Propočítané příklady CV.5, DÚ5

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Úloha 1. *Rozviňte funkci $f(z)$ do mocninné řady se středem v z_0 a určete parametry jejího kruhu konvergence.*

(a) $f(z) = \frac{z-3}{1-2z}$, $z_0 = 3$

(b) $f(z) = \frac{1}{(z+6)^2}$, $z_0 = -4$

(c) $f(z) = (z-2)^4 e^{3z}$, $z_0 = 2$

(d) $f(z) = \frac{(z+1)^5}{z^2+z-2}$, $z_0 = -1$

mega super

$$a)\ f(z) = \frac{z-3}{1-2z} = (z-3)\frac{1}{1-2(z-3+3)} = (z-3)\cdot\frac{1}{1-6-2(z-3)} = \frac{z-3}{-5-2(z-3)} = \frac{z-3}{5}\cdot\frac{1}{-1-\left(\frac{2}{5}(z-3)\right)}$$

$$= \frac{z-3}{-5}\cdot\frac{1}{1+\left(\frac{2}{5}(z-3)\right)} = \frac{z-3}{-5}\cdot\frac{1}{1-\underbrace{\left(-\frac{2}{5}(z-3)\right)}_{\text{kvocient}}} = \frac{z-3}{-5}\sum_{n=0}^{\infty}\left(-\frac{2}{5}(z-3)\right)^n =$$

$$= \sum_{n=0}^{\infty} 2^n\left(-\frac{1}{5}\right)^{n+1}(z-3)^{n+1}$$

$$\left|-\frac{2}{5}(z-3)\right| < 1$$

$$|z-3| < 2{,}5$$

$$R = 2{,}5,\ z_0 = 3$$

Výsledky

(a) $f(z) = \sum_{n=0}^{\infty} \frac{2^n}{(-5)^{n+1}}(z-3)^{n+1}$ pro $|z-3| < \frac{5}{2}$ (tj. $R = \frac{5}{2}$)

(b) $f(z) = \frac{1}{(z+6)^2}$, $z_0 = -4$

$$f(z) = \frac{1}{(z+6)^2} = \frac{d}{dz}\int\frac{1}{(z+6)^2}dz = \frac{d}{dz}\left[\left(-\frac{1}{z+6}\right)\right] = \frac{d}{dz}\left[-\frac{1}{2+(z+4)}\right] =$$

konstanta byla odderivována

$$= \frac{d}{dz}\left[-\frac{1}{2}\ \frac{1}{1-\left(-\frac{(z+4)}{2}\right)}\right] = \frac{d}{dz}\left[-\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{1}{10}(z+4)\right)^n\right] = \sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^{n+1} n (z+4)^{n-1}$$

při derivaci $0 \xrightarrow{n=} 1$

$$|z-4| < 10$$

$$R = 2$$

$$z_0 = 4$$

(b) $f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{n+1}} n(z+4)^{n-1}$ pro $|z+4| < 2$ (tj. $R = 2$)

(c) $f(z) = (z-2)^4 e^{3z}$, $z_0 = 2$

$$f(z) = (z-2)^4 e^{3(z-2+2)} = (z-2)^4 e^6 e^{3(z-2)} = (z-2)^4 e^6 \sum_{n=0}^{\infty}\frac{(3(z-2))^n}{n!} = e^6\sum_{n=0}^{\infty}\frac{3^n}{n!}(z-2)^{n+4}$$

(c) $f(z) = e^6 \sum_{n=0}^{\infty} \frac{3^n (z-2)^{n+4}}{n!}$ pro každé $z \in \mathbb{C}$ (tj. $R = \infty$)

$$R = \infty$$

$$z_0 = 2$$

(d) $f(z) = \frac{(z+1)^5}{z^2+z-2}$, $z_0 = -1$

$$(d)\ \ f(z) = \frac{(z+1)^5}{z^2+z-2},\ \ z_0 = -1$$

$$f(z) = (z+1)^5 \frac{1}{\left(z+\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2} = (z+1)^5 \cdot \frac{1}{\left(\left(z+\frac{1}{2}\right)+\frac{3}{2}\right)\left(\left(z+\frac{1}{2}\right)-\frac{3}{2}\right)} = (z+1)^5 \frac{1}{(z+2)(z-1)}$$

