

# QUANDLE VARIETIES, GENERALIZED SYMMETRIC SPACES, AND $\varphi$ -SPACES

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**Abstract.** We define a quandle variety as an irreducible algebraic variety  $Q$  endowed with an algebraically defined quandle operation  $\triangleright$ . It can also be seen as an analogue of a generalized affine symmetric space or a regular  $s$ -manifold in algebraic geometry.

Assume that  $Q$  is normal as an algebraic variety and that the action of its inner automorphism group  $\text{Inn}(Q)$  has a dense orbit. Then we show that there is an algebraic group  $G$  acting on  $Q$  with the same orbits as  $\text{Inn}(Q)$  such that each  $G$ -orbit is isomorphic to the quandle  $(G/H, \triangleright_\varphi)$  associated to the group  $G$ , an automorphism  $\varphi$  of  $G$  and a subgroup  $H$  of  $G^\varphi$ .

## 1. Introduction

### 1.1. Generalized symmetric spaces and $\varphi$ -spaces

A symmetric space is a Riemannian manifold with a symmetry  $s_x$  around any point  $x$ . Here, by a symmetry around  $x$ , we mean an isometry which fixes  $x$  and inverts tangent vectors at  $x$ . After the extensive studies by É. Cartan, the theory of symmetric spaces has been significantly generalized. One important direction was to regard them as manifolds endowed with the binary operation  $x \cdot y := s_x(y)$ . (See [SSS02, Intr. and §§1–7] for a survey of developments of the theory of generalized symmetric spaces.)

In [Lo67] (see also [Lo69]), Loos characterized symmetric spaces in terms of binary operations. For the brevity of the presentation, let us mean by the word “automorphism” either an isometry, a diffeomorphism preserving an affine connection, or just a diffeomorphism, according to the context. Then a symmetric space is a Riemannian manifold  $M$  endowed with a smooth binary operation  $\cdot : M \times M \rightarrow M$  subject to the following conditions:

- (1)  $x \cdot x = x$ .
- (2)  $s_x : M \rightarrow M; y \mapsto x \cdot y$  is an automorphism.

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$$(3) \quad x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z).$$

$$(4) \quad s_x \circ s_x = id_M \text{ and } x \text{ is an isolated fixed point of } s_x.$$

In the setting of affine differential geometry, the same conditions define an affine symmetric space, although Loos proved that the affine connection can be canonically constructed from the rest of the data.

The notion of symmetric space was generalized by relaxing the condition (4). Loos considered manifolds with an operation satisfying the conditions (1)–(3) and

$$(4R) \quad (\text{reflexivity}) \quad s_x \circ s_x = id_M,$$

and named them reflexion spaces.

Non-involutive symmetries were introduced to the field by Ledger ([Le67]). Then Kowalski and Fedenko considered the notions of regular  $s$ -manifold and generalized affine symmetric space. A regular  $s$ -manifold is defined as a manifold with an operation satisfying conditions (1)–(3) and

$$(4T) \quad (\text{tangential regularity}) \quad d_x s_x - \text{Id} \text{ is invertible on } T_x M.$$

A generalized affine symmetric space is an affine manifold which *admits* an operation satisfying conditions (1)–(3) and

$$(4I) \quad (\text{isolatedness}) \quad x \text{ is an isolated fixed point of } s_x.$$

In this case, the family  $\{s_x\}$  of automorphisms is called an admissible  $s$ -structure.

The conditions (4T) and (4I) are equivalent in the presence of a Riemannian metric or an affine connection, so a generalized affine symmetric space together with an admissible  $s$ -structure is a regular  $s$ -manifold.

Cartan proved that a symmetric space can be described as a special kind of homogeneous space. Loos, Fedenko and Kowalski gave generalizations of this result in their respective settings. In order to state their results, let us explain the notion of  $\varphi$ -space introduced by Vedernikov ([V65]).

**Definition 1.1.** Let  $G$  be a Lie group,  $\varphi$  an automorphism of  $G$ , and  $H$  a subgroup of  $G$  satisfying  $(G^\varphi)^\circ \subseteq H \subseteq G^\varphi$ . Then  $G/H$ , or the triple  $(G, H, \varphi)$ , is called a  $\varphi$ -space.

It is called a regular  $\varphi$ -space if  $d_{\bar{e}} \bar{\varphi} - \text{Id}$  is invertible, where  $\bar{e}$  is the residue class of the identity element and  $\bar{\varphi} : G/H \rightarrow G/H$  is the induced diffeomorphism.

*Remark 1.2.* (1) In the original definition by Vedernikov,  $\varphi$  was assumed to be an endomorphism, not necessarily an automorphism.

(2) In the current context, there seem to be two meanings for the term “regular.” For  $s$ -manifolds, the word “regular” refers to the condition  $x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$ , while for  $\varphi$ -spaces, it refers to the tangential regularity.

Extending the work of Loos on affine symmetric spaces (see Theorem 4.9), Stepanov, Fedenko, and Kowalski studied the relation between regular  $s$ -manifolds, generalized affine symmetric spaces, and regular  $\varphi$ -spaces. Their works show that these three notions are essentially equivalent to each other.

**Theorem 1.3** ([K80, Thms. II. 4 and II.25]).

(1) For a regular  $\varphi$ -space  $(G, H, \varphi)$  one can define an operation  $\cdot_\varphi$  by

$$xH \cdot_\varphi yH := x\varphi(x^{-1}y)H,$$

for which  $(G/H, \cdot_\varphi)$  is a regular  $s$ -manifold. It also admits a canonical affine connection which is invariant under the symmetries, so  $G/H$  is a generalized affine symmetric space.

- (2) Conversely, for a generalized affine symmetric space endowed with an admissible  $s$ -structure, one can construct a regular  $\varphi$ -space which gives the inverse correspondence to the above.
- (3) Finally, if  $M$  is a regular  $s$ -manifold, then there exists uniquely an affine connection on  $M$  which makes  $M$  a generalized affine symmetric space with an admissible  $s$ -structure  $\{s_x\}$ .

## 1.2. Knots and quandles

A quandle is a nonempty set  $Q$  endowed with a binary operation  $\triangleright$  which satisfies the following conditions.

- (1)  $q \triangleright q = q$ .
- (2)  $s_q : Q \rightarrow Q; r \mapsto q \triangleright r$  is a bijection. (We write  $q \triangleright^{-1} r$  for  $s_q^{-1}(r)$ .)
- (3)  $q \triangleright (r \triangleright s) = (q \triangleright r) \triangleright (q \triangleright s)$ .

In more descriptive words, it is a distributive groupoid.

While there were some previous works on similar algebraic structures, quandles began to attract wide attention after the works of Joyce([J82]) and Matveev([M82]) relating them to knots.

To any knot or a link they associated its knot quandle or fundamental quandle. The coloring numbers of a knot can be interpreted as the number of quandle homomorphisms from the knot quandle to certain finite quandles. More generally, for any finite quandle  $Q$  an invariant called the  $Q$ -coloring number is defined. It is an invariant explicitly calculable from the knot diagram. There is also a theory of quandle homology and cohomology, which gives rise to even more invariants([CJKLS03]).

As one can see by comparing the definitions, a regular  $s$ -manifold can be regarded as a quandle. Joyce already noted, citing [Lo67] and [Lo69], that a reflexion space is a quandle. However, little attention has been paid to regular  $s$ -manifolds in knot theory, probably because finite quandles already provide plenty of useful information. Recently, Rubinsztein defined the notion of topological quandle([R07]). He also defined an invariant topological space  $J_Q(L)$  for any topological quandle  $Q$  and any link  $L$ , which is a continuous generalization of the coloring numbers.

## 1.3. Quandle varieties

In this paper, we study quandles and generalized symmetric spaces in the category of algebraic varieties.

By a quandle variety, we mean an algebraic variety  $Q$  equipped with quandle operations  $\triangleright$  and  $\triangleright^{-1}$  which are regular maps of algebraic varieties. Here, an algebraic variety will be always assumed to be irreducible and reduced.

As a possible application to knot theory, we can associate the space of colorings to each link and each quandle variety, as in the case of topological quandles. The space of colorings is an algebraic scheme in this case, so it might give a finer information than just as a topological space, as pointed out in [R07, Rem. 6.2].

In this paper, we focus on the basic theory of quandle varieties themselves. Especially, we try to understand the structure of quandle varieties.

For any quandle  $Q$ , we have two important groups of automorphisms. The group  $\text{Inn}(Q)$  is defined as the group generated by the automorphisms  $s_q$ . The group  $\text{Tr}(Q)$  is the subgroup of  $\text{Inn}(Q)$  consisting of products of the same numbers of  $s_q$  and  $s_q^{-1}$ . It can be shown that  $\text{Inn}(Q)$ -orbits and  $\text{Tr}(Q)$ -orbits coincide, and we say that a quandle is  $\triangleright$ -connected if it is comprised of one orbit.

Our main theorem implies that a  $\triangleright$ -connected quandle variety is an algebraic  $\varphi$ -space  $G/H$ , except that  $H$  does not necessarily contain  $(G^\varphi)^\circ$ .

**Theorem 1.4** (See Theorem 4.1). *Let  $Q$  be a quandle variety which is irreducible and normal as a variety. Assume that  $Q$  contains an open  $\text{Inn}(Q)$ -orbit  $O$ .*

- (1) *The group  $\text{Tr}(Q)$  can be given a structure of a connected algebraic group  $\mathbf{Tr}(Q)$  such that the action of  $\mathbf{Tr}(Q)$  on  $Q$  is algebraic. If  $Q$  is quasi-affine, then  $\mathbf{Tr}(Q)$  is affine.*
- (2) *The open orbit  $O$  is  $\triangleright$ -connected, and there are natural isomorphisms  $\text{Tr}(O) \cong \text{Tr}(Q)$  and  $\text{Inn}(O) \cong \text{Inn}(Q)$ .*
- (3) *For any  $q \in Q$ , the conjugation map  $g \mapsto s_q g s_q^{-1}$  restricts to an automorphism  $\varphi_q$  of  $\mathbf{Tr}(Q)$ , the stabilizer  $\mathbf{Tr}(Q)_q$  is contained in  $\mathbf{Tr}(Q)^{\varphi_q}$  and the natural map  $\mathbf{Tr}(Q)/\mathbf{Tr}(Q)_q \rightarrow \text{Inn}(Q)_q$ ;  $g\mathbf{Tr}(Q)_q \mapsto gq$  is an isomorphism of quandle varieties.*

*Each orbit acts on another orbit, and this action can also be described using a certain self-map of  $\mathbf{Tr}(Q)$ .*

The plan of this paper is as follows. We recall the definitions and elementary results related to quandles in §2. In §3, we define quandle varieties and show several basic results on orbits which are similar to results on orbits of an algebraic group action. We state and prove the main theorem in §4. The key is to find a space with a  $Q$ -action such that  $\text{Tr}(Q)$  can be identified with an orbit. In the last section, we show a result on the relation between isolatedness of fixed points and  $\triangleright$ -connectedness, which gives a partial answer to an algebraic version of a question of Kowalski.

