

# Stochastic Processes

Kevin Chang

April 5, 2022

## 1 Laplace Transform

- $\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st} dt$
- Property
  - $tf(t) \leftrightarrow -F'(s)$
  - $\frac{f(t)}{t} \leftrightarrow \int_s^\infty F(\sigma) d\sigma$
  - $f'(t) \leftrightarrow sF(s) - f(0^-)$
  - $\int_0^t f(\tau) d\tau \leftrightarrow \frac{F(s)}{s}$
  - $e^{at}f(t) \leftrightarrow F(s-a)$
  - $f(t-a)u(t-a) \leftrightarrow e^{-as}F(s)$

## 2 Moment Generating Function

- Moment Generating Function:  $\mathbb{E}[e^{tX}]$ 
  - Property:
    - \*  $\mathbb{E}[e^{tX}] = \int_{-\infty}^\infty e^{tx} f_X(x) dx$
    - \*  $\mathbb{E}[e^{tX}] = \sum_{k=0}^\infty E[X^k] \frac{t^k}{k!}$ 
      - $e^{tx} = \sum_{k=0}^\infty \frac{(tx)^k}{k!}$
      - $E[e^{tX}] = E[\sum_{k=0}^\infty \frac{(tX)^k}{k!}] = \sum_{k=0}^\infty E[X^k] \frac{t^k}{k!}$
    - \*  $\frac{d\mathbb{E}[e^{tX}]}{dt} = \mathbb{E}[X]$
    - \*  $\mathbb{E}[e^{t(aX+b)}] = e^{tb} \mathbb{E}[e^{taX}]$
    - \* Not all random variables have Moment generating function
  - Characteristic Function:  $\mathbb{E}[e^{itX}]$ 
    - Property:
      - \* All random variables have Moment generating function
  - Joint Moment Generating Function:  $G(x, y) = \mathbb{E}[e^{xX} e^{yY}]$
  - Property:
    - (Joint) moment generating function uniquely determines the (joint) CDF
  - Example
    - Trapped miner's random walk
      - \* Miner has probability of  $\frac{1}{3}$  to waste 3 hours in vain,  $\frac{1}{3}$  to waste 5 hours in vain, and  $\frac{1}{3}$  to spend 2 hours to go out of the mine.
      - \*  $X$  is the random variables of the hours to go out of the mine
      - \*  $Y_i$  is the random variables of the hours for the  $i$ -th action.
      - \*  $\mathbb{E}[e^{tX}] = \mathbb{E}[e^{tX} | Y_1 = 2] + \mathbb{E}[e^{tX} | Y_1 = 3] + \mathbb{E}[e^{tX} | Y_1 = 5]$ 
        - $= \mathbb{E}[e^{2t}] + \mathbb{E}[e^{t(X+3)}] + \mathbb{E}[e^{t(X+5)}]$
      - \* Find expectation and variance by joint moment generating function

### 3 Expectation

- $N$  i.i.d. events, when  $N$  is a random variable
  - Suppose  $N$  is a integer random variable
  - Suppose  $X_1, \dots, X_i, \dots, X_N$  are i.i.d random variables with mean  $\mu$  and variance  $\sigma^2$
  - $Y = \sum_{i=1}^N X_i$
  - $\mathbb{E}[Y] = \mathbb{E}[N]\mu$ 
    - \*  $\mathbb{E}[Y] = \sum_{n=1}^{\infty} \mathbb{E}[\sum_{i=1}^N X_i | N = n] P[N = n]$   
 $= \mu \times \sum_{n=1}^{\infty} n P[N = n] = \mathbb{E}[N]\mu$
  - $\mathbb{E}[Y^2] = \mathbb{E}[N]\mathbb{E}[X^2] + \mathbb{E}[N^2]\mu^2 - \mathbb{E}[N]\mu^2$ 
    - \*  $\mathbb{E}[Y^2] = \sum_{n=1}^{\infty} \mathbb{E}[(\sum_{i=1}^N X_i)^2 | N = n] P[N = n] = \sum_{n=1}^{\infty} (n\mathbb{E}[X_i^2] + n(n-1)\mu^2) P[N = n]$   
 $= \mathbb{E}[N]\mathbb{E}[X^2] + \mathbb{E}[N^2]\mu^2 - \mathbb{E}[N]\mu^2$
  - $\text{Var}(Y) = \mathbb{E}[N]\sigma^2 + \text{Var}(N)\mu^2$
- Expectation by  $P[X > x]$ 
  - $\mathbb{E}[X] = \sum_x P[X > x]$ , when  $X$  is a non-negative discrete random variable
    - \*  $\mathbb{E}[X] = \sum_{x=0}^{\infty} x P[X = x] = \sum_{x=0}^{\infty} \sum_{y=0}^{x-1} P[X = x] = \sum_{y=0}^{\infty} \sum_{x=y+1}^{\infty} P[X = x] = \sum_{y=0}^{\infty} P[X > y]$
  - $\mathbb{E}[X] = \int_0^{\infty} P[X > x] dx$ , when  $X$  is a non-negative continuous random variable
    - \*  $\mathbb{E}[X] = \int_0^{\infty} x f_X(x) dx = \int_0^{\infty} \int_0^x f_X(x) dy dx = \int_0^{\infty} \int_y^{\infty} f_X(x) dx dy = \int_0^{\infty} P[X > y] dy$

