3.4.2 True solutions

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3.5 Parametric interval systems

When solving interval linear systems of equations $\boldsymbol{A}x = \boldsymbol{b}$ we have so far assumed that the matrix and the right-hand side entries vary inside the corresponding intervals independently. However, this is hardly true for real-life problems. Usually, the system reads as $A(p)x = b(p)$, where the entries of $A(p)$ and $b(p)$ depends on parameters $p_1, \dots, p_K$, whose domains are intervals $\boldsymbol{p}_1, \dots, \boldsymbol{p}_K$. Calculating the ranges (or their enclosures) of $\boldsymbol{A} := A(\boldsymbol{p})$ and $\boldsymbol{b} := b(\boldsymbol{p})$ leads to the relaxed system of standard interval linear equations $\boldsymbol{A}x = \boldsymbol{b}$, but the loss of dependence structure may cause a huge overestimation. Thus, appropriate methods for dependent linear systems are of high importance.

The solution set of a parametric interval system is defined as

$$\Sigma_{\mathrm{p}} = \{x \in \mathbb{R}^n;\ A(p)x = b(p) \text{ for some } p \in \boldsymbol{p}\}.$$

It may have a complicated structure and find a tight enclosure is very challenging.

In general, we may apply preconditioning

$$CA(\boldsymbol{p})x = Cb(\boldsymbol{p}).$$

Again, we may simply relax the system to standard interval linear equations, which is useful provided we are able to somehow simplify and tightly evaluate the entries of $CA(\boldsymbol{p})$ and $Cb(\boldsymbol{p})$.

A promising approach is to employ the residual form (Section 3.2.2). Let $x^* \in \mathbb{R}^n$, for example the solution of $A(p^c)x = b(p^c)$. Then we find an enclosure to Σ_{p} in the form of $x^* + \boldsymbol{y}$, where $\boldsymbol{y}$ encloses the solution set to

$$CA(\boldsymbol{p})y = C(b(\boldsymbol{p}) - A(\boldsymbol{p})x^*). \tag{3.25}$$

Now, it depends on the special structure of dependencies how effectively we are able to enclose both terms $C(b(\boldsymbol{p}) - A(\boldsymbol{p})x^*)$ and $CA(\boldsymbol{p})$. Since the dependencies may be very complex, no general recipe can easily be given.

Applying this approach to the residual Krawczyk method, for instance, we come up with the residual Krawczyk operator

$$K_r(\boldsymbol{y}) := C(b(\boldsymbol{p}) - A(\boldsymbol{p})x^*) + (I_n - CA(\boldsymbol{p}))\boldsymbol{y}.$$

The direct Krawczyk enclosure (3.14) used on the system (3.27), which was originally proposed by Skalna (2012), works as follows.

Proposition 3.53 (Skalna, 2012)**.** *Let $R := \operatorname{mag}(I_n - CA(\boldsymbol{p}))$. If $\rho(R) < 1$, then*

$$\Sigma_{\mathrm{p}} \subseteq x^* + [-1, 1](I_n - R)^{-1} \operatorname{mag}(C(b(\boldsymbol{p}) - A(\boldsymbol{p})x^*)).$$

As it is very hard to handle the parametric systems in the general form, we focus on particular case of a linear dependence structure.

3.5.1 Linear dependencies

We will consider a linear parametric structure, that is, a system

$$A(p)x = b(p),$$

where $A(p) = \sum_{k=1}^{K} A^k k p_k$, $b(p) = \sum_{k=1}^{K} b^k p_k$ and $p \in \boldsymbol{p}$ for some given interval vector $\boldsymbol{p} \in \mathbb{IR}^K$, matrices $A^1, \dots, A^K \in \mathbb{R}^{n \times n}$ and vectors $b^1, \dots, b^K \in \mathbb{R}^n$. This linear parametric case covers a wide area of interval systems with dependencies. For instance, the interval systems $\boldsymbol{A}x = \boldsymbol{b}$, where the constraint matrix is supposed to be symmetric, skew-symmetric, Toeplitz or Hankel.

