2.1. Rates of Changes and Tangents to Curves

Calculus is a tool that helps us understand how a change in one quantity is related to a change in another.

How does the speed of a falling object change as a function of time?
How does the level of water in a barrel change as a function of amount of liquid poured into it?

We will first start with average and instantaneous rates of change, and show that they are closely related to the slope of a curve at a point P on the curve.

2.1. Rates of Changes and Tangents to Curves

- **Average and Instantaneous Speed**
- A moving object's **average speed** during an interval of time is found by dividing the distance covered by the time elapsed. The unit of measure is length per unit time (such as kilometers per hour, meters per second).

2.1. Rates of Changes and Tangents to Curves

- **Example 1.** Physical experiments show that if a rock is dropped from rest near the surface of the earth, in the first t seconds it will fall a distance $y = 4.9t^2$ meters. What is its average speed of the falling rock
- (a) during the first 2 seconds of fall?
- (b) during the 1 second interval between second 1 and second 2?

2.1. Rates of Changes and Tangents to Curves

- If we want to determine the speed of the falling object **at a single instant** t_0, how can we do it?
- First let's examine what happens when we calculate the average speed over **shorter and shorter time intervals starting at** t_0. Let $t_0 = 1$, assume that h represents a very small positive real number and consider the interval $[t_0, t_0 + h]$. Since

$$Average\ Speed := \frac{\Delta y}{\Delta t}$$

- Then we obtain the following values;

h		**Average Speed**
1	$\longrightarrow$	14.7
0.1	$\longrightarrow$	10.29
0.01	$\longrightarrow$	9.849
0.001	$\longrightarrow$	9.8049.

- The average speed on interval starting at $t_0 = 1$ seems to approach a limiting value of 9.8 as the length of the interval decreases.
- **How can we confirm it algebrically?**

- **Average Rate of Changes and Secant Lines**
- **Definition** The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} := \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

- or if we let $h = x_2 - x_1$ then

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1 + h) - f(x_1)}{h}.$$

2.1. Rates of Changes and Tangents to Curves

- **Geometrically**, the rate of change of f over $[x_1, x_2]$ is the slope of the line through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$.
- In geometry, a line joining two points of a curve is a **secant** to the curve. Thus, the average rate of change of f from x_1 to x_2 is identical with the slope of secant PQ. Consider what happens as the point Q approaches the point P along the curve.

2.1. Rates of Changes and Tangents to Curves

- What is a tangent line?
- What is meant by the slope of a curve at a point P on that curve?
- **Example 2.** Can you consider a method to evaluate the slope of the parabola $y = x^2$ at the point $P(2, 4)$?

2.1. Rates of Changes and Tangents to Curves

- **Instantaneous Rates of Change and Tangent Lines:**
- The rate at which the rock in Example 1 was falling at the instant $t = 1$ or $t = 2$ is called the **instantaneous rate of change**.
- We will understand how it is evaluated in the following subsection.

2.2. Limit of a Function and Limit Laws

- How do the values of a function f behave when **its independent variable** x **gets closer and closer to a given value**?
- In some cases the answer is intuitively clear. For example; if $f(x) = x^2 + 1$ and x gets closer and closer to the number 2, it seems evident that $f(x)$ gets closer and closer to the number $f(2) = 5$.
- This situation is summarized by saying that

  "the limit of $f(x)$ as x approaches 2 is equal to 5."

- **Example 3.** Consider Example 1 and the question following it, then evaluate how fast is the rock at time $t = 1$?

2.2. Limit of a Function and Limit Lawss

- However, in other cases we can not find the limit so simply by making an easy substitution. It may happen that the function is not even defined at the point where the limit is evaluated.
- **Example 4.** (a)Describe the behaviour of the function $f(x) = \dfrac{x^2-1}{x-1}$ near $x = 1$.

