

Stochastic Processes

Kevin Chang

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1 Laplace Transform

- $\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st} dt$
- Property
 - $tf(t) \leftrightarrow -F'(s)$
 - $\frac{f(t)}{t} \leftrightarrow \int_s^\infty F(\sigma) d\sigma$
 - $f'(t) \leftrightarrow sF(s) - f(0^-)$
 - $\int_0^t f(\tau) d\tau \leftrightarrow \frac{F(s)}{s}$
 - $e^{at}f(t) \leftrightarrow F(s-a)$
 - $f(t-a)u(t-a) \leftrightarrow e^{-as}F(s)$

2 Moment Generating Function

- Moment Generating Function: $\mathbb{E}[e^{tX}]$
 - Property:
 - * $\mathbb{E}[e^{tX}] = \int_{-\infty}^\infty e^{tx} f_X(x) dx$
 - * $\mathbb{E}[e^{tX}] = \sum_{k=0}^\infty E[X^k] \frac{t^k}{k!}$
 - $e^{tx} = \sum_{k=0}^\infty \frac{(tx)^k}{k!}$
 - $E[e^{tX}] = E[\sum_{k=0}^\infty \frac{(tX)^k}{k!}] = \sum_{k=0}^\infty E[X^k] \frac{t^k}{k!}$
 - * $\frac{d\mathbb{E}[e^{tX}]}{dt} = \mathbb{E}[X]$
 - * $\mathbb{E}[e^{t(aX+b)}] = e^{tb} \mathbb{E}[e^{taX}]$
 - * Not all random variables have Moment generating function
- Characteristic Function: $\mathbb{E}[e^{itX}]$
 - Property:
 - * All random variables have Moment generating function
- Joint Moment Generating Function: $G(x, y) = \mathbb{E}[e^{xX} e^{yY}]$
- Property:
 - (Joint) moment generating function uniquely determines the (joint) CDF
- Example
 - Trapped miner's random walk
 - * Miner has probability of $\frac{1}{3}$ to waste 3 hours in vain, $\frac{1}{3}$ to waste 5 hours in vain, and $\frac{1}{3}$ to spend 2 hours to go out of the mine.
 - * X is the random variables of the hours to go out of the mine
 - * Y_i is the random variables of the hours for the i -th action.
 - * $\mathbb{E}[e^{tX}] = \mathbb{E}[e^{tX} | Y_1 = 2] + \mathbb{E}[e^{tX} | Y_1 = 3] + \mathbb{E}[e^{tX} | Y_1 = 5]$
 - $= \mathbb{E}[e^{2t}] + \mathbb{E}[e^{t(X+3)}] + \mathbb{E}[e^{t(X+5)}]$
 - * Find expectation and variance by joint moment generating function

3 Expectation

- N i.i.d. events, when N is a random variable
 - Suppose N is a integer random variable
 - Suppose $X_1, \dots, X_i, \dots, X_N$ are i.i.d random variables with mean μ and variance σ^2
 - $Y = \sum_{i=1}^N X_i$
 - $\mathbb{E}[Y] = \mathbb{E}[N]\mu$
 - * $\mathbb{E}[Y] = \sum_{n=1}^{\infty} \mathbb{E}[\sum_{i=1}^N X_i | N = n] P[N = n]$
 $= \mu \times \sum_{n=1}^{\infty} n P[N = n] = \mathbb{E}[N]\mu$
 - $\mathbb{E}[Y^2] = \mathbb{E}[N]\mathbb{E}[X^2] + \mathbb{E}[N^2]\mu^2 - \mathbb{E}[N]\mu^2$
 - * $\mathbb{E}[Y^2] = \sum_{n=1}^{\infty} \mathbb{E}[(\sum_{i=1}^N X_i)^2 | N = n] P[N = n] = \sum_{n=1}^{\infty} (n\mathbb{E}[X_i^2] + n(n-1)\mu^2) P[N = n]$
 $= \mathbb{E}[N]\mathbb{E}[X^2] + \mathbb{E}[N^2]\mu^2 - \mathbb{E}[N]\mu^2$
 - $\text{Var}(Y) = \mathbb{E}[N]\sigma^2 + \text{Var}(N)\mu^2$
- Expectation by $P[X > x]$
 - $\mathbb{E}[X] = \sum_x P[X > x]$, when X is a non-negative discrete random variable
 - * $\mathbb{E}[X] = \sum_{x=0}^{\infty} x P[X = x] = \sum_{x=0}^{\infty} \sum_{y=0}^{x-1} P[X = x] = \sum_{y=0}^{\infty} \sum_{x=y+1}^{\infty} P[X = x] = \sum_{y=0}^{\infty} P[X > y]$
 - $\mathbb{E}[X] = \int_0^{\infty} P[X > x] dx$, when X is a non-negative continuous random variable
 - * $\mathbb{E}[X] = \int_0^{\infty} x f_X(x) dx = \int_0^{\infty} \int_0^x f_X(x) dy dx = \int_0^{\infty} \int_y^{\infty} f_X(x) dx dy = \int_0^{\infty} P[X > y] dy$

4 Inequality

- Markov Inequality

Definition:

- Suppose $X \geq 0$, then $P[X \geq \epsilon] \leq \frac{\mathbb{E}[X]}{\epsilon}$

Proof:

1. $\mathbb{E}[X] = \int_0^{\infty} x f_X(x) dx \geq \int_{\epsilon}^{\infty} x f_X(x) dx \geq \epsilon \int_{\epsilon}^{\infty} f_X(x) dx = \epsilon P[X \geq \epsilon]$
2. $X(\omega) \geq \epsilon \mathbb{1}_{X(\omega) \geq \epsilon}, \forall \omega \in S$
 - Calculate expectation on both side.
 - $\mathbb{E}[X] \geq \epsilon P[X \geq \epsilon]$

Property:

- The equality happens when $P[X = k] = 0, \forall k \notin \{0, \epsilon\}$.

- Chebyshev Inequality

Definition:

- Suppose $m = \mathbb{E}[X], \sigma^2 = \text{Var}(X)$, then $P[|X - m| \geq \epsilon] \leq \frac{\sigma^2}{\epsilon^2}$

Proof:

- $P[|X - m| \geq \epsilon] = P[(X - m)^2 \geq \epsilon^2]$
- $P[(X - m)^2 \geq \epsilon^2] \leq \frac{\mathbb{E}[(X - m)^2]}{\epsilon^2}$ (by Markov Inequality)

Property:

- The equality happens when $P[X = k] = 0, \forall k \notin \{m - \epsilon, m, m + \epsilon\}$.
- Might be tighter than Markov Inequality since it requires m, σ

- Chernoff Inequality

Definition:

- Suppose X_1, \dots, X_n are independent identically distributed Bernoulli random variable with probability p and $X = \sum_{i=1}^n X_i$
- $P[X \geq \epsilon] \leq \frac{(pe^t + 1 - p)^n}{e^{t\epsilon}} \leq \frac{e^{np(e^t - 1)}}{e^{t\epsilon}}$
 - * $P[X \geq \epsilon] = P[e^{tX} \geq e^{t\epsilon}] \leq \frac{E[e^{tX}]}{e^{t\epsilon}} = \frac{(E[e^{tX_i}])^n}{e^{t\epsilon}} = \frac{(pe^t + 1 - p)^n}{e^{t\epsilon}} \leq \frac{e^{np(e^t - 1)}}{e^{t\epsilon}}$
- $P[X \geq np(1 + \epsilon)] \leq \left(\frac{e^\epsilon}{(1 + \epsilon)^{1 + \epsilon}} \right)^{np} \leq \begin{cases} e^{-\frac{\epsilon^2 np}{3}} & \text{if } 0 \leq \epsilon \leq 1 \\ e^{-\frac{\epsilon^2 np}{(2 + \epsilon)}} & \text{if } \epsilon \geq 1 \end{cases}$
 - * Substitute ϵ with $np(1 + \epsilon)$
 - * Substitute t with $\log(1 + \epsilon)$
 - * the last inequality is without proof
- $P[X \leq \epsilon] \leq \frac{(pe^{-t} + 1 - p)^n}{e^{-t\epsilon}} \leq \frac{e^{np(e^{-t} - 1)}}{e^{-t\epsilon}}$
 - * $P[X \leq \epsilon] = P[e^{-tX} \geq e^{-t\epsilon}] \leq \frac{E[e^{-tX}]}{e^{-t\epsilon}} = \frac{(E[e^{-tX_i}])^n}{e^{-t\epsilon}} = \frac{(pe^{-t} + 1 - p)^n}{e^{-t\epsilon}} \leq \frac{e^{np(e^{-t} - 1)}}{e^{-t\epsilon}}$
- $P[X \leq np(1 - \epsilon)] \leq \left(\frac{e^{-\epsilon}}{(1 - \epsilon)^{1 - \epsilon}} \right)^{np} \leq e^{-\frac{\epsilon^2 np}{2}}$
 - * Substitute ϵ with $np(1 - \epsilon)$
 - * Substitute t with $-\log(1 - \epsilon)$
 - * the last inequality is without proof