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## 2. Quandles

We recall the definition of quandles and basic results that will be used in the sequel. Most of the contents in this section are from [J82]. In this paper, we use the convention where  $Q$  “acts” from the left.

**Definition 2.1.** A quandle is a nonempty set  $Q$  equipped with a binary operation  $\triangleright$  satisfying the following conditions.

- (1) For any  $q \in Q$ ,  $q \triangleright q = q$  holds.
- (2) For any  $q, r \in Q$ , there exists a unique element  $r' \in Q$  such that  $q \triangleright r' = r$ .
- (3) For any  $q, r, s \in Q$ ,  $q \triangleright (r \triangleright s) = (q \triangleright r) \triangleright (q \triangleright s)$ .

We define  $s_q : Q \rightarrow Q$  by  $s_q(r) = q \triangleright r$ , which is bijective by (2). We write  $q \triangleright^{-1} r$  for  $s_q^{-1}(r)$ . For  $\sigma_1, \dots, \sigma_n \in \{\pm 1\}$ , we write  $q_n \triangleright^{\sigma_n} q_{n-1} \triangleright^{\sigma_{n-1}} \dots \triangleright^{\sigma_1} q_0$  for  $q_n \triangleright^{\sigma_n} (q_{n-1} \triangleright^{\sigma_{n-1}} (\dots \triangleright^{\sigma_1} q_0))$ .

**Definition 2.2.** Let  $Q$  and  $Q'$  be quandles. A homomorphism from  $Q$  to  $Q'$  is a map  $f : Q \rightarrow Q'$  such that  $f(q \triangleright r) = f(q) \triangleright f(r)$  holds for any  $q, r \in Q$ .

An isomorphism is a homomorphism which admits an inverse homomorphism. An isomorphism onto itself is called an automorphism.

An action of a group  $G$  on a quandle  $Q$  is an action of  $G$  on the underlying set of  $Q$  such that  $g(q \triangleright r) = (gq) \triangleright (gr)$  holds for any  $g \in G$  and  $q, r \in Q$ .

The following is immediate from the definitions.

**Proposition 2.3.**

- (1) If  $f$  is a homomorphism,  $f(q \triangleright^{-1} r) = f(q) \triangleright^{-1} f(r)$  holds.
- (2) A homomorphism is an isomorphism if and only if it is bijective.
- (3) The map  $s_q$  is an automorphism for any  $q \in Q$ .

**Example 2.4.** If  $Q$  is a nonempty set, then  $q \triangleright r := r$  defines the structure of a trivial quandle.

**Example 2.5.** Let  $X$  be a nonempty set and  $A$  an abelian group. Let  $F : X \times X \rightarrow A$  be a map. On  $Q = X \times A$ , the operation  $(x, a) \triangleright (y, b) := (y, b + F(x, y))$  gives a quandle structure if and only if  $F(x, x) = 0$  for any  $x$ .

**Example 2.6.** If  $G$  is a group, then  $g \triangleright h := g^{-1}hg$  gives a quandle structure on  $G$ . A conjugacy class in  $G$  is also a quandle.

**Example 2.7.** If  $G$  is a group,  $\varphi$  is an automorphism of  $G$ , and  $H$  is contained in  $G^\varphi$ , then  $xH \triangleright_\varphi yH := x\varphi(x^{-1}y)H$  gives a quandle structure on  $G/H$ .

In particular, if  $G = V$  is a vector space,  $\varphi$  is a linear automorphism, and  $H = 0$ , then  $\mathbf{v} \triangleright \mathbf{w} := \mathbf{v} + \varphi(\mathbf{w} - \mathbf{v})$  defines a quandle.

**Example 2.8.** If  $G$  is a group and  $\varphi$  is an automorphism of  $G$ , then  $x \triangleright'_\varphi y := x\varphi(yx^{-1})$  gives a quandle structure on  $G$ .

This is related to the previous example in the following way ([V65]). Consider the action of  $G$  on  $(G, \triangleright'_\varphi)$  defined by  $x \cdot a = xa\varphi(x^{-1})$ . Then  $G/G^\varphi$  can be identified with the orbit of  $e$  and the quandle operations coincide under this identification. More generally, the orbit of  $a$  can be identified with  $(G/G^{\varphi_a}, \triangleright_{\varphi_a})$ , where  $\varphi_a(x) := a\varphi(x)a^{-1}$ .

**Definition 2.9.** We define the automorphism group of  $Q$  as

$$\text{Aut}_\triangleright(Q) := \{f : Q \rightarrow Q \mid f(q \triangleright r) = f(q) \triangleright f(r) \text{ and } f \text{ is bijective}\}$$

and the inner automorphism group of  $Q$  as

$$\text{Inn}(Q) := \langle s_q \mid q \in Q \rangle.$$

Let

$$\text{Tr}_k(Q) := \{s_{b_1}^{-1} \circ \dots \circ s_{b_k}^{-1} \circ s_{a_k} \circ \dots \circ s_{a_1} \mid a_i, b_i \in Q\}$$

and define the group of transvections of  $Q$  to be

$$\mathrm{Tr}(Q) := \bigcup_{k \in \mathbb{N}} \mathrm{Tr}_k(Q).$$

We will see that  $\mathrm{Tr}(Q)$  is in fact a group. Moreover,  $\mathrm{Inn}(Q)$  and  $\mathrm{Tr}(Q)$  are normal subgroups of  $\mathrm{Aut}_{\triangleright}(Q)$ .

**Lemma 2.10.**

- (1) For any  $f \in \mathrm{Aut}_{\triangleright}(Q)$  and  $q \in Q$ , the equality  $f \circ s_q = s_{f(q)} \circ f$  holds.
- (2) For any nonnegative integers  $k$  and  $l$ , let  $\sigma_1, \dots, \sigma_{k+l} \in \{\pm 1\}$  be such that  $\#\{i \mid \sigma_i = 1\} = k$  and  $\#\{i \mid \sigma_i = -1\} = l$ . Define

$$S_{\sigma_1, \dots, \sigma_{k+l}} := \{s_{a_{k+l}}^{\sigma_{k+l}} \circ \dots \circ s_{a_1}^{\sigma_1} \mid a_1, \dots, a_{k+l} \in Q\}.$$

Then  $S_{\sigma_1, \dots, \sigma_{k+l}}$  depends only on  $k$  and  $l$ .

*Proof.* (1) This follows from

$$(f \circ s_q)(r) = f(q \triangleright r) = f(q) \triangleright f(r) = (s_{f(q)} \circ f)(r).$$

(2) Since  $s_a^{-1}$  is an automorphism, it follows from (1) that  $s_a^{-1} \circ s_b = s_{s_a^{-1}(b)} \circ s_a^{-1}$ . So  $s_{a_i}^{-1}$ 's can be gathered to the right.  $\square$

**Proposition 2.11.**  $\mathrm{Inn}(Q)$  and  $\mathrm{Tr}(Q)$  are normal subgroups of  $\mathrm{Aut}_{\triangleright}(Q)$ .

*Proof.* From (2) of the previous lemma, we see that  $\mathrm{Tr}(Q)$  is a subgroup. By (1) of the lemma, we have

$$f \circ s_{a_k}^{\sigma_k} \circ \dots \circ s_{a_1}^{\sigma_1} \circ f^{-1} = s_{f(a_k)}^{\sigma_k} \circ \dots \circ s_{f(a_1)}^{\sigma_1},$$

so  $\mathrm{Inn}(Q)$  and  $\mathrm{Tr}(Q)$  are normal subgroups.  $\square$

**Definition 2.12.** Let  $Q$  be a quandle and  $X$  a set. An action of  $Q$  on  $X$  is a map  $Q \times X \rightarrow X; (q, x) \mapsto q \triangleright x$  subject to the following conditions.

- (1) For any  $q \in Q$  and  $x \in X$ , there exists a unique element  $x' \in X$  such that  $q \triangleright x' = x$ .
- (2) For any  $q, r \in Q$  and  $x \in X$ ,  $q \triangleright (r \triangleright x) = (q \triangleright r) \triangleright (q \triangleright x)$ .

For any  $q \in Q$ , we define  $s_q : X \rightarrow X$  by  $s_q(x) = q \triangleright x$ . We write  $q \triangleright^{-1} x$  for  $s_q^{-1}(x)$ .

Let  $\mathrm{Op}(Q, X)$  be the group generated by  $\{s_q \mid q \in Q\}$ . Let  $\mathrm{Tr}_k(Q, X)$  and  $\mathrm{Tr}(Q, X)$  be defined as in the previous definition.

We can prove the following by the same arguments as in the proof of the previous proposition.

**Proposition 2.13.**  $\mathrm{Tr}(Q, X)$  is a normal subgroup of  $\mathrm{Op}(Q, X)$ .

*Remark 2.14.* (1) There is a natural action of  $Q$  on itself, for which  $\text{Op}(Q, Q) = \text{Inn}(Q)$  and  $\text{Tr}(Q, Q) = \text{Tr}(Q)$  hold.

(2) Let  $\text{As}(Q)$  be the group generated by  $\{g_q \mid q \in Q\}$  subject to the relations  $g_q g_r = g_{q \triangleright r} g_q$  for all  $q, r \in Q$ .

Then an action of  $Q$  on  $X$  is equivalent to a group action of  $\text{As}(Q)$  on  $X$ , and there is a natural surjective homomorphism  $\text{As}(Q) \rightarrow \text{Op}(Q, X)$ . Let  $\text{As}(Q)_0$  be the subgroup of  $\text{As}(Q)$  defined by

$$\text{As}(Q)_0 := \left\{ g_{q_1}^{e_1} \cdots g_{q_k}^{e_k} \mid \sum_{i=1}^k e_i = 0 \right\},$$

then the above homomorphism restricts to a surjective homomorphism  $\text{As}(Q)_0 \rightarrow \text{Tr}(Q, X)$ .

(3) Note that an action of  $Q$  on  $X$  does not necessarily induce a homomorphism  $\text{Inn}(Q) \rightarrow \text{Op}(Q, X)$ . In other words, a  $Q$ -action on  $X$  does not necessarily factor through  $\text{Inn}(Q)$ . For example, let  $Q = \mathbb{A}^1$  be endowed with the trivial operation  $q \triangleright r = r$ . Then  $Q$  acts on  $X = \mathbb{A}^1$  by  $q \triangleright x = x + 1$ . This action does not factor through  $\text{Inn}(Q) = \{id_Q\}$ .

(4) Similarly, if we make the trivial quandle  $Q = \mathbb{A}^1$  act on  $X = \mathbb{A}^1$  by  $q \triangleright x = x + q$ , then  $s_r^{-1} s_q(x) = x + q - r$  and so the action of  $\text{Tr}(Q, X)$  does not factor through  $\text{Tr}(Q) = \{id_Q\}$ .