### 4 Inequality

- Markov Inequality

Definition:

- Suppose  $X \geq 0$ , then  $P[X \geq \epsilon] \leq \frac{\mathbb{E}[X]}{\epsilon}$

Proof:

1.  $\mathbb{E}[X] = \int_0^{\infty} x f_X(x) dx \geq \int_{\epsilon}^{\infty} x f_X(x) dx \geq \epsilon \int_{\epsilon}^{\infty} f_X(x) dx = \epsilon P[X \geq \epsilon]$
2.  $X(\omega) \geq \epsilon \mathbb{1}_{X(\omega) \geq \epsilon}, \forall \omega \in S$ 
  - Calculate expectation on both side.
  - $\mathbb{E}[X] \geq \epsilon P[X \geq \epsilon]$

Property:

- The equality happens when  $P[X = k] = 0, \forall k \notin \{0, \epsilon\}$ .

- Chebyshev Inequality

Definition:

- Suppose  $m = \mathbb{E}[X], \sigma^2 = \text{Var}(X)$ , then  $P[|X - m| \geq \epsilon] \leq \frac{\sigma^2}{\epsilon^2}$

Proof:

- $P[|X - m| \geq \epsilon] = P[(X - m)^2 \geq \epsilon^2]$
- $P[(X - m)^2 \geq \epsilon^2] \leq \frac{\mathbb{E}[(X - m)^2]}{\epsilon^2}$  (by Markov Inequality)

Property:

- The equality happens when  $P[X = k] = 0, \forall k \notin \{m - \epsilon, m, m + \epsilon\}$ .
- Might be tighter than Markov Inequality since it requires  $m, \sigma$

- Chernoff Inequality

Definition:

- Suppose  $X_1, \dots, X_n$  are independent identically distributed Bernoulli random variable with probability  $p$  and  $X = \sum_{i=1}^n X_i$
- $P[X \geq \epsilon] \leq \frac{(pe^t + 1 - p)^n}{e^{t\epsilon}} \leq \frac{e^{np(e^t - 1)}}{e^{t\epsilon}}$ 
  - \*  $P[X \geq \epsilon] = P[e^{tX} \geq e^{t\epsilon}] \leq \frac{E[e^{tX}]}{e^{t\epsilon}} = \frac{(E[e^{tX_i}])^n}{e^{t\epsilon}} = \frac{(pe^t + 1 - p)^n}{e^{t\epsilon}} \leq \frac{e^{np(e^t - 1)}}{e^{t\epsilon}}$
- $P[X \geq np(1 + \epsilon)] \leq (\frac{e^\epsilon}{(1+\epsilon)^{1+\epsilon}})^{np} \leq \begin{cases} e^{-\frac{\epsilon^2 np}{3}} & \text{if } 0 \leq \epsilon \leq 1 \\ e^{-\frac{\epsilon^2 np}{(2+\epsilon)}} & \text{if } \epsilon \geq 1 \end{cases}$ 
  - \* Substitute  $\epsilon$  with  $np(1 + \epsilon)$
  - \* Substitute  $t$  with  $\log(1 + \epsilon)$
  - \* the last inequality is without proof
- $P[X \leq \epsilon] \leq \frac{(pe^{-t} + 1 - p)^n}{e^{-t\epsilon}} \leq \frac{e^{np(e^{-t} - 1)}}{e^{-t\epsilon}}$ 
  - \*  $P[X \leq \epsilon] = P[e^{-tX} \geq e^{-t\epsilon}] \leq \frac{E[e^{-tX}]}{e^{-t\epsilon}} = \frac{(E[e^{-tX_i}])^n}{e^{-t\epsilon}} = \frac{(pe^{-t} + 1 - p)^n}{e^{-t\epsilon}} \leq \frac{e^{np(e^{-t} - 1)}}{e^{-t\epsilon}}$
- $P[X \leq np(1 - \epsilon)] \leq (\frac{e^{-\epsilon}}{(1-\epsilon)^{1-\epsilon}})^{np} \leq e^{-\frac{\epsilon^2 np}{2}}$ 
  - \* Substitute  $\epsilon$  with  $np(1 - \epsilon)$
  - \* Substitute  $t$  with  $-\log(1 - \epsilon)$
  - \* the last inequality is without proof

