Stochastic Processes

Kevin Chang

February 1, 2022

1 Moment Generating Function

- Moment Generating Function: $\mathbb{E}[e^{tX}]$
  - Property:
    - $\mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx$
    - $\mathbb{E}[e^{tX}] = \sum_{k=0}^{\infty} E[X^k]\frac{t^k}{k!}$
      - $e^{tx} = \sum_{k=0}^{\infty} \frac{(tx)^k}{k!}$
      - $E[e^{tX}] = E[\sum_{k=0}^{\infty} \frac{(tX)^k}{k!}] = \sum_{k=0}^{\infty} E[X^k]\frac{t^k}{k!}$
    - $\frac{d\mathbb{E}[e^{tX}]}{dt} = \mathbb{E}[X]$
    - $\mathbb{E}[e^{t(aX+b)}] = e^t b\mathbb{E}[e^{taX}]$
    - Not all random variables have Moment generating function
- Characteristic Function: $\mathbb{E}[e^{itX}]$
  - Property:
    - All random variables have Moment generating function
- Joint Moment Generating Function: $G(x, y) = \mathbb{E}[e^{xX}e^{yY}]$
- Property:
  - (Joint) moment generating funciton uniquely determines the (joint) CDF
- Example
  - Trapped miner's random walk
    - Miner has probability of $\frac{1}{3}$ to waste 3 hours in vain, $\frac{1}{3}$ to waste 5 hours in vain, and $\frac{1}{3}$ to spend 2 hours to go out of the mine.
    - X is the random variables of the hours to go out of the mine
    - Y_i is the random variables of the hours for the i-th action.
    - $\mathbb{E}[e^{tX}] = \mathbb{E}[e^{tX}|Y_1 = 2] + \mathbb{E}[e^{tX}|Y_1 = 3] + \mathbb{E}[e^{tX}|Y_1 = 5]$
      $= \mathbb{E}[e^{2t}] + \mathbb{E}[e^{t(X+3)}] + \mathbb{E}[e^{t(X+5)}]$
    - Find expectation and variance by joint moment generating function

2 Expectation

- N i.i.d. events, when N is a random variable
  - Suppose N is a integer random variable
  - Suppose $X_1, \ldots, X_i, \ldots, X_N$ are i.i.d random variables with mean μ and variance σ^2
  - $Y = \sum_{i=1}^{N} X_i$
  - $\mathbb{E}[Y] = \mathbb{E}[N]\mu$
    - $\mathbb{E}[Y] = \sum_{n=1}^{\infty} \mathbb{E}[\sum_{i=1}^{N} X_i|N = n]P[N = n]$
      $= \mu \times \sum_{n=1}^{\infty} nP[N = n] = \mathbb{E}[N]\mu$

- $\mathbb{E}[Y^2] = \mathbb{E}[N]\mathbb{E}[X^2] + \mathbb{E}[N^2]\mu^2 - \mathbb{E}[N]\mu^2$
    - $\mathbb{E}[Y^2] = \sum_{n=1}^{\infty} \mathbb{E}[(\sum_{i=1}^{N} X_i)^2 | N = n] P[N = n] = \sum_{n=1}^{\infty}(n\mathbb{E}[X_i^2] + n(n-1)\mu^2)P[N = n]$ $= \mathbb{E}[N]\mathbb{E}[X^2] + \mathbb{E}[N^2]\mu^2 - \mathbb{E}[N]\mu^2$
- $Var(Y) = \mathbb{E}[N]\sigma^2 + Var(N)\mu^2$

- Expectation by $P[X > x]$
  - $\mathbb{E}[X] = \sum_x P[X > x]$, when X is a non-negative discrete random variable
    - $\mathbb{E}[X] = \sum_{x=0}^{\infty} xP[X = x] = \sum_{x=0}^{\infty}\sum_{y=0}^{x-1} P[X = x] = \sum_{y=0}^{\infty}\sum_{x=y+1}^{\infty} P[X = x] = \sum_{y=0}^{\infty} P[X > y]$
  - $\mathbb{E}[X] = \int_0^{\infty} P[X > x]dx$, when X is a non-negative continuous random variable
    - $\mathbb{E}[X] = \int_0^{\infty} xf_X(x)dx = \int_0^{\infty}\int_0^{x} f_X(x)dydx = \int_0^{\infty}\int_y^{\infty} f_X(x)dxdy = \int_0^{\infty} P[X > y]dy$

3 Inequality

- Markov Inequality

  Definition:
  - Suppose $X \geq 0$, then $P[X \geq \epsilon] \leq \frac{\mathbb{E}[X]}{\epsilon}$

  Proof:
  1. $\mathbb{E}[X] = \int_0^{\infty} xf_X(x) \geq \int_{\epsilon}^{\infty} xf_X(x) \geq \epsilon\int_{\epsilon}^{\infty} f_X(x) = \epsilon P[X \geq \epsilon]$
  2. $X(\omega) \geq \epsilon \mathbb{1}_{X(\omega)\geq\epsilon}, \forall\omega \in S$
     - Calculate expectation on both side.
     - $\mathbb{E}[X] \geq \epsilon P[X \geq \epsilon]$

  Property:
  - The equality happens when $P[X = k] = 0, \forall k \notin \{0, \epsilon\}$.

