Tutorial 2 - Fourier analysis

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Problem 1. Let $S \subseteq [n]$ and $\mathbb{Z}_2^n$ be a group.

a) Prove that the map $\chi_S(x) = (-1)^{\sum_{i \in S} x_i}$ is a character $\mathbb{Z}_2^n \to \mathbb{C}^*$,

b) for $T \subseteq [n], S \neq T$, show that $\chi_S \neq \chi_T$.

Problem 2. Generalize the previous problem into general case of finite abelian groups.

Problem 3. Let G be a finite abelian group and $a \in G$. Then compute:

a) $\widehat{\chi_a}$,
b) $\widehat{\delta_a}$,
c) $\widehat{\delta_{-1} + \delta_0 + \delta_1}$ for $G = \mathbb{Z}_n$,
d) $\widehat{\sum_{i=0}^{k-1} \delta_{il}}$ for $G = \mathbb{Z}_{kl}$.

Problem 4. Verify that:

a) $\widehat{f}(0) = \mathbb{E}[f]$ for every finite abelian group G and every function $f\colon G \to \mathbb{C}$,

b) $||\widehat{f}||_\infty \leq ||f||_1$.

Problem 5. Fix $p \in G$ and $c \in \mathbb{C}$. We define operators T_p and P_c over $\mathbb{C}G$ such that for every function $f\colon G \to \mathbb{C}$ it holds that $(T_p f)(x) = f(x+p)$, $(P_c f)(x) = cf(x)$.

In case G is a field and $p \neq 0$, we define $(S_p f)(x) = f(px)$. Prove that:

a) $\widehat{T_p f}(a) = \chi_a(p)\widehat{f}(a)$,
b) $\widehat{P_c f}(a) = c\widehat{f}(a)$,
c) $\widehat{S_p f}(a) = \widehat{f}(\frac{a}{p})$.

Problem 6. Consider group $\mathbb{Z}_n$ and matrix $M \in \mathbb{C}_{n \times n}$ such that for every $f \in \mathbb{Z}_n \to \mathbb{C}$ we have:

$$(\widehat{f}(0), \widehat{f}(1), \ldots, \widehat{f}(n-1))^T = M(f(0), f(1), \ldots f(n-1))^T.$$

Compute $\det M$ and reprove that Fourier transform is a bijection.

Problem 7. Draw a Cayley graph of $\mathbb{Z}_6$ with respect to the set $S = \{\pm 2, \pm 3\}$ and compute the eigenvalues of the adjacency matrix.