4.5. Indeterminate Forms and L'Hospital's Rule

- L'Hospital's Rule calculates limits of fractions, whose numerators and denominators both approach zero or $+\infty$, by using derivatives.

- **Indeterminate Form $\frac{0}{0}$** How the function

$$F(x) = \frac{x - \sin x}{x^3}$$

behaves near zero? Can we apply the quotient rule that we learned for limits? Such limits may or may not exist in general.

- If the continuous functions $f(x)$ and $g(x)$ are both zero at $x = a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

can not be found by substituting $x = a$. The substitution produces a meaningless expression which we can not evaluate.

4.5. Indeterminate Forms and L'Hospital's Rule

- Other meaningless expressions often occur such as

$$\frac{0}{0}, \frac{\infty}{\infty}, \ \infty \cdot 0, \ \infty - \infty, \ 0^o \text{ and } 1^\infty$$

which can not be evaluated in a consistent way; these are called indeterminate forms.

- **Theorem 6.** (L'Hospital's Rule) Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

assuming that the limit on the right hand side of this equation exists.

4.5. Indeterminate Forms and L'Hospital's Rule

- **Example 1.** Calculate the following limits;
- (a) $\lim_{x\to 0} \dfrac{3x - \sin x}{x}$
- (b) $\lim_{x\to 0} \dfrac{\sqrt{1+x} - 1 - x/2}{x^2}$
- (c) $\lim_{x\to 1} \dfrac{\ln x}{2x^2 - 3x + 1}$

4.5. Indeterminate Forms and L'Hospital's Rule

- **Remark** 1. To find
$$\lim_{x\to a} \frac{f(x)}{g(x)}$$
by l'Hospital's Rule, we continue to differentiate f and g, so long as we still get the form $0/0$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating.
- 2. L'Hospital's Rule applies to one sided limits as well.

4.5. Indeterminate Forms and L'Hospital's Rule

- **Example 2.** Calculate the following limits;
- $(a)\ \lim_{x\to 0} \dfrac{1-\cos x}{x+x^2}$
- $(b)\ \lim_{x\to 0^+} \dfrac{\sin x}{x^2}$
- $(c)\ \lim_{x\to 0^-} \dfrac{\sin x}{x^2}$

4.5. Indeterminate Forms and L'Hospital's Rule

Indeterminate Forms of $\dfrac{\infty}{\infty}$, $\infty\cdot 0$, $\infty-\infty$

If $f(x) \longrightarrow \pm\infty$ and $g(x) \longrightarrow \pm\infty$ as $x \longrightarrow a$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right hand side exists. In the notation $x \longrightarrow a$, a may be finite or infinite, and also $x \longrightarrow a$ may be replaced by the one-sided limits.

4.5. Indeterminate Forms and L'Hospital's Rule

- **Example 3**. Calculate the following limits;
- $(a)\ \lim_{x \to \frac{\pi}{2}} \dfrac{\sec x}{1 + \tan x}$
- $(b)\ \lim_{x \to 0^+} \dfrac{\ln(\sin x)}{\ln(\tan x)}$
- $(c)\ \lim_{x \to \infty} \dfrac{1 + x^2}{x} \sin \dfrac{1}{x}$

4.5. Indeterminate Forms and L'Hospital's Rule

- $(d)\ \lim_{x \to \infty} x \sin \dfrac{1}{x}$
- $(e)\ \lim_{x \to 0^+} \sqrt{x} \ln x$
- $(f)\ \lim_{x \to 0} \left(\dfrac{1}{\sin x} - \dfrac{1}{x} \right)$

- (g) $\lim_{x\to 0} (1-\cos x)\cot x$
- (h) $\lim_{x\to 0} \left(\cot x - \dfrac{1}{x}\right)$

- **Indeterminate Powers** Limits that lead to the indeterminate forms 0^o, 1^∞ and ∞^0 can sometimes be handled by first taking the logarithm of the function.
- We find the limit of the logarithm expression by using l'Hospital's Rule and then exponentiate the result to find the original function limit; i.e.
- $$\lim_{x\to a} \ln f(x) = L \Longrightarrow \lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^L.$$
- Here a may be either finite or infinite and also it is valid for one sided limits.

4.5. Indeterminate Forms and L'Hospital's Rule

- **Example 4**. Calculate the following limits;
- $(a)\ \lim_{x\to 0^+} (1+x)^{1/x}$
- $(b)\ \lim_{x\to 0^-} (1-e^x)^{\sin x}$
- $(c)\ \lim_{x\to 0^+} (\cot x)^{1/\ln x}$

4.5. Indeterminate Forms and L'Hospital's Rule

Proof for a Version of L'Hospital's Rule We consider a special case of the rule to provide some geometric insight for its reasonableness. Consider the two functions $f(x)$ and $g(x)$ having continuous derivatives and satisfying $f(a) = g(a) = 0$, $g'(a) \neq 0$. Sketch the graphs of $f(x)$ and $g(x)$ roughly, together with their linerizations $y = f'(a)(x-a)$ and $y = g'(a)(x-a)$.

4.5. Indeterminate Forms and L'Hospital's Rule

We know that near a the linerizations provide good approximations to the functions, i.e.

$$\begin{aligned} f(x) &= f'(a)\,\Delta x + \epsilon_1 \Delta x = f'(a)(x-a) + \epsilon_1 (x-a) \\ g(x) &= g'(a)\,\Delta x + \epsilon_2 \Delta x = g'(a)(x-a) + \epsilon_2 (x-a) \end{aligned}$$

where $\epsilon_1 \longrightarrow 0$ and $\epsilon_2 \longrightarrow 0$ as $x \longrightarrow a$. So from the geometric view

$$\begin{aligned} \lim_{x \longrightarrow a} \frac{f(x)}{g(x)} &= \lim_{x \longrightarrow a} \frac{f'(a)(x-a) + \epsilon_1 (x-a)}{g'(a)(x-a) + \epsilon_2 (x-a)} \\ &= \lim_{x \longrightarrow a} \frac{f'(a) + \epsilon_1}{g'(a) + \epsilon_2} = \frac{f'(a)}{g'(a)} = \lim_{x \longrightarrow a} \frac{f(x)}{g(x)} \end{aligned}$$

as asserted by the rule.

4.5. Indeterminate Forms and L'Hospital's Rule

- The proof of the general form is based on the following theorem and reading homework for you.
- **Theorem 7.** (Cauchy's Mean Value Theorem) Suppose functions f and g are continuous on $[a, b]$ and differentiable throughout $]a, b[$ and also suppose that $g'(x) \neq 0$ on $]a, b[$. Then there exists a number c in $]a, b[$ at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

4.6. Applied Optimization

- In this section we use derivatives to solve a variety of optimization problems in mathematics, physics, economics and business. For example; we try to find answers for the following questions:
- *What are the dimensions for the least expensive cylindirical can of a given volume?*
- *How many items should be produced for the most profitable production run?*

4.6. Applied Optimization

Solving Applied Optimization Problems

1. Understand the problem

$$\begin{cases} \text{What is the unknown quantity to be optimized?} \\ \text{What are the other quantities?} \\ \text{What are the given conditions?} \end{cases}$$

2. Draw a picture of the problem.

3. Introduce variables. List every relation in the picture and identify the unkown variables.

4. Write an equation for the unkown quantity. If possible, express the unkown as a function of a single variable.

5. Test the critical points and endpoints in the domain of the unkown. Find absolute max. and absolute min. value of the function