Propočítané příklady CV.1, DÚ1

Sunday, December 22, 2024 14:47

Úloha 1. *Určete reálnou a imaginární část komplexního čísla z. Dále určete velikost z.*

(a) $z = (3-i)^2 + \frac{1+i^{11}}{1+i}$

(b) $z = \frac{i^{12}}{(1+2i)^2}$

CV.1

a) $z = (9-6i-1) + \frac{1-i}{1+i} = 8-6i + \frac{(1-i)^2}{1+1} = 8-6i + \frac{1-2i-1}{2} = 8-6i-i = 8-7i$

$\mathrm{Re}(z) = 8$

$\mathrm{Im}(z) = -7$

$|z| = \sqrt{8^2+7^2} = \sqrt{64+49} = \sqrt{113}$

b) $z = \frac{i^{12}}{(1+2i)^2} = \frac{1}{1+4i-4} = \frac{1}{-3+4i} \cdot \frac{-3-4i}{-3-4i} = \frac{-3-4i}{9+16} = -\frac{3}{25} - \frac{4}{25}i$

$\mathrm{Re}\, z = -\frac{3}{25}$

$\mathrm{Im}\, z = -\frac{4}{25}$

$|z| = \frac{1}{25}\sqrt{3^2+4^2} = \frac{1}{5}$

Úloha 1: (a) $\mathrm{Re}\, z = 8$, $\mathrm{Im}\, z = -7$, $|z| = \sqrt{113}$

(b) $\mathrm{Re}\, z = -\frac{3}{25}$, $\mathrm{Im}\, z = -\frac{4}{25}$, $|z| = \frac{1}{5}$

Úloha 2. *Určete $r > 0$ a (nějaké) $\varphi \in \mathbb{R}$ tak, aby platilo $z = r(\cos\varphi + i\sin\varphi) = re^{i\varphi}$, kde*

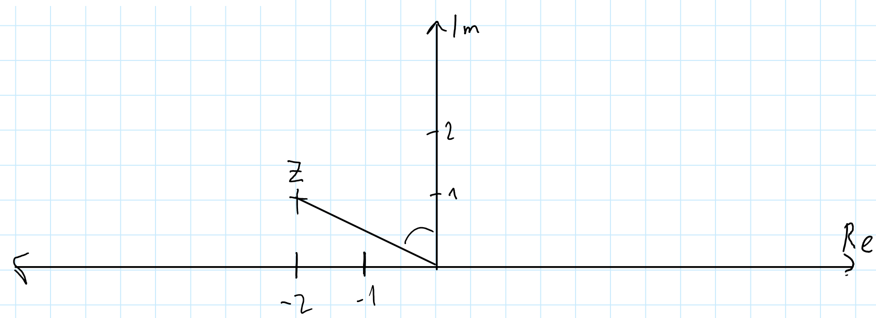
(a) $z = -2+i$;

(b) $z = -1+3i^{43}$;

(c) $z = \frac{i^{31}}{2-i}$.

a) $r = |z| = \sqrt{2^2+1^2} = \sqrt{5}$

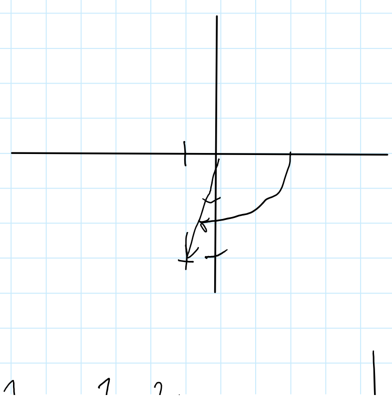
$\varphi = \arg(z) = \frac{\pi}{2} + \arctan 2$



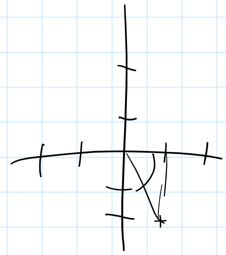
b) $z = -1+3i^{43} = -1-3i$

$r = \sqrt{1^2+3^2} = \sqrt{10}$

$\arg z = -\frac{\pi}{2} - \arctan\left(\frac{1}{3}\right)$



c) $z = \frac{i^{31}}{2-i} = \frac{-i}{2-i} = \frac{-2i+1}{4+1} = \frac{1}{5} - \frac{2}{5}i$



$$\frac{\frac{2}{5}}{\frac{1}{5}} = 2$$

$r = \sqrt{\frac{1}{25} + \frac{4}{25}} = \frac{\sqrt{5}}{5}$

$\arg z = -\arctan 2$

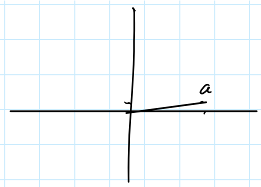
Úloha 3. *Určete velikost a v jakém leží kvadrantu komplexní číslo z. Přibližně ho zakreslete do komplexní roviny. Dále určete hlavní hodnotu argumentu čísla z.*

(a) $z = 5\big(\cos(-\frac{399}{200}\pi) + i\sin(-\frac{399}{200}\pi)\big)$;

(b) $z = (-3-3i)e^{\frac{\pi}{3}i}$;

(c) $z = (5-5i)^{11}$.

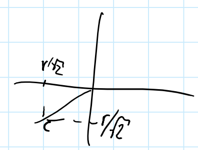
a) $r = 5$ I. kvadrant



$\arg z = \frac{1}{200}\pi$

b) $z = 3\sqrt{2}\, e^{-\frac{3}{4}\pi i} \cdot e^{\frac{\pi}{3}i} = \underbrace{3\sqrt{2}}_{r}\, e^{\underbrace{-\frac{5}{12}\pi i}_{\arg z}}$

c) $z = (5-5i)^{11} = (5-5i)^{11} = \left(5\sqrt{2}\, e^{-\frac{\pi}{4}i}\right)^{11} = (5\sqrt{2})^{11}\, e^{-\frac{11}{4}\pi i} = (5\sqrt{2})^{11}\, e^{-\frac{3}{4}\pi i}$



III. kvadrant

Úloha 4. *Nechť $z, w \in \mathbb{C} \setminus \{0\}$, $\varphi \in \operatorname{Arg} z$ a $\psi \in \operatorname{Arg} w$. Dokažte*[1]*, že $z = w$ tehdy a jen tehdy, když $|z| = |w|$ a $\varphi = \psi + 2k\pi$ pro nějaké $k \in \mathbb{Z}$.*

$z = w$

$|z| e^{i\varphi} = |w| e^{i\psi}$

$|w| e^{i\psi} = |z| e^{i(\psi + 2k\pi)}$

$|w| e^{i\psi} = |z| e^{i\psi + 2k\pi i}$

$|w| e^{i\psi} = |z| e^{i\psi} e^{2k\pi i}$

$|w| e^{i\psi} = |z| e^{i\psi} (e^{2\pi i})^k$, $k \in \mathbb{Z}$, $e^{2\pi i} = \underbrace{\cos 2\pi}_{1} + i \underbrace{\sin 2\pi}_{0} = 1$

$|w| \cancel{e^{i\psi}} = |z| \cancel{e^{i\psi}}$ $\quad e^{i\psi} \neq 0$

$$|w|e^{i\varphi} = |z|e^{i\varphi} \qquad e^{i\varphi} \neq 0$$
$$|w| = |z|$$

Úloha 5. *Nalezněte všechna řešení následujících binomických rovnic.*
(a) $z^4 = 81i$
(b) $z^5 = 1$
(c) $z^2 - 2 - 2i = 0$

a) $z^4 = 81i$

$$|z|^4 e^{4\varphi i} = 81 e^{\frac{\pi}{2}i}$$

$$|z|^4 = 81 \longrightarrow |z|^2 = 9$$
$$4\varphi = \frac{\pi}{2} + 2k\pi \qquad \underline{\underline{|z| = 3}}$$
$$\underline{\underline{\varphi = \frac{\pi}{8} + \frac{1}{2}k\pi \;..\; k \in \{0; 1; 2; 3\}}}$$

