Propočítáné příklady CV.7,DÚ7

Monday, December 30, 2024 16:41

Cvičení 7 a 8 – Komplexní analýza 2024/2025
Týden 8 a 9

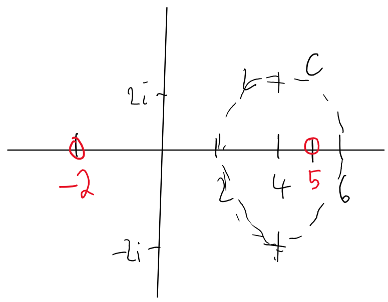
Úloha 2. *Spočtěte.*

(a)

$$\int_C \frac{z}{(z+2)(z-5)^2}\,dz, = 2\pi i \operatorname{res}_5\left(\frac{z}{(z+2)(z-5)^2}\right) = \frac{4}{49}\pi i$$

Úloha 2: (a) $\frac{4}{49}\pi i$

kde C je kladně orientovaná kružnice o rovnici $|z-4|=2$.



V $z=5$ má $\frac{z}{(z+2)(z-5)^2}$ pól řádu 2

$\boxed{k=2}$

$$\lim_{z\to 2}\left(\cancel{(z-5)^2}\frac{z}{(z+2)\cancel{(z-5)^2}}\right)' = \lim_{z\to 5}\left(\frac{z}{z+2}\right)' =$$

$$= \lim_{z\to 5}\left(\frac{z+2-z}{(z+2)^2}\right) = \frac{2}{49}$$

(b)

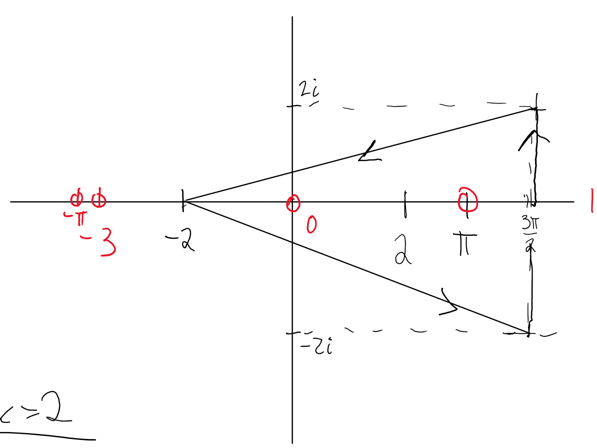
$$\int_C \frac{1}{z\sin z} + \frac{e^{\sin z}}{z+3}\,dz, = \int_C \frac{1}{z\sin z}dz + \underbrace{\int_C \frac{e^{\sin z}}{z+3}dz}_{0} = 2\pi i\left(\operatorname{res}_0\left(\frac{1}{z\sin z}\right) + \operatorname{res}_\pi\left(\frac{1}{z\sin z}\right)\right)$$

kde C je kladně orientovaná hranice trojúhelníka s vrcholy -2, $\frac{3\pi}{2}+2i$, $\frac{3\pi}{2}-2i$.

$$= 2\pi i \frac{1}{-\pi} = -2i \quad \text{yes}$$

Izol. singularity: 0, $k\pi$, $\cancel{-3}$

$\pi \to$ pól řádu 2



k=2

$f(z) = \frac{1}{z\sin z}$ v $z_0 = 0$ má izol. singul.

Pól řádu 2

Čit. nás. kor. $=0$

Jmen. nás. kor.:

$z\sin z|_{z=0} = 0$

$(z\sin z)' = \sin z + z\cos z|_{z=0} = 0$

$(z\sin z)'' = \cos z + \cos z - z\sin z|_{z=0} = 2 \neq 0 \to$ jmen. nás. kor.: 2

$$\operatorname{res}_0 f(z) = \lim_{z\to 0}\left((z)^2\frac{1}{z\sin z}\right)' =$$

$$= \lim_{z\to 0}\left(\frac{z}{\sin z}\right)' = \lim_{z\to 0}\left(\frac{\sin z - z\cos z}{\sin^2 z}\right) \overset{L'H}{\underset{\frac{0}{0}}{=}} \lim_{z\to 0}\left(\frac{\cos z - \cos z + z\sin z}{2\cos^2 z}\right) = 0$$

$$\operatorname{res}_\pi f(z) = \lim_{z\to\pi}\left((z-\pi)\frac{1}{z\sin z}\right) = \lim_{z\to\pi}\left(\frac{z-\pi}{z\sin z}\right) \overset{L'H}{=} \lim_{z\to\pi}\frac{1}{\sin z + z\cos z} = \frac{1}{-\pi}$$

pól řádu 1

(c)

$$\int_C \frac{1}{e^z-1} + \frac{z-\frac{\pi}{2}}{(\cos z)^2}\,dz, = 2\pi i \operatorname{res}_{\frac{\pi}{2}}\left(\frac{z-\frac{\pi}{2}}{(\cos z)^2}\right) = 2\pi i$$