$$\frac{1}{(z+2)(z-1)} = \frac{A}{z+2} + \frac{B}{z-1}$$

$$1 = Az - A + Bz + 2B$$

$$0 = A + B$$
$$1 = 2B - A$$
$$1 = 3B$$
$$B = \frac{1}{3}$$
$$A = -\frac{1}{3}$$

$$= (z+1)^5 \left[\frac{-\frac{1}{3}}{z+2} + \frac{\frac{1}{3}}{z-1} \right]$$

$$\to \frac{1}{3} \frac{1}{-2} \frac{1}{1 - \frac{z+1}{2}}$$

$$= (z+1)^5 \left[\frac{-\frac{1}{3}}{1-(-(z+1))} + \frac{\frac{1}{3}}{-2+(z+1)} \right]$$

$$= (z+1)^5 \left[-\frac{1}{3} \sum_{n=0}^{\infty} (-(z+1))^n + \frac{1}{3} \frac{-1}{2} \sum_{n=0}^{\infty} \left(\frac{z+1}{2}\right)^n \right]$$

$|z+1| < 1$

$$= -\frac{1}{3} \sum_{n=0}^{\infty} (-1)^n (z+1)^{n+5} - \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} (z+1)^{n+5}$$

$R = 1$

$z_0 = -1$

(d) $f(z) = -\frac{1}{3}\sum_{n=0}^{\infty}(-1)^n(z+1)^{n+5} - \frac{1}{3}\sum_{n=0}^{\infty}\frac{(z+1)^{n+5}}{2^{n+1}}$ pro $|z+1| < 1$ (tj. $R = 1$)

Úloha 2.

(a) Laurentova řada

$$\sum_{n=-\infty}^{\infty} a_n (z+4)^n$$

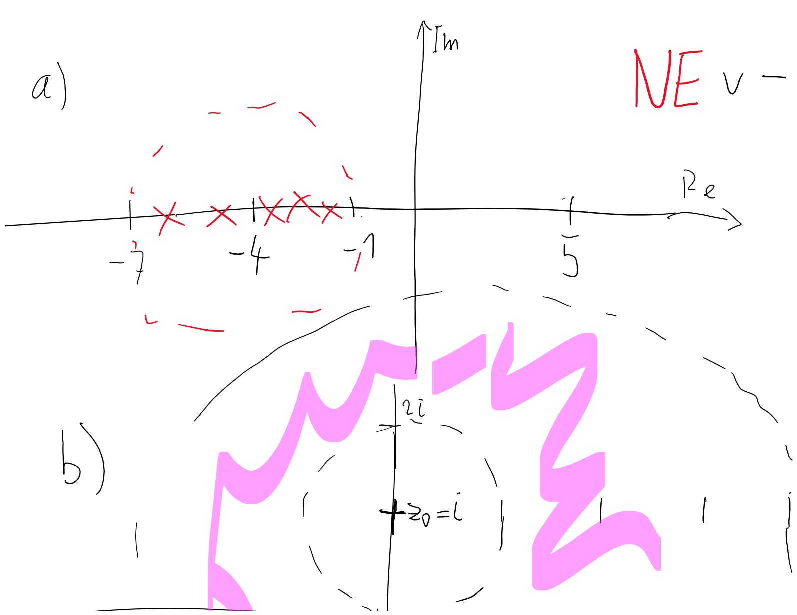
má vnitřní poloměr konvergence $r = 3$ a vnější $R = 9$. Konverguje v bodě $z = -6$?

(b) Laurentova řada

$$\sum_{n=-\infty}^{\infty} a_n (z-i)^n$$

má vnitřní poloměr konvergence $r = 1$ a vnější $R = 4$. Konverguje v bodě $z = 2$?

a)



NE v -6 Ř konverguje

b)

ANO v $z=2$

Ř konverguje

Úloha 3. *Rozviňte funkci*

$$f(z) = \frac{1}{(z+5)(z-3)^4(z^2+2z-15)^2}$$

do Laurentovy řady na maximálním prstencovým okolí bodu $z_0 = 3$ a určete jeho parametry.

