Stochastic Processes

Kevin Chang

February 4, 2022

1

Let X be a continuous random variable such that $\mathbb{P}(X<0)=0$. Show that $\mathbb{E}[X]=\int_0^\infty \mathbb{P}(X>t)dt$

- $\mathbb{E}[X]=\int_{-\infty}^{\infty} xf_X(x)dx=\int_0^\infty xf_X(x)dx$
  $=\int_0^\infty\int_0^x f_X(x)dtdx=\int_0^\infty\int_t^\infty f_X(x)dxdt=\int_0^\infty P[X>t]dt$

2

$p=1-q$

(i) Let X, Y be independent and identically distributed random variables that take values from the set $\{0,1,2,\ldots\}$ such that $\mathbb{P}(X=n)=pq^n$. Find
(a) $\mathbb{P}(X=Y)$ (b) $\mathbb{P}(X\geq 2Y)$
(ii) Show that $\mathbb{P}(X=k|X+Y=n)=\frac{1}{(n+1)}$

- (i)
  - (a)
    * $P[X=Y]=\sum_{n=0}^\infty P[X=Y|X=n]P[X=n]=\sum_{n=0}^\infty P[Y=n|X=n]P[X=n]$
      $=\sum_{n=0}^\infty P[Y=n]P[X=n]=\sum_{n=0}^\infty p^2\times q^{2n}=p^2\frac{1}{1-q^2}$
  - (b)
    * $P[X\geq 2Y]=\sum_{n=0}^\infty P[X\geq 2Y|Y=n]P[Y=n]$
      $=\sum_{n=0}^\infty P[X\geq 2n]P[Y=n]=\sum_{n=0}^\infty \frac{pq^{2n}}{1-q}pq^n=\frac{p^2q^{3n}}{(1-q)(1-q^3)}$
- (ii)
  - $P[X=k|X+Y=n]=\frac{P[X=k,X+Y=n]}{P[X+Y=n]}=\frac{P[X=k,Y=n-k]}{\sum_{i=0}^n P[X=i]P[Y=n-i]}$
    $=\frac{pq^k\times pq^{n-k}}{\sum_{i=0}^n pq^i\times pq^{n-i}}=\frac{1}{(n+1)}$

3

(i) Let the probability that a family has exactly n children be αp^n when $n\geq 1$, and $p_0=1-\alpha p(1+p+p^2+\ldots)$. Suppose that all the sex distributions of n children have the same probability. Show that for $k\geq 1$ the probability that a family has exactly k boys is $2\alpha p^k/(2-p)^{k+1}$
(ii) Given that a family includes at least one boy, what is the probability that there are two or more?

-

4

Let $\{X_n, n = 1, 2, \ldots\}$ be a sequence of independent Bernoulli random variables such that $\mathbb{P}(X_n = 1) = \frac{1}{n}$, $\mathbb{P}(X_n = 0) = 1 - \frac{1}{n}$. Define $A_n = \{X_{n-2} = 0, X_{n-1} = 1, X_n = 1\}$. Show that $\mathbb{P}(A_n \text{ i.o.}) = 0$.

- $P[A_n] = (1 - \frac{1}{n-2})(\frac{1}{n-1})(\frac{1}{n}) = \frac{n-3}{(n-2)(n-1)n}$
- $\sum_{n=1}^{\infty} P[A_n] < \infty \rightarrow P[A_n \ f.o.] = 1$
- $P[A_n \ i.o.] = 1 - P[A_n \ f.o.] = 0$

5

If X and Y are independent exponential random variables with parameters λ_1 and λ_2 respectively, compute the distribution of $Z = \min(X, Y)$. What is the conditional distribution of Z given that $Z = X$?

- $P[Z \leq z]$
  - $P[Z > z] = P[X > z, Y > z] = e^{-(\lambda_1+\lambda_2)z}$
  - $P[Z \leq z] = 1 - P[Z > z] = 1 - e^{-(\lambda_1+\lambda_2)z}$
- $P[Z \leq z | Z = X]$
  - $P[Z > z | Z = X] = \frac{P[Y > X > z]}{P[Z = X]} = \frac{\int_z^{\infty} e^{-\lambda_2 x} \lambda_1 e^{-\lambda_1 x} dx}{\int_0^{\infty} e^{-\lambda_2 x} \lambda_1 e^{-\lambda_1 x} dx} = e^{-(\lambda_1+\lambda_2)z}$
  - $P[Z \leq z | Z = X] = 1 - P[Z > z | Z = X] = 1 - e^{-(\lambda_1+\lambda_2)z}$

