8.4. Trigonometric Substitutions

- In some cases to replace the variable of the function by a trigonometric function transforms the asked integral into an easier one. The most common substitutions are

$$x = a\tan t,\ \ x = a\sin t \text{ and } x = a\sec t.$$

- These substitutions are effective in transforming integrals involving

$$\sqrt{a^2+x^2},\ \ \sqrt{a^2-x^2} \text{ and } \sqrt{x^2-a^2}$$

into integrals we can evaluate directly.

8.4. Trigonometric Substitutions

- Applying $x = a\tan t$ provides

$$a^2+x^2 = a^2 + a^2\tan^2 t = a^2\left(1+\tan^2 t\right) = a^2\sec^2 t,$$

- with $x = a\sin t$ we obtain

$$a^2-x^2 = a^2 - a^2\sin^2 t = a^2\left(1-\sin^2 t\right) = a^2\cos^2 t$$

- and with $x = a\sec t$ we obtain

$$x^2-a^2 = a^2\sec^2 t - a^2 = a^2\left(1-\sec^2 t\right) = a^2\tan^2 t.$$

any substitution we use in an integral should be reversible so that we can change back to the original variable afterward.

- For example; if we apply the substitution $x = a\tan t$, we want to be able to set $t = \tan^{-1}\left(\frac{x}{a}\right)$ after the ,integration takes place. Similarly we want to be able to set $t = \sin^{-1}\left(\frac{x}{a}\right)$ and $t = \sec^{-1}\left(\frac{x}{a}\right)$. For reversibility;
- $x = a\tan t$ *requires* $t = \tan^{-1}\left(\frac{x}{a}\right)$ *with* $-\frac{\pi}{2} < t < \frac{\pi}{2}$
- $x = a\sin t$ *requires* $t = \sin^{-1}\left(\frac{x}{a}\right)$ *with* $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$
- $x = a\sec t$ *requires* $t = \sec^{-1}\left(\frac{x}{a}\right)$ *with* $\begin{cases} 0 \leq t < \frac{\pi}{2} & \text{if } \frac{x}{a} \geq 1 \\ \frac{\pi}{2} < t \leq \pi & \text{if } \frac{x}{a} \leq 1 \end{cases}$.

- To simplify calculations with the substitution $x = a\sec t$, we will restrict its use to integrals in which $\frac{x}{a} \geq 1$. This will place t in $\left[0, \frac{\pi}{2}\right)$ and make $\tan t \geq 0$. We will then have

$$\sqrt{x^2 - a^2} = \sqrt{a^2\tan^2 t} = \left|a\tan t\right| = a\tan t,$$

free for absolute values, provided $a > 0$.

- **Procedure for a Trigonometric Substitution**
- 1. Write down the substitution, calculate the differential dx, specify the selected values of t for the substitution.
- 2. Substitute the trigonometric expression and the calculated dx then simplify the result algebraically.
- 3. Integrate the integral, keep in mind the restrictions on the angle t for reversibility.
- 4. Draw an appropriate reference triangle to reverse the substitution in the integration result and convert it back to the original variable x.

- **Example 7.** Evaluate the following integrals;
- $(a)\ \int \dfrac{x+8}{\sqrt{9-x^2}}dx$

$(b) \int \frac{dx}{(4-x^2)^{3/2}}$

$(c) \int \frac{x^2\,dx}{\sqrt{1-x^2}}$

(d) $\int \frac{dx}{x\sqrt{x^2-9}}$

(e) $\int \frac{dx}{\sqrt{25x^2-4}},\ x > \frac{2}{5}$

8.4. Trigonometric Substitutions

$(f)\ \int \frac{dx}{x^2\sqrt{x^2+4}}$

$(g)\ \int \frac{3dy}{\sqrt{1+9y^2}}$

8.4. Trigonometric Substitutions

- **Remark 1.** If the integrand includes the terms $\sqrt[n_i]{ax+b}$ for $i = 1, 2, \cdots, k$ then apply the substitution with $t^p = ax + b$ where p is the least common multiple of n_i.
- 2. If the integrand includes a rational expression of trigonometric functions applying the substitution $\tan\frac{x}{2} = t$ may provide a simple function.
- **Example 8.** Evaluate the following integrals;
- $(a)\ \int \frac{\sqrt[4]{x+1}+2}{\sqrt[6]{x+1}}dx$
- $(b)\ \int \frac{\sqrt[3]{x+3}+\sqrt[6]{x+3}}{\sqrt{x+3}}dx$