$$f(z) = \frac{1}{(z-3)^4}\,\frac{1}{(z+5)}\,\frac{1}{(z-3)^2(z+5)^2} = \frac{1}{(z-3)^6}\,\frac{1}{(z+5)^3} = \frac{1}{(z-3)^6}\,\frac{d^2}{dz^2}\left[\int\!\!\int \frac{1}{(z+5)^3}\,dz\,dz\right]$$

$$= \frac{1}{(z-3)^6}\,\frac{d^2}{dz^2}\left[\int -\frac{1}{2}\,\frac{1}{(z+5)^2}\,dz\right] = \frac{1}{(z-3)^6}\,\frac{d^2}{dz^2}\left[\frac{1}{2}\,\frac{1}{(z+5)}\right] = \frac{1}{(z-3)^6}\,\frac{d^2}{dz^2}\left(\frac{1}{2}\,\frac{1}{8+(z-3)}\right) =$$

$$= \frac{1}{(z-3)^6}\,\frac{d^2}{dz^2}\left(\frac{1}{2^4}\,\frac{1}{1-\left(-\frac{z-3}{8}\right)}\right) = \frac{1}{(z-3)^6}\,\frac{d^2}{dz^2}\left(\frac{1}{2^4}\sum_{n=0}^{\infty}\left(-\frac{z-3}{2^3}\right)^n\right) =$$

$$= \frac{1}{(z-3)^6}\,\frac{d^2}{dz^2}\left(\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)^{3n+4}(z-3)^n\right) = \frac{1}{(z-3)^6}\sum_{n=2}^{\infty}\frac{(-1)^n}{(2)^{3n+4}}\,n(n-1)(z-3)^{n-2}$$

$$= \sum_{n=2}^{\infty}\frac{(-1)^n}{(2)^{3n+4}}\,n(n-1)(z-3)^{n-8}$$

$|z-3| < 8$

$R = 8, r = 0$

$z_0 = 3$

Je to stejné akorát já mám mocniny dvojky

Úloha 3: $f(z) = \frac{1}{2}\sum_{n=2}^{\infty}\frac{(-1)^n}{8^{n+1}}n(n-1)(z-3)^{n-8}$ pro $0 < |z-3| < 8$ (tj. $r = 0$ a $R = 8$)

Úloha 4. *Klasifikujte typ izolované singularity funkce $f(z)$ v bodě z, je-li*

(a) $z = -2$ *a*

$$f(z) = -\frac{9}{(z+2)^5} + \frac{8}{(z+2)^3} - \frac{3}{(z+2)^2} + \sum_{n=-3}^{\infty} n^2(z+2)^{3n+4}, \quad z \in P(-2);$$

(b) $z = i$ *a*

$$f(z) = \frac{2}{(z-i)^3} + \frac{1}{z-i} + \sum_{n=-5}^{\infty}(n+3)(z-i)^{2n+7}, \quad z \in P(i).$$

a) $z = -2$

$9 \quad 8 \quad 3 \quad 9 \quad 4 \quad \sum^{\infty} n^2(z+2)^{3n+4} \quad z \in P(-2)$

a) $z=-2$

$$f(z) = -\cancel{\frac{9}{(z+2)^5}} + \underline{\frac{8}{(z+2)^3}} - \frac{3}{(z+2)^2} + \underbrace{\cancel{\frac{9}{(z+2)^5}}}_{n=-3} + \underbrace{\frac{4}{(z+2)^2}}_{n=-2} + \sum_{n=-1}^{\infty} n^2 (z+2)^{3n+4}, \quad z \in P(-2)$$

Pól řádu 3 v $z=-2$

b) $z=i$

$$f(z) = \cancel{\frac{2}{(z-i)^3}} + \cancel{\frac{1}{z-i}} - \underbrace{\cancel{\frac{2}{(z-i)^3}}}_{n=-5} - \underbrace{\cancel{\frac{1}{(z-i)}}}_{n=-4} + \sum_{n=-5}^{\infty} (n+3)(z-i)^{2n+7}$$

V $z=i$ má $f(z)$ odstranitelnou singularitu

Úloha 4: (a) Pól řádu 3.
(b) Odstranitelná singularita.

Úloha 5. *Určete koeficient $\alpha \in \mathbb{C}$ a exponent $k \in \mathbb{Z}$ tak, aby funkce*

$$f(z) = \frac{\alpha}{(z-1)^k} + \frac{7}{3(z-1)^2} + \sum_{n=1}^{\infty} \frac{(z-1)^{n-3}}{3^n}, \quad z \in P(1),$$

měla v bodě 1 jednoduchý pól.

$$f(z) = \frac{\alpha}{(z-1)^k} + \frac{7}{3(z-1)^2} + \underbrace{\frac{1}{3(z-1)^2}}_{n=1} + \ldots$$

$\boxed{\alpha = -\frac{8}{3}\ ;\ k=2}$ => Poté se odstraní 2. řád a zůstanou jen -1. mocniny

=> jednoduchý pól bude mít nenulový koef.

