PR01

Monday, September 23, 2024 08:46

Komplexní čísla

$i^2 = -1$, označujem mn. $\mathbb{C}$

$z = x + iy$

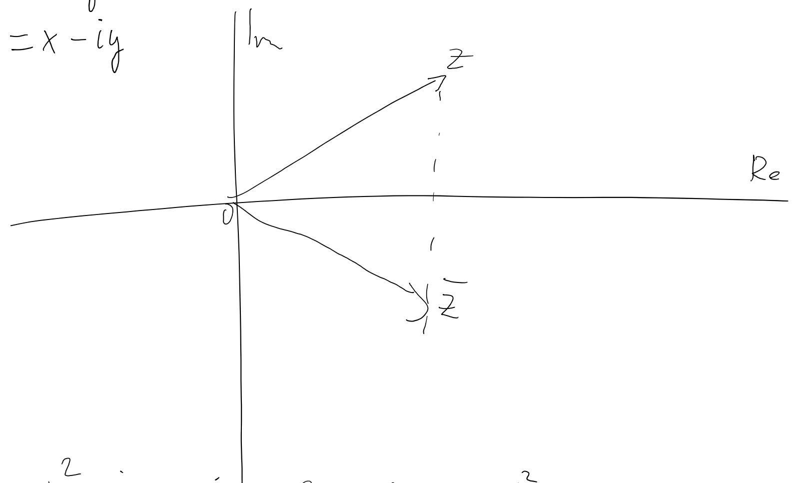
$\operatorname{Re} z = x$

$\operatorname{Im} z = y$

Sdružení komplexní číslo

$z = x + iy$

$\bar{z} = x - iy$



Př.:

$z\bar{z} = |z|^2$

$z = x + iy$

$\bar{z} = x - iy$

$$z \cdot \bar{z} = (x+iy)(x-iy) = x^2 - ixy + ixy + y^2 = \underline{x^2 + y^2} = |z|^2$$

$z, w \in \mathbb{C}$

$z \neq 0$

$$\frac{w}{z} = w \cdot \frac{1}{z} =$$

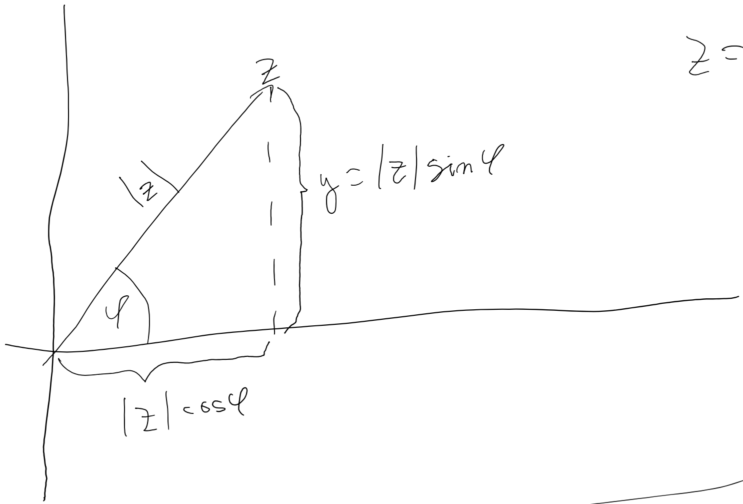
$$\frac{1}{z} z = 1 \quad / \cdot \bar{z}$$

$$\frac{z\bar{z}}{z} = \bar{z}$$

$$\frac{|z|^2}{z} = \bar{z}$$

$$\boxed{\frac{1}{z} = \frac{\bar{z}}{|z|^2}}$$

Goniometrický tvar



$$z = x + iy = |z| \cdot (\cos\varphi + i \sin\varphi)$$

Exponenciální tvar

$$e^{i\varphi} = (\cos\varphi + i\sin\varphi),\ \varphi \in \mathbb{R}$$

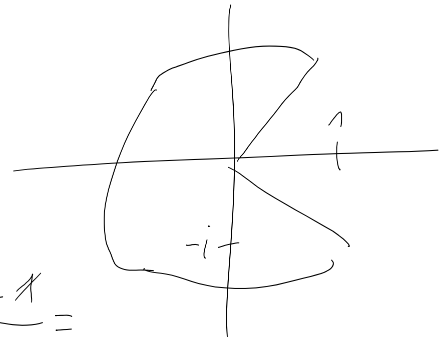
$$z = |z| \cdot e^{i\varphi}$$

Argument

$$\operatorname{Arg} z = \varphi,\ \boxed{\varphi \in \boxed{(-\pi, \pi\rangle}}$$

Úloha 1

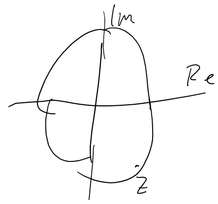
Určete Re a Im komplx. č. z, Určete |z|



(a) $z=(3-i)^2+\frac{1+i^{11}}{1+i}$

$z=9-6i-1+\frac{1-i}{1+i}\cdot\frac{1-i}{1-i}=8-6i+\frac{\cancel{1}-2i-\cancel{1}}{1+1}=$

$z=8-6i \quad -i=\underline{\underline{8-7i}}$



$\operatorname{Re} z=8$

$\operatorname{Im} z=-7$

$|z|=\sqrt{64+49}=\underline{\underline{\sqrt{103}}}$

(b) $\boxed{\times}$

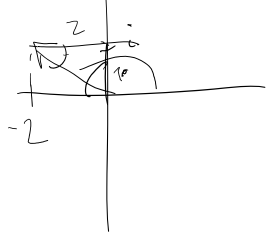
Úloha 2

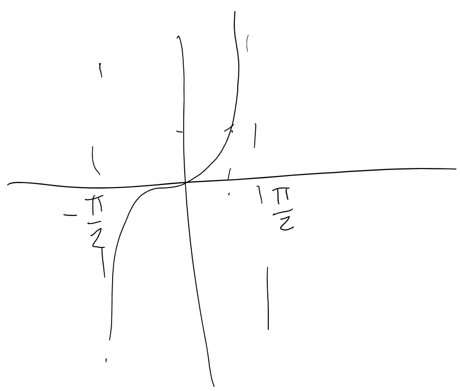
Gon. a exp. tvar

a) $z=-2+i$

$\varphi=\frac{\pi}{2}+\operatorname{arctg}2$

$\operatorname{tg}\varphi=\frac{2}{1}$



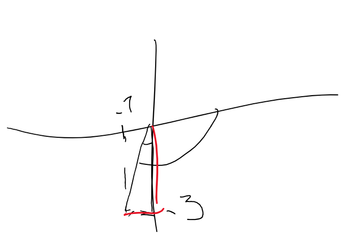


$\varphi=$

$|z|=\sqrt{4+1}=\sqrt{5}$

$z=\sqrt{5}\left(\cos\left(\frac{\pi}{2}+\operatorname{arctg}2\right)+i\sin\left(\frac{\pi}{2}+\operatorname{arctg}2\right)\right)=\sqrt{5}\,e^{i\varphi}$

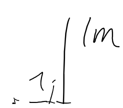
b) $-1+3i^{43}=-1-3i$



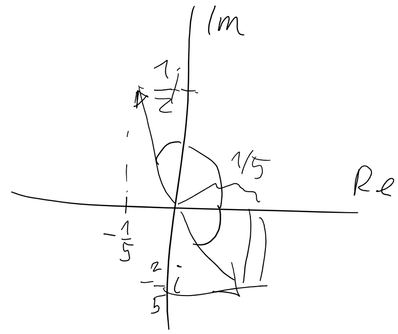
$|z|=\sqrt{10}$

$\varphi=-\frac{\pi}{2}-\operatorname{arctg}\frac{1}{3}$

$z=\sqrt{10}\left(\cos\left(-\frac{\pi}{2}-\operatorname{arctg}\left(\frac{1}{3}\right)\right)+i\sin\left(-\frac{\pi}{2}-\operatorname{arctg}\frac{1}{3}\right)\right)$



c) $z = \frac{i^{31}}{2-i} = \frac{-i}{2-i} = \frac{-2i+1}{4+1} = \frac{1}{5} - \frac{2}{5}i$



$|z| = \sqrt{\frac{5}{25}} = \frac{\sqrt{5}}{5}$

$\varphi = -\operatorname{arctg} 2$

$z = \sqrt{\frac{29}{100}}\left(\cos\varphi + i\sin\varphi\right)$

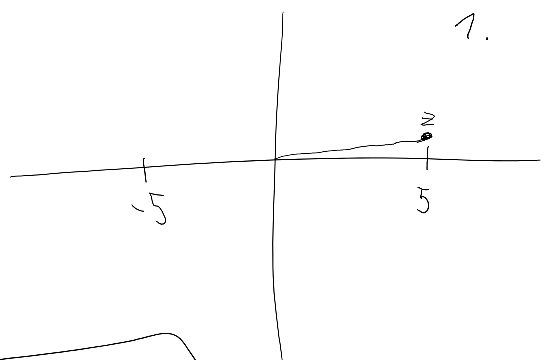
---

3)

a) $z = 5\left(\cos\left(-\frac{399}{200}\pi\right) + i\sin\left(-\frac{399}{200}\pi\right)\right)$

$\arg z = \frac{1}{200}\pi$

$|z| = 5$



b) $z = (-3-3i)\,e^{\frac{\pi}{3}i}$

kvadrant 4.

$z = \sqrt{18}\,e^{-\frac{3}{4}\pi i}\,e^{\frac{\pi}{3}i} = \sqrt{18}\,e^{-\frac{5}{12}\pi i}$

$\sqrt{18} \to |z|$

$\arg z = -\frac{5}{12}\pi$

PRO2

Tuesday, October 1, 2024 09:12

$\cdot\ f : D \subseteq \mathbb{C} \to \mathbb{C}$

$z = x + iy$ $\qquad f(z) = \underbrace{u(x,y)}_{\in \mathbb{R}} + i\, \underbrace{v(x,y)}_{\in \mathbb{R}}$

$u = \operatorname{Re} f$

$v = \operatorname{Im} f$

Př.: $f(z) = z^2 = (x+iy)^2$

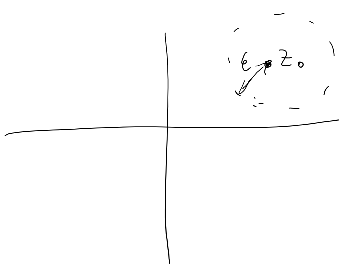
$z = x+iy) = x^2 + 2xyi - y^2$

$= \underbrace{x^2 - y^2}_{\operatorname{Re} f(z)} + i\,\underbrace{(2xy)}_{\operatorname{Im}(f(z))}$

Okolí bodu

$U(z_0, \varepsilon) = \{ z \in \mathbb{C} \mid |z - z_0| < \varepsilon \}$

$P(z_0, \varepsilon) = U(z_0, \varepsilon) \setminus \{z_0\}$

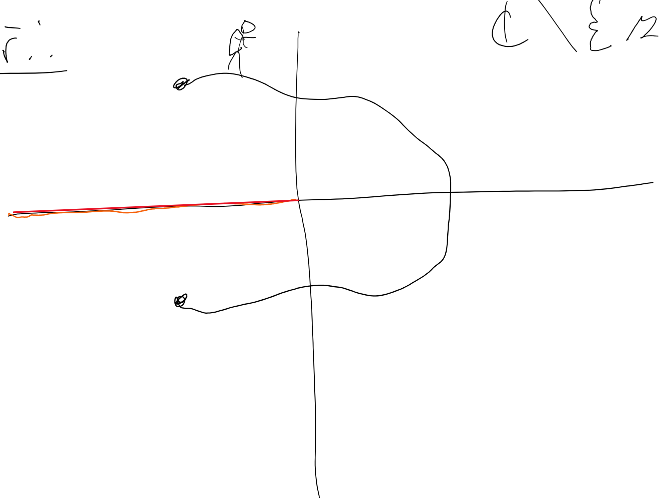


$\cdot\ r \subseteq \mathbb{C}$

r obecně

Oblast = souvislá otevřená množina

Př.:



$\mathbb{C} \setminus \{ z \in \mathbb{C} \mid \operatorname{Re}_z \le 0,\ \operatorname{Im}_z = 0 \}$

$z_0 \in \mathbb{C}$

$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(h)}{h} \in \mathbb{C}$

Př.: $n = 1$: $f(z) = z$

$f'(z_0) = \lim_{h \to 0} \frac{(z_0 + h) - z_0}{h} = 1$

$n \to n+1$: $f(z) = z^{n+1}$

$f'(z) = (z^{n+1})' =$

$= (z^n z)' = (z^n)' z + z^n$

$= n z^{n-1} z + z^n$

$= (n+1) z^n$

Cauchy-Riemannovy podm.

$z_0 = x_0 + i y_0$

$f'(z_0)$: $f(z) = u(x,y) + i v(x,y)$

$$f'(z_0) \in \mathbb{C} \iff \frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \quad (CR1)$$

&

$$\frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0) \quad (CR2)$$

Př.: a) $f(z) = Re(z) = x$

$z = x + iy$

$Re f = u(x,y) = x$
$Im f = v(x,y) = 0$

CR1 $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$

$1 \neq 0$ ✗

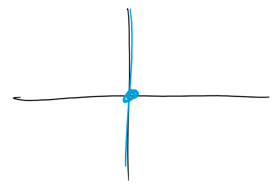
b) $f(z) = |z|^2$

$z = x + iy$

$f(z) = |x+iy|^2 = \left(\sqrt{x^2+y^2}\right)^2 = x^2+y^2$

$Re(f(z)) = x^2+y^2$

$Im(f(z)) = 0$



$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= 2x \\ \frac{\partial v}{\partial y} &= 0\end{aligned}\right\} \; x = 0$$

& $\Rightarrow$ je dif. v bodě $(0,0)$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\left.\begin{aligned}\frac{\partial u}{\partial y} &= 2y \\ -\frac{\partial v}{\partial x} &= -0\end{aligned}\right\} \; y = 0$$

$$f'(0) = \frac{\partial u}{\partial x}(0,0) + \frac{\partial v}{\partial x}(0,0)$$

$$= 2x + i\,0 \Big|_{x=y=0} = \underline{\underline{0}}$$

Holomorfní a celistvá funkce

(celá)

→ f má derivaci všude

→ f má derivaci na otevř. množině

Harmonické funkce

Laplace

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$u(x,y) : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$

je harmonická, jestliže

$u(x,y): \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$

u harmonická, jestliže

$u \in C^2(\Omega)$

2. parc. der. existují a jsou spojité, pro každé $(x,y) \in \Omega$

Př.:

$u(x,y) = x^2 - y^2$

$v(x,y) = 2xy$

$2 + (-2) = 0$ ✓

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$ ✓

$0 + 0 = 0$

$\forall (x,y) \rightarrow$ Je harmonická

Tvrzení: f je holomorfní na otevř. Ω

$\Rightarrow \operatorname{Re} f$ je harmon.

$\operatorname{Im} f$ je —"—

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) \quad \Big/ \frac{\partial}{\partial x}$$

$$\frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y) \quad \Big/ \frac{\partial}{\partial y}$$

$$\frac{\partial^2 u}{\partial x^2}(x,y) = \frac{\partial^2 v}{\partial y x}(x,y)$$

$+$

$$\frac{\partial^2 u}{\partial y^2}(x,y) = -\frac{\partial^2 v}{\partial y x}(x,y)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Př.:

$x^2 - y^2 + x$

a) $u(x,y)$ harmonická (sami)

b) funkce $v(x,y)$ aby $f(z) = u(x,y) + iv(x,y)$ celistvá

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} \quad \Big/ \int dy$$

$v(x,y) =$

CV2

Tuesday, October 1, 2024 16:16

1) a) $f(z) = \dfrac{\mathrm{Im}(z^2)}{i\bar{z}}$

$z = x + iy$

$$f(x+iy) = \frac{\mathrm{Im}((x+iy)^2)}{i(x-iy)} = \frac{\mathrm{Im}(x^2+2xyi-y^2)}{y+ix} = \frac{2xy}{y+ix}\cdot\frac{y-ix}{y-ix} = \frac{2xy^2-2ix^2y}{y^2+x^2}$$

$$= \underbrace{\boxed{\frac{2xy^2}{y^2+x^2}}}_{u(x,y)} - \underbrace{\frac{2x^2y}{y^2+x^2}}_{v(x,y)} i$$

$\mathrm{Re}(f) = \dfrac{2xy^2}{y^2+x^2}$

$\mathrm{Im}(f) = -\dfrac{2ix^2y}{y^2+x^2}$

b) $f(z) = |z+i|^2 + e^{z-\bar{z}}$

$$f(x+iy) = |x+i(y+1)|^2 + e^{2x} = x^2+(y+1)^2+e^{2yi} = x^2+(y+1)^2+\cos(2y)+i\sin(2y)$$

$\mathrm{Re}(f) = x^2+(y+1)^2+\cos(2y)$

$\mathrm{Im}(f) = \sin(2y)$

---

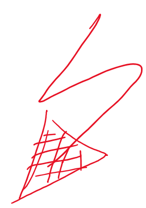
3) $f(z) = \mathrm{Re}(z^2) + i(z+\bar{z})^2 + 2i\,\mathrm{Im}(z)$

Určete kde je f dif.?

$$f(x+iy) = \mathrm{Re}(x^2+2iy-y^2) + i(2x) + 2i\cdot y$$
$$= x^2-y^2+2ix+2iy$$

$u(x,y) = x^2-y^2$

$v(x,y) = 2x+2y$



CR

$\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \rightarrow 2x = 2 \rightarrow x = 1$

$\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \rightarrow -2y = -2 \rightarrow y = 1$

F je dif. v bodě (1,4)

v (1+4i) není f difer...

