

Tutorial 2 - Fourier analysis

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Problem 1. Let $S \subseteq [n]$ and \mathbb{Z}_2^n be a group.

- a) Prove that the map $\chi_S(x) = (-1)^{\sum_{i \in S} x_i}$ is a character $\mathbb{Z}_2^n \rightarrow \mathbb{C}^*$,
- b) for $T \subseteq [n], S \neq T$, show that $\chi_S \neq \chi_T$.

Problem 2. Generalize the previous problem into general case of finite abelian groups.

Problem 3. Let G be a finite abelian group and $a \in G$. Then compute the Fourier transform of:

- a) χ_a ,
- b) δ_a ,
- c) $\delta_{-1} + \delta_0 + \delta_1$ for $G = \mathbb{Z}_n$,
- d) $\sum_{i=0}^{k-1} \delta_{il}$ for $G = \mathbb{Z}_{kl}$.

Problem 4. Verify that:

- a) $\widehat{f}(0) = \mathbb{E}[f]$ for every finite abelian group G and every function $f: G \rightarrow \mathbb{C}$,
- b) $\|\widehat{f}\|_\infty \leq \|f\|_1$.

Problem 5. Fix $p \in G$ and $c \in \mathbb{C}$. We define operators T_p and P_c over $\mathbb{C}G$ such that for every function $f: G \rightarrow \mathbb{C}$ it holds that $(T_p f)(x) = f(x + p)$, $(P_c f)(x) = cf(x)$.

In case G is a field and $p \neq 0$, we define $(S_p f)(x) = f(px)$. Prove that:

- a) $\widehat{T_p f}(a) = \chi_a(p)\widehat{f}(a)$,
- b) $\widehat{P_c f}(a) = c\widehat{f}(a)$,
- c) $\widehat{S_p f}(a) = \widehat{f}(\frac{a}{p})$.

Problem 6. Consider group \mathbb{Z}_n and matrix $M \in \mathbb{C}_{n \times n}$ such that for every $f \in \mathbb{Z}_n \rightarrow \mathbb{C}$ we have:

$$(\widehat{f}(0), \widehat{f}(1), \dots, \widehat{f}(n-1))^T = M(f(0), f(1), \dots, f(n-1))^T.$$

Compute $\det M$ and reprove that Fourier transform is a bijection.

Problem 7. Draw a Cayley graph of \mathbb{Z}_6 with respect to the set $S = \{\pm 2, \pm 3\}$ and compute the eigenvalues of the adjacency matrix.