CHAPTER 4. APPLICATIONS OF DERIVATIVES

4.1. Extreme Values of Functions
4.2. The Mean Value Theorem
4.3. Monotonic Functions and the First Derivative Test
4.4. Concavity and Curve Sketching

In the first section we will learn how to locate and identify extreme (maximum or minimum) values of a function from its derivative. Then we can solve a variety of optimization problems. The domains of the functions we consider are intervals or unions of seperate inervals.

4.1. Extreme Values of Functions

- **Definition** Let f be a function with domain D. Then f has an absolute maximum value on D at a point c if

$$f(x) \leq f(c) \text{ for all } x \in D,$$

- and an absolute minimum value on D at c if

$$f(x) \geq f(c) \text{ for all } x \in D.$$

- Maximum and minimum values are called extreme values of the function f. Absolute minima or maxima are also referred to global minima or maxima.

- **Example 1.** Find the extreme values of $f(x) = \sin x$ and $g(x) = \cos x$ on $I = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
- **Example 2.** Functions with the same defining rule or formula can have different extrema, depending on the domain. Consider $y = x^2$ on R (resp. $[0,3]$, $]0,3]$ and $]0,3[$)

- **<u>Theorem 1</u>** (Recall the "Extreme Value Theorem") If f is continuous on a closed interval $[a,b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a,b]$. That is,
- there are numbers x_1 and x_2 in $[a,b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a,b]$.

- **Example 3.** Find a function which takes on its maximum and minimum values at interior points of its domain (resp. at endpoints of its domain, maximum at an interior and minimum at an endpoint of its domain).

4.1. Extreme Values of Functions

- **Local (Relative) Extreme Values**
- **Definition** A function f has a **local maximum value** at a point c within its domain D if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing c.
- A function f has a **local minimum value** at a point c within its domain D if $f(x) \geq f(c)$ for all $x \in D$ lying in some open interval containing c.

4.1. Extreme Values of Functions

- If the domain of f is the closed interval $[a, b]$ then

(a) f has a local maximum at $x = a$ if $f(x) \leq f(a)$ for all x in some half-open interval $[a, a + \delta[$,
(b) f has a local maximum at an interior point $x = c$ if $f(x) \leq f(c)$ for all x in some open interval $]a - \delta, a + \delta[$,
(c) f has a local maximum at $x = b$ if $f(x) \leq f(b)$ for all x in some half-open interval $]b - \delta, b]$.

- An absolute maximum (or minimum) is a local maximum (or minimum), if exists.

4.1. Extreme Values of Functionss

Theorem 2 (The First Derivative Theorem for Local Extreme Values)
If f has a local maximum or minimum value at an interior point c of its domain and if f' is defined at c, then $f'(c) = 0$.

- Theorem 2 says that a function's first derivative is always zero at an interior point where the function has a local extreme value and the derivative is defined. If we recall that all the domains we consider are intervals or unions of separate intervals, the only places where a function f can possibly have an extreme value(local or global) are
- (a) interior points where $f' = 0$
- (b) interior points where f' is not defined
- (c) endpoints of the domain of f.

- **Definition** An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.
- **Remark 1.** The converse of this Theorem 2 is not true.
- (i) For a differentiable function f, we may evaluate $f'(c) = 0$ without having local extreme value there. For instance; $f(x) = x^3$ has a critical point at $x = 0$ but f does not have a local extremum at $x = 0$.
- (ii) A function which has a local extremum at a point may not be differentiable at that point. For instance; $f(x) = |x|$ has a local minimum at $x = 0$, but it is not differentiable at $x = 0$.
- **Remark 2.** Here we are considering functions defined on bounded closed intervals, and Theorem 1 assures us that extreme values exist for such functions and Theorem 2 tells us that extreme values are taken on only at critical points and endpoints. Of course, if the interval is not closed or not finite, we have seen that absolute extrema need not exist.

