CHAPTER 3. DERIVATIVES

3.5. Derivatives of Trigonometric Functions

- **Derivative of the Sine and Cosine Function**
- To calculate the derivative of $f(x) = \sin x$ and $g(x) = \cos x$, for x measured in radians, consider the angle sum identities

$$\begin{aligned}\sin(x+h) &= \sin x \cos h + \cos x \sin h, \\ \cos(x+h) &= \cos x \cos h - \sin x \sin h.\end{aligned}$$

- Then we can obtain that

$$\frac{d}{dx}(\sin x) = \cos x \text{ and } \frac{d}{dx}(\cos x) = -\sin x.$$

3.5. Derivatives of Trigonometric Functions

Prove that $\frac{d}{dx}(\sin x) = \cos x$.

3.5. Derivatives of Trigonometric Functions

- **Derivatives of the Other Basic Trigonometric Functions** Because $\sin x$ and $\cos x$ are differentiable functions of f the derivative of the related functions are as follows:
- $$\frac{d}{dx}(\tan x) = \sec^2 x, \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$
  $$\frac{d}{dx}(\sec x) = \sec x \tan x, \quad \frac{d}{dx}(\csc x) = -\csc x \cot x.$$

3.5. Derivatives of Trigonometric Functions

- **Example 15.** Find the derivatives of the following functions;
- $(a)\ y = e^x \sin x$
- $(b)\ y = \dfrac{\sin x}{x}$
- $(c)\ y = \dfrac{\cos x}{1 - \sin x}$
- $(d)\ y = t^2 - \sec t + 5e^t$
- $(e)\ p = \dfrac{\sin q + \cos q}{\cos q}$

3.5. Derivatives of Trigonometric Functions

- **Example 16.** Find the following limits
- $(a) \lim_{x \longrightarrow 2} \sin\left(\dfrac{1}{x} - \dfrac{1}{2}\right)$
- $(b) \lim_{\theta \longrightarrow \pi/4} \dfrac{\tan\theta - 1}{\theta - \frac{\pi}{4}}$

3.6. The Chain Rule

- How do we differentiate the composite $f \circ g$ of two functions $y = f(u)$ and $u = g(x)$; for example consider $F(x) = \sin(x^2 - 4)$?
- **Derivative of Composite Function** Think the derivative as a rate of change, and consider the simple function $y = \frac{3}{2}x$ as the composite of two functions; i.e.

$$f(u) = \frac{1}{2}u \text{ and } u = g(x) = 3x.$$

- Thus if $y = f(u)$ changes half as fast as u and $u = g(x)$ changes 3 times as fast as x. Then we expect y to change $3/2$ times as fast as x. This corresponds to formulate the derivative

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

3.6. The Chain Rule

- **Theorem 2.** (The Chain Rule) If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

- In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/dx is evaluated at $u = g(x)$.

3.6. The Chain Rule

Example 17. An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of t.

3.6. The Chain Rule

- **Outside-Inside Rule** To apply the Chain Rule with the Leibniz notation sometimes causes difficulty. In this cases it is useful to use the functional notation. If $y = f(g(x))$, then

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x).$$

- In words, differentiate the "**outside**" function f and evaluate it at the "**inside**" function $g(x)$ left alone; then multiply by the derivative of the "**inside function**".

3.6. The Chain Rule

- **Example 18.** Differentiate the following with respect to x
- (a) $y = \sin\left(x^2 + e^x\right)$
- (b) $y = e^{\cos x}$

3.6. The Chain Rule

- **Example 19.** Differentiate the following with respect to x

$$\begin{array}{ll} (a)\, y = \sin f\left(x\right) & (d)\, y = \left(\tan\sqrt{x}\right)^5 \\ (b)\, y = \tan g\left(x\right) & (e)\, y = \tan^3 x + \sin\left(3x^2\right) \\ (c)\, y = \sin 3x + \cos\left(x^2 + 1\right) & (f)\, y = \left(\sec 6x\right)^{3/2} \end{array}$$

