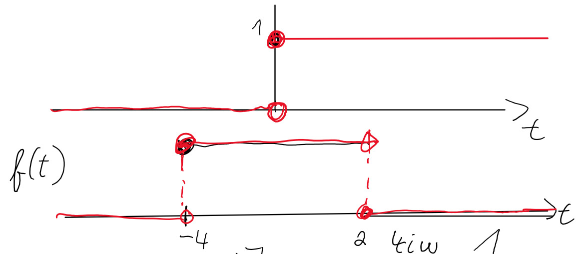

Úloha 1. *Spočtěte Fourierovu transformaci funkce*

$$f(t) = \mathbb{1}(t+4) - \mathbb{1}(t-2) = \begin{cases} 1 & \text{pokud } t \in [-4,2), \\ 0 & \text{pokud } t \in \mathbb{R} \setminus [-4,2), \end{cases}$$

kde $\mathbb{1}$ *je* Heavisideova funkce *definovaná jako*

$$\mathbb{1}(t) = \begin{cases} 1 & \text{pokud } t \geq 0, \\ 0 & \text{pokud } t < 0. \end{cases}$$

$$\mathcal{F}[\mathbb{1}(\underset{-4}{t+4}) - \mathbb{1}(t-2)](\omega) = \mathcal{F}[\mathbb{1}(t+4)](\omega) - \mathcal{F}[\mathbb{1}(t-2)](\omega) = e^{\underset{-(-4)}{4i\omega}} \cdot \frac{1}{i\omega} - e^{-2i\omega}\frac{1}{i\omega}$$

$\omega \neq 0$

$$\int_{-\infty}^{\infty} \mathbb{1}(t)\, e^{-i\omega t}\, dt = \int_{0}^{\infty} 1 \cdot e^{-i\omega t}\, dt = \left[\frac{e^{-i\omega t}}{-i\omega}\right]_0^{\infty} = 0 - \frac{1}{-i\omega} = \frac{1}{i\omega}$$

$$\hat{f}(\omega) = \begin{cases} \frac{e^{-2i\omega} - e^{4i\omega}}{\omega} i & \text{pokud } \omega \neq 0, \\ 6 & \text{pokud } \omega = 0. \end{cases}$$

$\omega = 0$ $\displaystyle\int_{-\infty}^{\infty} f(t)\, dt = \int_{-4}^{2} f(t)\, dt = 6 \cdot 1 = 6$

$$(a)\ \mathcal{F}[h(4t+5)](\omega) = e^{i\omega \cdot 5}\, \mathcal{F}[h(4t)](\omega) = e^{\frac{5\omega}{4} i} \cdot \frac{1}{4}\, \mathcal{F}[h(t)]\left(\frac{\omega}{4}\right)$$

(a) $\frac{1}{4} e^{\frac{5\omega}{4} i} \hat{h}(\frac{\omega}{4})$

$$\mathcal{F}[f(t-a)](\omega) = e^{-i\omega a} \cdot \mathcal{F}[f(t)](\omega)$$

$$\mathcal{F}[f(at)](\omega) = \frac{1}{|a|}\, \mathcal{F}[h(t)]\left(\frac{\omega}{a}\right)$$

$$\mathcal{F}[e^{iat} f(t)](\omega) = \mathcal{F}[f(t)](\omega - a)$$

b) $$(b)\ \mathcal{F}[e^{5it} h(t-2)](\omega) = \mathcal{F}[h(t-2)](\omega - 5) = e^{-i(\omega-5)\cdot 2}\, \hat{h}(\omega - 5)$$

(b) $e^{-2(\omega-5)i} \hat{h}(\omega - 5)$

$$(c)\ \mathcal{F}[h''(t)](\omega) = (i\omega)^2 \hat{h}(\omega) = -\omega^2 \hat{h}(\omega)$$

$$\mathcal{F}\left[\frac{d^n h(t)}{dt^n}\right](\omega) = (i\omega)^n \hat{h}(\omega)$$

(c) $-\omega^2 \hat{h}(\omega)$

(d) $\mathcal{F}[\sin(3t)h'(t+4)](\omega) = \mathcal{F}\left[\frac{e^{3it}-e^{-3it}}{2i} h'(t+4)\right](\omega)$