---

2) $F(z) = i(\mathrm{Re}\,z)^2 - (\mathrm{Im}\,z)^2, \ z \in \mathbb{C}$

$F(x+iy) = ix^2 - y^2$

$\dfrac{\partial u}{\partial \ } = \dfrac{\partial v}{\ } \rightarrow 0 = 0$ ✓

$$F(x+iy) = ix^2 - y^2$$

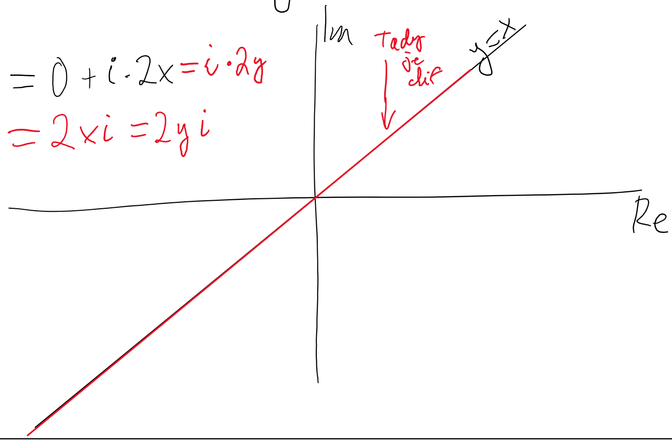
$$u(x,y) = -y^2$$
$$v(x,y) = x^2$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \rightarrow 0 = 0 \checkmark$$

&

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \rightarrow -2y = -2x \rightarrow y = x$$

$$F'(x+iy) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 + i \cdot 2x = i \cdot 2y$$
$$= 2xi = 2yi$$



4) $f(z) = x^2 + \alpha x + \alpha y + i(y^2 - 5\beta y + \beta x)$, $z = x+iy \in \mathbb{C}$; $\alpha, \beta \in \mathbb{R}$

$$u(x,y) = x^2 + \alpha x + \alpha y$$
$$v(x,y) = y^2 - 5\beta y + \beta x$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \rightarrow 2x + \alpha = 2y - 5\beta \rightarrow 2x - \beta = 2y - 5\beta$$

$$2x - 2y = -4\beta$$
$$x - y = -2\beta$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \rightarrow \alpha = -\beta$$

pro $z = 1+3i$

$$1 - 3 = -2\beta$$
$$\boxed{\beta = 1;\ \alpha = -1}$$

$$f(x+iy) = x^2 - x + y + i(y^2 - 5y + x)$$

$$f'(1+3i) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\Big|_{1+3i} =$$
$$= (2-1) + i = \boxed{1+i}$$

Param.: $\alpha = -1$
$\beta = 1$
$f'(1+3i) = 1+i$

5) Rozhodněte zda Re(F) harmonická

$$f(z) = (\mathrm{Im}(z^2))^3 + i\bar{z} = 8x^3y^3 + xi + y$$

$$i(x - iy) = xi - i^2 y = xi + y$$

$Re(f) = 8x^3y^3 + y$

$\frac{\partial u}{\partial x^2} = 24x^2y^3 \qquad \frac{\partial^2 u}{\partial x2} = 48xy^3$

není harmonická

$\frac{\partial u}{\partial y} = 24x^3y^2 + 1 \qquad \frac{\partial^2 u}{\partial y^2} = 48x^3y$

---

6) Nalezněte všechny hodnoty params α a $\beta \in \mathbb{R}$, tak aby u byla harmonická

$u(x,y) = e^y \sin(\alpha x) + 3xy + x^2y + \beta y^3, \; x,y \in \mathbb{R}$

S
C
-S
-C

$\frac{\partial u}{\partial x} = \alpha e^y \cos(\alpha x) + 3y + 2xy$

$\frac{\partial u}{\partial y} = e^y \sin(\alpha x) + 3x + 1 + 3\beta y^2$

$\frac{\partial^2 u}{\partial x^2} = -\alpha^2 e^y \cos(\alpha x) + 2 \quad = 0 \quad \Rightarrow \quad -\alpha^2 e^y \cos(\alpha x) + 2 = e^y \sin(\alpha x) + 6\beta y$

$\frac{\partial^2 u}{\partial y^2} = e^y \sin(\alpha x) + 6\beta y \quad = 0 \quad \Rightarrow$

CV03

Tuesday, October 8, 2024 16:39

1) a)

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x - 4y + 3 \ \Big/ \int dy$$

$$v(x,y) = 2xy - 2y^2 + 3y + C(x)$$

$$\frac{\partial v}{\partial x} \overset{(*)}{=} 2y + C'(x)$$

CR2

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y + 4x + 1$$

$$\cancel{2y} + 4x + 1 = \cancel{2y} + C'(x)$$

$$C'(x) = 4x + 1 \ \Big/ \int dx$$

$$C(x) = 2x^2 + x + C$$

$v(x,y) = 2xy - 2y^2 + 3y + 2x^2 + x + K$, kde $K \in \mathbb{R}$

$$f(i) = -2 + 4i$$

$$f(i) = u(0,1) + i\,v(0,1) = -2 + 4i$$

$$v(0,1) = 4$$

$$-2 + 3 + K = 4 \qquad \boxed{K = 3}$$

2) $u(x,y) = e^{2x}\cos(2y) + x^3 - 3xy^2, \ (x,y) \in \mathbb{R}^2$

a)

b) $f(1 + i\pi) = 1 + e^2 - 3\pi^2 + 3\pi i$

CR1 $$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2e^{2x}\cos 2y + 3x^2 - 3y^2 \ \Big/ \int dy$$

$$v(x,y) = \int 2e^{2x}\cos 2y + 3x^2 - 3y^2 \, dy = e^{2x}\sin 2y + 3x^2 y - y^3 + C(x)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2e^{2x}\sin(2y) + 6xy$$

$$\frac{\partial v}{\partial x} = 2e^{2x}\sin(2y) + 6xy + C'(x)$$

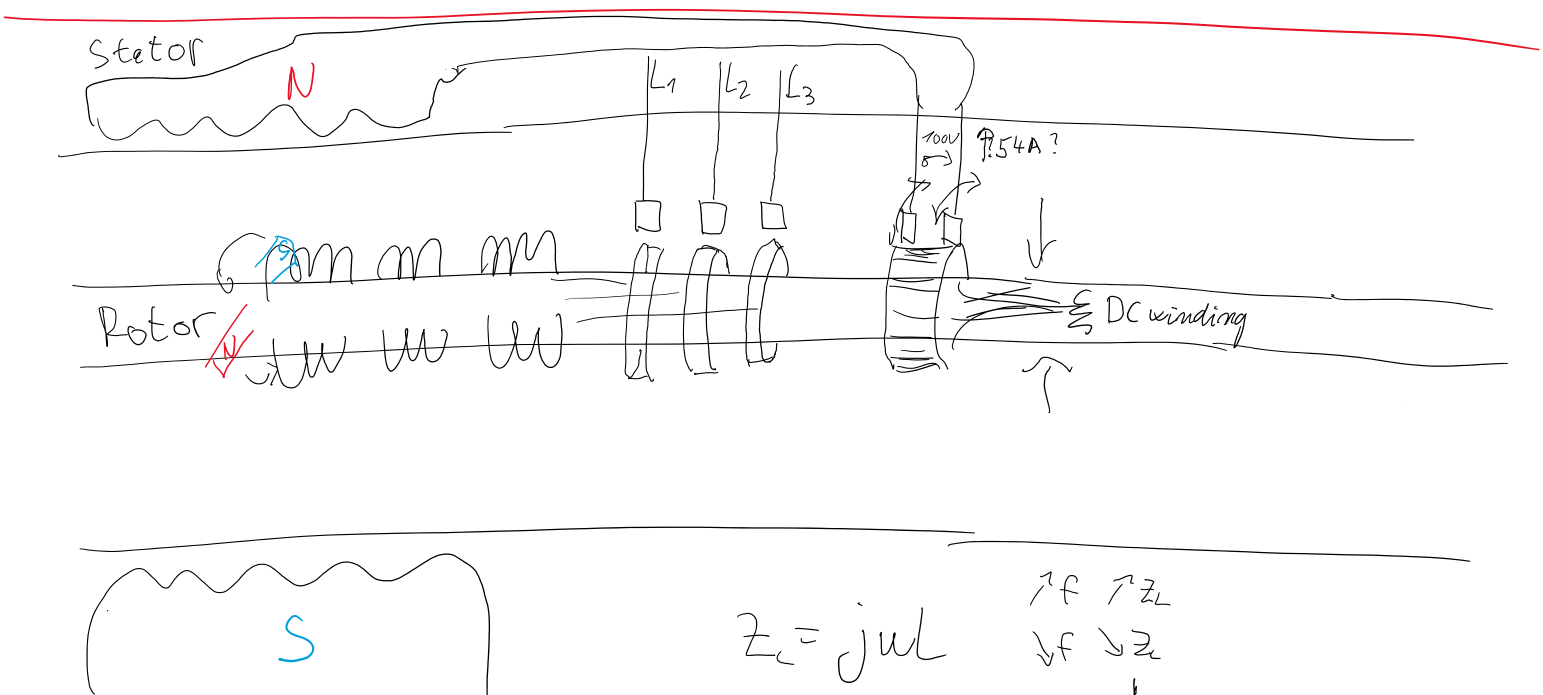
$$C'(x) = 0$$

$$C(x) = K, \text{ kde } K \in \mathbb{R}$$

$$v(x,y) = e^{2x}\sin 2y + 3x^2y - y^3 + K$$

$$f(1+i\pi) = u(1,\pi) + i\,v(1,\pi)$$

$$=$$



$Z_L = j\omega L$ $\quad \nearrow f \ \nearrow Z_L$

$\qquad\qquad\quad\ \searrow f \ \searrow Z_L$

$$z = e^{(2-3i)^2}$$

PR04

Tuesday, October 15, 2024 09:14

Mocninné řady $\in \mathbb{Z}$

Def.

proměnná

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

koef. střed řady

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

Př.:

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \dots$$

$|z| < 1$

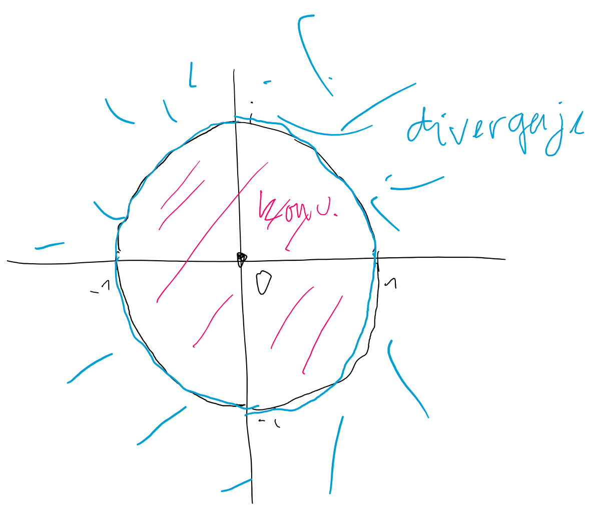
$$\sum_{n=0}^{N} z^n = 1 + \cancel{z} + \cancel{z^2} + \cancel{z^3} + \cancel{\dots} + \cancel{z^{N-1}} + \cancel{z^N}$$

$$-z\sum_{n=0}^{N} = -\cancel{z} - \cancel{z^2} - \cancel{z^3} - \cancel{z^4} \cancel{-\dots} - \cancel{z^N} - z^{N+1}$$

$+ \Big\rangle \quad (1-z)\sum_{n=0}^{N} z^n = 1 - z^{N+1}$

$|z| < 1$

$$\sum_{n=0}^{N} z^n = \frac{1 - z^{N+1}}{1-z} \overset{\lim_{N\to\infty}}{\Longrightarrow} \sum_{n=0}^{\infty} = \left(\frac{1 - \overset{0}{z^{\infty}}}{1-z}\right) = \frac{1}{1-z}$$



Obory mocn. řady konverguje na otevř. kruhu se středem z_0 a poloměru R

$\parallel$

poloměr konvergence

Př.:

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z+1)^{2n}}{3^n} = \sum^{\infty} \left(\frac{-(z+1)^2}{3}\right)^n$$

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \forall |z| < 1$$

$$f(z)=\sum_{n=0}^{\infty}\frac{(-1)^n(z+1)^{2n}}{3^n}=\sum_{n=0}^{\infty}\left(\frac{-(z+1)^2}{3}\right)^n$$

$$(z+1)=(z-\underbrace{(-1)}_{z_0})\quad\Big|=\frac{1}{1+\frac{(z+1)^2}{3}}=\frac{3}{3+(z+1)^2}$$

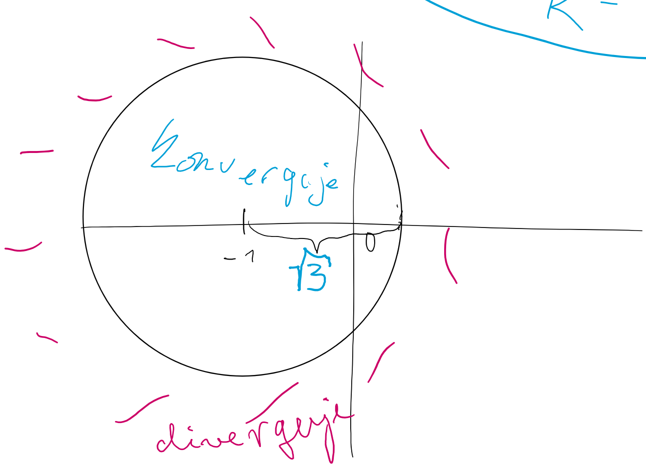
$$(z-z_0)\qquad \boxed{|z-z_0|<R}$$

$$\left|-\frac{(z+1)^2}{3}\right|<1 \quad /\cdot 3$$

$$|(z+1)^2|<3 \quad /\sqrt{\ }$$

$$\boxed{|z+1|<\underbrace{\sqrt{3}}}$$

$$R=\sqrt{3}$$



$$\boxed{\text{Vždy převedeme na } \sum_{n=0}^{\infty} z^n}$$

NIKDY NEROZNÁSOBOVAT ZÁVORKT V KANA

Derivace mocn. řady

Věta: $f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$

1) $f'(z)=\left(a_0+a_1(z-z_0)+a_2(z-z_0)^2+\dots\right)'$

$$=\left(a_1+2a_2(z-z_0)+3a_3(z-z_0)^2+\dots\right)$$

$$=\sum_{n=1}^{\infty}a_n n(z-z_0)^{n-1}$$

2) Zderivovaná řada má stejný R

3) $a_n=\frac{f^{(n)}(z_0)}{n!}$

$f(z_0)=a_0$

$f'(z_0)=a_1$

$f''(z_0)=2a_2$

$$f'''(z_0) = 6a_3$$

$\vdots$

Integrace mocn. řad

1) $$\sum_{n=0}^{\infty} a_n (z-z_0)^n \xrightarrow{\int} \underbrace{\sum_{n=0}^{\infty} a_n \frac{(z-z_0)^{n+1}}{n+1}}$$

$$F(z) \quad F'(z) = f(z)$$

2) $\int \check{R}$ má stejný R jako $\check{R}$

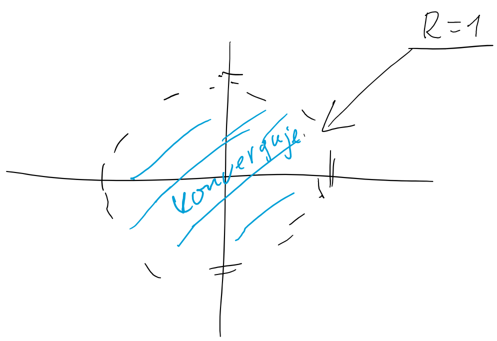
---

1) $$\sum_{n=1}^{\infty} (n+1) z^n \xrightarrow{\int} \sum_{n=1}^{\infty} \frac{\cancel{(n+1)} z^{n+1}}{\cancel{n+1}} = \sum_{n=0}^{\infty} z z^n = z \sum_{n=0}^{\infty} z^n = z \frac{1}{1-z} = \boxed{\frac{z}{1-z}}$$

součet $\int \check{R}$

$z_0 = 0$

$$\left(\frac{z}{1-z}\right)' = \frac{(1-z) - z(-1)}{(1-z)^2} = \frac{1}{(1-z)^2} \quad \text{pro } |z| < 1$$



---

2) $$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \xrightarrow{\frac{d}{dz}} \sum_{n=1}^{\infty} \frac{\cancel{n} z^{n-1}}{\cancel{n}} = \sum_{n=1}^{\infty} z^{n-1} = \sum_{n=1}^{\infty} \frac{z^n}{z} = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$$

$z_0 = 0$

$$\int \frac{1}{1-z} dz = -\ln(1-z) + C$$

$$\boxed{f(z) = -\ln(1-z) \quad \text{pro } |z| < 1}$$

$$f(z_0 = 0) = C = 0$$

Taylorova řada

$\sum^{\infty}$ (n) — n-tá derivace, střed

$$\sum_{n=0}^{\infty} \frac{f^{\boxed{(n)}}(z_0)}{n!}(z-z_0)^n$$

– n-tá derivace

střed

CV4

Tuesday, October 15, 2024 16:09

Nalezněte součet $f(z)$ mocninné řady na jejím kruhu ...

a) $\sum_{n=0}^{\infty} 3^n (z+i)^{2n+1} \Rightarrow \sum_{n=0} a_n (z+z_0)^n$

$\boxed{z_0 = -i}$ $\quad (z+i)\sum_{n=0}^{\infty} (3(z+i)^2)^n = \frac{z+i}{1-3(z+i)^2}$ $\quad |3(z+i)^2| < 1$

$|(z+i)^2| < \frac{1}{3}$

$|z+i| < \frac{\sqrt{3}}{3}$

$R = \frac{\sqrt{3}}{3}$

b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n! \cdot 4^{n+1}} z^{3n+2} = \frac{1}{4} z^2 \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-(z)^3}{4}\right)^n$ $\quad \boxed{e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}}$

$= \frac{z^2}{4} e^{-\frac{z^3}{4}}$ Konverguje na celém $\mathbb{C}$ ✓

c) $\sum_{n=0}^{\infty} \frac{(2n+1)}{n!} z^{2n+3} = \boxed{z^3} \sum_{n=0}^{\infty} \frac{2n+1}{n!} (z)^{2n} \xrightarrow{\int dz} \sum_{n=0}^{\infty} \frac{2n+1}{2n+1} \frac{z^{2n+1}}{n!}$ *

e^z

* $= z \sum_{n=0}^{\infty} \frac{(z^2)^n}{n!} = z\exp(z^2) + C \xrightarrow{\frac{d}{dz}} 1\exp(z^2) + 2z^2\exp(z^2)$ $\quad e^z = \exp(z)$

$z^3 \cdot (\exp(z^2) + 2z^2\exp(z^2))$

$= z^3(1+2z^2)\exp(z^2)$! $\forall z \in \mathbb{C}$!

d) $\sum^{\infty} \frac{(-1)^n (n+2)}{2^{n+1}} z^{n+5} = \boxed{\frac{1}{2} z^4} \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n (n+2)}{2^n} z^{n+1}} \xrightarrow{\int \square dz} \sum_{n=0}^{\infty} \frac{(-1)^n z^{n+2}}{2^n} = \frac{2z^2}{2+z}$

d) $\sum_{n=0} \frac{(-1)^n (n+2)}{2^{n+1}} z^{n+5} = \boxed{\frac{1}{2} z^4} \sum_{n=0} (-1)^n \frac{(n+2)}{2^n} z^{n+1}$ $\longrightarrow \sum_{n=0} \frac{(-1)^n z}{2^n} \frac{2z}{2+z}$

$\xrightarrow{\frac{d}{dz}} \frac{4z(2+z) - z^2}{(2+z)^2} \rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)}{2^{n+1}} z^{n+5} = \frac{1}{2} z^4 \frac{4z(2+z) - z^2}{(2+z)^2}$

$\left| -\frac{z}{2} \right| < 1$

$\boxed{\forall\ |z| < 2}$

---

e) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1) 5^{n+1}} z^{n+3} = \frac{z^2}{5} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)5^n} z^n \xrightarrow{\frac{d}{dz}} \sum_{n=0}^{\infty} \cancel{(n+1)} \frac{(-1)^n}{\cancel{(n+1)} 5^n} z^n =$ ☺

☺ $= \cancel{z} \sum_{n=0}^{\infty} \left(\frac{-z}{5}\right)^n \stackrel{!}{=} \cancel{z} \cdot \frac{1}{1-\left(-\frac{z}{5}\right)} \xrightarrow{\int dz} \int \frac{1}{1+\frac{z}{5}} dz = \int \frac{5}{5+z} dz =$

$= 5 \ln(5+z) + C \qquad z_0 = 0 \qquad 0 = 5 \ln 5 + C$

$C = -5 \ln 5$

$= \frac{z^2}{5}\left(5 \ln(5+z) + C\right)$

$= \frac{z^2}{\cancel{5}}\left(\cancel{5} \ln\left(\frac{5+z}{5}\right)\right) = z^2 \ln\left(\frac{5+z}{5}\right) \qquad \left| -\frac{z}{5} \right| < 1$

$\forall\ |z| < 5$

---

f) $\sum_{n=0} \frac{2^n}{n!(2n+2)} z^{2n+5} = \boxed{z^3} \sum_{n=0}^{\infty} \frac{2^n}{n!(2n+2)} z^{2n+2} \xrightarrow{\frac{d}{dz}}$

$n! \Rightarrow e^z = \ldots$

$\xrightarrow{\frac{d}{dz}} \sum_{n=0}^{\infty} \frac{2^n}{n!} z^{2n+1} = z \sum_{n=0}^{\infty} \frac{(2z^2)^n}{n!} \stackrel{!}{=} z e^{2z^2} \xrightarrow{\int dz}$

$n=0$ $\quad$ $n=0$

$$\int z e^{2z^2}\,dz = \left| \begin{array}{l} w = 2z^2 \\ dw = 4z\,dz \\ \frac{dw}{4} = z\,dz \end{array} \right| = \int \frac{e^w}{4}\,dw = \frac{1}{4}e^w + C = \frac{1}{4}e^{2z^2} + C$$

$z_0 = 0$

$0 = \frac{1}{4} + C$

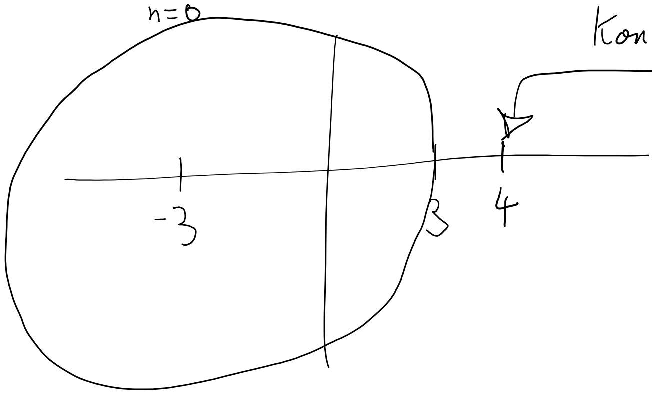
$C = -\frac{1}{4}$

$$\sum_{n=0}^{\infty} \frac{2^n}{n!\,(2n+2)} z^{2n+5} = \frac{z^3}{4}\cdot\left(e^{2z^2}+1\right) \; \forall z \in \mathbb{C}$$

---

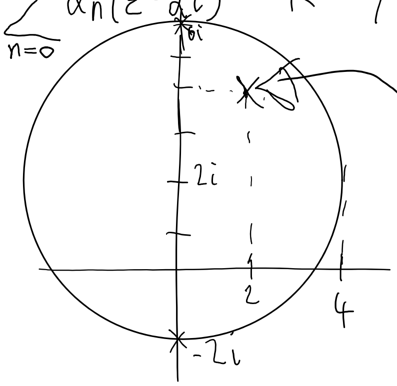
2) a) $\sum_{n=0}^{\infty} a_n (z+3)^n \quad R = 6,\ z_0 = -3$

Konv. v $z=4$? → Ne $4 > (|z+3| < 6)$



b) $\sum_{n=0}^{\infty} a_n (z-2i)^n \quad R = 4,\ z_0 = 2i$ $\quad$ Konv. v $z = 2+3i$?

ANO, konv. v $z = 2+3i$



3) Rozv. $f(z)$ do mocn. Ř. se stř. z_0, a urč. R

a) $f(z) = \dfrac{z-3}{1-2z}$, $z_0 = 3$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!} \cdot z^n$$

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(3)}{n!}\cdot z^{n}$$

PR5

Tuesday, October 22, 2024 09:15

1) $\frac{1}{1+z^2} = \frac{1}{1-(-z)^2} = \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n}$

$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad |z|<1$

$|-z^2|<1$

$|z|^2<1$

pro $|z|<1$

2) $\frac{1}{(2+z)^2} \quad z_0=3 \quad = \frac{1}{(5+(z-3))^2} = \frac{1}{(5+(z-3))^2}$

$\left(\frac{1}{5+(z-3)}\right)' = -\frac{1}{(5+(z-3))^2} \; *$

rozvineme

$\frac{1}{5+(z-3)} = \frac{1}{5}\,\frac{1}{1-\underbrace{\left(\frac{3-z}{5}\right)}_{\text{kvocient}}} = \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{3-z}{5}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{5^{n+1}}(z-3)^n \xrightarrow{\frac{d}{dz}} \sim \left(\sum_{n=0}^{\infty}\frac{(-1)^n}{5^{n+1}}(z-3)^n\right)' = *$

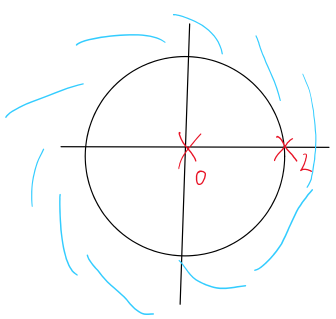
$* = -\sum_{n=1}^{\infty}\frac{(-1)^n}{5^{n+1}}\, n\,(z-3)^{n-1}$

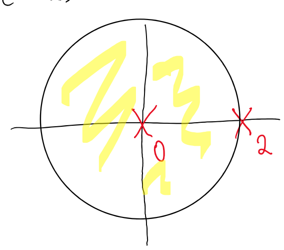
$\left|-\frac{z-3}{5}\right|<1$

pro $|z-3|<5$

Laurentovy řady

$\frac{1}{z(2-z)}$





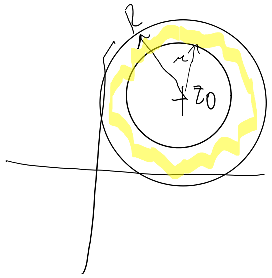
$\sum_{n=-\infty}^{\infty} \underbrace{a_n}_{\text{řada}} (z-\underbrace{z_0}_{\text{střed}})^n = \underbrace{\sum_{n=-\infty}^{-1} a_n (z-z_0)^n}_{\text{hlavní část}} + \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{regulární část}}$

ex. $R \in [0,\infty]$, že reg. konverguje pro $|z-z_0|<R$

$$w = \frac{1}{z-z_0} \longrightarrow \tilde{R} \in [0,\infty]$$

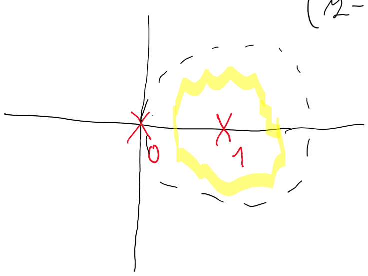
$\sum_{n=-\infty}^{-1} d_n w^{-n}$ konverguje pro $|w|<\tilde{R}$ $\quad \frac{1}{|z-z_0|}<\tilde{R} \quad |z-z_0|>\frac{1}{\tilde{R}}=r$

Mezikruží



$$f(z) = \frac{1}{z(1-z)} = -\frac{1}{z-1}\,\frac{1}{z} = -\frac{1}{z-1}\sum_{n=0}^{\infty}(-1)^n(z-1)^n = \sum_{n=0}^{\infty}(-1)^{n+1}(z-1)^{n-1} \text{ pro } |z-1|<1$$

$(z-1)$



$$\frac{1}{z} = \frac{1}{(z-1)+1} = \frac{1}{1-\underbrace{(-(z-1))}}$$

$$= \sum_{n=0}^{\infty}(-1)^n(z-1)^n$$

Rozvojová země

a) $\frac{z-3}{1-2z}, z_0=3$ $\quad f(z)=\frac{z-3}{1-2z}, z_0=3$

bezmyšlenkovitá myšlenka
— Zdenda 2024

$$\frac{1}{1-2z}=\frac{1}{1-2(z-3)-6}=\frac{1}{-5-2(z-3)}=-\frac{1}{5+2(z-3)}$$

$$=-\frac{1}{5}\frac{1}{1-\left(-\frac{2(z-3)}{5}\right)}=-\frac{1}{5}\sum_{n=0}^{\infty}\left(-\frac{2(z-3)}{5}\right)^n=-\frac{1}{5}\sum_{n=0}^{\infty}\frac{(-2)^n}{5^n}(z-3)^n$$

$$\left|-\frac{2(z-3)}{5}\right|<1$$

$$|z-3|<\frac{5}{2}$$

---

b) $f(z)=\frac{1}{(z+6)^2}, z_0=-4$

$\boxed{z+4}$ — Nedilitelnej faktor

$\infty > 10\,000 \quad / -10\,000$ $\qquad \infty = \lim_{x\to\infty} x$

$\infty > 0 \quad / \cdot i$

$8 > 0$

Měření se mi líbilo


Doslova já
$\frac{1}{z+2} = \frac{1}{(2+1)+1}$

PR5

Tuesday, November 5, 2024 09:15

$f(z)$ ... holomorfní na $U(z_0)$
z_0 je kořen násobnosti $k \in \mathbb{N}$, jestliže

$f(z_0) = 0$
$f'(z_0) = 0$
⋮
$f^{(n-1)}(z_0) = 0$
$f^{(n)}(z_0) \neq 0$

Věta: Je-li z_0 k-násobný $f(z_0)$
a l-násobný funkce $g(z)$,
potom z_0 je $(k+l)$ násobný kořen
$h(z) = f(z)g(z)$

dk: $h(z) = \tilde{f}(z)(z-z_0)^{l}\underbrace{\tilde{g}(z)}_{\neq 0}$
$= (z-z_0)^{k+l}\underbrace{\tilde{f}(z)\tilde{g}(z)}_{\neq 0}$

Př: $f(z) = z^{50}(1-\cos z)^2$

0 je 50 násobný kořen z^{50}

$(1-\cos z)^2$ 0 je $2 \cdot 2 = 4$ nás. kořen $(1-\cos z)^2$

0 je 54 násobný kořen $f(z)$
$\parallel$
$50+4$

$1-\cos z|_0 = 1-1 = 0$

$(1-\cos z_0)' = \sin z|_0 = 0$

$(1-\cos z_0)'' = \cos z|_0 = 1 \neq 0$

Nechť $f(z_0)$ má v bodě z_0 kořen násobnosti $m \in \mathbb{N}$
a $g(z)$ má v z_0 kořen násobnosti $n \in \mathbb{N}$.

Potom $h(z) = \frac{f(z)}{g(z)}$ má v z_0

$\frac{(z-z_0)^3}{(z-z_0)^3}$ $\frac{(z-z_0)^5}{(z-z_0)^{10}}$

odstranitelnou singularitu, pokud $m \geq n$

pól řádu $n-m$, pokud $m < n$

$\hat{h}(z_0)$ je holomorfní na $U(z_0)$

dk: $f(z) = (z-z_0)^m \tilde{f}(z)$
$\tilde{f}(z_0) \neq 0$

$g(z) = (z-z_0)^n \underbrace{\tilde{g}(z)}_{}$
$\tilde{g}(z_0) \neq 0$

$\frac{f(z)}{g(z)} = \frac{(z-z_0)^m}{(z-z_0)^n} \cdot \frac{\tilde{f}(z_0)}{\tilde{g}(z_0)}$

Pokud $m \geq n$, jak $\frac{f(z)}{g(z)} = (z-z_0)^{\overbrace{m-n}^{\geq 0}} \tilde{h}(z_0)$
má v z_0 od ...

$$\underbrace{(\;\;)}_{\tilde{g}(z_0)\neq 0}$$

má v z_0 od.........!

