Stochastic Processes

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April 21, 2022

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Suppose $B_1(t), B_2(t)$ are independent Brownian processes with variance parameters σ_1^2, σ_2^2 respectively. Define: $\forall t,\ X(t) = B_1(t) - B_2(t)$. Derive the mean and autocorrelation functions of $X(t)$.

i.e., Cov(X(s), X(t))

- $\mathbb{E}[X(t)] = \mathbb{E}[B_1(t)] - \mathbb{E}[B_2(t)] = 0$
- $Cov(X(u), X(t))$
  $= Cov(B_1(u), B_1(t)) - Cov(B_1(u), B_2(t)) - Cov(B_2(u), B_1(t)) + Cov(B_2(u), B_2(t))$
  $= \sigma_1^2 \min\{u, t\} + \sigma_2^2 \min\{u, t\}$

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In Problems 8.1, 8.2, and 8.3, let $\{X(t),\ t \geq 0\}$ denote a Brownian motion process.

8.1. Let $Y(t) = tX(1/t)$.

(a) What is the distribution of $Y(t)$?

(b) Compute $\text{Cov}(Y(s), Y(t))$.

(c) Argue that $\{Y(t),\ t \geq 0\}$ is also Brownian motion.

(d) Let

$$T = \inf\{t > 0: \quad X(t) = 0\}.$$

Using (c) present an argument that

$$P\{T = 0\} = 1.$$

- (a) since t is not a random variable $\rightarrow Y(t)$ is Gaussian distribution
- (b)
  - Suppose $s \geq t$
  - $Cov(Y(s), Y(t)) = Cov(sX(\frac{1}{s}), tX(\frac{1}{s})) + Cov(sX(\frac{1}{s}), t(X(\frac{1}{t}) - X(\frac{1}{s}))$
    $= t$
  - $Cov(Y(s), Y(t)) = \min\{s, t\}$

- (c) Since a Gaussian process is determined by its mean and covariance and $Y(t)$ has the same mean and covariance as the Brownian motion, $Y(t)$ is a Brownian motion.

- (d)

  - T
    $= inf\{t > 0 : X(t) = 0\} = inf\{t > 0 : Y(t) = 0\}$ (since $Y(t)$ is also a Brownian motion)
    $= inf\{t > 0 : tX(\frac{1}{t}) = 0\}$
  - $P[\lim_{t\to 0} tX(\frac{1}{t}) = 0] = 1$, therefore $P[inf\{t > 0 : tX(\frac{1}{t}) = 0\} = 0]1$

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8.2. Let $W(t) = X(a^2t)/a$ for $a > 0$. Verify that $W(t)$ is also Brownian motion.

b) Verify that $Y(t) = t^{\beta}\, X(t^{1-2\beta})$ is not Brownian motion unless $\beta \in \{0,1\}$.
In particular, plot together $X(t)$ and $\sqrt{t}\, X(1)$.

- (a)

  - $W(t) = \frac{X(a^t)}{a} = X(t)$
  - therefore $W(t)$ is a Brownian motion

- (b)

  - if $\beta < 0.5$
    * Suppose $s \leq t$
    * $Cov(Y(s), Y(t))$
      $= Cov(s^{\beta}X(s^{1-2\beta}), t^{\beta}X(s^{1-2\beta})) + Cov(s^{\beta}X(s^{1-2\beta}), t^{\beta}(X(t^{1-2\beta}) - X(s^{1-2\beta})))$
      $= s^{1-\beta}t^{\beta}$
    * $Cov(Y(s), Y(t)) = \min\{s, t\}^{1-\beta} \max\{s, t\}^{\beta}$
    * $Y(t)$ is Brownian motion if $t = 0$
  - if $\beta = 0.5$
    * $Cov(Y(s), Y(t)) = \sqrt{st}$
    * not a Brownian motion
  - if $\beta > 0.5$
    * Suppose $s \geq t$
    * $Cov(Y(s), Y(t))$
      $= Cov(s^{\beta}X(s^{1-2\beta}), t^{\beta}X(s^{1-2\beta})) + Cov(s^{\beta}X(s^{1-2\beta}), t^{\beta}(X(t^{1-2\beta}) - X(s^{1-2\beta})))$
      $= s^{1-\beta}t^{\beta}$
    * $Cov(Y(s), Y(t)) = \max\{s, t\}^{1-\beta} \min\{s, t\}^{\beta}$
    * $Y(t)$ is Brownian motion if $t = 1$
  - Overall $Y(t)$ is Brownian motion if $t \in \{0, 1\}$

