Propočítané příklady CV.4, DÚ4

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Úloha 1. *Nalezněte součet $f(z)$ mocninné řady na jejím kruhu konvergence a určete jeho parametry.*

(a) $\sum_{n=0}^{\infty} 3^n (z+i)^{2n+1} = (z+i)\sum_{n=0}^{\infty}\left(3(z+i)^2\right)^n = (z+i)\dfrac{1}{1-3(z+i)^2}$

$z_0 = -i$ $\quad = \dfrac{z+i}{1-3(z+i)^2}$ $\qquad \left|3(z+i)^2\right| < 1$

$R = \sqrt{\frac{1}{3}}$ $\qquad |z+i| < \sqrt{\frac{1}{3}}$

Úloha 1: (a) $\frac{z+i}{1-3(z+i)^2}$ pro $|z+i| < \frac{1}{\sqrt{3}}$ $(z_0 = -i,\ R = \frac{1}{\sqrt{3}})$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!4^{n+1}} z^{3n+2}$

$\sum_{n=0}^{\infty} \dfrac{z^n}{n!} = e^z$

$\sum_{n=0}^{\infty} \dfrac{(-1)^n}{n!\,4^{n+1}} z^{3n+2} = \frac{1}{4} z^2 \sum_{n=0}^{\infty} \left(\frac{-z^3}{4}\right)^n \cdot \frac{1}{n!}$

$= \dfrac{z^2}{4} e^{-\frac{z^3}{4}}$ $\qquad \left.\begin{array}{l} R = +\infty \\ z_0 = 0 \end{array}\right\}$ konverguje na $\mathbb{C}$

(b) $\frac{z^2}{4} e^{-\frac{z^3}{4}}$ pro každé $z \in \mathbb{C}$ $(z_0 = 0,\ R = \infty)$

(c) $\sum_{n=0}^{\infty} \frac{(2n+1)}{n!} z^{2n+3} = z^3 \sum \frac{2n+1}{n!} z^{2n} = *$

$\dfrac{d}{dz}\int \sum_{n=0}^{\infty} \dfrac{2n+1}{n!} z^{2n}\, dz = \dfrac{d}{dz}\sum_{n=0}^{\infty} \dfrac{z^{2n+1}}{n!} = \dfrac{d}{dz}\, z \sum_{n=0}^{\infty} \dfrac{(z^2)^n}{n!} = \dfrac{d}{dz}\left(z \cdot e^{z^2}\right)$

$* = z^3\left[\dfrac{d}{dz} z \cdot e^{z^2}\right] = z^3\left(e^{z^2} + 2z^2 e^{z^2}\right) = \underline{\underline{z^3 e^{z^2}(1+2z^2)}}$ $\quad z_0 = 0;\ R = \infty \Rightarrow$ konverguje na $\mathbb{C}$

(c) $z^3 e^{z^2}(2z^2+1)$ pro každé $z \in \mathbb{C}$ $(z_0 = 0,\ R = \infty)$

(d) $\sum_{n=0}^{\infty} \frac{(-1)^n (n+2)}{2^{n+1}} z^{n+5} = \frac{1}{2} z^4 \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (n+2) z^{n+1}$

$\boxed{z_0 = 0}$

$$= \frac{1}{2} z^4 \frac{d}{dz} \int \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (n+2) z^{n+1} dz = \frac{1}{2} z^4 \frac{d}{dz} \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n z^{n+2} =$$

$$= \frac{z^4}{2} \frac{d}{dz}\left(z^2 \cdot \sum_{n=0}^{\infty} \left(-\frac{z}{2}\right)^n\right) = \frac{z^4}{2} \frac{d}{dz}\left(\underline{\frac{z^2}{1+\frac{z}{2}}}\right) = \frac{z^4}{2} \cdot \frac{2z\left(1+\frac{z}{2}\right) - z^2 \cdot \frac{1}{2}}{\left(1+\frac{z}{2}\right)^2} =$$

$$\left|-\frac{z}{2}\right| < 1 \Rightarrow R = 2$$

$$= \frac{z^4}{2} \frac{2z + \frac{1}{2} z^2}{\left(1+\frac{z}{2}\right)^2} = \frac{z^5 + \frac{z^6}{4}}{\left(1+\frac{z}{2}\right)^2} \quad z_0 = 0, \; R = 2$$

