1.2. Combining Functions; Shifting and Scaling Graphs

Let's look at the main ways functions are combined or transformed to form new functions.

- Like numbers, functions can be added, subtracted, multiplied, and divided (except where the denominator is zero) to produce new functions. If f and g are functions, then for every x that belongs to the domains of both f and g (that is, for $x \in D_f \cap D_g$), we define functions $f+g$, $f-g$ and $f.g$ by the formulas:

$$\begin{aligned} (f+g)(x) &= f(x)+g(x), \\ (f-g)(x) &= f(x)-g(x), \\ (f.g)(x) &= f(x).g(x). \end{aligned}$$

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- At any point of $D_f \cap D_g$ at which $g(x) \neq 0$, we can also define the function f/g by the formula $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ where $g(x) \neq 0$.
- Functions can also be multiplied by constants: if c is a real number, then the function cf is defined for all x in the domain of f by

$$(cf)(x) = cf(x).$$

- **Example 9.** Let $f(x) = x$ and $g(x) = x-1$, sketch the graphs of these functions and consider the graph of $f+g$.

- **Definition** If f and g are functions, the **composite function** $f \circ g$ ("f composed with g") is defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.

- **Example 10.** Let $f(x) = x^2$ and $g(x) = 1 - \sqrt{x}$, find $(f \circ g)(x)$, $(g \circ f)(x)$, $(f \circ f)(x)$ and $(g \circ g)(x)$.

- **Shifting and Scaling Graphs**
- **Shifting a Graph of a Function** A common way to obtain a new function from an existing one is by adding a constant to each output of the existing function, or to its input variable.
- Vertical Shifts $y = f(x) + k$ Shifts the graph of f up k units if $k > 0$, shifts the graph of f down $|k|$ units if $k < 0$.
- Horizantal Shifts $y = f(x + h)$ Shifts the graph of f left h units if $h > 0$, shifts the graph of f right $|h|$ units if $h < 0$.

1.2. Combining Functions; Shifting and Scaling Graphs

- **Example 11.** If the real function f is given with the rule $f(x) = -x^2 + 4x - 3$ sketch the graph of the curve $y = f(x) + 3$.
- **Example 12.** Sketch the graph of

$$g(x) = (x-2)^3 \text{ and } h(x) = (x+1)^3$$

by using the graph of $f(x) = x^3$.

1.2. Combining Functions; Shifting and Scaling Graphs

- **Scaling and Reflecting a Graph of a Function**: To scale the graph of a function $y = f(x)$ is to stretch or compress it, vertically and horizontally. This is accomplished by multiplying the function f, or the independent variable x, by an appropriate constant c. Reflections across the coordinate axes are special cases where $c = -1$.
- **For $c > 1$ the graph is scaled:**
- $y = cf(x)$ $\longrightarrow$Stretches the graph of f vertically by a factor of c.
- $y = \dfrac{1}{c}f(x)$ $\longrightarrow$Compresses the graph of f vertically by a factor of c.
- $y = f(cx)$ $\longrightarrow$Compresses the graph of f horizontally by a factor of c.
- $y = f\left(\dfrac{x}{c}\right)$ $\longrightarrow$Stretches the graph of f horizontally by a factor of c.

- **For $c = -1$ the graph is reflected:**
- $y = -f(x) \longrightarrow$ Reflects the graph of f across the x-axis.
- $y = f(-x) \longrightarrow$ Reflects thegraph of f across the y-axis.

Example 13. Sketch the graph of $y = 2f(x)$, $y = \frac{1}{2}f(x)$, $y = f(2x)$, $y = f\left(\frac{1}{2}x\right)$, $y = -f(x)$ and $y = f(-x)$ where $f(x) = x^2 + 2x$.

Exercises related with 1.1 and 1.2

- **1**. Find the domain of the following functions;
- $(a)\ f(x) = \sqrt{1-|x|}$,
- $(b)\ g(x) = \sqrt{|x-1|-2}$,
- $(c)\ h(x) = \sqrt{|x|+4}$

Exercises related with 1.1 and 1.2

- $(d)\ t(x) = \sqrt{1-\lfloor x \rfloor}$
- $(e)\ s(x) = \dfrac{x}{\lfloor x \rfloor}$
- $(f)\ r(x) = \sqrt{x-\lfloor x \rfloor}$

Exercises related with 1.1 and 1.2

- $(g)\ f(x) = \dfrac{2^x + x}{\lfloor x \rfloor \lfloor x+2 \rfloor}$
- $(h)\ g(x) = \dfrac{1}{\lfloor x-2 \rfloor} + \sqrt{4-x^2}$
- $(i)\ h(x) = \dfrac{\sqrt{9 - |2x+1|}}{\lfloor \frac{x}{3} - 1 \rfloor}$

Exercises related with 1.1 and 1.2

- $(j)\ f(x) = sgn\left(\dfrac{x-2}{\sqrt{\lfloor x \rfloor^2 - 9}}\right)$
- $(k)\ s(x) = sgn\left(\dfrac{x-2}{x+2}\right)$
- $(l)\ r(x) = \dfrac{|x|\, sgnx}{\lfloor x \rfloor}$

Exercises related with 1.1 and 1.2

- $(m)\ g(x) = \sqrt{sgn(x+2)}$
- $(n)\ h(x) = \sqrt{x|x|}$
- $(o)\ t(x) = \dfrac{\sqrt{sgn(x+1)}}{\lfloor x-2 \rfloor}$

Exercises related with 1.1 and 1.2

- **2**. Sketch the graph of the following functions;
- $(a)\ f(x) = 2|x-1| + 3|x+1|$
- $(b)\ g(x) = |x+2| + 1$
- $(c)\ h(x) = \left\lfloor \dfrac{x}{2} \right\rfloor$, over $[-2,2]$

Exercises related with 1.1 and 1.2

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- $(d)\ s(x) = x^2 + 6x + 5$
- $(e)\ y = |s(x)|$
- $(f)\ y = s(|x|)$
- $(g)\ y = |s(|x|)|$

Exercises related with 1.1 and 1.2

- $(h)\ f(x) = sgn\left(x^2 + x - 6\right)$
- $(i)\ g(x) = |\lfloor x \rfloor|$, over $[-2, 2]$

Exercises related with 1.1 and 1.2

- $(j)\ f(x) = x + |x| + sgn(|x|)$
- $(k)\ s(x) = |x+1| - xsgn(x-2)$

Exercises related with 1.1 and 1.2

- $(l)\ r(x) = \lfloor |x| \rfloor$, over $[-2, 2]$
- $(m)\ g(x) = x^2 + sgn(x-1)$

Exercises related with 1.1 and 1.2

- (n) $h(x) = \left|x^2 - 1\right| + sgn\left(x^2 - 1\right)$
- (o) $t(x) = \dfrac{\lfloor xsgnx \rfloor}{\lfloor x - 2 \rfloor}$, over $[-2, 2]$

Exercises related with 1.1 and 1.2

- **3**. Use the graph of the function $f(x) = x^2 - 2x - 3$ and sketch the graph of the following;

$$\begin{array}{cccc} y = 2f(x), & y = f(x-1), & y = f(x+1), & y = -f(x) \\ y = -2f(x), & y = f(x) + 3, & y = f(x) - 1 & y = f(x-1) + 1 \end{array}$$