4.1. Extreme Values of Functions

- Finding the Absolute Extrema of a Continuous Function f on a Finite Closed Interval
- 1. Specify the critical points of f.
- 2. Evaluate f at all critical points and endpoints.
- 3. Take the largest or smallest of these values.
- **Example 4.** Find the extreme values of

$$f(x) = \begin{cases} -x, & -3 \leq x < -1 \\ x+2, & -1 \leq x < 0 \\ 2(x-1)^2, & 0 \leq x \leq 3 \end{cases}$$

4.1. Extreme Values of Functions

- **What we have learned until now:**
- Absolut and local maximum and minimum values
- If f is continuous on $[a, b]$, it attains its absolute max. and min. on $[a, b]$.
- If f has a local max. or min. at $c \in D_f$ and f' is defined at c, then $f'(c) = 0$. (But the converse is not true.)
- **Critical points** of f; c is an interior point of D_f, $f'(c) = 0$ or f' is not defined at c.

4.1. Extreme Values of Functions

- **Example 5.** Find the absolute maximum and minimum values of
- (a) $f(x) = 10x(2 - \ln x)$ on $[1, e^2]$,
- (b) $g(x) = 2x^3 - 9x^2 - 24x + 45$ on $[-5, 5]$,
- (c) $h(x) = x^{2/3}$ on $[-2, 3]$.

4.2. The Mean Value Theorem

- *Can we write a complicated function whose derivative is always zero?*
- *If two functions have identical derivatives over an interval, how are the functions related?*

4.2. The Mean Value Theorem

Theorem 3. (Rolle's Theorem) Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior $]a, b[$. If $f(a) = f(b)$, then there is at least one number c in $]a, b[$ at which $f'(c) = 0$.

4.2. The Mean Value Theorem

Remark 3. The hypothesis of Theorem 3 are essential. If they fail even at one point, the graph may not have a horizontal tangent.

4.2. The Mean Value Theorems

- **Example 6.** (a) Show that $x^3 + 3x + 1 = 0$ has exactly one real solution.
- (b) If $f(x) = x^4 - 5x^2 + 4$, find the bounds for the roots of f'.
- (c) Show that the equation $5x^4 - 4x + 1 = 0$ has at least one root in the interval $]0, 1[$.

4.2. The Mean Value Theorem

Question 1. Assume that f is continuous on $[a, b]$. How can we estimate the difference $f(b) - f(a)$?

4.2. The Mean Value Theorem

Theorem 4 (The Mean Value Theorem)
Suppose $y = f(x)$ is a continuous function over a closed interval $[a, b]$ and differentiable at every point of its interior $]a, b[$. Then there is at least one point $c \in]a, b[$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

4.2. The Mean Value Theorem

- **Question 2.** Observe Theorem 4, for $f(x) = x^2$ on $[0, 2]$.
- **Question 3.** Can you give a physical interpretation for Theorem 4?
- **Example 7.** If a car accelerating from zero takes 8 seconds to go 176 meter, its average velocity for the interval $[0, 8]$ is $176/8 = 22m/s$. What can we say by using the Mean Value Theorem?

4.2. The Mean Value Theorem

Corollary 1. If $f'(x) = 0$ at each point x of an open interval $]a, b[$, then there exists a constant c such that $f(x) = c$ for all $x \in]a, b[$ where c is a constant.

4.2. The Mean Value Theorem

- **Corollary 2.** If $f'(x) = g'(x)$ at each point x of an open interval $]a, b[$, then there exists a constant c such that $f(x) = g(x) + c$ for all $x \in]a, b[$. That is; $f - g$ is a constant function.
- **Example 8.** Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point $(0, 2)$.

4.2. The Mean Value Theorem

- **Finding Velocity and Position from Acceleration** We can use Corollary 2 to find the velocity and position functions of an object moving along a vertical line.
- Assume the object is falling freely from rest with acceleration $9.8m/s^2$. We assume the position $s(t)$ of the object is measured positive downward from the rest position (so the vertical coordinate line points downward, in the direction of the motion, with the rest point at zero). Find the velocity function and position function.
- **Homework** Investigate how can we use the Mean Value Theorem for the proofs of laws of logarithms.

4.3. Monotonic Functions and the First Derivative Test

- **Increasing and Decreasing Functions**
- **Corollary 3.** Suppose f is continuous on $[a, b]$ and differentiable on $]a, b[$.
- (a) If $f'(x) > 0$ at each point $x \in]a, b[$, then f is increasing on $[a, b]$.
- (b) If $f'(x) < 0$ at each point $x \in]a, b[$, then f is decreasing on $[a, b]$.

4.3. Monotonic Functions and the First Derivative Test

- **Remark 4.** Corollary 3 is valid for infinite intervals as well as finite intervals. For example; $f(x) = \sqrt{x}$ is increasing on $[0, b]$ for any $b > 0$ because $f'(x) = \frac{1}{2\sqrt{x}}$ is positive on $]0, b[$. The derivative does not exist at $x = 0$ but the Corollary 3 still applies and f is increasing on $[0, \infty[$.
- **Example 8.** Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the open intervals on which f is increasing and on which f is decreasing.

4.3. Monotonic Functions and the First Derivative Test

- **First Derivative Test for Local Extrema** Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across this interval from left to the right:
- **1**. If f' changes negative to positive at c, then f has a local minimum at c;
- **2**. If f' changes positive to negative at c, then f has a local maximum at c;
- **3**. If f' does not change sign at c, then f has a local extremum at c.

4.3. Monotonic Functions and the First Derivative Test

Example 9. Find the critical points of $f(x) = x^{1/3}(x-4)$. Identify the open intervals on which f is decreasing and increasing. Find the functions local and absolute extreme values.

4.3. Monotonic Functions and the First Derivative Test

Example 10. Find the critical points of $f(x) = \left(x^2-3\right)e^x$. Identify the open intervals on which f is decreasing and increasing. Find the functions local and absolute extreme values.

4.4 Concavity and Curve Sketching

- Sketch the graph of $f(x) = x^3$, observe the results that we can obtain by using the First Derivative Test and observe what can be obtained from the second derivative:
- **Definition** The graph of a differentiable function $y = f(x)$ is
- (a) **concave up** on an open interval I, if f' is increasing on I;
- (b) **concave down** on an open interval I, if f' is decreasing on I;
- If $y = f(x)$ has a second derivative, to obtain where f' is increasing or decreasing we can apply the Corollary 3 of the Mean Value Theorem and conclude that where f is concave up and concave down.

4.4 Concavity and Curve Sketching

- **The Second Derivative Test for Concavity** Let $y = f(x)$ be twice-differentiable on an interval I.
- (a) If $f'' > 0$ on I, the graph of f over I is **concave up**.
- (b) If $f'' < 0$ on I, the graph of f over I is **concave down**.

4.4 Concavity and Curve Sketching

- **Example 11.** Determine the concavity of the following functions;
- $(a)\ f(x) = x^3$
- $(b)\ g(x) = x^2$
- $(c)\ h(x) = 3 + \sin x$ on $[0, 2\pi]$

4.4 Concavity and Curve Sketching

- **<u>Points of Inflection</u>** The curve $y = 3 + \sin x$ in the previous example changes concavity at the point $(\pi, 3)$. The curve has a tangent line with slope
$$y' \Big|_{x=\pi} = (\cos x) \Big|_{x=\pi} = -1$$
at $x = \pi$. This point is called a point of inflection of the curve. Observe that the graph crosses its tangent line at this point and the second derivative $y'' = -\sin x$ has value zero when $x = \pi$.
- **<u>Definition</u>** A point $(c, f(c))$ where the graph of the function has a tangent line and where the concavity changes is **a point of inflection**.
- At a point of inflection $(c, f(c))$ either $f''(c) = 0$ or $f''(c)$ fails to exis

4.4 Concavity and Curve Sketchingt

- **Example 12.** Identify the inflection points of the following functions if exists;
- $(a)\ f(x) = x^{5/3}$
- $(b)\ g(x) = x^4$
- $(c)\ h(x) = x^{1/3}$

4.4 Concavity and Curve Sketching

- **Second Derivative Test for Local Extrema**
- **Theorem 5 (Second Derivative Test for Local Extrema)** Suppose f'' is continuous on an open interval that contains c.
- (a) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum value at $x = c$.
- (b) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum value at $x = c$.
- (c) If $f'(c) = 0$ and $f''(c) = 0$, then the test fails.

Procedure for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Identify any asymptotes that may exist.
3. Find the derivatives y' and y''.
4. Find the critical points of f, if any, and identify the functions behavior at each one.
5. Find where the curve is increasing and where it is decreasing.
6. Find the points of inflection, if any occur, and determine the concavity of the curve.
7. Plot key points, such as the intercepts and the points found in Steps 4,5 and 6.
8. Make a table and sketch the curve together with any asymptotes that exist.