CHAPTER 1. FUNCTIONS

1.1. Functions and Their Graphs

- Functions are a tool describing the real world in mathematical terms. All we know that

the area of a circle depends on its radius,
the price of an object depends on the supply and the demand.

- In each case, the value of one variable quantity, say y, depends on the value of another variable quantity, which we might call x. We say that "y is a function of x" and write symbolically as

$$y = f(x) \qquad (\text{"}y \text{ equals } f \text{ of } x\text{"}).$$

- In this notation,

the symbol f represents the function,

the letter x is the **independent variable** *representing the input variable of f* and

y is the **dependent variable** *or* **output value** *of f at x.*

1.1. Functions and Their Graphs

- **Definition** A **function** f from a set D to a set Y is a rule that assigns a unique (single) element $f(x) \in Y$ to each element $x \in D$.
- **Example 1.** (a) Is the area of a square, a function of its lenght of sides?
- (b) Is the area of a rectangle, a function of its diagonal?

1.1. Functions and Their Graphs

- The set D of all possible input values is called the **domain** of the function, usually denoted by D_f.
- When we define a function $y = f(x)$ with a formula and the domain is not stated explicitly or restricted by context, the domain is assumed to be **the largest set of real x-values for which the formula gives real y-values**, which is called the **natural domain**.

1.1. Functions and Their Graphs

- The set of all output values of $f(x)$ as x varies throughout D is called the **range** of the function which is denoted by R_f.
- The domain and the range of a function can be any set of objects, but often **in calculus** they are **sets of real numbers interpreted as points of a coordinate line**.
- When the range of a function is a set of real numbers, the function is said to be **real valued**.

1.1. Functions and Their Graphs

- The domains and ranges of most real-valued functions of a real variable we consider are **intervals or combinations of intervals**. The intervals may be open (such as $]1,2[$, $]2,\infty[$ or $]-\infty,2[$), closed (such as $[1,2]$, $]-\infty,2]$ or $[2,\infty[$) or half open (such as $]1,2]$ or $[1,2[$).
- **Example 2.** Identify the domain and the range of the following functions:
- (a) $f(x) = \dfrac{1}{x}$,
- (b) $h(t) = t^3 + 1$,

1.1. Functions and Their Graphs

- (c) $g(x) = \sqrt{1-x}$,
- (d) $q(s) = \sqrt{1-s^2}$.

1.1. Functions and Their Graphs

Graphs of Functions

- If f is a function with domain D, its graph consist of all points $(x, f(x))$ in the Cartesian plane whenever x varies in D. In set notation, the graph is
$$G_f = \{(x, f(x)) \mid x \in D\} \subset \mathbb{R}^2.$$
- **Example 3.** Graph the functions $f(x) = x + 2$ and $g(x) = x^2$ over
- (a) the interval $[-2, 2[$,
- (b) the interval $[0, 3]$,
- (c) the set of natural numbers less than 5.
- **Homework 1.** Can you represent a function numerically? What is a scatterplot?

The Vertical Line Test For a Function

Not every curve in the coordinate plane can be the graph of a function. A function f can only have one value $f(x)$ for each x, so no vertical line can intersect the graph of a function more than once. For example, a circle can not be the graph of a function.

Piecewise-Defined Functions

- Sometimes a function is described in pieces by using different formulas on different parts of its domain. One example is the absolute value function on reals

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

- Absolute value of a function, greatest integer function (or integer **floor function**), least integer function (or integer **ceiling function**) and signum function (or the **sign function**) are important examples of piecewise-defined functions.

1.1. Functions and Their Graphs

- **Absolute value of a function:** Let f be a real function. The function

$$|f|(x) = |f(x)| = \begin{cases} f(x), & \text{if } f(x) \geq 0 \\ -f(x), & \text{if } f(x) < 0 \end{cases}$$

is called the absolute value function of f.

- **Example 4.** Sketch the graph of $y = |f(x)|$ where $f(x) = x^2 - 3x - 4$.

1.1. Functions and Their Graphs

Graphing functions generated by the absolute value function:
Let f be a real function, then

- To plot the graph of the curve $y = f(|x|)$, sketch the graph of f for all $x \geq 0$ and also plot the symmetric about the y-axis.
- To plot the graph of the curve $y = |f(|x|)|$, sketch the graph of $y = f(|x|)$. The graph of $y = |f(|x|)|$ is the same as the graph of $y = f(|x|)$ whenever $y = f(|x|)$ is positive and the graph of $y = |f(|x|)|$ is the symmetric of the graph of $y = f(|x|)$ about x-axis whenever $y = f(|x|)$ is negative.

