Quantitative approach to strain modelling

using Python, Numpy and Matplotlib

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Outlines

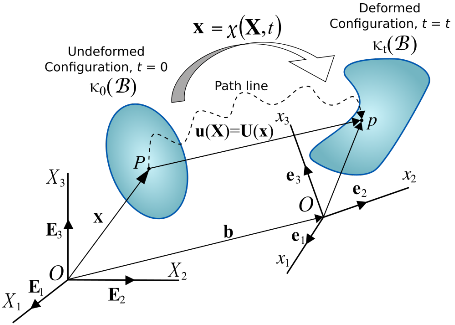
- Concept of homogeneous deformation
- From finite to continuous deformation
- Superposition of deformations

Deformation in terms of displacement

A change in the configuration of a continuum body results in a displacement from an **initial or undeformed configuration** to a current or **deformed configuration**. The displacement of a body has two components:

- **Rigid-body displacement**
  - Translation
  - Rotation
- **Deformation or strain**
  - Distortion - isochoric change in shape
  - Dilation - change in volume

  implies the change in shape and/or size of the body from an initial or undeformed configuration



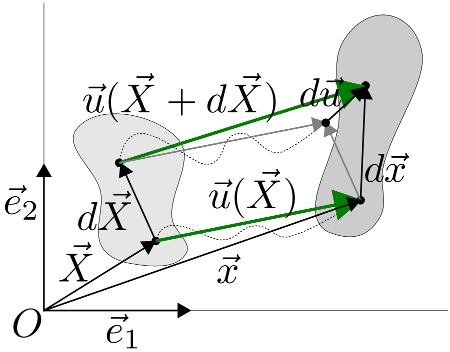
Kinematics of continuum body

The motion of a continuum body is a **continuous** time sequence of displacements. Thus, the material body will occupy **different configurations** at different times so that a particle occupies a series of points in space which describe a **pathline**.

There is **continuity** during deformation or motion of a continuum body in the sense that:

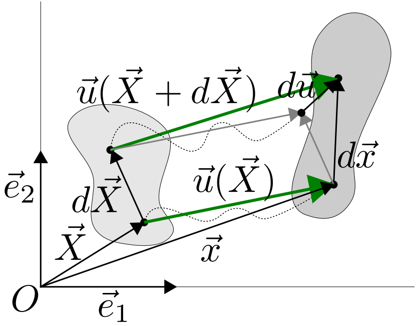
- The material points forming a closed curve at any instant will always form a closed curve at any subsequent time.
- The material points forming a closed surface at any instant will always form a closed surface at any subsequent time and the matter within the closed surface will always remain within.

It is convenient to identify a **reference configuration or initial condition** which all subsequent **deformed configurations** are referenced from. Often, the configuration at $t = 0$ is considered the reference configuration.



The components x_i of the position vector $\vec{x}$ of a particle, taken with respect to the reference configuration, are called the **material or reference coordinates**.

Kinematics of infinitesimal deformation I.



The displacement of first point is decribed as:

$$\vec{x} = \vec{X} + \vec{u}(\vec{X})$$

while displacement of second surrounding point is described as:

$$\vec{x} + d\vec{x} = \vec{X} + d\vec{X} + \vec{u}(\vec{X} + d\vec{X})$$

Substituting first equation into second we got:

$$\vec{X} + \vec{u}(\vec{X}) + d\vec{x} = \vec{X} + d\vec{X} + \vec{u}(\vec{X} + d\vec{X})$$

which simplifies to:

$$d\vec{x} = d\vec{X} + \vec{u}(\vec{X} + d\vec{X}) - \vec{u}(\vec{X})$$

Note that difference between displacements is: $d\vec{u} = d\vec{x} - d\vec{X}$.

Detour on Taylor's theorem

Taylor's theorem states that any function that is infinitely differentiable may be represented by a Taylor series expansion:

$$f(X+dX) = f(X) + \frac{f'(X)}{1!}dX + \frac{f''(X)}{2!}dX^2 + \ldots = \sum_{k=0}^{\infty} \frac{f^{(k)}(X)}{k!}dX^k$$

than

$$f(X+dX) - f(X) = \frac{f'(X)}{1!}dX + \frac{f''(X)}{2!}dX^2 + \ldots = \sum_{k=0}^{\infty} \frac{f^{(k)}(X)}{k!}dX^k$$

neglecting higher terms as $\left|d\vec{X}\right| \ll 1$ as dX^k is very small, it is:

$$f(X+dX) - f(X) = \frac{f'(X)}{1!}dX = (\boldsymbol{\nabla} \boldsymbol{f})dX$$

where $\boldsymbol{\nabla} \boldsymbol{f}$ is gradient of vector field.