- Chernoff/ Hoeffding Lemma

Definition:

- Suppose  $X_1, \dots, X_n$  are independent distributed random variable and  $a_i \leq X_i \leq b_i$
- Suppose  $X = \sum_{i=1}^n X_i$  and  $\mu = \mathbb{E}[X]$
- $P[|X - \mu| \geq \epsilon] \leq 2e^{\frac{-2\epsilon^2}{\sum_{i=1}^n (b_i - a_i)^2}}$  without proof

- Application:

- Balls in Bins

Definition: Throw  $n$  balls into  $n$  bins, find bounds for the maximum number of balls in all bins

- \*  $P[\text{maximum number of balls in all bins} \geq \epsilon]$   
 $= P[\cup_{i=1}^n \text{number of balls in } i\text{-th bin} \geq \epsilon]$   
 $\leq n \times P[\text{number of balls in one bin} \geq \epsilon]$
- \* By Markov inequality:
  - $P[\text{number of balls in one bin} \geq \epsilon] \leq \frac{1}{\epsilon} \rightarrow \text{useless}$
- \* By Chebyshev inequality:
  - $P[\text{number of balls in one bin} \geq \epsilon] \leq \frac{(1 - \frac{1}{n})}{\epsilon^2}$
  - $P[\text{maximum number of balls in all bins} \geq n^{\frac{1}{2} + \epsilon}] \leq \frac{(1 - \frac{1}{n})}{n^{2\epsilon}}$
  - when  $n \rightarrow \infty$ , the maximum number of balls should less than  $n^{\frac{1}{2} + \epsilon}$
- \* By Chernoff inequality:
  - $P[\text{number of balls in one bin} \geq 2 \log n] \leq \frac{e^{np(e^t - 1)}}{n^{2t}}$
  - $P[\text{maximum number of balls in all bins} \geq 2 \log n] \leq \frac{e^{np(e^t - 1)}}{n^{2t-1}}$
  - when  $t$  is a constant  $\geq 0.5$  and  $n \rightarrow \infty$ , the maximum number of balls should less than  $2 \log n$

## 5 Law of Large Numbers

- $\{X_i\}_{i=1}^\infty$  is a sequence of pairwise uncorrelated random variable with  $\mathbb{E}[X_i] = m$ ,  $\text{Var}(X_i) = \sigma_i^2$ .
- $M_n = \frac{1}{n} \sum_{i=1}^n X_i$
- $M_n \rightarrow m$  almost surely, in mean square and in probability.

## 6 Memoryless

- Definition:  $P[X > x_1 + x_2 | X > x_1] = P[X > x_2]$
- Property:
  - Exponential random variable is the only continuous memoryless random variable
  - Bernoulli random variable is the only discrete memoryless random variable

## 7 Famous Random Variable

- Poisson:
 
$$P[X = k] = \frac{\lambda^k}{k!} \exp(-\lambda)$$

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} \exp(-\lambda) = \sum_{k=0}^{\infty} \lambda \frac{\lambda^{k-1}}{(k-1)!} \exp(-\lambda) = \lambda$$

Interpretation:

- Cut total time into infinite period in Binomial random variable,  $n \rightarrow \infty, p \rightarrow \frac{\lambda}{n}$
- $\rightarrow P[X = k] = \lim_{n \rightarrow \infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} = \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n}\right)^n = \frac{\lambda^k}{k!} \exp(-\lambda)$
- Gaussian:  $N(m, \sigma^2)$

- $f_X[x] = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}, \forall x \in \mathbb{R}$
- $\mathbb{E}[e^{cX}] = e^{cm + \frac{c^2\sigma^2}{2}}$

- Erlang:

$$f_X(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!}, \forall x \in \mathbb{R}$$

