1.3. Trigonometric Functions

Angles

An **angle** is defined as the union of two rays (initial side and terminal side) with a common endpoint, the vertex. An angle whose vertex is the center of a circle is a **central angle**, and the arc of the circle through which the terminal side moves is the **intercepted arc**.

1.3. Trigonometric Functions

- There are two systems for measuring angles. One is the well known system of **degree measure**. The other is the strictly mathematical system called the **radian measure**.
- **Degree Measure:** To measure an angle in degrees, we imagine the circumference of a circle divided into 360 equal parts and we call each of those "equal" parts **a degree**. The full circle then will be 360 degrees which is usually written as 360°. The measure of an angle, then, will be as many degrees as its sides include.

Why are we using the number 360?

1.3. Trigonometric Functions

- **Radian Measure:** An arc of a circle with the same length as the radius of that circle corresponds to an angle of 1 **radian**. A full circle than corresponds to an angle of 2π radians. The magnitude in radians of an angle θ is equal to the ratio of the arc length s to the radious r of the circle, that is; $\theta = \frac{s}{r}$.
- From the definitions, clearly;

$$2\pi \ radians = 360^\circ.$$

- **Converting from degrees to radians**:

Degrees :	0°	30°	45°	60°	90°	120°	135°	150°	180°
Radians :	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	π

1.3. Trigonometric Functions

- **In Cartesian coordinate system**, an angle is defined by its two sides, with its vertex at the origin.
- The initial side is the positive x-axis and the terminal side defined by the measure from the initial side (anticlockwise).
- Positive angles represent rotations towards the positive y-axis (anticlockwise) and negative angles represent the rotations towards the negative y-axis (clockwise).
- A negative angle $-\theta$ is efficiently equal to an angle of **one full turn minus** θ

1.3. Trigonometric Functions

The Sine and Cosine Functions

- The two basic trigonometric functions the sine and cosine are defined using a unit circle. An angle of θ radians is measured counterclockwise around the circle from the point $(1,0)$ to the point $P(x,y)$. Then we define

$$x := \cos\theta \text{ and } y := \sin\theta.$$

1.3. Trigonometric Functions

The Sine and Cosine Functions

- Each real number corresponds to a directed length so the functions are defined from $\mathbb{R}$ to the closed interval $[-1,1]$. Since the equation of a unit circle is $x^2+y^2=1$, we have the following fundamental identity:

$$\cos^2\theta+\sin^2\theta=1.$$

- The domains of sine and cosine are all reals. The graphs of sine and cosine oscillate between -1 and 1.

1.3. Trigonometric Functions

The Tangent and Cotangent Function

- If θ is any number with $\cos\theta \neq 0$, we define the tangent function as

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

and the domain is

$$D = \{\theta \mid \cos\theta \neq 0\} = \left\{\theta \mid \theta \neq (2k+1)\frac{\pi}{2}, k \in \mathbb{Z}\right\}.$$

- $\tan\theta$ is the **slope** of the line passing through the origin and the point $P(\cos\theta, \sin\theta)$

1.3. Trigonometric Functions

The Tangent and Cotangent Function

- If $\sin\theta \neq 0$, we define the cotangent function as

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

and the domain is

$$D = \{\theta \mid \sin\theta \neq 0\} = \{\theta \mid \theta \neq k\pi, k \in \mathbb{Z}\}.$$

- The other two basic trigonometric functions secant and cosecant are defined as

$$\sec\theta = \frac{1}{\cos\theta} \text{ and } \csc\theta = \frac{1}{\sin\theta},$$

respectively and clearly secant has the same domain with tangent and cosecant has the same domain with cotangent.

Example 14. Define the trigonometric functions of an acute triangle in terms of the sides of a right triangle. Try to find the values of $\sin\theta$, $\cos\theta$ and $tan\theta$ for $\theta = \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$.

1.3. Trigonometric Functions

Example 15. Consider a circle centered at the origin with radius r and a ray making an angle of θ with the positive x-axis. If A is the intersection point of the terminal ray of θ radians and the circle, find the coordinates of A with respect to θ and r.

Values of $\sin\theta, \cos\theta$ **and** $tan\theta$ **for selected values of** θ **:**

θ(**radians**) :	$\mathbf{0}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin\boldsymbol{\theta}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\cos\boldsymbol{\theta}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\tan\boldsymbol{\theta}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	*undefined*	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0

Example 16. Investigate the signs of basic trigonometric functions via a unit circle.

- **Periodicity and Graphs of The Trigonometric Functions**
- When an angle of measure θ and an angle of measure $\theta + 2\pi$ are in standard position, their terminal rays coincide. The two functions therefore have the same trigonometric values. This is the result of periodicity.
- **Definition** A function f is periodic if there is a positive number p such that

$$f(x+p) = f(x)$$

for every value of x. If such a number exists we call f **periodic** and the smallest such value of p is called **the period of** f..