It is a tempting problem to determine a simple characterization of Σ_{p}. By using a time consuming Fourier–Motzkin elimination $\boxed{\text{REF}}$ we describe the parametric solution set by possibly double exponential number of nonlinear inequalities. The description, however, shows that the shape of Σ_{p} is characterizes by quadrics. For special classes of parametric interval systems, the explicit description of the solution set was developed by Hladík (2008b); Popova (2009). Shapes of the particular solution sets were first analyzed in Alefeld et al. (1997, 2003).

The following necessary characterization of Σ_{p} is by $\boxed{\text{REF}}$; see also Hladík (2012d).

Theorem 3.54. *If $x \in \Sigma_{\mathrm{p}}$, then it solves*

$$|A(p^c)x - b(p^c)| \le \sum_{k=1}^{K} p_k^{\Delta} |A^k x - b^k|. \tag{3.26}$$

Proof. Let $x \in \mathbb{R}^n$ be a solution to $A(p)x = b(p)$ for some $p \in \boldsymbol{p}$. Then,

$$\begin{aligned}|A(p^c)x - b(p^c)| &= \left|\sum_{k=1}^{K} p_k^c (A^k x - b^k)\right| = \left|\sum_{k=1}^{K} p_k^c (A^k x - b^k) - \sum_{k=1}^{K} p_k (A^k x - b^k)\right| \\ &= \left|\sum_{k=1}^{K} (p_k^c - p_k)(A^k x - b^k)\right| \le \sum_{k=1}^{K} |p_k^c - p_k||A^k x - b^k| \le \sum_{k=1}^{K} p_k^{\Delta} |A^k x - b^k|. \qquad \square\end{aligned}$$

Popova (2009) showed that it is the complete characterization of Σ_{p} as long as no interval parameter appears in more than one equation. Thus, it can serve as a generalization of the Oettli–Prager Theorem 3.1 and a basis for deriving direct enclosures to the parametric solution set; see Hladík (2012d).

Theorem 3.55 (Popova, 2009)**.** *Suppose that for every $k = 1, \dots, K$, the matrix $(A^k \mid b^k)$ has only one nonzero row. Then $x \in \Sigma_{\mathrm{p}}$ if and only if x solves* (3.26)*.*

Proof. By Theorem 3.54, we have necessity of (3.26). Here, we show sufficiency. Let $i \in \{1, \dots, n\}$ and $\mathcal{K}_i \subseteq \{1, \dots, K\}$ the set of parameters appearing in the ith row. From (3.26), there is $\alpha \in [0, 1]$ such that

$$|A(p^c)_{i*} x - b(p^c)_i| = \sum_{k \in \mathcal{K}_i} \alpha p_k^{\Delta} |A_{i*}^k x - b_i^k|.$$

Hence we can find $q_k \in [-p_k^{\Delta}, p_k^{\Delta}]$, $k \in \mathcal{K}_i$ such that

$$A(p^c)_{i*} x - b(p^c)_i = \sum_{k \in \mathcal{K}_i} q_k (A_{i*}^k x - b_i^k),$$

which after rearrangement reads

$$\sum_{k \in \mathcal{K}_i} A_{i*}^k (p_k^c - q_k) x = \sum_{k \in \mathcal{K}_i} b_i^k (p_k^c - q_k).$$

Therefore, x solves the realization $A(p)x = b(p)$ with $p = p^c - q \in \boldsymbol{p}$, where $q_k := 0$ for $k \notin \cup_i \mathcal{K}_i$. Since the sets $\mathcal{K}_i$, $i\{1, \dots, n\}$, are mutually disjoint, there is no conflict in the selestion of p. $\square$

Even though it is hard to describe the solution set in general, checking $x \in \Sigma_{\mathrm{p}}$ for a given $x \in \mathbb{R}^n$ is a polynomial problem carried out by a suitable linear programming solver since the constraints $A(p)x = b(p)$, $p \in \boldsymbol{p}$ are linear.