Pokud $m<n$ $\frac{f(z)}{g(z)} = \frac{1}{(z-z_0)^{n-m}} \cdot \sum_{n=1}^{\infty} a_n (z-z_0)^n$

$$= \frac{1}{(z-z_0)^m}\left(a_0 + a_1(z-z_0) \dots\right) *$$

$$* = \frac{a_0}{(z-z_0)^{n-m}} + \frac{a_1}{(z-z_0)^{n-m-1}} + \dots$$

z_0 je pól řádu $n-m$

$\neq 0$ protože $a_0 = \tilde{h}(z_0) \neq 0$

---

2) $f(z) = \frac{(1-\cos z)\sin z}{z^3}$ $\quad z_0 = 0$

- 0 je $3\times$ kořen jmenovatele
- 0 je $2+1=3\times$ kořen čitatele

$\Big\}$ 0 je odstranitelná singularita

$1-\cos z$ .... 0 je 2 násobný kořen

$\sin z$ ... 0 je 1 násobný kořen

$(\sin z_0)' = \cos z_0 |_0 = 1 \neq 0$

---

3) Zk. Pr.!

$$\frac{\sin z - z}{(1-e^{iz})^2}$$

$e^{iz} = 1$

$e^z = e^0$

$z = 0 + 2k\pi$

$z_k = 2k\pi, k \in \mathbb{Z}$

jmenovatel

$1 - e^{iz}|_{z_k\pi} = 0$

$-ie^{iz}|_{2k\pi} = -ie^{i2k\pi} = 1 \neq 0$

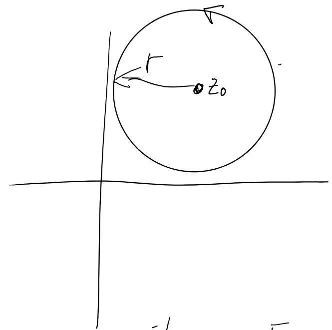
$(1-e^{iz})|_{z=0} = \cos z - 1 | = 0$

Dohromady:

Křivky v komplexní rovinně

Def.: $f(z)$ spojitá $C \subseteq \mathbb{C}$ s param. $\varphi\,[a,b] \to \mathbb{C}$

$$\int_C f(z)\,dz = \int_a^b f(\varphi(t)) \cdot \varphi'(t)\,dt$$



$$\int_C (z-z_0)^n\,dz = \int_0^{2\pi} (\underbrace{Re^{it}}_{z_e - z_0})^n Ri\,e^{it}\,dt$$

$\varphi(t) = z_0 + Re^{it}$, $t \in [0, 2\pi]$
$\quad -\pi, \pi$

$\varphi'(t) = Ri\,e^{it}$

$$= Ri \int_0^{2\pi} R^n e^{int} \cdot e^{it}\,dt$$

$$= R^{n+1} i \int_0^{2\pi} e^{i(n+1)t}\,dt \underset{n \neq -1}{=} iR^{n+1} \left[\frac{e^{i(n+1)t}}{i(n+1)} \right]_0^{2\pi} = iR^{n+1} \frac{1}{i(n+1)} \left(\underbrace{e^{2\pi(n+1)i}}_{1} - 1 \right)$$

$$= 0$$

$\hookrightarrow n = 1:\ \underbrace{R^{n+1}}_{1} i \int_0^{2\pi} 1\,dt = \underline{\underline{2\pi i}}$

CV5

Tuesday, November 5, 2024 16:16

$$1)\quad f(z) = \cancel{\frac{2}{(z-i)^3}} + \cancel{\frac{1}{z-i}} + \cancel{\frac{-2}{(z-i)^3}} = \cancel{\frac{1}{z-i}} + \underbrace{0 + \dots}_{\text{nezáporné mocniny } (z-i)}$$

nejsou single

$$2)\quad f(z) = \frac{\alpha}{(z-1)^4} + \frac{7}{3(z-1)^2} + \sum_{n=1}^{\infty} \frac{(z-1)^{n-3}}{3^n}, \quad z \in P(i)$$

$$= \frac{\alpha}{(z-1)^4} + \frac{7}{3(z-1)^2} + \frac{1}{3(z-1)^2} + \frac{1}{9(z-1)} + \underbrace{\frac{1}{27} + \dots}_{\text{nezáporné mocniny}}$$

$$= \frac{\alpha}{(z-1)^4} + \frac{8}{3(z-1)^2} + \frac{1}{9(z-1)} + \dots =$$

$$\alpha + \frac{8}{3} = 0$$

$$\alpha = -\frac{8}{3}$$

3)

$$a)\quad f(z) = \frac{\sin(z) + z - \pi}{z^2 (z-\pi)^4}$$

$z^2 \to 0$, $(z-\pi)^4 \to \pi$

Iz: $0, \pi$

J 0: 2 násobný kořen jmenovatele

Č 0: 0 kořenem čitatele není

$\to$ pól řádu $2 - 0 = 2$

J π: 4 násobný kořen jmenovatele

Č π: $\sin\pi + \pi - \pi = 0 + \pi - \pi = 0 \to$ 3 násobný kořen čitatele

$(\sin z + z - \pi)' = \cos z - 1 \big|_{z=\pi} = 0$

$-\sin z \big|_{\pi} = 0$

$\to$ pól řádu $4 - 3 = 1$

$-\sin z|_{\pi} = 0$

$-\cos z|_{\pi} = 1 \neq 0$

b) $f(z) = \dfrac{(e^z-1)(1-\cos z)^4}{z^{11}}$

Iz. 0: Jmenovatel 11 násobný

0 je 1 násobný kořen (e^z-1)

S
C
−S
−C

$(e^z - 1) = 0$

$e^z = 1$

$(1-\cos z)^4$: 0 je $4 \cdot 2 = 8$ násobný kořen

$(1-\cos z)|_0 = 0$

$\sin z|_0 = 0$

$\cos z|_0 = 1$

Čitatel: 0: $8+1=9$
Jmenovatel: 11

0 je pól řádu $11-9=2$ pro $f(z)$

---

c) $f(z) = \dfrac{1-\cos z}{z^5(1-e^{iz})}$

2π

Do zápočtáku:
- exponenciála rozvoj
- geometrická řada součet

Iz: $2k\pi$, 0

$k \in \mathbb{Z}$

$1-\cos(2k\pi) = 0$ Body $2k\pi$ je 2 násobné kořeny čitatele

$\sin(2k\pi) = 0$

$\cos(2k\pi) = 1$

0 je 2 násobný kořen $1-\cos z$

č: $1-\cos 0 = 0$

$\sin(0) = 0$

$\cos(0) = 1$

Jmenovatel:

z^5: 0 je 5-násobný kořen z^5

Celkem: 0 je pól řádu $6-2=4$

Body $2k\pi, k\neq 0$ jsou odstranitelné singularity

Čitatel tě může zachránit

---

d) $f(z) = \dfrac{e^{iz} - i - \cos z}{(1-\sin z)^2 (z - \frac{\pi}{2})}$

IZ: $\frac{\pi}{2} + 2k\pi$ pro $k \in \mathbb{Z}$

Čitatel: $e^{iz} - i - \cos z \Big|_{z=\frac{\pi}{2}+2k\pi} = e^{\overbrace{\frac{\pi}{2}i + 2k\pi i}^{=i}} - i - 0 = i - i - 0 = 0$

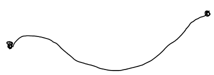
$(e^{iz} - i\cos z)' \Big|_{z=\frac{\pi}{2}+2k\pi} = ie^{iz} + \sin z \Big|_{z=\frac{\pi}{2}+2k\pi} = i^2 + 1 = 0$

$(\ )'' = (ie^{iz} + \sin z) = -e^{iz} + \cos z \Big|_{z=\frac{\pi}{2}+2k\pi} = -i + 1 \neq 0$

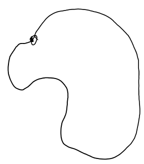
Jmenovatel:

$z - \frac{\pi}{2}$: $\frac{\pi}{2}$ je jednonásobný kořen

Reziduum



jednoduché křivky

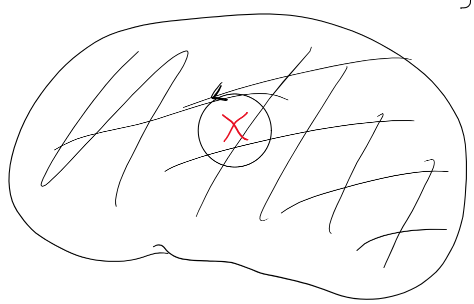


jednoduchá + uzavřená = jordanova křivka

~~Cauchyho věta~~

Definice: Řekneme, že oblast $\Omega \subseteq \mathbb{C}$ je jednoduše souvislá, jestliže platí $\forall$ Jordanovu křivku $C \subseteq \Omega$ platí $\operatorname{Int} C \subseteq \Omega$.

Ω je neděravá oblast = jednoduše souvislá



Věta: (Cauchyova)

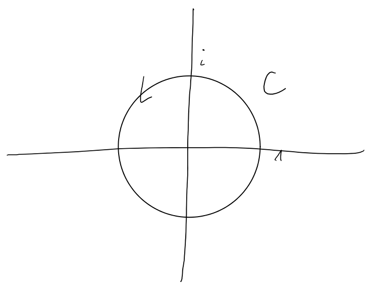
$\Omega \subseteq \mathbb{C}$ je jednoduše souvislá a f je holomorfní na Ω.

Potom $\int_C f = 0$ pro každou uzavřenou $C \subseteq \Omega$

Důkaz: Dokážeme pro $C = K_R(z_0)$

$$\int_{K_R(z_0)} f(z)\,dz = \int_{K_R(z_0)} \sum_{n=0}^{\infty} a_n (z-z_0)^n\,dz = \sum_{n=0}^{\infty} a_n \int_{K_R(z_0)} (z-z_0)^n\,dz = 0$$

$$f(z) = \sum a_n (z-z_0)^n \quad \forall z \in U(z_0),\ C \subseteq U(z_0)$$



Reziduum

$\forall z \in P(z_0)$ $\qquad \dots = \underbrace{a_{-1}}_{\text{reziduum}} \cdot 2\pi i$

Def: $f(z_0)$ má v $z_0 \in \mathbb{C}$ izolovanou singularitu

$$f(z) = \sum_{-\infty}^{\infty} a_n (z-z_0)^n, \quad z \in P(z_0)$$

<u>Reziduum</u> $f(z)$ v z_0 je koeficient a_{-1} $\quad \mathrm{res}_{z_0} f(z) = a_{-1}$

Věta: Pokud $f(z)$ má v $z_0 \in \mathbb{C}$ odstranitelnou singularitu, potom je reziduum nulové $= \mathrm{res}_{z_0} f(z) = 0$

dk: odstranitelná singularita $\Longrightarrow f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \; z \in P(z_0)$

$$\Longrightarrow \mathrm{res}_{z_0} f(z) = 0 \quad \boxtimes$$

Př.: $\mathrm{res}_0 \dfrac{e^z}{z^3} = \dfrac{1}{2}$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\frac{e^z}{z^3} = \sum_{n=0}^{\infty} \frac{z^{n-3}}{n!} \qquad a_{-1} = \frac{1}{n!}, \; n=2$$

$$a_{-1} = \frac{1}{2}$$

Věta: Nechť $f(z)$ má v z_0 pól řádu $k \in \mathbb{N}$.

Potom $\mathrm{res}_{z_0} f(z) = \lim_{z \to z_0} \left((z-z_0) f(z) \right)^{(k-1)}$

Dk.: $f(z)=\sum_{n=-k}^{\infty} a_n (z-z_0)^n, \; z\in P(z_0)$

$$\left((z-z_0)^k f(k)\right)^{(k-1)} = \left(\sum_{n=-k}^{\infty} a_n (z-z_0)^{n+k}\right)^{k-1} = \sum_{n=-1}^{\infty} a_n (n+k)(n+k-1)\dots(n+2)(z-z_0)^{n+1}$$

$$= a_{-1}\underbrace{(k-1)(k-2)\cdot 1}_{(k-1)!} + a_0 (\text{číslo})(z-z_0)$$

$+$ nezáporné mocniny $(z-z_0)^n$

$$\lim_{z\to z_0}\left((z-z_0)^k f(k)^{(k-1)}\right) = \boxed{a_{-1}}_{\operatorname{res}_{z_0} f(z)} (k-1)!$$

Př.:

1) $\operatorname{res}_{-1} \frac{1}{(z-1)(z+1)^2} = \lim_{z\to -1}\left((z+1)^2 \frac{1}{(z-1)(z+1)^2}\right)'$

$\boxed{-1 \text{ pól řádu } 2}$ $\lim_{z\to -1}\left(\frac{1}{z-1}\right)' = \lim_{z\to -1} (-1)\frac{1}{(z-1)^2} = -\frac{1}{4}$

2) $\operatorname{res}_0 \frac{\sin z}{z^2} = \lim_{z\to 0} z\frac{\sin z}{z^2} \overset{\langle\langle \frac{0}{0}\rangle\rangle}{\underset{=}{LH}} \lim_{z\to 0}\frac{\cos z}{1} = \langle\langle \cos 0\rangle\rangle = 1$

0 je dvojnásobný kořen jmenovatele

$\sin z|_{z=0} = 0$

$\sin z|_{z=0} = \cos z|_{z=0} = 1 \neq 0$

1-násobný kořen jmenovatele

0 je pól řádu $2-1 = 1$

Věta: Nechť $f(z)$ je holomorfní $U(z_0)$ a $g(z)$ má v z_0 jednonásobný kořen

$$\operatorname{res}_{z_0} \frac{f(z)}{g(z)} = \frac{f(z_0)}{g'(z_0)}.$$

Př.: $\operatorname{res}_\pi \frac{\overbrace{\cos(z-\pi)}^{f(z)}}{z\underbrace{(1+e^{iz})}_{g(z)}} = *$ můžem dosadit

$(1+e^{iz})|_{z=\pi} = 1-1=0$

$(1+e^{iz})'|_{z=\pi} = ie^{iz}|_{z=\pi} = -i \neq 0$

$$* = \frac{\frac{\cos(z-\pi)}{z}}{(1+e^{iz})} = \frac{\cos(0)}{\pi(1+e^{iz})'|_{\pi}} = \frac{1}{\pi(-i)} = \frac{i}{\pi}$$

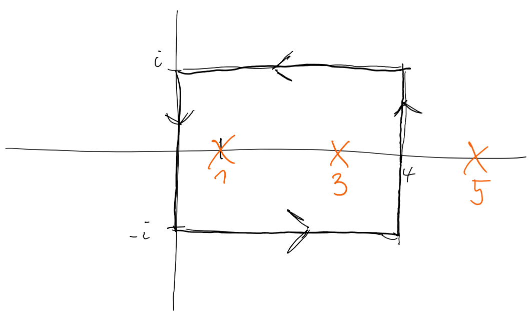
$$* = \frac{\overline{z}}{(1+e^{iz})} = \frac{\cos(0)}{\pi\,(1+e^{iz})'\big|_{z=\pi}} = \frac{1}{\pi(-i)} = \frac{i}{\pi}$$

Reziduová věta

$$\int_C f(z) = 2\pi i\left(res_{w_1} f(z) + res_{w_2} f(z) + res_{w_3} f(z)\right)$$

Př.:

$$\int_C \frac{1}{(z-1)(z-3)(z-5)^2}\,dz \overset{\text{rez. věta}}{=} 2\pi i\left(res_1 \frac{1}{(z-1)(z-3)(z-5)^2} + res_3 f(z)\right) = \maltese$$



$$res_1 \frac{1}{(z-1)(z-3)(z-5)^2} = \frac{1}{-2\cdot 16\,(z-1)'\big|_1} = -\frac{1}{32}$$

dosaz. metoda

$$res_3 \frac{1}{(z-3)(z-1)(z-5)^2} = \frac{1}{2\cdot 4\cdot 1} = \frac{1}{8}$$

$$\maltese = 2\pi i\left(\frac{4}{32} - \frac{1}{32}\right) = \frac{3}{16}\pi i$$

Před započtakem 2

Sunday, November 17, 2024 14:39

1. Určete koeficienty u mocnin $(z-i)^{-10}$, $(z-i)^{-1}$, $(z-i)^{1}$ a $(z-i)^{2}$ v Laurentově řadě
$$\sum_{n=-4}^{\infty} \frac{n+2}{2^n}(z-i)^{3n+8}.$$

$a_n (z-i)^{-10} \to 3n+8=-10$

$3n=-18$

$n=-6$

$a_n = \frac{n+2}{2^n}\Big|_{n=-6} \neq \frac{-6+2}{2^{-6}} = \frac{-4}{2^{-6}} = -4\cdot 2^6 = -2^2\cdot 2^6 = -2^8 = \cancel{-256}$

-1

$\cancel{-1\cdot 2^4\cdot 2^4}$, $\cancel{16\cdot 16}$, $\cancel{256}$

řada začíná $n=-4$, $n=-6$ v řadě není čili $\boxed{a_n=0}$

$(z-i)^{-1} \to 3n+8=-1 \to n=-3$

$a_{-3} = \frac{-3+2}{2^{-3}} = -1\cdot 8 = -8$

$3n+8=1$;

$n=-\frac{7}{3}$ → tento člen v řadě není ⇒ $a_n=0$

↳ $a_n=0$ ⇒ n není celé číslo!

