from $v(1)$ to $v(1)$ given by $\gamma(t) = H(1, t)$. Then $\beta \sim (\text{Rope } v)\gamma(\text{Rope } v)^{-1}$ which lies in P. Thus $G_v = P$. $\quad\square$

The next corollary follows from Theorem 16.1 and Corollary 15.3.

Corollary 16.2. *Let K be a knot with knot group G and knot quandle Q. Let P be a peripheral subgroup of G containing a meridian m. Then $(P \setminus G; m)$, as described in Section 7, is isomorphic to the knot quandle.* $\quad\square$

Thus, the knot quandle contains the same information as the triple (G, P, m) consisting of the knot group G, a peripheral subgroup P, and a meridian m in P.

Neuworth [7] remarks that if two tame knot groups are isomorphic by a map which sends a meridian to a meridian and the group system (the conjugate peripheral subgroups) of one onto the group system of the other, then the (unoriented) knots are equivalent. Conway and Gordon [4] use a slightly stronger principle to construct a group that classifies oriented knots. If two tame knot groups are isomorphic by a map which sends a meridian and corresponding longitude of one onto those of the other, then the oriented knots are equivalent. A proof of this principle may be found in Waldhausen [12].

Corollary 16.3. *If the knot quandles of two tame knots are isomorphic, then the (unoriented) knots are equivalent.* $\quad\square$

Other algebraic characterizations of knots have been described by Simon [10] and Whitten [13]. The constructions given by Conway and Gordon, Simon, and Whitten are not functorial, unlike the knot quandle.

17. The Alexander invariant

In this section we derive a correspondence between the Alexander invariant of a knot and the abelian knot quandle. Indeed, each determines the other.

Let K be a tame knot with knot group G. Let Y be the infinite cyclic cover of the complement of K. Then $\pi_1(Y) \cong G' = [G, G]$, and $H_1(Y) \cong G'/G''$. Conjugation by any meridian of K gives an automorphism of $H_1(Y)$ which is independent of choice of meridian. Thus, $H_1(Y)$ is a module over the Laurent polynomial ring $\Lambda = Z[t, t^{-1}]$. This Λ-module is called the *Alexander invariant* A of the knot K. The usual presentation of A is by means of a matrix. One takes independent generating cycles $a_1, \ldots, a_n$ for the homology of a Seifert surface for the knot. The matrix (v_{ij}) is called a *Seifert matrix* where v_{ij} is the linking number of a_i with a_j. The *Alexander matrix* is $P = (v_{ij} - t v_{ji})$. The entries of P lie in Λ. The Alexander invariant is presented as the cokernel

$$\Lambda^n \xrightarrow{\quad P \quad} \Lambda^n \to A \to 0.$$

The determinant of P is called the *Alexander polynomial* $\Delta(t)$ of the knot. $\Delta(t)$ is defined only up to a unit of Λ. One important property of $\Delta(t)$ is that $|\Delta(1)| = 1$.

We first show that A may be constructed from the abelian knot quandle AbQ. Since $\operatorname{Adconj} Q = G$, we may apply the remarks of Section 10 to conclude that $\operatorname{AbQ} = Q/N$ and $\operatorname{Adconj} \operatorname{AbQ} = G/N$, where N is the normal subgroup of G generated by elements of the form

$$ab^{-1}ca^{-1}bc^{-1} \quad \text{with } a, b, c \in \varepsilon(Q).$$

Lemma 17.1. $N = G''$.

Proof. $N \subseteq G''$: By the connectivity of Q, for a, b, c in Q there exist x, y in G such that $b = x^{-1}ax$ and $c = y^{-1}ay$. Hence,

$$ab^{-1}ca^{-1}bc^{-1} = (ax^{-1}a^{-1}x)(y^{-1}aya^{-1})(x^{-1}axy^{-1}a^{-1}y).$$

Since the first two terms of the parenthesized product lie in G', we may interchange them modulo G'', after which the product simplifies to 1. Therefore, $ab^{-1}ca^{-1}bc^{-1} \in G''$. Thus, $N \subseteq G''$.