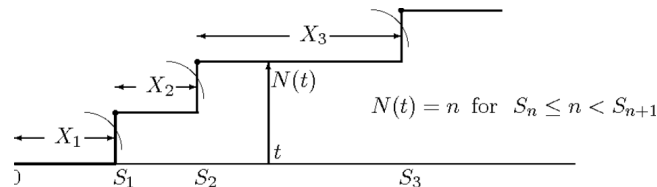
$$\mathbb{E}[X] = \frac{n}{\lambda}$$

Interpretation:

- Suppose  $X_1, X_2, \dots, X_n$  are i.i.d exponential random variable with  $\lambda$ .
- $X = \sum_{i=1}^n X_i$
- Proof by induction:  
Suppose  $n = 2$ ,  $f_X(x) = \int_0^x \lambda e^{-\lambda t} \lambda e^{-\lambda(x-t)} dt = \lambda^2 x e^{-\lambda x}$

## 8 Stochastic Processes

- Stochastic Process: a collection of random variable
- Arrival Process: a sequence of arriving event in continuous time



- $X_i$ : the time between the  $i$ -th event and the  $i - 1$ -th event
- $S_i$ : the time from start to  $i$ -th event
- $N(t)$ : the number of the arrived event at time  $t$
- $X$  and  $S$  Relation:
  - \*  $X_1 = S_1, X_i = S_i - S_{i-1}$
- $N$  and  $S$  Relation:

- \*  $N(t) < n \leftrightarrow S_{n+1} > t$
- \*  $N(t) \geq n \leftrightarrow S_n \leq t$
- \*  $N(t) = n \leftrightarrow S_n \leq t < S_{n+1}$
- \*  $N(t) = \max\{n : S_n \leq t\}$

– Renewal Process: an arrival process with i.i.d  $X_i$

Delayed Renewal Process: the process becomes a renewal process after several arrivals

$X_i$  Property

- \* if  $X_i$  is dependent on the interval states, then  $X_i$  might be dependent on  $X_{i-1} \rightarrow$  not renewal process

$S_i$  Property

- \*  $P[\lim_{n \rightarrow \infty} S_n = \infty] = 1$   
 Proof:  $\lim_{n \rightarrow \infty} P[S_n = \infty] = \lim_{n \rightarrow \infty} P[\sum_{i=1}^n X_i = n \times \mathbb{E}[X_i]] = 1$   
 Interpretation: infinite events do not take finite time

$N(t)$  Property

- \* for any  $t, P[N(t) < \infty] = 1$   
 Proof:  $P[\lim_{n \rightarrow \infty} S_n = \infty] = 1 \rightarrow$  for any  $t, P[\lim_{n \rightarrow \infty} S_{n+1} > t] = 1$   
 Interpretation: infinite events do not take finite time
- \*  $P[\lim_{t \rightarrow \infty} N(t) \rightarrow \infty] = 1$   
 Proof: if  $P[\lim_{t \rightarrow \infty} N(t) = k] > 0 \rightarrow P[X_{k+1} = \infty] > 0$   
 Interpretation: finite events do not take infinite time
- \*  $P[\lim_{t \rightarrow \infty} \frac{N(t)}{t} = \frac{1}{\mathbb{E}[X_i]}] = 1$   
 Proof:  $P[\lim_{t \rightarrow \infty} \frac{N(t)}{S_{N(t)+1}} \leq \lim_{t \rightarrow \infty} \frac{N(t)}{t}] = 1$  and  $P[\lim_{t \rightarrow \infty} \frac{N(t)}{S_{N(t)+1}} = \frac{1}{\mathbb{E}[X_i]}] = 1$   
 $P[\lim_{t \rightarrow \infty} \frac{N(t)}{t} \leq \lim_{t \rightarrow \infty} \frac{N(t)}{S_{N(t)}}] = 1$  and  $P[\lim_{t \rightarrow \infty} \frac{N(t)}{S_{N(t)}} = \frac{1}{\mathbb{E}[X_i]}] = 1$