Kinematics of infinitesimal deformation II.

Using that for infinitesimal deformation equation

$$d\vec{x} = d\vec{X} + \vec{u}(\vec{X} + d\vec{X}) - \vec{u}(\vec{X})$$

it could be written in terms of gradient as:

$$d\vec{x} = d\vec{X} + (\boldsymbol{\nabla u})d\vec{X}$$

where $\boldsymbol{\nabla u}$ is gradient of displacement field or **displacement gradient**.

Displacement gradient

The **displacement gradient** is the derivative of each component of the linear element displacement $d\vec{u}$ with respect to each component of the reference element $d\vec{X}$:

$$\boldsymbol{\nabla u} = u_{i,j} = \frac{\partial u_i}{\partial X_j} = \begin{bmatrix} \frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} & \frac{\partial u_1}{\partial X_3} \\ \frac{\partial u_2}{\partial X_1} & \frac{\partial u_2}{\partial X_2} & \frac{\partial u_2}{\partial X_3} \\ \frac{\partial u_3}{\partial X_1} & \frac{\partial u_3}{\partial X_2} & \frac{\partial u_3}{\partial X_3} \end{bmatrix}$$

and characterise the local change of the displacement field at a material point with position vector $\vec{X}$. Knowing that:

$$d\vec{u} = d\vec{x} - d\vec{X}$$

it could be also written as:

$$d\vec{u} = (\boldsymbol{\nabla u})d\vec{X}$$

Deformation gradient

Recalling that $d\vec{u} = d\vec{x} - d\vec{X}$

$$(\boldsymbol{\nabla u}) = \frac{\partial u_i}{\partial X_j} = \frac{\partial (x_i - X_i)}{\partial X_j} = \frac{\partial x_i}{\partial X_j} - \frac{\partial X_i}{\partial X_j} = \boldsymbol{F} - \boldsymbol{I}$$

where $\boldsymbol{F}$ is so called **deformation gradient**, i.e the derivative of each component of the deformed linear element $d\vec{x}$ with respect to each component of the reference element $d\vec{X}$:

$$\boldsymbol{F} = x_{i,j} = \frac{\partial x_i}{\partial X_j} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix}$$

and characterizes the local deformation at a material point with position vector $\vec{X}$, assuming continuity. As

$$d\vec{u} = d\vec{x} - d\vec{X} = (\boldsymbol{\nabla u})d\vec{X} = (\boldsymbol{F} - \boldsymbol{I})d\vec{X} = \boldsymbol{F}d\vec{X} - d\vec{X}$$

$$d\vec{x} = \boldsymbol{F}d\vec{X}$$

Properties of deformation gradient

Deformation gradient $\boldsymbol{F}$ contains all the required local information about the changes in length, volumes and angles due to the deformation as follows:

- When vector $\vec{N}$ in the reference configuration is deformed into the vector $\vec{n}$, these vectors are related as: $\vec{n} = \boldsymbol{F}\vec{N}$
- The ratio between the local volume of the deformed configuration to the local volume in the reference configuration is equal to the determinant of the deformation gradient tensor: $J = \det \boldsymbol{F}$
- Two infinitesimal areas with da and dA being their magnitudes and $\vec{n}$ and $\vec{N}$ are unit vectors perpendicular to them, then the relationship is given by: $(da)\vec{n} = \det(\boldsymbol{F})(dA)\boldsymbol{F}^{-T}\vec{N}$
- An isochoric deformation is a deformation preserving local volume, i.e., $\det \boldsymbol{F} = 1$ at every point
- A deformation is called homogeneous if $\boldsymbol{F}$ is constant at every point. Otherwise, the deformation is called non-homogeneous
- The physical restriction of possible deformation: $\det \boldsymbol{F} > 0$

Homogeneous deformation

A **homogeneous deformation** is one where the deformation gradient is uniform, i.e. independent of the coordinates, and the associated motion is termed **affine**. Every part of the material deforms as the whole does, and straight parallel lines in the reference configuration map to straight parallel lines in the deformed configuration.

$$x = aX + bY + t_X$$
$$y = cX + dY + t_Y$$

or in matrix form:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} t_X \\ t_Y \end{bmatrix}$$

Properties of homogeneous deformation are not spatially dependent.

Deformation gradient

Without translation the homogeneous deformation (rotation and strain) could be described as:

$$x = aX + bY$$
$$y = cX + dY$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

or

$$\vec{x} = \boldsymbol{F}\vec{X}$$

where $\mathbf{F}$ is so called **deformation gradient**.