$G'' \subseteq N$: Let $a, b \in G'$. Then a is of the form

$$a_1^{e_1} \cdots a_n^{e_n} \quad \text{with } a_i \in \varepsilon(Q) \quad \text{and} \quad \sum e_i = 0.$$

Note that if $x, y \in \varepsilon(Q)$, then $xy^{-1} = y^{-1}z$ where $z = yxy^{-1} \in \varepsilon(Q)$. Hence, a may be rewritten as

$$a = a_1 a_2^{-1} a_3 a_4^{-1} \cdots a_{n-1} a_n^{-1}$$

with each a_i in $\varepsilon(Q)$. We write b similarly. Using the fact that modulo N we have the congruences

$$xy^{-1}wz^{-1} \equiv wy^{-1}xz^{-1} \equiv wz^{-1}xy^{-1} \quad \text{for } w, x, y, z \in \varepsilon(Q),$$

we conclude that a commutes with b modulo N. Hence, $[a, b] \in N$. Thus, $G'' \subseteq N$. $\square$

Since $\text{Adconj AbQ} = G/N = G/G''$ is constructible from AbQ, so is its commutator G'/G''. The symmetry of an element in AbQ is an automorphism of AbQ which induces an automorphism on G'/G'', the required Λ-structure on $H_1(Y) \cong G'/G''$.

Theorem 17.2. *Let K be a tame knot with abelian knot quandle* AbQ. *Then the commutator subgroup B of* Adconj AbQ *is an abelian group. Also, B is a Λ-module where multiplication by t is given by conjugation by any element $\varepsilon(q)$ of* Adconj AbQ *with q in* AbQ. *Furthermore, as a Λ-module B is isomorphic the Alexander invariant of K.* $\square$

Next, we show that the Alexander invariant not only determines AbQ but has quandle structure itself which is isomorphic to AbQ.

Theorem 17.3. *Let the Alexander invariant A be given the quandle structure*

$$x \triangleright y = t(x - y) + y, \qquad x \triangleright^{-1} y = t^{-1}(x - y) + y.$$

Then with this structure, A is isomorphic to AbQ.

Proof. Let $a_0 \in Q$. Let P be the peripheral subgroup associated to the meridian $m = \varepsilon(a_0)$. By Corollary 16.2, we have $Q \cong (P \setminus G; m)$, or using s to denote conjugation by m, $Q \cong (P \setminus G; s)$. It follows from Lemma 17.1 that $\text{AbQ} \cong Q/N = Q/G''$. Hence,

$$\text{AbQ} \cong ((P/P \cap N) \setminus (G/N); s) \cong ((P/P \cap G'') \setminus (G/G''); s).$$

The map $G \to Q$ sending x to $a_0 x$ remains surjective upon restriction to G', so

$$\text{AbQ} \cong ((P \cap G'/P \cap G'') \setminus (G'/G''); s).$$

Let l be a longitude in P. Then $P \cap G'$ is the subgroup (l) generated by l. From the fact that $|\Delta(1)| = 1$, it can be shown that l lies in G''. Consequently, $P \cap G' = (l) = P \cap G''$. Therefore, $\text{AbQ} \cong (G'/G''; s)$.

The quandle structure in $(P \setminus G; m)$ is given by

$$Px \triangleright Py = Pxy^{-1}my = Pmxy^{-1}my = P(x \triangleright m)(y \triangleright m)^{-1}y.$$

In $(G'/G''; s)$, we have

$$x \triangleright y = (x \triangleright m)(y \triangleright m)^{-1}y$$

which in the Λ-module notation of A says

$$x \triangleright y = tx - ty + y = t(x - y) + y. \qquad \square$$

The Alexander invariant is insufficient to distinguish all knots from the trivial knot. For instance, the Alexander invariant of any doubled knot is trivial.

18. The cyclic invariants of a knot

The nth cyclic invariant is defined similarly to the Alexander invariant. The space Y_n is the n-fold branched cyclic cover rather than the infinite cyclic cover. The homology group $H_1(Y_n)$ has an automorphism induced by any meridian of the knot. Thus, $H_1(Y_n)$ is a module over $\Lambda_n = Z[t]/(t^n - 1)$, and as such is called the nth cyclic invariant, A_n. A quandle structure on A_n is given by

$$x \triangleright y = t(x - y) + y.$$

With this quandle structure A_n is an n-quandle, in fact, it is the largest quotient of the abelian knot quandle which is an n-quandle, which we may call the abelian knot n-quandle $\mathrm{AbQ}_n(K)$.

The order of $H_1(Y_2)$ is called the determinant of the knot and is found by evaluating the Alexander polynomial at -1, $\det K = |\Delta(-1)|$. Thus, the order of the involutory abelian knot quandle is $\det K$.