- Chebyshev Inequality

  Definition:
  - Suppose $m = \mathbb{E}[X], \sigma^2 = Var(X)$, then $P[|X - m| \geq \epsilon] \leq \frac{\sigma^2}{\epsilon^2}$

  Proof:
  - $P[|X - m| \geq \epsilon] = P[(X - m)^2 \geq \epsilon^2]$
  - $P[(X - m)^2 \geq \epsilon^2] \leq \frac{\mathbb{E}[(X-m)^2]}{\epsilon^2}$ (by Markov Inequality)

  Property:
  - The equality happens when $P[X = k] = 0, \forall k \notin \{m - \epsilon, m, m + \epsilon\}$.
  - Might be tighter than Markov Inequality since it requires m, σ

- Chernoff Inequality

  Definition:
  - Suppose $X_1, \ldots, X_n$ are independent identically distributed Bernoulli random variable with probability p and $X = \sum_{i=1}^{n} X_i$
  - $P[X \geq \epsilon] \leq \frac{(pe^t+1-p)^n}{e^{t\epsilon}} \leq \frac{e^{np(e^t-1)}}{e^{t\epsilon}}$
    - $P[X \geq \epsilon] = P[e^{tX} \geq e^{t\epsilon}] \leq \frac{E[e^{tX}]}{e^{t\epsilon}} = \frac{(E[e^{tX_i}])^n}{e^{t\epsilon}} = \frac{(pe^t+1-p)^n}{e^{t\epsilon}} \leq \frac{e^{np(e^t-1)}}{e^{t\epsilon}}$
  - $P[X \geq np(1+\epsilon)] \leq (\frac{e^{\epsilon}}{(1+\epsilon)^{1+\epsilon}})^{np} \leq \begin{cases} e^{\frac{-\epsilon^2 np}{3}} & \text{if } 0 \leq \epsilon \leq 1 \\ e^{\frac{-\epsilon^2 np}{(2+\epsilon)}} & \text{if } \epsilon \geq 1 \end{cases}$
    - Substitude ϵ with $np(1+\epsilon)$

    * Substitude t with $\log(1+\epsilon)$
    * the last inequality is without proof
  - $P[X \leq \epsilon] \leq \frac{(pe^t+1-p)^n}{e^{t\epsilon}} \leq \frac{e^{np(e^t-1)}}{e^{t\epsilon}}$
    * $P[X \leq \epsilon] = P[e^{-tX} \geq e^{-t\epsilon}] \leq \frac{E[e^{-tX}]}{e^{-t\epsilon}} = \frac{(E[e^{-tX_i}])^n}{e^{-t\epsilon}} = \frac{(pe^{-t}+1-p)^n}{e^{-t\epsilon}} \leq \frac{e^{np(e^{-t}-1)}}{e^{-t\epsilon}}$
  - $P[X \leq np(1-\epsilon)] \leq (\frac{e^{-\epsilon}}{(1-\epsilon)^{1-\epsilon}})^{np} \leq e^{\frac{-\epsilon^2 np}{2}}$
    * Substitude ϵ with $np(1-\epsilon)$
    * Substitude t with $-\log(1-\epsilon)$
    * the last inequality is without proof

- Chernoff/ Hoeffding Lemma
  Definition:
  - Suppose $X_1, \ldots, X_n$ are independent distributed random variable and $a_i \leq X_i \leq b_i$
  - Suppose $X = \sum_{i=1}^n X_i$ and $\mu = \mathbb{E}[X]$
  - $P[|X-\mu| \geq \epsilon] \leq 2e^{\frac{-2\epsilon^2}{\sum_{i=1}^n (b_i-a_i)^2}}$ without proof