3.6. The Chain Rule

$(g)\ y=(f(x))^3$

$(h)\ y=u^n$ where $u=\cos x$

$(i)\ y=e^{\sqrt{3x+1}}$

$(j)\ y=\dfrac{1}{(1-2x)^3}$

$(l)\ y=|x|$ whenever $x\neq 0$

$(l)\ y=te^{-t}+e^{t^2}$

$(m)\ y=\sqrt{1+\cos t^2}$

$(n)\ y=\left(t^{-3/4}\sin t\right)^{4/3}$

$(o)\ y=\sqrt{3t+\sqrt{2\sqrt{1-t}}}$

3.7. Implicit Differentiation

- Most of the functions we have dealt have been described by an equation of the form $y=f(x)$. Another situation occurs when we encounter equations like
$$x^3+y^3-9xy=0 \text{ or } y^2-x=0.$$
- These equations define an implicit relation between the variables x and y. In these case we may not put an equation $F(x,y)=0$ in the from $y=f(x)$ to differentiate it in the usual way.
- **Example 20.** Find dy/dx if $y^2=x$.
- **Example 21.** Find the slope of the circle $x^2+y^2=25$ at the point $(3,-4)$

3.7. Implicit Differentiation

- **Implicit Differentiation**
- (a) Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
- (b) Collect the terms with dy/dx on one side of the equation and solve for dy/dx.
- **Example 22.** (a) Find dy/dx for the following

$$y^2 = x^2 + \sin xy, \quad e^{x^2 y} = 2x + \tan(xy)$$
$$y \cos \frac{1}{y} = 1 - xy, \quad \cos r + \cot \theta = e^{r\theta}$$

3.7. Implicit Differentiation

- (b) Find $\dfrac{d^2y}{dx^2}$ if $2x^3 - 3y^2 = 8$.
- (c) Show that $(2,4)$ lies on the curve $x^3 + y^3 - 9xy = 0$. Then find the tangent and normal to the curve at this point.

3.8.Derivatives of Inverse Functions and Logarithms

- **Derivatives of Inverses of Differentiable Functions** The inverse of $f(x) = \frac{1}{2}x + 1$ is clearly $f^{-1}(x) = 2x - 2$. If we calculate their derivatives, we see that

$$\frac{d}{dx}f(x) = \frac{1}{2} \text{ and } \frac{d}{dx}f^{-1}(x) = 2.$$

- The derivatives are reciprocals of one another. This is not a special case. Reflecting any nonhorizantal and nonvertical line across the line $y = x$ always inverts the line's slope.
- If the original line has slope $m \neq 0$, the reflected line has slope $1/m$.

3.8.Derivatives of Inverse Functions and Logarithms

- The reciprocal relationship between the slopes of f and f^{-1} holds for other functions as well. But we must be careful to compare slopes at corresponding points; i.e. if the slope of

$$y = f(x) \text{ at } (a, f(a)) \text{ is } f'(a) \text{ and } f'(a) \neq 0,$$

- then the slope of

$$y = f^{-1}(x) \text{ at } (f(a), a) \text{ is the reciprocal } \frac{1}{f'(a)}.$$

- If we set $b = f(a)$, then

$$\left(f^{-1}\right)'(b) = \frac{1}{f'(a)} = \frac{1}{f'\left(f^{-1}(b)\right)}.$$

- If $y = f(x)$ has a horizontal tangent line at $(a, f(a))$, then the inverse function f^{-1} has a vertical tangent line at $(f(a), a)$, and so f^{-1} is not differentiable at $f(a)$.

3.8.Derivatives of Inverse Functions and Logarithms

- **Theorem 3.** (The Derivative Rule For Inverses) If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (the range of f). The value of $\left(f^{-1}\right)'$ at a point b in $D_{f^{-1}}$ is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$\left(f^{-1}\right)'(b) = \frac{1}{f'\left(f^{-1}(b)\right)}$$

or

$$\frac{df^{-1}}{dx}\Big|_{x=b} = \frac{1}{\frac{df}{dx}\Big|_{x=f^{-1}(b)}}.$$

- In addition, if we start with $f\left(f^{-1}(x)\right) = x$ and differentiate both sides with respect to x, then clearly $\frac{d}{dx}f^{-1}(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$

3.8.Derivatives of Inverse Functions and Logarithms

- **Example 23.** (a) Compare the derivative of $f(x) = x^2, x > 0$ and its inverse $f^{-1}(x) = \sqrt{x}$. Obtain the derivative of the inverse by using the last formula given above.
- (b) Let $f(x) = x^3 - 2$, $x > 0$. Find the value of $\dfrac{df^{-1}}{dx}$ at $x = 6$.

