Odvození Simpsonova pravidla

Postupujeme dle skript Numerické metody II od P. Přikrila a M. Brandnera. Pro jednoduchost uvažujme ekvidistantní uzly s kromem h.

$$
\begin{aligned}
L_2(x) &= \frac{(x-x_{k+1})(x-x_{k+2})}{(x_k-x_{k+1})(x_k-x_{k+2})}f(x_k) + \frac{(x-x_k)(x-x_{k+2})}{(x_{k+1}-x_k)(x_{k+1}-x_{k+2})}f(x_{k+1}) + \frac{(x-x_k)(x-x_{k+1})}{(x_{k+2}-x_k)(x_{k+2}-x_{k+1})}f(x_{k+2}) = \\
&= \frac{x^2-x(x_{k+1}+x_{k+2})+x_{k+1}x_{k+2}}{h(2h)}f(x_k) + \frac{x^2-x(x_k+x_{k+2})+x_kx_{k+2}}{h(-h)}f(x_{k+1}) + \frac{x^2-x(x_k+x_{k+1})+x_kx_{k+1}}{2h(h)}f(x_{k+2}) = \\
&= \frac{1}{h^2}\Bigg\{x^2\left[\frac{1}{2}f(x_k) - f(x_{k+1}) + \frac{1}{2}f(x_{k+2})\right] + x\left[-\frac{1}{2}(2x_k+3h)f(x_k) + (2x_k+2h)f(x_{k+1}) - \frac{1}{2}(2x_k+h)f(x_{k+2})\right] + \\
&\qquad + \left[\frac{1}{2}(x_k+h)(x_k+2h)f(x_k) - x_k(x_k+2h)f(x_{k+1}) + \frac{1}{2}x_k(x_k+h)f(x_{k+2})\right]\Bigg\}
\end{aligned}
$$

kde jsme využili

$$
\begin{aligned}
x_{k+1} &= x_k + h \\
x_{k+2} &= x_k + 2h
\end{aligned}
$$

Dále označme u kvadratického členu hranaté závorky jako A, u lineárního členu B a zbytek jako C, dostaneme

$$
L_2(x) = \frac{1}{h^2}\left(Ax^2 + Bx + C\right)
$$

Nyní již přistupme k samotnému odvození

$$
\begin{aligned}
\underline{\underline{\int_{x_k}^{x_{k+2}} f(x)\,\mathrm{d}x}} &\approx \int_{x_k}^{x_{k+2}} L_2(x)\,\mathrm{d}x = \frac{1}{h^2}\int_{x_k}^{x_{k+2}} (Ax^2+Bx+C)\,\mathrm{d}x = \frac{1}{h^2}\left\{\frac{A}{3}(x_{k+2}^3 - x_k^3) + \frac{B}{2}(x_{k+2}^2 - x_k^2) + C(x_{k+2}-x_k)\right\} = \\
&= \frac{x_{k+2}-x_k}{h^2}\left\{\frac{A}{3}(x_{k+2}^2 + x_{k+2}x_k + x_k^2) + \frac{B}{2}(x_{k+2}+x_k) + C\right\} \overset{(1)}{=} \frac{2}{h}\left\{\frac{A}{3}(3x_k^2 + 6x_kh + 4h^2) + \frac{B}{2}(x_k+h) + C\right\} \overset{(2)}{=} \\
&\overset{(2)}{=} \frac{2}{h}\Bigg\{\left[\frac{1}{2}x_k^2 + x_kh + \frac{2}{3}h^2 - \frac{1}{2}x_k(2x_k+3h) - \frac{1}{2}h(2x_k+3h) + \frac{1}{2}(x_k+h)(x_k+2h)\right]f(x_k) + \left[-x_k^2 - 2x_kh - \frac{4}{3}h^2 + x_k(2x_k+2h) + \right. \\
&\qquad \left. + h(2x_k+2h) - x_k(x_k+2h)\right]f(x_{k+1}) + \left[\frac{1}{2}x_k^2 + x_kh + \frac{2}{3}h^2 - \frac{1}{2}x_k(2x_k+h) - \frac{1}{2}h(2x_k+h) + \frac{1}{2}x_k(x_k+h)\right]f(x_{k+2})\Bigg\} = \\
&= \frac{2}{h}\left\{\frac{1}{6}h^2f(x_k) + \frac{2}{3}h^2f(x_{k+1}) + \frac{1}{6}h^2f(x_{k+2})\right\} = \underline{\underline{\frac{h}{3}\left[f(x_k) + 4f(x_{k+1}) + f(x_{k+2})\right]}}
\end{aligned}
$$

kde u rovnosti (1) jsme využili známé značení $x_{k+2} = x_k + 2h$. U rovnosti (2) jsme zpětně dosadili původní výrazy místo A, B, C a upravili je.