6

The conditional variance of X, given Y, is defined by

$$\mathrm{Var}(X|Y) = \mathbb{E}\left[(X - \mathbb{E}[X|Y])^2 | Y\right].$$

Show that

$$\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X|Y)] + \mathrm{Var}(\mathbb{E}[X|Y]).$$

- $Var(X|Y) = \mathbb{E}[(X - \mathbb{E}[X|Y])^2|Y] = \mathbb{E}[X^2 + \mathbb{E}[X|Y]^2 - 2X\mathbb{E}[X|Y]|Y]$
  $= \mathbb{E}[X^2|Y] + \mathbb{E}[\mathbb{E}[X|Y]^2|Y] - 2\mathbb{E}[X|Y] \times \mathbb{E}[X|Y] = \mathbb{E}[X^2|Y] - \mathbb{E}[X|Y]^2$
- $\mathbb{E}[Var(X|Y)] = \mathbb{E}[\mathbb{E}[X^2|Y] - \mathbb{E}[X|Y]^2] = \mathbb{E}[X^2] - \mathbb{E}[\mathbb{E}[X|Y]^2]$
- $Var(\mathbb{E}[X|Y]) = \mathbb{E}[\mathbb{E}[X|Y]^2] - \mathbb{E}[\mathbb{E}[X|Y]]^2 = \mathbb{E}[\mathbb{E}[X|Y]^2] - \mathbb{E}[X]^2$
- $\mathbb{E}[Var(X|Y)] + Var(\mathbb{E}[X|Y]) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = Var(X)$

7

An urn contains a white balls and b black balls. After a ball is drawn, it is returned to the urn if it is white; but if it is black, it is replaced by a white ball from another urn. Let M_n denote the expected number of white balls in the urn after the foregoing operation has been repeated n times.

(i) Derive the recursive equation

$$M_{n+1} = \left(1 - \frac{1}{a+b}\right) M_n + 1.$$

(ii) Use part (i) to prove that

$$M_n = a + b - b\left(1 - \frac{1}{a+b}\right)^n.$$

- (i)
  - Suppose the number of white balls in the urn after the n times foregoing operation is X_n
  - $P[X_{n+1} = k] = P[X_n = k]\frac{k}{a+b} + P[X_n = k-1]\frac{a+b-k+1}{a+b}$
  - $\mathbb{E}[X_{n+1}] = \sum_{k=a}^{a+b} kP[X_{n+1}] = \sum_{k=a}^{a+b} k(P[X_n = k]\frac{k}{a+b} + P[X_n = k-1]\frac{a+b-k+1}{a+b})$
    $= \sum_{k=a}^{a+b} P[X_n = k]\frac{k^2}{a+b} + \sum_{k=a}^{a+b} P[X_n = k](k+1)\frac{a+b-k}{a+b}) = \sum_{k=a}^{a+b} P[X_n = k](k - \frac{k}{a+b} + 1)$
    $= \mathbb{E}[X_n](1 - \frac{1}{a+b}) + 1$
  - $M_{n+1} = (1 - \frac{1}{a+b})M_n + 1$
- (ii)
  - $M_0 = a$
  - $M_{n+1} = (1 - \frac{1}{a+b})M_n + 1$
  - $M_n = 1 + (1 - \frac{1}{a+b}) + \cdots + (1 - \frac{1}{a+b})^{n-1} + a(1 - \frac{1}{a+b})^n = \frac{(1-\frac{1}{a+b})^n - 1}{(1-\frac{1}{a+b}) - 1} + a(1 - \frac{1}{a+b})^n$
    $= a + b - b(1 - \frac{1}{a+b})^n$

8

2.3. For a Poisson process show, for $s < t$, that

$$P\{N(s) = k \mid N(t) = n\} = \binom{n}{k}\left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}, \qquad k = 0, 1, \ldots, n.$$

- $P[N(s) = k | N(t) = n] = \frac{P[N(s)=k, N(t-s)=n-k]}{P[N(t)=n]} = \frac{\frac{(\lambda s)^k}{k!}e^{-\lambda s} \times \frac{(\lambda(t-s))^{n-k}}{(n-k)!}e^{-\lambda(t-s)}}{\frac{(\lambda t)^n}{n!}e^{-\lambda t}}$
  $= \binom{n}{k}(\frac{s}{t})^k(1 - \frac{s}{t})^{n-k}$