

Continuous Functions on Closed Intervals

- Continuous functions defined on closed intervals have very special properties.
- Continuity of a function is related to the possibility of tracing its graph without lifting the pen from the paper. The following theorem expresses this fact with precision.

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Warning:
This Theorem is so essential.

- **Theorem (Intermediate Value Theorem)** Suppose that the function f is continuous on the closed interval $[a, b]$. Then $f(x)$ assumes every intermediate value between $f(a)$ and $f(b)$. That is; if K is any number between $f(a)$ and $f(b)$, then

there exists at least one number c in $]a, b[$ such that $f(c) = K$.

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- **Example 29.** The discontinuous function

$$f(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0 \end{cases}$$

defined on $[0, 1]$ does not attain the intermediate value $\frac{1}{2}$. Is that result a contradiction?

- **Example 30.** Show that there is a root of the equation $x^3 - x - 1 = 0$ between 1 and 2.

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- **Example 31.** Prove that the equation

$$\sqrt{2x + 5} = 4 - x^2$$

has a solution.

- **Example 32.** Is the function $f(x) = x^3 - x - 2$ has a solution between $x = 1$ and $x = 2$?
- **Theorem (Bolzano's Theorem)** Suppose that f is continuous on the closed interval $[a, b]$. If $f(a)$ and $f(b)$ have opposite sign, then there exist a number $c \in]a, b[$ such that $f(c) = 0$.

Continuous Functions on Closed Intervals

- **Continuous Extension to a Point**

Sometimes the formula that describes a function f does not make sense at a point $x = c$. It might nevertheless be possible to extend the domain of f , to include $x = c$, creating a new function that is continuous at $x = c$.

- For example; the function $y = \frac{\sin x}{x}$ is continuous at every point except $x = 0$. Since $y = \frac{\sin x}{x}$ has a finite limit as $x \rightarrow 0$, we can extend the function's domain to include the point $x = 0$ in such a way that the extended function is continuous at $x = 0$. We define the new function

$$F(x) := \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

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- More generally, a function may have a finite limit at a point where it is not defined (such as a rational function). If $f(c)$ is not defined, but $\lim_{x \rightarrow c} f(x) = L$ exists, we can define a new function $F(x)$ by the rule

$$F(x) := \begin{cases} f(x), & \text{if } x \in D_f \\ L, & \text{if } x = c. \end{cases}$$

The function F is continuous at c . It is called the continuous extension of f to $x = c$.

- **Example 33.** Show that

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}, x \neq 2$$

has a continuous extension to $x = 2$, and find that extension.

Exercises

1. Define $g(3)$ in a way that extends $g(x) = \frac{x^2 - 9}{x - 3}$ to be continuous at $x = 3$.
2. For what values of b is

$$g(x) = \begin{cases} x, & x < -2 \\ bx^2, & x \geq -2 \end{cases}$$

continuous at every x ?

3. For what values of a and b are f and g continuous at every x ?

$$(a) f(x) = \begin{cases} 2 \cos x, & x < 0 \\ a \cos x + b, & 0 \leq x \leq \pi \\ -\sin x, & x > \pi \end{cases}$$

$$(b) \ g(x) = \begin{cases} (x^2)^a \sin^b(x^2) & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Exercises

4. At which points do the following functions fail to be continuous and at what points are they continuous? Give reasons for your answers.

$$(a) \ f(x) = \begin{cases} \lfloor x \rfloor + \lfloor -x \rfloor, & x \notin \mathbb{Z} \\ 0, & x \in \mathbb{Z} \end{cases}$$

$$(b) \ g(x) = x^2 + \operatorname{sgn}(x^2 - 1)$$

$$(c) \ h(x) = \begin{cases} \frac{2 \sin x}{|x|}, & x \neq 0 \\ 2, & x = 0 \end{cases}$$

$$(d) \ s(x) = x \lfloor x \rfloor$$

$$(e) f(x) = \frac{3^{1/x} + 2^{1/x}}{3^{1/x} - 2^{1/x}}$$

$$(f) g(x) = \begin{cases} \cos \frac{\pi x}{2}, & |x| \leq 1 \\ |x-1|, & |x| > 1 \end{cases}$$

