CHAPTER 3. DERIVATIVES

3.1. Tangents and the Derivative at a Points

- **Definition** The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \longrightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \text{ (provided the limit exists).}$$

- The tangent line to the curve at P is the line through P with this slope.

3.1. Tangents and the Derivative at a Points

- **Example 1.** (a) Find the slope of the curve $y = \frac{1}{x}$ at any point $x = a \neq 0$. What is the slope at $x = -1$?
- (b) Where does the slope equal $-1/4$?
- (c) What happens to the tangent to the curve at the point $\left(a, \frac{1}{a}\right)$ as a changes?

3.1. Tangents and the Derivative at a Points

- Rates of Change: Derivative at a Point
- The expression

$$\frac{f(x_0+h)-f(x_0)}{h}, \ h \neq 0$$

  is called **the difference quotient of** f **at** x_0 **with increment** h. If the difference quotient has a limit as h approaches zero, that is given a special name and notation.
- **Definition** The **derivative of a function** f at a point x_0, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \longrightarrow 0} \frac{f(x_0+h)-f(x_0)}{h}$$

  provided this limit exists.

3.1. Tangents and the Derivative at a Points

- If we interpret the difference quotient as the slope of a secant line, then the derivative gives the slope of the curve $y = f(x)$ at the point $P(x_0, f(x_0))$. For example; let $f(x) = mx + b$, the derivative of f at any point x_0 is simply the slope of the line, so $f'(x_0) = m$.
- If we interpret the difference quotient as an average rate of change (as we did in Chapter 2, Part 2.1.), the derivative gives the function's instantaneous rate of change with respect to x at the point $x = x_0$.
- **Example 2.** For a freely falling rock from rest near the surface of the earth, we know that the rock fell $y = 4.9t^2$ meters during the first t seconds. What is the rock's exact speed at $t = 1$?

3.1. Tangents and the Derivative at a Points

- **Summary** The following are all interpretations for the limit of the difference quotient,

$$\lim_{h \longrightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

- (1) The slope of the graph of $y = f(x)$ at $x = x_0$
- (2) The slope of the tangent to the curve $y = f(x)$ at $x = x_0$
- (3) The rate of change of $f(x)$ with respect to x at $x = x_0$
- (4) The derivative $f'(x_0)$ at a point

3.1. Tangents and the Derivative at a Points

- **Example 3. (Vertical Tangents)** Show that the graph of $y = x^{1/3}$ has a vertical tangent at origin but $y = x^{2/3}$ does not.

3.2. The Derivative as a Function

- We will investigate the derivative as a **function** derived from f by considering the limit of the difference quotient at each point x in the domain of f.
- **Definition** The derivative of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \longrightarrow 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists.

3.2. The Derivative as a Function

The domain of f' is the set of points in the domain of f for which the limit exists. This means that the domain may be the same as or smaller than the domain of f. If f' exists at a particular x, we say that f is **differentiable (has a derivative) at** x. If f' exists at every point in the domain of f, we call f **differentiable**.

3.2. The Derivative as a Function

If we write $z = x + h$, then $h = z - x$ and h approaches 0 if and only if z approaches x (obtain this equivalent definition through a graph). This equivalent formula is sometimes more convenient to use, that is

$$f'(x) = \lim_{z \longrightarrow x} \frac{f(z) - f(x)}{z - x}.$$

3.2. The Derivative as a Function

- **Calculating Derivatives from the Definition** The process of calculating a derivative is called differentiation. To emphasize the idea that differentiation is an operation performed on a function $y = f(x)$, we use the notation
$$\frac{d}{dx} f(x)$$
as another way to denote the derivative $f'(x)$.
- There are many ways to denote the derivative of a function $y = f(x)$, where the independent variable is x and the dependent variable is y:
$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = D(f)(x) = D_x f(x).$$

3.2. The Derivative as a Function

- Here the symbols $\frac{d}{dx}$ and D indicate the **operation of differentiation**.
- We read dy/dx as "**the derivative of** y **with respect to** x".
- The **prime** notations y' and f' comes from notations that Newton used for derivatives. The d/dx notations are similar to those used by Leibniz. To indicate the value of a derivative at a specified number $x = a$, we use the notation
$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx} f(x)\right|_{x=a}$$

3.2. The Derivative as a Function

- **Example 4.** Differentiate the following functions
- (a) $f(x) = \dfrac{1}{x}$, $x \neq 0$
- (b) $g(x) = \dfrac{x}{x-1}$, $x \neq 1$
- (c) $h(x) = \sqrt{x}$ for $x > 0$ and find the tangent line to the curve at $x = 4$.
- (d) $f(x) = x^3 + 2x$ and evaluate $f'(1)$.