- Chernoff/ Hoeffding Lemma

Definition:

- Suppose X_1, \dots, X_n are independent distributed random variable and $a_i \leq X_i \leq b_i$
- Suppose $X = \sum_{i=1}^n X_i$ and $\mu = \mathbb{E}[X]$
- $P[|X - \mu| \geq \epsilon] \leq 2e^{-\frac{2\epsilon^2}{\sum_{i=1}^n (b_i - a_i)^2}}$ without proof

- Application:

- Balls in Bins

Definition: Throw n balls into n bins, find bounds for the maximum number of balls in all bins

- * $P[\text{maximum number of balls in all bins} \geq \epsilon]$
 $= P[\cup_{i=1}^n \text{number of balls in } i\text{-th bin} \geq \epsilon]$
 $\leq n \times P[\text{number of balls in one bin} \geq \epsilon]$
- * By Markov inequality:
 - $P[\text{number of balls in one bin} \geq \epsilon] \leq \frac{1}{\epsilon} \rightarrow \text{useless}$
- * By Chebyshev inequality:
 - $P[\text{number of balls in one bin} \geq \epsilon] \leq \frac{(1 - \frac{1}{n})}{\epsilon^2}$
 - $P[\text{maximum number of balls in all bins} \geq n^{\frac{1}{2} + \epsilon}] \leq \frac{(1 - \frac{1}{n})}{n^{2\epsilon}}$
 - when $n \rightarrow \infty$, the maximum number of balls should less than $n^{\frac{1}{2} + \epsilon}$
- * By Chernoff inequality:
 - $P[\text{number of balls in one bin} \geq 2 \log n] \leq \frac{e^{np(e^t - 1)}}{n^{2t}}$
 - $P[\text{maximum number of balls in all bins} \geq 2 \log n] \leq \frac{e^{np(e^t - 1)}}{n^{2t - 1}}$
 - when t is a constant ≥ 0.5 and $n \rightarrow \infty$, the maximum number of balls should less than $2 \log n$

5 Law of Large Numbers

- $\{X_i\}_{i=1}^\infty$ is a sequence of pairwise uncorrelated random variable with $\mathbb{E}[X_i] = m$, $\text{Var}(X_i) = \sigma_i^2$.
- $M_n = \frac{1}{n} \sum_{i=1}^n X_i$
- $M_n \rightarrow m$ almost surely, in mean square and in probability.

6 Memoryless

- Definition: $P[X > x_1 + x_2 | X > x_1] = P[X > x_2]$
- Property:
 - Exponential random variable is the only continuous memoryless random variable
 - Bernoulli random variable is the only discrete memoryless random variable

7 Famous Random Variable

- Poisson:

$$P[X = k] = \frac{\lambda^k}{k!} \exp(-\lambda)$$

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} \exp(-\lambda) = \sum_{k=0}^{\infty} \lambda \frac{\lambda^{k-1}}{(k-1)!} \exp(-\lambda) = \lambda$$

Interpretation:

- Cut total time into infinite period in Binomial random variable, $n \rightarrow \infty, p \rightarrow \frac{\lambda}{n}$
- $\rightarrow P[X = k] = \lim_{n \rightarrow \infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} = \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n}\right)^n = \frac{\lambda^k}{k!} \exp(-\lambda)$

- Erlang:

$$f_X(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!}, \forall x \in \mathbb{R}$$

$$\mathbb{E}[X] = \frac{n}{\lambda}$$

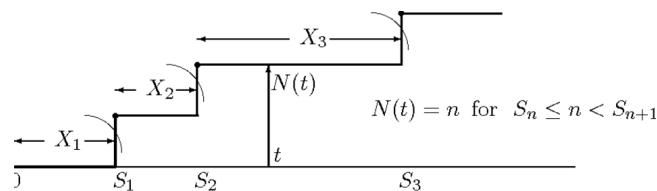
Interpretation:

- Suppose X_1, X_2, \dots, X_n are i.i.d exponential random variable with λ .
- $X = \sum_{i=1}^n X_i$
- Proof by induction:

Suppose $n = 2$, $f_X(x) = \int_0^x \lambda e^{-\lambda t} \lambda e^{-\lambda(x-t)} dt = \lambda^2 x e^{-\lambda x}$

8 Stochastic Processes

- Stochastic Process: a collection of random variable
- Arrival Process: a sequence of arriving event in continuous time



- X_i : the time between the i -th event and the $i - 1$ -th event
- S_i : the time from start to i -th event
- $N(t)$: the number of the arrived event at time t
- X and S Relation:
 - * $X_1 = S_1, X_i = S_i - S_{i-1}$
- N and S Relation:
 - * $N(t) < n \leftrightarrow S_{n+1} > t$
 - * $N(t) \geq n \leftrightarrow S_n \leq t$
 - * $N(t) = n \leftrightarrow S_n \leq t < S_{n+1}$
 - * $N(t) = \max\{n : S_n \leq t\}$

– Renewal Process: an arrival process with i.i.d X_i

Delayed Renewal Process: the process becomes a renewal process after several arrivals

X_i Property

- * if X_i is dependent on the interval states, then X_i might be dependent on $X_{i-1} \rightarrow$ not renewal process

S_i Property

- * $P[\lim_{n \rightarrow \infty} S_n = \infty] = 1$
Proof: $\lim_{n \rightarrow \infty} P[S_n = \infty] = \lim_{n \rightarrow \infty} P[\sum_{i=1}^n X_i = n \times \mathbb{E}[X_i]] = 1$
Interpretation: infinite events do not take finite time

$N(t)$ Property

- * for any $t, P[N(t) < \infty] = 1$
Proof: $P[\lim_{n \rightarrow \infty} S_n = \infty] = 1 \rightarrow$ for any $t, P[\lim_{n \rightarrow \infty} S_{n+1} > t] = 1$
Interpretation: infinite events do not take finite time
- * $P[\lim_{t \rightarrow \infty} N(t) \rightarrow \infty] = 1$
Proof: if $P[\lim_{t \rightarrow \infty} N(t) = k] > 0 \rightarrow P[X_{k+1} = \infty] > 0$
Interpretation: finite events do not take infinite time
- * $P[\lim_{t \rightarrow \infty} \frac{N(t)}{t} = \frac{1}{\mathbb{E}[X_i]}] = 1$
Proof: $P[\lim_{t \rightarrow \infty} \frac{N(t)}{S_{N(t)+1}} \leq \lim_{t \rightarrow \infty} \frac{N(t)}{t}] = 1$ and $P[\lim_{t \rightarrow \infty} \frac{N(t)}{S_{N(t)+1}} = \frac{1}{\mathbb{E}[X_i]}] = 1$
 $P[\lim_{t \rightarrow \infty} \frac{N(t)}{t} \leq \lim_{t \rightarrow \infty} \frac{N(t)}{S_{N(t)}}] = 1$ and $P[\lim_{t \rightarrow \infty} \frac{N(t)}{S_{N(t)}} = \frac{1}{\mathbb{E}[X_i]}] = 1$

