Convex Optimization 12/08/2021

Assignment 2 Jay Li

Chapter 2 Convex Sets

Convex Cones and generalized inequalities

2.28

By Sylvester's crieerion, $X \in S^n_+ \iff$ all principal minors of X have nonnegative determinants

Therefore, for S^1_+

$$x_1 \geq 0$$

for S^2_+,

$$x_1 \geq 0, \quad x_3 \geq 0, \quad x_1 x_3 - x_2^2 \geq 0$$

for S^3_+,

$$x_1 \geq 0, \; x_4 \geq 0, \; x_6 \geq 0$$

$$x_1 x_4 - x_2^2 \geq 0, \quad x_1 x_6 - x_3^2 \geq 0, \quad x_4 x_6 - x_5^2 \geq 0$$

$$x_1 (x_4 x_6 - x_5^2) - x_2 (x_2 x_6 - x_3 x_5) + x_3 (x_2 x_5 - x_3 x_4) \geq 0$$

$$\hookrightarrow x_1 x_4 x_6 - x_1 x_5^2 - x_2^2 x_6 + 2 x_2 x_3 x_5 - x_3^2 x_4 \geq 0.$$

▨

2.33

(a) A cone $K \subseteq \mathbb{R}^n$ is a proper cone if

1. K is convex.
2. K is closed
3. K is solid
- K is pointed

For convexity, let $X', X'' \in K_{m+}$.

$\theta X' + (1-\theta)X'' \in K_{m+}$, for $0 \le \theta \le 1$

because $\theta X_i' + (1-\theta)X_i'' \ge \theta X'_{i+1} + (1-\theta)X''_{i+1}$

Closedness comes from semi-inequalities

solid because we can find interior $X \in K_{m+}$

For pointedness,

if $X \in K_{m+}$, $-X \in K_{m+}$.

then $X_i \ge X_{i+1}$ and $-X_i \ge -X_{i+1}$ and $0 \ge X_n \ge 0$

which gives $X = 0$ zero-vector

(b)

dual cone $K^*_{m+} = \{ y \mid x^T y \ge 0 \text{ for all } x \in K_{m+} \}$

$x^T y = \sum_{i=1}^{n} x_i y_i$ = the identity in the book.

from which we can infer

$y_1 \ge 0$, $y_1 + y_2 \ge 0$, $y_1 + y_2 + y_3 \ge 0$ ..., $y_1 + \cdots + y_n \ge 0$

Chapter 3 Convex functions

Definition of Convexity

3.2

Not convex

Can show a line segment to be not convex

① For the first level sets

→ Can be convex because all line segments are above.

maybe not Not concave

Can be quasiconvex, because sublevel sets can be convex

Not quasiconcave, e.g. superlevel set $f(x) \geq 2$ is not convex.

② For the 2nd level sets

Not convex, as the ~~...~~ spacing increases as $1 \to 6$.

line segment ~~...~~ not strictly above the function.

Can be concave

Not quasiconvex, as sublevel set $f(x) \leq 3$ is not convex

Can be quasiconcave.

3.5

for any s, $f(sx)$ is convex
because it's a composition with an affine mapping

Then $\int_0^1 f(sx)\,ds$ is convex
because it is a nonnegative weighted integration.

let $t = sx$
then $F(x) = \int_0^x f(t) \frac{dt}{x}$

$= \frac{1}{x} \int_0^x f(t)\, dt$ for $x \in \mathbb{R}_{++}$

is convex in x.

3.6

① halfspace

The function is a plane.

i.e. affine

② convex cone

max of planes that intersects origin.

i.e. positively homogeneous $f(\alpha x) = \alpha f(x)$ for $\alpha \geq 0$.

③ polyhedron

max of planes.

i.e. piecewise-affine.

Examples

3.15

(a) By l'Hopital's rule

$$\lim_{\alpha \to 0} u_\alpha(x) = \lim_{\alpha \to 0} \frac{\ln(x)\, x^\alpha}{1}$$

$$= \ln(x)$$

$$= u_0(x)$$

(b) $u_\alpha(1) = \frac{1-1}{\alpha} = 0$ for $\alpha \neq 0$

$u_0(1) = 0 = \log 1$ for $\alpha = 0$.

For monotone increasy

$u'_\alpha(x) = \frac{\alpha x^{\alpha-1}}{\alpha} = x^{\alpha-1} \geq 0$ for $x \geq 0$, $0 < \alpha \leq 1$

$u'_0(x) = \frac{1}{x} \geq 0$. for $x > 0$.

i.e. first derivative ≥ 0.