**Proposition 2.15.** *The equality  $\text{Inn}(Q)q = \text{Tr}(Q)q$  holds for any  $q \in Q$ .*

*Proof.* It is clear that  $\text{Tr}(Q)q$  is contained in  $\text{Inn}(Q)q$ . For the other direction, write an element  $q'$  of  $\text{Inn}(Q)q$  as  $s_{a_k}^{\sigma_k} \circ \cdots \circ s_{a_1}^{\sigma_1}(q)$ . Then we have

$$q' = s_{q'}^{-\sum \sigma_i} \circ s_{a_k}^{\sigma_k} \circ \cdots \circ s_{a_1}^{\sigma_1}(q) \in \text{Tr}(Q)q. \quad \square$$

*Remark 2.16.* For an action of  $Q$  on a set  $X$ , the  $\text{Op}(Q, X)$ -orbits are not necessarily the same as the  $\text{Tr}(Q, X)$ -orbits. In the example of Remark 2.14(3), we have  $\text{Op}(Q, X)0 = \mathbb{Z}$  while  $\text{Tr}(Q, X)0 = \{0\}$ .

The study of  $\text{Tr}(Q, X)$ -orbits plays an important role in the proof of our main theorem. Let us introduce the following notation.

**Definition 2.17.** Let  $Q$  be a quandle acting on a set  $X$  and let  $Z$  be a subset of  $X$ .

For  $i \in \mathbb{N}$ , we define  $Q^i Z$  inductively by  $Q^0 Z = Z$  and  $Q^{i+1} Z = Q \triangleright Q^i Z$ .

Similarly, define  $Q^{-i} Z$  inductively by  $Q^0 Z = Z$  and  $Q^{-(i+1)} Z = Q \triangleright^{-1} Q^{-i} Z$ .

Define  $Q^{\pm i} Z$  by  $Q^{\pm 0} Z = Z$  and  $Q^{\pm(i+1)} Z = (Q \triangleright Q^{\pm i} Z) \cup (Q \triangleright^{-1} Q^{\pm i} Z)$ .

**Proposition 2.18.**

- (1)  $\text{Op}(Q, X)Z = \bigcup_{i \in \mathbb{N}} Q^{\pm i} Z$ .
- (2)  $\text{Tr}_i(Q, X) \subseteq \text{Tr}_{i+1}(Q, X)$ , hence  $\text{Tr}_i(Q, X)Z \subseteq \text{Tr}_{i+1}(Q, X)Z$ , for any  $i \in \mathbb{N}$ .
- (3) For the natural action of  $Q$  on  $X = Q$ , there are inclusions  $Q^i Z \subseteq Q^{i+1} Z$ ,  $Q^{-i} Z \subseteq Q^{-(i+1)} Z$  and  $Q^{\pm i} Z \subseteq Q^{\pm(i+1)} Z$ .
- (4) If  $f$  is an element of  $\text{Op}(Q, X)$ , then  $Q^i f(Z) = f(Q^i Z)$ ,  $Q^{-i} f(Z) = f(Q^{-i} Z)$  and  $\text{Tr}_i(Q) f(Z) = f(\text{Tr}_i(Q) Z)$  hold.
- (5) If  $Z \subseteq X$  is stable under the action of  $Q$ , i.e.,  $Q \triangleright Z \subseteq Z$  and  $Q \triangleright^{-1} Z \subseteq Z$ , then  $q \triangleright Z = q \triangleright^{-1} Z = Z$  holds for any  $q \in Q$ .

*Proof.* (1) is obvious from the definitions.

(2) Take an element  $q$  of  $Q$ , and then

$$\mathrm{Tr}_{i+1}(Q, X) \supseteq s_q s_q^{-1} \mathrm{Tr}_i(Q, X) = \mathrm{Tr}_i(Q, X)$$

holds by Lemma 2.10.

(3) For any  $q \in Q^i Z$ , we have  $q = q \triangleright q \in q \triangleright Q^i Z \subseteq Q^{i+1} Z$ , hence  $Q^i Z \subseteq Q^{i+1} Z$ . The assertions about  $Q^{-i} Z$  and  $Q^{\pm i} Z$  can be proven in the same way.

(4) Choose a lift of  $f$  in  $\mathrm{As}(Q)$  and let  $f_Q$  denote its image in  $\mathrm{Inn}(Q)$ . It follows from the axiom  $q \triangleright (r \triangleright x) = (q \triangleright r) \triangleright (q \triangleright x)$  that  $f(q \triangleright x) = f_Q(q) \triangleright f(x)$  and  $f(q \triangleright^{-1} x) = f_Q(q) \triangleright^{-1} f(x)$ . Since  $f_Q$  is an automorphism of  $Q$ , we have the assertion.

(5) We have  $q \triangleright Z \subseteq Z$  and  $q \triangleright^{-1} Z \subseteq Z$ . Applying  $s_q^{-1}$  to the former and  $s_q$  to the latter, we obtain the assertion.  $\square$

In this paper, we are mainly interested in the situation where the automorphism group is big enough.

**Definition 2.19.** A quandle  $Q$  is called homogeneous if the action of  $\mathrm{Aut}_{\triangleright}(Q)$  on  $Q$  is transitive.

A quandle  $Q$  is called  $\triangleright$ -connected if the action of  $\mathrm{Inn}(Q)$  on  $Q$  is transitive. By Proposition 2.15, this is equivalent to saying that the action of  $\mathrm{Tr}(Q)$  on  $Q$  is transitive.

For a discrete quandle  $Q$ , the homogeneity of  $Q$  implies that the quandle can be described in terms of a group automorphism as in Example 2.7.

**Proposition 2.20.** Let  $Q$  be a quandle and let  $G$  be  $\mathrm{Aut}_{\triangleright}(Q)$ ,  $\mathrm{Inn}(Q)$ , or  $\mathrm{Tr}(Q)$ .

- (1) For any element  $q$  of  $Q$ , let  $\varphi_q$  denote the map  $G \rightarrow G; g \mapsto s_q g s_q^{-1}$  and  $G_q$  the stabilizer of  $q$  in  $G$ . Then the natural map

$$\pi_q : G/G_q \rightarrow Gq; \quad gG_q \mapsto gq$$

is an isomorphism of quandles  $(G/G_q, \triangleright_{\varphi_q})$  and  $Gq$ .

In particular,

- If  $Q$  is homogeneous, then it is isomorphic to  $(\mathrm{Aut}_{\triangleright}(Q)/\mathrm{Aut}_{\triangleright}(Q)_q, \triangleright_{\varphi_q})$ .
  - If  $Q$  is  $\triangleright$ -connected, then it is isomorphic to  $(\mathrm{Inn}(Q)/\mathrm{Inn}(Q)_q, \triangleright_{\varphi_q})$  and  $(\mathrm{Tr}(Q)/\mathrm{Tr}(Q)_q, \triangleright_{\varphi_q})$ .
- (2) For any two elements  $q$  and  $r$  of  $Q$ , let  $\psi_{q,r}$  denote the map  $G \rightarrow G; g \mapsto s_q g s_r^{-1}$ . Then there is a well-defined action

$$\triangleright_{\psi_{q,r}} : G/G_q \times G/G_r \rightarrow G/G_r; \quad (gG_q, hG_r) \mapsto g\psi_{q,r}(g^{-1}h)G_r$$

compatible with the action of  $Gq$  on  $Gr$ .

*Proof.* These statements follow from the arguments of the proofs of Theorem 7.1 and Theorem 7.2 of [J82].

First of all, note that  $G$  is stable under the map  $g \mapsto s_q g s_r^{-1}$ . This is obvious if  $G$  is  $\mathrm{Aut}_{\triangleright}(Q)$  or  $\mathrm{Inn}(Q)$ . If  $G$  is  $\mathrm{Tr}(Q)$ , then its element  $g$  can be written as  $s_{a_1} \cdots s_{a_k} s_{b_1}^{-1} \cdots s_{b_k}^{-1}$ , and so is  $s_q g s_r^{-1}$ .



(1) The natural map  $G/G_q \rightarrow Gq$  is bijective, and the assertion that it is a homomorphism is a special case of (2), which we prove below. For the second assertion, note that  $\text{Tr}(Q)q$  is equal to  $\text{Inn}(Q)q$  by Proposition 2.15.

(2) From  $gs_q = s_{g(q)}g$ , we see that any element of  $G_q$  commutes with  $s_q$ . Similarly, any element of  $G_r$  commutes with  $s_r^{-1}$ . Thus, if  $g' = gg_1$  and  $h' = hh_1$  with  $g_1 \in G_q$  and  $h_1 \in G_r$ , we have

$$\begin{aligned} g'\psi_{q,r}((g')^{-1}h')G_r &= gg_1s_qs_g^{-1}g^{-1}hh_1s_r^{-1}G_r \\ &= gs_qg_1g_1^{-1}g^{-1}hs_r^{-1}h_1G_r \\ &= gs_qg^{-1}hs_r^{-1}G_r \\ &= g\psi_{q,r}(g^{-1}h)G_r, \end{aligned}$$

so the action is well-defined.

We have

$$\begin{aligned} \pi_r(gG_q \triangleright_{\psi_{q,r}} hG_r) &= g\psi_{q,r}(g^{-1}h)(r) \\ &= gs_qg^{-1}hs_r^{-1}(r) \\ &= s_{g(q)}gg^{-1}h(r) = s_{g(q)}h(r) = g(q) \triangleright h(r) \\ &= \pi_q(gG_q) \triangleright \pi_r(hG_r), \end{aligned}$$

so the actions are compatible.  $\square$

*Remark 2.21.* Let  $G$  be  $\text{Aut}_{\triangleright}(Q)$ ,  $\text{Inn}(Q)$ , or  $\text{Tr}(Q)$ . For any  $q \in Q$ , the map

$$\bar{s} : Q \rightarrow (G, \triangleright'_{\varphi_q}); x \mapsto s_x s_q^{-1}$$

is easily seen to be a quandle homomorphism.

If  $Q$  is homogeneous with  $G = \text{Aut}_{\triangleright}(Q)$  or  $Q$  is  $\triangleright$ -connected with  $G = \text{Inn}(Q)$  or  $\text{Tr}(Q)$ , this map is the same as the composite map

$$Q \cong (G/G_q, \triangleright_{\varphi_q}) \twoheadrightarrow (G/G^{\varphi_q}, \triangleright_{\varphi_q}) \hookrightarrow (G, \triangleright'_{\varphi_q}),$$

where the last map is the injective homomorphism in Example 2.8.

### 3. Quandle varieties and orbits of actions

**Definition 3.1.** Let  $k$  be an algebraically closed field of an arbitrary characteristic.

A quandle variety over  $k$ , or an algebraic quandle, is an algebraic (reduced and irreducible) variety  $Q$  over  $k$  equipped with a quandle operation  $\triangleright$  such that  $Q \times Q \rightarrow Q \times Q; (q, r) \mapsto (q, q \triangleright r)$  is an automorphism of a variety.

An action of a quandle variety  $Q$  on an algebraic variety  $X$  is called an algebraic action if  $Q \times X \rightarrow Q \times X; (q, x) \mapsto (q, q \triangleright x)$  is an automorphism of a variety. A variety with an action of  $Q$  will be called a  $Q$ -variety.

In the sequel, actions will be always algebraic when we are dealing with quandle varieties and algebraic varieties.