Enclosures of solutions. As we have already noticed, relaxing the dependencies we get an ordinary interval system of equations $\boldsymbol{A}x = \boldsymbol{b}$, where $\boldsymbol{A} := A(\boldsymbol{p})$ and $\boldsymbol{b} := b(\boldsymbol{p})$. Many interval system solvers use preconditioning, however, is is not effective to precondition the relaxed system $C\boldsymbol{A}x = C\boldsymbol{b}$. Provably tighter relaxation is obtained by preconditioning the original system and after relaxing, which leads to the interval system $\boldsymbol{A'}x = \boldsymbol{b'}$ with

$$\boldsymbol{A}' := \sum_{k=1}^{K} \boldsymbol{p}_k(CA^k), \quad \boldsymbol{b}' := \sum_{k=1}^{K} \boldsymbol{p}_k(Cb^k).$$

As mentioned in the general case, it seems convenient to seek for an enclosure by using the preconditioned residual form (3.25), which takes the form

$$\left(\sum_{k=1}^{K} \boldsymbol{p}_k(CA_k)\right) y = \sum_{k=1}^{K} \boldsymbol{p}_k(C(b_k - A_k x^*)). \tag{3.27}$$

The important aspect of this form is that the right-hand side interval vector of (3.27), and sometimes the interval matrix, too, are generally tighter than simple relaxation $\boldsymbol{A} := A(\boldsymbol{p})$, $\boldsymbol{b} := b(\boldsymbol{p})$ and transformation to

$$\boldsymbol{A}y = C(\boldsymbol{b} - \boldsymbol{A}x^*).$$

This is because of the sub-distributivity law. On the other hand, using A_k and b_k, $k = 1, \ldots, K$, explicitly may be time consuming. When K is large, the matrices A_k are often sparse, and one can think of evaluating $C(b_k - A_k x^*)$ without explicitly constructing A_k and b_k. This will be the case in Section 3.5.2.

Evaluating the interval matrix and right-hand side, (3.27) becomes a standard interval linear system, which may be solved by any method presented in Section 3.2, depending on the requirements on running time and tighness of an enclosure.

In particuler, the residual Krawczyk operator applied on (3.27) draws

$$K_r(\boldsymbol{y}) = \sum_{k=1}^{K} C(b_k - A_k x^*)\boldsymbol{p}_k + \left(I_n - \sum_{k=1}^{K} (CA_k)\boldsymbol{p}_k\right)\boldsymbol{y}.$$

The direct enclosure formula (3.14) is applied below; in a similar form, it was observed in Hladík (2012d); Skalna (2006).

Proposition 3.56. *Let* $R := \operatorname{mag}\left(I_n - \sum_{k=1}^{K}(CA_k)\boldsymbol{p}_k\right)$. *If* $\rho(R) < 1$, *then*

$$\Sigma_{\mathrm{p}} \subseteq x^* + [-1, 1](I_n - R)^{-1} \operatorname{mag}\left(\sum_{k=1}^{K} C(b_k - A_k x^*)\boldsymbol{p}_k\right).$$

Linear parametric systems appear in diverse problems. We show an example from stucture mechanics.

Example 3.57 (Displacements of a truss structure (Skalna, 2006))**.** Consider a 7-bar truss structure as depicted in Figure 3.5. Supposing a downward force at some nodes, the displacements of the nodes are computed by solving the linear system of equations

$$Kd = f, \tag{3.28}$$

where is the stiffness matrix, and f is the vector of forces at particular nodes. The stiffness matrix has the structure

$$K = \begin{pmatrix}
\frac{s_{12}}{2} + s_{13} & -\frac{s_{12}}{2} & -\frac{s_{12}}{2} & -s_{13} & 0 & 0 & 0 \\
-\frac{s_{21}}{2} & \frac{s_{21}+s_{23}}{2} + s_{24} & \frac{s_{21}-s_{23}}{2} & -\frac{s_{23}}{2} & \frac{s_{23}}{2} & -s_{24} & 0 \\
-\frac{s_{21}}{2} & \frac{s_{21}-s_{23}}{2} & \frac{s_{21}+s_{23}}{2} & \frac{s_{23}}{2} & -\frac{s_{23}}{2} & 0 & 0 \\
-s_{31} & -\frac{s_{32}}{2} & \frac{s_{32}}{2} & s_{31} + \frac{s_{32}+s_{34}}{2} + s_{35} & \frac{s_{34}-s_{32}}{2} & -\frac{s_{34}}{2} & -\frac{s_{34}}{2} \\
0 & \frac{s_{32}}{2} & -\frac{s_{32}}{2} & \frac{s_{34}-s_{32}}{2} & \frac{s_{34}+s_{32}}{2} & -\frac{s_{34}}{2} & -\frac{s_{34}}{2} \\
0 & -s_{42} & 0 & -\frac{s_{43}}{2} & -\frac{s_{43}}{2} & s_{42} + \frac{s_{43}+s_{45}}{2} & 0 \\
0 & 0 & 0 & -\frac{s_{43}}{2} & -\frac{s_{43}}{2} & 0 & \frac{s_{43}+s_{45}}{2}
\end{pmatrix}.$$