(d) $\frac{z^6+4z^5}{(z+2)^2}$ pro $|z| < 2$ $(z_0 = 0,\ R = 2)$

$$= \frac{\frac{1}{4}\left(z^6 + 4z^5\right)}{\left(1+\frac{z}{2}\right)^2} = \frac{z^6+4z^5}{4\left(1+\frac{z}{2}\right)^2} = \frac{z^6+4z^5}{2^2\left(1+\frac{z}{2}\right)^2} = \frac{z^6+4z^5}{\left(2\left(1+\frac{z}{2}\right)\right)^2} = \underline{\frac{z^6+4z^5}{(2+z)^2}}$$

(e) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)5^{n+1}} z^{n+3}$ {Ven s $\frac{1}{5}$ a se z^2}

$$= \frac{z^2}{5} \sum_{n=0}^{\infty} \left(-\frac{1}{5}\right)^n \frac{z^{n+1}}{n+1} = \left| \text{Připraveno na derivování} \right| = \frac{z^2}{5}\left(\int \left[\frac{d}{dz}\left(\sum_{n=0}^{\infty} \left(-\frac{1}{5}\right)^n \frac{z^{n+1}}{n+1}\right)\right] dz\right) = \left| \text{zevnitř ven} \right|$$

$$\int \frac{1}{\xi} d\xi = \ln|\xi|$$

$$= \frac{z^2}{5}\left(\int \Bigg(\underbrace{\sum_{n=0}^{\infty} \left(-\frac{1}{5}\right)^n z^n}_{\left(-\frac{z}{5}\right)^n \; z_0 = 0}\Bigg) dz\right) = \frac{z^2}{5} \int \frac{1}{1+\left(\frac{z}{5}\right)} dz = \left| \begin{array}{l} \xi = 1+\frac{z}{5} \\ d\xi = \frac{1}{5} dz \end{array} \right| = z^2 \left(\ln(\xi) + C\right)$$

$$=$$

$$\left|-\frac{z}{5}\right| < 1$$

$$\boxed{\begin{array}{l} R = 5 \\ z_0 = 0 \end{array}}$$

$$\int \frac{1}{\xi} = \ln(\xi) + C \quad \Big| \xi = 1 + \frac{z}{5} \quad \Rightarrow C = 0$$

$$\ln\left(\frac{1}{5}(5+z)\right) = -\ln(5) + \ln(5+z) \Rightarrow \boxed{\begin{array}{l} \sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}(n+1)} z^{n+3} = z^2\left(\ln(5+z) - \ln 5\right) \\ R = 5 \\ z_0 = 0 \end{array}}$$

ACH JO ZASE ZDENDA VYTÝKAL BOŽE MŮJ

INT. Konst. je $C = 0$ (dosadíme za $z = z_0 = 0$)

(e) $z^2(\ln(z+5) - \ln 5)$ pro $|z| < 5$ $(z_0 = 0,\ R = 5)$

(e) $z^2(\ln(z+5)-\ln 5)$ pro $|z|<5$ ($z_0=0$, $R=5$)

(f) $\sum_{n=0}^{\infty} \frac{2^n}{n!(2n+2)} z^{2n+5}$

Potřebujeme sežrat $(2n+2)^{-1}$

→ budeme derivovat $\frac{\partial}{\partial} \longleftrightarrow \int$ 

+ Fucktoriál detected → $\sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$

$$= z^3 \sum_{n=0}^{\infty} \frac{2^n}{n!} \cdot \frac{z^{2n+2}}{2n+2} = z^3 \int \left(\frac{d}{dz} \left[\sum_{n=0}^{\infty} \frac{2^n}{n!} \frac{z^{2n+2}}{2n+2} \right] \right) dz =$$

$$= z^3 \int \left(\underbrace{\sum_{n=0}^{\infty} \frac{2^n}{n!} z^{2n+1}}_{z \sum_{n=0}^{\infty} \frac{(2z^2)^n}{n!}} \right) dz = z^3 \int z \cdot e^{2z^2} dz \overset{SUB}{=} \left| \begin{matrix} \xi = 2z^2 \\ d\xi = 4z\,dz \end{matrix} \right| = \frac{z^3}{4} \int e^{\xi} d\xi =$$

