## 3.11. Linerization and Differentials

Sometimes we can approximate complicated functions with simpler ones. That give the accuracy we want for specific applications and are easier to work with. The approximating functions discussed in this section are called **linerizations**, and they are based on tangent lines.



# 3.11. Linerization and Differentials

- **Linerization** Consider tangent to the curve $y = x^2$ at the point $(1, 1)$. The tangent lies close to the curve near the point of tangency. When we consider smaller and smaller intervals about $x = 1$, the $y$-values along the tangent line give good approximations to the $y$-values on the curve.
- Observe this by zooming in on the graph at the point of tangency or by looking at tables of values for the difference between $f(x)$ and its tangent line near the $x$ coordinate of the point of tangency.
- This phenomenon is true not just parabolas; every differentiable curve behaves locally like its tangent line.

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- In general, the tangent to $y = f(x)$ at a point $x = a$, where $f$ is differentiable, passes through the point $(a, f(a))$, so its **point-slope equation** is
$$y = f(a) + f'(a)(x - a).$$
- Thus this tangent line is the graph of the linear function
$$L(x) = f(a) + f'(a)(x - a).$$
This line remains close to the graph of $f$ as we move off the point of tangency, $L(x)$ gives a good approximation to $f(x)$.



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- **Definition** If $f$ is differentiable at $x = a$, then the approximating function
$$L(x) = f(a) + f'(a)(x - a)$$
is the **linerization** of $f$ at $a$. The approximation
$$f(x) \approx L(x)$$
of $f$ by $L$ is the **standard linear approximation** of $f$ at $a$. The point $x = a$ is the **center** of the approximation.

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- **Example 33.** Find the linerization of $f(x) = \sqrt{1+x}$ at $x = 0$ and $x = 3$, respectively.
- **Example 34.** Find the linerization of $f(x) = \cos x$ at $x = \dfrac{\pi}{2}$.



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- **Example 35.** $(a)$ Find the linerization of $f(x) = e^x$ $x = 0$ and $g(x) = \ln x$ at $x = 1$.
- $(b)$ Show that the linerization of $f(x) = (1+x)^k$ at $x = 0$ is $L(x) = 1 + kx$.

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- **Differentials** We know that; contrary to its appearence, the Leibniz notation $dy/dx$ is not a ratio. We now introduce two new variables $dx$ and $dy$ with the property that when their ratio exists, it is equal to the derivative.
- **Definition** Let $y = f(x)$ be a differentiable function. The **differential** $dx$ is an independent variable. The **differential** $dy$ is

$$dy = f'(x)\,dx.$$

- Unlike the independent variable $dx$, the variable $dy$ is always a dependent variable. It depends on both $x$ and $dx$.



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- **Example 36.** Find $dy$ if $y = x^2 + 7x$.
- If $dx$ is given a specific value and $x$ is a particular number in the domain of the function, then these values determine the numerical value of $dy$. Often the variable $dx$ is choosen to be $\Delta x$, the change in $x$.
- **Example 37.** Consider Example 36 and find the value of $dy$ when $x = 1$ and $dx = 0.2$.

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- **The Geometric Meaning of Differential** Let $x = a$ and set $dx = \Delta x$. The corresponding change in $y = f(x)$ is

$$dy = f(a + dx) - f(a).$$

- The corresponding change in the tangent line $L$ is

$$\begin{aligned} \Delta L &= L(a + dx) - L(a) \\ &= f(a) + f'(a)[(a + dx) - a] - f(a) \\ &= f'(a)\,dx. \end{aligned}$$

- That is, the change in the linerization of $f$ is precisely the value of the differential $dy$ when $x = a$ and $dx = \Delta x$. Therefore, $dy$ represents the amount the tangent line rises or falls when $x$ changes by an amount $dx = \Delta x$.