Inspection Paradox

- * $\mathbb{E}[X_{N(t)+1}] \geq \mathbb{E}[X_i]$: inspection paradox
Interpretation:
 - $f_{X_{N(t)+1}}(x) = \lambda x f_{X_i}(x)$
 - when selecting t with equal probability, we tend to choose X_i with longer period
- * $P[\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (S_{N(t)+1} - s) ds = \frac{\mathbb{E}[X_i^2]}{2\mathbb{E}[X_i]}] = 1$
Proof:
 $P[\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=i}^{N(t)} \frac{\mathbb{E}[X_i^2]}{2} \leq \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (S_{N(t)+1} - s) ds] = 1$ and $P[\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=i}^{N(t)} \frac{\mathbb{E}[X_i^2]}{2} = \frac{\mathbb{E}[X_i^2]}{2\mathbb{E}[X_i]}] = 1$
 $P[\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (S_{N(t)+1} - s) ds \leq \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=i}^{N(t)+1} \frac{\mathbb{E}[X_i^2]}{2}] = 1$ and $P[\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=i}^{N(t)+1} \frac{\mathbb{E}[X_i^2]}{2} = \frac{\mathbb{E}[X_i^2]}{2\mathbb{E}[X_i]}] = 1$
- * $P[\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (s - S_{N(t)}) ds = \frac{\mathbb{E}[X_i^2]}{2\mathbb{E}[X_i]}] = 1$
Proof: similar to above
- * $P[\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t X_{N(t)} ds = \frac{\mathbb{E}[X_i^2]}{\mathbb{E}[X_i]}] = 1$
Proof: $P[\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t X_{N(t)} ds = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (S_{N(t)+1} - S_{N(t)}) ds] = 1$
- * $\mathbb{E}[X_{N(t)+1}] = \frac{\mathbb{E}[X_i^2]}{\mathbb{E}[X_i]}$
Proof: $P[\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t X_{N(t)} ds = \frac{\mathbb{E}[X_i^2]}{\mathbb{E}[X_i]}] = P[\mathbb{E}[X_{N(t)+1}] = \frac{\mathbb{E}[X_i^2]}{\mathbb{E}[X_i]}] = 1$

Central Limit Theorem

- * $\mu = \mathbb{E}[X_i]$
- * $\sigma = \sqrt{\text{Var}(X_i)}$
- * $Z \sim \text{Normal}(0,1)$
- * $\lim_{t \rightarrow \infty} P[N(t) \leq \frac{t}{\mu} + k \frac{\sigma\sqrt{t}}{\sqrt{\mu^3}}] = P[Z \leq k]$
Proof:
 1. Suppose $n(t) = \frac{t}{\mu} + k \frac{\sigma\sqrt{t}}{\sqrt{\mu^3}}$
 2. $P[N(t) \geq n(t)] = P[S_{n(t)} \leq t] = P[\frac{S_{n(t)} - n\mu}{\sigma\sqrt{n}} \leq \frac{t - n\mu}{\sigma\sqrt{n}}]$
 3. When $t \rightarrow \infty, \frac{t - n\mu}{\sigma\sqrt{n}} \rightarrow k$
 4. By law of large number, $\lim_{t \rightarrow \infty} P[\frac{S_{n(t)} - n\mu}{\sigma\sqrt{n}} \leq k] = P[Z \leq k]$

Interpretation:

- $\frac{t}{\mu}$ is approximately the mean of $N(t)$
- $k \frac{\sigma\sqrt{t}}{\sqrt{\mu^3}}$ is $k\sigma\sqrt{n}$ after dividing by μ , the ratio between t and $N(t)$ and changing n with $\frac{t}{\mu}$

Wald's Identity

- * Stopping Times: a random variable τ s.t. $\{\tau = n\}$ is independent of $\{X_i\}_{i=n+1}^\infty$
- * Stopping Condition: a condition to stop if we can consider $\tau = \min\{n : \text{condition}(n) = \top\}$
- * Example: $N(t) + 1$ is a stopping times and can be consider $N(t) + 1 = \min\{n : S_n > t\}$
- * $\mathbb{E}[\sum_{i=1}^\tau X_i] = \mathbb{E}[\tau]\mathbb{E}[X_i]$ if $\mathbb{E}[X_i] < \infty$ and $\mathbb{E}[N] < \infty$
- Proof:
 1. $\mathbb{E}[\sum_{i=1}^\tau X_i] = \sum_{i=1}^\infty \mathbb{E}[X_i \times \mathbb{1}_{i \leq \tau}]$ (by Fubin's Theorem without proof)
(if $\mathbb{E}[X_i] < \infty$ and $\mathbb{E}[N] < \infty$)
 2. $\sum_{i=1}^\infty \mathbb{E}[X_i \times \mathbb{1}_{i \leq \tau}] = \mathbb{E}[X_i] \sum_{i=1}^\infty \mathbb{E}[\mathbb{1}_{i \leq \tau}]$ (by $P[\tau \geq i] = 1 - P[\tau < i]$ is independent of X_i)
 3. $\mathbb{E}[X_i] \sum_{i=1}^\infty \mathbb{E}[\mathbb{1}_{i \leq \tau}] = \mathbb{E}[\tau]\mathbb{E}[X_i]$
- * $\lim_{t \rightarrow \infty} \frac{\mathbb{E}[N(t)]}{t} = \frac{1}{\mathbb{E}[X_i]}$

Proof:

- Suppose $\mu = \mathbb{E}[X_i]$
- $\frac{\mathbb{E}[N(t)]}{t} = \frac{\mathbb{E}[S_{N(t)+1}]}{t \times \mu} - \frac{1}{t}$ (by considering $N(t) + 1$ as the stopping time)
- $\lim_{t \rightarrow \infty} \frac{\mathbb{E}[N(t)]}{t} \geq \frac{1}{\mu}$ (by $\mathbb{E}[S_{N(t)+1}] > t$)
- Suppose $\hat{X}_n = \min\{X_n, T\}$, where T is a constant
- $\frac{\mathbb{E}[N(t)]}{t} \leq \frac{\mathbb{E}[\hat{N}(t)]}{t} = \frac{\mathbb{E}[S_{\hat{N}(t)+1}]}{t \times \hat{\mu}} - \frac{1}{t} \leq \frac{t+T}{t \times \hat{\mu}} - \frac{1}{t}$
- $\lim_{n=\sqrt{t}, t \rightarrow \infty} \frac{\mathbb{E}[N(t)]}{t} \leq \frac{1}{\mu}$

Blackwell's Theorem

- * $\mathbb{E}[N(t)] = F_{X_i}(t) + \int_0^t \mathbb{E}[N(t-x)]f_{X_i}(t)dt$
- Proof: $\mathbb{E}[N(t)] = \int_0^t \mathbb{E}[N(t)|X_1 = x]f_{X_1}(x)dx$
 $= \int_0^t \mathbb{E}[N(t-x) + 1]f_{X_1}(x)dx = F_{X_i}(t) + \int_0^t \mathbb{E}[N(t-x)]f_{X_i}(t)dt$
- * $\mathcal{L}\{\mathbb{E}[N(t)]\}(s) = \frac{\mathcal{L}\{f_{X_i}\}(s)}{s(1-\mathcal{L}\{f_{X_i}\}(s))}$
- Proof: Laplace transform both sides
- * Lattice/ Non-Lattice: $N(t)$ is lattice iff X_i only takes on values that are $nd, n \in \mathbb{N}, d \in \mathbb{R}$
- * For a non-lattice process: $\lim_{t \rightarrow \infty} \mathbb{E}[N(t+\delta) - N(t)] = \frac{\delta}{\mathbb{E}[X_i]}$, for any δ
- Proof: Without Proof
- Interpretation: $\mathbb{E}[N(t)]$ will converge to be linear
- * For a lattice process and period d : $\lim_{n \rightarrow \infty} \mathbb{E}[\# \text{ events at } t = nd] = \frac{d}{\mathbb{E}[X_i]}$
- Proof: Without Proof
- Interpretation: $\mathbb{E}[N(t)]$ will converge to be stairs with width d and height $\frac{d}{\mathbb{E}[X_i]}$

– Renewal-Reward Process:

Definition

- * A renewal process $N(t)$ and $\{R_i\}_{i=1}^\infty$ such that (X_i, R_i) are i.i.d.
 $(X_i, R_j, i \neq j)$ are independent, but X_i, R_i might be dependent

