4.8. Antiderivatives

- How can we recover a function from its known derivative? When we start with a function f, can we find a function F whose derivative is f? If such a function F exists, it is called an antiderivative of f. Antiderivatives are the link connecting derivatives and definite integrals.
- **Finding Antiderivatives**
- **Definition** A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.
- The process of recovering a function $F(x)$ from its derivative $f(x)$ is called antidifferentiation.

4.8. Antiderivatives

- **Example 1.** Find an antiderivative for each of the following functions;
- (a) $f(x) = 2x$
- (b) $g(x) = \cos x$
- (c) $h(x) = \dfrac{1}{x} + 2e^{2x}$

4.8. Antiderivatives

- **Theorem 8.** If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.
- Thus the most general antiderivative of f on I is a family of functions $F(x) + C$ whose graphs are vertical translations of one another. A particular antiderivative can be choosen by assigning a specific value to C.
- **Example 2.** Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

4.8. Antiderivatives

- **Some Antiderivative Formulas**

Function	General Antiderivative
x^n	$\frac{1}{n+1}x^{n+1} + C,\ n \neq -1$
$\sin kx$	$-\frac{1}{k}\cos kx + C$
$\cos kx$	$\frac{1}{k}\sin kx + C$
$\frac{1}{x}$	$\ln\lvert x\rvert + C,\ x \neq 0$
e^{kx}	$\frac{1}{k}e^{kx} + C$

4.8. Antiderivatives

- **Antiderivative Linearity Rule**
- 1. Constant Multible Rule: $kf(x) \longrightarrow kF(x) + C$
- 2. Negative Rule: $-f(x) \longrightarrow -F(x) + C$
- 3. Sum or Difference Rule: $f(x) \pm g(x) \longrightarrow F(x) \pm G(x) + C$
- **Example 4.** Find the general antiderivative of $f(x) = \dfrac{3}{\sqrt{x}} + \sin 2x$

4.8. Antiderivatives

- **Initial Value Problems and Differential Equations** Finding an antiderivative for a function $f(x)$ is the same problem as finding a function $y(x)$ that satisfies the equation

$$\frac{dy}{dx} = f(x).$$

- This is called a differential equation since it is an equation involving an unknown function y that is being differentiated. We can fix the arbitrary constant by specifying an initial condition $y(x_0) = y_0$.
- The combination

$$\frac{dy}{dx} = f(x), \; y(x_0) = y_0$$

is called an **initial value problem**. The most general antiderivative $F(x) + C$ of $f(x)$ gives the **general solution** $y = F(x) + C$ of the differential equation. In addition, we solve the initial value problem by finding the **particular solution** that satisfies the initial condition $y(x_0) = y_0$.

Example 5. A hot air balloon ascending at the rate of 3.6 m/s is at a height 24.5 m above the ground when a package is dropped. How long does it take to reach the ground?

- **Indefinite Integrals**
- **Definition** The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x, and is defined by

$$\int f(x)\,dx.$$

- The symbol $\int$ is an integral sign. The function f is the integrand of the integral, and x is the variable of integration. So if F is an antiderivative of f then

$$\int f(x)\,dx = F(x) + C.$$

- **Example 6.** Evaluate $\int 2x dx$, $\int \cos x dx$ and $\int \left(\frac{1}{x} + 2e^{2x}\right) dx$.

