# Introduction to Computational Origami: Modern Art and Science

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June 3, 2016

# Overview

# PART I: BACKGROUND OF ORIGAMI

# History

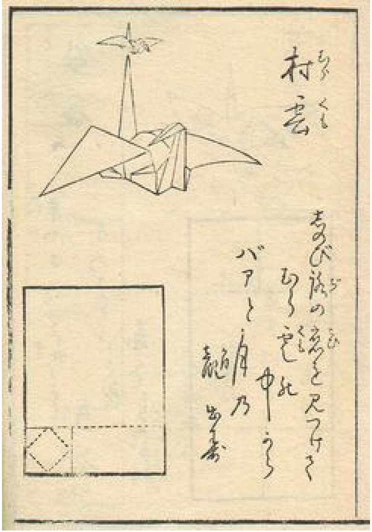
Origmi is a kind of ancient art which is widely believed to origins from China. But it is Japan that developed it into a independent art form, and the word "origami" comes from Japanese.

Japanese origami began sometime after Buddhist monks carried paper to Japan during the 6th century. The first Japanese origami is dated from this period and was used for religious ceremonial purposes only, due to the high price of paper. Then in 1797 the first known origami book was published in Japan: Senbazuru orikata.

# History

The father of the art of paper folding is consider to be Akira Yoshizawa, who invented a language systems to guide people to fold. Since then, the art of origami spread all over the world quickly and become more and more popular.

# History



# Development

Now, origami has became an independent form of art. And here is a universally accepted constrain in circle of origami: One squre paper and no cuts.

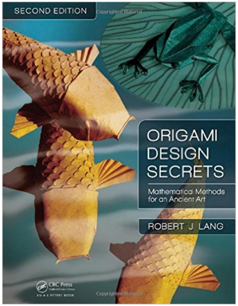
Amazingly, even with this constrain, origami can be very very very complex. Here are some examples:

# Development

# Development

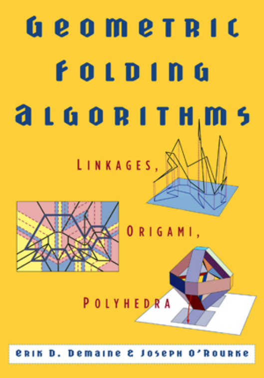
# Folding Science

Here is a person, who changed pattern of origami and make folding a new kind of science, and he is Robert J.Lang, who used to be a scientists of NASA.
He studied math behind origami and made software *TreeMaker* to help design origami.




# Folding Science

And here is another person, Erik D.Demaine develop a set of theory of folding, he wrote a book: *Geometric Folding Algorithms*




# PART II: FLAT FOLDING

# Vertex Flat Foldability

## Definations

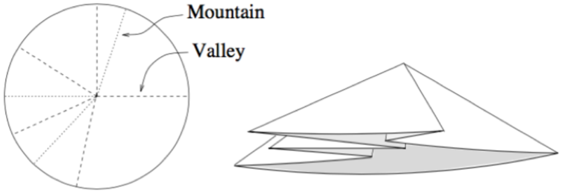
**Definition 1.** A **crease pattern** is a finite planar straight-line graph drawn on a convex planar region (the paper). A crease is an edge of the planar graph.

**Definition 2.** An **embedding** is a continuous, one-to-one mapping of a crease pattern to $\mathbf{R}^3$. The mapping must be smooth (differentiable) everywhere except along creases.

**Definition 3.** A **flat origami** is an infinite sequence of embeddings of the same crease pattern, such that the images of each crease converge to a line segment and the images of each face converge to a planar polygonal region, congruent to the face. (Convergence is not just pointwise, but sufficiently strong that metric properties converge as well.) Moreover, the dihedral along each crease must converge uniformly to either $\pi$ or $-\pi$.

# Vertex Flat Foldability

**Definition 4.** A **vertex-flat assignment** is an MV- assignment to a crease pattern such that for each vertex this assignment is the assignment of a flat single-vertex origami.
**Definition 5.** **VERTEX FLAT FOLDABILITY** is the problem of determining whether or not a given crease pattern has a vertex-flat assignment.



# Vertex Flat Foldability

## Necessary Conditions

**Theorem 1 (Necessary Conditions)** Let $v$ be an interior vertex of crease pattern $C$, and there is a flat origami with crease pattern C. Then following conditions satisfy:
**(K1)** The sum of alternate angles around $v$ is 0.
**(M)** The number of mountain folds minus the number of valley folds meeting at v is either 2 or -2.
**(K2)** If $\alpha_i < \alpha_{i1}$ and $\alpha_i < \alpha_{i+1}$, then $e_i$ and $e_i + 1$ must have opposite assignments.