Property

- * $P[\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=1}^{N(t)} R_i = \frac{\mathbb{E}[R_i]}{\mathbb{E}[X_i]}] = 1$
- Proof: $P[\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=1}^{N(t)} R_i = \lim_{t \rightarrow \infty} \sum_{i=1}^{N(t)} \frac{R_i}{N(t)} \times \lim_{t \rightarrow \infty} \frac{N(t)}{t}] = 1$

– Poisson Process: a renewal process with $X_i \sim \text{Exponential}(\lambda)$

S_i Property

- * S_i is an Erlang random variable
 Erlang is the sum of the Exponential random variables
- * Joint Distribution $f_{S_1, \dots, S_n}(s_1, \dots, s_n) = \lambda^n e^{-\lambda s_n}$
 Prove by induction.
 Induce by $f_{S_1, \dots, S_n}(s_1, \dots, s_n) = f_{S_1, \dots, S_{n-1}}(s_1, \dots, s_{n-1}) \times f_{S_n|S_1, \dots, S_{n-1}}(s_n, s_1, \dots, s_{n-1})$

$N(t)$ Property

- * $N(t) \sim \text{Poisson}(\lambda t)$, $P[N(t) = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$
Prove by $P[N(t) = n] = P[S_n \leq t \text{ and } S_{n+1} > t]$
- * Conditioned on $N(t) = n$, the set of arrival times $\{s_1, \dots, s_n\}$ have the same distribution with a set of n sorted i.i.d. $\text{Uniform}(0, t)$ random variables
Prove by $f_{S_1, \dots, S_n | N(t)}(s_1, \dots, s_n, n) = \frac{f_{S_1, \dots, S_n}(s_1, \dots, s_n) P[X_{n+1} > t - s_n]}{P[N(t) = n]} = \frac{n!}{t^n}$

Property

- * Z is the interval from t to the first arrival $\rightarrow Z$ is exponential random variable with same λ and independent of $N(t)$ and the arrival time before t
Proof:

$$P[Z > z] = \sum_{n=0}^{\infty} \int_0^{\infty} \dots \int_0^{\infty} P[Z > z | N(t) = n, S_1 = s_1, \dots, S_n = s_n] ds_1 \dots ds_n$$

$$= \sum_{n=0}^{\infty} \int_0^{\infty} \dots \int_0^{\infty} P[X_{n+1} > z + t - s_n | N(t) = n, S_1 = s_1, \dots, S_n = s_n] ds_1 \dots ds_n$$

$$= \sum_{n=0}^{\infty} \int_0^{\infty} \dots \int_0^{\infty} P[X_{n+1} > z + t - s_n | X_{n+1} > t - s_n] ds_1 \dots ds_n = e^{-\lambda z}$$
- * Stationary Increments: $N(t_1 + t_2) - N(t_1)$ and $N(t_2)$ share the same distribution
Without Proof
- * Independent Increments: $\forall 0 < t_1 < t_2 < \dots, t_k, N(t_1), N(t_2) - N(t_1), \dots$ are independent
Without Proof
- * Any arrival process with stationary and independent increments must be a Poisson process
Without Proof

Exercise

- * $\mathbb{E}[S_i | N(t) = n] = \frac{t \times i}{n+1}$

$$\cdot \mathbb{E}[S_i | N(t) = n] = i \times \mathbb{E}[X_1 | N(t) = n] = i \int_0^t \int_0^{s_n} \dots \int_0^{s_2} s_1 \times \frac{n!}{t^n} ds_1 \dots ds_{n-1} ds_n = \frac{t \times i}{n+1}$$
- * $\mathbb{E}[\sum_{i=0}^{N(t)} S_i] = \frac{\lambda t^2}{2}$

$$\cdot \mathbb{E}[\sum_{i=0}^{N(t)} S_i] = \sum_{n=0}^{\infty} \mathbb{E}[\sum_{i=0}^n S_i | N(t) = n] P[N(t) = n]$$

$$= \sum_{n=0}^{\infty} \frac{n t^2}{2} P[N(t) = n] = \frac{\lambda t^2}{2}$$

2D Poisson Process

- * Definition:
 - For any region R : number of points in R is a Poisson random variable
 - number of points in the non-overlapping region is independent

Combining Poisson Process

- * $N^1(t), N^2(t)$ are two independent Poisson process with λ_1, λ_2
- * X_i is the first arrival of X_i^1, X_i^2
- * Property
 - X_i is independent of $\{X_i^1 < X_i^2\}$ and $\{X_i^1 > X_i^2\}$
Proof: $P[X_1^1 < X_1^2] = \frac{\lambda_1}{\lambda_1 + \lambda_2}$

$$P[X_1 > x] = P[X_1^1 > x, X_1^2 > x] = e^{-(\lambda_1 + \lambda_2)x}$$

$$P[X_1 > x, X_1^1 < X_1^2] = P[X_1 > x] P[X_1^1 < X_1^2]$$
 - X_i is a Poisson Process with $\lambda = \lambda_1 + \lambda_2$
 - $\min(X_1, X_2)$ is an exponential random variable with $\lambda = \lambda_1 + \lambda_2$

Splitting Poisson Process

- * $N^1(t), N^2(t)$ are two independent Poisson process with λ_1, λ_2
- * $N(t)$ is a random process with $\lambda = \lambda_1 + \lambda_2$
 - $N^{1*}(t)$ is the process of the first event
when $N(t)$ arrives consider it as first event with probability $\frac{\lambda_1}{\lambda_1 + \lambda_2}$
 - $N^{2*}(t)$ is the process of the second event
when $N(t)$ arrives consider it as second event with probability $\frac{\lambda_2}{\lambda_1 + \lambda_2}$
- * $N^i(t)$ and $N^{i*}(t)$ share the same distribution
- * Proof:
 - $B_n(k)$ is a Binomial random variable with $p = \frac{\lambda_1}{\lambda_1 + \lambda_2}$
 - $P[N^{1*}(t) = m, N^{2*}(t) = n] = P[N(t) = m + n, B_{m+n}(m)] = P[N^1(t) = m, N^2(t) = n]$

Compound Poisson Process

- * $N(t)$ is a Poisson Process
- * A_n is a sequence of cost
- * $A(t) = \sum_{n=0}^{N(t)} A_n$ is the summation of cost over Poisson Process

Non-Homogeneous Poisson Process

- * $N(t) - N(s) \sim \text{Poisson}(\int_s^t \lambda(x) dx)$

Queueing Theory

- * Definition: *Arrival_Process/Service_Process/number_of_services*
 - M : memoryless (Poisson) process
 - D : deterministic process
 - G : general renewal process
- * T : the random variable of the processing time for each customer
- * $Y(t)$: number of cutomers in the service
 - $Y(t) \sim \text{Poisson}(\lambda \int_0^t P[T > x] dx)$
 - Proof:
Consider $Y(t)$ is a splitting Poisson Process. Since the distribution for the arrival given $N(t)$ is universal, the probability the arrival is still in service: $\frac{1}{t} \int_0^t P[T > t-x] dx = \frac{1}{t} \int_0^t P[T > x] dx$

9 Markov Chain

- Definition
 - Model with states and transition probability matrix
 - States: $\{X_n\}_{n=0}^{\infty}$
 - Transition Probability Matrix: $[P]_{ij} = P[X_{n+1} = j | X_n = i]$
 - Terminology
 - $p^n = [P[X_n = 0], P[X_n = 1], \dots]^T$: distribution at step n
 - $T_i = \min\{n \geq 1 : X_n = i\}$: a random variable of the minimum time step to go to state i
 - $f_{ij} = P[T_j < \infty | X_0 = i]$: the probability of starting at i and ever reaching j
 - $\mu_{ij} = \mathbb{E}[T_j | X_0 = i]$
 - $i \rightarrow j$ iff $f_{ij} > 0$: j is reachable from i with probability greater than 0
 - $N_i(n)$: number of visits to i by time n
 - Irreducible: $i \leftrightarrow j, \forall$ states i, j
 - aperiodic: period of $X_n = i$ is 1, \forall states i
 - Property
 - Consider a given distribution as an event $\tau : [P[X_n = 0 | \tau], P[X_n = 1 | \tau], \dots]^T$
 - Updating distribution
 - * $p^n = p^0 P^n$
 - Markovian: transition probability depend only on current state
 - * $P[X_{n+1} = j | X_n = i, \dots, X_0 = x_0] = [P]_{ij}$
 - Stationary Distribution: p s.t. if $p^n = p \rightarrow p^{n+1} = p$
- Property from renewal process
- * consider $X_n = j$ as a event \rightarrow Markov Chain becomes a delayed renewal process
 - * If $i \leftrightarrow j$ and the model starts from i , then following holds
 - * $P[\lim_{n \rightarrow \infty} \frac{N_j(n)}{n} = \frac{1}{\mu_{jj}}] = 1$
 - * $\lim_{n \rightarrow \infty} \frac{\mathbb{E}[N_j(n)]}{n} = \frac{1}{\mu_{jj}}$
 - * if the period of $X_n = j$ is $d \rightarrow \lim_{n \rightarrow \infty} p_j^{nd} = \frac{d}{\mu_{jj}}$

