

- Continuous functions defined on closed intervals have very special properties.
- Continuity of a function is related to the possibility of tracing its graph without lifting the pen from the paper. The following theorem expresses this fact with precision.

Continuous Functions on Closed Intervals

Warning:
This Teorem is so essential.

- **Theorem (Intermediate Value Theorem)** Suppose that the function f is continuous on the closed interval $[a, b]$. Then $f(x)$ assumes every intermediate value between $f(a)$ and $f(b)$. That is; if K is any number between $f(a)$ and $f(b)$, then

there exists at least one number c in $]a, b[$ such that $f(c) = K$.

Continuous Functions on Closed Intervals

- **Example 29.** The discontinuous function

$$f(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0 \end{cases}$$

defined on $[0, 1]$ does not attain the intermediate value $\frac{1}{2}$. Is that result a contradiction?

- **Example 30.** Show that there is a root of the equation

$$x^3 - x - 1 = 0$$
 between 1 and 2.

Continuous Functions on Closed Intervals

- **Example 31.** Prove that the equation

$$\sqrt{2x + 5} = 4 - x^2$$

has a solution.

- **Example 32.** Is the function $f(x) = x^3 - x - 2$ has a solution between $x = 1$ and $x = 2$?

- **Theorem (Bolzano's Theorem)** Suppose that f is continuous on the closed interval $[a, b]$. If $f(a)$ and $f(b)$ have opposite sign, then there exist a number $c \in]a, b[$ such that $f(c) = 0$.

Continuous Functions on Closed Intervals

- **Continuous Extension to a Point**

Sometimes the formula that describes a function f does not make sense at a point $x = c$. It might nevertheless be possible to extend the domain of f , to include $x = c$, creating a new function that is continuous at $x = c$.

- For example; the function $y = \frac{\sin x}{x}$ is continuous at every point except $x = 0$. Since $y = \frac{\sin x}{x}$ has a finite limit as $x \rightarrow 0$, we can extend the function's domain to include the point $x = 0$ in such a way that the extended function is continuous at $x = 0$. We define the new function

$$F(x) := \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}.$$

Continuous Functions on Closed Intervals

- More generally, a function may have a finite limit at a point where it is not defined (such as a rational function). If $f(c)$ is not defined, but $\lim_{x \rightarrow c} f(x) = L$ exists, we can define a new function $F(x)$ by the rule

$$F(x) := \begin{cases} f(x), & \text{if } x \in D_f \\ L, & \text{if } x = c. \end{cases}$$

The function F is continuous at c . It is called the continuous extension of f to $x = c$.

- **Example 33.** Show that

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}, x \neq 2$$

has a continuous extension to $x = 2$, and find that extension.

Exercises

1. Define $g(3)$ in a way that extends $g(x) = \frac{x^2 - 9}{x - 3}$ to be continuous at $x = 3$.

2. For what values of b is

$$g(x) = \begin{cases} x, & x < -2 \\ bx^2, & x \geq -2 \end{cases}$$

continuous at every x ?

3. For what values of a and b are f and g continuous at every x ?

$$(a) f(x) = \begin{cases} 2 \cos x, & x < 0 \\ a \cos x + b, & 0 \leq x \leq \pi \\ -\sin x, & x > \pi \end{cases}$$

$$(b) g(x) = \begin{cases} (x^2)^a \sin^b (x^2) & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Exercises

4. At which points do the following functions fail to be continuous and at what points are they continuous? Give reasons for your answers.

$$(a) f(x) = \begin{cases} \lfloor x \rfloor + \lfloor -x \rfloor, & x \notin \mathbb{Z} \\ 0, & x \in \mathbb{Z} \end{cases}$$

$$(b) g(x) = x^2 + \operatorname{sgn}(x^2 - 1)$$

$$(c) h(x) = \begin{cases} \frac{2 \sin x}{|x|}, & x \neq 0 \\ 2, & x = 0 \end{cases}$$

$$(d) s(x) = x \lfloor x \rfloor$$

Exercises

$$(e) f(x) = \frac{3^{1/x} + 2^{1/x}}{3^{1/x} - 2^{1/x}}$$

$$(f) g(x) = \begin{cases} \cos \frac{\pi x}{2}, & |x| \leq 1 \\ |x-1|, & |x| > 1 \end{cases}$$