Example 5. Sketch the graph of $y = \left|x^2\right| - 2\left|x\right|$ and $y = \left|\left|x^2\right| - 2\left|x\right|\right|$.

- **Greatest integer (floor) function:** The function whose value at any real number x is the largest integer which is not greater than x, is called the floor function. It is denoted

$$floor\left(x\right) = \lfloor x \rfloor.$$

- **Example 6.** Sketch the graph of the following functions:
- (a) $f\left(x\right) = \lfloor x \rfloor$ over the interval $\left[-3, 3\right]$,

(b) $g(x) = x - f(x)$ over the interval $[-2, 2]$,

(c) $h(x) = \lfloor x^2 \rfloor$ over the interval $[0, 2]$.

1.1. Functions and Their Graphs

- $(d)\ s(x) = \left\lfloor \frac{x}{2} \right\rfloor$ over the interval $[-2, 2]$
- **Homework 2.** Make an investigation about the definition of the ceiling function, and solve Example 4. by replacing the floor function with the ceiling function

1.1. Functions and Their Graphs

- **Signum (sign) function:** For a given real function f, its sign function is defined as

$$g(x) := \begin{cases} \frac{|f(x)|}{f(x)}, & \text{if } f(x) \neq 0 \\ 0, & \text{if } f(x) = 0 \end{cases}.$$

- the sign function of any given function f is denoted as $sgnf$, and simply

$$sgnf(x) = \begin{cases} 1, & \text{if } f(x) > 0 \\ 0, & \text{if } f(x) = 0 \\ -1 & \text{if } f(x) < 0 \end{cases}.$$

Example 7. Sketch the graph of $sgnf$ where $f(x) = x^2 - 2x - 3$.

Increasing and Decreasing Functions

- **Definition** Let f be a function defined on an interval I.
- (a) f is called increasing on I if $f(a) < f(b)$ for all $a, b \in I$ satisfying $a < b$.
- (b) f is called decreasing on I if $f(a) > f(b)$ for all $a, b \in I$ satisfying $a < b$.

1.1. Functions and Their Graphs

- A subset A of reals is called symmetric if for all $x \in A$, $-x$ is also in A.
- Therefore, the interval $[-1, 1]$, the set of integers and also $\mathbb{R}$ itself is symmetric.
- **Definition** A function f defined on a symmetric set is
- *(a) an even function of x if $f(x) = f(-x)$ for every x in D_f,*
- *(b) an odd function of x if $f(-x) = -f(x)$ for every x in D_f.*
- The graph of an even function is **symmetric about the y-axis** and the graph of an odd function is **symmetric about the origin**.
- **Example 8.** Give an example of an even (respectively; odd/neither even nor odd/both even and odd) function.

1.1. Functions and Their Graphs

Common Functions

- **Linear Functions** A function of the form $f(x) = mx + b$, for constants m and b, is called a **linear function**.
- Lines through the origin, constant functions are linear functions
- **Definition** Two variables y and x are **proportional** (to one another) if one is always a constant multiple of the other; that is, if

$$y = kx$$

for some none zero constant k.
- If the variable y is proportional to the reciprocal $1/x$, then sometimes it is said that y is **inversely proportional** to x.

- **Power Functions** A function $f(x) = x^a$, where a is constant, is called a **power function**. There are several important cases to consider.
- (a) If $a = n$ is a positive integer sketch the grapshs of f for $n = 1, 2, 3, 4, 5$ and compare.
- (b) If $a = -1$ or $a = -2$ sketch the graphs and compare.
- (c) If $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$ and $\frac{2}{3}$ sketch the graphs and compare.

- **Polynomials** A function p is a **polynomial** if

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are real constants (called **coefficients** of the polynomial).
- All polynomials have domain $(-\infty, \infty)$. If the **leading coefficient** $a_n \neq 0$ and $n > 0$, then n is called the **degree** of the polynomial.
- Lineer functions with $m \neq 0$ are polynomials of degree 1, polynomials of degree 2, usually written as $p(x) = ax^2 + bx + c$, are called quadratic functions, and also cubic functions are polynomials of degree 3.

- **Rational Functions** A **rational function** is a quotient or ratio $f(x) = p(x)/q(x)$, where p and q are polynomials. The domain of a rational function is the set of reals for which $q(x) \neq 0$.
- **Algebraic Functions** Any function constructed from polynomials using algebraic operations lies within the class of **algebraic functions**, such as

$$y = \frac{x^{1/3}}{x-4} \text{ and } z = s(1-s)^{2/3}.$$

1.1. Functions and Their Graphs
Common Functions

Transcendental Functions
These are functions that are not algebraic. They include the trigonometric, inverse trigonometric, exponential and logarithmic functions. These functions will be investigated particularly in the sequal.