$3n+8=2$; $n=-2$ } $a_n = \frac{-2+2}{2^{-2}} = 0$

VERIFIED

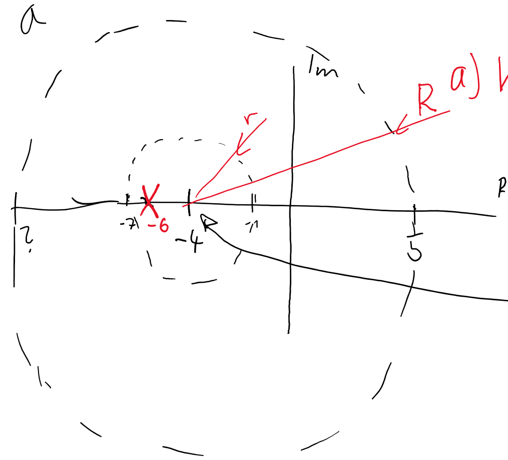
1. Ať a_n je koeficient u $(z-i)^n$. Pak $a_{-10}=0$, $a_{-1}=-8$, $a_1=0$ a $a_2=0$.

---

2) 2. (a) Laurentova řada
$$\breve{R} = \sum_{n=-\infty}^{\infty} a_n(z+4)^n$$
má vnitřní poloměr konvergence $r=3$ a vnější $R=9$. Konverguje v bodě $z=-6$?

(b) Laurentova řada
$$\breve{R}_b = \sum_{n=-\infty}^{\infty} a_n(z-i)^n$$
má vnitřní poloměr konvergence $r=1$ a vnější $R=4$. Konverguje v bodě $z=2$?

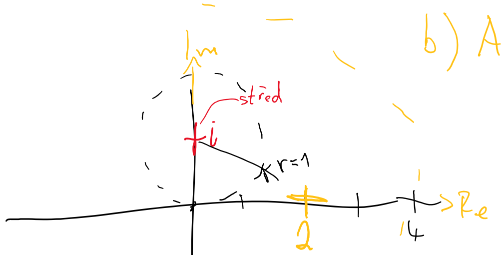
a



a) V bodě $z=-6$ řada $\breve{R}$ nekonverguje = diverguje

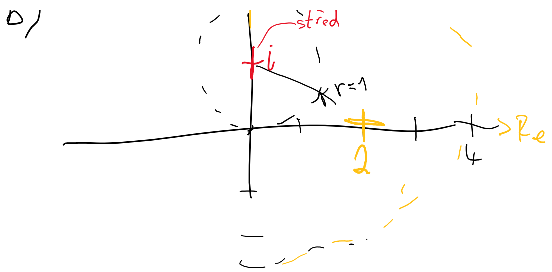
$(z+4)^n$ ⇒ střed v -4, protože $\left\{ (z+4)\Big|_{z=-4} = 0 \right\}$

b)



b) Ano v bodě $z=2$ řada $\breve{R}_b$ konverguje

{ $\breve{R}$ je placeholder pro $\sum \ldots$ } abych to nemusel opisovat b)



$\{ \bar{R}$ je placeholder pro $\sum \ldots \}$ abych to nemusel opisovat

VERIFIED

2. (a) Ne.
   (b) Ano.

---

3. Nalezněte rozvoj funkce

$$f(z) = \frac{e^{2z}}{(z-1)^3}$$

do Laurentovy řady na co největším prstencovém okolí bodu 1 a toto okolí určete.

$\to$ exponenciála $(e^z)^2$

$$f(z) = \frac{e^{2z}}{(z-1)^3} = \frac{e^{2w+2}}{w^3} = \frac{e^2}{w^3} \cdot \sum_{n=0}^{\infty} \frac{(2w)^n}{n!} = \sum_{n=0}^{\infty} \frac{e^2 \cdot 2^n w^n}{n!(w)^{3}} = \sum_{n=0}^{\infty} \frac{e^2 \cdot 2^n (z-1)^{n-3}}{n!} =$$

$w = z-1$

$z = w+1$

$$= \sum_{n=-3}^{\infty} \frac{e^2 2^{n+3} (z-1)^n}{(n+3)!} \qquad 0 < |z-1|$$

Připomínka:

$$\boxed{e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}}$$

VERIFIED

3. $\sum_{n=-3}^{+\infty} \frac{2^{n+3} e^2}{(n+3)!} (z-1)^n$ pro $0 < |z-1|$.

---

4. Rozviňte funkci

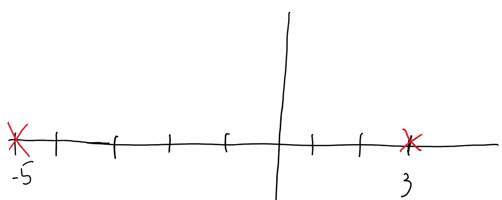
$$f(z) = \frac{1}{(z+5)(z-3)^4(z^2+2z-15)^2}$$

do Laurentovy řady na maximálním prstencovém okolí bodu $z_0 = 3$ a určete jeho parametry.

$$z^2 + 2z - 15 = 0$$

$$z_{1,2} = \frac{-2 \pm \sqrt{4 + 4 \cdot 15}}{2} = -1 \pm 4 \; \begin{cases} 3 \\ -5 \end{cases}$$

$$(z^2+2z-15)^2 = ((z-3)(z+5))^2$$



$R = 8$

$$f(z) = \frac{1}{(z+5)^3 \cdot (z-3)^6} =$$

$$\frac{1}{z+5} = \frac{1}{z+5-8+8} = \frac{1}{(z-3)+8} = \frac{1}{8} \frac{1}{1+\left(\frac{z-3}{8}\right)} = \frac{1}{8} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-3}{8}\right)^n = \bar{R}$$

$$\left(\frac{1}{z+5}\right)'' = \left(\frac{-1}{(z+5)^2}\right)' = \left(\frac{2}{(z+5)^3}\right)$$

$$\left(\frac{1}{z+5}\right)' = \left(\frac{-\,\cdot}{(z+5)^2}\right) = \left(\frac{\quad}{(z+5)^3}\right)$$

$$\check{h}'' = \left(\frac{1}{8}\sum_{n=0}^{\infty}(-1)^n \cdot \frac{1}{8^n}\cdot n\,(z-3)^{n-1}\right)' = \frac{1}{8}\sum_{n=0}^{\infty}(-1)^n\cdot\frac{1}{8^n}\cdot n(n-1)\,(z-3)^{n-3}$$

$$= \sum_{n=3}^{\infty}(-1)^{n+3}\cdot\frac{1}{8^{n+4}}\,(n+3)(n+2)\,(z-3)^n$$

CV6

Tuesday, November 19, 2024 16:40

$\alpha = -2 < 0$

$$4(c)\quad \int_{-\infty}^{\infty} \frac{e^{-2iz}}{(z^2+1)(z^2+9)}\,dx \overset{\alpha=-2}{=} -2\pi i \left(\operatorname{res}_{-i} \frac{e^{-2iz}}{(z^2+1)(z^2+9)}, + \operatorname{res}_{-3i} \frac{e^{-2iz}}{(z^2+1)(z^2+9)} \right)$$

$=$

$z^2+1=0$

$z=\pm i$

$z^2+3=0$

$z=\pm 3i$

$$\operatorname{res}_{-i} \frac{e^{-2iz}}{(z^2+1)(z^2+9)} = \frac{e^{-2i(-i)}}{((-i)^2+9)\,2z}\Big|_{z=-i} = \frac{e^{-2}}{8\cdot(-2i)} = -\frac{e^{-2}}{16i}$$

Zakrývací met.

$$\operatorname{res}_{-3i} \frac{e^{-2iz}}{(z^2+1)(z^2+9)} = \frac{e^{-2i(-3i)}}{((-3i)^2+1)\,2z}\Big|_{z=-3i} = \frac{e^{-6}}{(-8)(-6i)} = \frac{e^{-6}}{48i}$$

$$= -2\pi i \left(-\frac{e^{-2}}{16i} + \frac{e^{-6}}{48i} \right)$$

$\alpha = 1 > 0$

$$d)\quad \int_{-\infty}^{\infty} \frac{e^{1ix}}{(z^2-2z+2)^2}\,dz = 2\pi i \operatorname{res}_{1+i} \frac{e^{iz}}{(z^2-2z+2)^2}$$

$z^2-2z+1+1$

$(z-1)^2=-1$

$z-1=\pm i$

$z=1\pm i$

pól $k=2$

$$= \lim_{z\to 1+i} \left((z-1-i)^2 \frac{e^{iz}}{z^2-2z+2} \right)'$$

$$= \lim_{z\to 1+i} \left(\cancel{(z-1-i)} \frac{e^{iz}}{\cancel{(z-1-i)}(z-1+i)} \right)'$$

$$= \lim_{z\to 1+i} \left(\frac{e^{iz}}{(z-1+i)^2} \right)' = \lim_{z\to 1+i} \frac{ie^{iz}(z-1+i) - e^{iz}}{(z-1+i)^2}$$

$$= \ldots = e^{-1+i}\,\frac{-4}{-8i} = -\frac{1}{2}\, i\, e^{-1+i}$$

$$\text{res?} = 2\pi i \frac{e^{-1+i}}{2i} = \pi e^{-1+i}$$

$$4a)\quad \int_{-\infty}^{\infty} \frac{1}{z^2-6z+25}\,dz \overset{\alpha=0}{=} \int_{-\infty}^{\infty} \frac{1}{\underbrace{(z-3)^2+4^2}}\,dz \overset{\alpha=0}{=} +2\pi i \operatorname{res}_{3+4i} \frac{1}{z^2-6z+25} = 2\pi i \cdot \frac{1}{8i} = \frac{\pi}{4}$$

e^{0iz} $\quad \alpha = 0$

kořen:

$z = 3 \pm 4i$

$$\operatorname{res}_{3+4i} \frac{1}{z^2-6z+25} = \frac{1}{2z-6}\Big|_{z=3+4i}$$

e^{0iz} $\quad d=0$ $\qquad$ kořen: $z = 3 \pm 4i$

$$\operatorname{res}_{3+4i} z^2-6z+25 \qquad 2z-6\Big|_{z=3+4i}$$

$$= \frac{1}{6+8i-6} = \frac{1}{8i}$$

---

b) $$\int_{-\infty}^{\infty} \frac{z^2}{(z^2+4)^2}\,dx \overset{d=0\geq 0}{=} 2\pi i\left(\operatorname{res}_{+2i} \frac{z^2}{(z^2+4)^2}\,dx\right) = \underline{\underline{\pi}}$$

$$\lim_{z\to 2i}\left((z-2i)^2 \frac{z^2}{(z^2+4)^2}\right)' = \lim_{z\to 2i}\left(\frac{z^2}{(z+2i)^2}\right)' = \lim_{z\to 2i} \frac{2z(z+2i)^2 - 2(z+2i)z^2}{(z+2i)^4} =$$

$$= \lim_{z\to 2i} 2z\,\frac{z+2i-z}{(z+2i)^3} = \lim_{z\to 2i} \frac{4zi}{(z+2i)^3} = \frac{-8}{-4i} = \underline{\frac{1}{2i}}$$

Heavisideova Funkce

$\hat{f}(t) : \mathbb{R} \to \mathbb{C}$

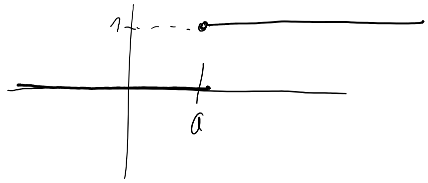
$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\,dt$$

real!

$\mathbb{1}(t) : \mathbb{R} \to \mathbb{R}$

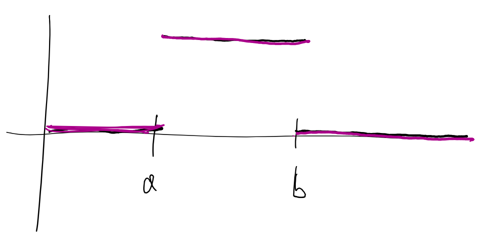
$a \in \mathbb{R}$

$\mathbb{1}(t-a)$:

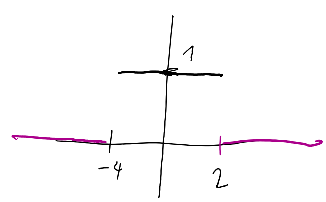


$a < b$

$\mathbb{1}(t-a) - \mathbb{1}(t-b)$;



„...pustili jsme tam záporný světlo."
—Zdenda 2024



1) $f(t) = \mathbb{1}(t+4) - \mathbb{1}(t-2)$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\,dt = \int_{-4}^{2} e^{-i\omega t}\,dt = \left[\frac{e^{-i\omega t}}{-i\omega}\right]_{-4}^{2} = -i\omega\left(e^{-2i\omega} - e^{4i\omega}\right)$$

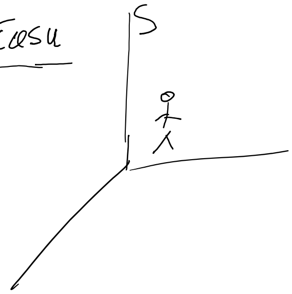
$\omega \neq 0$

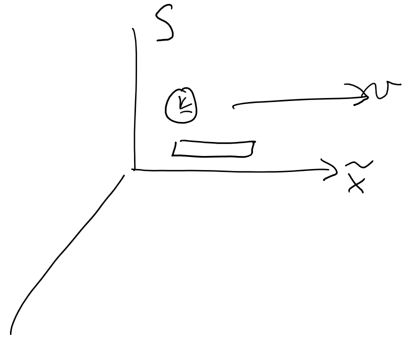
PR08

Wednesday, November 20, 2024 14:31

Kontrakce délek & dilatace času

$$\frac{dl}{dt} = const$$





$$\tilde{t} = \frac{t - vx/c^2}{\sqrt{1-\frac{v^2}{c^2}}} \qquad t = \frac{\tilde{t} + \frac{v\tilde{x}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\tilde{x} = \frac{t - vt}{\sqrt{1-\frac{v^2}{c^2}}} \qquad x = \frac{\tilde{x} + v\tilde{t}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Delta\tilde{t} = \frac{\Delta t - \frac{v\Delta x}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \Delta\tilde{x} = \frac{\Delta x - v\Delta t}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Delta t = \frac{\Delta\tilde{t} + \frac{v\Delta\tilde{x}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \Delta x = \frac{\Delta\tilde{x} + v\Delta\tilde{t}}{\sqrt{1-\frac{v^2}{c^2}}}$$

1) hodiny

$\Delta\tilde{t} = \Delta t_0$

Δt časový interval u pozorovatele

$\Delta\tilde{x} = 0$

$\Delta x \neq 0$

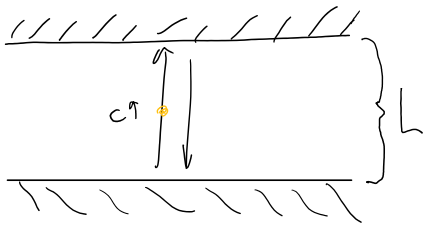
$\frac{v\Delta\tilde{x}}{c^2} = 0$ ve $\tilde{S}$ se hodiny nepohybují

$$\boxed{\Delta t = \frac{\Delta\tilde{t}}{\sqrt{1-\frac{v^2}{c^2}}}}$$ dilatace času

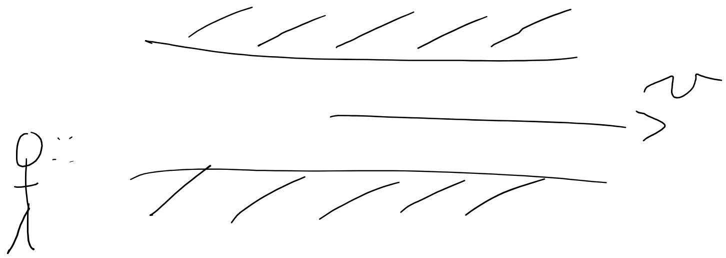
$$\Delta t = 0!$$

$$\boxed{l = l_0 \sqrt{1 - \frac{v^2}{c^2}}}$$ kontrakce délek

Realizace letících hodin

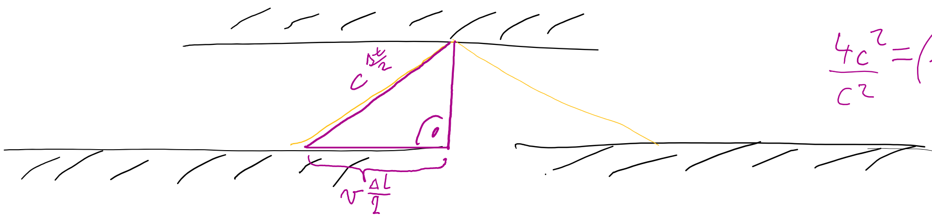


$$\Delta t_0 = \frac{2L}{c}$$



$$L^2 + \frac{v^2 \Delta t^2}{4} = c^2 \frac{\Delta t^2}{4}$$

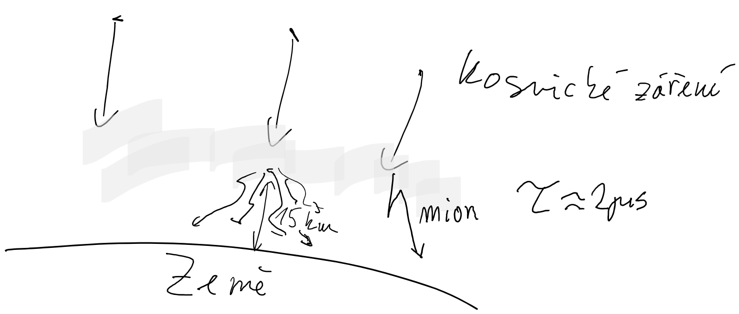
$$\frac{4c^2}{c^2} = \left(1 - \frac{v^2}{c^2}\right) \Delta t^2$$