Inspection Paradox

- \*  $\mathbb{E}[X_{N(t)+1}] \geq \mathbb{E}[X_i]$ : inspection paradox  
 Interpretation:  
 ·  $f_{X_{N(t)+1}}(x) = \lambda x f_{X_i}(x)$   
 · when selecting  $t$  with equal probability, we tend to choose  $X_i$  with longer period
- \*  $P[\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (S_{N(t)+1} - s) ds = \frac{\mathbb{E}[X_i^2]}{2\mathbb{E}[X_i]}] = 1$   
 Proof:  
 $P[\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=i}^{N(t)} \frac{\mathbb{E}[X_i^2]}{2} \leq \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (S_{N(t)+1} - s) ds] = 1$  and  $P[\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=i}^{N(t)} \frac{\mathbb{E}[X_i^2]}{2} = \frac{\mathbb{E}[X_i^2]}{2\mathbb{E}[X_i]}] = 1$   
 $P[\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (S_{N(t)+1} - s) ds \leq \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=i}^{N(t)+1} \frac{\mathbb{E}[X_i^2]}{2}] = 1$  and  $P[\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=i}^{N(t)+1} \frac{\mathbb{E}[X_i^2]}{2} = \frac{\mathbb{E}[X_i^2]}{2\mathbb{E}[X_i]}] = 1$
- \*  $P[\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (s - S_{N(t)}) ds = \frac{\mathbb{E}[X_i^2]}{2\mathbb{E}[X_i]}] = 1$   
 Proof: similar to above
- \*  $P[\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t X_{N(t)} ds = \frac{\mathbb{E}[X_i^2]}{\mathbb{E}[X_i]}] = 1$   
 Proof:  $P[\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t X_{N(t)} ds = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (S_{N(t)+1} - S_{N(t)}) ds] = 1$
- \*  $\mathbb{E}[X_{N(t)+1}] = \frac{\mathbb{E}[X_i^2]}{\mathbb{E}[X_i]}$   
 Proof:  $P[\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t X_{N(t)} ds = \frac{\mathbb{E}[X_i^2]}{\mathbb{E}[X_i]}] = P[\mathbb{E}[X_{N(t)+1}] = \frac{\mathbb{E}[X_i^2]}{\mathbb{E}[X_i]}] = 1$

Central Limit Theorem

- \*  $\mu = \mathbb{E}[X_i]$
- \*  $\sigma = \sqrt{\text{Var}(X_i)}$
- \*  $Z \sim \text{Normal}(0,1)$
- \*  $\lim_{t \rightarrow \infty} P[N(t) \leq \frac{t}{\mu} + k \frac{\sigma\sqrt{t}}{\sqrt{\mu^3}}] = P[Z \leq k]$   
 Proof:  
 1. Suppose  $n(t) = \frac{t}{\mu} + k \frac{\sigma\sqrt{t}}{\sqrt{\mu^3}}$   
 2.  $P[N(t) \geq n(t)] = P[S_{n(t)} \leq t] = P[\frac{S_{n(t)} - n\mu}{\sigma\sqrt{n}} \leq \frac{t - n\mu}{\sigma\sqrt{n}}]$

3. When  $t \rightarrow \infty$ ,  $\frac{t-n\mu}{\sigma\sqrt{n}} \rightarrow k$

4. By law of large number,  $\lim_{t \rightarrow \infty} P[\frac{S_{n(t)}-n\mu}{\sigma\sqrt{n}} \leq k] = P[Z \leq k]$

Interpretation:

- $\frac{t}{\mu}$  is approximately the mean of  $N(t)$
- $k \frac{\sigma\sqrt{t}}{\sqrt{\mu^3}}$  is  $k\sigma\sqrt{n}$  after dividing by  $\mu$ , the ratio between  $t$  and  $N(t)$  and changing  $n$  with  $\frac{t}{\mu}$

Wald's Identity

- \* Stopping Times: a random variable  $\tau$  s.t.  $\{\tau = n\}$  is independent of  $\{X_i\}_{i=n+1}^\infty$
- \* Stopping Condition: a condition to stop if we can consider  $\tau = \min\{n : \text{condition}(n) = \top\}$
- \* Example:  $N(t) + 1$  is a stopping times and can be consider  $N(t) + 1 = \min\{n : S_n > t\}$
- \*  $\mathbb{E}[\sum_{i=1}^\tau X_i] = \mathbb{E}[\tau]\mathbb{E}[X_i]$  if  $\mathbb{E}[X_i] < \infty$  and  $\mathbb{E}[N] < \infty$

Proof:

1.  $\mathbb{E}[\sum_{i=1}^\tau X_i] = \sum_{i=1}^\infty \mathbb{E}[X_i \times \mathbb{1}_{i \leq \tau}]$  (by Fubin's Theorem without proof)  
(if  $\mathbb{E}[X_i] < \infty$  and  $\mathbb{E}[N] < \infty$ )
  2.  $\sum_{i=1}^\infty \mathbb{E}[X_i \times \mathbb{1}_{i \leq \tau}] = \mathbb{E}[X_i] \sum_{i=1}^\infty \mathbb{E}[\mathbb{1}_{i \leq \tau}]$  (by  $P[\tau \geq i] = 1 - P[\tau < i]$  is independent of  $X_i$ )
  3.  $\mathbb{E}[X_i] \sum_{i=1}^\infty \mathbb{E}[\mathbb{1}_{i \leq \tau}] = \mathbb{E}[\tau]\mathbb{E}[X_i]$
- \*  $\lim_{t \rightarrow \infty} \frac{\mathbb{E}[N(t)]}{t} = \frac{1}{\mathbb{E}[X_i]}$

Proof:

- Suppose  $\mu = \mathbb{E}[X_i]$
- $\frac{\mathbb{E}[N(t)]}{t} = \frac{\mathbb{E}[S_{N(t)+1}]}{t \times \mu} - \frac{1}{t}$  (by considering  $N(t) + 1$  as the stopping time)
- $\lim_{t \rightarrow \infty} \frac{\mathbb{E}[N(t)]}{t} \geq \frac{1}{\mu}$  (by  $\mathbb{E}[S_{N(t)+1}] > t$ )
- Suppose  $\hat{X}_n = \min\{X_n, T\}$ , where  $T$  is a constant
- $\frac{\mathbb{E}[N(t)]}{t} \leq \frac{\mathbb{E}[\hat{N}(t)]}{t} = \frac{\mathbb{E}[S_{\hat{N}(t)+1}]}{t \times \mu} - \frac{1}{t} \leq \frac{t+T}{t \times \mu} - \frac{1}{t}$
- $\lim_{n=\sqrt{t}, t \rightarrow \infty} \frac{\mathbb{E}[N(t)]}{t} \leq \frac{1}{\mu}$

Blackwell's Theorem

- \*  $\mathbb{E}[N(t)] = F_{X_i}(t) + \int_0^t \mathbb{E}[N(t-x)]f_{X_i}(x)dx$   
Proof:  $\mathbb{E}[N(t)] = \int_0^t \mathbb{E}[N(t)|X_1 = x]f_{X_1}(x)dx$   
 $= \int_0^t \mathbb{E}[N(t-x) + 1]f_{X_1}(x)dx = F_{X_i}(t) + \int_0^t \mathbb{E}[N(t-x)]f_{X_i}(x)dx$
- \*  $\mathcal{L}\{\mathbb{E}[N(t)]\}(s) = \frac{\mathcal{L}\{f_{X_i}\}(s)}{s(1-\mathcal{L}\{f_{X_i}\}(s))}$   
Proof: Laplace transform both sides
- \* Lattice/ Non-Lattice:  $N(t)$  is lattice iff  $X_i$  only takes on values that are  $nd, n \in \mathbb{N}, d \in \mathbb{R}$
- \* For a non-lattice process:  $\lim_{t \rightarrow \infty} \mathbb{E}[N(t+\delta) - N(t)] = \frac{\delta}{\mathbb{E}[X_i]}$ , for any  $\delta$   
Proof: Without Proof  
Interpretation:  $\mathbb{E}[N(t)]$  will converge to be linear
- \* For a lattice process and period  $d$ :  $\lim_{n \rightarrow \infty} \mathbb{E}[\# \text{ events at } t = nd] = \frac{d}{\mathbb{E}[X_i]}$   
Proof: Without Proof  
Interpretation:  $\mathbb{E}[N(t)]$  will converge to be stairs with width  $d$  and height  $\frac{d}{\mathbb{E}[X_i]}$

– Renewal-Reward Process:

Definition

- \* A renewal process  $N(t)$  and  $\{R_i\}_{i=1}^\infty$  such that  $(X_i, R_i)$  are i.i.d.  
( $X_i, R_j, i \neq j$  are independent, but  $X_i, R_i$  might be dependent)

Property

- \*  $P[\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=1}^{N(t)} R_i = \frac{\mathbb{E}[R_i]}{\mathbb{E}[X_i]}] = 1$   
Proof:  $P[\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=1}^{N(t)} R_i = \lim_{t \rightarrow \infty} \sum_{i=1}^{N(t)} \frac{R_i}{N(t)} \times \lim_{t \rightarrow \infty} \frac{N(t)}{t}] = 1$

– Poisson Process: a renewal process with  $X_i \sim \text{Exponential}(\lambda)$

$S_i$  Property

- \*  $S_i$  is an Erlang random variable  
Erlang is the sum of the Exponential random variables