19. The involutory knot quandle

The involutory knot quandle $Q_2(K)$ results from imposing the identity $(x \triangleright y) \triangleright y = x$ on the knot quandle $Q(K)$.

Example 19.1. Refer to the figure-8 knot described in Section 15 and illustrated in Fig. 5. The involutory knot quandle for the figure-8 knot is presented as

$$Q_2(K) = (x_1, x_2, x_3, x_4 : x_1 \triangleright x_3 = x_2, x_2 \triangleright x_4 = x_3,$$

$$x_3 \triangleright x_1 = x_4, x_4 \triangleright x_2 = x_1).$$

This quandle contains five elements and is isomorphic to Core Z_5. It is abelian as well as involutory.

Example 19.2. The knot 10_{124} has determinate 1, so $\mathrm{AbQ}_2(K)$ is trivial. A few computations show that $Q_2(K)$ is nontrivial and has order 30. It may be faithfully represented on a sphere as the 30 midpoints of the edges of a dodecahedron projected onto the sphere.

Theorem 19.3. *The link quandle is not an invariant of the complement of the link.*

Proof. We examine the involutory link quandles of the links K_1 and K_2 displayed in Fig. 10. The complements of K_1 and K_2 are homeomorphic as described in [9, p. 49].

$$Q_2(K_1) = (a, b, c, d, e : a \triangleright c = b, b \triangleright d = a, c \triangleright a = d, \ d \triangleright c = e, e \triangleright a = c)$$

$$\cong \mathrm{Core}\, Z_4,$$
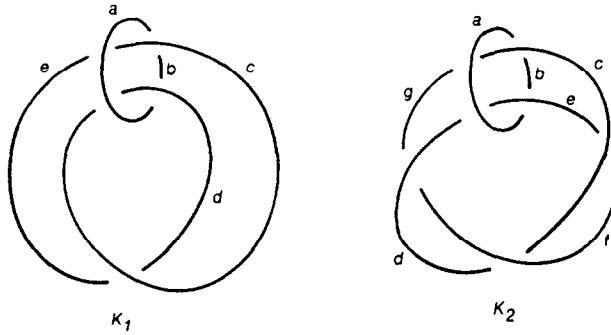
Fig. 10. Links K_1 and K_2.

$$Q_2(K_2) = (a, \ldots, g : a \rhd c = b, b \rhd e = a, c \rhd f = d, d \rhd a = e,$$

$$e \rhd c = f, f \rhd d = g, g \rhd a = c).$$

$Q_2(K_2)$ has order 8. It may be represented with geodesics as the quandle in Fig. 3 of Section 11. Since the involutory quandles of K_1 and K_2 are distinct, so are their quandles. $\square$

Acknowledgment

References

[1] J.W. Alexander and G.B. Briggs, On types of knotted curves, Ann. of Math. 28 (1926) 562–586.

[2] R.H. Bruck, A Survey of Binary Systems (Springer, Berlin, 1958).

[3] C. Burstin and W. Mayer, Distributive Gruppen, J. Reine Angew. Math. 160 (1929) 111–130.

[4] J.H. Conway and C. McA. Gordon, A group to classify knots, Bull. London Math. Soc. 7 (1975) 84–88.

[5] O. Loos, Spiegelungsraüme and homogene symmetrische Raüme, Math. Z. 99 (1967) 141–170.

[6] O. Loos, Symmetric Spaces (Benjamin, New York, 1969).

[7] L.P. Neuworth, Knot Groups (Princeton Univ. Press, Princeton, NJ, 1965).

[8] K. Reidemeister, Knotentheorie (Chelsea, New York, 1948) (reprint).

[9] D. Rolfsen, Knots and Links (Publish or Perish, Berkeley, CA, 1976).

[10] J. Simon, An algebraic classification of knots in S^3, Ann. of Math. 93 (1973) 1–13.

[11] M. Takasaki, Abstractions of symmetric functions, Tohoku Math. J. 49 (1943) 143–207 (Japanese), Math. Rev. 9, p. 8.

[12] F. Waldhausen, On irreducible 3-manifolds which are sufficiently large, Ann. of Math. 87 (1968) 56–88.

[13] W. Whitten, Characterization of knots and links, Bull. Amer. Math. Soc. 80 (1974) 1265–1270.