$\Rightarrow$ monotone nondecreasing

For concavity

$u''_\alpha(x) = (\alpha - 1)\, x^{\alpha-2} \leq 0$ for $x \geq 0$, $0 < \alpha \leq 1$

$u''_0(x) = -x^{-2} \leq 0$ for $x > 0$

i.e second derivative ≤ 0.

$\Rightarrow$ concavity.

3.16

(b) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$

its Hessian $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is indefinite

$\Rightarrow$ Neither Convex nor Concave

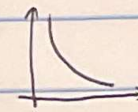
sublevel sets not convex.

e.g. $x_1 x_2 \leq 1$.

$\Rightarrow$ not quasiconvex.

However, superlevel sets are convex,

as can be seen from plots of $x_2 = c/x_1$.



$\Rightarrow$ quasi concave

$\boxed{4}$

(c) $f(x_1, x_2) = \frac{1}{x_1 x_2}$

Not concave,

because if it is concave, then

$x_1 x$ being $\frac{1}{x_1 x_2}$ is a nonincreasing convex func

of a concave function, would be convex

Convex though.

if we compute Hessian, we get $\begin{bmatrix} \frac{2}{x_1^3 x_2} & \frac{1}{x_1^2 x_2^2} \\ \frac{1}{x_1^2 x_2^2} & \frac{2}{x_1 x_2^3} \end{bmatrix}$

~~the~~ which is $\succeq 0$.

Therefore quasi-convex as well

From graph, see that it is not quasi-concave.

$\boxed{4}$

$f(x_1, x_2) = \frac{x_1}{x_2}$

(d) From graph, see that it is neither convex nor ~~co~~ concave,

Also can ~~been~~ be seen from Hessian

It is quasi-convex.

proof: let $\frac{x_1}{x_2} \le \alpha$, $\frac{y_1}{y_2} \le \alpha$.

then $\frac{\theta x_1 + (1-\theta) y_1}{\theta x_2 + (1-\theta) y_2} \le \frac{\theta \alpha x_2 + (1-\theta) \alpha y_2}{\theta x_2 + (1-\theta) y_2} = \alpha$

i.e. sublevel sets are convex.

similarly can prove for quasi-concavity

sublevel and superlevel sets are halfspaces.

$f(x_1, x_2) = \frac{x_1^2}{x_2}$

(e) Compute Hessian $\rightarrow \begin{bmatrix} \frac{2}{x_2} & -\frac{2x_1}{x_2^2} \\ -\frac{2x_1}{x_2^2} & \frac{2x_1^2}{x_2^3} \end{bmatrix} \succeq 0$

Therefore convex, and not concave

$\Rightarrow$ quasi-convex

From graph, see that it is not quasi-concave

3.18

(b) $f(X) = (\det X)^{1/n}$ prove concave on $\operatorname{dom} f = S^n_{++}$

Define $g(t) = f(Z + tV)$ where $Z \succ 0$ and $V \in S^n$

$$g(t) = \left(\det(Z + tV)\right)^{1/n}$$
$$= \left(\det Z^{1/2} \det(I + t Z^{-1/2} V Z^{-1/2}) \det Z^{1/2}\right)^{1/n}$$
$$= (\det Z)^{1/n} \left(\prod_{i=1}^{n} (1 + t\lambda_i)\right)^{1/n}$$

where λ_i are the eigenvalues of $Z^{-1/2} V Z^{-1/2}$

$\det Z > 0$, and geometric mean $\left(\prod_{i=1}^{n} x_i\right)^{1/n}$ is concave on $\mathbb{R}^n_{++}$

$\Rightarrow$ ~~f(X) is~~ $g(t)$ is concave on $\{t \mid Z + tV \succ 0\}$

$\Rightarrow$ $f(X)$ is concave on S^n_{++}

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what is unknown is how $Z + tV \succ 0$

$\Rightarrow$ $1 + t\lambda_i$ are positive.

3.24

(f) quartile(x) = $\inf\{\beta \mid \text{prob}(x \le \beta) \ge 0.25\}$

Not convex or concave,

because the function is discrete.