Úloha 5: $k = 2$, $a = -\frac{8}{3}$

Úloha 1. *Laurentova řada*

$$\sum_{n=-\infty}^{\infty} a_n (z-5i)^n$$

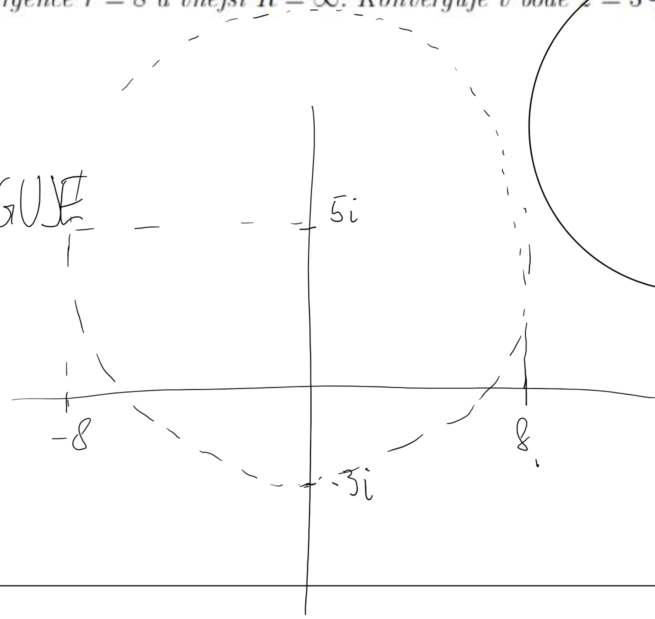
má vnitřní poloměr konvergence $r = 8$ a vnější $R = \infty$. Konverguje v bodě $z = 5 + 8i$?

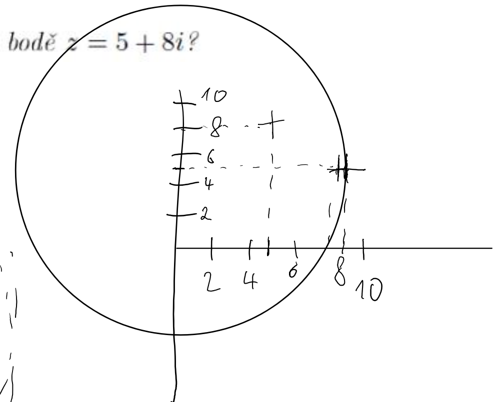
$$\sum_{n=-\infty} \ldots$$

má vnitřní poloměr konvergence $r = 8$ a vnější $R = \infty$. Konverguje v bodě $z = 5 + 8i$?

$z_0 = 5i$

NEKONVERGUJE





Úloha 2. *Rozviňte funkci*

$$f(z) = \frac{(z+i)^3}{(3z-2)^2}$$

do mocninné řady se středem $z_0 = -i$ na maximálním okolí bodu $-i$ a určete jeho parametry.

$$f(z) = (z+i)^3 \frac{1}{(3z-2)^2} = (z+i)^3 \frac{d}{dz}\int \frac{1}{(3z-2)^2}\,dz = \left| \begin{matrix} w = 3z-2 \\ dw = 3dz \end{matrix} \right| = (z+i)^3 \frac{d}{dz}\left(-\frac{1}{3}\,\frac{1}{3z-2}\right)$$

$$= (z+i)^3 \frac{d}{dz}\left(-\frac{1}{3}\,\frac{1}{3(z-\frac{2}{3})}\right) = (z+i)^3 \frac{d}{dz}\left(-\frac{1}{3^2}\,\frac{1}{(-\frac{2}{3}-i)+(z+i)}\right)$$

$$= (z+i)^3 \frac{d}{dz}\left(\frac{1}{3^2}\,\frac{1}{(\frac{2}{3}-i)}\,\frac{1}{\left(1-\left(-\left(\frac{z+i}{-\frac{2}{3}-i}\right)\right)\right)}\right) = (z+i)^3\left(\frac{1}{3^2}\,\frac{1}{(\frac{2}{3}+i)}\right)\frac{d}{dz}\left[\sum_{n=0}^{\infty}\left(-\frac{z+i}{-i-\frac{2}{3}}\right)^n\right] =$$

$$\left|\frac{z+i}{i-\frac{2}{3}}\right| < 1$$

$$z+i < \left|i-\frac{2}{3}\right|$$

$$z+i < \sqrt{\frac{9}{9}+\frac{4}{9}} = \frac{\sqrt{13}}{3}$$

$$R = \sqrt{13}/3 \quad z_0 = -i$$

$$= \frac{1}{3}\,\frac{(z+i)^3}{2+3i}\sum_{n=1}^{\infty} -\frac{n}{-i-\frac{2}{3}}\left(-\frac{z+i}{-i-\frac{2}{3}}\right)^{n-1}$$