Note, that as we excluded translation, the origin of coordinates do not change during deformation:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \boldsymbol{F} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Displacement gradient

Displacement of particle is vector between initial and final postion, i.e:

$$u = x - X = aX + bY - X = (a-1)X + bY$$
$$v = y - Y = cX + dY - Y = cX + (d-1)Y$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a-1 & b \\ c & d-1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

or

$$\vec{u} = (\boldsymbol{F} - \boldsymbol{I})\vec{X} = \boldsymbol{\nabla u}\vec{X}$$

where $\boldsymbol{\nabla}\mathbf{u}$ is so called **displacement gradient**.

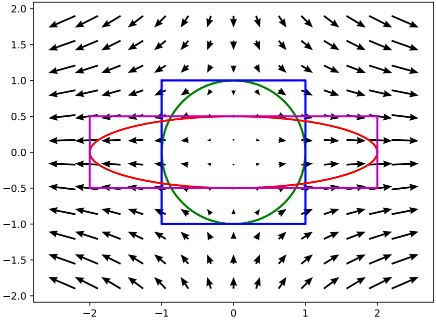
Examples of pure shear

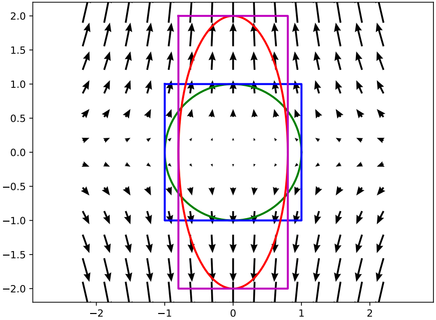
$$\mathbf{F} = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$\nabla\mathbf{u} = \begin{bmatrix} 1 & 0 \\ 0 & -0.5 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 0.8 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\nabla\mathbf{u} = \begin{bmatrix} -0.2 & 0 \\ 0 & 1 \end{bmatrix}$$





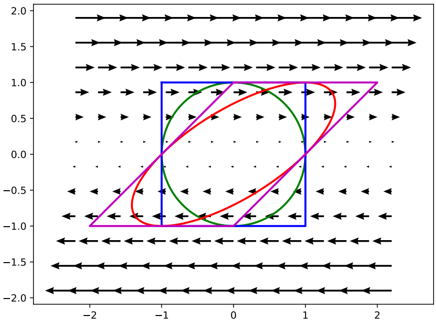
Examples of simple shear

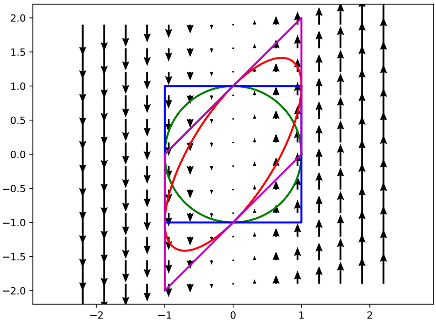
$$\mathbf{F} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\nabla \mathbf{u} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\nabla \mathbf{u} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$





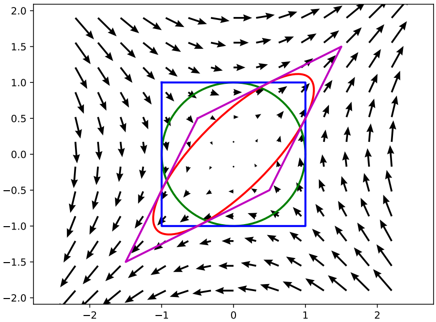
Examples of general shear

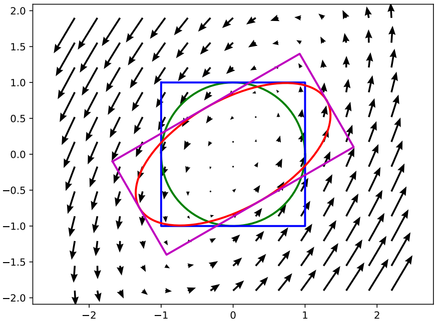
$$\mathbf{F} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

$$\nabla\mathbf{u} = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 1.300 & -0.375 \\ 0.750 & 0.650 \end{bmatrix}$$

$$\nabla\mathbf{u} = \begin{bmatrix} 0.300 & -0.375 \\ 0.750 & -0.350 \end{bmatrix}$$





Time to think...

$\boldsymbol{F}$ maps any undeformed vector into its deformed state. This vector can also be a position vector of a point. Therefore $\boldsymbol{F}$ also maps any point into its new position after deformation. Considering two successive deformations $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ write transformation equation....