$\sin(t) = \frac{e^{it}-e^{-it}}{2i}$ $= \mathcal{F}\left[\frac{e^{3it}}{2i} h'(t+4)\right](\omega) - \mathcal{F}\left[\frac{e^{-3it}}{2i} h'(t+4)\right](\omega)$

$= \frac{1}{2i}\left(\mathcal{F}[h'(t+4)](\omega-3) - \mathcal{F}[h'(t+4)](\omega+3)\right) \neq \cancel{\frac{e^{4i(\omega+3)}}{2i}}\left(\mathcal{F}[h'(t)](\omega-3) - \mathcal{F}[h'(t)](\omega+3)\right)$

$\neq \cancel{\frac{e^{-4i(\omega+3)}}{2i}}\left(\cancel{i}(\omega-3)\hat{h}(\omega-3) - \cancel{i}(\omega+3)\hat{h}(\omega+3)\right) = \frac{1}{2}\left(e^{4i(\omega-3)}(\omega-3)\hat{h}(\omega-3) - e^{4i(\omega+3)}(\omega+3)\hat{h}(\omega+3)\right)$

$+3$ $\quad -3$

(d) $\frac{1}{2}(e^{4(\omega-3)i}(\omega-3)\hat{h}(\omega-3) - e^{4(\omega+3)i}(\omega+3)\hat{h}(\omega+3))$

(a) $\mathcal{F}[te^{-9t^2}](\omega) = i\frac{d}{d\omega}\mathcal{F}[e^{-9t^2}](\omega) = i\frac{d}{d\omega}\left(\frac{\sqrt{\pi}}{3} e^{-\frac{\omega^2}{36}}\right) = i\frac{\sqrt{\pi}}{3}\left(-\frac{2\omega}{36}\right)e^{-\frac{\omega^2}{36}}$

$= -i\frac{\sqrt{\pi}}{54}e^{-\frac{\omega^2}{36}}$ $\quad -i\frac{\sqrt{\pi}}{54}e^{-\frac{\omega^2}{36}}$

$-a = -9 \Rightarrow a = 9$

$\mathcal{F}[t f(t)] = i\frac{d}{d\omega}\hat{f}(t)$ $\qquad \mathcal{F}[e^{-at^2}] = \sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}}$

(b) $\mathcal{F}[e^{-4(2t-3)^2}](\omega) \overset{1)}{=} \frac{1}{2}\mathcal{F}[e^{-4(t-3)^2}]\left(\frac{\omega}{2}\right) \overset{2)}{=} \frac{e^{-\frac{3}{2}\omega i}}{2}\mathcal{F}[e^{-4t^2}]\left(\frac{\omega}{2}\right) =$

$= \frac{e^{-\frac{3}{2}\omega i}}{2}\cdot\sqrt{\frac{\pi}{4}}\, e^{\frac{-\left(\frac{\omega}{2}\right)^2}{4\cdot 4}} = \frac{\sqrt{\pi}}{4} e^{-\frac{3}{2}\omega i - \frac{1}{64}\omega^2}$ ✓

1) škálování $\quad$ 2) posun vzoru $\quad$ 3) Gauss

(b) $\frac{\sqrt{\pi}}{4}e^{-\frac{3\omega}{2}i}e^{-\frac{\omega^2}{64}}$

Úloha 4. *Nalezněte Fourierův obraz $\hat{y}(\omega)$ řešení diferenciální rovnice*

$$y'''(t) + 2y'(t) + 3y(t) = \frac{1}{1+t^2}. \qquad / \mathcal{F}$$

[Využijte faktu, že $\mathcal{F}[\frac{1}{1+t^2}](\omega) = \pi e^{-|\omega|}$.]

$$(i\omega)^3\hat{y}(t) + 2i\omega\hat{y}(t) + 3\hat{y}(\omega) = \pi e^{-|\omega|}$$

$$\hat{y}(t) = \frac{\pi e^{-|\omega|}}{-i\omega^3 + 2i\omega + 3}$$

Úloha 4: $\hat{y}(\omega) = \frac{\pi e^{-|\omega|}}{-i\omega^3+2i\omega+3}$

Úloha 5. *Určete řešení $y(t)$ diferenciální rovnice, víte-li že její Fourierův obraz je*
(a) $\hat{y}(\omega) = \frac{1}{\omega^2-2\omega+5}$;
(b) $\hat{y}(\omega) = \frac{i}{(\omega-2i)^2(\omega+i)}$.

