Time to think...

$\boldsymbol{F}$ maps any undeformed vector into its deformed state. This vector can also be a position vector of a point. Therefore $\boldsymbol{F}$ also maps any point into its new position after deformation. Considering two successive deformations $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ write transformation equation....

$$\vec{x}_1 = \mathbf{F_1} \cdot \vec{X}$$

$$\vec{x}_2 = \mathbf{F_2} \cdot \vec{x}_1$$

Substitute first equation to second gives:

$$\vec{x}_2 = \mathbf{F_2} \cdot \mathbf{F_1} \cdot \vec{X}$$

so

$$\vec{x}_2 = \mathbf{F} \cdot \vec{X}$$

where

$$\mathbf{F} = \mathbf{F_2} \cdot \mathbf{F_1}$$

Polar Decomposition I.

In last example the object has clearly been stretched and rotated. But by how much? the following two-step process of deformation followed by rigid body rotation gets you there...

$$\mathbf{F} = \begin{bmatrix} 1.300 & -0.375 \\ 0.750 & 0.650 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.500 \\ 0.500 & 0.866 \end{bmatrix} \begin{bmatrix} 1.50 & 0.00 \\ 0.00 & 0.75 \end{bmatrix}$$

or

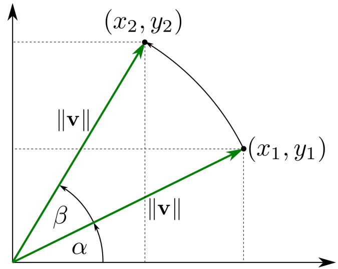
$$\mathbf{F} = \mathbf{R} \cdot \mathbf{U}$$

where $\mathbf{R}$ is the **rotation matrix**, and $\mathbf{U}$ is the **right stretch tensor** that is responsible for all the problems in life: stress, strain, fatigue, cracks, fracture, etc. Note that the process is read from right to left, not left to right. $\mathbf{U}$ is applied first, then $\mathbf{R}$.

This partitioning of the **deformation gradient** into the product of a **rotation matrix** and **stretch tensor** is known as a **polar decomposition**.

Derivation of 2D rotation matrix

In order to rotate vector $\mathbf{v} = (x_1, y_1)$ to vector (x_2, y_2), we can write following equations for $\cos(\alpha)$, $\sin(\alpha)$, $\cos(\alpha + \beta)$ and $\sin(\alpha + \beta)$:



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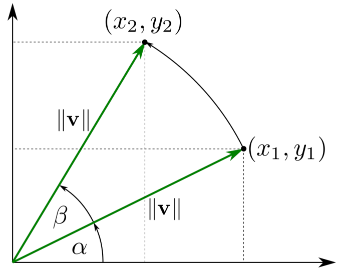

$$\cos(\alpha) = \frac{x_1}{\|\mathbf{v}\|}, \ \sin(\alpha) = \frac{y_1}{\|\mathbf{v}\|}$$

$$\cos(\alpha+\beta) = \frac{x_2}{\|\mathbf{v}\|}, \ \sin(\alpha+\beta) = \frac{y_2}{\|\mathbf{v}\|}$$

Substituting to the angle sum identities:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

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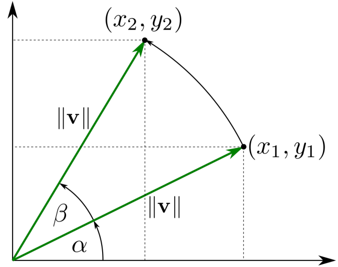

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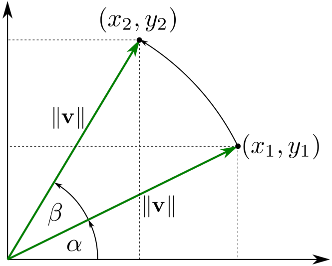
$$\frac{x_2}{\|\mathbf{v}\|} = \frac{x_1}{\|\mathbf{v}\|}\cos\beta - \frac{y_1}{\|\mathbf{v}\|}\sin\beta \rightarrow x_2 = x_1\cos\beta - y1\sin\beta$$

$$\frac{y_2}{\|\mathbf{v}\|} = \frac{y_1}{\|\mathbf{v}\|}\cos\beta + \frac{x_1}{\|\mathbf{v}\|}\sin\beta \rightarrow y_2 = x_1\sin\beta + y_1\cos\beta$$

Derivation of 2D rotation matrix

In order to rotate vector $\mathbf{v}$ by angle β, we can use the **rotation matrix** $\mathbf{R}$:



$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

or

$$\mathbf{u} = \mathbf{R} \cdot \mathbf{v}$$