Theorem of an irreducible, aperiodic Markov Chain

- * Either
 - All states have $\mu_{ii} = \infty$
 - All states have $\mu_{ii} < \infty$ and $p_i = \frac{1}{\mu_{ii}}$ is the unique stationary distribution
- * Proof
 - From if the period of $X_n = j$ is $d \rightarrow \lim_{n \rightarrow \infty} p_j^{nd} = \frac{d}{\mu_{jj}}$
 - Proof: $\lim_{n \rightarrow \infty} p_j^{nd} = \lim_{n \rightarrow \infty} \mathbb{E}[\# \text{ events at } nd]$

Theorem of an irreducible, aperiodic Markov Chain

- * All states have $\mu_{ii} < \infty$ and $p_i = \frac{1}{\mu_{ii}}$ is the unique stationary distribution

p can be calculated as the eigenvector corresponds to eigenvalue 1 of P^T

– Detailed Balance

Definition:

- * Given a distribution π
- * $\pi_i P_{ij} = \pi_j P_{ji}, \forall i, j$

Property:

- * distribution π satisfying Detailed Balance is the stationary distribution p
- * symmetric transition probability matrix \rightarrow uniform stationary distribution

– Reversible

Definition: A Markov Chain with stationary distribution p is reversible if it satisfies detailed balance

Interpretation

- * Transitions forward and backward in the stationary distribution have the same probability
- * $P[X_{n+1} = j | X_n = i] = P_{ij}$
- * $P[X_{n-1} = j | X_n = i] = \frac{P[X_{n-1}=j, X_n=i]}{P[X_n=i]} = \frac{p_j P_{ji}}{p_i} = P_{ij}$

– Metropolis Update Rule

Definition

- * Given a Markov Chain and distribution p' , find P' such that p' is the stationary distribution

Procedure

- * For each pair (i, j) , $P'_{ij} = P_{ij} \times \min\{1, \frac{p'_j P_{ji}}{p'_i P_{ij}}\}$
- * construct self loop to satisfy $\sum_j P'_{ij} = 1$

Proof

- * To satisfy detailed balance, for each pair (i, j) , we should set $p'_i P'_{ij} = \min\{p'_i P_{ij}, p'_j P_{ji}\}$

– Distance between Probability Measure

Definition:

- * Total Variation Distance between P_1 and P_2 is: $d_{TV}(P_1, P_2) = \frac{1}{2} \sum_{\omega} |P_1[\omega] - P_2[\omega]|$

Interpretation:

- * consider the distributions as events τ_1, τ_2
- * $P_i[\omega] = P[\omega | \tau_i]$
- * $d_{TV}(P_1, P_2) = \frac{1}{2} \sum_{\omega} |P[\omega | \tau_1] - P[\omega | \tau_2]| = \sum_{\omega} |P[\omega \wedge \tau_1] - P[\omega \wedge \tau_2]|$

– Mixing Time

Definition

- * Mixing time τ is the least t such that for all initial state p^0 , $d_{TV}(p, p^0 P^t) \leq \frac{1}{2e}$

Interpretation

- * the factor $\frac{1}{2e}$ is set such that $d_{TV}(p, p^0 P^t) \leq \epsilon$ if $t \geq \tau \times \log(\frac{1}{\epsilon})$
- Without proof

– Example

Random Walk on Graph

- * Definition: move from vertex i to vertex j with probability $P_{ij} = \begin{cases} 0 & \text{if } (i, j) \notin E \\ \frac{1}{\text{degree}(i)} & \text{if } (i, j) \in E \end{cases}$
- * Distribution π , $\pi_i = \frac{\text{degree}(i)}{2|E|}$ satisfies detailed balance
- * If we want stationary distribution to be uniform $\rightarrow P'_{ij} = \begin{cases} \frac{1}{\text{degree}(i)} & \text{if } \text{degree}(i) \geq \text{degree}(j) \\ \frac{1}{\text{degree}(j)} & \text{if } \text{degree}(i) < \text{degree}(j) \end{cases}$

Random graph coloring

- * Given a graph with V vertices, maximum degree Δ and q colors, to color each vertex one color such that adjacent vertex do not share the same color
- * Assume $q > 4\Delta$
- * Markov Chain Transition:
 - Pick random vertex and random color, if the color is changeable then change
- * Property
 - Aperiodic: there exist self loops
 - Symmetric: symmetric transition
 - Irreducible
- * Mixing time is $O(V \log V)$
- Proof:
 - Assume X is a event s.t. Markov Chain starts with any valid coloring and Y is a event s.t. Markov Chain starts with uniform distribution
 - Apply same transition on both X and Y
 - D_n is a random variable for the number of vertices in different colors in X and Y at time n
 - Good moves: number of vertices in different colors decrease $\geq D_n \times (q - 2\Delta) \geq (2\Delta + 1)D_n$ (vertices with different colors \times color that is different with any adjacent color in X and Y)
 - Bad moves: number of vertices in different colors increase $\leq (D_n \Delta) \times 2$ (vertices adjacent to different colors vertices \times color of the different colors vertices)
 - $\mathbb{E}[D_{n+1} - D_n] \leq V(1 - \frac{1}{qV})^n$
 - $\mathbb{E}[D_n] \leq V(1 - \frac{1}{qV})^n$
 - $P[D_n \geq 1] \leq V(1 - \frac{1}{qV})^n$

• Hidden Markov Chain

- Definition: output is a function of the state
- Interpretation: if the model is not markovian, then reformulate the model as a hidden markov chain by complicating the states and rendering the output as a function of the state

10 Continuous Markov Chain

• Interpretation

- v_i : coefficient of exponential distribution, where time in state i before next step is $\sim \text{Exponential}(v_i)$

• Definition

- Model with states and transition rate matrix
- States: $X(t), \forall 0 \leq t < \infty$
- Transition Probability Matrix R

• $P_{ij}(t)$

- Definition: $P_{ij}(t) = P[X(t) = j | X(0) = i]$
- Chapman-Kolmogorov Equation
 - * Definition: $P(s+t) = P(s) \times P(t)$
 - * Proof

