Stochastic Processes

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1 Moment Generating Function

- Moment Generating Function: $\mathbb{E}[e^{tX}]$

    - Property:
        * $\mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx$
        * $\mathbb{E}[e^{tX}] = \sum_{k=0}^{\infty} E[X^k] \frac{t^k}{k!}$
            · $e^{tx} = \sum_{k=0}^{\infty} \frac{(tx)^k}{k!}$
            · $E[e^{tX}] = E[\sum_{k=0}^{\infty} \frac{(tX)^k}{k!}] = \sum_{k=0}^{\infty} E[X^k] \frac{t^k}{k!}$
        * $\frac{d\mathbb{E}[e^{tX}]}{dt} = \mathbb{E}[X]$
        * $\mathbb{E}[e^{t(aX+b)}] = e^t b \mathbb{E}[e^{taX}]$
        * Not all random variables have Moment generating function

- Characteristic Function: $\mathbb{E}[e^{itX}]$

    - Property:
        * All random variables have Moment generating function

- Joint Moment Generating Function: $G(x, y) = \mathbb{E}[e^{xX} e^{yY}]$

- Property:

    - (Joint) moment generating funciton uniquely determines the (joint) CDF

- Example

    - Trapped miner's random walk
        * Miner has probability of $\frac{1}{3}$ to waste 3 hours in vain, $\frac{1}{3}$ to waste 5 hours in vain, and $\frac{1}{3}$ to spend 2 hours to go out of the mine.
        * X is the random variables of the hours to go out of the mine
        * Y_i is the random variables of the hours for the i-th action.
        * $\mathbb{E}[e^{tX}] = \mathbb{E}[e^{tX}|Y_1 = 2] + \mathbb{E}[e^{tX}|Y_1 = 3] + \mathbb{E}[e^{tX}|Y_1 = 5]$
          $= \mathbb{E}[e^{2t}] + \mathbb{E}[e^{t(X+3)}] + \mathbb{E}[e^{t(X+5)}]$
        * Find expectation and variance by joint moment generating function