**Example 3.2.** (1) In Example 2.5, if  $X$  is an algebraic variety,  $A$  is a connected commutative algebraic group, and  $F$  is a regular map, then  $X \times A$  is a quandle variety.

(2) If  $G$  is a connected algebraic group in Example 2.6, then the variety  $G$  with the conjugation operation is a quandle variety.

(3) In Example 2.7, we have a quandle variety  $(G/H, \triangleright_\varphi)$  if  $G$  is a connected algebraic group,  $\varphi$  is an algebraic automorphism, and  $H$  is a closed subgroup.

(4) In Example 2.8, we have a quandle variety  $(G, \triangleright'_\varphi)$  if  $G$  is a connected algebraic group and  $\varphi$  is an algebraic automorphism. The quandle  $(G/G^\varphi, \triangleright_\varphi)$  from the previous example can be embedded into this quandle.

**Definition 3.3.** Let  $G$  be an algebraic group,  $\varphi$  an automorphism of  $G$ , and  $H$  a subgroup of  $G$  satisfying  $H \subseteq G^\varphi$ . Then  $(G/H, \triangleright_\varphi)$ , or the triple  $(G, H, \varphi)$ , is called a weak algebraic  $\varphi$ -space.

If  $H$  contains the connected component  $(G^\varphi)^\circ$  of  $G^\varphi$  containing the identity element, then  $G/H$  is called an algebraic  $\varphi$ -space.

We say that  $G/H$  is a regular algebraic  $\varphi$ -space if  $d_{\bar{e}}\bar{\varphi} - \text{Id}$  is invertible, where  $\bar{e}$  is the residue class of the identity element and  $\bar{\varphi} : G/H \rightarrow G/H$  is the induced automorphism.

The group  $\text{Tr}(Q)$  plays an important role in the theory of generalized symmetric spaces. A key fact is that it is a connected Lie group. Now we will show a partial analogue of this fact for quandle varieties. First, we prove that orbits of  $\text{Tr}(Q)$ -actions enjoy a number of properties similar to those satisfied by orbits of algebraic group actions.

Let  $Q$  be a quandle variety acting on a variety  $X$ . Recall from Proposition 2.18(2) that  $\{\text{Tr}_i(Q, X)Z\}$  is an increasing sequence for any subset  $Z$  of  $X$ .

**Lemma 3.4.**

- (1) Let  $Z$  be an irreducible subset of  $X$ . If  $i_0$  is a natural number such that  $\dim \text{Tr}_{i_0}(Q, X)Z = \dim \text{Tr}_{i_0+1}(Q, X)Z$ , then the closure  $\overline{\text{Tr}_i(Q, X)Z}$  of  $\text{Tr}_i(Q, X)Z$  in  $X$  is equal to  $\overline{\text{Tr}(Q, X)Z}$  for  $i \geq i_0$ .  
In particular,  $\overline{\text{Tr}_i(Q, X)Z}$  is equal to  $\overline{\text{Tr}(Q, X)Z}$  for  $i \geq \dim X - \dim Z$ .
- (2) Let  $Z$  be a constructible subset of  $X$ , i.e., a finite union of locally closed subsets of  $X$ . Then  $\overline{\text{Tr}_i(Q, X)Z}$  is equal to  $\overline{\text{Tr}(Q, X)Z}$  for  $i \geq \dim X$ .

*Proof.* (1) Since  $\{\overline{\text{Tr}_i(Q, X)Z}\}$  is an increasing sequence of irreducible closed sets, the equality of dimensions implies  $\overline{\text{Tr}_{i_0}(Q, X)Z} = \overline{\text{Tr}_{i_0+1}(Q, X)Z}$ . Let  $W$  denote this set. It is obviously contained in  $\overline{\text{Tr}(Q, X)Z}$ .

Since

$$s_q^{-1}s_r W \subseteq \overline{s_q^{-1}s_r \text{Tr}_{i_0}(Q, X)Z} \subseteq \overline{\text{Tr}_{i_0+1}(Q, X)Z} = W$$

holds for any  $q, r \in Q$ , the set  $W$  is stable under the action of  $\text{Tr}(Q, X)$ , hence contains  $\text{Tr}(Q, X)Z$ . Since it is closed, it contains  $\overline{\text{Tr}(Q, X)Z}$ .

(2) This is an immediate consequence of (1).  $\square$

**Proposition 3.5.** Let  $z_0$  be a point of  $X$ . Then  $\text{Tr}(Q, X)z_0$  is a locally closed subvariety of  $X$ .

If  $i_0$  is a natural number such that  $\dim \mathrm{Tr}_{i_0}(Q, X)z_0 = \dim \mathrm{Tr}_{i_0+1}(Q, X)z_0$ , then  $\mathrm{Tr}_i(Q, X)z_0 = \mathrm{Tr}(Q, X)z_0$  holds for  $i \geq 2i_0$ .

In particular,  $\mathrm{Tr}_i(Q, X)z_0 = \mathrm{Tr}(Q, X)z_0$  holds for  $i \geq 2 \dim X$ .

*Proof.* Let  $W = \overline{\mathrm{Tr}(Q, X)z_0}$ . For a subset  $V$  of  $W$ , let  $V^\circ$  denote the relative interior of  $V$  in  $W$ .

By the previous lemma and Chevalley's theorem on the image of a morphism,  $\mathrm{Tr}_{i_0}(Q, X)z_0$  contains a nonempty open subset of  $W$ . In general, if a group acts on a topological space by homeomorphisms and an orbit contains a nonempty open subset, then this orbit is easily seen to be open. Thus  $\mathrm{Tr}(Q, X)z_0$  is open in  $W$ , hence locally closed in  $X$ .

Let  $z_1$  be an element of  $\mathrm{Tr}(Q, X)z_0$  and write  $z_1$  as  $f(z_0)$  with  $f \in \mathrm{Tr}(Q, X)$ . By Proposition 2.18 (4),  $\mathrm{Tr}_{i_0}(Q, X)z_1$  is equal to  $f(\mathrm{Tr}_{i_0}(Q, X)z_0)$ , which is dense in  $W$ . It follows that  $\mathrm{Tr}_{i_0}(Q, X)z_0$  and  $\mathrm{Tr}_{i_0}(Q, X)z_1$  have nonempty intersection and therefore that  $z_1$  is contained in  $\mathrm{Tr}_{2i_0}(Q, X)z_0$ .  $\square$

Thus, we see that  $X$  is divided into locally closed  $\mathrm{Tr}(Q, X)$ -orbits.

Recall that an open subgroup of a connected algebraic group is the whole group. As an analogy, for an open subquandle  $U$  of  $Q$ , it is hoped that  $\mathrm{Inn}(U) \cong \mathrm{Inn}(Q)$  and  $\mathrm{Tr}(U) \cong \mathrm{Tr}(Q)$ . The main theorem asserts that this is true if  $Q$  has an open orbit. In general, the following holds.

**Proposition 3.6.** *Let  $Q$  be an arbitrary quandle variety and  $U$  an open subquandle. Then there are natural injective homomorphisms  $\mathrm{Inn}(U) \rightarrow \mathrm{Inn}(Q)$  and  $\mathrm{Tr}(U) \rightarrow \mathrm{Tr}(Q)$ .*

*Proof.* If we write an element of  $\mathrm{Inn}(U)$  as a product of  $s_q^{\pm 1}$ 's, then the same expression defines an element of  $\mathrm{Inn}(Q)$ , and this gives a well-defined homomorphism  $\mathrm{Inn}(U) \rightarrow \mathrm{Inn}(Q)$  since  $U$  is dense in  $Q$ . It is injective since  $U \subseteq Q$ . By restriction, we obtain an injective homomorphism  $\mathrm{Tr}(U) \rightarrow \mathrm{Tr}(Q)$ .  $\square$

Although we do not know if these homomorphisms are surjective in general, the following proposition shows that at least they give rise to the same orbits.

**Proposition 3.7.** *Let  $Q$  be a quandle variety,  $U$  an open subquandle and  $X$  a  $Q$ -variety. Then  $\mathrm{Op}(U, X)z_0 = \mathrm{Op}(Q, X)z_0$  and  $\mathrm{Tr}(U, X)z_0 = \mathrm{Tr}(Q, X)z_0$  hold, where  $z_0 \in X$ .*

*Proof.* Write  $G$  for  $\mathrm{Tr}(Q, X)$  and  $H$  for  $\mathrm{Tr}(U, X)$ . Since  $U$  is dense in  $Q$ , the set  $\mathrm{Tr}_i(U, X)z_0$  is dense in  $\mathrm{Tr}_i(Q, X)z_0$ . For  $i \gg 0$ , they are equal to  $H z_0$  and  $G z_0$ , respectively, and are locally closed in  $X$  by Proposition 3.5. So  $H z_0$  is open in  $G z_0$ . If we take a point  $z$  of  $G z_0$ , then  $H z$  is open in  $G z$  by the same arguments, and the latter is equal to  $G z_0$ . Thus  $H z_0$  and  $H z$  intersect, and  $z$  is contained in  $H z_0$ .

Choose a point  $q \in U$ , then  $\mathrm{Op}(Q, X)$  is equal to  $\bigcup_{n \in \mathbb{Z}} \mathrm{Tr}(Q, X)s_q^n$ . Thus we have  $\mathrm{Op}(U, X)z_0 = \mathrm{Op}(Q, X)z_0$ .  $\square$

**Corollary 3.8.** *A  $\triangleright$ -connected quandle variety  $Q$  contains no nontrivial open subquandle.*

*Proof.* Let  $U$  be an open subquandle. For any  $q \in U$ , we have  $U \supseteq \text{Inn}(U)q = \text{Inn}(Q)q = Q$  by the proposition.  $\square$

For the natural  $Q$ -action on  $Q$  itself, the  $\text{Inn}(Q)$ -orbits are equal to the  $\text{Tr}(Q)$ -orbits by Proposition 2.15. They are also equal to the “forward” orbits in the case of quandle varieties.

**Lemma 3.9.**

- (1) Let  $Z$  be an irreducible subset of  $Q$ . If  $i_0$  is a natural number such that  $\dim Q^{i_0}Z = \dim Q^{i_0+1}Z$ , then  $\overline{Q^iZ}$  is equal to  $\overline{\text{Inn}(Q)Z}$  for  $i \geq i_0$ .  
In particular,  $\overline{Q^iZ}$  is equal to  $\overline{\text{Inn}(Q)Z}$  for  $i \geq \dim Q - \dim Z$ .
- (2) Let  $Z$  be a constructible subset of  $Q$ . Then  $\overline{Q^iZ}$  is equal to  $\overline{\text{Inn}(Q)Z}$  for  $i \geq \dim Q$ .

By symmetry of  $\triangleright$  and  $\triangleright^{-1}$ , similar statements hold for  $\overline{Q^{-i}Z}$ .

*Proof.* (1) Using Proposition 2.18 (3) this time, the equality of dimensions implies that  $\overline{Q^iZ}$  stabilizes to an irreducible closed set  $W$  for  $i \geq i_0$ . Obviously we have  $W \subseteq \overline{\text{Inn}(Q)Z}$ .

