Institute of Computer Science
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A Method for Handling Dependent Data in Interval Linear Systems

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Technical report No. 911

July 7, 2004

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Abstract:

This report contains a verbatim copy of author's informal text from 1999 which was aimed at outlining some general ideas of solving systems of interval linear equations with dependent coefficients, not at exact formulations of results.

Keywords:
Interval linear systems, dependent data.

1. Origin of the text. In a letter dated December 23, 1985, Arnold Neumaier proposed to me the problem of computing the "symmetric interval hull" of a system of interval linear equations with a symmetric interval matrix. In June 1986, after many unsuccessful attempts to prove a symmetric version of the Oettli-Prager theorem, I started looking for another way and I found out that an enclosure of the symmetric solution set could be found by using the signs (if constant) of the partial derivatives of the variables along the coefficients of the system, and that this idea could be easily extended to more general kinds of dependences. But I did not pursue these ideas any further, refraining from any publication, at that time being interested in other matters. I only gave a talk on this topic at an Oberwolfach conference in February 1990 [8]. I returned to the problem in 1999, motivated by Rump's paper [9] and mainly by a series of papers by Alefeld, Kreinovich and Mayer [4], [5], [1], [2], [3] dedicated to various descriptions of solution sets that, however, in my view did not show a way how to compute efficiently an enclosure of the hull taking into account data dependences. According to my notes, on April 21, 1999 I wrote the text included here in Section 2, and I sent it to Günter Mayer on June 5, 1999. In March 2001 I informed the community over the *reliable-computing* net that the text was downloadable from my web page. In the meantime, there appeared another papers by Popova [7], [6], and Shary [10] dedicated to this theme. In particular, Sergey Shary in a letter to me truly complained that my text having never been published, it was difficult to quote it. This had finally convinced me that it should be made available in some form, and I decided to publish it in this report version. The text which follows is a verbatim copy of the text from April 21, 1999 which was originally meant as a part of a letter and was aimed at outlining the ideas, not at exact formulations of results; no word or symbol has been changed, although nowadays I would formulate several parts in a more clear, or more detailed, form. I would only remark that Δ^k means the kth radius matrix, not a power od Δ, and that the same holds for δ^ℓ.

2. The text. I consider a system of linear interval equations with dependent coefficients in the form

$$A(t)x = b(\tau) \tag{1}$$

with

$$\begin{aligned} A(t) &= A + \sum_{k=1}^{p} t_k \Delta^k, \\ b(\tau) &= b + \sum_{\ell=1}^{q} \tau_\ell \delta^\ell, \end{aligned}$$

where $A, \Delta^k \in R^{n\times n}$, $b, \delta^\ell \in R^n$, $\Delta^k \geq 0$, $\delta^\ell \geq 0$ and $t_k \in [-1,1]$, $\tau_\ell \in [-1,1]$ ($k = 1,\ldots,p$, $\ell = 1,\ldots,q$). The variables $t_1,\ldots,t_p,\tau_1,\ldots,\tau_q$ are assumed to be mutually independent. The solution set is defined by

$$X = \{x;\ A(t)x = b(\tau),\ t_k \in [-1,1],\ \tau_\ell \in [-1,1]\ \forall k \forall \ell\}$$

and we are interested in computing

$$\begin{aligned}\underline{x}_i &= \min_X x_i,\\ \overline{x}_i &= \max_X x_i,\end{aligned}$$

or in an enclosure of X which takes into account the dependences given.