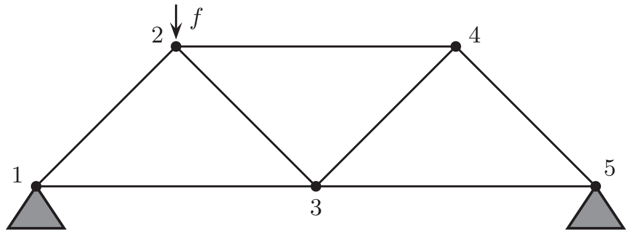

Figure 3.5: (Example 3.57) A truss structure.

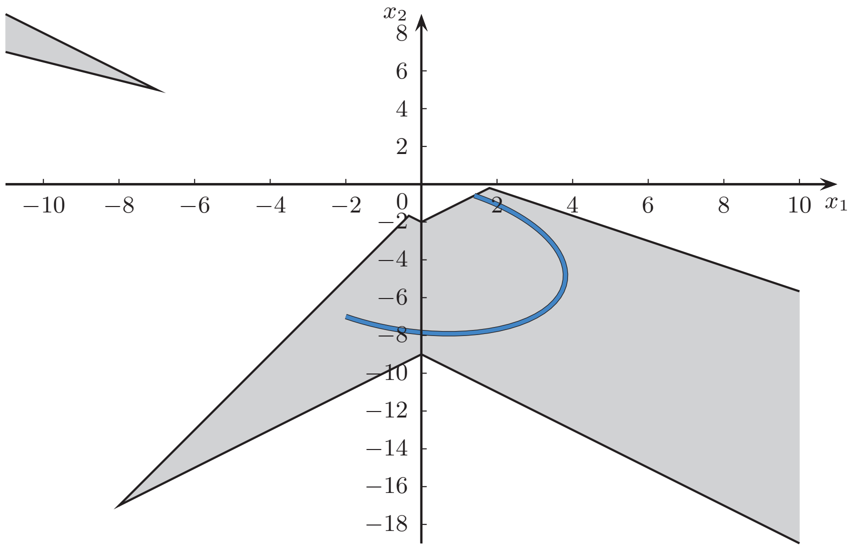

Figure 3.6: (Example 3.58) Solution set of a linear parametric system and its relaxation.

The stiffness of a particular bar (i, j) is computed as $s_{ij} = E\sigma/\ell$, where E is Young's modulus, σ is the cross-sectional area, and ℓ is the length of the bar. Since Young's modulus is usually given only approximately with some error ΔE, the stiffnesses s_{ij} of bars are uncertain and known to lie in some intervals only. Thus, (3.28) has a form of a linear parametric interval system. □

Example 3.58. Consider the linear parametric system $A(p)x = b(p)$ with

$$A(p) = \begin{pmatrix} 1-2p & 1 \\ 2 & 4p-1 \end{pmatrix}, \quad b(p) = \begin{pmatrix} 7p-9 \\ 3-2p \end{pmatrix}, \quad p \in \boldsymbol{p} = [0, 1].$$

The parametric solution set Σ_{p} is a curve depicted in Figure 3.6. Direct relaxation of the parametric system leads to a standard interval system $\boldsymbol{A}x = \boldsymbol{b}$, where

$$\boldsymbol{A} = \begin{pmatrix} [-1, 1] & 1 \\ 2 & [-1, 3] \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} -[2, 9] \\ [1, 3] \end{pmatrix}.$$

the solution set of which, however, is unbounded. □

3.5.2 The symmetric solution set

To understand the idea to handle dependencies, we focus on the symmetric case now. The corresponding solution set is

$$\{x \in \mathbb{R}^n;\ Ax = b \text{ for some symmetric } A \in \boldsymbol{A}\}.$$

This interval systems obeys the linear parametric structure mentioned above since the symmetry can be simply modelled by using $\frac{1}{2}n(n-1)$ interval parameters.

