CHAPTER 1. FUNCTIONS

1.5. Inverse Functions and Logarithms

One-to-One Functions

A function f is called one-to-one (injective) if f preserves distinctness; that is, it never maps distinct elements of its domain to the same element of its codomain. Formally;

$$\begin{aligned} f \text{ is one-to-one } &\Leftrightarrow \left(\forall a, b \in D_f \ni f(a) = f(b) \Longrightarrow a = b\right) \\ &\Leftrightarrow \left(\forall a, b \in D_f \ni a \neq b \Longrightarrow f(a) \neq f(b)\right). \end{aligned}$$

Some functions are one-to-one on their entire natural domains some not. But buy restricting a function to a smaller domain we can create one-to-one functions.

Example 21. Which of the following functions are one-to-one;

$$f(x) = x^2, g(x) = x^3, h(x) = \sqrt{x}$$

the exponential function and the six basic trigonometric functions.

Example 22. Show that, if f is strictly increasing, then f is one-to-one.

The Horizantal Line Test for One-to-One Functions: A function $y = f(x)$ is one-to-one if its graph intersects each horizontal line at most once.

Image and Preimage of a Subset: For a real function f; the **image of a subset** A of the domain D_f is the subset $f(A)$ defined by

$$f(A) = \{f(x) \mid x \in D_f\}$$

and, the **preimage of a subset** B of reals is the subset $f^{-1}(B)$ which is defined by

$$f^{-1}(B) = \{x \mid f(x) \in B\}.$$

Inverse Functions

Suppose that f is a one-to-one function on a domain D_f with range R_f (in this case f is also an **onto (surjective)** function). The inverse function f^{-1} is defined by

$$f^{-1}(b) = a \text{ if } f(a) = b.$$

The fact which makes this definition meaningful is that; each output of a one-to-one function comes from just one input. The domain f^{-1} is R_f and the range of f^{-1} is D_f. We read the symbol f^{-1} as "f inverse".

Composing a function with its inverse has the same effect as doing nothing.

$$\begin{aligned}\left(f^{-1}\circ f\right)(x) &= x \text{ for all } x \text{ in the domain of } f\\ \left(f\circ f^{-1}\right)(y) &= y \text{ for all } y \text{ in the domain of } f^{-1}.\end{aligned}$$

Finding Inverses

Th graph of f and f^{-1} are related. To graph f^{-1} by using the graph of f, consider the points (a,b) on the graph of f and interchange the order. The more usual way is to reflect the graph of f across the line $y=x$. Passing from f to f^{-1} :

1. Solve the equation $y=f(x)$ for x, and obtain a formula $x=f^{-1}(y)$.

2. Interchange x and y, obtaining a formula $y=f^{-1}(x)$ where f^{-1} is expressed in the usual format with x as the independent variable and y as the dependent variable.

Example 23. Find the inverse of the following functions and express them as a function of x.

$$\begin{aligned}f(x) &= \frac{1}{3}x+2, x\in\mathbb{R}\\ g(x) &= x^2, x\geq 0\\ f(x) &= x^3+1, x\in\mathbb{R}.\end{aligned}$$

Logarithmic Function

Definition Let a be a positive real number other than 1. The **logarithm function with base** a, $y=\log_a x$, is the inverse of the base a exponential function $y=a^x$ $(a>0, a\neq 1)$.

The domain of $\log_a x$ is $]0,\infty[$, the range of a^x. The range of $\log_a x$ is $]-\infty,\infty[$, the domain of a^x.

Example 24. Sketch the graph of $y=\log_a x$.

Logarithms with base 2 are commonly used in compurter science. Logarithms with base e and base 10 are so important in applications and they have their own special notations and names:

$\log_e x$ is written as $\ln x$ and called the **natural logarithm function**
$\log_{10} x$ is written as $\log x$ and called the **common logarithm function**.

For the natural logarithm

$$\ln x = y \Leftrightarrow e^y = x.$$

In particular, if we set $x=e$, we obtain $\ln e = 1$.

Properties of Logarithms

Algebraic Poroperties of the Natural Logarithm: For any numbers $a > 0$ and $b > 0$, the natural logarithm satisfies the following rules:

(a) $\ln ab = \ln a + \ln b$

(b) $\ln \dfrac{a}{b} = \ln a - \ln b$

(c) $\ln \dfrac{1}{b} = -\ln b$

(d) $\ln a^b = b \ln a$

Inverse Properties for a^x and $\log_a x$:

(a) $a^{\log_a x} = x, \log_a a^x = x, a > 0, a \neq 1, x > 0$

(b) $e^{\ln x} = x, \ln e^x = x, x > 0.$

Change of Base Formula: Every logarithmic function is a constant multiple of the natural logarithm;

$$\log_a x = \frac{\ln x}{\ln a} \ (a > 0, a \neq 1).$$

Example 25. If \$1000 is invested in an account that earns 5.25% interest compounded annually, how long will it take the account to reach \$2500?