I will write the solution of a particular system (1), which obviously depends on t and τ, alternatively as $x(t,\tau)$. My approach consists in computing and employing the partial derivatives $\frac{\partial x(t,\tau)}{\partial t_k}$ and $\frac{\partial x(t,\tau)}{\partial \tau_\ell}$ ($k = 1,\dots,p$, $\ell = 1,\dots,q$). First, taking the ith equation in (1)

$$\sum_{h=1}^{n}(a_{ih} + \sum_{k=1}^{p} t_k \Delta_{ih}^k)x_h = b_i + \sum_{\ell=1}^{q} \tau_\ell \delta_i^\ell$$

and taking the partial derivative $\frac{\partial}{\partial t_j}$ on both sides, we obtain

$$\sum_{h=1}^{n}\left[\Delta_{ih}^j x_h + (a_{ih} + \sum_{k=1}^{p} t_k \Delta_{ih}^k)\frac{\partial x_h}{\partial t_j}\right] = 0. \tag{2}$$

Introducing the vector

$$\frac{\partial x}{\partial t_j} = \left(\frac{\partial x_h}{\partial t_j}\right)_{h=1}^{n}$$

we can write (2) as

$$(\Delta^j x)_i + \left(A(t)\frac{\partial x}{\partial t_j}\right)_i = 0$$

for $i = 1,\dots,n$, hence

$$\Delta^j x + A(t)\frac{\partial x}{\partial t_j} = 0$$

and finally

$$\frac{\partial x(t,\tau)}{\partial t_j} = -(A(t))^{-1}\Delta^j x(t,\tau).$$

It is better to write it now in terms of the original variable t_k:

$$\frac{\partial x(t,\tau)}{\partial t_k} = -(A(t))^{-1}\Delta^k x(t,\tau). \tag{3}$$

Second, from (1) we have

$$x = (A(t))^{-1}(b + \sum_{\ell=1}^{q} \tau_\ell \delta^\ell)$$

which gives

$$\frac{\partial x(t,\tau)}{\partial \tau_\ell} = (A(t))^{-1}\delta^\ell \tag{4}$$

where again the left-hand side is the vector with components $\frac{\partial x_h}{\partial \tau_\ell}$, $h = 1, \ldots, n$. Now, to compute an enclosure of the solution set X, I proceed in the following way: first, take $\Delta = \sum_{k=1}^{p} \Delta^k$, $\delta = \sum_{\ell=1}^{q} \delta^\ell$ and compute by some classical method (e.g., [preconditioned] Gaussian algorithm) an enclosure $[x]$ of the solution set of

$$[A - \Delta, A + \Delta]x = [b - \delta, b + \delta]$$

(which is a usual linear interval system where the dependences are *not* taken into account), and an enclosure $[B]$ of the inverse interval matrix $\{(A')^{-1};\ A' \in [A - \Delta, A + \Delta]\}$. Now, for a given i, evaluate the interval

$$(-[B]\Delta^k[x])_i \tag{5}$$

in interval arithmetic. If its upper bound is ≤ 0, then according to (3) we have $\frac{\partial x_i}{\partial t_k} \leq 0$ in the whole region, hence $\overline{x}_i$ is achieved for $t_k = -1$; if the lower bound of (5) is ≥ 0, then $\overline{x}_i$ is achieved for $t_k = 1$. Similarly, according to (4) we can set $\tau_\ell = -1$ if the upper bound of

$$([B]\delta^\ell)_i \tag{6}$$

is nonpositive, and $\tau_\ell = 1$ if its lower bound is nonnegative. In this way we have fixed some of the values t_k, τ_ℓ. Denote

$$\begin{aligned} K_- &= \{k;\ t_k \text{ was fixed at } -1\} \\ K_+ &= \{k;\ t_k \text{ was fixed at } 1\} \end{aligned}$$

and similarly L_-, L_+ for τ_ℓ. Then $\overline{x}_i$ is achieved at the solution of some system (1) of the form

$$(A - \sum_{k \in K_-} \Delta^k + \sum_{k \in K_+} \Delta^k + \sum_{k \notin K_- \cup K_+} t_k \Delta^k)x = b - \sum_{\ell \in L_-} \delta^\ell + \sum_{\ell \in L_+} \delta^\ell + \sum_{\ell \notin L_- \cup L_+} \tau_\ell \delta^\ell.$$

Hence if we define

$$\begin{aligned} A' &= A - \sum_{k \in K_-} \Delta^k + \sum_{k \in K_+} \Delta^k, \\ \Delta' &= \sum_{k \notin K_- \cup K_+} \Delta^k, \\ b' &= b - \sum_{\ell \in L_-} \delta^\ell + \sum_{\ell \in L_+} \delta^\ell, \\ \delta' &= \sum_{\ell \notin L_- \cup L_+} \delta^\ell, \end{aligned}$$

and if we compute by any classical method an enclosure $[[x]]$ of the system

$$[A' - \Delta', A' + \Delta']x = [b' - \delta', b' + \delta'],$$

then the upper bound of $[[x]]_i$ is also an upper bound on $\overline{x}_i$ which, since the dependences have been taken into account, can be expected to be smaller than the upper bound of $[x]_i$.