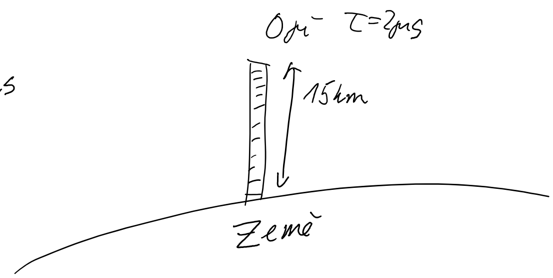
$$\Delta t^2 = \frac{4c^2/c^2}{1 - \frac{v^2}{c^2}} = \frac{2L/c}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Kontrakce délek fyzikálně



$\tau \approx 2\mu s$



$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}^{\tilde{S}} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

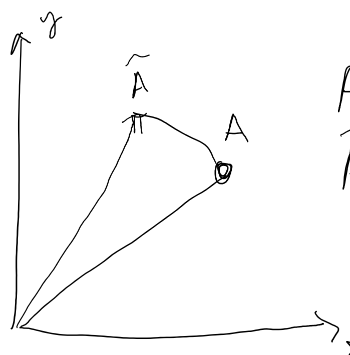
$$v \to -v$$
$$(\gamma \to \gamma)$$
$$(\beta \to -\beta)$$

$$\Lambda^{-1} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Lambda \cdot \Lambda^{-1} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma^2-\gamma^2\beta^2 ; & \cancel{\gamma^2\beta-\gamma^2\beta} ; & 0 ; & 0 \\ \cancel{-\gamma^2\beta+\gamma^2\beta} ; & -\gamma^2\beta^2+\gamma^2 , & 0 , & 0 \\ 0 , & 0 , & 1 , & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \underbrace{\gamma^2(1-\beta^2)}_{1} & 0 & 0 & 0 \\ 0 & \underbrace{\gamma^2(1-\beta^2)}_{1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Běžné rotace



$$A = (x, y) = x + iy$$
$$\tilde{A} = (\tilde{x}, \tilde{y}) = \tilde{x} + i\tilde{y}$$
$$\tilde{A} = A e^{i\alpha}$$

$$\tilde{x} + i\tilde{y} = (x+iy)e^{i\alpha}$$
$$\tilde{x} + i\tilde{y} = (x+iy)(\cos\alpha + i\sin\alpha)$$

$$\mathrm{Re}: \tilde{x} = x\cos\alpha - y\sin\alpha$$
$$\mathrm{Im}: \tilde{y} = x\sin\alpha + y\cos\alpha$$

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}}_{\mathbb{R}} \begin{pmatrix} x \\ y \end{pmatrix}$$

α velmi malé $\cos\alpha \approx 1$
$\sin\alpha \approx \alpha$

$$\boxed{d\mathbb{R} = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$$

$$\underbrace{\qquad\qquad}_{\mathbb{R}}$$

$$\boxed{d\mathbb{R} = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$$

$$\det \mathbb{R} = \cos^2\alpha + \sin^2\alpha = \underline{\underline{1}}$$

$$\Lambda = \begin{array}{cc} \begin{array}{cccc} t & x & & \end{array} & \\ \left(\begin{array}{cc|cc} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) & \begin{array}{c} t \\ x \\ \\ \\ \end{array} \end{array}$$

$\gamma = \cos\alpha$

$\gamma\beta = \sin\alpha$

$$\Lambda = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \text{ nebude fungovat}$$

$$\left.\begin{array}{l} \det \Lambda = \cos^2\alpha - \sin^2\alpha \neq 1 \\ \quad ch^2\alpha - ch^2\alpha = 1 \end{array}\right\} 2/1 \quad \frac{\gamma\beta}{\gamma} = \frac{sh\,u}{ch\,u}$$

$\gamma = \cosh u$

$\gamma\beta = \sin u$

$$\Lambda = \begin{pmatrix} ch\,u & -sh\,u & 0 & 0 \\ -sh\,u & ch\,u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\det \Lambda = 1 \cdot 1 \cdot (ch^2 u - sh^2 u) = 1$$

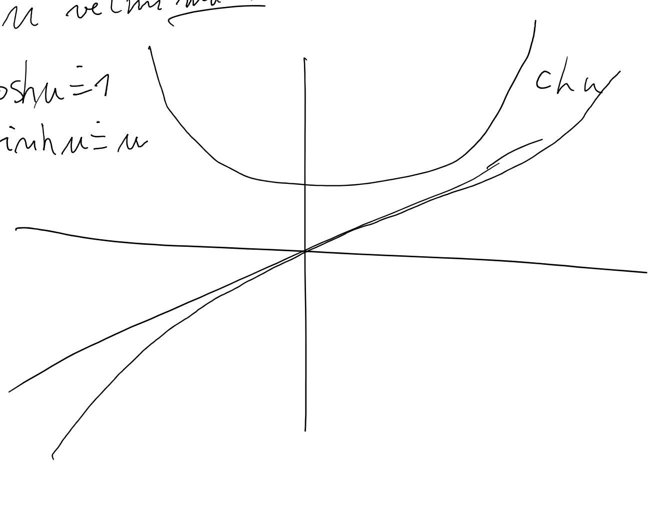
$th\,u = \beta$

$$\boxed{u = \operatorname{argth} \beta}$$

u velmi <u>malé</u>

$\cosh u \doteq 1$

$\sinh u \doteq u$



$$\Lambda = \begin{pmatrix} ch\,u & -sh\,u & & \\ -sh\,u & ch\,u & & \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & -u & 0 & 0 \\ -u & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} - u \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Čtyřvektory

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$ t → sekundy; x, y, z → metr

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix}$$

$c = 2{,}98 \ldots 10^8 \, m s^{-1}$

$c = 1$

$2{,}98 \cdot 10^8 \frac{m}{s} = 1$

$2{,}98 \cdot 10^8 \, m = 1 s$

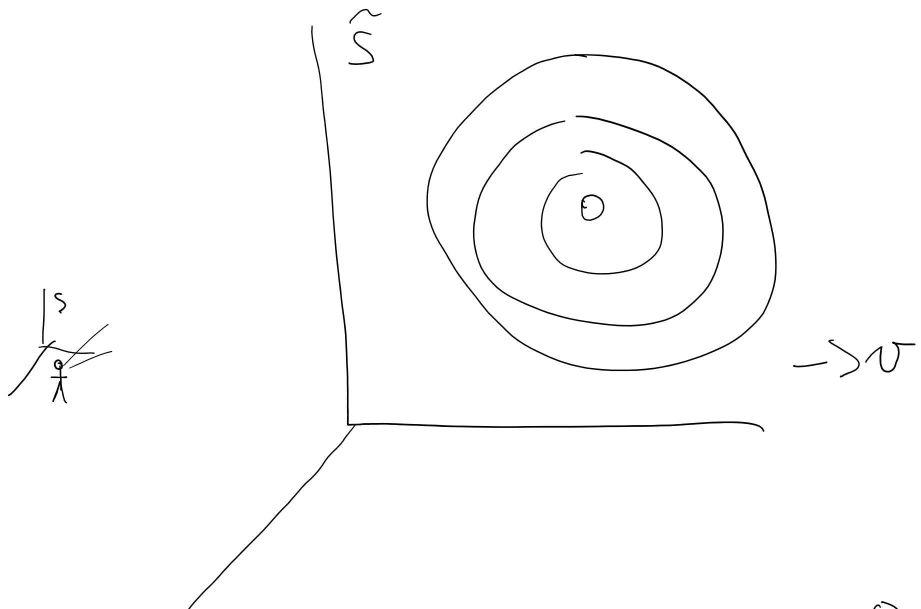
$X \equiv \begin{pmatrix} ct \\ \vec{x} \end{pmatrix}$ událost (ct → kdy, $\vec{x}$ → kde)

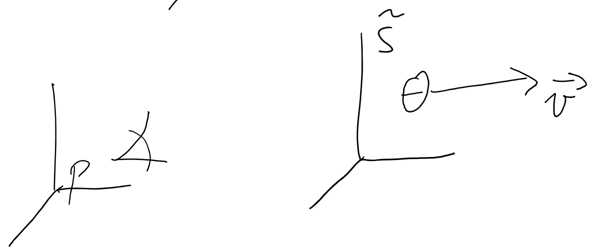
$P \equiv \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix}$ čtyřhybnost

$K = \begin{pmatrix} \omega/c \\ \vec{k} \end{pmatrix}$ vlnový čtyřvektor

$A = \begin{pmatrix} \phi/c \\ \vec{A} \end{pmatrix}$ čtyřpotenciál

$\begin{pmatrix} \rho_Q c \\ \vec{j}_Q \end{pmatrix}$ čtyřtok





$$\phi = \frac{Q}{4\pi\varepsilon_0 r}$$

$$\vec{A} = (0,0,0)$$

$\vec{A} \cdot \vec{B} = A_1 B_1 + A_2 B_2 + A_3 B_3 =$

$$\vec{A}\cdot\vec{B} = A_1B_1 + A_2B_2 + A_3B_3 =$$

$$\frac{dl}{dt} = c \qquad dx^2+dy^2+dz^2 = c^2dt^2$$

$$-c^2dt^2+dx^2+dy^2+dz^2 = 0$$

$$\frac{\sqrt{dx^2+dy^2+dz^2}}{dt} = c^2 \qquad \boxed{-dx_0^2+dx_1^2+dx_2^2+dx_3^2 = 0}$$

$$\vec{A}\cdot\vec{A} = A_1^2+A_2^2+A_3^2$$

v rel. $A\cdot B = -A_0B_0 + A_1B_1 + A_2B_2 + A_3B_3$

$$K\cdot X = -\frac{\omega}{c}ct + k_x\cdot x + k_y\cdot y + k_z\cdot z = -\omega t + \vec{k}\vec{x}$$

$$= \underline{\underline{\vec{k}\vec{x} - \omega t}}$$

$$K = \begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix} \qquad X = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Interval

$B\ (t_B, x_B, y_B, z_B)$

$A\ (t_A, x_A, y_A, z_A)$

$\Delta t = t_B - t_A$ v různu ?????

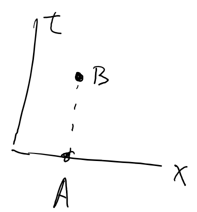
$\Delta x = x_B - x_A$ — || —

světlo:

$$\underbrace{-c^2dt^2+dx^2+dy^2+dz^2}_{\equiv ds^2 \text{ interval}} = 0$$

světlo $\boxed{ds^2 = 0}$

$$\Delta s^2 = -c^2(t_B-t_A)^2 + (x_B-x_A)^2 + (y_B-y_A)^2 + (z_B-z_A)^2 \neq 0$$



$$d\tilde{s}^2 = -c^2d\tilde{t}^2 + d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2 \qquad d\tilde{s}^2 = -c^2\left[\gamma\left(dt - \frac{v\,dx}{c^2}\right)\right]^2 + \left[\gamma(dx - v\,dt)\right]^2$$

$$d\tilde{s}^2 = -c^2 d\tilde{t}^2 + d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2$$

$$\tilde{t} = \frac{t - v_x/c^2}{\sqrt{}} = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$\tilde{x} = \frac{x - vt}{\sqrt{}} = \gamma(x - vt)$$

$$d\tilde{t} = \gamma\left(dt - \frac{v\,dx}{c^2}\right)$$

$$d\tilde{x} = \gamma(dx - v\,dt)$$

$$d\tilde{y} = dy$$

$$d\tilde{z} = dz$$

$$d\tilde{s}^2 = -c^2\left[\gamma\left(dt - \frac{v\,dx}{c^2}\right)\right]^2 + \left[\gamma(dx - v\,dt)\right]^2 + dy^2 + dz^2$$

$$= -c^2\left[\gamma^2\left(dt^2 - 2\frac{v}{c^2}dt\,dx + \frac{v^2}{c^4}dx^2\right)\right] + \gamma^2\left(dx^2 - 2v\,dx\,dt + v^2 dt^2\right) + dy^2 + dz^2$$

$$= \left[-c^2\gamma^2 + \gamma^2 v^2\right]dt^2 + \left[-c^2\gamma^2\frac{v^2}{c^4} + \gamma^2\right]dx^2$$

$$+ \underbrace{\left[+2c^2\gamma^2\frac{v}{c^2} - 2\gamma^2 v\right]}_{0} dx\,dt + dy^2 + dz^2$$

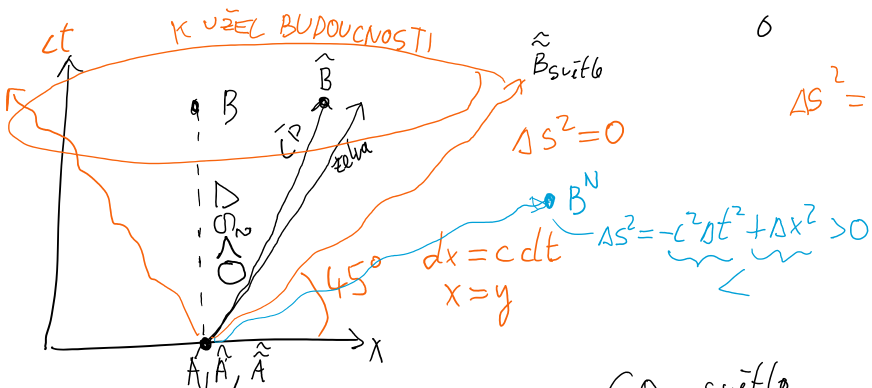
$$= -c^2\underbrace{\gamma^2\left(1 - \frac{v^2}{c^2}\right)}\,dt^2 + \underbrace{\gamma^2\left(1 - \frac{v^2}{c^2}\right)}\,dx^2 + dy^2 + dz^2$$

$$\left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\right)^2\left(1 - \frac{v^2}{c^2}\right) = 1$$

$$= -c^2dt^2 + dx^2 + dy^2 + dt^2$$

Kužele

$$s^2 = -c^2dt^2 + \underbrace{dx^2 + dy^2 + dz^2}_{0} = -c^2dt^2 < 0$$



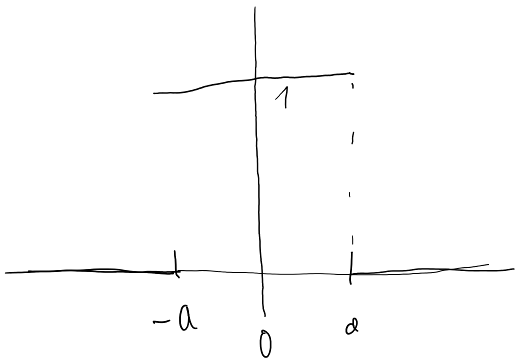
$$\Delta s^2 = -c^2dt^2 + dx^2$$

$$c^2dt^2 = dx^2$$

$$c = \frac{dx}{dt}$$

$$\Delta s^2 = \begin{cases} 0 & \text{- světlo} \\ < 0 & \text{- kauzalita} \\ > 0 & \text{- nejsou kauzálně spojené} \end{cases}$$

Fourierova transformace



$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt = \int_{-a}^{a} e^{-i\omega t} dt =$$

$$= \left[\frac{e^{-i\omega t}}{-i\omega} \right]_{t=-a}^{a} = -\frac{1}{i\omega}\left(e^{-a\omega} - e^{ia\omega}\right)$$

$$= \frac{2\sin(a\omega)}{\omega}, \omega \neq 0$$

$$\hat{f}(0) = \int_{-a}^{a} 1\, dt = 2a$$

Př.: $\mathcal{F}\left[\frac{1}{1+t^2}\right](\omega) = \underbrace{\int_{-\infty}^{\infty} \frac{1}{1+t^2} e^{-i\omega t} dt}_{I} =$ $\quad \alpha = -\omega$

$$= 2\pi i \cdot \operatorname{res}_{z=i} \left(\frac{e^{-i\omega z}}{1+z^2} \right) = 2\pi i \frac{e^{-i\omega(i)}}{2z\big|_{z=i}} = \pi e^{\omega}$$

b) $\omega > 0 \quad \alpha < 0$

$$-2\pi i \operatorname{res}_{z=-i} \left(\frac{e^{-i\omega z}}{1+z^2} \right) = -2\pi i \frac{e^{-i\omega(-i)}}{2z\big|_{z=-i}} = \pi e^{-\omega}$$

Obraz Gaussovy funkce

$a > 0$

$$\mathcal{F}\left[e^{-at^2}\right](\omega) = \sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}}$$

Posuny vzoru

$\alpha \in \mathbb{R}$

1) $\mathcal{F}[f(t-\alpha)](\omega) = e^{-i\omega\alpha} \mathcal{F}[f(t)](\omega)$

1) Př.:

$$\mathcal{F}\left[\frac{1}{1+(t-3)^2}\right](\omega) =$$

1) $\mathcal{F}[f(t-a)](\omega) = e^{\dots} \mathcal{F}[f(t)](\omega)$

Dk.:

$$\mathcal{F}[f(t-a)] = \int_{-\infty}^{\infty} f(t-a)\, e^{-i\omega t}\, dt \overset{\substack{t-a=s \\ dt=ds}}{=}$$

$$= \int_{-\infty}^{\infty} f(t)\, e^{-i\omega(s+a)}\, dt = e^{-ia\omega} \int_{-\infty}^{\infty} f(s)\, e^{-\omega s}\, ds$$

$$= e^{-a\omega}\, \mathcal{F}[f(t)](\omega)$$

$$\mathcal{F}\left[\frac{1}{1+(t-3)^2}\right](\omega) =$$

$f(t) = \frac{1}{1+t^2}$

$$= \mathcal{F}[f(t-3)](\omega) = e^{-i3\omega}\, \mathcal{F}\left[\frac{1}{1+t^2}\right](\omega) = e^{-3i\omega}\, \pi\, e^{-|\omega|}$$

Škálování

3) $a \neq 0,\ a \in \mathbb{R}$

$$\mathcal{F}[f(at)](\omega) =$$

$at = s$
$a\,dt = ds$

$a>0$: ∞; $a<0$: $-\infty$

$$\mathcal{F}[f(at)](\omega) = \int_{-\infty}^{\infty} f(at)\, e^{-i\omega t}\, dt = \frac{1}{a} \int_{-\infty}^{\infty} f(s)\, e^{-i\omega\left(\frac{s}{a}\right)}\, ds = \frac{1}{|a|} \int_{-\infty}^{\infty} f(t)\, e^{-i\frac{\omega}{a}s}\, ds$$

$$= \frac{1}{|a|}\, \mathcal{F}[f(t)]\left(\boxed{\frac{\omega}{a}}\right)$$

Př.: 2) $\mathcal{F}\left[\frac{1}{1+(2t-3)^2}\right](\omega)$

1.
2.

$f(t) = \frac{1}{1-(t-3)^2}$

$$= \mathcal{F}[f(2t)](\omega) \overset{a=2}{=} \frac{1}{|2|}\, \mathcal{F}\left[\frac{1}{1-(t-3)^2}\right]\left(\frac{\omega}{|2|}\right) =$$

$$= \frac{1}{2}\, e^{-3i\frac{\omega}{2}}\, \pi\, e^{-\left|\frac{\omega}{2}\right|}$$

Pravidlo

2) $\mathcal{F}[e^{iat} f(t)](\omega)$

$$= \dots \int_{-\infty}^{\infty} f(t)\, e^{iat} \dots \int_{-\infty}^{\infty} \dots e^{-i(\omega-a)t} \dots$$

2) $\mathcal{F}[e^{\ } \ f(t)](\omega)$

$$\Gamma\ \mathcal{F}[e^{iat}f(t)](\omega)=\int_{-\infty}^{\infty} e^{iat} f(t)\, e^{-i\omega t}\,dt = \int_{-\infty}^{\infty} f(t)e^{-i(\omega-a)t}\,dt=\mathcal{F}[f(t)](\omega-a)$$

$$f(t)=\frac{1}{1+(2t-3)^2}$$

<u>Př.:</u> $\mathcal{F}\left[\frac{e^{it}}{1+(2t-3)^2}\right](\omega) \overset{a=1}{=} \mathcal{F}\left[\frac{1}{1+(2t-3)^2}\right](\omega-1)$

$$=\frac{1}{2}e^{-3i\left(\frac{\omega-1}{2}\right)}\pi e^{-\frac{|\omega-1|}{2}} = \underline{\underline{\pi e^{-\omega}}}\ ?$$

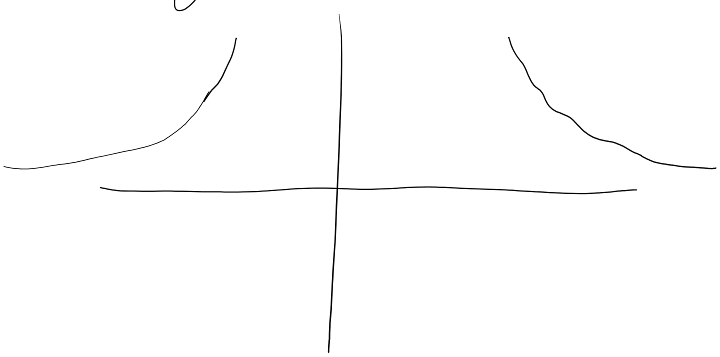
Obraz derivace

$$\mathcal{F}[f^{(n)}](\omega)=(i\omega)^n\,\mathcal{F}[f(t)](\omega)$$

<u>dk:</u>

$$\mathcal{F}[f'(t)](\omega)=\int_{-\infty}^{\infty} f'(t)e^{-i\omega t}\,dt = \left[f(t)e^{-i\omega t}\right]_{t=-\infty}^{\infty} = \int_{-\infty}^{\infty} f(t)(-i\omega)e^{-\omega t}\,dt =$$

$$=\underbrace{[\quad\ ]}_{=0} + i\omega\,\mathcal{F}[f(t)](\omega)$$



<u>Derivace obrazu</u>

$$\mathcal{F}[t f(t)](\omega)=i\frac{d}{d\omega}\mathcal{F}[f(t)](\omega)$$

$$\Gamma\ \frac{d}{d\omega}\int_{-\infty}^{\infty} f(t)\,e^{-i\omega t}\,dt = \int_{-\infty}^{\infty} f(t)(-it)e^{-i\omega t}\,dt = -i\int_{-\infty}^{\infty} t f(t)\,e^{-i\omega t}\,dt =$$

derivování dovnitř

$$\underline{\underline{=-i\ \mathcal{F}[t\cdot f(t)](\omega)}}$$

Př. – derivace obrazu, Gaussova funkce

$$\mathcal{F}[t \cdot \underbrace{e^{-4t^2}}_{f(t)}](\omega) = i \frac{d}{d\omega} \hat{f}(\omega) = i \frac{d}{d\omega}\left(\frac{\sqrt{\pi}}{2} e^{-\frac{\omega^2}{16}}\right) = \frac{\sqrt{\pi}}{2} i \left(-\frac{2\omega}{16} e^{-\frac{\omega^2}{16}}\right)$$

$$\mathcal{F}[e^{-\overset{a=4}{4}t^2}](\omega) = \sqrt{\frac{\pi}{4}}\, e^{-\frac{\omega^2}{16}} =$$

Př. – diferenciální rovnice

$$y'''(t) + 2y''(t) + y(t) = \frac{1}{1+t^2} \quad / \mathcal{F}$$

$$(i\omega)^3 \hat{y}(\omega) + 2(i\omega)^2 \hat{y}(\omega) + \hat{y}(\omega) = \mathcal{F}\left[\frac{1}{1+t^2}\right](\omega)$$

$$-i\omega^3 \hat{y}(\omega) - 2\omega^2 \hat{y}(\omega) + \hat{y}(\omega) = \pi e^{-|\omega|}$$

$$\hat{y}(\omega) = \frac{\pi e^{-|\omega|}}{-i\omega^3 - 2\omega^2 + 1}$$

Inverzní Fourierova transformace a věta o inverzi

$f: \mathbb{R} \to \mathbb{C}$

$$\mathcal{F}^{-1}[f(t)](\omega)$$

$\check{f}: \mathbb{R} \to \mathbb{C}$

$$\check{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$$

Věta o inverzi:

Pro pěkné funkce f:

$$\mathcal{F}^{-1}[\hat{f}(\omega)](t) = f(t)$$

1) $\mathcal{F}[e^{-|t|}](\omega)$

$$\mathcal{F}\left[\frac{1}{1+t^2}\right](\omega) = \pi e^{-|\omega|}$$

$$\mathcal{F}^{-1}: \frac{1}{1+t^2} = \mathcal{F}^{-1}(\pi e^{-|\omega|})(t)$$

$$\frac{1}{\quad} = \mathcal{F}[\pi e^{-|\omega|}](-t)$$

$$\mathcal{F}: \frac{1}{1+t^2} = \mathcal{F}(\pi e)(t)$$

$$= \frac{1}{2\cancel{\pi}} \mathcal{F}[\cancel{\pi} e^{-|\omega|}](-t)$$

<u>Pr.:</u>

CV9

Tuesday, November 26, 2024 16:16

2) a) $\mathcal{F}[h(4t+5)](\omega) \overset{f(t)=h(t+5)}{\underset{\text{škálování}}{=}} \mathcal{F}[f(4t)](\omega) =$

$\underset{a=4}{=} \frac{1}{4}\mathcal{F}[\underbrace{h(t+5)}_{t-(-5)}]\left(\frac{\omega}{4}\right) = \frac{1}{4} e^{-i\frac{\omega}{4}(-5)} \hat{h}\left(\frac{\omega}{4}\right) =$

$\boxed{a=-5}$ posun vzoru

b) $\mathcal{F}[e^{5it}h(t-2)](\omega) =$ posun obrazu

$f(t)=h(t-2)$

$\mathcal{F}[h(t-2)](\omega-5) = e^{-i(\omega-5)\cdot 2}\hat{h}(\omega-5)$

posun vzoru $a=2$

c) $\mathcal{F}[h''(t)](\omega) = i\omega^2 \cdot \hat{h}(\omega)$

$= -\omega^2 \hat{h}(\omega)$

d) $\mathcal{F}[\sin(3t)h'(t+4)](\omega)$

$\sin(3t) = \frac{e^{3it}-e^{-3it}}{2i} = \frac{1}{2i}\left[\mathcal{F}[e^{3it}h'(t+4)(\omega)] - \mathcal{F}[e^{-3it}h'(t+4)](\omega)\right]$

$\mathcal{F}[e^{3it}h'(t+4)](\omega) = \mathcal{F}[h'(\underbrace{t+4}_{t-(-4)})](\omega-3) = e^{-i(\omega-3)(-4)}\mathcal{F}[h'(t)](\omega-3) =$

$a=3$ posun obrazu

$\boxed{a=-4}$ posun vzoru

$= e^{4i(\omega-3)} i(\omega-3)\hat{h}(\omega-3)$

$\mathcal{F}[e^{-3it}h'(t+4)](\omega) = \mathcal{F}[h'(t+4](\omega+3) = \ldots = e^{4i(\omega+3)} i(\omega+3)\hat{h}(\omega+3)$

$a-3$ posun obrazu

3) a) $\mathcal{F}[t\,\overbrace{e^{-9t^2}}^{f(t)}](\omega) \underset{4}{=} i\frac{d}{d\omega}\mathcal{F}[e^{-9t^2}]$

„derivovátko" >.<

$$3)\ a)\ \mathcal{F}\Big[\underbrace{t\, e^{-9t^2}}_{f(t)}\Big](\omega) = i\frac{d}{d\omega}\mathcal{F}\left[e^{-9t^2}\right]$$

derivace obrazu

$$\bullet\ \mathcal{F}\left[e^{-9t^2}\right](\omega) = \sqrt{\frac{\pi}{9}}\, e^{-\frac{\omega^2}{36}} = \frac{\sqrt{\pi}}{3} e^{-\frac{\omega^2}{36}}$$

$$= i\frac{\sqrt{\pi}}{3}\left(e^{-\frac{\omega^2}{36}}\right)'$$

„derivovátko" >.<

$$= i\frac{\sqrt{\pi}}{3}\left(\frac{-\omega}{18} e^{-\frac{\omega^2}{36}}\right)$$

$$3)\ b)\ \mathcal{F}\left[e^{-4(2t-3)^2}\right](\omega) = \cancel{\sqrt{\frac{\pi}{4}}\, e^{-\frac{(2t-3)^2}{16}}} = \frac{1}{2}\mathcal{F}\left[e^{-4(t-3)^2}\right]\left(\frac{\omega}{2}\right) = \boxed{\ }$$

škálování

posun

$$\boxed{\ } = \frac{1}{2} e^{-i\frac{\omega}{2}\cdot 3}\, \mathcal{F}\left[e^{-4t^2}\right]\left(\frac{\omega}{2}\right) = \frac{1}{2}\cdot e^{-i\frac{\omega}{2}\cdot 3}\cdot\sqrt{\frac{\pi}{4}}\, e^{-\frac{\left(\frac{\omega}{2}\right)^2}{16}}$$

$$4)\ y'''(t) + 2y'(t) + 3y(t) = \frac{1}{1+t^2}\quad /\ \mathcal{F}$$

$$i\omega^3\cdot\hat{y}(\omega) + 2i\omega\,\hat{y}(\omega) + 3\hat{y}(t) = \mathcal{F}\left[\frac{1}{1+t^2}\right](\omega)$$

$$(i\omega^3 + 2i\omega + 3)\,\hat{y}(\omega) = \pi e^{-|\omega|}$$

$$\hat{y}(\omega) = \frac{\pi e^{-|\omega|}}{i\omega^3 + 2i\omega + 3}$$

$$\hat{y}(\omega) = \frac{\pi e^{-|\omega|}}{-i\omega^3 + 2i\omega + 3}$$

5) a) $\hat{y}(w) = \dfrac{1}{w^2 - 2w + 5}$

$$y(t) = \mathcal{F}^{-1}\left[\frac{1}{w^2-2w+5}\right](t) = \frac{1}{2\pi}\underbrace{\int_{-\infty}^{\infty} \frac{1}{w^2-2w+5} e^{iwt}\, dw}_{I}$$

$t = \alpha$

a) $\underline{t \geq 0}$

dosadime

$$I = 2\pi i \operatorname{res}_{z=1+2i}\left(\frac{e^{izt}}{z^2-2z+5}\right) = 2\pi i\left(\frac{e^{it-2t}}{2z-2\Big|_{z=1+2i}}\right) = 2\pi i\left(\frac{e^{it-2t)}}{4i}\right)$$

zderivujeme

$$= \underline{\underline{\frac{\pi}{2} e^{it-2t}}}$$

$z^2 - 2z + 5$

$(z-1)^2 - 1 + 5 = 0$

$z - 1 = \pm 2i$

$z = 1 \pm 2i$

b) $\underline{t < 0}$

$$I = -2\pi i \operatorname{res}_{z=1-2i} \frac{e^{izt}}{z^2-2z+5} = -2\pi i \cdot \frac{e^{it+2t}}{2z-2\Big|_{z=1-2i}} = -2\pi i \frac{e^{it+2t}}{-4i} = \underline{\underline{\frac{\pi}{2} e^{it+2t}}}$$

$$y(t) = \begin{cases} \frac{1}{4} e^{it-2t} \ \ldots\ t > 0 \\ \frac{1}{4} e^{it+2t} \ \ldots\ t < 0 \end{cases}$$

---

5 b) $\hat{y}(w) = \dfrac{i}{(w-2i)^2(w+i)} \quad / \mathcal{F}^{-1}$

$$y(t) = \mathcal{F}^{-1}\left[\frac{i}{(w-2i)^2(w+i)}\right](t) = \frac{1}{2\pi}\underbrace{\int_{-\infty}^{\infty} \frac{i}{(w-2i)^2(w+i)} e^{iwt}\, dw}_{I}$$

a) $t \geq 0$

b) $t < 0$

a) $t \geq 0$

$$I = 2\pi i \operatorname{res}_{z=2i}\left(\frac{e^{izt}}{(z-2i)^2(z+i)}\right)$$

$$\operatorname{res}_{z=2i}\left(\frac{e^{izt}}{(z-2i)^2(z+i)}\right)=\lim_{z\to 2i}\left((z-2i)^2 \text{———}\right)'=\lim_{z\to 2i}\left(\frac{e^{izt}}{z+i}\right)'=\lim_{z\to 2i}\frac{ite^{izt}(z+i)-e^{izt}}{(z+i)^2}$$

$$=e^{-2t}\cdot\frac{-3t+1}{-9}=2\pi i\,\frac{3t+1}{9}e^{-2t}$$

b) $\underline{t<0}$:

$$I=-2\pi i\,\operatorname{res}_{z=-i}\frac{e^{izt}}{(z-2i)^2(z+i)}$$

$$y(t)=\begin{cases}\dfrac{-3t+1}{9}e^{-2t}; & t\geq 0\\[2mm] -\dfrac{1}{9}e^{t}; & t<0\end{cases}$$

PR10

Tuesday, December 3, 2024 09:10

Minule pokrač. – Fourierova transformace

$$\underbrace{y''(t)} - y(t) = -\frac{e^{-|t|}}{2} \quad / \mathcal{F}$$

$$i\omega^2 \hat{y}(\omega) - \hat{y}(\omega) = \mathcal{F}\left[-\frac{e^{-|t|}}{2}\right](\omega)$$

$$(-\omega^2 - 1)\hat{y}(\omega) = -\frac{1}{1+\omega^2}$$

$$\hat{y}(\omega) = \frac{1}{(1+\omega^2)^2}$$

$$y(t) = \mathcal{F}^{-1}\left[\frac{1}{(1+\omega^2)^2}\right](t)$$

$$= \frac{1}{2\pi} \underbrace{\int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)^2} e^{it\omega} \, d\omega}_{I}$$

α

a) $t \geq 0$:

$$I = 2\pi i \operatorname{res}_{z=i} \frac{e^{itz}}{(1+z^2)^2}$$

$$= 2\pi i \lim_{z\to i} \left(\cancel{(z-i)^2} \frac{e^{itz}}{(z+i)^2 \cancel{(z-i)^2}}\right)' =$$

$$= 2\pi i \lim_{z\to i} \left(\frac{e^{itz}}{(z-i)^2}\right)'$$

$$= 2\pi i \lim_{z\to i} \frac{ite^{itz}(z+i)^{\cancel{2}} - 2\cancel{(z+i)}e^{itz}}{(z+i)^{\cancel{4}\,3}}$$

$$= 2\pi i e^{-t} \frac{\cancel{2}\cdot 2t + \cancel{2}\cdot 2}{\cancel{-}8i}$$

$$= \pi e^{-t} \frac{t+1}{2}$$

$t < 0$:

$$I = -2\pi i \operatorname{res}_{z=-i} \frac{e^{itz}}{(1+z^2)^2} = \dots = -\frac{\pi}{2}(t-1)e^{t}$$

$$y(t) = \begin{cases} e^{-t}\frac{t+1}{4} & t \geq 0 \\ -\frac{t-1}{4}e^{t} & t < 0 \end{cases}$$

Konvoluce

Def.: $f: \mathbb{R} \to \mathbb{C}$, $g: \mathbb{R} \to \mathbb{C}$ $\;\to\; (f * g)\; \mathbb{R} \to \mathbb{C}$

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$$

Věta: $\mathcal{F}[f * g](\omega) = \hat{f}(\omega) \cdot \hat{g}(\omega)$

Př.: $$y'(t) + \underbrace{\int_{-\infty}^{\infty} e^{-\pi\tau^2} y(t-\tau)\, dt} = e^{-\pi t^2}$$

$$\underbrace{y'(t)} + (e^{-\pi t^2} * y)(t) = e^{-\pi t^2} \quad / \mathcal{F}$$

$$i\omega\hat{y}(\omega) + \mathcal{F}[e^{-\pi t^2}](\omega)\,\hat{y}(\omega) = \mathcal{F}[e^{-\pi t^2}](\omega)$$

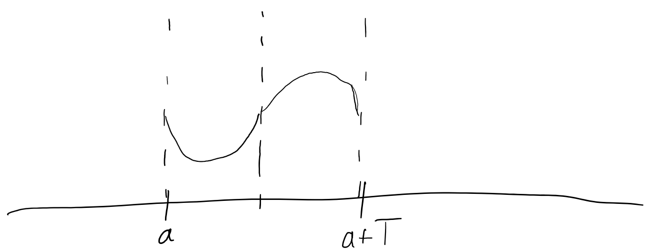
$$\underbrace{y'(t)}_{} + (e^{-\pi t^2} * y(t)) = e^{-\pi t^2} \quad / \mathcal{F}$$

$$i\omega \hat{y}(\omega) + \mathcal{F}[e^{-\pi t^2}](\omega) \cdot \hat{y}(\omega) = \mathcal{F}[e^{-\pi t^2}](\omega)$$

$$i\omega \hat{y}(\omega) + e^{-\frac{\omega^2}{4\pi}} \hat{y}(\omega) = e^{-\frac{\omega^2}{4\pi}}$$

$$\left(i\omega + e^{-\frac{\omega^2}{4\pi}}\right) \hat{y}(\omega) = e^{-\frac{\omega^2}{4\pi}}$$

$$\hat{y}(\omega) = \frac{e^{-\frac{\omega^2}{4\pi}}}{i\omega + e^{-\frac{\omega^2}{4\pi}}}$$



$$c_n = \frac{1}{T} \int_a^{a+T} f(t) e^{-i\frac{2\pi n t}{T}} dt$$

$$= \frac{1}{T} \int_{-\infty}^{\infty} f(t) e^{-i\frac{2\pi n}{T} t} dt = \frac{1}{T} \hat{f}\left(\frac{2\pi n}{T}\right)$$

Laplaceova transformace

F. připomenutí

$f: \mathbb{R} \to \mathbb{C}$

$\hat{f}: \mathbb{R} \to \mathbb{C}$

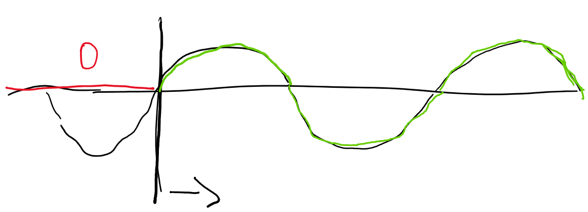
Definice:

$f: [0; \infty) \longrightarrow \mathbb{C}$

$$\mathcal{L}[f(t)](s) = F(s) = \int_0^{\infty} f(t) e^{-st} dt, \quad s \in \mathbb{C}$$

Laplaceova transformace hledí jen do budoucnosti

f stotožňuje s $f \cdot 1$



Př.:

1) $\alpha \in \mathbb{C}$

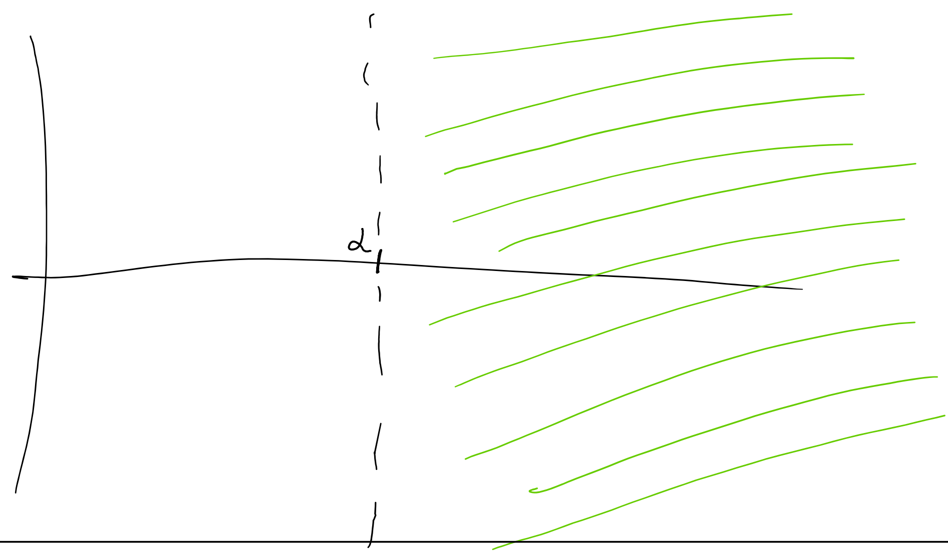
$$\mathcal{L}[e^{\alpha t}](s) = \int_0^\infty e^{\alpha t} e^{-st}\, dt = \int_0^\infty e^{(\alpha - s)t}\, dt = \left[-\frac{e^{-(s-\alpha)t}}{s-t}\right]_t^\infty =$$

$$= \underbrace{\lim_{t\to\infty} -\frac{e^{-(s-\alpha)t}}{-(s-\alpha)}}_{\overset{?}{=}0} + \frac{1}{s-\alpha}$$

$\to$ oscilující e^x

$$\left|e^{-(s-\alpha)t}\right| = e^{-(\mathrm{Re}\, s - \mathrm{Re}\,\alpha)t} \xrightarrow{t\to\infty} 0$$

$\mathrm{Re}\, s > \mathrm{Re}\,\alpha$



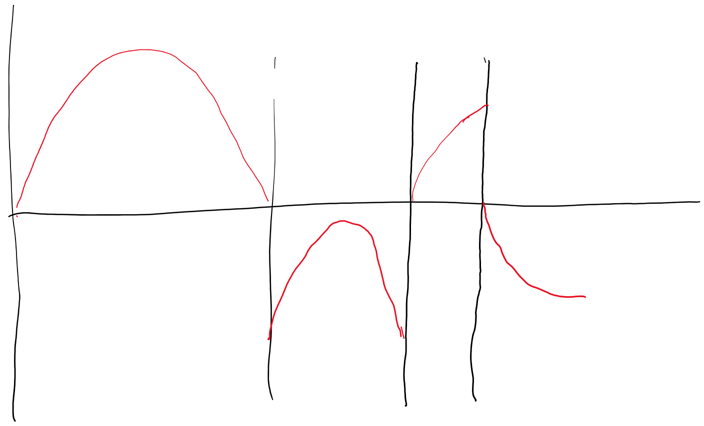
$\omega \in \mathbb{C}$

3) $\mathcal{L}[\sin(\omega t)](s) = \frac{1}{2i}\left[\mathcal{L}[e^{i\omega t}](s) - \mathcal{L}[e^{-i\omega t}](s)\right]$

$$\sin(\omega t) = \frac{e^{i\omega t} - e^{i\omega t}}{2i} = \frac{1}{2i}\left[\frac{1}{s-i\omega} - \frac{1}{s+i\omega}\right] = \frac{1}{\cancel{2i}}\,\frac{\cancel{s}+i\omega - \cancel{s}+i\omega}{(s-i\omega)(s+i\omega)}$$

$$= \underline{\underline{\frac{2\omega}{s^2+\omega^2}}}$$

Po částech spojitá fce!



$\exists M > 0 \ \exists \alpha \in \mathbb{R}$

$|f(t)| \leq M e^{\alpha t} \ \forall t \geq 0$

L_0

1) $\mathcal{L}[f(at)](s) = \int_0^\infty f(at) e^{-st} dt = \left| \begin{matrix} at = x \\ a\,dt = dx \end{matrix} \right| = \frac{1}{a} \int_0^\infty f(x) e^{-\frac{s}{a}x} dx =$

$0 \rightsquigarrow 0$
$\infty \rightsquigarrow \infty$

$= \frac{1}{a} F\left(\frac{s}{a}\right)$

$a \in \mathbb{C}$

2) $\mathcal{L}[e^{at} f(t)](s) = \int_0^\infty e^{at} e^{-st} dt = \int_0^\infty f(t) e^{-(s-a)t} dt = F(s-a$

$$\mathcal{F}\left[\left(te^{-\frac{(3t+2)^2}{2}}\right) * \left(e^{-2it} h'''(t)\right)\right](\omega) = \underbrace{\mathcal{F}\left[\left(te^{-\frac{(3t+2)^2}{2}}\right)\right](\omega)}_{F_1} \cdot \underbrace{\mathcal{F}\left[\left(e^{-2it} h'''(t)\right)\right](\omega)}_{F_2}$$

$$* \longrightarrow \cdot$$

$$F_1 = i \frac{d}{d\omega} \mathcal{F}\left[e^{-\frac{(3t+2)^2}{2}}\right](\omega) =$$

$$F_1 = \frac{i\sqrt{2\pi}}{3}\left(e^{\frac{2}{3}\omega i - \frac{\omega^2}{18}}\right)' = \frac{i\sqrt{2\pi}}{3} \cdot \left(\frac{2}{3} i - \frac{\omega}{9}\right) e^{\frac{2}{3}\omega i - \frac{\omega^2}{18}}$$

$$\mathcal{F}\left[e^{-\frac{(3t+2)^2}{2}}\right](\omega) \underset{a=3}{\overset{\text{škálování}}{=}} \frac{1}{3}\mathcal{F}\left[e^{-\frac{(t+2)^2}{2}}\right]\left(\frac{\omega}{3}\right)$$

$$\underset{a=-2}{\overset{\text{posun vzoru}}{=}} \frac{1}{3} e^{-i\frac{\omega}{3}(-2)} \mathcal{F}\left[e^{-\frac{t^2}{2}}\right]\left(\frac{\omega}{3}\right)$$

$$= \frac{1}{3} e^{\frac{2}{3}\omega i} \sqrt{2\pi}\, e^{-\frac{(\omega/3)^2}{2}}$$

$$F_2 \underset{a=-2}{\overset{\text{posun obrazu}}{=}} \mathcal{F}[h'''](\omega+2) = \left(i(\omega+2)\right)^3 \hat{h}(\omega+2)$$

7) $$y''(t) + \int_{-\infty}^{\infty} e^{-|\tau|} y(t-\tau)\, d\tau = e^{-\frac{t^2}{4}} \quad / \mathcal{F}$$

$$(i\omega)^2 \hat{y}(\omega) + \frac{2}{1+\omega^2} \cdot \hat{y}(\omega) = \mathcal{F}\left[e^{-\frac{t^2}{4}}\right](\omega)$$

8) b) $$\hat{g}(\omega) = \frac{1}{\omega^2 - 2i\omega - 1}$$

$$\hat{f} \cdot \hat{g} = \widehat{f * g}(\omega)$$

$\uparrow$ vím $\quad\uparrow$ vím

$$\hat{f} = \frac{\widehat{f*g}(\omega)}{\hat{g}}$$

$$\hat{f} = \frac{\omega^2 - 2i\omega - 1}{(\omega - i)^4}$$

$$z^2 - 2iz - 1 = (z-i)^2$$

$$f(t) = \mathcal{F}^{-1}\left[\frac{1}{(\omega-i)^2}\right](t) = \frac{1}{2\pi} \underbrace{\int_{-\infty}^{\infty} \frac{1}{(\omega-i)^2} e^{i\omega t} dt}_{I} = -te^{-t} \quad t \geq 0$$

a) $t \geq 0$:

$$I = 2\pi i \operatorname{res}_{z=i} \frac{e^{izt}}{(z-i)^2} = \lim_{z\to i}\left((z-i)^2 \underline{\quad\quad}\right)'$$

$\uparrow$ pól ř. 2

$$= 2\pi i \lim_{z\to i}\left(e^{izt}\right)' = 2\pi i \lim_{z\to i} ite^{izt} = -2\pi \cdot te^{-t}$$

b) $t < 0$:

$$I = 0$$

Některý věci jsou si podobný, ale nejsou stejný...

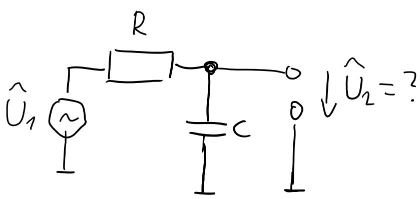
— Zdenda 2024

$$\boxed{\frac{d}{dt} f(t) g(t) = ?}$$

Laplace T

a) $\mathcal{L}[e^{-it}t^2](s) \overset{\text{Posun obrazu}}{\underset{a=-i}{=}}$

$\mathcal{L}[t^2](s+i) =$



$\hat{U}_1 = |\hat{U}_1| e^{j\varphi t}$

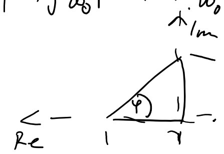
$u_1 = \sin(\omega t)$

$\mathcal{L}(u_1)(s) = \frac{\omega}{s^2+\omega^2}$

$$\hat{H} = \frac{\hat{U}_2}{\hat{U}_1} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{\cancel{\frac{1}{j\omega C}}}{\frac{j\omega RC + 1}{\cancel{j\omega C}}}$$

$$\hat{H} = \frac{1}{1 + j\omega \underbrace{(RC)}_{\omega_0^{-1}}} \Rightarrow \left|1 + j\frac{\omega}{\omega_0}\right| = 1^2 + \frac{\omega^2}{\omega_0^2}$$

$$\hat{H}^{-1} = \left(1 + \frac{\omega^2}{\omega_0^2}\right) e^{\frac{\omega}{\omega_0} j}$$



$$\hat{H} = \frac{1}{\underbrace{1 + \left(\frac{\omega}{\omega_0}\right)^2}} e^{-j\frac{\omega}{\omega_0}} \varphi ?$$

$$\log_{10}|\hat{H}| = \log_{10} \frac{1}{1 + \left(\frac{\omega}{\omega_0}\right)^2}$$

$\omega = 10\,\omega_0$

$\omega = \omega_0$

$\omega = \omega_0/10$

$\underline{\omega = \omega_0}$:

$$\log_{10} \frac{1}{1+1}$$

$$\underbrace{\log_{10} 1 - \log 2}$$

$\cdot 20 \cdot (-0{,}3) = -6\ \text{dB}$