- Application:
  - Balls in Bins
    Definition: Throw n balls into n bins, find bounds for the maximum number of balls in all bins
    * $P[$ maximum number of balls in all bins $\geq \epsilon]$
      $= P[\cup_{i=1}^n$ number of balls in i-th bin $\geq \epsilon]$
      $\leq n \times P[$ number of balls in one bin $\geq \epsilon]$
    * By Markov inequality:
      · $P[$ number of balls in one bin $\geq \epsilon] \leq \frac{1}{\epsilon} \rightarrow$ useless
    * By Chebyshev inequality:
      · $P[$ number of balls in one bin $\geq \epsilon] \leq \frac{(1-\frac{1}{n})}{\epsilon^2}$
      · $P[$ maximum number of balls in all bins $\geq n^{\frac{1}{2}+\epsilon}] \leq \frac{(1-\frac{1}{n})}{n^{2\epsilon}}$
      · when $n \rightarrow \infty$, the maximum number of balls should less than $n^{\frac{1}{2}+\epsilon}$
    * By Chernoff inequality:
      · $P[$ number of balls in one bin $\geq 2\log n] \leq \frac{e^{np(e^t-1)}}{n^{2t}}$
      · $P[$ maximum number of balls in all bins $\geq 2\log n] \leq \frac{e^{np(e^t-1)}}{n^{2t-1}}$
      · when t is a constant ≥ 0.5 and $n \rightarrow \infty$, the maximum number of balls should less than $2\log n$

4 Law of Large Numbers

- $\{X_i\}_{i=1}^{\infty}$ is a sequence of pairwise uncorrelated random variable with
  $\mathbb{E}[X_i] = m$, $Var(X_i) = \sigma_i^2$.
- $M_n = \frac{1}{n}\sum_{i=1}^n X_i$
- $M_n \rightarrow m$ almost surely, in mean square and in probability.

5 Memoryless

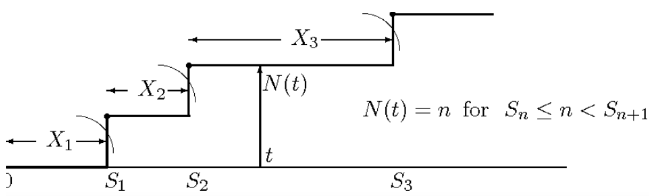
- Definition: $P[X > x_1 + x_2 | X > x_1] = P[X > x_2]$
- Property:
  - Exponential random variable is the only continuous memoryless random variable
  - Bernoulli random variable is the only discrete memoryless random variable

6 Famous Random Variable

- Poisson:
  $P[X = k] = \frac{\lambda^k}{k!} \exp(-\lambda)$
  $\mathbb{E}[X] = \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} \exp(-\lambda) = \sum_{k=0}^{\infty} \lambda \frac{\lambda^{k-1}}{(k-1)!} \exp(-\lambda) = \lambda$
  Interpretation:
  - Cut total time into infinite period in Binomial random variable, $n \to \infty, p \to \frac{\lambda}{n}$
  - $\to P[X = k] = \lim_{n\to\infty} \binom{n}{k} (\frac{\lambda}{n})^k (\frac{n-\lambda}{n})^{n-k} = \frac{\lambda^k}{k!}(1 - \frac{\lambda}{n})^n = \frac{\lambda^k}{k!} \exp(-\lambda)$
- Erlang:
  $f_X(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!}, \forall x \in \mathbb{R}$
  $\mathbb{E}[X] = \frac{n}{\lambda}$
  Interpretation:
  - Suppose $X_1, X_2, ..., X_n$ are i.i.d exponential random variable with λ.
  - $X = \sum_{i=1}^{n} X_i$
  - Proof by induction:
    Suppose $n = 2$, $f_X(x) = \int_0^x \lambda e^{-\lambda t} \lambda e^{-\lambda(x-t)} dt = \lambda^2 x e^{-\lambda x}$

7 Stochastic Processes

- Stochastic Process: a collection of random variable
  Arrival Process: a sequence of arriving event in continuous time



  - X_i: the time between the i-th event and the $i-1$-th event
  - S_i: the time from start to i-th event
  - $N(t)$: the number of the arrived event at time t
  - X and S Relation:
    * $X_1 = S_1, X_i = S_i - S_{i-1}$
  - N and S Relation:
    * $N(t) < n \leftrightarrow S_{n+1} > t$
    * $N(t) \geq n \leftrightarrow S_n \leq t$
    * $N(t) = n \leftrightarrow S_n \leq t < S_{n+1}$
    * $N(t) = \max\{n : S_n \leq t\}$
  - Renewal Process: an arrival process with i.i.d X_i
  - Poisson Process: a renewal process with $X_i \sim$ Exponential(λ)
    S_i Property
    * S_i is an Erlang random variable
      Erlang is the sum of the Exponential random variables
    * Joint Distribution $f_{S_1,...,S_n}(s_1, \ldots, s_n) = \lambda^n e^{-\lambda s_n}$
      Prove by induction.
      Induce by $f_{S_1,...,S_n}(s_1, \ldots, s_n) = f_{S_1,...,S_{n-1}}(s_1, \ldots, s_{n-1}) \times f_{S_n|S_1,...,S_{n-1}}(s_n, s_1, \ldots, s_{n-1})$

