Introduction to Computational Origami: Modern Art and Science

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Abstract. Origami is not only an ancient art, but also a science. It's about math, computer, machinery and has been applied to fields of Robots, Medicine, Spaceflight and so on.
In this paper, I will introduce some foundation of theory in computational origami, which contains flat folding, crease assignment and a algorithm of designing origami model by computer and discuss their complexity.

Keywords: Origami, Complexity, Crease, Design

1 Introduction

Origmi is a kind of ancient art which is widely believed to origins from China. But it is Japan developed it into a independent art form, and the word "origami" comes from Japanese.
Japanese origami began sometime after Buddhist monks carried paper to Japan during the 6th century. The first Japanese origami is dated from this period and was used for religious ceremonial purposes only, due to the high price of paper. Then in 1797 the first known origami book was published in Japan: Senbazuru orikata.
The father of the art of paper folding is consider to be Akira Yoshizawa, who invented a language systems to guide people to fold. Since then, the art of origami spread all over the world and gradually transited to modern form: One square paper, and no cuts.
As some general designing method founded, more and more complex models emerged. People started realizing that mathematic plays a very important roly in origami. And until now, origami has became a new kind of independent science. One of an important concept abstract from origami is "flat folding". It is a natural derivation from real origami. And the whole paper is under this constrain. In this paper, we will introduce some key theorems in flat folding, discuss the complexity of finding a way to fold a crease pattern into flat state. And at the end, we will have a overlook on *Tree Theory*, a method of designing origami by computer.

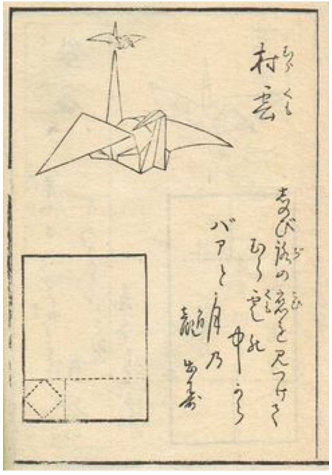

Fig. 1. The folding of two origami cranes linked together from the first known book on origami Hiden senbazuru orikata published in Japan in 1797.

Fig. 2. Akira Yoshizawa and his work.

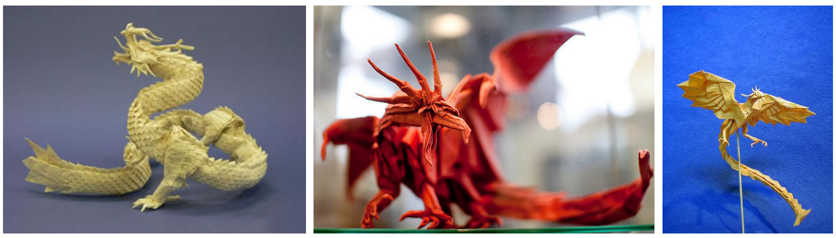

Fig. 3. Some really complex origami models. They all fold by a single square paper without any cuts.

2 Flat Folding

2.1 General Definitions

Definition 1. A **crease pattern** is a finite planar straight-line graph drawn on a convex planar region (the paper). A crease is an edge of the planar graph.

Definition 2. An **embedding** is a continuous, one-to-one mapping of a crease pattern to $\mathbf{R}^3$. The mapping must be smooth (differentiable) everywhere except along creases.

Definition 3. A **flat origami** is an infinite sequence of embeddings of the same crease pattern, such that the images of each crease converge to a line segment and the images of each face converge to a planar polygonal region, congruent to the face. (Convergence is not just pointwise, but sufficiently strong that metric properties converge as well.) Moreover, the dihedral along each crease must converge uniformly to either π or $-\pi$.

2.2 Vertex Flat Foldability

In this section, we will concentrate on **VERTEX FLAT FOLDABILITY**, language of crease patterns whose creases all start from the same vertex. So it is convenient to constrain the paper to a neighborhood (sphere) of a single vertex. Let v be an interior vertex of crease pattern C with crease $e_1,\ e_2,\ ...,\ e_n$, we will give the necessary conditions and suffcient conditions for **VERTEX FLAT FOLDABILITY**.
First, we provide some definitions.

Definition 4. A **vertex-flat assignment** is an MV- assignment to a crease pattern such that for each vertex this assignment is the assignment of a flat single-vertex origami.

Definition 5. VERTEX FLAT FOLDABILITY is the problem of determining whether or not a given crease pattern has a vertex-flat assignment.

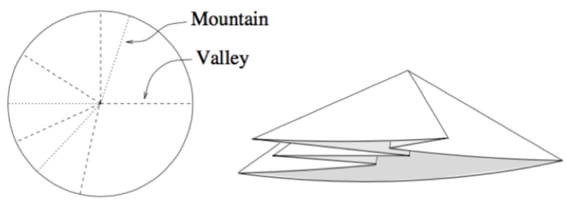

Fig. 4. An example of single vertex flat folding

Theorem 1 (Necessary Conditions) Let v be an interior vertex of crease pattern C, and there is a flat origami with crease pattern C. Then following conditions satisfy:
(K1) The sum of alternate angles around v is π.
(M) The number of mountain folds minus the number of valley folds meeting at v is either 2 or 2.
(K2) If $\alpha_i < \alpha_{i1}$ and $\alpha_i < \alpha_{i+1}$, then e_i and $e_i + 1$ must have opposite assignments.

Theorem 2 (Sufficient Condition) (Kawasaki,Justin) Let D be a crease pattern drawn on a disk, consisting of a single vertex v at the center of the disk, along with some number of creases, each a radius of the disk. Then D is flat foldable if the sum of alternate angles around v is π.

The main idea in proof of Theorem 2 is to use (K2) condition to merge adjacent angles recussively.
Using Theorem 2, we can fold a foldable single vertex in this way: scan all angles around the vertex and put the local minimal angle into a queue. By (K2) condition, their two edges have opposite directions. So we can put the first angle of the queue out one by one, fold it and scan some $O(1)$ angles near it to find some new local minimal angles and put them to the end of the queue. When the queue become empty, the algorithm ends. Obviously, this alogrithm consume $O(n)$ time, where n is number of angles around the vertex.
So we have the following theorem:

Theorem 3 There is a linear-time algorithm for **VERTEX FLAT FOLDABILITY**.

2.3 Flat Foldability

In this section, we will discuss global foldability of a given crease pattern and the complexity of the algorithm of finding a foldable crease assignment.
As what we usually do to prove a language to be **NP**-hard, we may reduce some version **SAT** to the language. This time, we will use **Not-All-Equal 3-SAT** and the definition is as follow:

Definition 6. Not-All-Equal 3-SAT is given by a collection of clauses, each containing exactly three literals. The problem is to determine whether or not there exists a truth assignment such that each clause has either one or two true literals.

It is obvious that this language is **NP**-complete.
Now we provide the definition of the origami problem we will discuss.

Definition 7. Flat Foldability is the problem of determining whether or not a given crease pattern is the crease pattern of a flat origami.

As usual in reductions to versions of SAT, we construct gadgets for boolean variables and clauses, which we interconnect by wires. For us, a wire will be two closely-spaced parallel creases. The spacing is close enough that in any flat folding the two creases in a wire must have opposite assignments, forming a pleat. In order to distinguish left from right, we shall label the wires in our gadgets with directions. These directions serve only as expository devices; they are not part of the crease pattern. We shall call a wire in an MV- assignment true (respectively, false) if the valley crease lies to the right (left) of the mountain crease when facing along the wires direction.

We will use three gadgets to construct a crease pattern that are mapped from an instance of **Not-All-Equal 3-SAT**: clause gadget, reflector gadget and crossover gadget. And we have following lemmas:

Lemma 1 The clause crease pattern is flat foldable if and only if one or two of the incoming wires are true.

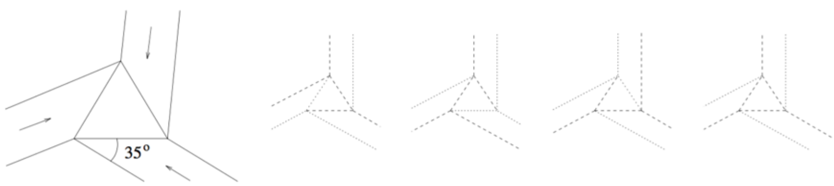

Fig. 5. Clause gadget. The first three can be folded flat and the last one can't.

Lemma 2 The reflector crease pattern is flat foldable if and only if the incoming wire agrees with the outgoing broad wire and disagrees with the outgoing narrow wire.

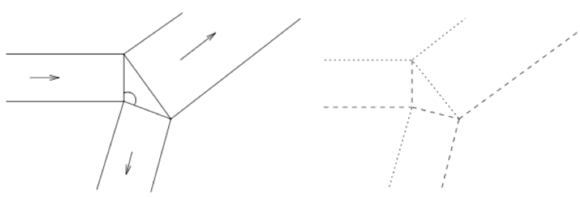

Fig. 6. Reflector gadget and its possible crease pattern.