5. Show that the equation $8x^3 - 12x^2 - 2x + 3 = 0$ has at least one root in $[-1, 0[$, $]0, 1[$ and $]1, 2[$.

6. Evaluate the following limits

$$(a) \lim_{x \rightarrow 0} \sqrt{\tan x + 3 \sec x}$$

$$(b) \lim_{x \rightarrow 0} \sin\left(\frac{\pi}{2} \cos(\tan x)\right).$$

2.6 Limits Involving Infinity, Asymptotes of Graphs

- Consider the graph
- Definition** (a) For a given real function f

$$\lim_{x \rightarrow \infty} f(x) = L$$

means that given any $\varepsilon > 0$ there exists a corresponding real number M such that for all x ,

$$x > M \implies |f(x) - L| < \varepsilon$$

and we say that " $f(x)$ has limit the L as x approaches infinity".

2.6 Limits Involving Infinity, Asymptotes of Graphs

- Consider the graph
- (b) For a given real function f

$$\lim_{x \rightarrow -\infty} f(x) = L$$

means that given any $\varepsilon > 0$ there exists a corresponding real number N such that for all x ,

$$x < N \implies |f(x) - L| < \varepsilon$$

and we say that " $f(x)$ has the limit L as x approaches minus infinity".

2.6 Limits Involving Infinity, Asymptotes of Graphs

- **Example 18.** Show that
- (a) $\lim_{x \rightarrow \infty} \frac{x+4}{x} = 1$ and $\lim_{x \rightarrow -\infty} \frac{x+4}{x} = 1$,
- (b) $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$ and $\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$.
- **Remark** The Theorem giving Limit Laws is also valid when we replace $\lim_{x \rightarrow c}$ by $\lim_{x \rightarrow \infty}$ or $\lim_{x \rightarrow -\infty}$.

2.6 Limits Involving Infinity, Asymptotes of Graphs

- **Example 19.** Find the following limits

- (a) $\lim_{x \rightarrow \infty} \frac{x}{[x]}$

- (b) $\lim_{x \rightarrow \infty} \frac{2x + 5}{4x^2 + 8x + 1}$

- (c) $\lim_{x \rightarrow \infty} \frac{x^3 + 1}{2x^2 + 3}$

2.6 Limits Involving Infinity, Asymptotes of Graphs

- (d) $\lim_{x \rightarrow \infty} \frac{5x^2 + 1}{2x^2 + 3x - 3}$

- (e) $\lim_{x \rightarrow \infty} (\sqrt{x^2 + x} - x)$

- (f) $\lim_{x \rightarrow -\infty} (\sqrt{x^2 + x} - x)$

- **Example 20.** Try to find a way to evaluate $\lim_{x \rightarrow 2} \frac{x-3}{x^2 - 4}$
(Homework).

2.6 Limits Involving Infinity, Asymptotes of Graphs

• Horizontal Asymptotes

If the distance between the graph of a function and some fixed line approaches zero, as a point on the graph moves increasingly far from the origin, we say that the graph approaches the line **asymptotically** and the line is an **asymptote** of the graph.

