HW1

Kevin Chang

September 12, 2025

Terminology

- System state Y: an unknown random variable.
- Measurement X: an observed random variable statistically related to Y.
- Estimator $\hat{Y}(X)$: a random variable defined as a function of X.
- Probability:
  - Prior: $P[Y]$
  - Posterior: $P[Y \mid X]$
  - Likelihood: $P[X \mid Y]$
- Objective (Risk):
$$R[\hat{Y}] = \mathbb{E}\big[loss(\hat{Y}(X), Y)\big]$$
- Optimal Estimator (Posterior form):
$$\hat{Y}(x) = \mathbb{1}\left\{P[Y = 1 \mid X = x] \;\geq\; \frac{loss(1,0) - loss(0,0)}{loss(0,1) - loss(1,1)}\, P[Y = 0 \mid X = x]\right\}$$
  - Proof:
$$\mathbb{E}[loss(\hat{Y}(X), Y)] = \int_{-\infty}^{\infty} \mathbb{E}[loss(\hat{Y}(X), Y) \mid X = x] f_X(x)\, dx$$
$$= \int_{-\infty}^{\infty} \Big(\mathbb{E}[loss(\hat{Y}(X), 1) \mid X = x]\, P[Y = 1 \mid X = x] + \mathbb{E}[loss(\hat{Y}(X), 0) \mid X = x]\, P[Y = 0 \mid X = x]\Big) f_X(x)\, dx$$
  - Thus, $\hat{Y}(x)$ is chosen according to the label (0 or 1) that minimizes the conditional expected loss.
- Optimal Estimator (Likelihood ratio form):
$$\hat{Y}(x) = \mathbb{1}\left\{\frac{p(x \mid Y = 1)}{p(x \mid Y = 0)} \;\geq\; \frac{p_0\,\big(loss(1,0) - loss(0,0)\big)}{p_1\,\big(loss(0,1) - loss(1,1)\big)}\right\}$$
  - Proof by rearrangement of the posterior condition.
  - This corresponds to a *likelihood ratio test*.

Types of errors and successes

- True Positive Rate: $P[\hat{Y} = 1 | Y = 1]$
- False Negative Rate: $P[\hat{Y} = 0 | Y = 1]$
- False Positive Rate: $P[\hat{Y} = 1 | Y = 0]$
- True Negative Rate: $P[\hat{Y} = 0 | Y = 0]$
- Precision: $P[Y = 1 | \hat{Y} = 1]$

Receiver Operating Characteristic(ROC) curve

- Example

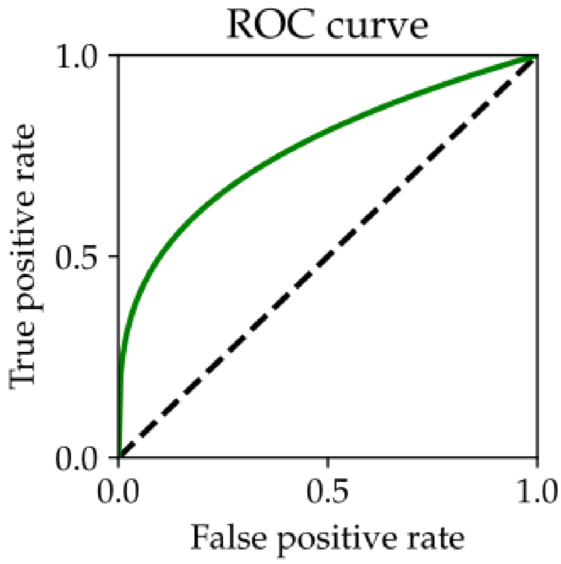

Figure 1: The ROC curve is plotted in the FPR–TPR plane.

- Lemma 2 (Neyman–Pearson Lemma) Suppose the likelihood functions $p(x \mid y)$ are continuous. Then the optimal probabilistic predictor that maximizes TPR subject to an upper bound on FPR is a deterministic likelihood ratio test.