For any  $q \in W$ , we have

$$q \triangleright W = q \triangleright \overline{Q^{i_0}Z} \subseteq \overline{q \triangleright Q^{i_0}Z} \subseteq \overline{Q^{i_0+1}Z} = W.$$

Since  $q \triangleright W$  is an irreducible closed set with  $\dim(q \triangleright W) = \dim W$ , the equality  $q \triangleright W = W$  holds. Applying  $s_q^{-1}$ , we also have  $W = q \triangleright^{-1} W$ . Thus the set  $W$  is stable under the action of  $Q$ , hence contains  $\text{Inn}(Q)Z$ . Since it is closed, it contains  $\overline{\text{Inn}(Q)Z}$ .

(2) is immediate from (1).  $\square$

**Proposition 3.10.** Let  $z_0$  be a point of  $Q$ . Then  $\text{Inn}(Q)z_0$  is a locally closed subvariety of  $Q$ .

If  $i_1$  and  $i_2$  are natural numbers for which  $\dim Q^{i_1}z_0 = \dim Q^{i_1+1}z_0$  and  $\dim Q^{-i_2}z_0 = \dim Q^{-(i_2+1)}z_0$  hold, then  $Q^{i_1}z_0 = Q^{-i_2}z_0 = \text{Inn}(Q)z_0$  holds for  $i \geq i_1 + i_2$ .

In particular,  $Q^i z_0 = Q^{-i} z_0 = \text{Inn}(Q)z_0$  holds for  $i \geq 2 \dim Q$ .

*Proof.* We write  $W = \overline{\text{Inn}(Q)z_0}$  and denote the relative interior of  $V \subseteq W$  by  $V^\circ$ .

By the previous lemma and Chevalley’s theorem,  $Q^{i_1}z_0$  contains a nonempty open subset of  $W$ . Therefore  $\text{Inn}(Q)z_0$  is open in  $W$ , hence locally closed in  $Q$ .

If  $z_1 = f(z_0)$  with  $f \in \text{Inn}(Q)$ , then  $Q^{-i_2}z_1$  is equal to  $f(Q^{-i_2}z_0)$  by Proposition 2.18 (4), so it is dense in  $W$ . It follows that  $Q^{i_1}z_0$  and  $Q^{-i_2}z_1$  have nonempty intersection and therefore that  $z_1$  is contained in  $Q^{i_1+i_2}z_0$ .  $\square$

*Remark 3.11.* For an action on an arbitrary  $X$ , the example in Remark 2.16 shows that  $\text{Op}(Q, X)z_0$  is not necessarily constructible.

*Remark 3.12.* For an arbitrary irreducible closed subset  $Z$  of  $Q$ , it is not necessarily true that  $\text{Inn}(Q)Z$  is locally closed. For example, let  $Q = \text{SL}_2(k) \times \mathbb{A}^1$  be endowed with the operation  $(h, t) \triangleright (g, s) = (h^{-1}gh, s)$ . Let  $Z = \left\{ \left( \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, s \right) \mid s \in \mathbb{A}^1 \right\}$ .

Then the fiber of  $\text{Inn}(Q)Z$  over  $s \in \mathbb{A}^1$  is  $\{A \in \text{SL}_2(k) \mid (A - I)^2 = O, A \neq I\}$  if  $s \neq 0$  and is  $\{I\}$  if  $s = 0$ . It follows that  $\overline{\text{Inn}(Q)Z} = \{A \in \text{SL}_2(k) \mid (A - I)^2 = O\} \times \mathbb{A}^1$  and that  $\text{Inn}(Q)Z$  is not open in its closure.

In showing  $\triangleright$ -connectedness of a quandle, the following proposition is useful.

**Proposition 3.13.** *For a quandle variety, the following conditions are equivalent.*

- (1)  $Q$  is  $\triangleright$ -connected.
- (2)  $Q^i q = Q$  for any  $q \in Q$  and  $i \geq 2 \dim Q$ .
- (2')  $Q^i q = Q$  for some  $q \in Q$  and  $i \geq 2 \dim Q$ .
- (3)  $Q^i q = Q$  for any  $q \in Q$  and  $i \gg 0$ .
- (3')  $Q^i q = Q$  for some  $q \in Q$  and  $i \gg 0$ .
- (4)  $\overline{Q^i q} = Q$  for any  $q \in Q$  and  $i \geq \dim Q$ .
- (5)  $\overline{Q^i q} = Q$  for any  $q \in Q$  and  $i \gg 0$ .

*Proof.* Assume the condition (1). Then (2) and (4) follow from the previous proposition and lemma.

The implications from (2) to (2')–(3'), from (2)–(3') to (1), and from (4) to (5) are obvious.

Assume that (5) holds. Then for any  $q, q' \in Q$ , there exists  $i$  such that  $Q^i q$  and  $Q^i q'$  are both dense. By Chevalley's lemma, they contain nonempty open subsets, so they have nonempty intersection. Thus  $q$  and  $q'$  can be connected by an inner automorphism.  $\square$

**Corollary 3.14.** *If  $Q$  is nonsingular and  $d_q s_q - \text{Id}$  is invertible for any  $q \in Q$ , then  $Q$  is  $\triangleright$ -connected.*

*Proof.* Let  $t_q : Q \rightarrow Q$  be defined by  $t_q(r) = r \triangleright q$ . From  $r \triangleright r = r$ , we see that  $d_q t_q + d_q s_q = \text{Id}$ . Thus the assumption means that  $d_q t_q$  is invertible. It follows that  $t_q$  is dominant, and  $Q$  is  $\triangleright$ -connected by the proposition.  $\square$

*Remark 3.15.* The converse does not hold. Consider the quandle

$$Q = \left\{ A^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} A \mid A \in \text{SL}_2(k) \right\}$$

with the conjugation operation.

We show that  $Q$  is  $\triangleright$ -connected. Note that an element of  $Q$  can be written as  $M = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$  with  $xw - yz = 1$ ,  $x + w = 2$ , and  $M \neq I_2$ . Let  $J_t$  denote the matrix  $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ . Then we have  $M \triangleright J_1 = \begin{pmatrix} 1 + zw & w^2 \\ -z^2 & 1 - zw \end{pmatrix}$ . Since  $z$  and  $w$  can be considered as algebraically independent functions, we see that  $Q \triangleright J_1$  is dense. Using the transitive action of  $\text{SL}_2(k)$ , we see that  $Q \triangleright M$  is dense for any  $M \in Q$ . Now Proposition 3.13 tells us that  $Q$  is  $\triangleright$ -connected.

Alternatively, note that the  $\text{Inn}(Q)$ -orbits are equal to the orbits by the conjugacy action of the group  $G$  generated by  $Q$ . We see that  $Q \cdot \{J_t \mid t \neq 0\}$ , and therefore  $G$ , contains a nonempty open subset of  $\text{SL}_2(k)$  by an argument similar to the above. Such a subgroup must be equal to  $\text{SL}_2(k)$ , hence the orbit of  $J_1$  is equal to  $Q$ .

On the other hand, the fixed point locus of  $s_{J_1}$  is  $\{J_t \mid t \neq 0\}$ . Again using the  $\mathrm{SL}_2(k)$ -action, we see that the fixed point locus of  $s_M$  is a 1-dimensional set containing  $M$  for any  $M \in Q$ . In particular,  $d_M s_M - \mathrm{Id}$  is singular.

**Corollary 3.16.** *If an orbit  $U := \mathrm{Inn}(Q)q_0$  is dense in  $Q$ , then it is open and is a  $\triangleright$ -connected quandle.*

*Proof.* Openness follows from Proposition 3.10. For any  $q \in U$ , we have  $Q^i q = U$  for some  $i$  by the same proposition. Hence  $U^i q$  is dense in  $U$ , and  $U$  is  $\triangleright$ -connected by the previous proposition.  $\square$

*Remark 3.17.* In general, an  $\mathrm{Inn}(Q)$ -orbit is not necessarily  $\triangleright$ -connected. In Example 2.5, let  $X = A = \mathbb{A}^1$  and  $F(x, y) = y - x$ . Then  $Q = \mathbb{A}^2$  with  $(x, a) \triangleright (y, b) = (y, b + y - x)$ , and the orbit of  $(0, 0)$  is  $\{0\} \times \mathbb{A}^1$ , which is a trivial quandle.

4. Main Theorem

In this section, we work over an algebraically closed field  $k$  of characteristic 0. The word “variety” will always mean an irreducible and reduced variety.

We denote the category of algebraic varieties over  $k$  by  $(\mathrm{Var}_k)$  and the category of groups by  $(\mathrm{Grp})$ . For a variety  $S$ , denote by  $\mathrm{Aut}_S(S \times Q)$  the group of automorphisms of  $S \times Q$ , as a variety, commuting with the first projection. For a quandle variety  $Q$ , let  $\mathcal{Tr}(Q) : (\mathrm{Var}_k) \rightarrow (\mathrm{Grp})$  be the functor defined by

$$\mathcal{Tr}(Q)(S) := \{f \in \mathrm{Aut}_S(S \times Q) \mid f|_{\{s\} \times Q} \in \mathrm{Tr}(Q) \text{ for any closed point } s \text{ of } S\}.$$

The following is our main theorem.

**Theorem 4.1.** *Let  $Q$  be a quandle variety which is irreducible and normal as a variety. Assume that  $Q$  contains an open  $\mathrm{Inn}(Q)$ -orbit  $O$ .*

- (1) *The group  $\mathrm{Tr}(Q)$  can be given a structure of a connected algebraic group  $\mathbf{Tr}(Q)$  such that the action map  $\alpha : \mathbf{Tr}(Q) \times Q \rightarrow \mathbf{Tr}(Q) \times Q; (f, q) \mapsto (f, f(q))$  is a regular map. If  $Q$  is quasi-affine, then  $\mathbf{Tr}(Q)$  is affine. The pair  $(\mathbf{Tr}(Q), \alpha)$  represents the functor  $\mathcal{Tr}(Q)$ .*
- (2) *The open orbit  $O$  is  $\triangleright$ -connected, and there are natural isomorphisms  $\mathrm{Tr}(O) \cong \mathrm{Tr}(Q)$  and  $\mathrm{Inn}(O) \cong \mathrm{Inn}(Q)$ .*
- (3) *For any two elements  $q$  and  $r$  of  $Q$ , let*

$$\varphi_q : \mathbf{Tr}(Q) \rightarrow \mathbf{Tr}(Q); g \mapsto s_q g s_q^{-1}$$

and

$$\psi_{q,r} : \mathbf{Tr}(Q) \rightarrow \mathbf{Tr}(Q); g \mapsto s_q g s_r^{-1}$$

*be the maps from Proposition 2.20. Then  $\varphi_q$  is an automorphism of an algebraic group and  $\psi_{q,r}$  is an automorphism of an algebraic variety. The natural map*

$$\begin{aligned} \pi_q : \mathbf{Tr}(Q)/\mathbf{Tr}(Q)_q &\rightarrow \mathrm{Inn}(Q)q, \\ g\mathbf{Tr}(Q)_q &\mapsto gq \end{aligned}$$

of Proposition 2.20 is an isomorphism of quandle varieties compatible with actions between orbits.

