2.4. One-Sided Limits

If f is given with the rule

$$f(x) = \frac{|x-a|}{x-a} = \begin{cases} 1, & \text{if } x > a \\ -1, & \text{if } x < a \end{cases}$$

then consider the graph of the function and observe how it behaves at the right of a and at the left of a.

2.4. One-Sided Limits

- It jumps from -1 to 1 as its input value x goes from the left of a to the right of a, and therefore f has no limit at a.
- However, f would approach a limiting value at a, if the approach of x to a was from one side only. For then we could ignore the behavior of f on the other side of a. Thus, we say that the function f has one sided limits at a, even though the ordinary limit of f at a does not exist.
- If the variable point x approaches the fixed point a from the right, taking only values larger then a $(x > a)$, we write

$$x \longrightarrow a^+,$$

- while if x approaches the point from the left taking only values smaller then a $(x < a)$, we write

$$x \longrightarrow a^-.$$

2.4. One-Sided Limits

For the above function; $f(x) \longrightarrow 1$ as $x \longrightarrow a^+$ and $f(x) \longrightarrow -1$ as $x \longrightarrow a^-$ equivalently

$$\lim_{x \longrightarrow a^+} f(x) = 1 \text{ and } \lim_{x \longrightarrow a^-} f(x) = -1.$$

The first one is called "*the right-hand limit of* f *at* a" and the second is called "*the left-hand limit of* f *at* a".

2.4. One-Sided Limits

- **Formal Definition** If f is a function defined for all x near a except possibly at a itself, then
- (a) $f(x) \longrightarrow L$ as $x \longrightarrow a^+$ means that for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$a < x < a + \delta \Longrightarrow |f(x) - L| < \varepsilon,$$

- and we write

$$\lim_{x \longrightarrow a^+} f(x) = L.$$

- (b) $f(x) \longrightarrow M$ as $x \longrightarrow a^-$ means that for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

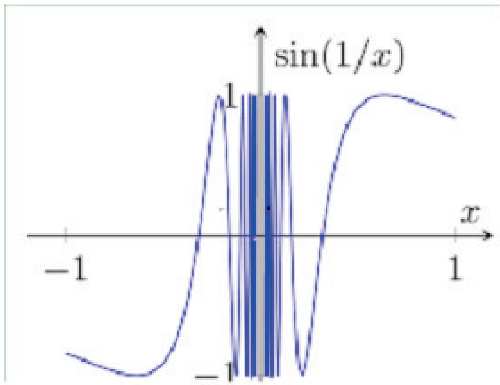
$$a - \delta < x < a \Longrightarrow |f(x) - M| < \varepsilon,$$

- and we write

$$\lim_{x \longrightarrow a^-} f(x) = M.$$

- **Example 16.** Find the right-hand and left-hand limits of the following functions at the given points.
- $(a)\ f(x) = \dfrac{x}{|x|}$ at $x = 0$,
- $(b)\ f(x) = \begin{cases} x^2 - 2x + 3, & \textit{if } x \geq 1 \\ -x^2 + 2x - 2, & \textit{if } x < 1 \end{cases}$

Example 17. Show that $y = \sin\left(\dfrac{1}{x}\right)$ has no limit as x approaches zero from either side.



2.4. One-Sided Limits

- **Corollary 1.** If f is a real function defined on an open interval $]b,c[$ and $a \in]b,c[$, then

$$\lim_{x \longrightarrow a^-} f(x) \text{ exists} \iff \lim_{x \longrightarrow a^+} f(x) \text{ and } \lim_{x \longrightarrow a^-} f(x)$$

both exists and equal.

- **2.** If we know that $\lim_{x \longrightarrow a} f(x) = L$, then

$$\lim_{x \longrightarrow a^+} f(x) = \lim_{x \longrightarrow a^-} f(x) = L.$$

- **3.** If we know that $\lim_{x \longrightarrow a^+} f(x) = \lim_{x \longrightarrow a^-} f(x) = L$ then

$$\lim_{x \longrightarrow a} f(x) = L.$$

2.4. One-Sided Limits

- **Example 18.** Prove that $\lim_{x \longrightarrow 0^+} \sqrt[n]{x} = 0$.

2.4. One-Sided Limits

Theorem (Limit of the Ratio $\sin\theta/\theta$ as $\theta \to 0$)

$$\lim_{\theta \longrightarrow 0} \frac{\sin\theta}{\theta} = 1, \ (\theta \text{ in radians}).$$

2.4. One-Sided Limits

- **Example 19.** Find the following limits
- (a) $\lim_{\theta \longrightarrow 0} \dfrac{\tan\theta}{\theta}$
- (b) $\lim_{\theta \longrightarrow 0} \theta \cot\theta$
- (c) $\lim_{x \longrightarrow 0} \dfrac{\sin ax}{\sin bx}$

2.4. One-Sided Limits

$(d)\ \lim_{x\longrightarrow 0} \dfrac{1-\cos 2x}{x^2}$

2.4. One-Sided Limits

Exercise 1. Evaluate the following limits.