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**Observe The Geometric Meaning of Differential**

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- If $dx \neq 0$, then the quotient of the differential $dy$ by the differential $dx$ is equal to the derivative $f'(x)$ because

$$dy \div dx = \frac{f'(x)\,dx}{dx} = f'(x) = \frac{dy}{dx}.$$

- We sometimes write

$$df = f'(x)\,dx$$

in place of $dy = f'(x)\,dx$, calling $df$ the differential of $f$. For instance, if $f(x) = 3x^2 - 6$ then

$$df = d\left(3x^2 - 6\right) = 6x dx.$$

Every differentiation formula have a corresponding differential form like

$$d(u+v) = du + dv \text{ or } d(\sin u) = \cos u du.$$



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**Example 38.** Find the differentials of the following functions
$(a)$ $f(x) = \tan x$
$(b)$ $g(x) = \dfrac{x}{x+1}$

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<ins>**Estimating with Differentials**</ins> Suppose we know the value of a differentiable function $f(x)$ at a point $a$ and want to estimate how much this value will change if we move to a nearby point $a+dx$. If $dx=\Delta x$ is small, then we can observe that $\Delta y$ is approximately equal to the differential $dy$. Since

$$f(a+dx)=f(a)+\Delta y$$

the differential approximation gives

$$f(a+dx)\approx f(a)+dy$$

when $dx=\Delta x$. Thus the approximation $\Delta y\approx dy$ can be used to estimate $f(a+dx)$ when $f(a)$ is known, $dx$ is small, and $dy=f'(a)\,dx$.



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**Example 39** The radius $r$ of a circle increases from $a=10m$ to $10.1m$. Use $dA$ to estimate the increase in circle's area $A$. Estimate the area of the enlarged circle and compare your estimate to the true area found by direct calculation.

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- **Example 40.** Use differentials to estimate
- $(a)$ $\sqrt[3]{7.97}$
- $(b)$ $\sin\left(\frac{\pi}{6}+0.01\right)$



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- **Error in Differential Approximation** Let $f(x)$ be differentiable at $x=a$ and suppose that $dx=\Delta x$ is an increment of $x$. We have two ways to describe the change in $f$ as $x$ changes from $a$ to $a+\Delta x$ :

$$\begin{aligned}&\textit{The true change}: \Delta f = f(a+\Delta x) - f(a)\\ &\textit{The differential estimate}: df = f'(a)\Delta x.\end{aligned}$$

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- **How does $df$ approximate $\Delta f$?** We measure the approximation error by subtracting $df$ from $\Delta f$ :

$$\begin{aligned} \textit{Approximation error} &= \Delta f - df \\ &= \Delta f - f'(a)\,\Delta x \\ &= f(a+\Delta x) - f(a) - f'(a)\,\Delta x \\ &= \left(\frac{f(a+\Delta x) - f(a)}{\Delta x} - f'(a)\right)\Delta x. \end{aligned}$$

Let $\varepsilon = \dfrac{f(a+\Delta x) - f(a)}{\Delta x} - f'(a)$.



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- As $\Delta x \longrightarrow 0$, the difference quotient

$$\frac{f(a+\Delta x) - f(a)}{\Delta x}$$

approach $f'(a)$, so $\varepsilon$ becomes a very small number. In fact, $\varepsilon \longrightarrow 0$ as $\Delta x \longrightarrow 0$. When $\Delta x$ is small, the approximation error $\varepsilon\Delta x$ is smaller still. In addition;

$$\Delta f = f'(a)\,\Delta x + \varepsilon\Delta x.$$

- **Change in $y = f(x)$ near $x = a$** If $y = f(x)$ is differentiable at $x = a$ and $x$ changes from $a$ to $a + \Delta x$, the change $\Delta y$ in $f$ is given by

$$\Delta y = f'(a)\,\Delta x + \varepsilon\Delta x$$

in which $\varepsilon \longrightarrow 0$ as $\Delta x \longrightarrow 0$.

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- **Sensitivity to Change** The equation $df = f'(x)\,dx$ tells how sensitive the output of $f$ to a change in input at different values of $x$. The larger the value of $f'$ at $x$, the greater the effect of a given change $dx$. As we move from $a$ to a nearby point $a + dx$, we can describe the change in $f$ in three ways:

| | *TRUE* | *ESTIMATED* |
|---|---|---|
| *Absolute change :* | $\Delta f$ | $df$ |
| *Relative change :* | $\Delta f / f(a)$ | $df / f(a)$ |
| *Percentage change :* | $(\Delta f / f(a)) \times 100$ | $(df / f(a)) \times 100$ |



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**Example 41.** You want to calculate the depth of a well from the equation $s = 4.9t^2$ by timing how long it takes a heavy stone you drop to splash into the water below. How sensitive will your calculations be to a $0.1 - s$ error in measuring time?