In this way, by a repeated use of this method for $i = 1, \ldots, n$ we can compute better bounds that reflect the dependences in the data. The disadvantage of this procedure consists in the fact that each i must be handled separately; but we must take into account that this is a difficult problem. The whole process may be used repeatedly until no further change in the bounds occurs.

It is worth mentioning that in a special case when all t_k and τ_ℓ have been fixed, i.e. $K_- \cup K_+ = \{1, \ldots, p\}$ and $L_- \cup L_+ = \{1, \ldots, q\}$, we have $\overline{x}_i = x_i$, where x is the solution of

$$(A - \sum_{k \in K_-} \Delta^k + \sum_{k \in K_+} \Delta^k)x = b - \sum_{\ell \in L_-} \delta^\ell + \sum_{\ell \in L_+} \delta^\ell. \tag{7}$$

A few more words about the symmetric case. Let us have a linear interval system $[A - \Delta, A + \Delta]x = [b - \delta, b + \delta]$ with symmetric A, Δ, and we are interested in bounding the solution set

$$X = \{x;\ A'x = b',\ A' \in [A - \Delta, A + \Delta],\ b' \in [b - \delta, b + \delta],\ A' \text{ symmetric}\}.$$

Here we can employ the form (1) with

$$A(t) = A + \sum_{k=1}^{n} t_{kk} \Delta_{kk} e_k e_k^T + \sum_{k<j} t_{kj} \Delta_{kj} (e_k e_j^T + e_j e_k^T), \tag{8}$$

$$b(\tau) = b + \sum_{\ell=1}^{n} \tau_\ell \delta_\ell e_\ell \tag{9}$$

where e_k is the kth column of the unit matrix I. Here by (3), (4)

$$\begin{aligned}
\frac{\partial x_i}{\partial t_{kk}} &= -((A(t))^{-1} \Delta_{kk} e_k e_k^T x)_i = -\Delta_{kk} (A(t))^{-1}_{ik} x_k, \\
\frac{\partial x_i}{\partial t_{kj}} &= -((A(t))^{-1} \Delta_{kj} (e_k e_j^T + e_j e_k^T) x)_i \\
&= -\Delta_{kj} ((A(t))^{-1}_{ik} x_j + (A(t))^{-1}_{ij} x_k), \\
\frac{\partial x_i}{\partial \tau_\ell} &= ((A(t))^{-1} \delta_\ell e_\ell)_i = \delta_\ell (A(t))^{-1}_{i\ell},
\end{aligned}$$

hence (5) has the form

$$-\Delta_{kk} [B]_{ik} [x]_k$$

for t_{kk} and

$$-\Delta_{kj} ([B]_{ik} [x]_j + [B]_{ij} [x]_k)$$

for t_{kj}, $k < j$, and (6) has the form

$$\delta_\ell [B]_{i\ell},$$

so that the evaluation of the signs of their bounds is easy.

In the particular case when all the t_k's and τ_ℓ's have been fixed, the equation (7) in view of (8) and (9) takes on the form

$$(A - T \circ \Delta)x = b + \tau \circ \delta$$

where $T_{ij} = t_{ij}$ for $i \leq j$ and $T_{ji} = T_{ij}$ for $j > i$, and "$\circ$" is the Hadamard product of matrices (i.e., $(T \circ \Delta)_{ij} = T_{ij}\Delta_{ij}$ etc.).

Other cases (as skew-symmetric, Toeplitz etc.) can be handled in a similar way.

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