Before starting the proof of the theorem, let us recall how the corresponding fact in differential geometry was proven (see Theorem 1.3). To prove that a regular  $s$ -manifold  $M$  is a regular  $\varphi$ -space, one first constructs a canonical affine connection which turns  $M$  into a generalized affine symmetric space. This rigidifies the manifold  $M$ , in the sense that a diffeomorphism which preserves the affine connection is determined by the image of a point and the tangent map there. So  $\text{Tr}(M)$  can be realized as a subset of a finite-dimensional space. Then, it can be proven that  $\text{Tr}(M)$  is a Lie group and that  $M$  is a homogeneous space.

Accordingly, our proof is divided into two parts. First, we find a certain structure which rigidifies the space and embed  $\text{Tr}(Q)$  into an algebraic variety. Second, we show that the image of this embedding is an algebraic group satisfying the conditions.

In view of Proposition 2.20, one might hope to extend our result to quandle varieties which are homogeneous in the sense that the action of  $\text{Aut}_{\triangleright}(Q)$  is transitive. However, it seems difficult to find a good algebraic subgroup of  $\text{Aut}_{\triangleright}(Q)$  without any assumption on the action of  $\text{Inn}(Q)$ . For example, consider  $Q = \mathbb{A}^n$  with the trivial quandle operation. Then the group  $\text{Aut}_{\triangleright}(Q)$  is equal to  $\text{Aut}(\mathbb{A}^n)$ , which is a very complicated group.

Our plan is to embed  $\text{Tr}(Q)$  into a variety  $X$  with a “good”  $Q$ -action. Specifically, we consider the following condition on  $X$ .

**Condition 4.2.**

- (1) There exists a natural transformation  $\Phi$  from  $\mathcal{T}r(Q)$  to

$$\mathcal{A}ut(X) : (\text{Var}_k) \rightarrow (\text{Grp}); S \mapsto \mathcal{A}ut_S(S \times X)$$

such that  $\Phi(pt)$  commutes with the natural surjective homomorphisms

$$\mathcal{A}S(Q)_0 \rightarrow \text{Tr}(Q) \quad \text{and} \quad \mathcal{A}S(Q)_0 \rightarrow \text{Tr}(Q, X).$$

- (2) There exists a point  $x \in X$  such that the stabilizer  $\text{Tr}(Q)_x$  of  $x$  in  $\text{Tr}(Q)$  is trivial.

*Remark 4.3.* By condition (1), we have a natural homomorphism  $\Phi(pt) : \text{Tr}(Q) \rightarrow \text{Tr}(Q, X)$ . It is an isomorphism by condition (2).

The natural transformation  $\Phi$  is unique if it exists. In fact, let  $f$  be a family of transvections over  $S$ . Then  $\Phi(S)(f)$  must be  $(s, x) \mapsto (s, (\Phi(pt)(f_s))(x))$ .

The theorem can be reduced to the following propositions.

**Proposition 4.4.** *Let  $Q$  be a quandle variety and assume that  $Q$  contains an open  $\text{Inn}(Q)$ -orbit  $O$ . Let  $n$  be a positive integer and  $X = \text{Hom}_{O^n}(k^{\dim Q} \times O^n, TO^n) \times O$ , where  $\text{Hom}$  denotes the vector bundle of homomorphisms over  $O^n$ . Then  $X$  satisfies Condition 4.2 for  $n \gg 0$ . More precisely, the following hold.*

- (1) *For any  $n$ , there is a natural transformation from  $\mathcal{A}ut_{\triangleright}(Q)$  to  $\mathcal{A}ut(X)$ . Hence  $\mathcal{A}ut_{\triangleright}(Q)$  acts on  $X$ , and  $Q$  acts algebraically on  $X$  via  $\text{Inn}(Q)$ .*
- (2) *For  $n \gg 0$ , there exists a point  $x \in X$  such that  $\mathcal{A}ut_{\triangleright}(Q)_x = \{\text{id}\}$ .*

**Proposition 4.5.** *Let  $Q$  be a normal quandle variety and  $X$  a  $Q$ -variety satisfying Condition 4.2 with a point  $x$  as in the condition. Write  $\mathbf{Tr}(Q)$  for  $\mathrm{Tr}(Q)x$ .*

- (1) *If  $U$  is an open subquandle, then there are natural isomorphisms  $\mathrm{Tr}(U) \cong \mathrm{Tr}(Q)$  and  $\mathrm{Inn}(U) \cong \mathrm{Inn}(Q)$ .*
- (2) *The action map  $\alpha : \mathbf{Tr}(Q) \times Q \rightarrow \mathbf{Tr}(Q) \times Q$  is an automorphism of a variety.*
- (3) *The pair  $(\mathbf{Tr}(Q), \alpha)$  represents the functor  $\mathcal{T}r(Q)$ .*
- (4) *Under the identification of  $\mathbf{Tr}(Q)$  and  $\mathrm{Tr}(Q)$  by the natural bijection,  $\mathbf{Tr}(Q)$  with the induced multiplication is an algebraic group.*
- (5) *For any  $q, r \in Q$ , the map  $\psi_{q,r} : \mathbf{Tr}(Q) \rightarrow \mathbf{Tr}(Q); g \mapsto s_q g s_r^{-1}$  is an automorphism of  $\mathbf{Tr}(Q)$  as an algebraic variety.*

Note that  $\mathbf{Tr}(Q)$  can be considered as a variety since it is a locally closed subset of  $X$  by Proposition 3.5.

*Proof of Theorem 4.1.* Assertions (1) and (2) are immediate consequences of Proposition 4.4, Proposition 4.5, and Corollary 3.16.

It also follows that  $\mathbf{Tr}(Q)/\mathbf{Tr}(Q)_q$  together with the operation induced by  $\varphi_q$  is a quandle variety. Since the natural map  $\mathbf{Tr}(Q)/\mathbf{Tr}(Q)_q \rightarrow \mathrm{Inn}(Q)q$  is a one-to-one regular map of nonsingular varieties over an algebraically closed field of characteristic 0, it is an isomorphism of varieties. It is an isomorphism of quandles compatible with actions between orbits by Proposition 2.20. Thus (3) holds.  $\square$

Now let us prove the propositions.

*Proof of Proposition 4.4.* (1) Note that  $\mathrm{Hom}_{O^n}(k^{\dim Q} \times O^n, TO^n)$  can be identified with the fiber product of  $\dim Q$  copies of  $TO^n$  over  $O^n$ . Then the first assertion follows from the following functorial constructions.

- (a) A family of quandle automorphisms on  $Q \times S$  induces an automorphism of  $O \times S$  over  $S$ .
- (b) Let  $f$  and  $g$  be automorphisms of  $V \times S$  and  $W \times S$ , respectively, over  $S$ . Then  $f \times_S g$  is an automorphism of  $(V \times S) \times_S (W \times S) \cong (V \times W) \times S$  over  $S$ .
- (c) If  $f$  is an automorphism of  $V \times S$  with  $V$  nonsingular, then  $d_{V \times S/S} f$  is an automorphism of  $TV \times S$  which commutes with  $f$ .
- (d) Generalizing (b), if  $V$  and  $W$  come with morphisms to  $Y$ , and  $f$  and  $g$  are automorphisms of  $V \times S$  and  $W \times S$  over  $S$  commuting with an automorphism  $h$  of  $Y \times S$  over  $S$ , then  $f \times_S g$  restricts to an automorphism of  $(V \times S) \times_{Y \times S} (W \times S) \cong (V \times_Y W) \times S$ .

The second assertion follows from the first assertion. Note that the action of  $Q$  on  $Q$  gives an element of  $\mathcal{A}ut_{\triangleright}(Q)(Q)$ , hence an element of  $\mathcal{A}ut(X)(Q)$ , i.e., an automorphism of  $Q \times X$  over  $Q$ . This gives a family  $\{s_{q,X}\}_{q \in Q}$  of automorphisms of  $X$ . The homomorphism  $\Phi(pt) : \mathcal{A}ut_{\triangleright}(Q) \rightarrow \mathcal{A}ut(X)$  sends  $s_q$  to  $s_{q,X}$ , as we see by mapping a point to  $q$  and using the naturality of  $\Phi$ . Thus we have an algebraic action of  $Q$  on  $X$  which factors through  $\mathcal{A}ut_{\triangleright}(Q)$ , hence through  $\mathrm{Inn}(Q)$ .

(2) Let

$$\mu : Q^n \times Q \rightarrow Q; ((q_1, \dots, q_n), q) \mapsto q_n \triangleright \dots \triangleright q_1 \triangleright q$$



be the multiplication map. Let  $q_0$  be a point of  $O$ . By Proposition 3.10 and the assumption that  $O$  is an open orbit, we see that

$$\mu_1 : Q^n \rightarrow O; \mathbf{q} \mapsto \mu(\mathbf{q}, q_0)$$

is surjective for  $n \gg 0$ . Since we are working over a field of characteristic 0, there exists a point  $\mathbf{q}_0 \in O^n$  such that  $d_{\mathbf{q}_0}\mu_1$  is surjective. Define  $\mu_2$  by

$$\mu_2 : Q \rightarrow Q; q \mapsto \mu(\mathbf{q}_0, q).$$

Let  $q' := \mu(\mathbf{q}_0, q_0) = \mu_1(\mathbf{q}_0) = \mu_2(q_0)$ , and choose a linear map  $\varphi : k^d \rightarrow T_{\mathbf{q}_0}O^n$  such that  $d_{\mathbf{q}_0}\mu_1 \circ \varphi : k^d \rightarrow T_{q'}O$  is an isomorphism. Then  $x := ((\mathbf{q}_0, \varphi), q_0)$  defines a point of  $X$ . Now we claim that  $\text{Aut}_{\triangleright}(Q)_x$  is trivial.

To show the claim, we first associate a vector field  $\tilde{\mathbf{v}}$  on  $O$  to each  $\mathbf{v} \in k^d$ . This is what “rigidifies”  $Q$ . Informally, this is defined by

$$\mu(\mathbf{q}_0 + \epsilon\varphi(\mathbf{v}), q) = \mu(\mathbf{q}_0, q) + \epsilon\tilde{\mathbf{v}}(\mu(\mathbf{q}_0, q)) = \mu_2(q) + \epsilon\tilde{\mathbf{v}}(\mu_2(q)).$$

Since  $\mu_2$  is an automorphism, this defines a vector field on  $O$ . Formally, we associate to any  $\mathbf{v} \in k^d$  the constant vector field  $(\varphi(\mathbf{v}), 0)$  to  $O^n \times O$  along  $O' := \mathbf{q}_0 \times O$ . By mapping it by the homomorphism  $d\mu : T(O^n \times O) \rightarrow \mu^*(TO)$ , we obtain a section of  $\mu^*(TO)|_{O'}$ . Since  $\mu|_{Q'} = \mu_2$  is an isomorphism, it defines a vector field  $\tilde{\mathbf{v}}$  on  $O$ . From  $\tilde{\mathbf{v}}(q') = d_{\mathbf{q}_0}\mu_1 \circ \varphi(\mathbf{v})$ , it follows that  $\tilde{\mathbf{e}}_1, \dots, \tilde{\mathbf{e}}_d$  form a local frame of  $TO$  at  $q'$ .

