

# 1.3. Trigonometric Functions

## Angles

An **angle** is defined as the union of two rays (initial side and terminal side) with a common endpoint, the vertex. An angle whose vertex is the center of a circle is a **central angle**, and the arc of the circle through which the terminal side moves is the **intercepted arc**.

## 1.3. Trigonometric Functions

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- Degree Measure: To measure an angle in degrees, we imagine the circumference of a circle divided into 360 equal parts and we call each of those “equal” parts **a degree**. The full circle then will be 360 degrees which is usually written as  $360^\circ$ . The measure of an angle, then, will be as many degrees as its sides include.

*Why are we using the number 360?*

## 1.3. Trigonometric Functions

- **Radian Measure:** An arc of a circle with the same length as the radius of that circle corresponds to an angle of 1 **radian**. A full circle than corresponds to an angle of  $2\pi$  radians. The magnitude in radians of an angle  $\theta$  is equal to the ratio of the arc length  $s$  to the radius  $r$  of the circle, that is;  $\theta = \frac{s}{r}$ .

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- **Converting from degrees to radians:**

<b>Degrees :</b>	$0^\circ$	$30^\circ$	$45^\circ$	$60^\circ$	$90^\circ$	$120^\circ$	$135^\circ$	$150^\circ$	$180^\circ$
<b>Radians :</b>	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\pi$

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- A negative angle  $-\theta$  is efficiently equal to an angle of **one full turn minus  $\theta$**

# 1.3. Trigonometric Functions

## The Sine and Cosine Functions

- The two basic trigonometric functions the sine and cosine are defined using a unit circle. An angle of  $\theta$  radians is measured counterclockwise around the circle from the point  $(1, 0)$  to the point  $P(x, y)$ . Then we define

$$x := \cos \theta \text{ and } y := \sin \theta.$$

# 1.3. Trigonometric Functions

## The Sine and Cosine Functions

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- The domains of sine and cosine are all reals. The graphs of sine and cosine oscillate between  $-1$  and  $1$ .

# 1.3. Trigonometric Functions

## The Tangent and Cotangent Function

- If  $\theta$  is any number with  $\cos \theta \neq 0$ , we define the tangent function as

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

and the domain is

$$D = \{\theta \mid \cos \theta \neq 0\} = \left\{ \theta \mid \theta \neq (2k+1)\frac{\pi}{2}, k \in \mathbb{Z} \right\}.$$

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- $\tan \theta$  is the **slope** of the line passing through the origin and the point  $P(\cos \theta, \sin \theta)$

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- If  $\sin \theta \neq 0$ , we define the cotangent function as

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

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- The other two basic trigonometric functions secant and cosecant are defined as

$$\sec \theta = \frac{1}{\cos \theta} \text{ and } \csc \theta = \frac{1}{\sin \theta},$$

respectively and clearly secant has the same domain with tangent and cosecant has the same domain with cotangent.

## 1.3. Trigonometric Functions

**Example 14.** Define the trigonometric functions of an acute triangle in terms of the sides of a right triangle. Try to find the values of  $\sin \theta$ ,  $\cos \theta$  and  $\tan \theta$  for  $\theta = \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$ .

## 1.3. Trigonometric Functions

**Example 15.** Consider a circle centered at the origin with radius  $r$  and a ray making an angle of  $\theta$  with the positive  $x$ -axis. If  $A$  is the intersection point of the terminal ray of  $\theta$  radians and the circle, find the coordinates of  $A$  with respect to  $\theta$  and  $r$ .

# 1.3. Trigonometric Functions

Values of  $\sin \theta$ ,  $\cos \theta$  and  $\tan \theta$  for selected values of  $\theta$  :

$\theta$ (radians) :	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\pi$
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\tan \theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0

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**Example 16.** Investigate the signs of basic trigonometric functions via a unit circle.

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- When an angle of measure  $\theta$  and an angle of measure  $\theta + 2\pi$  are in standard position, their terminal rays coincide. The two functions therefore have the same trigonometric values. This is the result of periodicity.
- **Definition** A function  $f$  is periodic if there is a positive number  $p$  such that

$$f(x + p) = f(x)$$

for every value of  $x$ . If such a number exists we call  $f$  **periodic** and the smallest such value of  $p$  is called **the period of  $f$** .

