Stochastic Processes

Kevin Chang

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A flashlight needs two batteries to be operational. Consider a flash light along with a set of n such batteries-battery 1, battery 2, ..., battery n. Initially, battery 1 and 2 are installed. Whenever a battery fails, it is immediately replaced by the lowest numbered functional battery that has not yet been put into use. Suppose that the life time of different batteries are independent exponential random variables each having a rate λ. At a random time, call it T, a battery will fail and our stockpile will be empty. At that moment exactly one of the batteries-which we call battery X-will not yet have failed.
(a). What is $\mathbb{P}(X = n)$?
(b). What is $\mathbb{P}(X = 1)$?
(c). Find $\mathbb{E}[T]$.
(d). What is the distribution of T?

- Consider the process as one Poisson process with 2λ and split the process with $p = \frac{1}{2}$
- (a) $P[X = n]$
  - as long as $n-1$-th fail and $n-2$-th fail are in different batteries holder $\rightarrow X = n$
  - $P[X = n] = \frac{1}{2}$
- (b)
  - $P[X = 1] = (\frac{1}{2})^{n-1}$
- (c)
  - $\mathbb{E}[T] = \frac{n-1}{\lambda}$
- (d)
  - a Poisson distribution

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Consider a train that starts from the LA-Union station. Let I_m denote the number of people that get into the train from the station m. Assume that I_m are independent and Poisson with parameter λ_m. Each person entering from the station m will, independent of everything else, get off at the station n with probability $P_{m,n}$. Also, $\sum_{n>m} P_{m,n} = 1$. Let G_n be the number of people getting off the train at the station n.
(a). Calculate $\mathbb{E}[G_n]$.
(b). What is the distribution of G_n?.
(c). What is the joint distribution of G_n and G_k?.

- G_n is a Poisson process with $\lambda = \sum_{i=0}^{n-1} \lambda_m \times P_{m,i}$

- (a)
  - $\mathbb{E}[G_n] = \sum_{i=0}^{n-1} \lambda_m \times P_{m,i}$
- (b)
  - G_n is a Poisson process with $\lambda = \sum_{i=0}^{n-1} \lambda_m \times P_{m,i}$
- (c)
  - G_n and G_k is indepedent
  - G_n is a Poisson process with $\lambda_n = \sum_{i=0}^{n-1} \lambda_m \times P_{m,i}$
  - G_k is a Poisson process with $\lambda_k = \sum_{i=0}^{k-1} \lambda_m \times P_{m,i}$
  - $F_{G_n,G_k}(i,j) = \frac{\lambda_n^i}{i!} \exp(-\lambda_n) \times \frac{\lambda_k^j}{j!} \exp(-\lambda_k)$

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2.10. Buses arrive at a certain stop according to a Poisson process with rate λ. If you take the bus from that stop then it takes a time R, measured from the time at which you enter the bus, to arrive home. If you walk from the bus stop then it takes a time W to arrive home. Suppose that your policy when arriving at the bus stop is to wait up to a time s, and if a bus has not yet arrived by that time then you walk home.

(a) Compute the expected time from when you arrive at the bus stop until you reach home.

(b) Show that if $W < 1/\lambda + R$ then the expected time of part (a) is minimized by letting $s = 0$; if $W > 1/\lambda + R$ then it is minimized by letting $s = \infty$ (that is, you continue to wait for the bus); and when $W = 1/\lambda + R$ all values of s give the same expected time.

(c) Give an intuitive explanation of why we need only consider the cases $s = 0$ and $s = \infty$ when minimizing the expected time.