3.8.Derivatives of Inverse Functions and Logarithms

- **Derivative of the Natural Logarithm Function** Since the exponential function $f(x) = e^x$ is differentiable everywhere, we can apply Theorem 3 to find the derivative of its inverse $f^{-1}(x) = \ln x$.

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, x > 0.$$

- The chain rule extends this formula to positive functions $u(x)$:

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}, u > 0.$$

3.8.Derivatives of Inverse Functions and Logarithms

- **Example 24.** (a) Find the derivatives of $y = \ln 2x$, $y = \ln\left(x^2 + 3\right)$ and $y = \ln|x|$.
- (b) A line with slope m passes through the origin and is tangent to the graph of $y = \ln x$. What is the value of m?

3.8.Derivatives of Inverse Functions and Logarithms

- **The Derivative of a^u and $\log_a u$** If $a > 0$, then a^x is differentiable and
$$\frac{d}{dx}a^x = a^x \ln a.$$
- If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and
$$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}.$$

3.8.Derivatives of Inverse Functions and Logarithms

- For $a > 0$ and $a \neq 1$,
$$\frac{d}{dx}\log_a u = \frac{1}{u \ln a}\frac{du}{dx}.$$
- **Example 25.** Find the derivatives of $y = 3^x$, $y = 3^{-x}$ and $y = 3^{\sin x}$, $y = \log_2\left(x^4 + x^2\right)$, $y = \log\left(e^{\cos x}\right)$.

3.8.Derivatives of Inverse Functions and Logarithms

- **Logarithmic Differentiation** To find the derivatives of functions defined as $y = [f(x)]^{g(x)}$ we have to take the logarithm of the equality. Then we obtain

$$\ln y = g(x) \ln f(x).$$

If we evaluate the derivative of both side, then we obtain dy/dx. This process is called logarithmic differentiation. We may also use this method to evaluate the derivatives of functions given by formulas that involve products and quotients.

- **Irrational Exponents and the Power Rule (General Version)** For any $x > 0$ and for any real number n, $x^n = e^{n \ln x}$. Therefore

$$\frac{d}{dx}x^n = nx^{n-1}.$$

If $x \leq 0$, then the formula holds whenever the derivative, x^n and x^{n-1} all exists.

3.8.Derivatives of Inverse Functions and Logarithms

- **Example 26.** Find dy/dx if
- (a) $f(x) = x^{\sin x}$, $x > 0$
- (b) $g(x) = \left(1 + x^2\right)^x$
- (c) $h(x) = \dfrac{\left(x^2+1\right)(x+3)^{1/2}}{x-1}, x > 1$
- (d) $f(x) = x^x$ (do not use logarithmic differentiation)

3.8.Derivatives of Inverse Functions and Logarithms

- **The Number e Expressed as a Limit** The number e can be defined as the base value for which the exponential function $y = a^x$ has slope 1 when it crosses the y-axis at $(0, 1)$. Thus e is the constant that satisfies the equation
$$\lim_{h \longrightarrow 0} \frac{e^h - 1}{h} = \ln e = 1.$$
- **Theorem 4** The number e can be calculated as the limit
$$e = \lim_{x \longrightarrow 0} (1 + x)^{1/x} .$$

3.9. Inverse Trigonometric Functions

Derivatives of Inverse Trigonometric Functions The inverse function $y = \sin^{-1} x$ is differentiable through the interval $-1 < x < 1$ and not differentiable at $x = -1$ and $x = 1$ because the tangents to the graph are vertical at these points.

