Stochastic Processes

Kevin Chang

January 25, 2022

1 Moment Generating Function

- Moment Generating Function: $\mathbb{E}[e^{tX}]$
  - Property:
    - $\mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx$
    - $\mathbb{E}[e^{tX}] = \sum_{k=0}^{\infty} E[X^k]\frac{t^k}{k!}$
      - $e^{tx} = \sum_{k=0}^{\infty} \frac{(tx)^k}{k!}$
      - $E[e^{tX}] = E[\sum_{k=0}^{\infty} \frac{(tX)^k}{k!}] = \sum_{k=0}^{\infty} E[X^k]\frac{t^k}{k!}$
    - $\frac{d\mathbb{E}[e^{tX}]}{dt} = \mathbb{E}[X]$
    - $\mathbb{E}[e^{t(aX+b)}] = e^t b\mathbb{E}[e^{taX}]$
    - Not all random variables have Moment generating function
- Characteristic Function: $\mathbb{E}[e^{itX}]$
  - Property:
    - All random variables have Moment generating function
- Joint Moment Generating Function: $G(x,y) = \mathbb{E}[e^{xX}e^{yY}]$
- Property:
  - (Joint) moment generating funciton uniquely determines the (joint) CDF
- Example
  - Trapped miner's random walk
    - Miner has probability of $\frac{1}{3}$ to waste 3 hours in vain, $\frac{1}{3}$ to waste 5 hours in vain, and $\frac{1}{3}$ to spend 2 hours to go out of the mine.
    - X is the random variables of the hours to go out of the mine
    - Y_i is the random variables of the hours for the i-th action.
    - $\mathbb{E}[e^{tX}] = \mathbb{E}[e^{tX}|Y_1 = 2] + \mathbb{E}[e^{tX}|Y_1 = 3] + \mathbb{E}[e^{tX}|Y_1 = 5]$
      $= \mathbb{E}[e^{2t}] + \mathbb{E}[e^{t(X+3)}] + \mathbb{E}[e^{t(X+5)}]$
    - Find expectation and variance by joint moment generating function

2 Expectation

- N i.i.d. events, when N is a random variable
  - Suppose N is a integer random variable
  - Suppose $X_1, \ldots, X_i, \ldots, X_N$ are i.i.d random variables with mean μ and variance σ^2
  - $Y = \sum_{i=1}^{N} X_i$
  - $\mathbb{E}[Y] = \mathbb{E}[N]\mu$
    - $\mathbb{E}[Y] = \sum_{n=1}^{\infty} \mathbb{E}[\sum_{i=1}^{N} X_i|N = n]P[N = n]$
      $= \mu \times \sum_{n=1}^{\infty} nP[N = n] = \mathbb{E}[N]\mu$

- $\mathbb{E}[Y^2] = \mathbb{E}[N]\mathbb{E}[X^2] + \mathbb{E}[N^2]\mu^2 - \mathbb{E}[N]\mu^2$
  * $\mathbb{E}[Y^2] = \sum_{n=1}^{\infty} \mathbb{E}[(\sum_{i=1}^{N} X_i)^2 | N = n] P[N = n] = \sum_{n=1}^{\infty} (n\mathbb{E}[X_i^2] + n(n-1)\mu^2) P[N = n]$
    $= \mathbb{E}[N]\mathbb{E}[X^2] + \mathbb{E}[N^2]\mu^2 - \mathbb{E}[N]\mu^2$
- $Var(Y) = \mathbb{E}[N]\sigma^2 + Var(N)\mu^2$

- Expectation by $P[X > x]$
  - $\mathbb{E}[X] = \sum_x P[X > x]$, when X is a non-negative discrete random variable
    * $\mathbb{E}[X] = \sum_{x=0}^{\infty} xP[X = x] = \sum_{x=0}^{\infty} \sum_{y=0}^{x-1} P[X = x] = \sum_{y=0}^{\infty} \sum_{x=y+1}^{\infty} P[X = x] = \sum_{y=0}^{\infty} P[X > y]$
  - $\mathbb{E}[X] = \int_0^{\infty} P[X > x]dx$, when X is a non-negative continuous random variable
    * $\mathbb{E}[X] = \int_0^{\infty} x f_X(x)dx = \int_0^{\infty} \int_0^{x} f_X(x)dydx = \int_0^{\infty} \int_y^{\infty} f_X(x)dxdy = \int_0^{\infty} P[X > y]dy$

3 Inequality

- Markov Inequality

  Definition:
  - Suppose $X \geq 0$, then $P[X \geq \epsilon] \leq \frac{\mathbb{E}[X]}{\epsilon}$

  Proof:
  1. $\mathbb{E}[X] = \int_0^{\infty} x f_X(x) \geq \int_{\epsilon}^{\infty} x f_X(x) \geq \epsilon \int_{\epsilon}^{\infty} f_X(x) = \epsilon P[X \geq \epsilon]$
  2. $X(\omega) \geq \epsilon \mathbb{1}_{X(\omega) \geq \epsilon}, \forall \omega \in S$
     - Calculate expectation on both side.
     - $\mathbb{E}[X] \geq \epsilon P[X \geq \epsilon]$

  Property:
  - The equality happens when $P[X = k] = 0, \forall k \notin \{0, \epsilon\}$.