3.2. The Derivative as a Function

- **Graphing the Derivative** We can often make a reasonable plot of the derivative of $y = f(x)$ by estimating the slopes on the graph of f. That is, we plot the points $(x, f'(x))$ in the xy-plane and connect them with a smooth curve, which represents $y = f'(x)$.
- **What can we learn from the graph of** $y = f'(x)$? We can see
- (a) where the rate of change of f is positive, negative or zero,
- (b) the rough size of the growth rate at any x and its size in relation to the size of $f(x)$,
- (c) where the rate of change itself is increasing or decreasing.

3.2. The Derivative as a Function

- **Differentiable on an Interval; One-Sided Derivatives**
- A function $y = f(x)$ is differentiable on an open interval (finite or infinite) if it has a derivative at each point of the interval. It is differentiable on a closed interval $[a, b]$ if it is differentiable on the interval $]a, b[$ and if the limits

$$\lim_{h \longrightarrow 0^+} \frac{f(x+h) - f(x)}{h} \text{ right-hand derivative at } a$$

$$\lim_{h \longrightarrow 0^-} \frac{f(x+h) - f(x)}{h} \text{ left-hand derivative at } a$$

exist at the endpoints.

3.2. The Derivative as a Function

- **Example 5.** (a) Show that the function $f(x) = |x|$ is differentiable on $\mathbb{R} - \{0\}$.
- (b) Does $f(x) = \sqrt{x}$ have a derivative at $x = 0$?

3.2. The Derivative as a Function

- When does a function not have a derivative at a point?
- A function can fail to have a derivative at a point for many reasons, including the existence of the points where the graph has:
- (a) a **corner**; where the one-sided derivatives differ.
- (b) a **cusp**; where the slope of the secant line approaches ∞ from one side and $-\infty$ from the other.
- (c) a **vertical tangent**; where the slope of the secant line approaches ∞ from both sides or approaches $-\infty$ from both sides.
- (d) a **discontinuity**.
- In addition, when the functions slope near the point is oscillating rapidly, as $f(x) = \sin(1/x)$ near zero, where it is discontinuous.

3.2. The Derivative as a Function

- **Theorem 1. (Differentiability Implies Continuity)** If f has a derivative at $x = c$, then f is continuous at $x = c$.
- **Example 6.** The converse of Theorem 1 is false. Give a counter example.

3.3. Differentiation Rules

- **Derivative of a Constant Function** If f has the constant value $f(x) = c$, then
$$\frac{df}{dx} = \frac{d}{dx}(c) = 0$$
- **Derivative of a Positive Integer Power** If n is a positive integer, then
$$\frac{d}{dx}x^n = nx^{n-1}.$$
- **Derivative of a General Power** If n is any real number, then
$$\frac{d}{dx}x^n = nx^{n-1}$$
for all x where the powers x^n and x^{n-1} are defined.

3.3. Differentiation Rules

- **Derivative Constant Multiple Rule** If u is a differentiable function of x, and c is a constant, then
$$\frac{d}{dx}(cu) = c\frac{du}{dx},$$
in particular, if n is any real number, then
$$\frac{d}{dx}(cx^n) = cnx^{n-1}.$$
- **Derivative Sum Rule** If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,
$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}.$$

3.3. Differentiation Rules

- **Example 7**. Differentiate the following functions;
- $(a)\ f(x) = x^3$
- $(b)\ f(x) = x^{\sqrt{3}}$
- $(c)\ f(x) = x^{-1/2}$
- $(d)\ f(x) = \sqrt{x^{\sqrt{2}}}$
- $(e)\ f(x) = \sqrt{3}x^3$
- $(f)\ f(x) = x^3 + \frac{2}{3}x^2 - 4x + 5$
- (g) Does the curve $y = x^4 - 2x^2 + 2$ have any horizantal tangents? If so, where?

3.3. Differentiation Rules

- **Derivatives of Exponential Functions** If we apply the definition of the derivative to $f(x) = a^x$, we get

$$\frac{d}{dx}(a^x) = (\ln a)\, a^x$$

and in particular

$$\frac{d}{dx}(e^x) = e^x.$$

- **Example 8.** Find an equation for a line that is tangent to the graph of $y = e^x$ and goes through the origin

3.3. Differentiation Rules

- **Products and Quotients** The derivative of the product of two functions is **not** the product of their derivatives. Clearly

$$\frac{d}{dx}(x.x) = \frac{d}{dx}\left(x^2\right) = 2x, \text{ while } \frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1.$$

- If u and v are differentiable at x, then so is their **product** uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

- If u and v are differentiable at x and if $v(x) \neq 0$, then the **quotient** u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