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1 - x^2}}, \quad |x| < 1.$$

3.9. Inverse Trigonometric Functions

- If u is a differentiable function of x with $|u| < 1$, we apply the Chain Rule to get the general formula

$$\frac{d}{dx}\left(\sin^{-1} u\right) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad |u| < 1.$$

- If u is a differentiable function of x, then

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2} \text{ and } \frac{d}{dx}\left(\tan^{-1} u\right) = \frac{1}{1+u^2}\frac{du}{dx}$$

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$$\begin{aligned}\frac{d}{dx}\left(\sec^{-1} x\right) &= \frac{1}{|x|\sqrt{x^2-1}}, |x| > 1\\ \frac{d}{dx}\left(\sec^{-1} u\right) &= \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, |u| > 1\end{aligned}$$

3.9. Inverse Trigonometric Functions

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$$\begin{aligned}\frac{d}{dx}\left(\cos^{-1} x\right) &= -\frac{1}{\sqrt{1-x^2}}, \quad |x| < 1\\ \frac{d}{dx}\left(\cos^{-1} u\right) &= -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad |u| < 1\end{aligned}$$

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$$\begin{aligned}\frac{d}{dx}\left(\cot^{-1} u\right) &= -\frac{1}{1+u^2}\frac{du}{dx}\\ \frac{d}{dx}\left(\csc^{-1} u\right) &= -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, |u| > 1\end{aligned}$$

3.9. Inverse Trigonometric Functions

- **Example 27.** Find the derivatives of
- (a) $y = \operatorname{arcsec} e^x$
- (b) $y = \tan^{-1}\dfrac{1}{x}$
- (c) $y = x\cos^{-1}x - \sqrt{1-x^2}$
- (d) $y = x\sqrt{a^2-x^2} + a^2 \arcsin\dfrac{x}{a}$

3.10. Related Rates

In this section we look at problems that ask for the rate at which some variable changes when it is known how the rate of some other related variable changes. The problem of finding a rate of change from other known rates of change is called a related rates problem.

3.10. Related Rates

- **Related Rates Equations** Suppose we are pumping air into a spherical balloon. Both the volume and radius of the balloon are increasing over time. If V is the volume and r is the radius of the balloon at an instant time, then

$$V = \frac{4}{3}\pi r^3.$$

- Using the Chain Rule, we differentiate both sides with respect to t to find an equation relating the rate of changes of V and r,

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

- So, if we know the radius r of the balloon and the rate dV/dt at which the volume is increasing at a given instant of time, then we can solve this last equation for dr/dt to find how fast the radius is increasing at that instant. The related rates equation allow us to calculate dr/dt from dV/dt.

3.10. Related Rates

- **Related Rates Problem Strategy**
- **1. Draw a picture that shows the geometric relation**s between the variables, name the variables and constants. Use t for time. Assume that all variables are differentiable functions of t.
- **2. Write down the numerical information** in terms of the symbols you have choosen.
- **3. Write down what you are asked to find** (usually a rate, expressed as a derivative).
- **4. Write an equation that relates the variables.** You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.
- **5. Differentiate with respect to** t. Then express the rate you want in terms of the rates and variables whose values you know.
- **6. Use known values to find the unknown rate.**

Exercises

1. Find the derivatives of the following functions;

$(a)\ y = x^2 \sin x \cos x$ $(b)\ s = \tan t - e^{-t}$ $(c)\ y = \ln t^2$
$(d)\ y = \ln(\sec t + \tan t)$ $(e)\ y = t \sin(\log_7 t)$ $(f)\ y = 3\log_8(\log_2 t)$
$(g)\ y = (\tan t)\sqrt{2t+1}$ $(h)\ y = (\ln x)^{\ln x}$ $(i)\ y = t^{\sqrt{t}}$
$(j)\ y = \ln\left(\tan^{-1} x\right)$ $(k)\ y = \cos^{-1}\left(e^{-t}\right)$ $(l)\ y \sin\left(\frac{1}{y}\right) = 1 - xy$
$(m)\ e^{x^2 y} = 2x + 2y$ $(n)\ y = (t \tan t)^{10}$

Exercises

2. Is there a value of b that will make

$$g(x) = \begin{cases} x + b, & x < 0 \\ \cos x, & x \geq 0 \end{cases}$$

continuous at $x = 0$? Differentiable at $x = 0$?

3. Find the tangent and normal lines to $y = \left(\dfrac{x-1}{x+1}\right)^2$ at $x = 0$ and to $x \sin 2y = y \cos 2x$ at $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$.

4. Find d^2y/dx^2 for $y^2 = e^{x^2} + 2x$ and $xy + y^2 = 4$