Let $n \in \mathbb{N}$. Find closed form of following series

$$\binom{4n}{2}+\binom{4n}{6}+\binom{4n}{10}+\cdots+\binom{4n}{4n-6}+\binom{4n}{4n-2}.$$

SOLUTION:

We use binomial identity (expansion) for $(1-1)^n, \quad (1+1)^n, \quad (1+\mathrm{i})^n$ where i is imaginary unit. So we have

$$(1-1)^n = \binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots \quad = \quad 0 \tag{1}$$

$$(1+1)^n = \binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots \quad = \quad 2^n \tag{2}$$

$$(1+\mathrm{i})^n = \binom{n}{0}+\mathrm{i}\binom{n}{1}-\binom{n}{2}-\mathrm{i}\binom{n}{3}+\cdots \quad = \quad (\sqrt{2})^n\cdot\Big(\cos\frac{\pi n}{4}+\mathrm{i}\sin\frac{\pi n}{4}\Big) \tag{3}$$

Adding (1) to (2) and dividing 2 we get

$$\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots = 2^{n-1} \tag{4}$$

Further we'll need a real part of (3). Which is

$$\Re\Big[(1+i)^n\Big] = \binom{n}{0}-\binom{n}{2}+\binom{n}{4}-\cdots = (\sqrt{2})^n\cdot\Big(\cos\frac{\pi n}{4}\Big). \tag{5}$$

Substracting (5) from (4) and dividing 2 yield

$$\binom{n}{2}+\binom{n}{6}+\cdots+\binom{n}{n-2} = 2^{n-2}-\frac{(\sqrt{2})^n}{2}\cdot\cos\frac{\pi n}{4}.$$

Hence

$$\sum_{k=1}^{n}\binom{4n}{4k-2} = 2^{4n-2}-2^{2n-1}\cdot\cos(\pi n)$$

OTHER SOLUTION:

Similarly as previous case, we use binomial theorem on $(1+1)^{4n}$ and we get sum of combinational number. The coefficients in front of these combinational numbers will be 1. Analogously for $(1-1)^{4n}$ with difference that coefficients in front of combinational numbers will be others. Instead of quadruplet coefficients $1, 1, 1, 1$, it will be repeat quadruplet coefficients $1, -1, 1, -1$. This same is valid for $(1+\mathrm{i})^{4n}$, where quadruplet coefficients are $1, \mathrm{i}, -1, -\mathrm{i}$ and for $(1-\mathrm{i})^{4n}$ is $1, -\mathrm{i}, -1, \mathrm{i}$.

It could be expect that combination of these quadruplet coefficients we get wanted sum total. To figure out algebraic system of equations we have

$$\frac{1}{4}(1,1,1,1)+\frac{1}{4}(1,-1,1,-1)-\frac{1}{4}(1,\mathrm{i},-1,-\mathrm{i})-\frac{1}{4}(1,-\mathrm{i},-1,\mathrm{i}) = (0,0,1,0)$$

Abviously

$$\begin{aligned}\sum_{k=1}^{n}\binom{4n}{4k-2} =& \frac{1}{4}(1+1)^{4n}+\frac{1}{4}(1-1)^{4n}-\frac{1}{4}(1+\mathrm{i})^{4n}-\frac{1}{4}(1-\mathrm{i})^{4n} = \\ =& \frac{1}{4}\cdot 2^{4n}-\frac{1}{4}(-4)^n-\frac{1}{4}(-4)^n = \\ =& 2^{4n-2}-\frac{1}{2}(-4)^n = \\ =& 2^{4n-2}-2^{2n-1}(-1)^n\end{aligned}$$