Convex Optimization 09/20/2021

Assignment 1 Jay Li

Chapter 2 Convex Sets

Definition of convexity

2.1

Prove by induction.

① For $k=2$, the statement is true by definition of convex set.

② Assume true for k, now prove for $k+1$.

i.e. given $x_1, \dots, x_{k+1} \in C$,

show $\theta_1 x_1 + \dots + \theta_{k+1} x_{k+1} \in C$ where $\theta_1 + \dots + \theta_{k+1} = 1$, $\theta_i \geq 0$

$$\theta_1 x_1 + \dots + \theta_{k+1} x_{k+1} = \theta_1 x_1 + \dots + \theta_k x_k + \theta_{k+1} x_{k+1}$$
$$= (1-\theta_{k+1}) \underbrace{\left[\frac{\theta_1}{1-\theta_{k+1}} x_1 + \dots \frac{\theta_k}{1-\theta_{k+1}} x_k \right]}_{\#} + \theta_{k+1} x_{k+1}$$

Now for #, $\frac{\theta_1}{1-\theta_{k+1}} + \dots + \frac{\theta_k}{1-\theta_{k+1}} = \frac{\theta_1 + \dots + \theta_k}{1-\theta_{k+1}} = 1$. $\frac{\theta_i}{1-\theta_{k+1}} \geq 0$

and $x_1, \dots, x_k \in C$.

By inductive assumption, $\# \in C$.

Therefore $(1-\theta_{k+1}) \# + \theta_{k+1} x_{k+1} \in C$. $1-\theta_{k+1} \geq 0$.

because $\# \in C$, $x_{k+1} \in C$, and $1-\theta_{k+1} + \theta_{k+1} = 1$

(definition of convex set).

i.e.

$\theta_1 x_1 + \dots + \theta_{k+1} x_{k+1} \in C$

③ By induction, the statement is true for all $k \geq 2$ ////

2.2

First, show the statement for affine sets.

① prove: a set is affine $\longrightarrow$ its intersection with any line is affine

Let $x_1, x_2 \in C$ which is affine

and let $x_1 = \theta_1 x' + (1-\theta_1) x''$ for some θ_1, θ_2

$x_2 = \theta_2 x' + (1-\theta_2) x''$ where x' and x'' are arbitrary.

now show $x^* = \psi x_1 + (1-\psi) x_2 \in \cap$ intersection

for any ψ.

$x^* \in C$ by definition of affine set.

Also $x^* = \underbrace{[\psi \theta_1 + (1-\psi)\theta_2]}_{a} x' + \underbrace{[\psi(1-\theta_1) + (1-\psi)(1-\theta_2)]}_{b} x''$

$a+b = \cancel{\psi\theta_1} + \cancel{\theta_2} - \cancel{\psi\theta_2} + \cancel{\psi} - \cancel{\psi\theta_1} + 1 - \cancel{\psi} - \cancel{\theta_2} + \cancel{\psi\theta_2}$

$= 1$

i.e. $x^* \in$ line. $\Rightarrow x^* \in \cap$.

② prove: If a set's intersection with any line is affine $\longrightarrow$ this set is affine

Let $x_1, x_2 \in C$.

now form the line $\theta x_1 + (1-\theta) x_2$

$\rightarrow x_1, x_2 \in \cap$.

because its affine. $\psi x_1 + (1-\psi) x_2 \in \cap$.

i.e. $\psi x_1 + (1-\psi) x_2 \in C$

which means C is affine.

$\blacksquare$

The proof for convex set is very similar, except with the extra condition that $1 \geq \psi \geq 0$

Examples

2.5

Let the first hyperplane be A.
second hyperplane be B.

$x' \in A$, $x'' \in B$.

the distance between these two parallel hyperplanes is

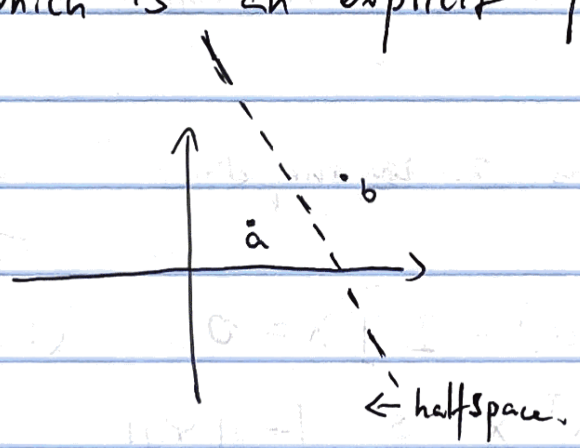
$$\|a^T(x'-x'')\|_2 / \|a\|_2$$
$$= \|b_1 - b_2\|_2 / \|a\|_2$$
$$= |b_1 - b_2| / \|a\|_2$$

