We woud like to count the following series

$$\sum_{k=1}^{\infty} \arctan \frac{1}{k^2}$$

Note this sum is convergent (necessary condition is fulfill), even absolutely convergent (thx to compare criterium). So we want to find result in closed form.

If we show validity of equality bellow written, then putting $x = y = \pi/\sqrt{2}$ yield previous series.

$$\sum_{k=1}^{\infty} \arctan \frac{2xy}{k^2\pi^2 - x^2 + y^2} = \arctan \frac{y}{x} - \arctan \frac{\tanh y}{\tan x}$$

Hence

$$\boxed{\sum_{k=1}^{\infty} \arctan \frac{1}{k^2} = \frac{\pi}{4} - \arctan \frac{\tanh \frac{\pi}{\sqrt{2}}}{\tan \frac{\pi}{\sqrt{2}}}}$$

Now thanks to formula for $\arctan$ function

$$\arctan u - \arctan v = \arctan \frac{u - v}{1 + uv}$$

we get

$$\sum_{k=1}^{\infty} \arctan \frac{2xy}{k^2\pi^2 - x^2 + y^2} = \sum_{k=1}^{\infty} \left\{ \arctan \frac{y}{k\pi - x} - \arctan \frac{y}{k\pi + x} \right\}$$

and series on right side is equals $\arctan \frac{y}{x} - \arctan \frac{\tanh y}{\tan x}$.
If we proof this we are done. So lets start. In following we're going to be label $\Im$ imaginary part of complex number and i for imaginary unit.

$$\Im \log \left[\left(1 + \frac{\mathrm{i}y}{x}\right) \prod_{k=1}^{\infty} \left(1 - \frac{\mathrm{i}y}{k\pi - x}\right)\left(1 + \frac{\mathrm{i}y}{k\pi + x}\right) \right] = \Im \left[\log\left(1 + \frac{\mathrm{i}y}{x}\right) + \sum_{k=1}^{\infty} \log\left(1 - \frac{\mathrm{i}y}{k\pi - x}\right) + \sum_{k=1}^{\infty} \log\left(1 + \frac{\mathrm{i}y}{k\pi + x}\right) \right] =$$

$$= \arctan \frac{y}{x} + \sum_{k=1}^{\infty} \left\{ -\arctan \frac{y}{k\pi - x} + \arctan \frac{y}{k\pi + x} \right\},$$

where we used Taylor series

$$\log(1 + t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \dots, \quad t = \frac{\mathrm{i}y}{x}$$

$$\arctan t = t - \frac{t^3}{3} + \frac{t^5}{5} - \dots, \quad t = \frac{\mathrm{i}y}{k\pi - x} \text{ and } t = \frac{\mathrm{i}y}{k\pi + x}$$

On the other hand, to hold

$$\Im \log \left[\left(1 + \frac{\mathrm{i}y}{x}\right) \prod_{k=1}^{\infty} \left(1 - \frac{\mathrm{i}y}{k\pi - x}\right)\left(1 + \frac{\mathrm{i}y}{k\pi + x}\right) \right] = \Im \log \frac{\sin(x + \mathrm{i}y)}{\sin x} = \Im \log \left(\cos(\mathrm{i}y) + \frac{\sin(\mathrm{i}y)}{\tan x} \right) =$$

$$= \Im \log \left(\cosh y + \mathrm{i}\frac{\sinh y}{\tan x} \right) = \Im \log \left(1 + \mathrm{i}\frac{\tanh y}{\tan x} \right) = \arctan \frac{\tanh y}{\tan x}.$$

where we used Taylor series on logarithm again and $\cos(\mathrm{i}y) = \cosh y$, $\sin(\mathrm{i}y) = \mathrm{i}\sinh y$. Thus

$$\arctan \frac{\tanh y}{\tan x} = \arctan \frac{y}{x} + \sum_{k=1}^{\infty} \left\{ -\arctan \frac{y}{k\pi - x} + \arctan \frac{y}{k\pi + x} \right\}$$

and up to rearrange

$$\sum_{k=1}^{\infty} \left\{ \arctan \frac{y}{k\pi - x} - \arctan \frac{y}{k\pi + x} \right\} = \arctan \frac{y}{x} - \arctan \frac{\tanh y}{\tan x}.$$