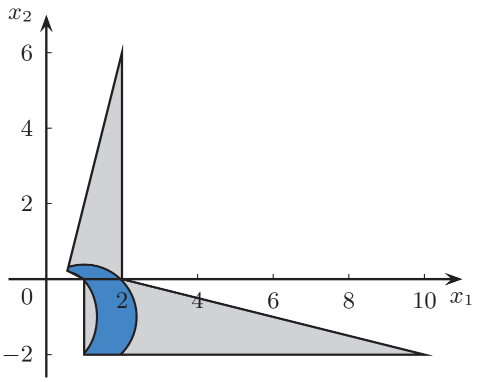

Figure 3.7: (Example 3.59) The solution set arbitrarily larger than the symmetric solution set, $a = 4$.

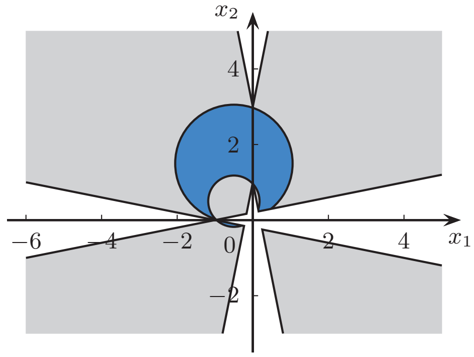

Figure 3.8: (Example 3.59) The solution set is unbounded, but the symmetric solution set is bounded.

Example 3.59. The symmetric interval system in two dimensional space draws $A(\boldsymbol{p})x = \boldsymbol{b}$, where

$$A(\boldsymbol{p}) = \boldsymbol{a}_{11}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \boldsymbol{a}_{12}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \boldsymbol{a}_{22}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}.$$

By Hladík (2008b), the symmetric solution set is described by

$$\begin{aligned}
|a^c_{11}x_1 + a^c_{12}x_2 - b^c_1| &\le a^\Delta_{11}|x_1| + a^\Delta_{12}|x_2| + b^\Delta_1\\
|a^c_{21}x_1 + a^c_{22}x_2 - b^c_2| &\le a^\Delta_{21}|x_1| + a^\Delta_{22}|x_2| + b^\Delta_2,\\
|-a^c_{11}x_1^2 + a^c_{22}x_2^2 + b^c_1x_1 - b^c_2x_2| &\ge a^\Delta_{11}x_1^2 + a^\Delta_{22}x_2^2 + b^\Delta_1|x_1| + b^\Delta_2|x_2|,
\end{aligned}$$

where the first two equations are the Oettli–Prager ones. Figure 3.7 illustrates the solution set (light grey color) and symmetric solution set (grey color) for the system

$$\boldsymbol{A} = \begin{pmatrix}[1,2] & [0,a]\\ [0,a] & -1\end{pmatrix},\ \boldsymbol{b} = \begin{pmatrix}2\\ 2\end{pmatrix},$$

in which the solution set can be arbitrarily larger than the symmetric one, depending on the real parameter $a > 0$. Figure 3.8 illustrates the case when the solution set is unbounded whereas the symmetric solution set is bounded. The corresponding data are

$$\boldsymbol{A} = \begin{pmatrix}-1 & [-5,5]\\ [-5,5] & 1\end{pmatrix},\ \boldsymbol{b} = \begin{pmatrix}1\\ [1,3]\end{pmatrix}. \qquad \square$$