- $P_{ij}(s+t) = P[X(s+t) = j | X(0) = i]$

$$= \sum_k P[X(s+t) = j | X(s) = k, X(0) = i] P[X(s) = k | X(0) = i]$$

$$= \sum_k P[X(s+t) = j | X(s) = k] P[X(s) = k | X(0) = i] = \sum_k P_{kj}(t) P_{ik}(s)$$
 - Kolmogorov's Differential Equation
 - * Forward: $\frac{dP(t)}{dt} = P(t)R$
Interpretation:
 - Change of distribution at t equals the distribution at $t \times R$
 - Proof:
 - $\frac{dP(t)}{dt} = \lim_{\delta \rightarrow 0} \frac{P(t+\delta) - P(t)}{\delta} = P(t) \lim_{\delta \rightarrow 0} \frac{P(\delta) - P(0)}{\delta} = P(t)R$
 - * Backward: $\frac{dP(t)}{dt} = RP(t)$
Interpretation:
 - Change of distribution at t equals the distribution at $t = 0 \times P(t)$
 - Proof:
 - $\frac{dP(t)}{dt} = \lim_{\delta \rightarrow 0} \frac{P(t+\delta) - P(t)}{\delta} = \lim_{\delta \rightarrow 0} \frac{P(\delta) - P(0)}{\delta} P(t) = RP(t)$
 - * Solution: $P(t) = e^{Rt}$
- R
 - Definition:
 - * $R_{ij} = \left. \frac{dP_{ij}(t)}{dt} \right|_{t=0}$
 - * $R_{ij} = \begin{cases} -v_i & \text{if } i = j \\ v_i P_{ij} & \text{if } i \neq j \end{cases}$ (if there is no self-transition)
 - Interpretation
 - * πR is the change of distribution of π (by Kolmogorov's Differential Equation)
 - * simulation by transition from state i to j when $e^{-R_{ij}t}$ event arrives
 - Proof
 - $\frac{dP_{ii}(t)}{dt} = R_{ii}P_{ii}(t) \rightarrow P_{ii}(t) = e^{-R_{ii}t}$
 - simulate the transition out of state i by $e^{-R_{ii}t}$ and transition to j state by probability $\frac{R_{ij}}{R_{ii}}$ is the same as transition from state i to j when $e^{-R_{ij}t}$ event arrives
 - Property
 - Continuous Markov Chain with same R are of the same functionality
 - Property:
 - * $\sum_j R_{ij} = 0$: sum of element is a row of R is 0
- Property
 - Self Transition:
 - * Since R defines the Markov Chain, we can modify v_i to conduct self transition without changing R
 - Uniformization:
 - * Since R defines the Markov Chain, we can modify v_i such that v_i are the same for all states without changing R
 - Stationary Distribution: p s.t. $pR = 0, pe^{Rt} = p$
Interpretation:
 - * $\frac{dpP(t)}{dt} = p \frac{dP(t)}{dt} = pRP(t) = 0$
 - * p is the eigenvector of eigenvalue 0 of R , then p is the eigenvector of eigenvalue 1 of $e^{Rt} \rightarrow$ the distribution would not change, if start with p
 - Trick:
 - * cluster states such that every state in the cluster share the same R_{ij} to calculate the stationary distribution of the cluster
 - * assume distribution is independent of the cluster and check $pR = 0$ after the calculation
 - Poisson process is a special case of Continuous Markov Chain

- * $v_i = \lambda, \forall i$
- * i -th state transition to $i + 1$ -th state
- Exploding process: only if $v_i \rightarrow \infty$
 - * exploding process: traverse infinite states in finite time

11 Martingales

- Definition

- Discrete

General Discrete Martingales

- * $\{Z_i\}_{i=0}^\infty$ such that
 1. $\mathbb{E}[|Z_n|] < \infty$
 2. $\mathbb{E}[Z_{n+1}|Z_0, \dots, Z_n] = Z_n$
 - sub-martingales: $\mathbb{E}[Z_{n+1}|Z_0, \dots, Z_n] \geq Z_n$
 - super-martingales: $\mathbb{E}[Z_{n+1}|Z_0, \dots, Z_n] \leq Z_n$

Discrete Martingales with respect to X_i

- * $\{Z_i\}_{i=0}^\infty$ such that
 1. $\mathbb{E}[|Z_n|] < \infty$
 2. $\mathbb{E}[Z_{n+1}|X_0, \dots, X_n] = Z_n$
 - sub-martingales: $\mathbb{E}[Z_{n+1}|X_0, \dots, X_n] \geq Z_n$
 - super-martingales: $\mathbb{E}[Z_{n+1}|X_0, \dots, X_n] \leq Z_n$
- * $\mathbb{E}[Z_{n+1}|X_0, \dots, X_n] = Z_n$ implies $\mathbb{E}[Z_{n+1}|Z_0, \dots, Z_n] = Z_n$
 - Z_n is a function of X_0, \dots, X_n
 - $\mathbb{E}[Z_{n+1}|Z_0, \dots, Z_n] = \mathbb{E}[\mathbb{E}[Z_{n+1}|X_0, \dots, X_n, Z_0, \dots, Z_n]|Z_0, \dots, Z_n]$
 $= \mathbb{E}[\mathbb{E}[Z_{n+1}|X_0, \dots, X_n]|Z_0, \dots, Z_n] = \mathbb{E}[Z_n|Z_0, \dots, Z_n] = Z_n$

- Continuous Martingales with respect to $N(t)$

- * $Y(t)$ such that
 1. $\mathbb{E}[|Y(t)|] < \infty$
 2. $\mathbb{E}[Y(t)|\{N(s)|0 \leq s \leq \tau\}] = Y(\tau), \forall \tau \leq t$
 - sub-martingales: $\mathbb{E}[Y(t)|\{N(s)|0 \leq s \leq \tau\}] \geq Y(\tau), \forall \tau \leq t$
 - super-martingales: $\mathbb{E}[Y(t)|\{N(s)|0 \leq s \leq \tau\}] \leq Y(\tau), \forall \tau \leq t$

- Property

- $\mathbb{E}[Z_n] = \mathbb{E}[Z_1]$
 Proof: $\mathbb{E}[Z_{n+1} - Z_n] = \mathbb{E}[\mathbb{E}[Z_{n+1} - Z_n|Z_0, \dots, Z_n]] = 0$
- $\mathbb{E}[Z_n|\{Z_i|i \in S\}] = Z_{\max_{i \in S} i}$, where $\forall i \in S, i < n$
 Proof: $\mathbb{E}[Z_n|Z_i] = \mathbb{E}[\mathbb{E}[Z_n|Z_0, \dots, Z_{n-1}]|Z_i] = \mathbb{E}[Z_{n-1}|Z_i]$
- Azuma's Inequality
 - * $\mu = \mathbb{E}[Z_0]$
 - * $-a_i \leq Z_i - Z_{i-1} \leq b_i$
 - * $P[|Z_n - \mu| \geq \delta] \leq 2e^{-\frac{2\delta^2}{\sum_{i=1}^n (b_i + a_i)^2}}$
- Kolmogorov's sub-martingales inequality
 - * $P[\sup_{n \geq 1} Z_n \geq a] \leq \frac{\mathbb{E}[Z_1]}{a}$
- Martingales Stopping Theorem
 - * Stopping Times: a random variable τ s.t. $\{\tau = n\}$ is independent of $\{X_i\}_{i=n+1}^\infty$
 - * Stopping Condition: a condition to stop if we can consider $\tau = \min\{n : \text{condition}(n) = \top\}$
 - * $\mathbb{E}[Z_\tau] = \mathbb{E}[Z_0]$ if the either of the following holds
 1. $P[\tau \leq k] = 1$