a) $$\mathcal{F}^{-1}[\hat{y}(w)](t) = \mathcal{F}^{-1}\left[\frac{1}{w^2-2w+5}\right](t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{itw}}{w^2-2w+5}\,dw =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{itw}}{(w-1+2i)(w-1-2i)}\,dw = I$$

pro $t \geq 0$

$$I = +\frac{1}{\cancel{2\pi}}\cdot \cancel{2\pi} i\left(\mathrm{res}_{w=1+2i} \frac{e^{itw}}{(w-1+2i)(w-1-2i)}\right)$$

$$I = i\left(\frac{e^{it-2t}}{4i}\right) = \underline{\underline{\frac{e^{it-2t}}{4}}} \quad t \geq 0$$

$t < 0$

$$I = -\frac{1}{2\pi}\cdot 2\pi i\left(\mathrm{res}_{w=1-2i} \frac{e^{itw}}{(w-1+2i)(w-1-2i)}\right) = \cancel{-i}\, \frac{e^{it+2t}}{\cancel{-4i}} = \underline{\underline{\frac{e^{it+2t}}{4}}}$$

$$I = \begin{cases} t \geq 0 & \frac{e^{it-2t}}{4} \\ t < 0 & \frac{e^{it+2t}}{4} \end{cases}$$

Úloha 5: (a) $y(t) = \begin{cases} \frac{e^{-2t+it}}{4} & \text{pokud } t \geq 0, \\ \frac{e^{2t+it}}{4} & \text{pokud } t < 0. \end{cases}$

(b) $\hat{y}(\omega) = \frac{i}{(\omega-2i)^2(\omega+i)}$. $/ \mathcal{F}^{-1}$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{i\cdot e^{itw}}{(w-2i)^2(w+i)}\,dw$$

pro $t \geq 0$

$$y(t) = \frac{1}{\cancel{2\pi}}\cdot \cancel{2\pi} i \cdot \mathrm{res}_{2i}\left(\frac{i e^{itw}}{\underbrace{(w-2i)^2}_{2i \text{... násk. } 2}(w+i)}\right)$$

$$\mathrm{res}_{w=2i} \hat{f}(t) = \lim_{w\to 2i}\left(\cancel{(w-2i)^2} \frac{i e^{itw}}{\cancel{(w-2i)^2}(w+i)}\right)'$$

$$= \lim_{w\to 2i} \frac{-e^{itw}(w+i) - ie^{itw}}{(w+i)^2} = \frac{-e^{-2t}\cdot 3i - ie^{-2t}}{-9} = \frac{3ie^{-2t}+ie^{-2t}}{9}$$

$t < 0$

$$f(t) = \frac{1}{2\pi}(-2\pi i)\,\mathrm{res}_{w=-i} \frac{i e^{itw}}{(w-2i)^2(w+i)}$$

$$= -i\cdot \frac{i e^{t}}{-9}$$

$$= \underline{\underline{-\frac{e^{t}}{9}}} \quad t < 0$$

$t \geq 0 \,|\, -\underline{\frac{3te^{-2t}+e^{-2t}}{9}}$

$$w \to 2i \qquad (w+i)^2 \qquad -9 \qquad = \frac{\quad}{9}$$

$$f(t) = i \cdot \frac{3ite^{-2t} + ie^{-2t}}{9} = -\frac{3te^{-2t} + e^{-2t}}{9} \quad t \geq 0$$

$$f(t) = \begin{cases} t \geq 0 & -\dfrac{3te^{-2t} + e^{-2t}}{9} \\ t < 0 & -\dfrac{e^{t}}{9} \end{cases}$$

(b) $y(t) = \begin{cases} -\frac{3t+1}{9}e^{-2t} & \text{pokud } t \geq 0, \\ -\frac{e^t}{9} & \text{pokud } t < 0. \end{cases}$

Úloha 6. *Pomocí Fourierovy transformace „dostatečně pěkné“ funkce* $h(t) \in L^1(\mathbb{R})$ *vyjádřete*

$$\mathcal{F}[(te^{-\frac{(3t+2)^2}{2}}) * (e^{-2it}h'''(t))](\omega).$$

$$= \mathcal{F}\left[te^{-\frac{(3t+2)^2}{2}}\right](\omega) \cdot \mathcal{F}\left[e^{-2it}h'''(t)\right](\omega)$$