3.3. Differentiation Rules

- **Example 9.** Find the derivative of
- $(a)\ y = \frac{1}{x}\left(x^2 + e^x\right)$
- $(b)\ y = e^{2x}$
- $(c)\ y = x^2\left(3^x + x^{1/2}\right)$
- $(d)\ y = \frac{x^2+1}{x^3+5}$
- $(e)\ y = \frac{t^2-1}{t^3+1}$
- $(f)\ y = e^{-t}\left(\frac{t^2}{t^3+1}\right)$
- $(g)\ y = \frac{(1-x)\left(x^2-2x\right)}{x^4}$

3.3. Differentiation Rules

- **Second- and Higher-Order Derivatives**
- If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function.
- If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f''. So $f'' = (f')'$. The function f'' is called the **second derivative** of f because it is the derivative of the first derivative. It is written in several ways:

$$f'(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y' = D^2(f)(x) = D_x^2 f(x).$$

3.3. Differentiation Rules

- If y'' is differentiable, its derivative, $y''' = dy''/dx = d^3y/dx^3$, is the **third derivative** of y with respect to x. The names continue as you imagine, with

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^ny}{dx^n} = D^ny$$

denoting the *nth* **derivative** of y with respect to x for any positive integer n.
- **Example 10.** Find the first four derivatives of $y = x^3 - 3x^2 + 2$.

3.4. The Derivative as a Rate of Change

- If we interpret the difference quotient

$$\frac{f(x+h) - f(x)}{h}$$

as the average rate of change in f over the interval from x to $x+h$, we can interpret its limit as $h \longrightarrow 0$ as the rate at which f is changing at the point x.

- **Definition** The **instantaneous rate of change** of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{h \longrightarrow 0} \frac{f(x_0+h) - f(x_0)}{h},$$

provided the limit exists. Instantaneous rates are limits of average rates.

3.4. The Derivative as a Rate of Change

- Using the word isntantaneous may lead you to think that we are considering the change with respect to time, but for any other independent variable we also use it to understand the change with respect to the instantaneous increment given to that variable. Usually we omit the word, and say "**rate of change**" instead of **instantaneous rate of change**.
- **Example 11.** The area A of a circle is related to its diameter by the equation

$$A = \frac{\pi}{4} D^2.$$

How fast does the area change with respect to the diameter when the diameter is $10m$?

3.4. The Derivative as a Rate of Change

- **Motion Along a Line**
- Suppose that an object is moving along a coordinate line (an s-axis), usually horizontal or vertical, so its position s on that line is a function of time t, that is;

$$s = f(t).$$

- The displacement of the object over the time interval from t to $t + \Delta t$ is

$$\Delta s = f(t + \Delta t) - f(t),$$

and the average velocity of the object over that time interval is

$$v_{av} = \frac{displacement}{travel\ time} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

- To find the object's velocity at the exact instant t, we take the limit of the average velocity over the interval from t to $t + \Delta t$ as Δt approaches to zero. This limit is the derivative of f with respect to t.

3.4. The Derivative as a Rate of Change

- **Definition Velocity** is the derivative of position with respect to time. If an object's position at time t is $s = f(t)$, then the object's velocity at time t is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \longrightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

- **Definition Speed** is the absolute value of velocity,

$$Speed = |v(t)| = \left| \frac{ds}{dt} \right|.$$

- **Homework** *What is difference between velocity and speed?*

3.4. The Derivative as a Rate of Change

- The rate at which an object's velocity changes is the object's **acceleration**. The acceleration measures how quickly the object picks up or loses speed. A sudden change in acceleration is called a **jerk**. When a ride in a car is jerky, it is not that the accelerations involved are necessarily large but that the changes in acceleration are abrupt.
- <u>**Definition**</u> **Acceleration** is the derivative of velocity with respect to time. If an object's position at time t is $s = f(t)$, then the object's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Jerk is the derivative of acceleration with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

3.4. The Derivative as a Rate of Change

- **Example 12.** A dynamite blast blows a heavy rock straight up with a launch velocity of 49 $m/s(176.4\ km/h)$. It reaches a height of $s = 49t - 4.9t^2$ m after t seconds.
- (a) How high does the rock go?
- (b) What are the velocity and speed of the rock when it is 78.4 m above the ground on the way up? On the way down?
- (c) What is the acceleration of the rock at any time t during its flight (after the blast)?
- (d) When does the hit the ground again?
- **Homework** Write a short essay about derivatives in economics, explain the marjinal cost of production.

3.4. The Derivative as a Rate of Change

- **Sensitivity to Change**
- When a small change in x produces a large change in the value of a function $f(x)$, we say that the function is relatively **sensitive** to changes in x. The derivative $f'(x)$ is a measure of this sensitivity.
- **Example 13.** If p $(0 < p < 1)$ is frequency of the gene for smooth skin in peas (dominant) and $(1 - p)$ is the frequency of the gene for wrinkled skin in peas, then the proportion of smooth-skinned peas in the next generation will be

$$y = 2p(1 - p).$$

Investigate the sensitivity to changes in p.