2. $P[\max_{i \leq \tau} |Z_i| \leq k] = 1$
 3. $\mathbb{E}[\tau] < k$ and $\mathbb{E}[|Z_{n+1} - Z_n| | Z_0, \dots, Z_n] < k$
- Application for generating Martingales
 - Sum of iid. random variables
 - * $\{X_i\}_{i=1}^\infty$ are iid. random variables
 - * $Z_n = \sum_{i=1}^n X_i - n\mathbb{E}[X_i]$ is a martingales.
 - * Proof: $\mathbb{E}[Z_{n+1} | Z_0, \dots, Z_n] = \mathbb{E}[Z_n + X_{n+1} - \mathbb{E}[X_i] | Z_0, \dots, Z_n] = Z_n$
 - Square of sum of iid. random variables
 - * $\{X_i\}_{i=1}^\infty$ are iid. random variables and $\mathbb{E}[X_i] = 0$
 - * $Z_n = (\sum_{i=1}^n X_i)^2 - n\mathbb{E}[X_i^2]$ is a martingales.
 - * Proof: $\mathbb{E}[Z_{n+1} | Z_0, \dots, Z_n] = \mathbb{E}[Z_n + X_{n+1}^2 + 2X_{n+1}(\sum_{i=1}^n X_i) - \mathbb{E}[X_i^2] | Z_0, \dots, Z_n] = Z_n$
 - Product of iid. random variables
 - * $\{X_i\}_{i=1}^\infty$ are iid. random variables
 - * $Z_n = \frac{\prod_{i=1}^n X_i}{\mathbb{E}[X_i]^n}$ is a martingales.
 - * Proof: $\mathbb{E}[Z_{n+1} | Z_0, \dots, Z_n] = \mathbb{E}[Z_n \frac{X_{n+1}}{\mathbb{E}[X_i]} | Z_0, \dots, Z_n] = Z_n$
 - Poisson Process
 - * $N(t)$ is a poisson process
 - * $Y(t) = N(t) - \lambda t$ is a martingales.
 - * Proof: $\mathbb{E}[Y(t) | \{N(s) | 0 \leq s \leq \tau\}] = \mathbb{E}[Y(\tau) + Y(t) - Y(\tau) | \{N(s) | 0 \leq s \leq \tau\}]$
 $= Y(\tau) + \mathbb{E}[N(t) - N(\tau) + \lambda(t - \tau) | \{N(s) | 0 \leq s \leq \tau\}] = Y(\tau)$
 - Doob-type Martingales
 - * $X, \{Y_i\}_{i=1}^\infty$ are random variables
 - * $Z_n = \mathbb{E}[X | Y_1, Y_2, \dots, Y_n]$ is a martingales
 - * Proof: $\mathbb{E}[Z_{n+1} | Y_1, \dots, Y_n] = \mathbb{E}[\mathbb{E}[X | Y_1, Y_2, \dots, Y_n, Y_{n+1}] | Y_1, Y_2, \dots, Y_n]$
 $= \mathbb{E}[X | Y_1, Y_2, \dots, Y_n] = Z_n$
 - Example
 - Symmetric Random Walk
 - * $p = 0.5$
 - * $\tau = \min\{i | \sum_{i=0}^n X_i \in \{-a, b\}\}$
 - * $Z_n = \sum_{i=0}^n X_i$, by second rule of Martingales Stopping Theorem: $\mathbb{E}[Z_\tau] = 0$
 $\rightarrow P[Z_\tau \text{ at } a] = \frac{b}{a+b}, P[Z_\tau \text{ at } b] = \frac{a}{a+b}$
 - * $Z_n = (\sum_{i=0}^n X_i)^2 - n$, by third rule of Martingales Stopping Theorem: $\mathbb{E}[Z_\tau] = 0$
 $\rightarrow \mathbb{E}[\tau] = ab$
 - Unbiased Random Walk
 - * $\tau = \min\{i | \sum_{i=0}^n X_i \in \{-a, b\}\}$
 - * $Z_n = (\frac{1-p}{p})^{\sum_{i=0}^n X_i}$, by second rule of Martingales Stopping Theorem: $\mathbb{E}[Z_\tau] = 0$
 $P[Z_\tau \text{ at } a] = \frac{(\frac{1-p}{p})^b - 1}{(\frac{1-p}{p})^b - (\frac{1-p}{p})^{-a}}, P[Z_\tau \text{ at } b] = \frac{1 - (\frac{1-p}{p})^{-a}}{(\frac{1-p}{p})^b - (\frac{1-p}{p})^{-a}}$
 - * $Z_n = \sum_{i=0}^n X_i - n\mathbb{E}[X_0]$, by third rule of Martingales Stopping Theorem: $\mathbb{E}[Z_\tau] = 0$
 $\rightarrow \mathbb{E}[\tau] = \frac{\mathbb{E}[\sum_{i=0}^\tau X_i]}{\mathbb{E}[X_0]}$

12 Random Walk

- Definition
 - $X_i = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1 - p \end{cases}$
 - $S_n = \sum_{i=0}^n X_i$

- The monkey at the cliff

$$- P_k = P[\exists n \text{ such that } S_n = k] = \begin{cases} 1 & \text{if } p \geq \frac{1}{2} \\ (\frac{p}{1-p})^k & \text{if } p < \frac{1}{2} \end{cases} \quad \text{where } k \in \mathbb{N}$$

Proof

$$* P_k = P_1^k \text{ by memoryless property}$$

$$* P_1 = p + q \times P_2 \rightarrow P_1 = 1 \text{ or } \frac{p}{1-p}$$

$$* \text{ if } p \geq 0.5 \rightarrow P_1 = 1 \text{ (} P_1 \leq 1 \text{)}$$

$$* \text{ if } p < 0.5 \rightarrow P_1 = \frac{p}{1-p}$$

Since $P_1 \leq \frac{p}{1-p}$ by induction on n to ∞ for $P_1(n) = P[S_n = k]$

$$- \mathbb{E}_k = \mathbb{E}[\min\{n : S_n = k\}] = \begin{cases} \infty & \text{if } p \leq \frac{1}{2} \\ \frac{k}{2p-1} & \text{if } p > \frac{1}{2} \end{cases} \quad \text{where } k \in \mathbb{N}$$

Proof

$$* \mathbb{E}_k = \mathbb{E}_1 \times k \text{ by memoryless property}$$

$$* \mathbb{E}_1 = 1 + 0 \times p + \mathbb{E}_2 \times (1-p)$$

$$* \text{ if } p < 0.5 \rightarrow P_1 = \frac{p}{1-p} \rightarrow \mathbb{E}_1 = \infty$$

$$* \text{ if } p = 0.5 \rightarrow \mathbb{E}_1 = 1 + \mathbb{E}_1 \text{ (no solution)} \rightarrow \mathbb{E}_1 = \infty$$

$$* \text{ if } p > 0.5 \rightarrow \mathbb{E}_1 = \frac{1}{2p-1}$$

$$- P_0 = P[\exists n \text{ such that } S_n = 0] = 1 - |2p - 1| \text{ where } k \in \mathbb{N}$$

$$* P_0 = p \times P_{-1} + (1-p) \times P_1$$

$$- \mathbb{E}_0 = \mathbb{E}[\min\{n : S_n = 0\}] = \infty$$

$$* \text{ if } p \neq \frac{1}{2} \rightarrow P_0 \neq 1 \rightarrow \mathbb{E}_0 = \infty$$

$$* \text{ if } p = \frac{1}{2} \rightarrow \mathbb{E}_0 = 1 + \frac{1}{2}\mathbb{E}_{-1} + \frac{1}{2}\mathbb{E}_1 = \infty$$

- The Gambler's Ruin

$$- \text{Definition: } \tau = \min\{i | S_i \in \{-a, b\}\}$$

$$- A_k = P[S_\tau = b | X_0 = k]$$

$$* A_k = pA_{k+1} + (1-p)A_{k-1}$$

$$- A_0 = \begin{cases} \frac{a}{a+b} & \text{if } p = \frac{1}{2} \\ \frac{(\frac{1-p}{p})^a - 1}{(\frac{1-p}{p})^{a+b} - 1} & \text{if } p \neq \frac{1}{2} \end{cases}$$

Solved by previous recursive equation

$$- E_k = \mathbb{E}[\tau | X_0 = k]$$

$$* E_k = 1 + pE_{k+1} + (1-p)E_{k-1}$$

$$- E_0 = \begin{cases} ab & \text{if } p = \frac{1}{2} \\ \frac{a}{1-2p} - \frac{a+b}{1-2p} \times \frac{(\frac{1-p}{p})^a - 1}{(\frac{1-p}{p})^{a+b} - 1} & \text{if } p \neq \frac{1}{2} \end{cases}$$

Solved by previous recursive equation

- Observation

$$- S_n = O(n)$$

$$\text{Upperbound: } \lim_{n \rightarrow \infty} P[S_n \leq k\sqrt{n}] = \int_{-\infty}^k \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$\text{Lowerbound: } P[|S_n| \geq k\sqrt{n}] \leq 2e^{-\frac{k^2}{2}}$$

