

HW1

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September 25, 2025

Terminology

- System state Y : an unknown random variable.
- Measurement X : an observed random variable statistically related to Y .
- Estimator $\hat{Y}(X)$: a random variable defined as a function of X .
- Probability:
 - Prior: $P[Y]$
 - Posterior: $P[Y | X]$
 - Likelihood: $P[X | Y]$

- Objective (Risk):

$$R[\hat{Y}] = \mathbb{E}[\text{loss}(\hat{Y}(X), Y)]$$

- Optimal Estimator (Posterior form):

$$\hat{Y}(x) = \mathbb{1} \left\{ P[Y = 1 | X = x] \geq \frac{\text{loss}(1, 0) - \text{loss}(0, 0)}{\text{loss}(0, 1) - \text{loss}(1, 1)} P[Y = 0 | X = x] \right\}$$

- Proof:

$$\begin{aligned} \mathbb{E}[\text{loss}(\hat{Y}(X), Y)] &= \int_{-\infty}^{\infty} \mathbb{E}[\text{loss}(\hat{Y}(X), Y) | X = x] f_X(x) dx \\ &= \int_{-\infty}^{\infty} (\mathbb{E}[\text{loss}(\hat{Y}(X), 1) | X = x] P[Y = 1 | X = x] + \mathbb{E}[\text{loss}(\hat{Y}(X), 0) | X = x] P[Y = 0 | X = x]) f_X(x) dx \end{aligned}$$

- Thus, $\hat{Y}(x)$ is chosen according to the label (0 or 1) that minimizes the conditional expected loss.

- Optimal Estimator (Likelihood ratio form):

$$\hat{Y}(x) = \mathbb{1} \left\{ \frac{p(x | Y = 1)}{p(x | Y = 0)} \geq \frac{p_0 (\text{loss}(1, 0) - \text{loss}(0, 0))}{p_1 (\text{loss}(0, 1) - \text{loss}(1, 1))} \right\}$$

- Proof by rearrangement of the posterior condition.
 - This corresponds to a *likelihood ratio test*.

Types of errors and successes

- True Positive Rate: $P[\hat{Y} = 1 | Y = 1]$
- False Negative Rate: $P[\hat{Y} = 0 | Y = 1]$
- False Positive Rate: $P[\hat{Y} = 1 | Y = 0]$
- True Negative Rate: $P[\hat{Y} = 0 | Y = 0]$
- Precision: $P[Y = 1 | \hat{Y} = 1]$

Receiver Operating Characteristic(ROC) curve

- Example

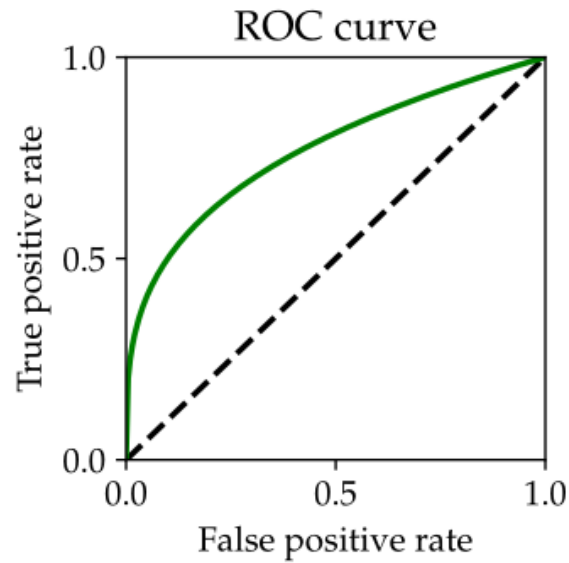


Figure 1: The ROC curve is plotted in the FPR–TPR plane.

- Lemma 2 (Neyman–Pearson Lemma) Suppose the likelihood functions $p(x | y)$ are continuous. Then the optimal probabilistic predictor that maximizes TPR subject to an upper bound on FPR is a deterministic likelihood ratio test.
- Properties
 - always passes through $(0,0)$ and $(1,1)$,
 - must lie above the main diagonal,
 - is concave.