Is quasi-convex

let $q(p) \le a_k \iff p_1 + \cdots + p_k \ge 0.25$

$q(p') \le a_k \iff p_1' + \cdots + p_k' \ge 0.25$

$\Rightarrow \theta p_1 + \cdots + \theta p_k$
$+ (1-\theta) p_1' + \cdots + (1-\theta) p_k' \ge 0.25$

i.e. sublevel set is convex.

Is also quasi-concave.

let $q(p) \ge a_k \iff p_1 + \cdots + p_{k-1} < 0.25$

$q(p') \ge a_k \iff p_1' + \cdots + p_{k-1}' < 0.25$.

$\Rightarrow \sum \theta p_i + (1-\theta) p_i' < 0.25$

$\Rightarrow q(\theta p + (1-\theta) p') \ge a_k$

i.e. superlevel sets also convex.

cardinality with prob $\geq 90\%$.

(g) Not convex or concave because discrete function

Not quasi-convex.

because eg $P_1 = 100\%$, $P_2 = 0\%$. $C = 1$

$P_1' = 0\%$, $P_2' = 100\%$. $C' = 1$

$C(\frac{1}{2}P + \frac{1}{2}P') = 2$.

sublevel set not convex.

Is quasi-concave.

because $\theta p + (1-\theta)p'$ would make the prob distribution more ~~spd~~ spread out.

therefore more elements to get 90%.

(h) min width interval w/ prob 90%

$W(p) = \inf\{\beta - \alpha \mid \text{prob}(\alpha \le x \le \beta) \ge 90\%\}$.

Not convex or concave because discrete valued.

Not quasi-convex.

eg $a_1 = 0$, $a_2 = 1$

$P_1 = 100\%$, $P_2 = 0\%$ $\rightarrow$ $W = 0$

$P_1' = 0\%$, $P_2' = 100\%$ $\rightarrow$ $W = 0$

but $W(\frac{1}{2}P + \frac{1}{2}P') = 1$

sublevel set not convex.

~~Not~~

Is quasi-concave

eg. $a_1 = 0$, $a_2 = 1$, $a_3 = 2$, $a_4 = 3$

$P_1 = 11\%$, $P_3 = 89\%$ $\rightarrow$ $W = 2$

$P_2' = 89\%$, $P_4' = 11\%$ $\rightarrow$ $W = 2$

$W(\frac{1}{2}P + \frac{1}{2}P')$

Again linear comb of probs spreads the distribution

$\Rightarrow$ min width interval increases.

i.e. superlevel set convex.

Conjugate functions

3.36 Derive the conjugates

(a)

Max function $f(x) = \max_i x_i$ on $\mathbb{R}^n$

conjugate function $f^*(y) = \sup_{x \in \mathrm{dom} f} (y^T x - f(x))$.

case 1. $\sum y_i > 1$

we can have $x_i = \max x_i$

$\Rightarrow f^*(y) \to \infty$

case 2. ~~simply~~ Any $y_i < 0$

we can have $x_i \to -\infty$ and other $x_j = 0$.

for which $f^*(y) \to \infty$

case 3 $y_i \geq 0$, $\sum y_i < 1$

we can have $x_i = x_j \to -\infty$ for all i, j.

then $f^*(y) \to \infty$

case 4 $y_i \geq 0$, $\sum y_i = 1$

$\Rightarrow y^T x \leq \max x_i$ for all x

$\Rightarrow f^*(y) = 0$

i.e. $f^*(y)$ is the indicator function of $\{y_i \geq 0 \mid \sum y_i = 1\}$

probability simplex

$\boxtimes$

(d) power function $f(x) = x^p$ on $\mathbb{R}_{++}$.

$$f^* = \sup_{x \in \mathbb{R}_{++}} (yx - x^p)$$

① $p > 1$.

for $y \le 0$, $f^* \to 0$

for $y > 0$, let $\frac{df^*}{dx} = 0 = y - px^{p-1}$

$$\Rightarrow x = \left(\frac{y}{p}\right)^{\frac{1}{p-1}}$$

substitute, get $f^* = y\left(\frac{y}{p}\right)^{\frac{1}{p-1}} - \left(\frac{y}{p}\right)^{\frac{p}{p-1}}$.

② $p < 0$.

for $y > 0$, $f^* \to \infty$

for $y = 0$, $f^* \to 0$

for $y < 0$, again let $\frac{df^*}{dx} = 0$.

$$\Rightarrow f^* = y\left(\frac{y}{p}\right)^{\frac{1}{p-1}} - \left(\frac{y}{p}\right)^{\frac{p}{p-1}}$$