- **Definition** A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \rightarrow \infty} f(x) = b \text{ or } \lim_{x \rightarrow -\infty} f(x) = b.$$

2.6 Limits Involving Infinity, Asymptotes of Graphs

- **Example 21.** Find the horizontal asymptotes of the graphs of

- (a) $f(x) = \frac{5x^2 + 8x - 3}{3x^2 + 2}$

- (b) $g(x) = \frac{x^3 - 2}{|x|^3 + 1}$

- (c) $h(x) = e^x$

2.6 Limits Involving Infinity, Asymptotes of Graphs

- (d) $s(x) = \sin\left(\frac{1}{x}\right)$
- (e) $p(x) = x \sin\left(\frac{1}{x}\right)$
- (f) $\lim_{x \rightarrow 0^-} e^{1/x}$

2.6 Limits Involving Infinity, Asymptotes of Graphs

- (g) $f(x) = 2 + \frac{\sin x}{x}$
- (h) $g(x) = x - \sqrt{x^2 + 16}$

2.6 Limits Involving Infinity, Asymptotes of Graphs

• Oblique Asymptotes

If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an **oblique** or **slant**

line asymptote. We find an equation for the asymptote by dividing numerator by denominator to express f as a linear function plus a remainder that goes to zero as $x \rightarrow \pm\infty$.

• Example 22. Find the oblique asymptotes of the graphs of

- (a) $f(x) = \frac{x^2 - 3}{2x - 4}$
- (b) $g(x) = \frac{x^3 + 1}{x}$

2.6 Limits Involving Infinity, Asymptotes of Graphs

• Infinity/Negative Infinity as Limits (Vertical Asymptotes)

Consider the graph

- **Definition** (a) $\lim_{x \rightarrow x_0} f(x) = \infty$ means that for every positive real number B there exists a corresponding $\delta > 0$ such that for all x ,

$$0 < |x - x_0| < \delta \implies f(x) > B$$

and we say

$f(x)$ approaches infinity as x approaches x_0 .

2.6 Limits Involving Infinity, Asymptotes of Graphs

- Consider the graph
- (b) $\lim_{x \rightarrow x_0} f(x) = -\infty$ means that for every negative real number K there exists a corresponding $\delta > 0$ such that for all x ,

$$0 < |x - x_0| < \delta \implies f(x) < K$$

and we say

$f(x)$ approaches negative infinity as x approaches x_0 .

2.6 Limits Involving Infinity, Asymptotes of Graphs

- Show that $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$, $\lim_{x \rightarrow 1^+} \frac{4}{x-1} = \infty$ and $\lim_{x \rightarrow 1^-} \frac{4}{x-1} = -\infty$.

2.6 Limits Involving Infinity, Asymptotes of Graphs

- **Example 23.** Evaluate the following limits

- (a) $\lim_{x \rightarrow 2} \frac{x-3}{x^2 - 4}$,

- (b) $\lim_{x \rightarrow 1^+} \frac{4}{x-1}$,

- (c) $\lim_{x \rightarrow 1^-} \frac{4}{x-1}$,

2.6 Limits Involving Infinity, Asymptotes of Graphs

- (d) $\lim_{x \rightarrow 0} \frac{1}{x}$

2.6 Limits Involving Infinity, Asymptotes of Graphs

- **Example 24.** Rational functions can behave in various ways near zeros of the denominator:

- (a) $\lim_{x \rightarrow 2} \frac{(x-2)^2}{x^2 - 4}$
- (b) $\lim_{x \rightarrow 2} \frac{x-2}{x^2 - 4}$
- (c) $\lim_{x \rightarrow 2^+} \frac{x-3}{x^2 - 4}$

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- (d) $\lim_{x \rightarrow 2^-} \frac{x-3}{x^2 - 4}$
- (e) $\lim_{x \rightarrow 2} \frac{x-3}{x^2 - 4}$
- (f) $\lim_{x \rightarrow 2} \frac{2-x}{(x-2)^3}$

2.6 Limits Involving Infinity, Asymptotes of Graphs

- **Vertical Asymptotes**

Definition A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \rightarrow a^+} f(x) = \pm\infty \text{ or } \lim_{x \rightarrow a^-} f(x) = \pm\infty.$$

- **Example 25.** Find the horizontal and vertical asymptotes of the following curves

- (a) $y = \frac{x+3}{x+2}$

2.6 Limits Involving Infinity, Asymptotes of Graphs

- (b) $y = -\frac{8}{x^2 - 4}$
- (c) $y = \ln x$
- (d) $y = \sec x$ and $y = \tan x$