$\int x^n dx = \frac{1}{n+1}x^{n+1} + C,\ n \neq -1$

$\int \sin x dx = -\cos x + C$

$\int \cos x dx = \sin x + C$

$\int \sec^2 x dx = \tan x + C$

$\int \csc^2 x dx = -\cot x + C$

$\int \sec x \tan x dx = \sec x + C$

$\int \csc x \cot x dx = -\csc x + C$

$\int e^x dx = e^x + C$

$\int \frac{1}{x} dx = \ln |x| + C,\ x \neq 0$

$\int a^x dx = \frac{1}{\ln a} a^x + C$

$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$

$\int \frac{1}{1+x^2} dx = \arctan x + C$

$\int \frac{1}{\sqrt{1+x^2}} dx = \ln\left(x + \sqrt{1+x^2}\right) + C$

$\int \frac{1}{\sqrt{x^2-1}} dx = \ln\left(x + \sqrt{x^2-1}\right) + C$

8.1. Using Basic Integration Formulas

- **Example 1.** Evaluate the following integrals;
- $(a)\ \int \left(2\sin x + \sqrt[3]{x^2} + \frac{2}{x}\right) dx$
- $(b)\ \int \left(\dfrac{4}{\cos^2 x} + 3e^x - \dfrac{2}{1+x^2}\right) dx$
- $(c)\ \int \dfrac{2t^2 + t^2\sqrt{t} - 1}{t^3} dt$

8.1. Using Basic Integration Formulas

- **The Substitution Rule** Sometimes we can not evaluate a given integral directly, then we use several kinds of methods to simplify the given one. This first method depends on finding a substitution that changes an integral which we can not evaluate directly into one that we can. If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

- **Example 2.** Evaluate the following integrals;
- (a) $\int (1-4x)^6\, dx$
- (b) $\int \dfrac{(4+\ln x)^5}{x} dx$
- (c) $\int \dfrac{e^{\sqrt{x}}}{\sqrt{x}} dx$

8.1. Using Basic Integration Formula

- (d) $\int \tan x dx$
- (e) $\int \dfrac{x^4}{1+x^{10}} dx$
- (f) $\int \dfrac{u'(x)}{u(x)} dx$
- (g) $\int \dfrac{\sin x - \cos x}{\cos x + \sin x} dx$

(h) $\int [u(x)]^{\alpha} u'(x)\, dx$, $\alpha \neq -1$

(i) $\int \sqrt{\sin^5 x} \cos x dx$

(j) $\int \cos(ax)\, dx$

(k) $\int e^{ax} dx$

(l) $\int \frac{dx}{a^2 + x^2}$

(m) $\int \frac{dx}{\sqrt{a^2 - x^2}}$, $a > 0$

(n) $\int \frac{dx}{\sqrt{3 - 2x - x^2}}$

(o) $\int x^2 \sqrt{x - 2} dx$

(p) $\int x^3 \cos\left(x^4 + 2\right) dx$

(q) $\int \dfrac{xdx}{\sqrt{1-4x^2}}$

(r) $\int \dfrac{dt}{\sqrt{t}\left(1+\sqrt{t}\right)^2}$

(s) $\int \dfrac{\cos\sqrt{t}}{\sqrt{t}\sin^2\sqrt{t}}dt$

Integration by Parts Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)\, g(x)\, dx.$$

It is useful when f can be differentiated repeatedly and g can be integrated repeatedly without difficulty. The integrals

$$\int x \cos x dx \text{ and } \int x^2 e^x dx$$

are such integrals because $f(x) = x$ or $f(x) = x^2$ can be differentiated repeatedly to become zero, and $g(x) = \cos x$ or $g(x) = e^x$ can be integrated repeatedly without difficulty. Integration by parts also applies to integrals like

$$\int \ln x dx \text{ and } \int e^x \cos x dx.$$

8.2. Integration by Parts

- **Product Rule in Integral Form** If f and g are differentiable functions of x, the product rule says that
$$\frac{d}{dx}\left[f\left(x\right)g\left(x\right)\right]=f'\left(x\right)g\left(x\right)+f\left(x\right)g'\left(x\right)$$
- If we integrate both sides
$$\int\frac{d}{dx}\left[f\left(x\right)g\left(x\right)\right]dx=\int f'\left(x\right)g\left(x\right)dx+\int f\left(x\right)g'\left(x\right)dx$$
and so we obtain
$$\begin{aligned}\int f\left(x\right)g'\left(x\right)dx &= \int\frac{d}{dx}\left[f\left(x\right)g\left(x\right)\right]dx-\int f'\left(x\right)g\left(x\right)dx\\ &= f\left(x\right)g\left(x\right)-\int f'\left(x\right)g\left(x\right)dx\end{aligned}$$
which is called the **integration by parts** formula.
- If we let $u=f\left(x\right)$ and $v=g\left(x\right)$ then the formula can be represented as
$$\int udv=uv-\int vdu.$$