13 Brownian Motion

- Interpretation: generalize discrete time and space to continuous time and space

$$- S_t = \delta_x(\sum_{i=0}^{\frac{t}{\delta_t}} X_i)$$

$$- \text{let } \delta_x = \sqrt{\delta_t} \text{ and } \delta_x \rightarrow 0$$

- $\mathbb{E}[S_t] = 0$
- $Var(S_t) = \delta_x^2 \frac{t}{\delta_t} = t$
- Definition:
 - $X(0) = 0$
 - $X(t) \sim N(0, \sigma^2 = t)$
 - $X(t)$ has independent, stationary increment
 - * independent: $X(t_{i_2}) - X(t_{i_1})$ and $X(t_{i_1}) - X(t_{i_0})$ are independent
 - * stationary: $X(s+t) - X(t) = X(s)$
- Property
 - Distribution self-similarity
 - * $X(t) \sim N(0, t)$
 - * $\sqrt{k}X(\frac{t}{k}) \sim N(0, t)$
 - Nowhere Differentiable
 - * With probability 1, $X(t)$ is nowhere differentiable
 - * $\lim_{\delta_t \rightarrow 0} \frac{X(t+\delta_t) - X(t)}{\delta_t} = \lim_{\delta_t \rightarrow 0} \frac{N(0, \delta_t)}{\delta_t} = \lim_{\delta_t \rightarrow 0} N(0, \frac{1}{\delta_t})$
 - Unbounded Variation
 - * Length of distance $\rightarrow \infty$ in finite time t
 - * $\lim_{n \rightarrow \infty} \sum_{j=1}^n |X(\frac{jt}{n}) - X(\frac{(j-1)t}{n})| = \infty$
 - Proof: $\lim_{n \rightarrow \infty} \sum_{j=1}^n |X(\frac{jt}{n}) - X(\frac{(j-1)t}{n})| = \lim_{n \rightarrow \infty} \sum_{j=1}^n |X(\frac{t}{n})| = n \times \sqrt{\frac{2}{\pi} \frac{t}{n}} = \infty$
 - Hitting Time
 - * The monkey at the cliff
 - $\tau = \min\{t \geq 0 : X(t) = B\}$
 - $P[\tau < \infty] = 1$
 - Prove by discretization $X(t) = \delta_x(\sum_{i=0}^{\frac{t}{\delta_t}} X_i)$
 - $\mathbb{E}[\tau] = \infty$
 - Prove by discretization $X(t) = \delta_x(\sum_{i=0}^{\frac{t}{\delta_t}} X_i)$
 - * The Gambler's Ruin
 - $\tau = \min\{t \geq 0 : X(t) \in \{-A, B\}\}$
 - $P[X(\tau) = B] = \frac{A}{A+B}$ Prove By discretization $X(t) = \delta_x(\sum_{i=0}^{\frac{t}{\delta_t}} X_i) \rightarrow P[X(\tau) = B] = \frac{\frac{A}{\delta_x}}{\frac{A}{\delta_x} + \frac{B}{\delta_x}}$
 - $\mathbb{E}[\tau] = AB$
 - Prove By discretization $X(t) = \delta_x(\sum_{i=0}^{\frac{t}{\delta_t}} X_i) \rightarrow \mathbb{E}[\tau] = \delta_t \frac{A}{\delta_x} \frac{B}{\delta_x}$
 - Diffusion Equation
 - * Forward Diffusion Equation: $\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial^2 f}{\partial x^2}$
 - * Backward Diffusion Equation: $\frac{\partial f}{\partial t} = -\frac{1}{2} \frac{\partial^2 f}{\partial x^2}$
 - * $f(X(t_2) = x | X(t_1) = k)$ satisfies Forward Diffusion equation
 - * $f(X(t_2) = k | X(t_1) = x)$ satisfies Backward Diffusion equation
- Gaussian Process
 - Definition: A stochastic process $\{X(t) : t \geq 0\}$ such that for every $\{t_i\}_{i=1}^n, [X(t_1), X(t_2), \dots, X(t_n)]$ is a joint Gaussian distribution
 - * Defined by
 - $\mathbb{E}[X(t)], \forall t$
 - $Cov(X(s), X(t)), \forall s, t$
 - Property
 - * Brownian Motion is a Gaussian Process with $\mathbb{E}[X(t)] = 0, Cov(X(s), X(t)) = \min\{s, t\}$

$$\cdot \text{Cov}(X(s), X(t)) = \min(s, t) \text{ (by } X(t) = X(s) + X(t-s) \text{ if } t > s)$$

- Brownian Bridge

- Definition: the distribution of t_1 given the result of the future $X(t_2)$

- Property

$$f_{X(t_1)|X(t_2)}(x_1, x_2) = \frac{f_{X(t_1), X(t_2)}(x_1, x_2)}{f_{X(t_2)}(x_2)} \sim N\left(\frac{t_1}{t_2}x_2, \frac{t_1(t_2-t_1)}{t_2}\right)$$

- * let $s = t_2 - t_1$

$$\begin{aligned} * f_{X(t_1)|X(t_2)}(x_1, x_2) &= \frac{f_{X(t_1), X(t_2)}(x_1, x_2)}{f_{X(t_2)}(x_2)} \\ &= \frac{f_{X(t_1), X(s)}(x_1, x_2 - x_1)}{f_{X(t_2)}(x_2)} \text{ (By transformation of 2-D random variables)} \end{aligned}$$

$$= \frac{f_{X(t_1)}(x_1)f_{X(s)}(x_2 - x_1)}{f_{X(t_2)}(x_2)} = \frac{\frac{1}{\sqrt{2\pi t_1}} e^{-\frac{x_1^2}{2t_1}} \frac{1}{\sqrt{2\pi(t_2-t_1)}} e^{-\frac{(x_2-x_1)^2}{2(t_2-t_1)}}}{\frac{1}{\sqrt{2\pi t_2}} e^{-\frac{x_2^2}{2t_2}}}$$

$$= \frac{1}{\sqrt{2\pi \frac{t_1(t_2-t_1)}{t_2}}} e^{-\frac{(x_1 - \frac{t_1}{t_2}x_2)^2}{2 \frac{t_1(t_2-t_1)}{t_2}}} \rightarrow X(t_1) - \frac{t_1}{t_2}X(t_2) \sim N(0, \frac{t_1(t_2-t_1)}{t_2})$$

- * $\mathbb{E}[X(t_1)|X(t_2)] = \frac{t_1}{t_2}X(t_2)$

- * $\text{Var}(X(t_1)|X(t_2)) = \frac{t_1(t_2-t_1)}{t_2}$

- * $Y(t_1) = X(t_1) - \frac{t_1}{t_2}X(t_2)$ share the same distribution as $X(t_1)|X(t_2) = 0$

$$\text{Cov}(X(t_1), X(t_2)|X(t_3)) = \frac{t_1(t_3-t_2)}{t_3}$$

- * $\text{Cov}(X(t_1), X(t_2)|X(t_3))$

$$= \mathbb{E}[X(t_1)X(t_2)|X(t_3)] - \mathbb{E}[X(t_1)|X(t_3)] \times \mathbb{E}[X(t_2)|X(t_3)]$$

$$= \mathbb{E}[X(t_1)^2 + X(t_1)(X(t_2) - X(t_1))|X(t_3)] - \frac{t_1 t_2}{t_3^2} X(t_3)^2$$

$$= \mathbb{E}[X(t_1)(X(t_2) - X(t_1))|X(t_3)] + \frac{t_1(t_1-t_2)}{t_3^2} X(t_3)^2 + \frac{t_1(t_3-t_1)}{t_3}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \frac{t_1(t_3-t_1)}{t_3}}} e^{-\frac{(x_1 - \frac{t_1}{t_3}X(t_3))^2}{2 \frac{t_1(t_3-t_1)}{t_3}}} \mathbb{E}[x_1(X(t_2) - x_1)|X(t_3), X(t_1) = x_1] dx_1 + \frac{t_1(t_1-t_2)}{t_3^2} X(t_3)^2 + \frac{t_1(t_3-t_1)}{t_3}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \frac{t_1(t_3-t_1)}{t_3}}} e^{-\frac{(x_1 - \frac{t_1}{t_3}X(t_3))^2}{2 \frac{t_1(t_3-t_1)}{t_3}}} x_1(-x_1 + X(t_3)) \frac{t_2-t_1}{t_3-t_1} dx_1 + \frac{t_1(t_1-t_2)}{t_3^2} X(t_3)^2 + \frac{t_1(t_3-t_1)}{t_3}$$

$$= \frac{t_1(t_2-t_1)}{t_3^2} X(t_3)^2 - \frac{t_1(t_2-t_1)}{t_3} + \frac{t_1(t_1-t_2)}{t_3^2} X(t_3)^2 + \frac{t_1(t_3-t_1)}{t_3}$$

$$= \frac{t_1(t_3-t_2)}{t_3}$$