5. Show that the equation $8x^3 - 12x^2 - 2x + 3 = 0$ has at least one root in $] -1, 0[$, $] 0, 1[$ and $] 1, 2[$.

6. Evaluate the following limits

$$(a) \lim_{x \rightarrow 0} \sqrt{\tan x + 3 \sec x}$$

$$(b) \lim_{x \rightarrow 0} \sin \left(\frac{\pi}{2} \cos(\tan x) \right).$$

2.6 Limits Involving Infinity, Asymptotes of Graphs

- Consider the graph

- Definition** (a) For a given real function f

$$\lim_{x \rightarrow \infty} f(x) = L$$

means that given any $\varepsilon > 0$ there exists a corresponding real number M such that for all x ,

$$x > M \implies |f(x) - L| < \varepsilon$$

and we say that " $f(x)$ has limit the L as x approaches infinity".

2.6 Limits Involving Infinity, Asymptotes of Graphs

- Consider the graph
- (b) For a given real function f

$$\lim_{x \rightarrow -\infty} f(x) = L$$

means that given any $\varepsilon > 0$ there exists a corresponding real number N such that for all x ,

$$x < N \implies |f(x) - L| < \varepsilon$$

and we say that " $f(x)$ has the limit L as x approaches minus infinity".

2.6 Limits Involving Infinity, Asymptotes of Graphs

- **Example 18.** Show that
- (a) $\lim_{x \rightarrow \infty} \frac{x+4}{x} = 1$ and $\lim_{x \rightarrow -\infty} \frac{x+4}{x} = 1$,
- (b) $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$ and $\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$.
- **Remark** The Theorem giving Limit Laws is also valid when we replace $\lim_{x \rightarrow c}$ by $\lim_{x \rightarrow \infty}$ or $\lim_{x \rightarrow -\infty}$.

2.6 Limits Involving Infinity, Asymptotes of Graphs

- **Example 19.** Find the following limits

- (a) $\lim_{x \rightarrow \infty} \frac{x}{[x]}$
- (b) $\lim_{x \rightarrow \infty} \frac{2x + 5}{4x^2 + 8x + 1}$
- (c) $\lim_{x \rightarrow \infty} \frac{x^3 + 1}{2x^2 + 3}$

2.6 Limits Involving Infinity, Asymptotes of Graphs

- (d) $\lim_{x \rightarrow \infty} \frac{5x^2 + 1}{2x^2 + 3x - 3}$
- (e) $\lim_{x \rightarrow \infty} (\sqrt{x^2 + x} - x)$
- (f) $\lim_{x \rightarrow -\infty} (\sqrt{x^2 + x} - x)$
- **Example 20.** Try to find a way to evaluate $\lim_{x \rightarrow 2} \frac{x - 3}{x^2 - 4}$
(Homework).

2.6 Limits Involving Infinity, Asymptotes of Graphs

- **Horizontal Asymptotes**

If the distance between the graph of a function and some fixed line approaches zero, as a point on the graph moves increasingly far from the origin, we say that the graph approaches the line **asymptotically** and the line is an **asymptote** of the graph.