$$\frac{d\xi}{4} = z\,dz$$

$$= \frac{z^3}{4}(e^{\xi} + C)$$

Abych zjistil C musím dosadit do

$$0 = e^{\xi} + C \Rightarrow 0 = \frac{1}{4}(e^{2\cdot 0^2} + C) = \frac{1}{4} + C \Rightarrow C = -\frac{1}{4}$$

součet řady pro $z_0 = 0$

$$\sum_{n=0}^{\infty} \frac{2^n}{n!(2n+2)} z^{2n+5} = \frac{z^3}{4}\left(e^{2z^2} - 1\right) \quad R = \infty,\ z_0 = 0$$

(f) $\frac{z^3}{4}(e^{2z^2}-1)$

Úloha 2.

(a) Víme, že mocninná řada

$$\sum_{n=0}^{\infty} a_n (z+3)^n$$

má poloměr konvergence $R = 6$. Konverguje tato mocninná řada v bodě $z = 4$?

(b) Víme, že mocninná řada

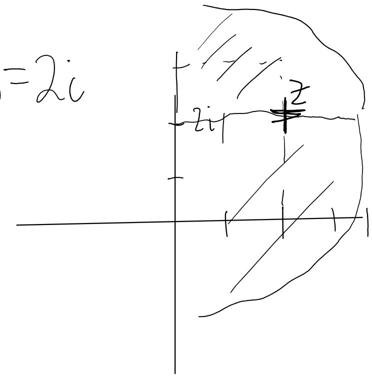
$$\sum_{n=0}^{\infty} a_n (z-2i)^n$$

má poloměr konvergence $R = 4$. Konverguje tato mocninná řada v bodě $z = 2+3i$?

a) $z_0 = -3$; $z = 4 = -3 - (-3-4) = -3 + \underset{>R}{7}$

V bodě $z=4$ Ř. diverguje

b) $z_0 = 2i$



Ano v $z = 2+3i$ Ř. konverguje

Úloha 3. *Rozviňte funkci $f(z)$ do mocninné řady se středem v z_0 a určete parametry jejího kruhu konvergence.*

(a) $f(z) = \frac{z-3}{1-2z}$, $z_0 = 3$
(b) $f(z) = \frac{1}{(z+6)^2}$, $z_0 = -4$
(c) $f(z) = (z-2)^4 e^{3z}$, $z_0 = 2$
(d) $f(z) = \frac{(z+1)^5}{z^2+z-2}$, $z_0 = -1$

a) $\frac{z-3}{1-2z}$, $z_0 = 3$ $f(z) = (z-3)\frac{1}{1-2z} = \frac{z-3}{1-2(z-3+3)} = \frac{z-3}{1-6-2(z-3)} = \frac{z-3}{-5-2(z-3)} =$

$$= \frac{z-3}{-5} \cdot \frac{1}{1-\underbrace{\left(-\frac{2(z-3)}{5}\right)}_{\text{kvocient}}} = -\frac{z-3}{5}\sum_{n=0}^{\infty}\left(-\frac{2}{5}(z-3)\right)^n = -\frac{z-3}{5}\sum_{n=0}^{\infty}\left(-\frac{2}{5}\right)^n (z-3)^n$$

$$= \sum^{\infty}\left(-\frac{1}{5}\right)^{n+1} \cdot 2^n \cdot (z-3)^{n+1}$$

$$\left|-\frac{2}{5}(z-3)\right| < 1 \Rightarrow |z-3| < \frac{2}{5} \Rightarrow R = \frac{2}{5}$$

Úloha 3: (a) $f(z) = \sum_{n=0}^{\infty} \frac{2^n}{(-5)^{n+1}}(z-3)^{n+1}$ pro $|z-3| < \frac{5}{2}$ (tj. $R = \frac{5}{2}$)