Inverse Trigonometric Functions

The six basic trigonometric functions are not one-to-one. However we can restrict their domains to intervals on which they are one-to-one. While doing that we always want to keep $\left[0.\frac{\pi}{2}\right]$ in the domain because most useful angles are acute.

1. The Arcsine and Arccosine Functions

By restricting the domain of the sine function to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we obtain a one-to-one function

$$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$$

which is invertible. We denote the inverse as

$$\sin^{-1} = \arcsin : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

and

$$y = \arcsin x \ \textit{is the number in} \ \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \ \textit{for which} \ \sin y = x.$$

By restricting the domain of the cosine function to the interval $[0, \pi]$, we obtain a one-to-one function

$$\cos : [0, \pi] \longrightarrow [-1, 1]$$

which is invertible. We denote the inverse as

$$\cos^{-1} = \arccos : [-1, 1] \longrightarrow [0, \pi],$$

and

$$y = \arccos x \textit{ is the number in } [0, \pi] \textit{ for which } \cos y = x.$$

Example 26. Sketch the graphs of the $y = \sin^{-1} x$ and $y = \cos^{-1} x$.

Example 27. (a) Evaluate $\sin^{-1}\left(-\frac{1}{2}\right)$ and $\cos^{-1}\left(\frac{1}{2}\right)$.

(b) If $arcsin 1 = x$, find x.

(c) If $\arcsin x = \frac{\pi}{3}$, find x.

2. The Arctangent and Arccotangent Functions

By restricting the domain of the tangent function to the interval $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, we obtain a one-to-one function

$$\tan : \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\longrightarrow \mathbb{R}$$

which is invertible. We denote the inverse as

$$\tan^{-1} = \arctan : \mathbb{R} \longrightarrow \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[,$$

and

$$y = \arctan x \textit{ is the number in } \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\textit{ for which } \tan y = x.$$

By restricting the domain of the cotangent function to the interval $]0, \pi[$, we obtain a one-to-one function

$$\cot :]0, \pi[\longrightarrow \mathbb{R}$$

which is invertible. We denote the inverse as

$$\cot^{-1} = \operatorname{arccot} : \mathbb{R} \longrightarrow]0, \pi[,$$

and

$$y = \operatorname{arccot} x \textit{ is the number in }]0, \pi[\textit{ for which } \cot y = x.$$

Example 28. Find $arctan\left(-\sqrt{3}\right)$, $\arctan 1$ and $\operatorname{arccot}\sqrt{3}$.

Exercises:

1. Find the domains of the following functions;

$$\begin{array}{ll} (a)\ \ y=\arcsin\frac{x}{3}, & (b)\ \ y=\arccos 3x \\ (c)\ \ y=\arctan\frac{x-1}{x} & (d)\ \ y=\arctan\sqrt{x-1} \end{array}$$

2. Find the domains of the following functions;

$$\begin{array}{ll} (a)\ \ f(x)=\ln\left(x^2-9\right) & (b)\ \ f(x)=\ln\left(\sqrt{x-4}+\sqrt{6-x}\right) \\ (c)\ \ f(x)=\ln\left(\sin\pi x\right) & (d)\ \ f(x)=\arcsin\left(\ln x\right) \\ (e)\ \ f(x)=\ln\left(\ln\left(1+x^2\right)\right) & (f)\ \ f(x)=\arcsin\left(\log\frac{x}{10}\right) \\ (g)\ \ f(x)=\log_2\left(\log_3\left(\log_4 x\right)\right) & (h)\ \ f(x)=\log\left(1-\log\left(x^2-5x+16\right)\right) \end{array}$$

3. Show that the function $f(x)=\ln\left(x+\sqrt{1+x^2}\right)$ is odd.

4. Using the graph of the curve $y=\sin x$, sketch the graphs of

$$y=|\sin x|\,,\ y=\sin|x|\,,\ y=\sin\left(x-\tfrac{\pi}{4}\right),$$
$$y=\sin 2x \text{ and } y=1+\sin 2x.$$

5. Prove the following relations;

$$\begin{aligned} \sin\left(\arctan x\right) &= \frac{x}{\sqrt{1+x^2}} \\ \cos\left(\arctan x\right) &= \frac{1}{\sqrt{1+x^2}} \\ \tan\left(\arccos x\right) &= \frac{\sqrt{1-x^2}}{x} \\ \arcsin\left(\cos x\right) &= \frac{\pi}{2}-x \end{aligned}$$

Referencess:

G.B. Thomas Jr., M.D. Weir, J. Heil and A. Behn, Thomas' CALCULUS Early Transcendentals Thirteenth Edition in SI Units. Pearson Educational Limited 2016