(c) $2\pi i$

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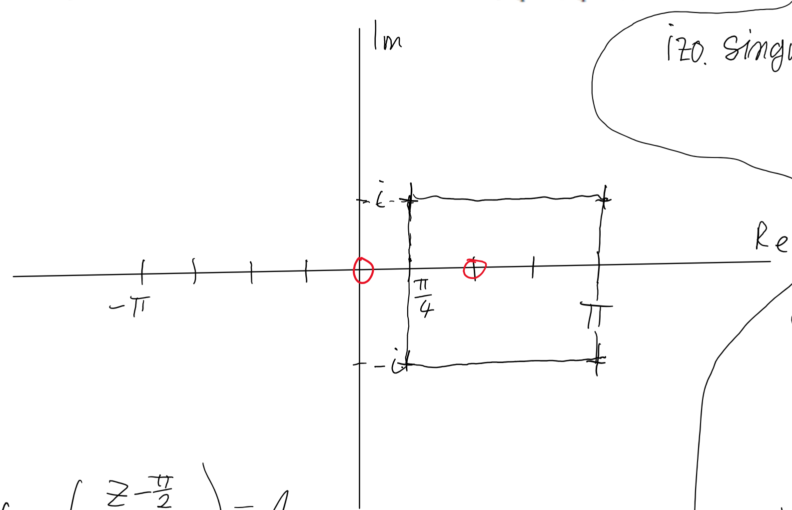
$$\int_C \frac{1}{e^z-1} + \frac{z-\frac{\pi}{2}}{(\cos z)^2}\,\mathrm{d}z, = 2\pi i \operatorname{res}_{\frac{\pi}{2}}\left(\frac{z-\frac{\pi}{2}}{(\cos z)^2}\right) = \underline{\underline{2\pi i}}$$

(c) $2\pi i$

kde C je kladně orientovaná hranice obdélníka s vrcholy $\frac{\pi}{4}+i$, $\frac{\pi}{4}-i$, $\pi+i$, $\pi-i$.

izo. singul.: $2k\pi,\ \frac{\pi}{2}+k\pi,\ k\in\mathbb{Z}$

Zajímá nás pouze Izol. singul. v $\frac{\pi}{2}$



$$\check{C}:\ z-\frac{\pi}{2}\Big|_{z=\frac{\pi}{2}} = 0$$

$$\left(z-\frac{\pi}{2}\right)' = 1 \neq 0 \ \to \text{násobnost} = 1$$

$$J:\ \cos z\Big|_{z=\frac{\pi}{2}} = 0$$

$$(\cos z)' = -\sin z\Big|_{z=\frac{\pi}{2}} = -1 \neq 0 \ \to \text{násobnost } 1\cdot\underline{2}$$

V $\frac{\pi}{2}$ pól řádu $2-1=1$

$$\operatorname{res}_{\frac{\pi}{2}}\left(\frac{z-\frac{\pi}{2}}{(\cos z)^2}\right) = 1$$

$$= \lim_{z\to\frac{\pi}{2}}\left(\left(z-\frac{\pi}{2}\right)\frac{z-\frac{\pi}{2}}{(\cos z)^2}\right) = \lim_{z\to\frac{\pi}{2}} \frac{\left(z-\frac{\pi}{2}\right)^2}{\cos^2 z}$$

$$= \left(\frac{0}{0}\right) \overset{L'H}{=} \lim_{z\to\frac{\pi}{2}} \frac{2\left(z-\frac{\pi}{2}\right)}{-2(\sin z)(\cos z)} = \left(\frac{0}{0}\right) \overset{L'H}{=} \lim_{z\to\frac{\pi}{2}} \frac{2}{-(\sin 2z)'} = \lim_{z\to\frac{\pi}{2}} \frac{2}{-2\cos 2z} = \underline{1}$$

$$(a)\ \int_{-\infty}^{\infty} \frac{1}{x^2-6x+25}\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{e^{i0x}}{x^2-6x+25} = \int_{-\infty}^{\infty} \frac{e^{i0x}}{(x-3)^2+4^2} = \int_{-\infty}^{\infty} \frac{e^{i0x}}{((x-3)-4i)((x-3)+4i)}$$

$$\underline{\alpha = 0} \ \to\ \int_{-\infty}^{\infty} \boxminus = 2\pi i \sum_{w\in S^+} \operatorname{res}_w \boxminus$$