- Chebyshev Inequality

  Definition:
  - Suppose $m = \mathbb{E}[X], \sigma^2 = Var(X)$, then $P[|X - m| \geq \epsilon] \leq \frac{\sigma^2}{\epsilon^2}$

  Proof:
  - $P[|X - m| \geq \epsilon] = P[(X - m)^2 \geq \epsilon^2]$
  - $P[(X - m)^2 \geq \epsilon^2] \leq \frac{\mathbb{E}[(X-m)^2]}{\epsilon^2}$ (by Markov Inequality)

  Property:
  - The equality happens when $P[X = k] = 0, \forall k \notin \{m - \epsilon, m, m + \epsilon\}$.
  - Might be tighter than Markov Inequality since it requires m, σ

- Chernoff Inequality

  Definition:
  - Suppose $X_1, \ldots, X_n$ are independent identically distributed Bernoulli random variable with probability p and $X = \sum_{i=1}^{n} X_i$
  - $P[X \geq \epsilon] \leq \frac{(pe^t + 1 - p)^n}{e^{t\epsilon}} \leq \frac{e^{np(e^t - 1)}}{e^{t\epsilon}}$
    * $P[X \geq \epsilon] = P[e^{tX} \geq e^{t\epsilon}] \leq \frac{E[e^{tX}]}{e^{t\epsilon}} = \frac{(E[e^{tX_i}])^n}{e^{t\epsilon}} = \frac{(pe^t + 1 - p)^n}{e^{t\epsilon}} \leq \frac{e^{np(e^t - 1)}}{e^{t\epsilon}}$
  - $P[X \geq np(1 + \epsilon)] \leq (\frac{e^{\epsilon}}{(1+\epsilon)^{1+\epsilon}})^{np} \leq \begin{cases} e^{\frac{-\epsilon^2 np}{3}} & \text{if } 0 \leq \epsilon \leq 1 \\ e^{\frac{-\epsilon^2 np}{(2+\epsilon)}} & \text{if } \epsilon \geq 1 \end{cases}$

    * Substitude ϵ with $np(1+\epsilon)$
    * Substitude t with $\log(1+\epsilon)$
    * the last inequality is without proof
  - $P[X \le \epsilon] \le \frac{(pe^t+1-p)^n}{e^{t\epsilon}} \le \frac{e^{np(e^t-1)}}{e^{t\epsilon}}$
    * $P[X \le \epsilon] = P[e^{-tX} \ge e^{-t\epsilon}] \le \frac{E[e^{-tX}]}{e^{-t\epsilon}} = \frac{(E[e^{-tX_i}])^n}{e^{-t\epsilon}} = \frac{(pe^{-t}+1-p)^n}{e^{-t\epsilon}} \le \frac{e^{np(e^{-t}-1)}}{e^{-t\epsilon}}$
  - $P[X \le np(1-\epsilon)] \le (\frac{e^{-\epsilon}}{(1-\epsilon)^{1-\epsilon}})^{np} \le e^{\frac{-\epsilon^2 np}{2}}$
    * Substitude ϵ with $np(1-\epsilon)$
    * Substitude t with $-\log(1-\epsilon)$
    * the last inequality is without proof

- Chernoff/ Hoeffding Lemma
  Definition:
  - Suppose $X_1, \ldots, X_n$ are independent distributed random variable and $a_i \le X_i \le b_i$
  - Suppose $X = \sum_{i=1}^n X_i$ and $\mu = \mathbb{E}[X]$
  - $P[|X-\mu| \ge \epsilon] \le 2e^{\frac{-2\epsilon^2}{\sum_{i=1}^n (b_i-a_i)^2}}$ without proof

- Application:
  - Balls in Bins
    Definition: Throw n balls into n bins, find bounds for the maximum number of balls in all bins
    * $P[$ maximum number of balls in all bins $\ge \epsilon]$
      $= P[\cup_{i=1}^n$ number of balls in i-th bin $\ge \epsilon]$
      $\le n \times P[$ number of balls in one bin $\ge \epsilon]$
    * By Markov inequality:
      · $P[$ number of balls in one bin $\ge \epsilon] \le \frac{1}{\epsilon} \to$ useless
    * By Chebyshev inequality:
      · $P[$ number of balls in one bin $\ge \epsilon] \le \frac{(1-\frac{1}{n})}{\epsilon^2}$
      · $P[$ maximum number of balls in all bins $\ge n^{\frac{1}{2}+\epsilon}] \le \frac{(1-\frac{1}{n})}{n^{2\epsilon}}$
      · when $n \to \infty$, the maximum number of balls should less than $n^{\frac{1}{2}+\epsilon}$
    * By Chernoff inequality:
      · $P[$ number of balls in one bin $\ge 2\log n] \le \frac{e^{np(e^t-1)}}{n^{2t}}$
      · $P[$ maximum number of balls in all bins $\ge 2\log n] \le \frac{e^{np(e^t-1)}}{n^{2t-1}}$
      · when t is a constant ≥ 0.5 and $n \to \infty$, the maximum number of balls should less than $2\log n$

4 Law of Large Numbers

- $\{X_i\}_{i=1}^{\infty}$ is a sequence of pairwise uncorrelated random variable with
  $\mathbb{E}[X_i] = m$, $Var(X_i) = \sigma_i^2$.
- $M_n = \frac{1}{n}\sum_{i=1}^n X_i$
- $M_n \to m$ almost surely, in mean square and in probability.

5 Memoryless

- Definition: $P[X > x_1 + x_2 | X > x_1] = P[X > x_2]$
- Property:
  - Exponential random variable is the only continuous memoryless random variable
  - Bernoulli random variable is the only discrete memoryless random variable