Lemma 3 The crossover crease patterns are flat foldable if and only if each opposite pair of incoming and outgoing wires agree.

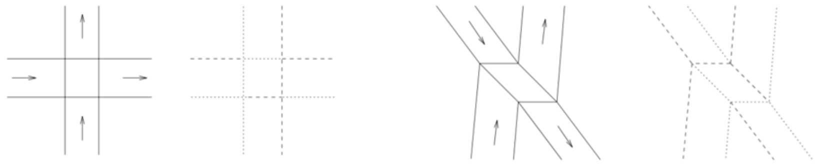

Fig. 7. Crossover gadget.

Now let's put them together. The following figure is an example to show how to construct crease pattern.

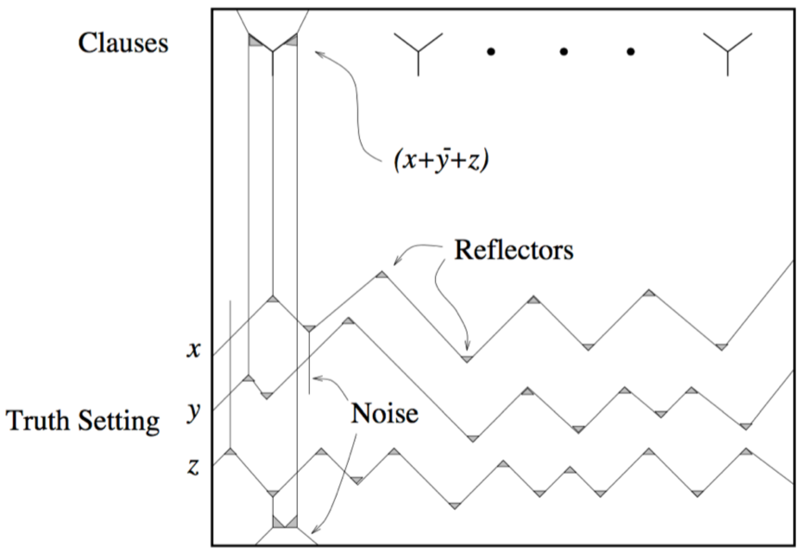

Fig. 8. Schematic of the entire reduction.

We put clause gadgets at the top of the paper representing the clause of 3CNF and each variable forms a zig-zag wire at the bottom of paper. We change True-False state of wires using reflactor gadget according to Lemma 2.

So we can reduce any instance of **Not-All-Equal 3SAT** to a crease pattern like above, which implies the following theorem:

Theorem 4 FLAT FOLDABILITY is **NP**-hard.

Whether **FLAT FOLDABILITY** belongs to **NP**? We has no answer now. Because varifying a certificate (a assigned crease pattern) is also very hard according the following theorem:

Theorem 5 ASSIGNED FLAT FOLDABILITY is **NP**-hard.

The proof can be found in [3] and we provide precise definition of **ASSIGNED FLAT FOLDABILITY**:

Definition 8. ASSIGNED FLAT FOLDABILITY is the problem of determining whether or not there exists a flat origami with a given assigned crease pattern.

3 Tree Theory

In this section, we will introduce a very powerful origami designing method: *Tree Method*. It is inspired by a method called Circle/River Packing. And we will discuss the complexity of Circle/River Packing.

3.1 Backgroud

The main idea of Circle/River Packing is representing a flap of an origami base by a circle in the paper and a gap between two groups of flaps by a river with equal width. The details of this method can be found in *Chapter* 9 of [4].
Robert J.Lang changed it a little and transfered the Circle Packing problem to an optimation problem defined as follows:

$$\begin{aligned} &\max\ \lambda \\ &s.t. \quad \begin{cases} \frac{Distance\ in\ shadow\ tree\ between\ leaf\ i\ and\ leaf\ j}{Euclidean\ distance\ between\ point\ i\ andpointj} \geq \lambda,\ \forall i \neq j \\ every\ point\ i\ lies\ within\ the\ convex\ piece\ of\ paper. \end{cases} \end{aligned}$$

The optimal value λ is called the largest scale between origami base and paper, i.e, the ratio of model and paper.