- \* Joint Distribution  $f_{S_1, \dots, S_n}(s_1, \dots, s_n) = \lambda^n e^{-\lambda s_n}$   
Prove by induction.  
Induce by  $f_{S_1, \dots, S_n}(s_1, \dots, s_n) = f_{S_1, \dots, S_{n-1}}(s_1, \dots, s_{n-1}) \times f_{S_n|S_1, \dots, S_{n-1}}(s_n, s_1, \dots, s_{n-1})$

$N(t)$  Property

- \*  $N(t) \sim \text{Poisson}(\lambda t)$ ,  $P[N(t) = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$   
Prove by  $P[N(t) = n] = P[S_n \leq t \text{ and } S_{n+1} > t]$
- \* Conditioned on  $N(t) = n$ , the set of arrival times  $\{s_1, \dots, s_n\}$  have the same distribution with a set of  $n$  sorted i.i.d.  $\text{Uniform}(0, t)$  random variables  
Prove by  $f_{S_1, \dots, S_n|N(t)}(s_1, \dots, s_n, n) = \frac{f_{S_1, \dots, S_n}(s_1, \dots, s_n) P[X_{n+1} > t - s_n]}{P[N(t) = n]} = \frac{n!}{t^n}$

Property

- \*  $Z$  is the interval from  $t$  to the first arrival  $\rightarrow Z$  is exponential random variable with same  $\lambda$  and independent of  $N(t)$  and the arrival time before  $t$   
Proof:  
$$P[Z > z] = \sum_{n=0}^{\infty} \int_0^{\infty} \dots \int_0^{\infty} P[Z > z | N(t) = n, S_1 = s_1, \dots, S_n = s_n] ds_1 \dots ds_n$$
$$= \sum_{n=0}^{\infty} \int_0^{\infty} \dots \int_0^{\infty} P[X_{n+1} > z + t - s_n | N(t) = n, S_1 = s_1, \dots, S_n = s_n] ds_1 \dots ds_n$$
$$= \sum_{n=0}^{\infty} \int_0^{\infty} \dots \int_0^{\infty} P[X_{n+1} > z + t - s_n | X_{n+1} > t - s_n] ds_1 \dots ds_n = e^{-\lambda z}$$
- \* Stationary Increments:  $N(t_1 + t_2) - N(t_1)$  and  $N(t_2)$  share the same distribution  
Without Proof
- \* Independent Increments:  $\forall 0 < t_1 < t_2 < \dots, t_k, N(t_1), N(t_2) - N(t_1), \dots$  are independent  
Without Proof
- \* Any arrival process with stationary and independent increments must be a Poisson process  
Without Proof

Exercise

- \*  $\mathbb{E}[S_i | N(t) = n] = \frac{t \times i}{n+1}$   
 $\cdot \mathbb{E}[S_i | N(t) = n] = i \times \mathbb{E}[X_1 | N(t) = n] = i \int_0^t \int_0^{s_n} \dots \int_0^{s_2} s_1 \times \frac{n!}{t^n} ds_1 \dots ds_{n-1} ds_n = \frac{t \times i}{n+1}$
- \*  $\mathbb{E}[\sum_{i=0}^{N(t)} S_i] = \frac{\lambda t^2}{2}$   
 $\cdot \mathbb{E}[\sum_{i=0}^{N(t)} S_i] = \sum_{n=0}^{\infty} \mathbb{E}[\sum_{i=0}^n S_i | N(t) = n] P[N(t) = n]$   
 $= \sum_{n=0}^{\infty} \frac{nt}{2} P[N(t) = n] = \frac{\lambda t^2}{2}$

2D Poisson Process

- \* Definition:
  - For any region  $R$ : number of points in  $R$  is a Poisson random variable
  - number of points in the non-overlapping region is independent

Combining Poisson Process

- \*  $N^1(t), N^2(t)$  are two independent Poisson process with  $\lambda_1, \lambda_2$
- \*  $X_i$  is the first arrival of  $X_i^1, X_i^2$
- \* Property
  - $X_i$  is independent of  $\{X_i^1 < X_i^2\}$  and  $\{X_i^1 > X_i^2\}$   
Proof:  $P[X_1^1 < X_1^2] = \frac{\lambda_1}{\lambda_1 + \lambda_2}$   
 $P[X_1 > x] = P[X_1^1 > x, X_1^2 > x] = e^{-(\lambda_1 + \lambda_2)x}$   
 $P[X_1 > x, X_1^1 < X_1^2] = P[X_1 > x] P[X_1^1 < X_1^2]$
  - $X_i$  is a Poisson Process with  $\lambda = \lambda_1 + \lambda_2$
  - $\min(X_1, X_2)$  is an exponential random variable with  $\lambda = \lambda_1 + \lambda_2$

