## 1.5. Inverse Functions and Logarithms

- **<u>One-to-One Functions</u>**
- A function $f$ is called **one-to-one (injective)** if $f$ preserves distinctness; that is, it never maps distinct elements of its domain to the same element of its codomain.
- Formally;

$$\begin{aligned} f \text{ is one-to-one} \quad &\Leftrightarrow \quad (\forall a, b \in D_f \ni f(a) = f(b) \Longrightarrow a = b) \\ &\Leftrightarrow \quad (\forall a, b \in D_f \ni a \neq b \Longrightarrow f(a) \neq f(b)). \end{aligned}$$

- Some functions are one-to-one on their entire natural domains some not. But buy restricting a function to a smaller domain we can create one-to-one functions.



## 1.5. Inverse Functions and Logarithms

- **Example 21.** Which of the following functions are one-to-one;

$$f(x) = x^2, g(x) = x^3, h(x) = \sqrt{x}$$

  the exponential function and the six basic trigonometric functions.
- **Example 22.** Show that, if $f$ is strictly increasing, then $f$ is one-to-one.

## 1.5. Inverse Functions and Logarithms

- **The Horizantal Line Test for One-to-One Functions**:
- A function $y = f(x)$ is one-to-one if its graph intersects each horizontal line at most once.



## 1.5. Inverse Functions and Logarithms

- **Image of a Subset:** For a real function $f$; the **image of a subset** $A$ of the domain $D_f$ is the subset $f(A)$ defined by

$$f(A) = \{f(x) \mid x \in A\}.$$

- **Preimage of a Subset:** The **preimage of a subset** $B$ of reals is the subset $f^{-1}(B)$ which is defined by

$$f^{-1}(B) = \{x \mid f(x) \in B\}.$$

# 1.5. Inverse Functions and Logarithms

- **Inverse Functions** Suppose that $f$ is a one-to-one function on a domain $D_f$ with range $R_f$ (in this case $f$ is also an **onto (surjective)** function). The inverse function $f^{-1}$ is defined by

$$f^{-1}(b) = a \text{ if } f(a) = b.$$

- The fact which makes this definition meaningful is that; each output of a one-to-one function comes from just one input. The domain $f^{-1}$ is $R_f$ and the range of $f^{-1}$ is $D_f$. We read the symbol $f^{-1}$ as "$f$ inverse".
- Composing a function with its inverse has the same effect as doing nothing.

$$\begin{aligned} \left(f^{-1} \circ f\right)(x) &= x \text{ for all } x \text{ in the domain of } f \\ \left(f \circ f^{-1}\right)(y) &= y \text{ for all } y \text{ in the domain of } f^{-1}. \end{aligned}$$



# 1.5. Inverse Functions and Logarithms

- **Finding Inverses** The graph of $f$ and $f^{-1}$ are related. To graph $f^{-1}$ by using the graph of $f$, consider the points $(a, b)$ on the graph of $f$ and interchange the order.
- The more usual way is to reflect the graph of $f$ across the line $y = x$.
- Passing from $f$ to $f^{-1}$ :
  1. Solve the equation $y = f(x)$ for $x$, and obtain a formula $x = f^{-1}(y)$.
  2. Interchange $x$ and $y$, obtaining a formula $y = f^{-1}(x)$ where $f^{-1}$ is expressed in the usual format with $x$ as the independent variable and $y$ as the dependent variable

- **Example 23.** Find the inverse of the following functions and express them as a function of $x$.
- $(a)$ $f(x) = \frac{1}{3}x + 2, x \in \mathbb{R}$
- $(b)$ $g(x) = x^2, x \geq 0$
- $(c)$ $f(x) = x^3 + 1, x \in \mathbb{R}$.





- **Logarithmic Function**
- **Definition** Let $a$ be a positive real number other than 1. The **logarithm function with base** $a$,

$$y = \log_a x,$$

is the inverse of the base $a$ exponential function $y = a^x$ $(a > 0, a \neq 1)$.
- The domain of $\log_a x$ is $]0, \infty[$, the range of $a^x$. The range of $\log_a x$ is $]-\infty, \infty[$, the domain of $a^x$.

**Example 24.** Sketch the graph of $y = \log_a x$.





- Logarithms with base 2 are commonly used in compurter science.
- Logarithms with base $e$ and base 10 are so important in applications and they have their own special notations and names:

  $\log_e x$ is written as $\ln x$ and called the **natural logarithm func.**
  $\log_{10} x$ is written as $\log x$ and called the **common logarithm func**.

- For the natural logarithm

$$\ln x = y \Leftrightarrow e^y = x.$$

- In particular, if we set $x = e$, we obtain $\ln e = 1$.

## 1.5. Inverse Functions and Logarithms

- **Algebraic Poroperties of the Natural Logarithm:**
- For any numbers $a > 0$ and $b > 0$, the natural logarithm satisfies the following rules:
- $(a)$ $\ln ab = \ln a + \ln b$
- $(b)$ $\ln \frac{a}{b} = \ln a - \ln b$
- $(c)$ $\ln \frac{1}{b} = -\ln b$
- $(d)$ $\ln a^b = b \ln a$



## 1.5. Inverse Functions and Logarithms

- **Inverse Properties for $a^x$ and $\log_a x$:**
- $(a)$ $a^{\log_a x} = x, \log_a a^x = x, a > 0, a \neq 1, x > 0$
- $(b)$ $e^{\ln x} = x, \ln e^x = x, x > 0.$
- **Change of Base Formula:** Every logarithmic function is a constant multiple of the natural logarithm;

$$\log_a x = \frac{\ln x}{\ln a} \ (a > 0, a \neq 1).$$

**Example 25.** If \$1000 is invested in an account that earns 5.25% interest compounded annually, how long will it take the account to reach \$2500?