1.3. Trigonometric Functions

Example 16. Find the period of the function $f(x) = x - \lfloor x \rfloor$ if it is periodic and sketch the graph of f.

1.3. Trigonometric Functions

Example 17. Find the periods of the six basic trigonometric function and sketch their graphs

1.3. Trigonometric Functions

Trigonometric Identities

With the help of the fundamental identity $\cos^2\theta + \sin^2\theta = 1$ we obtain

$$\begin{aligned} 1 + \tan^2\theta &= \sec^2\theta \\ 1 + \cot^2\theta &= \csc^2\theta. \end{aligned}$$

For angles α and β,

Addition Formulas : $\cos(\alpha \mp \beta) = \cos\alpha\cos\beta \pm \sin\alpha\sin\beta$

$\sin(\alpha \mp \beta) = \sin\alpha\cos\beta \mp \cos\alpha\sin\beta$

1.3. Trigonometric Functions

Trigonometric Identities

$$\underline{\textit{Double} - \textit{Angle Formulas}:}\quad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$
$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\underline{\textit{Half} - \textit{Angle Formulas}:}\quad \cos^2\alpha = \frac{1+\cos 2\theta}{2}$$
$$\sin^2\alpha = \frac{1-\cos 2\theta}{2}$$

1.3. Trigonometric Functions

The Laws of Cosine
If a, b and c are sides of a triangle ABC and if θ is the angle opposite to c, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

This equation is called the **law of cosines.**

Example 18. Give a proof for the law of cosine.

1.3. Trigonometric Functions

- **Two Special Inequalities**
- For an angle θ measured in radians, the sine and cosine functions satisfy

$$-|\theta| \leq \sin\theta \leq |\theta| \; and \; -|\theta| \leq 1-\cos\theta \leq |\theta|.$$

- **Reading Homework** Make an investigation about Graphing with Software.

1.3. Trigonometric Functions

Exercises:
1. Find the domain of $f(x) = \cos\sqrt{x}$.
2. Investigate periodicity for the following functions;

$$\begin{aligned} f(x) &= x + \sin x \\ g(x) &= |\cos x| \\ h(x) &= \sin x^2. \end{aligned}$$

1.4. Exponential Functions

- Exponential functions are among the most important in mathematics and occur in a wide variety of applications, including interest rates, radioactive decay, population growth, the spread of disease and consumption of natural resources.
- **Exponential Behavior**
- When a positive quantity P doubles, it increases by a factor of 2 and the quantity becomes $2P$. If it doubles again, it becomes

$$2(2P) = 2^2 P,$$

and a third doubling gives

$$2\left(2^2 P\right) = 2^3 P.$$

Continuing to double in this way leads us to consider the function $f(x) = 2^x$. We will call this an exponential function.

1.4. Exponential Functions

- In general, if $a \neq 1$ **is a positive constant**, the function

$$f(x) = a^x, a > 0$$

is the **exponential function with base** a.
- The domain of f is all reals and the range is positive real numbers.

1.4. Exponential Functions

If $a > 0$ and $b > 0$, the following rules hold true for all real numbers x and y.

$$\begin{aligned} a^x a^y &= a^{x+y} \\ \frac{a^x}{a^y} &= a^{x-y} \\ (a^x)^y &= (a^y)^x = a^{xy} \\ a^x b^x &= (ab)^x \\ \frac{a^x}{b^x} &= \left(\frac{a}{b}\right)^x. \end{aligned}$$

1.4. Exponential Functions

Example 19. Sketch the graph of the exponential function $f(x) = a^x$ ($a \neq 0$ and $a > 0$).

- **The Natural Exponential Function**
- The most important exponential function used for modelling natural, physical and economic phenomena is the **natural exponential function** whose base is the special number e, that is

$$f(x) = e^x$$

The number e is irrational, and its value is 2.718281828 to nine decimal places.

1.4. Exponential Functions

- **Exponential Growth and Decay**
- The exponential functions $y = e^{kx}$, where k is a nonzero constant, are frequently used for modelling exponential growth or decay. The function

$$y = y_0 e^{kx}$$

is a model for **exponential growth** if $k > 0$ and a model for **exponential decay** if $k < 0$. Here y_0 represents a constant.

1.4. Exponential Functions

- **Example 20.** Eliminating a disease: Suppose that in any given year the number of cases of a disease is reduced by 20%. If there are 10000 cases today, how many years will it take
- (a) to reduce the number of cases to 1000?
- (b) to eliminate the disease; that is, to reduce the number of the cases to less than 1?