# Vertex Flat Foldability

## Sufficient Conditions

**Theorem 2 (Sufficient Condition)** (Kawasaki,Justin) Let $D$ be a crease pattern drawn on a disk, consisting of a single vertex $v$ at the center of the disk, along with some number of creases, each a radius of the disk. Then $D$ is flat foldable if the sum of alternate angles around $v$ is $\pi$.

The main idea of proof of Theorem 2 is to use (K2) condition to merge angles recussively.

So, we have:

**Theorem 3** There is a linear-time algorithm for **VERTEX FLAT FOLDABILITY**.

# Flat Foldability

## Definitions

**Definition 6. Not-All-Equal 3-SAT** is given by a collection of clauses, each containing exactly three literals. The problem is to determine whether or not there exists a truth assignment such that each clause has either one or two true literals.

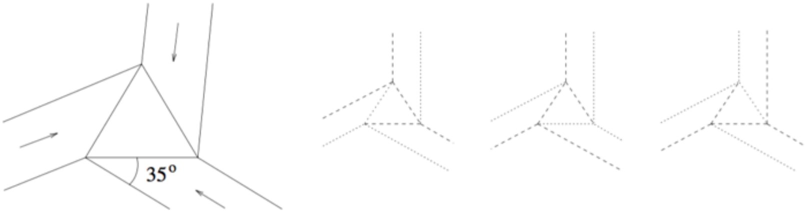
**Definition 7. Flat Foldability** is the problem of determining whether or not a given crease pattern is the crease pattern of a flat origami.

# Flat Foldability

**Theorem 4 FLAT FOLDABILITY** is **NP**-hard.

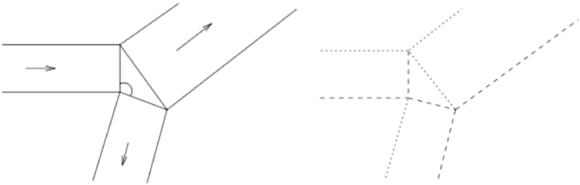
The key of reduction is to construct gadgets like method used in graph. Here are three gadgets that we will use:

# Flat Foldability




**Lemma 1** The clause crease pattern is flat foldable if and only if one or two of the incoming wires are true.

# Flat Foldability



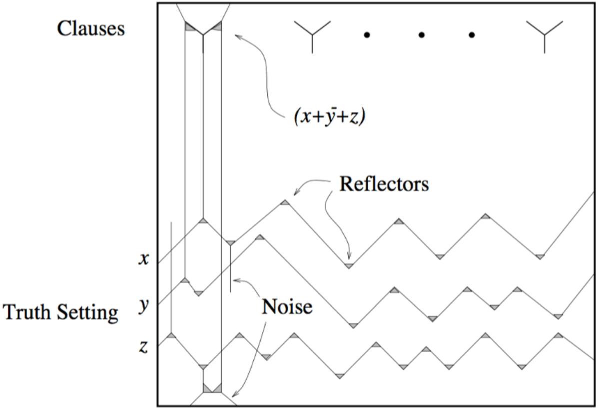
**Lemma 2** The reflector crease pattern is flat foldable if and only if the incoming wire agrees with the outgoing broad wire and disagrees with the outgoing narrow wire.

# Flat Foldability

**Lemma 3** The crossover crease patterns are flat foldable if and only if each opposite pair of incoming and outgoing wires agree.

# Flat Foldability

## Proof of Theorem 4



# Assigned Flat Foldability

**Definition 8. ASSIGNED FLAT FOLDABILITY** is the problem of determining whether or not there exists a flat origami with a given assigned crease pattern.

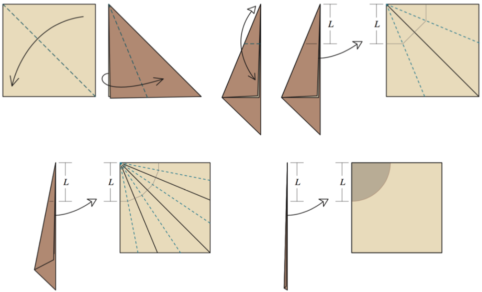
**Theorem 5** **ASSIGNED FLAT FOLDABILITY** is **NP**-hard.

# PART III: METHODS OF DESIGN

# Tree Method

## History and Development

The key idea of this method is to map flaps of model to circles in paper. The circle of a flap is the minimum paper that a flap needs.