And he prove that:

Theorem 6(Lang 1996). There is an algorithm that, given any convex piece of paper P and any metric tree T, constructs a crease pattern that folds P into

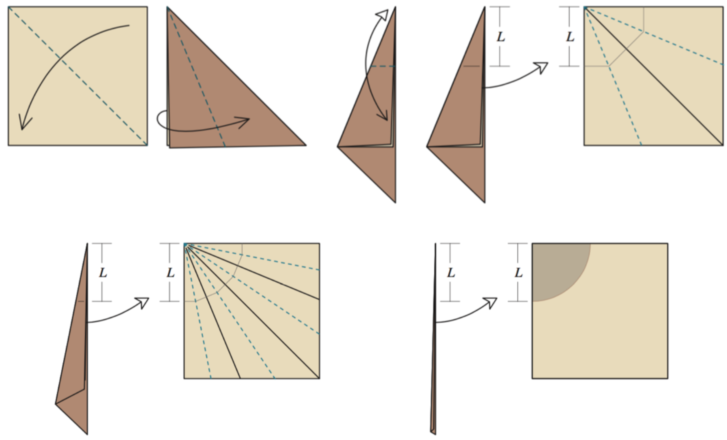

Fig. 9. A circle is the limit state of a flap

a uniaxial base whose shadow tree is the largest possible scaled copy of T, ignoring possible self-intersection of the paper.
The precise definations of uniaxial base and shadow tree can be found in *Chapter* 16 of [5], which is not the key of our paper.

Theorem 6 guarantees that the solution of the optimal solution of largest scale problem is exactly the position assignment of circles and rivers that you want.

Here is an example of origami model designed by tree method:

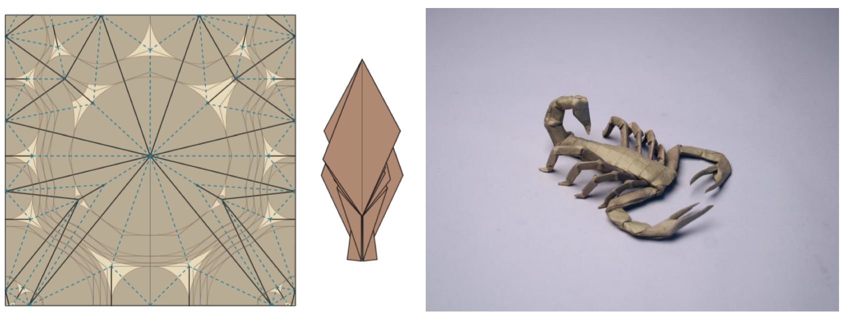

Fig. 10. An origami model of Scorpion.

3.2 The Complexity of Circle/River Packing

On the other hand, Theorem 6 also guarantees that given a set of circles, there must exist a way to pack them into a convex paper.

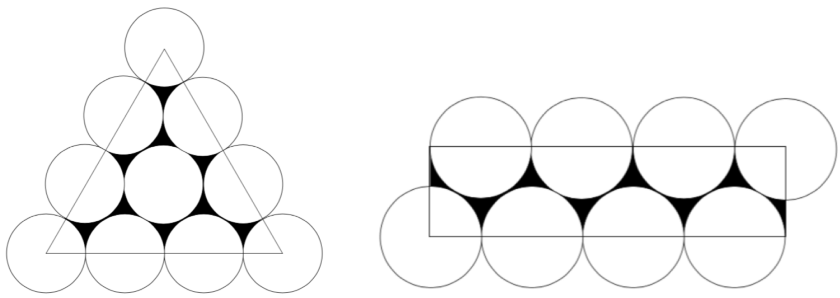

Fig. 11. Circle/River Packing over triangle and rectangle paper.

Here are some main results, the proof of which can be found in [6]:

Theorem 7 Circle/river origami design for triangular paper is **NP**-hard.

Corollary It is **NP**-hard to decide whether a given set of circles can be packed into an equilateral triangle.

Theorem 8 Circle/river origami design for rectangular paper is **NP**-hard.

Corollary It is **NP**-hard to decide whether a given set of circles can be packed into a given rectangle.

Theorem 9 Circle/river origami design for square paper is **NP**-hard.

Corollary It is **NP**-hard to decide whether a given set of circles can be packed into a given square.

4 Conclusion

In this paper, we provide some results on complexity of some problems in origami. We prove that **FLAT FOLDABILITY** is **NP**-hard. And provide some results interesting, such as **ASSIGNED FLAT FOLDABILITY** is **NP**-hard and Circle/River Origami Design is **NP**-hard.

As we can see, there are a lot of problems that is **NP**-hard. And since the science of origami is a very new science, there are a lot of open question in this field. But this is truely a useful and powerful science, which has been applied in fields of Aerospace Engineering, Medical Machinery and so on. You will learn more examples in video [7].

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