$(a)\ \lim_{x\longrightarrow 1} \left(2x^3-5x+4\right)$ $(b)\ \lim_{u\longrightarrow 1} \dfrac{u^2-9}{u^3+u}$

$(c)\ \lim_{x\longrightarrow -1} \dfrac{x+1}{x^2-x-2}$ $(d)\ \lim_{x\longrightarrow 3^+} \dfrac{\lfloor x\rfloor-1}{\lfloor x\rfloor-2x}$

$(e)\ \lim_{x\longrightarrow 0^+} \dfrac{|x|}{x}$ $(f)\ \lim_{x\longrightarrow 0} \dfrac{\sqrt{x+4}-2}{x}$

$(g)\ \lim_{x\longrightarrow 3} \left(x-\lfloor x\rfloor\right)$ $(h)\ \lim_{x\longrightarrow 1} sgn\left(x^2-1\right)$

$(k)\ \lim_{x\longrightarrow \frac{1}{2}^+} \lfloor \arccos x\rfloor$ $(l)\ \lim_{x\longrightarrow 0} \dfrac{x\sin x}{x+\sin x}$

$(m)\ \lim_{x\longrightarrow 0^+} \dfrac{\sin x}{\sqrt{x}}$ $(n)\ \lim_{x\longrightarrow 0} \dfrac{1-\cos x}{\sin x}$

$(o)\ \lim_{x\longrightarrow 0} \dfrac{\sin x-\tan x}{x^3}$ $(p)\ \lim_{x\longrightarrow 0} x^2\sin\dfrac{1}{x}$

$(q)\ \lim_{x\longrightarrow a} \dfrac{\tan x-\tan a}{x-a}$ $(r)\ \lim_{x\longrightarrow \frac{\pi}{2}} \dfrac{\sin\left(\cos x\right)}{\cos x}$

2.5. Continuity

- **Definition** Let f be a function and c be an interior point of D_f. The function f is **continuous at** c if

$$\lim_{x \longrightarrow c} f(x) = f(c).$$

- The function f is right-continuous at c (or continuous from the right) if

$$\lim_{x \longrightarrow c^+} f(x) = f(c).$$

- The function f is left-continuous at c (or continuous from the left) if

$$\lim_{x \longrightarrow c^-} f(x) = f(c).$$

- Thus f is continuous at c if and only if it is both right continuous and left continuous at c.
- We say that a function is **continuous over a closed interval** $[a, b]$ if it is right-continuous at a, left-continuous at b and continuous at all interior points of $[a, b]$.

2.5. Continuity

- If a function is not continuous at a point c, we say that f is **discontinous** at c and c is a point of discontinuity. Intuitively, a discontinuity of f is a point where the graph of f has **a gap or jump of some sort.**
- **Example 20**. The function $f(x) = \sqrt{4 - x^2}$ is continuous over its domain.
- **Example 21.** The unit step function is right-continuous at $x = 0$, but neither left-continuous nor continuous there.

2.5. Continuity

Continuity Test
A function f is continuous at $x = a$ if it meets the following three conditions:

$$f \text{ is defined at } x = a$$
$$\lim_{x \longrightarrow a^-} f(x) \text{ exists}$$
$$\lim_{x \longrightarrow a^-} f(x) = f(a).$$

2.5. Continuity

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- **Example 22.** Identify the continuity of the following functions at $x = 0$.
- (a) $f(x) = \begin{cases} -x^2, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ x + 1, & \text{if } x > 0 \end{cases}$
- (b) $g(x) = \lfloor x \rfloor + 2x.$

2.5. Continuity

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- **Formal Definition** Let f be a function and c be an interior point of D_f. The function f is **continuous at** c if for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$|x - c| < \delta \Longrightarrow |f(x) - f(c)| < \varepsilon.$$

- **Example 23.** Show that the functions $f(x) = 2x + 3$ and $g(x) = \sin x$ are continuous on reals

2.5. Continuity

- **Example 24.** Are the following functions continuous or not?
- (a) $f(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q} \\ 1, & \text{if } x \notin \mathbb{Q} \end{cases}$

2.5. Continuity

$$(b)\ g(x) = \begin{cases} x, & if\ x \in \mathbb{Q} \\ 0, & if\ x \notin \mathbb{Q} \end{cases}$$

2.5. Continuity

- **Remark** For a real function f, if
- (1) $\lim_{x \longrightarrow a^-} f(x)$ exists but $\lim_{x \longrightarrow a^-} f(x) \neq f(a)$ or the function f is not defined at a, then we say that f has removable discontinuity at a.
- (2) the one sided limits exist but have different values, then we say that f has jump discontinuity at a.
- (3) One of the one sided limits is $-\infty$ or ∞, then we say that f has infinite discontinuity at a.

2.5. Continuity

- **Example 25.** Find the points where the given function is discontinuous.
- $(a)\ f(x) = \dfrac{x^2 - x - 2}{x - 2}$
- $(b)\ g(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2}, & \text{if } x \neq 2 \\ 1, & \text{if } x = 2 \end{cases}$

2.5. Continuity

- **Theorem** If f and g are continuous functions at $x = a$ and k is a constant then the following functions are also continuous at $x = a$:

$$f + g, f - g, kf, |f| \text{ and } f/g \text{ if } g(a) \neq 0.$$

- **Example 26.** (a) Every polynomial function is continuos.
- (b) Every rational function is continuous at $x = a$ if it is defined at $x = a$.

2.5. Continuity

- **Inverse Functions and Continuity** The inverse function of any function **continuous on an interval** is continuous over its domain.
- **Theorem** If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.
- **Example 27.** Show that the following functions are continuous on their natural domains.
- (a) $y = \sqrt{x^2 - 2x - 5}$
- (b) $y = \dfrac{x^{2/3}}{1 + x^4}$
- (c) $y = \sin\left(x^2\right)$

2.5. Continuity

Theorem (Limits of Continuous Functions) If g is continous at the point b and $\lim_{x \longrightarrow c} f(x) = b$, then

$$\lim_{x \longrightarrow c} g(f(x)) = g(b) = g\left(\lim_{x \longrightarrow c} f(x)\right).$$

- **Example 28.** Evaluate the following limits
- (a) $\lim_{x\longrightarrow\pi/2}\cos\left(2x+\sin\left(\frac{3\pi}{2}+x\right)\right)$
- (b) $\lim_{x\longrightarrow 0}\sqrt{x+1}e^{\tan x}$