Izol singularity v

$$\underline{x = 3+4i}$$

$$x = 3-4i$$

$$\operatorname{res}_{3+4i} \frac{1 \to \text{násobnost}=0}{\underbrace{(x-3-4i)}_{\text{násobnost}=1}(x-3+4i)} \underset{\text{dosad}}{=} \frac{1}{\underset{\text{derivace}}{1}\cdot(8i)} = \frac{1}{8i}$$

$$\int_{-\infty}^{\infty} \frac{1}{x^2-6x+25}\,\mathrm{d}x = 2\pi i\cdot\frac{1}{4i} = \underline{\underline{\frac{1}{4}\pi}}$$

(a) $\frac{\pi}{4}$

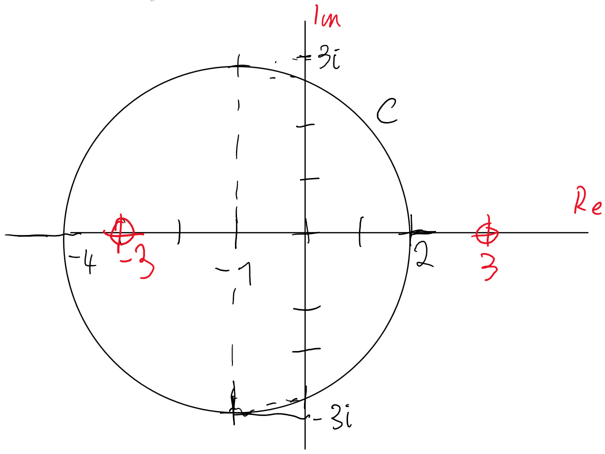
Úloha 3. *Určete, čemu se rovná* $f(z)$

$\int_{-\infty}$

Úloha 3. *Určete, čemu se rovná*

$$\int_C \underbrace{\left(\frac{2}{(z+3)^5} + \cancel{\frac{3}{z-3}} + \frac{4}{z+3}\right)}_{f(z)} \mathrm{d}z, \; = 2\pi i \left(res_{-3}\left(\frac{2}{(z+3)^5}\right) + res_{-3}\left(\frac{4}{z+3}\right)\right)$$

kde C je kladně orientovaná kružnice o rovnici $|z+1| = 3$.



singularity v

$z=-3$, $z=3$

$res_{-3} f(z) = 4$

$$\int_C \frac{2}{(z+3)^5} + \frac{4}{z+3} + \frac{3}{z-3}\, dz = \underbrace{\int_C \frac{2}{(z+3)^2} + \boxed{\frac{4}{z+3}} dz}_{\text{rozvoj}} + \underbrace{\int_C \frac{3}{z-3} dz}_{0} = 2\pi i \cdot 4 = 8\pi i$$

Úloha 3: $8\pi i$

residuum=4

singularita v $z=3$ není uvnitř C

(b) $\int_{-\infty}^{\infty} \frac{x^2 e^{i0\pi}}{(x^2+4)^2} \mathrm{d}x = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)^2} dx = res_{+2i}\Big(\underbrace{\frac{x^2}{(x^2+4)^2}}_{f(x)}\Big)$

izol. singularita v $x = \pm 2i$

$\alpha = 0$ beru izol. singul. s kl. imaginar. hodnotami

Ve $+2i$ je pól řádu 2 $k=2$

$$res_{2i} f(x) = \lim_{x\to 2i}\left(\cancel{(x-2i)^2}\frac{x^2}{(x+2i)^2\cancel{(x-2i)^2}}\right)' = \lim_{x\to 2i} \frac{2x(x+2i)^2 - x^2 \cdot 2(x+2i)}{(x+2i)^4}$$

$$= \frac{4i \cdot 16i^2 - 4i^2 \cdot 8i}{4^4} = \frac{-i(64-32)}{256} = -i\frac{32}{256} = -\frac{i}{8}$$

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)^2} dx = 2\pi i \cdot \frac{-i}{8} = \frac{\pi}{4}$$

(b) $\frac{\pi}{4}$