## 1.3. Trigonometric Functions

**Example 16.** Find the period of the function  $f(x) = x - \lfloor x \rfloor$  if it is periodic and sketch the graph of  $f$ .

## 1.3. Trigonometric Functions

**Example 17.** Find the periods of the six basic trigonometric function and sketch their graphs

# 1.3. Trigonometric Functions

## Trigonometric Identities

With the help of the fundamental identity  $\cos^2 \theta + \sin^2 \theta = 1$  we obtain

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 + \cot^2 \theta = \csc^2 \theta.$$

For angles  $\alpha$  and  $\beta$ ,

Addition Formulas :  $\cos(\alpha \mp \beta) = \cos \alpha \cos \beta \pm \sin \alpha \sin \beta$   
 $\sin(\alpha \mp \beta) = \sin \alpha \cos \beta \mp \cos \alpha \sin \beta$

# 1.3. Trigonometric Functions

## Trigonometric Identities

Double – Angle Formulas :

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$$
$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

Half – Angle Formulas :

$$\cos^2 \alpha = \frac{1 + \cos 2\theta}{2}$$
$$\sin^2 \alpha = \frac{1 - \cos 2\theta}{2}$$

## 1.3. Trigonometric Functions

### The Laws of Cosine

If  $a$ ,  $b$  and  $c$  are sides of a triangle  $ABC$  and if  $\theta$  is the angle opposite to  $c$ , then

$$c^2 = a^2 + b^2 - 2ab \cos \theta.$$

This equation is called the **law of cosines**.

**Example 18.** Give a proof for the law of cosine.

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- For an angle  $\theta$  measured in radians, the sine and cosine functions satisfy

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- **Reading Homework** Make an investigation about Graphing with Software.

# 1.3. Trigonometric Functions

## Exercises:

1. Find the domain of  $f(x) = \cos \sqrt{x}$ .
2. Investigate periodicity for the following functions;

$$f(x) = x + \sin x$$

$$g(x) = |\cos x|$$

$$h(x) = \sin x^2.$$

## 1.4. Exponential Functions

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- Exponential Behavior**
- When a positive quantity  $P$  doubles, it increases by a factor of 2 and the quantity becomes  $2P$ . If it doubles again, it becomes

$$2(2P) = 2^2 P,$$

and a third doubling gives

$$2(2^2 P) = 2^3 P.$$

Continuing to double in this way leads us to consider the function  $f(x) = 2^x$ . We will call this an exponential function.

## 1.4. Exponential Functions

- In general, if  $a \neq 1$  is a **positive constant**, the function

$$f(x) = a^x, a > 0$$

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- The domain of  $f$  is all reals and the range is positive real numbers.

## 1.4. Exponential Functions

If  $a > 0$  and  $b > 0$ , the following rules hold true for all real numbers  $x$  and  $y$ .

$$a^x a^y = a^{x+y}$$

$$\frac{a^x}{a^y} = a^{x-y}$$

$$(a^x)^y = (a^y)^x = a^{xy}$$

$$a^x b^x = (ab)^x$$

$$\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x.$$

## 1.4. Exponential Functions

**Example 19.** Sketch the graph of the exponential function  $f(x) = a^x$  ( $a \neq 0$  and  $a > 0$ ).

## 1.4. Exponential Functions

- The Natural Exponential Function

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- The Natural Exponential Function
- The most important exponential function used for modelling natural, physical and economic phenomena is the **natural exponential function** whose base is the special number  $e$ , that is

$$f(x) = e^x$$

The number  $e$  is irrational, and its value is 2.718281828 to nine decimal places.

## 1.4. Exponential Functions

- Exponential Growth and Decay

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- **Exponential Growth and Decay**

- The exponential functions  $y = e^{kx}$ , where  $k$  is a nonzero constant, are frequently used for modelling exponential growth or decay. The function

$$y = y_0 e^{kx}$$

is a model for **exponential growth** if  $k > 0$  and a model for **exponential decay** if  $k < 0$ . Here  $y_0$  represents a constant.

## 1.4. Exponential Functions

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## 1.4. Exponential Functions

- **Example 20.** Eliminating a disease: Suppose that in any given year the number of cases of a disease is reduced by 20%. If there are 10000 cases today, how many years will it take
  - (a) to reduce the number of cases to 1000?
  - (b) to eliminate the disease; that is, to reduce the number of the cases to less than 1?