- **Inverse Trigonometric Functions**
- The six basic trigonometric functions are not one-to-one. However we can restrict their domains to intervals on which they are one-to-one. While doing that we always want to keep $\left[0.\frac{\pi}{2}\right]$ in the domain because most useful angles are acute.

## 1.5. Inverse Functions and Logarithms

- **1. The Arcsine and Arccosine Functions** By restricting the domain of the sine function to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we obtain a one-to-one function

$$\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$$

which is invertible.

- We denote the inverse as

$$\sin^{-1} = \arcsin : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

and

$$y = \arcsin x \textit{ is the angle in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \textit{ for which } \sin y = x.$$



## 1.5. Inverse Functions and Logarithms

- By restricting the domain of the cosine function to the interval $[0, \pi]$, we obtain a one-to-one function

$$\cos : [0, \pi] \longrightarrow [-1, 1]$$

which is invertible.

- We denote the inverse as

$$\cos^{-1} = \arccos : [-1, 1] \longrightarrow [0, \pi],$$

and

$$y = \arccos x \textit{ is the angle in } [0, \pi] \textit{ for which } \cos y = x.$$

## 1.5. Inverse Functions and Logarithms

- **Example 26.** Sketch the graphs of the $y = \sin^{-1} x$ and $y = \cos^{-1} x$.
- **Example 27.** $(a)$ Evaluate $\sin^{-1}\left(-\frac{1}{2}\right)$ and $\cos^{-1}\left(\frac{1}{2}\right)$.
- $(b)$ If $arcsin 1 = x$, find $x$.
- $(c)$ If $\arcsin x = \frac{\pi}{3}$, find $x$.



## 1.5. Inverse Functions and Logarithms

- **2. The Arctangent and Arccotangent Functions** By restricting the domain of the tangent function to the interval $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, we obtain a one-to-one function

$$\tan : \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[ \longrightarrow \mathbb{R}$$

which is invertible.
- We denote the inverse as

$$\tan^{-1} = \arctan : \mathbb{R} \longrightarrow \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[,$$

and

$$y = \arctan x \textit{ is the angle in } \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[ \textit{ for which } \tan y = x.$$

## 1.5. Inverse Functions and Logarithms

- By restricting the domain of the cotangent function to the interval $]0, \pi[$, we obtain a one-to-one function

$$\cot : ]0, \pi[ \longrightarrow \mathbb{R}$$

which is invertible.

- We denote the inverse as

$$\cot^{-1} = \text{arccot} : \mathbb{R} \longrightarrow ]0, \pi[,$$

and

$$y = \text{arccot}\, x \textit{ is the angle in } ]0, \pi[ \textit{ for which } \cot y = x.$$

- **Example 28.** Find *arctan* $\left(-\sqrt{3}\right)$, $\arctan 1$ and $\text{arccot}\,\sqrt{3}$.



## Exercises

**1.** Find the domains of the following functions;

$$\begin{array}{llll} (a) & y = \arcsin \frac{x}{3}, & (b) & y = \arccos 3x \\ (c) & y = \arctan \frac{x-1}{x} & (d) & y = \arctan \sqrt{x-1} \end{array}$$

## Exercises

**2.** Find the domains of the following functions;

| | |
|---|---|
| $(a)\ f(x) = \ln\left(x^2 - 9\right)$ | $(b)\ f(x) = \ln\left(\sqrt{x-4} + \sqrt{6-x}\right)$ |
| $(c)\ f(x) = \ln(\sin \pi x)$ | $(d)\ f(x) = \arcsin(\ln x)$ |
| $(e)\ f(x) = \ln\left(\ln\left(1 + x^2\right)\right)$ | $(f)\ f(x) = \arcsin\left(\log \frac{x}{10}\right)$ |
| $(g)\ f(x) = \log_2(\log_3(\log_4 x))$ | $(h)\ f(x) = \log\left(1 - \log\left(x^2 - 5x + 16\right)\right)$ |



## Exercises

**3.** Show that the function $f(x) = \ln\left(x + \sqrt{1 + x^2}\right)$ is odd.

**4.** Using the graph of the curve $y = \sin x$, sketch the graphs of

$$y = |\sin x|\,,\ y = \sin|x|\,,\ y = \sin\left(x - \frac{\pi}{4}\right),$$
$$y = \sin 2x \text{ and } y = 1 + \sin 2x.$$

# Exercises

**5.** Prove the following relations;

$$\begin{aligned}
\sin(\arctan x) &= \frac{x}{\sqrt{1+x^2}} \\
\cos(\arctan x) &= \frac{1}{\sqrt{1+x^2}} \\
\tan(\arccos x) &= \frac{\sqrt{1-x^2}}{x} \\
\arcsin(\cos x) &= \frac{\pi}{2} - x
\end{aligned}$$