Now let  $f \in \text{Aut}_{\triangleright}(Q)$  be such that  $f(x) = x$ , i.e.,  $f(\mathbf{q}_0) = \mathbf{q}_0$ ,  $d_{\mathbf{q}_0}f \circ \varphi = \varphi$ , and  $f(q_0) = q_0$ . Since  $f$  is a quandle homomorphism, we have

$$\begin{aligned} f(\mu(\mathbf{q}_0 + \epsilon\varphi(\mathbf{v}), q)) &= f(\mu(\mathbf{q}_0, q) + \epsilon\tilde{\mathbf{v}}(\mu(\mathbf{q}_0, q))) \\ &= f(\mu_2(q)) + \epsilon(d_{\mu_2(q)}f)(\tilde{\mathbf{v}}(\mu_2(q))) \end{aligned}$$

and

$$\begin{aligned} f(\mu(\mathbf{q}_0 + \epsilon\varphi(\mathbf{v}), q)) &= \mu(f(\mathbf{q}_0 + \epsilon\varphi(\mathbf{v})), f(q)) \\ &= \mu(f(\mathbf{q}_0) + \epsilon d_{\mathbf{q}_0}f(\varphi(\mathbf{v})), f(q)) \\ &= \mu(\mathbf{q}_0 + \epsilon\varphi(\mathbf{v}), f(q)) \\ &= \mu(\mathbf{q}_0, f(q)) + \epsilon\tilde{\mathbf{v}}(\mu(\mathbf{q}_0, f(q))) \\ &= f(\mu_2(q)) + \epsilon\tilde{\mathbf{v}}(f(\mu_2(q))), \end{aligned}$$

hence

$$(d_{\mu_2(q)}f)(\tilde{\mathbf{v}}(\mu_2(q))) = \tilde{\mathbf{v}}(f(\mu_2(q))).$$

Since  $\mu_2$  is an automorphism, we conclude that  $f_*(\tilde{\mathbf{v}}) = \tilde{\mathbf{v}}$ . From  $f(q') = q'$ ,  $f_*(\tilde{\mathbf{e}}_i) = \tilde{\mathbf{e}}_i$  for  $i = 1, \dots, d$ , and the fact that  $\tilde{\mathbf{e}}_1, \dots, \tilde{\mathbf{e}}_d$  form a local frame at  $q'$ , it follows that  $f$  is the identity morphism on the formal neighborhood of  $q'$ , hence on  $Q$ .  $\square$

*Proof of Proposition 4.5.* (1) By Proposition 3.6, we have injective homomorphisms  $\text{Inn}(U) \rightarrow \text{Inn}(Q)$  and  $\text{Tr}(U) \rightarrow \text{Tr}(Q)$ . One can identify  $\text{Tr}(U)$  and  $\text{Tr}(Q)$  with  $\text{Tr}(U, X)x$  and  $\text{Tr}(Q, X)x$ , and they are equal by Proposition 3.7. So  $\text{Tr}(U) \cong \text{Tr}(Q)$ . Since  $\text{Inn}(Q)$  is generated by  $\text{Tr}(Q)$  and any one of  $s_q$ , we have  $\text{Inn}(U) \cong \text{Inn}(Q)$ .

(2) Write  $G$  for  $\mathbf{Tr}(Q)$ . For a positive integer  $n$ , write  $\mathbf{a}$  for an  $n$ -ple  $(a_1, \dots, a_n)$ , etc., and

$$s_{\mathbf{a}, \mathbf{b}} = s_{b_1}^{-1} \circ \dots \circ s_{b_n}^{-1} \circ s_{a_n} \circ \dots \circ s_{a_1}$$

for the automorphism of  $G$  (as a variety) or  $Q$ , depending on the context. Define

$$\tau_n : Q^{2n} \rightarrow G; (\mathbf{a}, \mathbf{b}) \mapsto s_{\mathbf{a}, \mathbf{b}}(x)$$

and

$$\alpha_n : Q^{2n} \times Q \rightarrow G \times Q \times Q; (\mathbf{a}, \mathbf{b}, q) \mapsto (\tau_n(\mathbf{a}, \mathbf{b}), q, s_{\mathbf{a}, \mathbf{b}}(q)).$$

By Proposition 3.5,  $\tau_n$  is surjective for  $n \gg 0$ . It follows that  $\text{Im } \alpha_n$  is the graph of the action map under the identification of  $\text{Tr}(Q)$  and  $G$ . We show that  $\text{Im } \alpha_n$  is a subvariety and that the action map is a regular map.

Note that one can always replace  $n$  by a larger number. We use the following two lemmas.

**Lemma 4.6.** *For  $n \gg 0$  and any  $g \in G$ , there exists a point  $(\mathbf{a}, \mathbf{b}) \in (Q_{\text{sm}})^{2n}$  such that  $\tau_n(\mathbf{a}, \mathbf{b}) = g$  and  $d_{(\mathbf{a}, \mathbf{b})}\tau_n$  is surjective.*

*Proof.* By the surjectivity of  $\tau_n$ , there exists  $(\mathbf{a}_0, \mathbf{b}_0) \in (Q_{\text{sm}})^{2n}$  such that  $d_{(\mathbf{a}_0, \mathbf{b}_0)}\tau_n$  is surjective. Let  $g_0 = \tau_n(\mathbf{a}_0, \mathbf{b}_0)$ . Since  $G$  is a  $\text{Tr}(Q)$ -orbit and  $\text{Tr}(Q) = \text{Tr}(Q_{\text{sm}})$  by Proposition 4.5 (1), there exist  $n' \in \mathbb{N}$  and  $(\mathbf{a}_1, \mathbf{b}_1) \in (Q_{\text{sm}})^{2n'}$  such that  $s_{\mathbf{a}_1, \mathbf{b}_1}(g_0) = g$ . Let  $\tau' : Q^{2n+2n'} \rightarrow G$  be defined by  $\tau'(\mathbf{a}, \mathbf{b}, \mathbf{a}', \mathbf{b}') = s_{\mathbf{a}', \mathbf{b}'} \circ s_{\mathbf{a}, \mathbf{b}}(x)$ . Then  $\tau'(\mathbf{a}_0, \mathbf{b}_0, \mathbf{a}_1, \mathbf{b}_1) = g$  and  $d_{(\mathbf{a}_0, \mathbf{b}_0, \mathbf{a}_1, \mathbf{b}_1)}\tau'$  is surjective.

Using  $s_a^{-1} \circ s_b = s_{a \triangleright^{-1} b} \circ s_a^{-1}$ , we see that there exists an automorphism  $f$  of  $Q^{2n+2n'}$  as a variety such that  $\tau' = \tau_{n+n'} \circ f$ . Thus  $d\tau_{n+n'}$  is surjective at some smooth point over  $g$ .

By openness of the smooth locus and noetherian induction, one can find  $n$  that works for any  $g \in G$ .  $\square$

**Lemma 4.7.** *Let  $S$  and  $U$  be normal varieties,  $T$  a variety, and  $f : S \rightarrow T$  and  $g : T \rightarrow U$  morphisms. Assume that  $g \circ f$  is smooth and surjective and that  $g$  is birational. Then  $g$  is an isomorphism.*

*Proof.* Let  $u \in U$  be an arbitrary point, and choose  $s \in S$  such that  $g(f(s)) = u$ . Let  $T'$  be the normalization of  $T$  and  $f' : S \rightarrow T'$  and  $g' : T' \rightarrow U$  the induced morphisms.

We claim that  $g'$  is étale at  $f'(s)$ . We have homomorphisms

$$\hat{O}_{U,u} \rightarrow \hat{O}_{T',f'(s)} \rightarrow \hat{O}_{S,s}.$$

Since  $g \circ f$  is smooth,  $\hat{O}_{S,s}$  can be identified with the power series ring over  $\hat{O}_{U,u}$  with generators  $h_1, \dots, h_d$ . So we have homomorphisms

$$\hat{O}_{U,u} \xrightarrow{\varphi} \hat{O}_{T',f'(s)} \xrightarrow{\psi} \hat{O}_{U,u} \cong \hat{O}_{S,s}/(h_1, \dots, h_d),$$

where  $\psi \circ \varphi = id$ . Then  $\psi$  is surjective. Since  $T'$  and  $U$  are normal,  $\widehat{\mathcal{O}}_{T', f'(s)}$  and  $\widehat{\mathcal{O}}_{U, u}$  are integral domains of dimension  $\dim U([Z48])$ , so  $\psi$  is also injective. It follows that  $\varphi$  is an isomorphism, hence the claim.

This means that  $f'(S)$  is contained in the étale locus  $V$  of  $g'$ , so  $g'|_V$  is surjective. By [Gr67, Lemma 18.10.18],  $g'|_V$  is an open immersion. Thus it is an isomorphism. It follows that  $V = T'$  and  $T = T'$ , so  $g$  is an isomorphism.  $\square$

Let us return to the proof of (2). Let  $S = Q^{2n} \times Q$ ,  $T = \overline{\text{Im } \alpha_n}$ ,  $U = G \times Q$ ,  $f = \alpha_n$  and  $g = (p_1 \times p_2)|_T$ , where  $p_i$  is the projection from  $G \times Q \times Q$  to the  $i$ -th factor. The restriction of  $g$  to  $\text{Im } \alpha_n$  is bijective and  $\text{Im } \alpha_n$  is a dense constructible subset of  $T$ . This implies that  $g : T \rightarrow U$  is birational, since we are working in characteristic 0. We also know that any point of  $U$  is in the image of the smooth locus of  $g \circ f$ .

By the previous lemma,  $g$  is an isomorphism. It also follows that  $\text{Im } \alpha_n$  is equal to  $T$ . Thus the action map, which is equal to  $p_3 \circ g^{-1} : G \times Q \rightarrow Q$ , is a morphism. We can see that  $\alpha^{-1}$  is also a morphism by the same arguments.

(3) If  $f : S \times Q \rightarrow S \times Q$  is an algebraic family of transvections, then it induces an automorphism  $\Phi(S)(f) : S \times X \rightarrow S \times X$  over  $S$  by Condition 4.2 (1). By sending the constant section  $S \times \{x\}$  by this morphism and projecting to  $X$ , we obtain a morphism  $\phi_f : S \rightarrow X$ . It is straightforward to see that its image is contained in  $\mathbf{Tr}(Q)$  and that the pullback of  $\alpha$  by  $\phi_f$  is equal to  $f$ .

(4) This follows from (3) since, as is well known, an object representing a group functor is a group object in a natural way. For example, to show that the multiplication map is a morphism, let  $\alpha_1, \alpha_2 \in \text{Aut}(\mathbf{Tr}(Q) \times \mathbf{Tr}(Q) \times Q)$  be defined by  $(g_1, g_2, q) \mapsto (g_1, g_2, g_1q)$  and  $(g_1, g_2, q) \mapsto (g_1, g_2, g_2q)$ . Then  $\alpha_1 \circ \alpha_2$  is an element of  $\mathcal{T}r(Q)(\mathbf{Tr}(Q) \times \mathbf{Tr}(Q))$ , so it corresponds to a morphism  $\mathbf{Tr}(Q) \times \mathbf{Tr}(Q) \rightarrow \mathbf{Tr}(Q)$ , which is nothing but the multiplication map.