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④ For each of the following processes, compute $\mathbb{E} X_t$ and $Cov(X_s, X_t)$.

($B(t)$ is standard Brownian motion.)

a) $X_t = \int_0^t B(u)du$

b) $X_t = \int_0^t u \cdot B(u)du$

c) $X_t = \int_0^t u^2 \cdot B(u)du$

Note: Since B_u is a Gaussian process, so is each of these X_ts (being sum/integrals of Gaussians). Therefore they are fully determined by the means and covariances.

- (a)
  - $\int_0^t B(u)du = \lim_{n\to\infty} \sum_{i=1}^n B(\frac{i}{n}t)\frac{1}{n}$
  - $\mathbb{E}[\int_0^t B(u)du] = \mathbb{E}[\lim_{n\to\infty} \sum_{i=1}^n B(\frac{i}{n}t)\frac{1}{n}] = 0$
  - $Cov(X(u), X(t))$
    * $Var(X(t)) = Var(\lim_{n\to\infty} \sum_{i=1}^n B(\frac{i}{n}t)\frac{1}{n}) = \lim_{n\to\infty} \frac{1}{n^2}\sum_{i=1}^n (2n+1-2i)\frac{it}{n}$
      $= \lim_{n\to\infty} \frac{1}{n^2}\frac{(2n+1)(n+1)t}{2} - \frac{(n+1)(2n+1)t}{3}$
      $= \lim_{n\to\infty} \frac{1}{n^2}\frac{(2n+1)(n+1)t}{6} = \frac{t}{3}$
    * $Cov(\int_0^u B(s)ds, B(u)) = Cov(\lim_{n\to\infty} \sum_{i=1}^n B(\frac{i}{n}t)\frac{1}{n}, B(u))$
      $= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \frac{it}{n} = \frac{t}{2}$
    * Suppose $u \le t$
    * $Cov(X(u), X(t)) = Cov(\int_0^u B(s)ds, \int_0^t B(s)ds)$
      $= Cov(\int_0^u B(s)ds, \int_0^u B(s)ds) + Cov(\int_0^u B(s)ds, \int_u^t (B(s)-B(u))ds$
      $+ Cov(\int_0^u B(s)ds, (t-u)B(u))$
      $= \frac{u}{3} + (t-u)\frac{u}{2}$
    * $Cov(X(u), X(t)) = \frac{\min\{u,t\}}{3} + |t-u|\frac{\min\{u,t\}}{2}$
- (b)
  - $\int_0^t uB(u)du = \lim_{n\to\infty} \sum_{i=1}^n \frac{i}{n}tB(\frac{i}{n}t)\frac{1}{n}$
  - $\mathbb{E}[\int_0^t uB(u)du] = 0$
  - $Cov(X(u), X(t))$
    * $Var(X(t)) = Var(\lim_{n\to\infty} \sum_{i=1}^n \frac{i}{n}tB(\frac{i}{n}t)\frac{1}{n})$
    * $Cov(\int_0^u sB(s)ds, B(u))$
    * $Cov(X(u), X(t)) = Cov(\int_0^u sB(s)ds, \int_0^t sB(s)ds)$
      $= Cov(\int_0^u sB(s)ds, \int_0^u sB(s)ds) + Cov(\int_0^u sB(s)ds, \int_u^t s(B(s)-B(u))ds$
      $+ Cov(\int_0^u sB(s)ds, \frac{t^2-u^2}{2}B(u))$