$$= \frac{1}{3}\,\frac{(z+i)^3}{2+3i}\sum_{n=1}^{\infty}\frac{3n}{3i+2}\left(\frac{3}{3i+2}\right)^{n-1}(z+i)^{n-1}$$

$$= \sum_{n=1}^{\infty}\frac{3^{n-1}}{(2+3i)^{n+1}}\,n\,(z+i)^{n+2}$$

$$f(z) = (z+i)^3\sum^{\infty}\frac{3^{n-1}}{(3i+2)^{n+1}}n(z+i)^{n-1} = \sum^{\infty}\frac{3^{n-1}}{(3i+2)^{n+1}}n(z+i)^{n+2}$$

$$f(z) = (z+i)^3 \sum_{n=1}^{\infty} \frac{3^{n-1}}{(3i+2)^{n+1}} n(z+i)^{n-1} = \sum_{n=1}^{\infty} \frac{3^{n-1}}{(3i+2)^{n+1}} n(z+i)^{n+2}$$

pro $|z+i| < \frac{\sqrt{13}}{3}$ *(poloměr konvergence je* $R = \frac{\sqrt{13}}{3}$*).*

Úloha 3. *Rozviňte funkci*

$$f(z) = \frac{1}{(z^2+z-6)^3}$$

do Laurentovy řady na maximálním $\underbrace{\text{prstencovým}}_{r=0}$ *okolí bodu* $z_0 = 2$ *a určete jeho parametry.*

$$f(z) = \frac{1}{((z-2)(z+3))^3} = \frac{1}{(z-2)^3} \cdot \frac{1}{(z+3)^3} = \frac{1}{(z-2)^3} \frac{d^2}{dz^2} \int\left(\int \frac{1}{(z+3)^3} dz\right) dz$$

$$= \frac{1}{(z-2)^3} \frac{d^2}{dz^2} \left(\frac{1}{2} \frac{1}{(z+3)} \right) = \frac{1}{(z-2)^3} \frac{d^2}{dz^2} \left(\frac{1}{2\cdot 5} \frac{1}{1-\left(-\left(\frac{z-2}{5}\right)\right)} \right) = \frac{1}{(z-2)^3} \frac{d^2}{dz^2} \left(\frac{1}{2\cdot 5} \sum_{n=0}^{\infty} \left(-\frac{z-2}{5}\right)^n \right)$$

$|z-2|<5$

$$= \frac{1}{(z-2)^3} \frac{d^2}{dz^2} \left(\frac{1}{2\cdot 5} \sum_{n=0}^{\infty} \left(-\frac{1}{5}\right)^n (z-2)^n \right) = \frac{1}{(z-2)^3} \cdot \frac{1}{2\cdot 5} \cdot \sum_{n=2}^{\infty} \left(-\frac{1}{5}\right)^n n(n-1)(z-2)^{n-2}$$

$$= \frac{1}{2} \sum_{n=2}^{\infty} \frac{(-1)^n}{5^{n+1}} n(n-1)(z-2)^{n-5} \quad z_0 = 0;\ r = 0;\ R = 5$$

$$= \frac{1}{2} \sum_{n=2}^{\infty} \frac{(-1)^n}{5^{n+1}} n(n-1)(z-2)^{n-5}$$ *pro* $|z-2| < 5$. *(prstencové okolí bodu 2 má poloměr* $R = 5$*).*

Úloha 4. *Klasifikujte typ izolované singularity funkce*

$$f(z) = \frac{4}{(z-1)^7} + \frac{3}{(z-1)^4} + \frac{2}{(z-1)^2} + \sum_{n=-4}^{\infty} n(z-1)^{3n+5}, \ z \in P(1)$$

v bodě 1.

$$f(z) = \cancel{\frac{4}{(z-1)^7}} + \cancel{\frac{3}{(z-1)^4}} + \frac{2}{(z-1)^2} + \cancel{\underbrace{\frac{-4}{(z-1)^7}}_{n=-4}} + \cancel{\underbrace{\frac{-3}{(z-1)^4}}_{n=-3}} + \underbrace{\frac{-2}{z-1}}_{n=-2} + \sum_{n=-1}^{\infty} n(z-1)^{3n+5}$$

V bodě $z = 1$ má $f(z)$ pól řádu 2