2.7

expand out the norm inequality.

$$(x-a)^T(x-a) \le (x-b)^T(x-b)$$
$$\Rightarrow x^T x - 2a^T x + a^T a \le x^T x - 2b^T x + b^T b$$
$$\Rightarrow 2(b-a)^T x \le b^T b - a^T a.$$

which is an explicit form for halfspaces.



2.8

(a) It is a polyhedra.

First, for the case where a_1, a_2 are not collinear.

we try to solve for

$$[a_1 \; a_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x$$

$$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = (A^TA)^{-1}A^T x$$ which gives the closest point to x on the subspace spanned by $\{a_1, a_2\}$

And because x is on the span.

$$\Rightarrow \underline{[A(A^TA^{-1})A^T - I]\, x = 0}$$

the conditions on y_1, y_2 gives

$$\underline{\begin{bmatrix} -1 \\ -1 \end{bmatrix} \le (A^TA)^{-1}A^T x \le \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

For the case where a_1, a_2 are collinear.

let $a_1 = \varphi a_2$

we have similar equations & inequalities

$$[a_1(a_1^Ta_1)^{-1}a_1^T - I]\, x = 0$$

$$-1-\|\varphi\| \le (a_1^Ta_1)^{-1}a_1^T x \le 1+\|\varphi\|$$

Could be more explicitly expressed as the $\cap$ of 3 sets. (1 plane + 2 slabs)

(b) It is a polyhedra. The equalities & inequalities are already linear. They can be reorganized as

$$-Ix \le 0 \quad , \quad \begin{bmatrix} 1 & \cdots & 1 \\ a_1 & \cdots & a_n \\ a_1^2 & \cdots & a_n^2 \end{bmatrix} x = \begin{bmatrix} 1 \\ b_1 \\ b_2 \end{bmatrix}.$$

(c) It is not a polyhedra.

It is $\frac{1}{2^n}$ section of a unit sphere in $\mathbb{R}^n$, which is the intersection of an infinite number of hyperplanes.

(d) It is a polyhedra.

$$S = \{ x \in \mathbb{R}^n \mid x \succeq 0,\ x \preceq \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \mathbf{1} \}$$

we can prove by the following 2 parts.

① If $x \preceq \mathbf{1}$, $\rightarrow$ for all y with $\sum |y_i| = 1$, $x^T y \leq 1$.

$\Rightarrow$

$$x^T y = \sum x_i y_i \leq \sum y_i \leq \sum |y_i| = 1$$

② If for all y with $\sum |y_i| = 1$, $x^T y \leq 1$.

$\rightarrow x \preceq \mathbf{1}$.

$\Rightarrow$

Now suppose $x_k > 1$, and $y_i = 0$ $(i \neq k)$, and $y_k = 1$

then

$$x^T y = \sum_{i \neq k} x_i y_i + x_k y_k = x_k y_k > 1.$$ contradiction

therefore x_k cannot be > 1 for any k.

i.e. $x \preceq \mathbf{1}$.

2.11

Let $x', x'' \in C$ hyperbolic set.

Now show that $\theta x' + (1-\theta) x'' \in C$ for $0 \le \theta \le 1$

$$\theta x' + (1-\theta) x'' = \begin{bmatrix} \theta x_1' + (1-\theta) x_1'' \\ \theta x_2' + (1-\theta) x_2'' \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ge \begin{bmatrix} x_1'^{\theta} \; x_1''^{(1-\theta)} \\ x_2'^{\theta} \; x_2''^{(1-\theta)} \end{bmatrix}$$ by Jensen's inequality.

$$x_1 x_2 \ge (x_1' x_2')^{\theta} (x_1'' x_2'')^{1-\theta}$$

$$\ge 1$$ because $x_1' x_2' \ge 1$, $x_1'' x_2'' \ge 1$.

i.e. $\theta x' + (1-\theta) x'' \in C$.

$\blacksquare$

The proof for the generalization is similar, by apply the more general form of Jensen's inequality.

$\Rightarrow$ If $a_1, \ldots a_n \ge 0$, and $0 \le \theta_i \le 1$, $\Sigma \theta_i = 1$

then $\prod a_i^{\theta_i} \le \Sigma \theta_i a_i$

2.12

(a). A slab is convex, because it's a polyhedron.