(5) We have an element  $(id \times s_q) \circ \alpha \circ (id \times s_r^{-1})$  of  $\mathcal{T}r(Q)(\mathbf{Tr}(Q))$ , so it induces a morphism  $\mathbf{Tr}(Q) \rightarrow \mathbf{Tr}(Q)$ .  $\square$

*Remark 4.8.* The group  $\text{Inn}(Q)$  is not necessarily an algebraic group. For example, if  $Q = \mathbb{A}^1$  and  $x \triangleright y = x + a(y - x)$  with  $a \neq 0$ , then  $Q$  is  $\triangleright$ -connected if and only if  $a \neq 1$ . It is easy to see that  $\text{Inn}(Q)$  consists of automorphisms  $x \mapsto a^n x + b$  ( $n \in \mathbb{Z}, b \in k$ ), so it does not have a natural structure of an algebraic group unless  $a$  is a root of unity.

As for non- $\triangleright$ -connected quandles, the following is known in the setting of differential geometry.

**Theorem 4.9** ([Lo67]). *Let  $(M, \cdot)$  be a reflexion space. Then there exists a regular  $\varphi$ -space  $(G, H, \varphi)$  with  $\varphi$  involutive and a manifold  $F$  with an  $H$ -action such that  $(M, \cdot)$  is isomorphic to  $G \times_H F$  with the operation induced from  $\cdot_\varphi$ .*

Note that the orbits are  $\triangleright$ -connected in this case. It might be possible to obtain a similar result for quandle varieties with this property.

## 5. Questions related to $\triangleright$ -connectedness

Kowalski posed the following problem.

**Problem 5.1** ([K80, Problem II.47]). *Can one obtain any reasonable theory of “regular  $s$ -manifolds” starting from the following conditions?*

- (1)  $x \cdot x = x$ .
- (2)  $s_x : M \rightarrow M; y \mapsto x \cdot y$  is a diffeomorphism.
- (3)  $x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$ .
- (4I)  $x$  is an isolated fixed point of  $s_x$ .

We give a partial answer to its analogue in algebraic geometry.

**Proposition 5.2.** *Let  $Q$  be a quandle variety such that  $q$  is an isolated fixed point of  $s_q$  for any  $q \in Q$ . Then the following hold.*

- (1)  $Q \triangleright q$  is dense for a general point  $q$ , hence  $Q$  contains an open orbit  $O$ .
- (2) For a point  $q \in O$ , the stabilizer  $\mathbf{Tr}(O)_q$  contains  $(\mathbf{Tr}(O)^{\varphi_q})^\circ$ , so  $O$  is an algebraic  $\varphi$ -space.
- (3) The quandle homomorphism  $\bar{s} : Q \rightarrow (\mathbf{Tr}(Q), \triangleright'_{\varphi_q}); x \mapsto s_x s_q^{-1}$  of Remark 2.21 is quasi-finite.

*Proof.* (1) Define  $f : Q \times Q \rightarrow Q \times Q$  by  $f(q, r) = (q \triangleright r, r)$ . Let  $\Delta \subseteq Q \times Q$  be the diagonal set and denote by  $p_1, p_2$  the projection maps. Then, since  $(q, q) = \Delta \cap p_1^{-1}(q)$  is isolated in  $f^{-1}\Delta \cap p_1^{-1}(q)$ ,  $\Delta$  is an irreducible component of  $f^{-1}\Delta$ . It follows that there exists a point  $r$  such that  $(r, r) = \Delta \cap p_2^{-1}(r)$  is isolated in  $f^{-1}\Delta \cap p_2^{-1}(r)$ . If we define  $t_r : Q \rightarrow Q$  by  $t_r(q) := q \triangleright r$ , this means that  $r$  is isolated in  $t_r^{-1}(r)$ . Then  $Q \triangleright r$  is dense and the orbit  $\text{Inn}(Q)r$  is open.

(2) If  $s_q g s_q^{-1} = g$  holds for an automorphism  $g$  of  $Q$ , then  $s_q = s_{g(q)}$  holds. In particular,  $g(q)$  is a fixed point of  $s_q$ . If  $g$  is contained in  $(\mathbf{Tr}(O)^{\varphi_q})^\circ$ , then we must have  $g(q) = q$  by the assumption that  $q$  is an isolated fixed point of  $s_q$ .

(3) Similarly, if a fiber of  $\bar{s}$  were to contain a positive-dimensional variety  $Z$ , then  $Z$  would be contained in the fixed point locus of  $s_z$  for any  $z \in Z$ , a contradiction. Thus  $\bar{s}$  is quasi-finite.  $\square$

How various regularity conditions are related seems to be a subtle question.

**Problem 5.3.** *What implications hold between the following conditions?*

- (T)  $Q$  is nonsingular and  $d_q s_q - \text{Id}$  is invertible for any  $q \in Q$ .
- (I)  $q$  is an isolated fixed point of  $s_q$  for any  $q$ .
- (D)  $Q \triangleright q$  is dense for any  $q$ .
- (C)  $Q$  is  $\triangleright$ -connected.
- ( $\Phi$ )  $Q$  is an algebraic  $\varphi$ -space.

We know the following implications.

- (T)  $\Rightarrow$  (I), (D), (C) and ( $\Phi$ ).
- (D)  $\Rightarrow$  (C).
- (I) and (C)  $\Rightarrow$  (D) and ( $\Phi$ ).

As we saw in Remark 3.15, the condition (D) does not imply condition (I). The same example also shows that (D) and ( $\Phi$ ) do not imply (I). In fact, let  $G := \text{SL}_2(k)$  act on  $Q$  by conjugation and define an automorphism  $\varphi$  of  $G$  by  $M \mapsto J_1^{-1} M J_1$ . Then  $G_{J_1}$  and  $G^\varphi$  are both equal to  $\{M \in \text{SL}_2(k) \mid M J_1 = J_1 M\}$ , and  $Q$  is isomorphic to  $(G/G^\varphi, \triangleright_\varphi)$ .

As a related question, what happens if  $d_q s_q = \text{Id}$  for any  $q \in Q$ ? Does there exist a  $\triangleright$ -connected quandle variety  $Q$  with  $d_q s_q = \text{Id}$  for any  $q \in Q$ ? The following example shows that the orbits can be of codimension 1.

**Example 5.4.** In this example,  $Q$  is a weak algebraic  $\varphi$ -space (Definition 3.3),  $d_q s_q = \text{Id}$  for any  $q \in Q$  and  $\dim(\text{Inn}(Q)q) = n - 1$  where  $n = \dim Q$ . More precisely, the dimension of  $Q^k q$  is  $k$  for  $k \leq n - 2$  and  $n - 1$  for  $k \geq n - 1$ .

In Example 2.5, let  $X = \mathbb{A}^1$ ,  $A = \mathbb{A}^{n-1}$ , and  $F(x, y) = ((y-x)^2, (y-x)^3, \dots, (y-x)^n)$ . Then  $Q = \mathbb{A}^n$  with

$$(x_1, \dots, x_n) \triangleright (y_1, \dots, y_n) = (y_1, y_2 + (y_1 - x_1)^2, y_3 + (y_1 - x_1)^3, \dots, y_n + (y_1 - x_1)^n).$$

Then  $d_q s_q = \text{Id}$  everywhere, but the orbit of  $(0, 0)$  is  $\{0\} \times \mathbb{A}^{n-1}$ .

To see that  $Q$  is a weak  $\varphi$ -space, note that the map  $f_{\alpha_1, \alpha_2(x_1), \dots, \alpha_n(x_1)} : Q \rightarrow Q$  defined by

$$(x_1, \dots, x_n) \mapsto (x_1 + \alpha_1, x_2 + \alpha_2(x_1), \dots, x_n + \alpha_n(x_1))$$

is an automorphism of  $Q$  for any  $\alpha_1 \in k$  and  $\alpha_i(x_1) \in k[x_1]$ , and that

$$G = \{f_{\alpha_1, \alpha_2(x_1), \dots, \alpha_n(x_1)} \mid \deg \alpha_i(x_1) \leq i - 1 (i = 2, \dots, n)\}$$

is a subgroup of  $\text{Aut}(Q)$  acting transitively on  $Q$ . We also see that the conjugation by  $s_{(0,0)}$  leaves  $G$  invariant, and therefore defines an automorphism  $\varphi$  of  $G$ . Then one can show that  $Q$  is isomorphic to  $(G/G_{(0,0)}, \triangleright_\varphi)$ .

## References

- [CJKLS03] J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford, M. Saito, *Quandle cohomology and state-sum invariants of knotted curves and surfaces*, Trans. Amer. Math. Soc. **355** (2003), 3947–3989.
- [Gr67] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas (Quatrième Partie)*, Inst. Hautes Études Sci. Publ. Math. No. 32 (1967).
- [J82] D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra **23** (1982), no. 1, 37–65.
- [K80] O. Kowalski, *Generalized Symmetric Spaces*, Lecture Notes in Mathematics, Vol. 805, Springer-Verlag, Berlin-New York, 1980.
- [Le67] A. J. Ledger, *Espaces de Riemann symétriques généralisés*, C. R. Acad. Sci. Paris Sér. A–B **267** (1967), A947–A948.
- [Lo67] O. Loos, *Spiegelungsräume und homogene symmetrische Räume*, Math. Z. **99** (1967), 141–170.
- [Lo69] O. Loos, *Symmetric Spaces. I: General Theory*, Benjamin, New York, 1969. Russian transl.: O. Лоос, *Симметрические пространства*, Наука, М., 1985.
- [M82] С. В. Матвеев, *Дистрибутивные группоиды в теории узлов*, Матем. сб., **119(161)** (1982), ном. 1(9), 78–88. Engl. translation: S. V. Matveev, *Distributive groupoids in knot theory*, Math. USSR-Sb. **47** (1984), no. 1, 73–83.

- [R07] R. L. Rubinsztein, *Topological quandles and invariants of links*, J. Knot Theory Ramifications **16** (2007), no. 6, 789–808.
- [SSS02] L. Sabinin, L. Sabinina, L. Sbitneva, *From symmetric to transsymmetric spaces*, 8th International Conference on Differential Geometry and its Applications (Opava, 2001). Differential Geom. Appl. **17** (2002), no. 2–3, 197–208.
- [V65] В. И. Ведерников, *Симметрические пространства и сопряжённые связности*, Учён. зап. Казан. гос. ун-та **125** (1965), no. 1, 7–59. [V. I. Vedernikov, *Symmetric spaces and conjugate connections*, Kazan. Gos. Univ. Učen. Zap. **125** (1965), no. 1, 7–59 (Russian).]
- [Z48] O. Zariski, *Analytical irreducibility of normal varieties*, Ann. of Math. **49** (1948), no. 2, 352–361.