(b) $f(z) = \frac{1}{(z+6)^2}$, $z_0 = -4$

$\left| \begin{array}{l} t = z+6 \\ dt = dz \end{array} \right|$

$$\frac{d}{dz}\int (f(z))\,dz = \frac{d}{dz}\int \frac{1}{(z+6)^2}\,dz = \frac{d}{dz}\int t^{-2}\,dt = \frac{d}{dz}\left(-\frac{1}{(z+6)}\right)$$

$$= \frac{d}{dz}\left(\frac{-1}{6+z}\right) = \frac{d}{dz}\left(\frac{-1}{2+(z+4)}\right) = \frac{d}{dz}\left(-\frac{1}{2}\,\frac{1}{1+\left(\frac{z+4}{2}\right)}\right)$$

| Když derivujeme

$$= \frac{d}{dz}\left(-\frac{1}{2}\cdot\frac{1}{1-\underbrace{\left(-\left(\frac{z+4}{2}\right)\right)}}\right) = \frac{d}{dz}\left(-\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z+4}{2}\right)^n\right)$$

Když derivujeme mocn. řadu tak přijdeme o nultý člen

$$\Rightarrow \left(\sum_{n=0}^{\infty}\right)' \to \sum_{n=1}^{\infty}$$

$$\left|-\left(\frac{z+4}{2}\right)\right| < 1$$

$$|z+4| < 2 = R$$

$$= \frac{d}{dz}\left(\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{n+1}(z+4)^n\right)$$

$$= \sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^{n+1} n (z+4)^{n-1} \quad R = 2$$

(b) $f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{n+1}} n(z+4)^{n-1}$ pro $|z+4| < 2$ (tj. $R = 2$)

(c) $f(z) = \underbrace{(z-2)^4}_{\text{MEGA SUPER}} e^{3z}$, $z_0 = 2$

$$e^{z} = \sum_{n=0}^{\infty}\frac{z^n}{n!}$$

$$f(z) = (z-2)^4 e^{3(z-2)+6}$$

$$= (z-2)^4 e^6 \sum_{n=0}^{\infty}\frac{3^n (z-2)^n}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{3^n e^6 (z-2)^{n+4}}{n!} \quad R = \infty$$

konverguje na $\mathbb{C}$

(c) $f(z) = e^6 \sum_{n=0}^{\infty} \frac{3^n (z-2)^{n+4}}{n!}$ pro každé $z \in \mathbb{C}$ (tj. $R = \infty$)

(d) $f(z) = \frac{(z+1)^5}{z^2+z-2}$, $z_0 = -1$

MEGASUPER

$$(z+1)^5 \cdot \frac{1}{(z+2)(z-1)} \quad \Bigg| \quad \frac{1}{(z+2)(z-1)} = \frac{A}{z+2} + \frac{B}{z-1}$$

$$1 = A(z-1) + B(z+2)$$

$$\left.\begin{aligned} 0 &= A + B \\ 1 &= 2B - A \end{aligned}\right\} + \quad > \quad 3B = 1$$

$$B = \frac{1}{3} \quad A = -\frac{1}{3}$$

$$\left.\begin{array}{l} 0 = A + B \\ 1 = 2B - A \end{array}\right\} + \Rightarrow 3B = 1 \quad B = \frac{1}{3} \quad A = -\frac{1}{3}$$

$$f(z) = (z+1)^5 \left[\frac{-\frac{1}{3}}{z+2} + \frac{\frac{1}{3}}{z-1} \right]$$

$$\triangleright \frac{-\frac{1}{3}}{z+2} = \frac{-\frac{1}{3}}{z+1+1} = \frac{-\frac{1}{3}}{1-(-(z+1))} = -\frac{1}{3} \sum_{n=0}^{\infty} (-(z+1))^n \quad R = 1$$

$$\triangleright \frac{\frac{1}{3}}{z-1} = \frac{\frac{1}{3}}{-2+(z+1)} = \frac{\frac{1}{3}}{-2\left(1-\left(\frac{z+1}{2}\right)\right)} = \frac{-\frac{1}{3\cdot 2}}{1-\left(\frac{z+1}{2}\right)}$$

$$= -\frac{1}{3\cdot 2} \sum_{n=0}^{\infty} \left(\frac{z+1}{2}\right)^n \quad R = 2$$

$$f(z) = \underbrace{\left(-\frac{1}{3} \sum_{n=0}^{\infty} (-1)^n (z+1)^{n+5}\right)}_{R=1} - \underbrace{\left(\frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} (z+1)^{n+5}\right)}_{R=2}$$

$$z_0 = -1 \quad \underline{\underline{R = 1}}$$

pro $|z+1| < 1$ (tj. $R = 1$)