Now, let us focus on computing an enclosure of the symmetric solution set. As in the previous section, we consider an enclosure in the form of $\boldsymbol{x} = x^* + \boldsymbol{y}$, where x^* is any real vector (e.g., the solution of $A^cx = b^c$) and $\boldsymbol{y} \in \mathbb{IR}^n$ encloses the solution set of

$$\boldsymbol{A}y = C(\boldsymbol{b} - \boldsymbol{A}x^*).$$

Here, the key idea is to delicately evaluate the right-hand side vector. The direct evaluation by interval arithmetic will ignore dependencies, leading to needless overestimation. Taking into account the symmetry of $\boldsymbol{A}$, the ℓ-th entry of the right-hand side is evaluated as

$$\boldsymbol{z}_\ell := \sum_{j=1}^n C_{\ell j}(\boldsymbol{b}_j - \boldsymbol{A}_{jj}x_j^*) + \sum_{i<j}(C_{\ell i}x_j^* + C_{\ell j}x_i^*)\boldsymbol{A}_{ij}.$$

Since each interval quantity appears at most once in the above expression, by Theorem 2.13, $\boldsymbol{z}_\ell$ is the optimal interval. That is,

$$\boldsymbol{z} = \square\{C(b - Ax^*);\ b \in \boldsymbol{b}, A \in \boldsymbol{A}, A = A^T\}.$$

Therefore, the enclosing interval vector $\boldsymbol{y}$ is computed by employing any (non-parametric) interval equation solver for

$$(C\boldsymbol{A})y = \boldsymbol{z}.$$

On the other hand, the constraint matrix $C\boldsymbol{A}$ can hardly by computed tighter than by direct evaluation.

The idea of delicate evaluation of the right-hand side vector $\boldsymbol{z}$ is originally due to Jansson (1991). He solved the system $(C\boldsymbol{A})y = \boldsymbol{z}$ by utilizing the (residual) Krawczyk operator

$$K_r(\boldsymbol{y}) \coloneqq C(\boldsymbol{b} - \boldsymbol{A}x^*) + (I_n - C\boldsymbol{A})\boldsymbol{y}.$$

Nevertheless, we can call any other method as well.

Similar rearrangements of $C(b(\boldsymbol{p}) - A(\boldsymbol{p})x^*)$ can be done for other specific linear parametric dependencies such as Toeplitz or Hankel matrices.

3.5.3 Regularity

Even for checking regularity of parametric interval matrices, we have to develop special techniques since simple relaxation leads to overestimation. For example, let

$$\boldsymbol{A} \coloneqq \begin{pmatrix} 1 & [-1,1] \\ [-1,1] & -1 \end{pmatrix}.$$

This interval matrix is irregular as it contains the singluar matrix $\left(\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}\right)$. On the other hand, there is no symmetric matrix inside that would be singular. That is, the symmetric interval matrix $\boldsymbol{A}^S$ is regular. Moreover, conditions for testing regularity of standard interval matrices are not straightforwardly applicable. For example, both vertex matrices

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

are nonsingular and have the same determinant, so direct adaptation of Theorem 3.45(3) fails.

Notes

The first paper on parametric interval systems with a special structure is by Jansson (1991). The general problem of interval parameter dependent linear systems was first treated in Rump (1994).

An effective method for a special class of parametric systems was proposed in Neumaier and Pownuk (2007). Interval Cholesky method for enclosing the symmetric solution set was dealt with in Alefeld and Mayer (1993, 2008); Garloff (2012), however, contrary to the non-interval case, it is not so effective. Direct methods for computing enclosures to the parametric solution set were studied in Hladík (2012d); Kolev (2006b); Skalna (2006). Iterative methods include parametrized Gauss-Seidel iteration Popova (2001), a fixed-point method by Rump (1994, 2010), and the approach by Kolev (2004b). Monotonicity approach Popova (2006b); Rohn (2004); Skalna (2008) may substantially improve accuracy of enclosures since it reduces domains of some parameters to the endpoints.

More general systems with nonlinear dependences between interval quantities were handled e.g. by Garloff et al. (2009); Kolev (2004a); Popova (2007).

A small selection of broad area of applications include that of structural mechanics Garloff et al. (2009), mechanical systems Dessombz et al. (2001), or tolerance analysis in linear circuits Kolev (1993).

A Mathematica package for solving parametric interval systems was introduced in Popova (2004), a C-XCS implementation in Popova and Krämer (2007); Zimmer et al. (2012), and an interactive service for computations webComputing in Popova (2006a).