- Suppose B is the time the bus arrived at the stop
- Suppose T is the time to reach home
- (a)
  - $\mathbb{E}[T] = (W+s) \times P[B > s] + \int_0^s (R+t)\lambda e^{-\lambda t} dt$
    $= (W+s)e^{-\lambda s} + R(1-e^{-\lambda s}) + \frac{1}{\lambda} + e^{-\lambda s}(-s + \frac{-1}{\lambda})$
    $= (W - R - \frac{1}{\lambda})e^{-\lambda s} + R + \frac{1}{\lambda}$
- (b)
  - $\frac{d\mathbb{E}[T]}{ds} = (W - R - \frac{1}{\lambda}) \times (-\lambda)e^{-\lambda s}$
  - if $(W - R - \frac{1}{\lambda}) < 0$, $\mathbb{E}[T]$ increases when s increases $\rightarrow$ minimum happens when $s = 0$
  - if $(W - R - \frac{1}{\lambda}) > 0$, $\mathbb{E}[T]$ decreases when s increases $\rightarrow$ minimum happens when $s = \infty$
  - if $(W - R - \frac{1}{\lambda}) = 0$, the derivative of $\mathbb{E}[T]$ is constant $\rightarrow$ minimum happens for any s
- (c)
  - since the exponential distribution is memoryless, if $\mathbb{E}[T]$ is minimized when $s = t$, then, at $s = t$, $\mathbb{E}[T]$ is minimized when $s = t + t = 2t$. Therefore, $t = \infty$

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2.11. Cars pass a certain street location according to a Poisson process with rate λ. A person wanting to cross the street at that location waits until she can see that no cars will come by in the next T time units. Find the expected time that the person waits before starting to cross. (Note, for instance, that if no cars will be passing in the first T time units then the waiting time is 0.)

- X_i: waiting time for i-th bus
- N: number of cars the person has to wait
- P: the time that the person has to wait
- $P[N=n] = (e^{-\lambda T})(1-e^{-\lambda T})^n$
- $\mathbb{E}[P|N=n] = n \times \mathbb{E}[X_i|X_i < T] + \mathbb{E}[X_i|X_i > T]$
  $= n(-Te^{-\lambda T} + \frac{1-e^{-\lambda T}}{\lambda}) + Te^{-\lambda T} + \frac{e^{-\lambda T}}{\lambda} = (1-n)(Te^{-\lambda T} + \frac{e^{-\lambda T}}{\lambda}) + \frac{n+1}{\lambda}$
- $\mathbb{E}[P] = \sum_{n=0}^{\infty} \mathbb{E}[P|N=n]P[N=n]$
  $= \sum_{n=0}^{\infty} \mathbb{E}[P|N=n]P[N=n]$
  $= (1 - \frac{1-e^{-\lambda T}}{e^{-\lambda T}})(Te^{-\lambda T} + \frac{e^{-\lambda T}}{\lambda}) + \frac{1}{\lambda e^{-\lambda T}}$
  $= (\frac{2e^{-\lambda T}-1}{e^{-\lambda T}})(Te^{-\lambda T} + \frac{e^{-\lambda T}}{\lambda}) + \frac{1}{\lambda e^{-\lambda T}}$

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In this problem you will simulate a 3D Poisson process, with two dimensions of space and a third dimension of time.

The space domain is the unit ball $\{\vec{x} \in \mathbf{R}^2 : \|\vec{x}\| \leq 1\}$, and time goes from 0 to 1.

On this domain, simulate a 3D Poisson process with rate $\lambda = 100$. You might imagine this as modeling raindrops falling into a pool of water.

Your answer should include an explanation of how and why your simulation works, in math and English. It should include your source code, commented as necessary. Finally, show the results of your simulation in an image that shows the 2D position of each event/raindrop, *color-coded* according to its time, from $t = 0$ to $t = 1$. Also print how many raindrops fell.

The sample code below should help get you started with plotting in python. More details on the functions can be found with Google.

- Code

```python
import numpy as np
import matplotlib.pyplot as plt


plt.figure(figsize=(10,10))
plt.gca().set_aspect(aspect='equal')
plt.gca().add_patch(plt.Circle((0,0), 1, fill=False))


numpoints = np.random.poisson(100)
#  numpoints = 3

alpha = 2 * np.pi * np.random.random(numpoints)
# random radius
r = np.sqrt(np.random.random(numpoints))
# calculating coordinates
xcoods = r * np.cos(alpha)
ycoods = r * np.sin(alpha)
colors = np.linspace(0,1, num = numpoints, endpoint = True)

plt.scatter(xcoods, ycoods, c=colors, marker='o', s = 36)
plt.savefig("result.png")
```

- Explanation
  - We first simulate n points from a Poisson process with $\lambda = 100$
  - We then unifomly sample n points uniformly in the 3D space
- Result: