CS 281A - Homework 1

August 28, 2025

This assignment is due on **September 12, 2025 at 11:59PM**. You should submit your solutions through **Peerceptiv**. You should already have access to the Peerceptiv course if you are registered on bCourses.

Errata 9/5/25:

- 1.1: You should expect strict equality.
- 2.3: Should be x_j (rather than x_k, as in the initial pset.)
- 3.2: M is full-rank.
- 3.3: h has a strictly (not monotonically) increasing derivative.

Problem 1

Let ξ_i be a sequence of independent random variables and X_n a function of the first n elements of the sequence. Define

$$\mathbb{E}_k[X_n] := \mathbb{E}[X_n|\xi_1, \dots, \xi_k]\,.$$

1. Show that
$$\mathbb{E}[(X_n - \mathbb{E}[X_n])^2] = \sum_{k=1}^{n} \mathbb{E}\left[(\mathbb{E}_k[X_n] - \mathbb{E}_{k-1}[X_n])^2\right]\,.$$

2. Suppose there is a constant B such that $|\mathbb{E}_k[X_n] - \mathbb{E}_{k-1}[X_n]| \le B$ with probability 1. Prove that
$$\lim_{n\to\infty} \frac{1}{n^2}\mathbb{E}[(X_n - \mathbb{E}[X_n])^2] = 0$$

3. Let Z_i be a sequence of independent, identically distributed random variables that take values in $[0, 1]$. Let $\mu = \mathbb{E}[Z_1]$ Use the above parts to prove
$$\lim_{n\to\infty} \mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n} Z_i - \mu\right)^2\right] = 0\,.$$

Problem 2

An ordering of the the set $\{1, \dots, T\}$ is a sequence of integers $\{\sigma_1, \dots \sigma_T\}$ where every number between 1 and T appears exactly once. A sequence of random variables $x_1, \dots, x_T$ is called exchangeable if the distribution of $x_1, \dots, x_T$ is equal to the distribution $x_{\sigma_1}, \dots, x_{\sigma_T}$ for any ordering of the indices.

1. Show that a sequence is exchangeable if and only if for any i and j, the distribution is the same when elements x_i and x_j are swapped.
2. Show that any iid sequence is exchangeable.
3. Show that for any indices i and j, $\mathbb{E}[x_j] = \mathbb{E}[x_i]$.

4. Let's suppose we want to predict values in an exchangeable sequence. Compute the expected mean squared error of using the mean of the first $T-1$ elements of the sequence to predict the last. That is, compute

$$\mathbb{E}\left[\left(x_T - \frac{1}{T-1}\sum_{t=1}^{T-1} x_t\right)^2\right].$$

when x_t is an exchangable sequence.

Problem 3

A symmetric, $n \times n$ matrix A is called positive definite if for all nonzero vectors $x \in \mathbb{R}^n$, $x^T A x > 0$. We say a function $f : \mathbb{R}^n \to \mathbb{R}$ has positive curvature if its Hessian matrix, $\nabla^2 f(x)$ is positive definite for all x.

1. Suppose f has positive curvature. For any vectors x and z, we can define a function that maps $\mathbb{R}$ to $\mathbb{R}$ by $g(t) := f(x + tz)$. Show the second derivative of g is positive for all t.
2. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ has positive curvature. Let M be a $n \times d$ dimensional **full-rank** matrix and v be an n-dimensional vector. Define $h : \mathbb{R}^d \to \mathbb{R}$ by $g(u) = f(Mu + v)$. Show that g has positive curvature.
3. Suppose f has positive curvature and h is a real valued function that is monotonically increasing and whose derivative is ~~monotonically~~ **strictly** increasing. Show that the composition function $h \circ f$, where $h \circ f(x) = h(f(x))$, has positive curvature.

Problem 4

Let A be an $n \times n$ positive definite matrix and let v be a n-vector with Euclidean norm 1. Define $x_0 = v$ and, for integers $k \geq 1$, set $x_k = Ax_{k-1}$.

1. Suppose all of the eigenvalues of A are less than 1. Show that $\|x_k\| \leq \beta^k$ for some number $\beta \in [0, 1]$. Here
$$\|z\| = \left(\sum_{k=1}^{n} z_k^2\right)^{1/2}.$$
2. Suppose one of the eigenvalues of A is equal to 2. Can the sequence $\|x_k\|$ converge to zero? Explain your answer.

Problem 5

Please submit a short (less than 250 words) final project proposal. The proposal should describe the project idea as well its connections to the course material. This proposal is flexible and can change as the semester continues. Choose a topic that is of interest to you and related to your research goals. You will submit something about the project along with every problem set, so you need to have an active, but potentially changing, project. The proposal should have the format and flow of the abstract of a conference or journal paper. This project should have the potential to gather real data to solve some prediction problem. Please state your data source and the prediction problem you are interested in studying.

Teams of 1, 2 or 3 are allowed for these projects. **Each project team should submit their information here so we know who is working together on what.** All members of a team should include the same proposal in their homework submission.