Cvičení 4 – Komplexní analýza 2024/2025
Dobrovolná domácí cvičení

Úloha 1. *Nalezněte součet $f(z)$ mocninné řady na jejím kruhu konvergence a určete jeho parametry.*

(a) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!2^{n+2}} (z+6)^{4n+3}$

(b) $\sum_{n=0}^{\infty} \frac{3^{n+2} z^{2n+6}}{2n+2}$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n (n+2)}{4^{n+1}} \frac{(z-2i)^{n+5}}{n!}$

$$a) \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\underline{n!}\, 2^{n+2}} (z+6)^{4n+3} = -\frac{1}{2^2} (z+6)^3 \sum_{n=0}^{\infty} \left(\frac{-(z+6)^4}{2}\right)^n \cdot \frac{1}{n!} = -\frac{(z+6)^3}{4} e^{-\frac{(z+6)^4}{2}} \quad R = \infty$$

konverguje pro $\forall z \in \mathbb{C}$

$$b) \sum_{n=0}^{\infty} \frac{3^{n+2} z^{2n+6}}{\boxed{2n+2}} = 3^2 \cdot z^4 \sum_{n=0}^{\infty} 3^n \frac{z^{2n+2}}{2n+2} = 3^2 z^4 \int \frac{d}{dz} \left(\sum_{n=0}^{\infty} 3^n \frac{z^{2n+2}}{2n+2} \right) dz$$

$$= 3^2 z^4 \int z \sum_{n=0}^{\infty} (3z^2)^n \, dz$$

$n=0$

$$= 3^2 z^4 \int \sum_{n=0}^{\infty} 3^n z^{2n+1}\, dz = 3^2 z^4 \int z \sum_{n=0}^{\infty} (3z^2)^n dz = 3^2 z^4 \int z \sum_{n=0}^{\infty} (3z^2)^n\, dz$$

$|3z^2| < 1$

$|z| < \sqrt{\frac{1}{3}}$

$$= 3^2 z^4 \int z \cdot \frac{1}{1-3z^2}\, dz = 3^2 z^4 \int \frac{-1}{6t}\, dt = -\frac{2}{3} z^4 (\ln(1-3z^2) + C)$$

$t = 1-3z^2$

$dt = -6z\, dz$

$C = 0$

$$\breve{R} = -\frac{2}{3} z^4 \ln(1-3z^2);\ |z| < \sqrt{\tfrac{1}{3}};\ \underline{z_0 = 0}$$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n (n+2)}{4^{n+1}} \frac{(z-2i)^{n+5}}{n!}$ $\quad z_0 = 2i$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (n+2)}{4^{n+1}} \frac{(z-2i)^{n+5}}{n!} = \frac{(z-2i)^4}{4} \sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n \frac{(n+2)}{n!} (z-2i)^{n+1} =$$

$$= \frac{(z-2i)^4}{4} \frac{d}{dz}\left[\int \left(\sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n \frac{n+2}{n!} (z-2i)^{n+1} \right) dz \right] = \frac{(z-2i)^4}{4} \frac{d}{dz}\left[\sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n \frac{(z-2i)^{n+2}}{n!} \right]$$

$$= \frac{(z-2i)^4}{4} \frac{d}{dz}\left[(z-2i)^2 \sum_{n=0}^{\infty} \frac{\left(-\frac{z-2i}{4}\right)^n}{n!} \right] = \frac{(z-2i)^4}{4} \frac{d}{dz}\left[(z-2i)^2 e^{-\frac{z-2i}{4}} \right]$$

$$= \frac{(z-2i)^4}{4} \left(2(z-2i)\, e^{-\frac{z-2i}{4}} + (z-2i)^2 \cdot \left(-\frac{z-2i}{4}\right) e^{-\frac{z-2i}{4}} \right) =$$

$$= \frac{(z-2i)^4}{4} e^{-\frac{z-2i}{4}} \left(2(z-2i) - \frac{(z-2i)^3}{4} \right)$$

$z_0 = 2i \quad R = \infty$