8.4. Trigonometric Substitutions

- $(c)\ \int \dfrac{1+\sin x}{(1+\cos x)\sin x}dx$
- $(d)\ \int \dfrac{dx}{5+\cos x}$

8.5. Integration of Rational Functions by Partial Fractions

- In this section we will learn to express a rational function as a sum of simpler fractions, called partial fractions, which are easily integrated.
- For instance the rational function

$$\frac{5x-3}{x^2-2x-3}$$

- can be written as

$$\frac{5x-3}{x^2-2x-3} = \frac{2}{x+1} + \frac{3}{x-3}.$$

- Therefore

$$\begin{aligned} \int \frac{5x-3}{x^2-2x-3}dx &= \int \frac{2}{x+1}dx + \int \frac{3}{x-3}dx \\ &= 2\ln|x+1| + 3\ln|x-3| + C. \end{aligned}$$

- The method of rewriting rational functions as a sum of simpler fractions is called **the method of partial fractions**. It consist of finding constants A and B such that

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}.$$

- Now pretend for the moment that we do not know what A and B.
- The fractions $\frac{A}{x+1}$ and $\frac{B}{x-3}$ are called **partial fractions** because their denominators are only part of the original denominator.
- We call A and B **undetermined coefficients** until suitable values for them have been found.

8.5. Integration of Rational Functions by Partial Fractions

- *How do we find A and B?*
- **General Description of the Method** To write a rational function $f(x)/g(x)$ as a sum of partial fractions depends on the following:
- (a) The degree of $f(x)$ must be less than the degree of $g(x)$. That is, the fraction must be proper. If is isn't, divide $f(x)$ by $g(x)$ and work with the remainder term.
- (b) Find the factors of $g(x)$. Theoretically, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors. Sometimes, the factors may be hard to find.

8.5. Integration of Rational Functions by Partial Fractions

- **Method of Partial Fractions When $f(x)/g(x)$ is Proper**
  Consider that we don't know the factors of $g(x)$. A quadratic polynomial is irreducible if it can not be written as the product of two linear factors with real coefficients. That is, the polynomial has no real roots.
- **1.** Let $x-r$ be a linear factor of $g(x)$. Suppose that $(x-r)^m$ is the highest power of $x-r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:
- $$\frac{A_1}{x-r}+\frac{A_2}{(x-r)^2}+\cdots+\frac{A_m}{(x-r)^m}.$$
- Do this for each distinct linear factor of $g(x)$. What does happen whenever $g(x)=(x-r_1)(x-r_2)\cdots(x-r_m)$?

8.5. Integration of Rational Functions by Partial Fractions

- **2.** Let x^2+px+q be an irreducible quadratic factor of $g(x)$ so that x^2+px+q has no real roots. Suppose that $\left(x^2+px+q\right)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:
- $$\frac{B_1x+C_1}{x^2+px+q}+\frac{B_2x+C_2}{(x^2+px+q)^2}+\cdots+\frac{B_nx+C_n}{(x^2+px+q)^n}.$$
- Do this for each quadratic factor of $g(x)$. What does happen whenever $g(x)=\left(x^2+px+q\right)\left(x^2+ax+b\right)$ whenever both of the polynomials are irreducible?

- **3.** Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x. What does happen whenever $g(x) = (x - r)(x^2 + px + q)$ whenever $x^2 + px + q$ is irreducible?
- **4.** Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Factors in denominator	*Terms in Partial Fraction Decomposition*
$ax + b$	$\frac{A}{ax+b}$
$(ax + b)^n$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$
$ax^2 + bx + c$ (*irreducible*)	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2 + bx + c)^m$	$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_mx+B_m}{(ax^2+bx+c)^m}$

Integration of Irrational Functions

1. For $\int \dfrac{dx}{\sqrt{ax^2+bx+c}}$, if $b^2-4ac>0$ and $a<0$, ax^2+bx+c can be written in the form k^2-u^2 where k is a constant and u is a linear expression, and if if $b^2-4ac>0$ and $a>0$, ax^2+bx+c can be written in the form u^2-p where a is a constant and u is a linear expression.

2.

$$\begin{aligned}\int \frac{mx+n}{\sqrt{ax^2+bx+c}}dx &= \frac{m}{2a}\int \frac{2ax+2a\frac{n}{m}}{\sqrt{ax^2+bx+c}}dx \\ &= \frac{m}{2a}\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}dx \\ &\quad + \left(n-\frac{mb}{2a}\right)\int \frac{dx}{\sqrt{ax^2+bx+c}}\end{aligned}$$

and handle with it by using 1.