2.2. Limit of a Function and Limit Laws

- **An Informal Definition for Limit**
- If $f(x)$ is defined for all x near a except possibily at a itself and, if we can ensure that $f(x)$ is as close as we want to a L by taking x close enough to a, we say that
- "f approaches the limit L as x approaches a"
- and write

$$\lim_{x \longrightarrow a} f(x) = L.$$

2.2. Limit of a Function and Limit Laws

- **Example 5.** (a) If f is the identity function $f(x) = x$, then for any value of c

$$\lim_{x \longrightarrow c} f(x) = \lim_{x \longrightarrow c} x = c.$$

- (b) If f is the constant function $f(x) = k$(k is a constant), then for any value of c

$$\lim_{x \longrightarrow c} f(x) = \lim_{x \longrightarrow c} k = k.$$

2.2. Limit of a Function and Limit Laws

- **Example 6.** Evaluate the following limits;
- (a) $\lim\limits_{x \longrightarrow -2} \dfrac{x^2 + x - 2}{x^2 + 5x + 6}$
- (b) $\lim\limits_{x \longrightarrow a} \dfrac{\frac{1}{x} - \frac{1}{a}}{x - a}$
- (c) $\lim\limits_{x \longrightarrow 4} \dfrac{\sqrt{x} - 2}{x^2 - 16}$

2.2. Limit of a Function and Limit Laws

- **Example 7.** The limit value of a function does not depend on how the function is defined at the point being approached. Consider the following functions; $f(x) = \dfrac{x^2-1}{x-1}$, $g(x) = x+1$ and

$$h(x) = \begin{cases} x+1 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}.$$

2.2. Limit of a Function and Limit Laws

- **Example 8.** A function may not have a limit at a particular point. The following are some situations that limits can fail to exist: the limit of
- $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$, at $x = 0$.
- $g(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$, at $x = 0$.

2.2. Limit of a Function and Limit Laws

- $h(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \sin\frac{1}{x} & \text{if } x > 0 \end{cases}$ at $x = 0$.

2.2. Limit of a Function and Limit Laws

- **The Limit Laws**
- **Theorem 1.** If L, M, c and k are real numbers and

$$\lim_{x \longrightarrow c} f(x) = L \text{ and } \lim_{x \longrightarrow c} g(x) = M$$

then

- $\lim_{x \longrightarrow c} (f(x) \mp g(x)) = L \mp M$
- $\lim_{x \longrightarrow c} (kf(x)) = k.L$
- $\lim_{x \longrightarrow c} (f(x).g(x)) = L.M$
- $\lim_{x \longrightarrow c} \frac{f(x)}{g(x)} = \frac{L}{M}; M \neq 0$
- $\lim_{x \longrightarrow c} (f(x))^n = L^n, n \in \mathbb{Z}^+$
- $\lim_{x \longrightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{L}, n \in \mathbb{Z}^+$ (if n is even we assume that $L > 0$)

- **Example 9.** Find the following limits:
- $(a)\ \lim\limits_{x\longrightarrow 1} \dfrac{x^3+2x^2-3x}{x^2-1}$
- $(b)\ \lim\limits_{t\longrightarrow 0} \dfrac{\sqrt{t^2+9}-3}{t^2}$

2.2. Limit of a Function and Limit Laws

- **Theorem 2.** (Limits of Polynomials)
- If

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

- then

$$\begin{aligned} \lim_{x\longrightarrow c} p(x) &= a_n c^n + a_{n-1}c^{n-1} + \cdots + a_2 c^2 + a_1 c + a_0 \\ &= p(c). \end{aligned}$$

- **Theorem 3.** (Limits of Rational Functions) If p and q are two polynomials and $q(c) \neq 0$, then

$$\lim_{x\longrightarrow c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}.$$

2.2. Limit of a Function and Limit Laws

- **Example 10.** Find the following limits:
- $(a)\ \lim\limits_{x\longrightarrow 3}\dfrac{x+3}{x+6}$
- $(b)\ \lim\limits_{x\longrightarrow 1}\dfrac{x^2+x-2}{x^2-x}$

2.2. Limit of a Function and Limit Laws

Theorem 4. (The Squeeze Theorem) Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at c itself. Suppose also that

$$\lim_{x\longrightarrow c} g(x) = \lim_{x\longrightarrow c} h(x) = L.$$

Then $\lim\limits_{x\longrightarrow c} f(x) = L$.

