数学模型 HW1

2018 年 3 月 9 日

1 第一题

1.1

Conservation law:

$$u_t + \left(\frac{u^2}{2} - \mu u_x\right)_x = 0, \quad x \in \mathbb{R}, \quad t > 0.$$

Flux function: $\frac{u^2}{2} - \mu u_x$.

1.2

Hopf-Cole 变换：

$$u = -a\frac{\partial}{\partial x}\ln\phi = -a\frac{\phi_x}{\phi}.$$

将 Hopf-Cole 变换带入到原方程，并利用 $a = 2\mu$,

$$-2\mu\frac{\phi\phi_{xt} - \phi_x\phi_t}{\phi^2} + 4\mu^2\frac{\phi\phi_x\phi_{xx} - \phi_x^3}{\phi^3} = -2\mu^2\frac{\phi^2(\phi_x\phi_{xx} + \phi\phi_{xxx} - 2\phi_x\phi_{xx}) - 2\phi\phi_x(\phi\phi_{xx} - \phi_x^2)}{\phi^4},$$

化简可得

$$-2\mu\frac{\phi\phi_{xt} - \phi_x\phi_t}{\phi^2} = -2\mu^2\frac{\phi\phi_{xxx} - \phi_x\phi_{xx}}{\phi^2},$$

即

$$\left(\frac{\phi_t}{\phi} - \mu\frac{\phi_{xx}}{\phi}\right)_x = 0.$$

因此

$$\frac{1}{\phi}\phi_t - \mu\frac{1}{\phi}\phi_{xx} = g(t)$$

是一个只关于 t 的函数。

2 第二题

2.1

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial t} + \frac{\partial f}{\partial p} \cdot \frac{\partial p}{\partial t}.$$

根据守恒律，我们有

$$\frac{\partial f}{\partial t} + \frac{\partial (f\dot{q})}{\partial q} + \frac{\partial (f\dot{p})}{\partial p} = 0.$$

将上式展开，我们有

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial t} + \frac{\partial f}{\partial p} \cdot \frac{\partial p}{\partial t} + f\left(\frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{p}}{\partial p}\right) = 0.$$

又由

$$\dot{p} = \frac{\partial H}{\partial q}, \quad \dot{q} = -\frac{\partial H}{\partial p},$$

因此

$$\frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{p}}{\partial p} = -\frac{\partial^2 H}{\partial p \partial q} + \frac{\partial^2 H}{\partial q \partial p} = 0.$$

因此我们有

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial t} + \frac{\partial f}{\partial p} \cdot \frac{\partial p}{\partial t} = 0.$$

2.2

对于任意区域 Ω，

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega f(t,q,p)\,\mathrm{d}q\,\mathrm{d}p = \int_\Omega \frac{\partial f}{\partial t}\,\mathrm{d}q\,\mathrm{d}p = \int_\Omega -\nabla\cdot(f\dot{q}, f\dot{p})\,\mathrm{d}q\,\mathrm{d}p = -\int_{\partial\Omega}(f\dot{q}, f\dot{p})\cdot\overrightarrow{\boldsymbol{n}}\,\mathrm{d}S$$

当 Ω 包含了 f 的支集时，根据上式我们可得

$$\frac{\mathrm{d}}{\mathrm{d}t}\int f(t,q,p)\,\mathrm{d}q\,\mathrm{d}p = 0.$$

3 第三题

3.1

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^2\,\mathrm{d}x &= \int \frac{\partial}{\partial t}|u|^2\,\mathrm{d}x \\ &= \int (u_t\bar{u} + u\bar{u}_t)\,\mathrm{d}x \\ &= \int \left(\frac{\hat{H}}{i\hbar}u\bar{u} - u\frac{\bar{\hat{H}}}{i\hbar}\bar{u}\right)\,\mathrm{d}x \\ &= 0\end{aligned}$$

其中最后一步利用了 $\hat{H}$ 是一个自伴算子。

3.2

由上题我们可知

$$\int |u(x,t)|^2 \, \mathrm{d}x = \int |u(x,0)|^2 \, \mathrm{d}x, \quad \forall t \geq 0.$$

因此所求的 ε 满足

$$1 = \int_{\mathbb{R}^d} a^2 e^{-2|x|^2/\varepsilon} \, \mathrm{d}x = a^2 \left(\frac{\varepsilon\pi}{2} \right)^{d/2}.$$

(此处利用了 $\int_{\mathbb{R}^d} e^{-A|x|^2} \, \mathrm{d}x = (\pi/A)^{d/2}$.)

因此

$$\varepsilon = \frac{2}{\pi a^{4/d}}.$$