    $N(t)$ Property

* $N(t) \sim \text{Poisson}(\lambda t), P[N(t) = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$
  Prove by $P[N(t) = n] = P[S_n \leq t \text{ and } S_{n+1} > t]$
* Conditioned on $N(t) = n$, the set of arrival times $\{s_1, \ldots, s_n\}$ have the same distribution with a set of n sorted i.i.d. Uniform$(0, t)$ random variables
  Prove by $f_{S_1,\ldots,S_n|N(t)}(s_1, \ldots, s_n, n) = \frac{f_{S_1,\ldots,S_n}(s_1,\ldots,s_n)P[X_{n+1}>t-s_n]}{P[N(t)=n]} = \frac{n!}{t^n}$

Property

* Z is the interval from t to the first arrival $\rightarrow$ Z is exponential random variable with same λ and independent of $N(t)$ and the arrival time before t
  Proof:
  $P[Z > z] = \sum_{n=0}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} P[Z > z | N(t) = n, S_1 = s_1, \ldots, S_n = s_n] ds_1 \ldots ds_n$
  $= \sum_{n=0}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} P[X_{n+1} > z + t - s_n | N(t) = n, S_1 = s_1, \ldots, S_n = s_n] ds_1 \ldots ds_n$
  $= \sum_{n=0}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} P[X_{n+1} > z + t - s_n | X_{n+1} > t - s_n] ds_1 \ldots ds_n = e^{-\lambda z}$
* Stationary Increments: $N(t_1 + t_2) - N(t_1)$ and $N(t_2)$ share the same distribution
  Without Proof
* Independent Increments: $\forall 0 < t_1 < t_2 < \ldots, t_k, N(t_1), N(t_2) - N(t_1), \ldots$ are independent
  Without Proof
* Any arrival process with stationary and independent increments must be a Poisson process
  Without Proof

Exercise

* $\mathbb{E}[S_i | N(t) = n] = \frac{t \times i}{n+1}$
  * $\mathbb{E}[S_i | N(t) = n] = i \times \mathbb{E}[X_1 | N(t) = n] = i \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} s_1 \times \frac{n!}{t^n} ds_1 \ldots ds_{n-1} ds_n = \frac{t \times i}{n+1}$
* $\mathbb{E}[\sum_{i=0}^{N(t)} S_i] = \frac{\lambda t^2}{2}$
  * $\mathbb{E}[\sum_{i=0}^{N(t)} S_i] = \sum_{n=0}^{\infty} \mathbb{E}[\sum_{i=0}^{n} S_i | N(t) = n] P[N(t) = n]$
    $= \sum_{n=0}^{\infty} \frac{nt}{2} P[N(t) = n] = \frac{\lambda t^2}{2}$

2D Poisson Process

* Definition:
  * For any region R: number of points in R is a Poisson random variable
  * number of points in the non-overlapping region is independent

Combining Poisson Process

* $N^1(t), N^2(t)$ are two independent Poisson process with λ_1, λ_2
* X_i is the first arrival of X_i^1, X_i^2
* Property
  * X_i is independent of $\{X_i^1 < X_i^2\}$ and $\{X_i^1 > X_i^2\}$
    Proof: $P[X_1^1 < X_1^2] = \frac{\lambda_1}{\lambda_1 + \lambda_2}$
    $P[X_1 > x] = P[X_1^1 > x, X_1^2 > x] = e^{-(\lambda_1 + \lambda_2)x}$
    $P[X_1 > x, X_1^1 < X_1^2] = P[X_1 > x] P[X_1^1 < X_1^2]$
  * X_i is a Poisson Process with $\lambda = \lambda_1 + \lambda_2$

Splitting Poisson Process

* $N^1(t), N^2(t)$ are two independent Poisson process with λ_1, λ_2
* $N(t)$ is a random process with $\lambda = \lambda_1 + \lambda_2$
  * $N^{1*}(t)$ is the process of the first event
    when $N(t)$ arrives consider it as first event with probability $\frac{\lambda_1}{\lambda_1 + \lambda_2}$
  * $N^{2*}(t)$ is the process of the second event
    when $N(t)$ arrives consider it as second event with probability $\frac{\lambda_2}{\lambda_1 + \lambda_2}$
* $N^i(t)$ and $N^{i*}(t)$ share the same distribution
* Proof:
  * $B_n(k)$ is a Binomial random variable with $p = \frac{\lambda_1}{\lambda_1 + \lambda_2}$
  * $P[N^{1*}(t) = m, N^{2*}(t) = n] = P[N(t) = m + n, B_{m+n}(m)] = P[N^1(t) = m, N^2(t) = n]$