(b) rectangle $= \{ x \in \mathbb{R}^n \mid \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \le I x \le \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} \}$.

~~polyh~~ polyhedron $\rightarrow$ convex.

(c) $\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} x \le \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

polyhedra $\rightarrow$ convex.

(d) prove it's convex.

let $x_1 \in C$, $x_2 \in C$

$\|\theta x_1 + (1-\theta) x_2 - x_0\|_2$ $\quad 0 \le \theta \le 1$

$= \|\theta (x_1 - x_0) + (1-\theta)(x_2 - x_0)\|_2$

$\le \theta \|x_1 - x_0\| + (1-\theta)\|x_2 - x_0\|_2$ triangle inequality.

$\le \theta$

expand the inequality

$$(x-x_0)^T(x-x_0) \le (x-y)^T(x-y)$$

$$\Rightarrow x^T x + x_0^T x_0 - 2x_0^T x \le x^T x + y^T y - 2y^T x$$

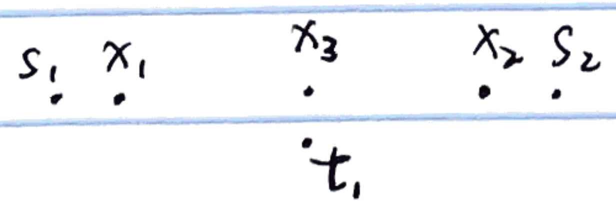
$$\Rightarrow 2(y-x_0)^T x \le y^T y - x_0^T x_0$$

which is a half space.

~~The~~ Therefore. because the set is the intersection of a (possibly infinite) number of convex sets.

it is convex.

(e) It is not convex. We can construct a case as following:



$S = \{s_1, s_2\}$

$T = \{t_1\}$.

$x_1, x_2 \in C$, but $\frac{x_1 + x_2}{2} = x_3 \notin C$.

~~(e)~~ (f) prove it is convex.

Let $x_1, x_2 \in C$.

$\theta x_1 + (1-\theta)x_2 + S_2$ $\quad 0 \leq \theta \leq 1$

$= \theta(x_1 + S_2) + (1-\theta)(x_2+S_2)$

$\subseteq S_1$, because $x_1 + S_2 \subseteq S_1$, $x_2 + S_2 \subseteq S_1$, and S_1 convex.

$\Rightarrow$ C is convex

(g) Graphically, I see that it is convex, but I have no proof.

Expanding the inequality, we can show that the set is in fact a ball.

i.e. convex

2.15

(a) It is convex.

Let $p_1, p_2 \in C$.

$p_3 = \theta p_1 + (1-\theta) p_2$. prove $p_3 \in C$, $0 \leq \theta \leq 1$

$$E f(x)\Big|_{p_3} = \Sigma [\theta p_{1i} + (1-\theta) p_{2i}] f(a_i)$$
$$= \theta \Sigma p_{1i} f(a_i) + (1-\theta) \Sigma p_{2i} f(a_i)$$
$$= \#$$

$\theta\alpha + (1-\theta)\alpha \leq \# \leq \theta\beta + (1-\theta)\beta$

$\Rightarrow \quad \alpha \leq \# \leq \beta$ ▭

(b) Convex

$\text{prob}(x > \alpha) \leq \beta \iff \sum_{k}^{n} p_i \leq \beta$, k is min i where $a_i > \alpha$

Can prove convexity similar as above.

Too lazy to prove from this point.

Will do by intuition & rough thinking.

(C) Not convex × A linear inequality $\sum_{i=1}^{n} p_i (|a_i|^3 - \alpha |a_i|) \leq 0$.

i.e. a halfspace

(d) Convex / linear inequality $\Sigma p_i a_i^2 \leq \alpha$

(e) Convex $\Sigma p_i a_i^2 \geq \alpha$

(f) ~~Convex~~ Convex × Not convex $\text{Var}(x) = Ex^2 - (Ex)^2 = \Sigma p_i a_i^2 - (\Sigma p_i a_i)^2 \leq \alpha$.

(g) Not convex × Convex. Can find a counter-example

Did not understand solution

(h) Convex ✓ with same logic, can convert to linear inequality.

(i) Convex.

$\sum_{i=1}^{k} p_i < 0.25$

with k chosen appropriately.

$\sum_{i=1}^{k} p_i \geq 0.25$.