8.2. Integration by Parts

- **Example 3.** Evaluate the following integrals;
- (a) $\int xe^{3x}dx$
- (b) $\int x\sin 2xdx$
- (c) $\int\frac{xdx}{\cos^2 x}$
- (d) $\int x^5\ln xdx$

- (e) $\int \ln x dx$
- (f) $\int (3x+1)^2 \log(2x)\, dx$
- (g) $\int \arctan x dx$
- (h) $\int e^x \cos x dx$

- (k) $\int e^{-2x} \sin 2x dx$
- (l) $\int x^2 e^{-x} dx$

8.2. Integration by Parts

- $(m)\ \int (x+1)\, e^{2x}\, dx$
- $(n)\ \int \dfrac{xe^x}{(x+1)^2}\, dx$

8.3. Trigonometric Integrals

- Trigonometric integrals involve algebraic combinations of six basic trigonometric functions. The general idea is to transform the integrals we have to find into integrals that are easier to work with.
- **Products of Powers of Sines and Cosines** For the integrals of the form
$$\int \sin^m x \cos^n x dx$$
where m and n are nonnegative integers; we have three cases.
- (i) If m is odd to obtain a simple integral make the substitution $\cos x = t$ and use the identity $\cos^2 x + \sin^2 x = 1$.
- (ii) If n is odd to obtain a simple integral make the substitution $\sin x = t$ and use the identity $\cos^2 x + \sin^2 x = 1$.
- (iii) If both m and n are even we substitute $\sin^2 x = \dfrac{1-\cos 2x}{2}$ or $\cos^2 x = \dfrac{1+\cos 2x}{2}$ to reduce the integrand to one in lower powers of $\cos 2x$.

8.3. Trigonometric Integrals

- **Example 4.** Evaluate the following integrals;
- $(a)\ \int \sin^5 x \cos^2 x dx$
- $(b)\ \int \sin^4 x \cos^3 x dx$
- $(c)\ \int \sin^9 x \cos^3 x dx$

8.3. Trigonometric Integrals

- $(d)\ \int \cos^5 x dx$
- $(e)\ \int \sin^2 x \cos^2 x dx$
- $(f)\ \int \sin^4 x \cos^2 x dx$

Products of Powers of $\tan x$ and $\sec x$

For the integrals of the form

$$\int \tan^m x \sec^n x dx$$

where m and n are nonnegative integers; we use the identity

$$\tan^2 x = \sec^2 x - 1$$

and integrate by parts when necessary to reduce the higher powers to lower powers. (For the products of powers of $\cot x$ and $\csc x$, use the identity $\csc^2 x = 1 + \cot^2 x$.)

8.3. Trigonometric Integrals

- **Example 5.** Evaluate the following integrals;
- (a) $\int \tan^6 x \sec^4 x dx$
- (b) $\int \sec^2 x dx$
- (c) $\int \tan^3 x dx$
- (d) $\int \tan^2 x \sec x dx$
- (e) $\int \cot^3 x \csc^4 x d$

- **Products of Sines and Cosines** The integrals

$$\int \sin mx \sin nx dx, \ \int \sin mx \cos nx dx \text{ and } \int \cos mx \cos nx dx$$

can be evaluated through integration by parts. In addition, to use the following identities is simpler:

$$\begin{aligned} \sin mx \sin nx &= \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right] \\ \sin mx co x nx &= \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right] \\ \cos mx \cos nx &= \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right] \end{aligned}$$

- **Example 6.** Evaluate the following integrals;
- (a) $\int \sin 4x \sin 7x dx$
- (b) $\int \cos 4x \cos 3x dx$
- (c) $\int \cos 3x \sin 5x dx$