2.2. Limit of a Function and Limit Laws

- **Example 11.**(a) Given that $3 - x^2 \leq u(x) \leq 3 + x^2$ for all $x \neq 0$, find $\lim\limits_{x \longrightarrow 0} u(x)$.
- (b) Show that if $\lim\limits_{x \longrightarrow a} |f(x)| = 0$, then $\lim\limits_{x \longrightarrow a} f(x) = 0$.
- (c) Show that $\lim\limits_{\theta \longrightarrow 0} \sin\theta = 0$

2.2. Limit of a Function and Limit Laws

- **Theorem 5.** If $f(x) \leq g(x)$ for all x in some open interval containing c, except possibly at c itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x \longrightarrow c} f(x) \leq \lim_{x \longrightarrow c} g(x).$$

- **Homework** Can we replace the less then or equal to inequality by the strict less then inequality in Theorem 5?

2.2. Limit of a Function and Limit Laws

Example 12. Does the function $g(x) = \sqrt{1-x^2}$ have a limit at $x = 1$ and $x = -1$? (Homework)

2.3. The Precise Definition of a Limit

"closeness of $f(x)$ to L" means "smallness of $|f(x) - L|$"
"closeness of x to c" means "smallness of $|x - c|$"

2.3. The Precise Definition of a Limit

- **Example 13.** Consider the function $y = 2x - 1$ near 4. Intuitively it appears that y is close to 7 when x is close to 4, so $\lim_{x \longrightarrow 4} (2x - 1) = 7$.
- However, **how close to** 4 **does** x **have to be,** so that $y = 2x - 1$ differs from 7 by 2 units? In other words; for what values of x is $|y - 7| < 2$?

2.3. The Precise Definition of a Limit

- **Example 14.** Calculate $\lim\limits_{x \longrightarrow 0} \dfrac{\sin x}{x}$.
- $f(x) = \dfrac{\sin x}{x}$ is not defined at $x = 0$ and we can not make any algebraic operations to simplfy it as we did before. But by using a calculator, for typical values of x close enough to zero, we can evaluate the value of $\dfrac{\sin x}{x}$.
-

x *Radians*	$f(x)$
1.0	0.77670
0.8	0.84147
0.6	0.94107
0.4	0.97355
0.2	0.99335

- Thus it seems that $\lim\limits_{x \longrightarrow 0} \dfrac{\sin x}{x} = 1$ but it is not a rigorous proof.

2.3. The Precise Definition of a Limit

- **Definition** Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the limit of $f(x)$ as x approaches c is the number L, and write

$$\lim_{x \longrightarrow c} f(x) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - c| < \delta \Longrightarrow |f(x) - L| < \varepsilon.$$

2.3. The Precise Definition of a Limit

- $\forall \varepsilon > 0 : \exists \delta > 0$ s.t. for all x, $0 < |x - c| < \delta \Longrightarrow |f(x) - L| < \varepsilon$.
- It must be possible to make $f(x)$ as close to as we want to L for all x sufficiently close to c, whether x lies to the left or to the right of c.

2.3. The Precise Definition of a Limit

- Note that this definition does not help to calculate limits but helps us to verify that a suspected limit is correct.
- **Example 15.** (a) Show that $\lim_{x\longrightarrow 4}(2x-1)=7$.

2.3. The Precise Definition of a Limit

(b) For the limit $\lim_{x\longrightarrow 5}\sqrt{x-1}=2$, find a $\delta>0$ that works for $\varepsilon=1$.

2.3. The Precise Definition of a Limit

(c) Prove that $\lim_{x \longrightarrow 2} f(x) = 4$ if $f(x) = \begin{cases} x^2, & \textit{if } x \neq 2 \\ 1, & \textit{if } x = 2 \end{cases}$

2.3. The Precise Definition of a Limit

Homework Consider $\lim\limits_{x \longrightarrow c} f(x) = L$. How can you find algebrically a $\delta > 0$ for f, L, c and $\varepsilon > 0$