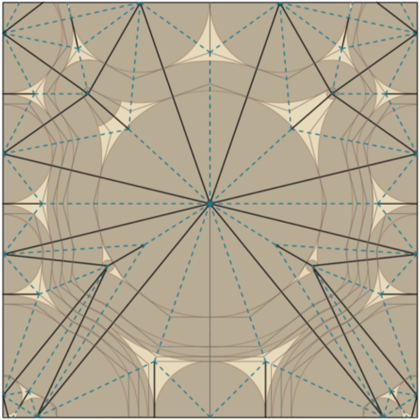


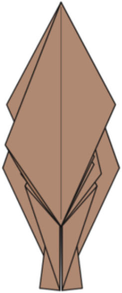
# Tree Method

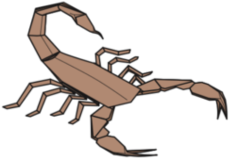
A flap will be mapped to a circle which radius is exactly the length of the flap. A stick between roots of two groups of flaps will be mapped to a river which width is exactly the length of the stick. in the paper. If there are two flap with same end, their circles will be tangent to each other. And circles and rivers of connected flaps and sticks are tangent to each other, too.

So given a stick figure with flaps you want to get (representing by a tree graph, called shadow tree), the key step is to assign the position of circles and rivers.

# Tree Method







It is sufficient to assign the centers of circles according to distance between circles.

If there is no stick between two flaps, the distance between centers of these two circles will be equal to sum of their radius.

If there is a stick or more between two flaps, the distance between centers of these two circles will be equal to sum of their radius and widths of rivers.

That the key point of Tree Method.

Robert J.Lang proved that the assignment problem can be transfered into a optimation problem. This make it is possible to use computer to compute this problem. And Robert J.Lang has write a software *Tree Maker* to automaticly assign the positions of circles.

**Theorem 6**(Lang 1996). There is an algorithm that, given any convex piece of paper $P$ and any metric tree $T$, constructs a crease pattern that folds $P$ into a uniaxial base whose shadow tree is the largest possible scaled copy of $T$, ignoring possible self-intersection of the paper.

## Mathematical Definition

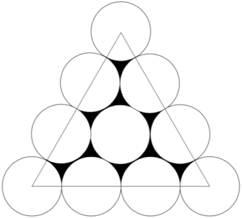
The problem can be expressed as follows.

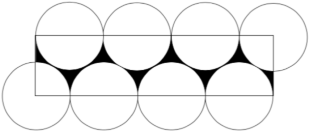
$$\max \ \lambda$$

$$s.t. \quad \begin{cases} \frac{Distance\ \ in\ \ shadow\ \ tree\ \ between\ \ leaf\ \ i\ \ and\ \ leaf\ \ j}{Euclidean\ \ distance\ \ between\ \ point\ \ i\ \ and point j} \geq \lambda,\ \forall i \neq j \\ every\ \ point\ \ i\ \ lies\ \ within\ \ the\ \ convex\ \ piece\ \ of\ \ paper. \end{cases}$$

# Circle/River Packing

On the other hand, **Theorem 6** guarantees that given a set of circles, we can pack them into a given a circle. And this make the problem more interesting. And we just provide some interesting results.





# Circle/River Packing

**Theorem 7** Circle/river origami design for triangular paper is **NP**-hard.

**Corollary** It is **NP**-hard to decide whether a given set of circles can be packed into an equilateral triangle.

**Theorem 8** Circle/river origami design for rectangular paper is **NP**-hard.

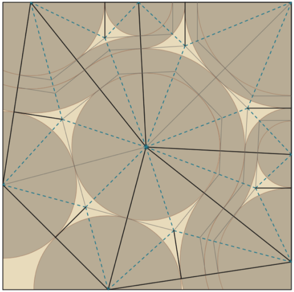
**Corollary** It is **NP**-hard to decide whether a given set of circles can be packed into a given rectangle.

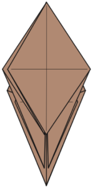
**Theorem 9** Circle/river origami design for square paper is **NP**-hard.

**Corollary** It is **NP**-hard to decide whether a given set of circles can be packed into a given square.

# Tree Method

Some art works designed by Tree Method





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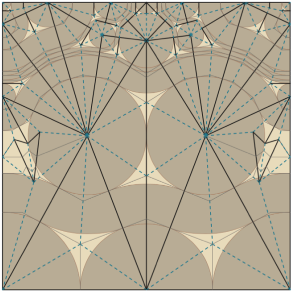

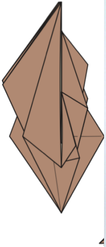

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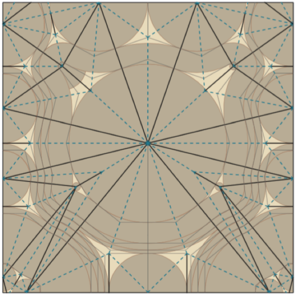

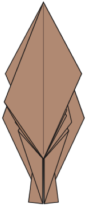

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