- **Definition** A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \rightarrow \infty} f(x) = b \text{ or } \lim_{x \rightarrow -\infty} f(x) = b.$$

2.6 Limits Involving Infinity, Asymptotes of Graphs

- **Example 21.** Find the horizontal asymptotes of the graphs of

- (a) $f(x) = \frac{5x^2 + 8x - 3}{3x^2 + 2}$

- (b) $g(x) = \frac{x^3 - 2}{|x|^3 + 1}$

- (c) $h(x) = e^x$

2.6 Limits Involving Infinity, Asymptotes of Graphs

- (d) $s(x) = \sin\left(\frac{1}{x}\right)$
- (e) $p(x) = x \sin\left(\frac{1}{x}\right)$
- (f) $\lim_{x \rightarrow 0^-} e^{1/x}$

2.6 Limits Involving Infinity, Asymptotes of Graphs

- (g) $f(x) = 2 + \frac{\sin x}{x}$
- (h) $g(x) = x - \sqrt{x^2 + 16}$

2.6 Limits Involving Infinity, Asymptotes of Graphs

- **Oblique Asymptotes**

If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an **oblique** or **slant line asymptote**. We find an equation for the asymptote by dividing numerator by denominator to express f as a linear function plus a remainder that goes to zero as $x \rightarrow \pm\infty$.

- **Example 22.** Find the oblique asymptotes of the graphs of

- (a) $f(x) = \frac{x^2 - 3}{2x - 4}$
- (b) $g(x) = \frac{x^3 + 1}{x}$

2.6 Limits Involving Infinity, Asymptotes of Graphs

- **Infinity/Negative Infinity as Limits (Vertical Asymptotes)**

Consider the graph

- **Definition** (a) $\lim_{x \rightarrow x_0} f(x) = \infty$ means that for every positive real number B there exists a corresponding $\delta > 0$ such that for all x ,

$$0 < |x - x_0| < \delta \implies f(x) > B$$

and we say

$f(x)$ approaches infinity as x approaches x_0 .

2.6 Limits Involving Infinity, Asymptotes of Graphs

- Consider the graph
- (b) $\lim_{x \rightarrow x_0} f(x) = -\infty$ means that for every negative real number K there exists a corresponding $\delta > 0$ such that for all x ,

$$0 < |x - x_0| < \delta \implies f(x) < K$$

and we say

$f(x)$ approaches negative infinity as x approaches x_0 .

2.6 Limits Involving Infinity, Asymptotes of Graphs

- Show that $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$, $\lim_{x \rightarrow 1^+} \frac{4}{x-1} = \infty$ and $\lim_{x \rightarrow 1^-} \frac{4}{x-1} = -\infty$.

2.6 Limits Involving Infinity, Asymptotes of Graphs

- **Example 23.** Evaluate the following limits

- (a) $\lim_{x \rightarrow 2} \frac{x-3}{x^2-4},$

- (b) $\lim_{x \rightarrow 1^+} \frac{4}{x-1},$

- (c) $\lim_{x \rightarrow 1^-} \frac{4}{x-1},$

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- (d) $\lim_{x \rightarrow 0} \frac{1}{x}$

2.6 Limits Involving Infinity, Asymptotes of Graphs

- **Example 24.** Rational functions can behave in various ways near zeros of the denominator:

- (a) $\lim_{x \rightarrow 2} \frac{(x-2)^2}{x^2-4}$

- (b) $\lim_{x \rightarrow 2} \frac{x-2}{x^2-4}$

- (c) $\lim_{x \rightarrow 2^+} \frac{x-3}{x^2-4}$

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- (d) $\lim_{x \rightarrow 2^-} \frac{x-3}{x^2-4}$

- (e) $\lim_{x \rightarrow 2} \frac{x-3}{x^2-4}$

- (f) $\lim_{x \rightarrow 2} \frac{2-x}{(x-2)^3}$

2.6 Limits Involving Infinity, Asymptotes of Graphs

- **Vertical Asymptotes**

Definition A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \rightarrow a^+} f(x) = \pm\infty \text{ or } \lim_{x \rightarrow a^-} f(x) = \pm\infty.$$

- **Example 25.** Find the horizontal and vertical asymptotes of the following curves

- (a) $y = \frac{x+3}{x+2}$

2.6 Limits Involving Infinity, Asymptotes of Graphs

- (b) $y = -\frac{8}{x^2 - 4}$
- (c) $y = \ln x$
- (d) $y = \sec x$ and $y = \tan x$