b) $z^5 = 1$

$$|z|^5 e^{5i\varphi} = 1 e^0$$
$$5\varphi = 0 + 2k\pi$$
$$\varphi = \frac{2}{5}k\pi \quad k \in \{0; 1; 2; 3; 4\}$$
$$|z| = 1$$
$$z = e^{\frac{2}{5}k\pi i} \quad k \in \{0; 1; 2; 3; 4\}$$

c) $z^2 - 2 - 2i = 0$

$$z^2 = 2 + 2i$$
$$|z|^2 e^{2\varphi i} = 2\sqrt{2}\, e^{\frac{\pi}{4}i}$$
$$|z| = \sqrt[4]{8}$$
$$2\varphi = \frac{\pi}{4} + 2k\pi$$
$$\varphi = \frac{\pi}{8} + k\pi, \; k \in \{0; 1\}$$
$$z = \sqrt[4]{8}\, e^{\left(\frac{\pi}{8} + k\pi\right)i}$$

DÚ 1:

Úloha 1. *Určete reálnou a imaginární část komplexního čísla*

$$z = \frac{6+2i}{-1-2i} + i^{81}.$$

Dále převeďte číslo z do goniometrického a exponenciálního tvaru, tj. určete $r > 0$ a (nějaké) $\varphi \in \mathbb{R}$ tak, aby platilo

$$z = r(\cos\varphi + i\sin\varphi) = re^{i\varphi}$$

$$\begin{array}{r} 21 \\ \cdot\; 21 \\ \hline 21 \\ 42 \\ \hline 441 \end{array}$$

$$z = \frac{\;\cdot\;}{-1-2i} + i\ .$$

Dále převeďte číslo z do goniometrického a exponenciálního tvaru, tj. určete $r > 0$ a (nějaké) $\varphi \in \mathbb{R}$ tak, aby platilo

$$z = r(\cos\varphi + i\sin\varphi) = re^{i\varphi}.$$

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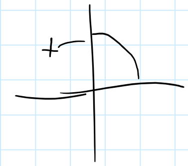

$$z = \frac{6+2i}{-1-2i} + i = \frac{(6+2i)(-1+2i)}{1+4} + i = \frac{-10+10i}{5} + i = -2+3i$$

$\operatorname{Re} z = -2$

$\operatorname{Im} z = \frac{21}{5}$

$r = \sqrt{4+4} = \sqrt{13}$

$\arg z = \frac{\pi}{2} + \operatorname{arctg}\frac{2}{3}$

$z = -2+2i = \sqrt{8}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = \sqrt{8}\, e^{i\frac{3\pi}{4}}$

Úloha 2. *Určete velikost a v jakém leží kvadrantu komplexní číslo*

$$z = (-2-2i)^{13}(3+3i)^{20}.$$

$$z = \sqrt{8}^{13} e^{13\left(-\frac{3}{4}\pi\right)i} \cdot \sqrt{18}^{10} e^{20i\left(\frac{\pi}{4}\right)} = 12\, e^{-\frac{19}{4}\pi i} = 12^{-\frac{3}{4}\pi i}$$

$2\sqrt{2} \cdot 3\sqrt{2} = 6 \cdot 2 = 12$

III. Kvadrant

$|z| = (\sqrt{8})^{13}(\sqrt{18})^{20}$ $\quad \arg z = -\frac{3}{4}\pi$

Úloha 3. *Nalezněte všechna řešení binomické rovnice*

$$z^3 = -5i.$$

$z^3 = -5i$

$|z|^3 e^{3\varphi i} = 5e^{-\frac{\pi}{2}i}$

$|z| = \sqrt[3]{5}$

$3\varphi = -\frac{\pi}{2} + 2k\pi$

$\varphi = -\frac{\pi}{6} + \frac{2}{3}k\pi,\ k \in \{0;1;2\}$

$$\boxed{z = \sqrt[3]{5}\, e^{\left(-\frac{\pi}{6} + \frac{2}{3}k\pi\right)i},\ k \in \{0;1;2\}}$$