Splitting Poisson Process

- \*  $N^1(t), N^2(t)$  are two independent Poisson process with  $\lambda_1, \lambda_2$
- \*  $N(t)$  is a random process with  $\lambda = \lambda_1 + \lambda_2$ 
  - $N^{1*}(t)$  is the process of the first event  
when  $N(t)$  arrives consider it as first event with probability  $\frac{\lambda_1}{\lambda_1 + \lambda_2}$
  - $N^{2*}(t)$  is the process of the second event  
when  $N(t)$  arrives consider it as second event with probability  $\frac{\lambda_2}{\lambda_1 + \lambda_2}$
- \*  $N^i(t)$  and  $N^{i*}(t)$  share the same distribution

\* Proof:

- $B_n(k)$  is a Binomial random variable with  $p = \frac{\lambda_1}{\lambda_1 + \lambda_2}$
- $P[N^{1*}(t) = m, N^{2*}(t) = n] = P[N(t) = m + n, B_{m+n}(m)] = P[N^1(t) = m, N^2(t) = n]$

Compound Poisson Process

- \*  $N(t)$  is a Poisson Process
- \*  $A_n$  is a sequence of cost
- \*  $A(t) = \sum_{n=0}^{N(t)} A_n$  is the summation of cost over Poisson Process

Non-Homogeneous Poisson Process

- \*  $N(t) - N(s) \sim \text{Poisson}(\int_s^t \lambda(x) dx)$

Queueing Theory

\* Definition: *Arrival\_Process/Service\_Process/number\_of\_services*

- $M$ : memoryless (Poisson) process
- $D$ : deterministic process
- $G$ : general renewal process
- \*  $T$ : the random variable of the processing time for each customer
- \*  $Y(t)$ : number of cutomers in the service
  - $Y(t) \sim \text{Poisson}(\lambda \int_0^t P[T > x] dx)$
  - Proof:  
Consider  $Y(t)$  is a splitting Poisson Process. Since the distribution for the arrival given  $N(t)$  is universal, the probability the arrival is still in service:  $\frac{1}{t} \int_0^t P[T > t-x] dx = \frac{1}{t} \int_0^t P[T > x] dx$

## 9 Markov Chain

### • Definition

- Model with states and transition probability matrix
- States:  $\{X_n\}_{n=0}^{\infty}$
- Transition Probability Matrix:  $[P]_{ij} = P[X_{n+1} = j | X_n = i]$

### • Terminology

- $p^n = [P[X_n = 0], P[X_n = 1], \dots]^T$ : distribution at step  $n$
- $T_i = \min\{n \geq 1 : X_n = i\}$ : a random variable of the minimum time step to go to state  $i$
- $f_{ij} = P[T_j < \infty | X_0 = i]$ : the probability of starting at  $i$  and ever reaching  $j$
- $\mu_{ij} = \mathbb{E}[T_j | X_0 = i]$
- $i \rightarrow j$  iff  $f_{ij} > 0$ :  $j$  is reachable from  $i$  with probability greater than 0
- $N_i(n)$ : number of visits to  $i$  by time  $n$
- Irreducible:  $i \leftrightarrow j, \forall$  states  $i, j$
- aperiodic: period of  $X_n = i$  is 1,  $\forall$  states  $i$

### • Property

- Consider a given distribution as an event  $\tau : [P[X_n = 0 | \tau], P[X_n = 1 | \tau], \dots]^T$
- Updating distribution
  - \*  $p^n = p^0 P^n$
- Markovian: transition probability depend only on current state
  - \*  $P[X_{n+1} = j | X_n = i, \dots, X_0 = x_0] = [P]_{ij}$
- Stationary Distribution:  $p$  s.t. if  $p^n = p \rightarrow p^{n+1} = p$

Property from renewal process

- \* consider  $X_n = j$  as a event  $\rightarrow$  Markov Chain becomes a delayed renewal process
- \* If  $i \leftrightarrow j$  and the model starts from  $i$ , then following holds



- \*  $P[\lim_{n \rightarrow \infty} \frac{N_j(n)}{n} = \frac{1}{\mu_{jj}}] = 1$
- \*  $\lim_{n \rightarrow \infty} \frac{\mathbb{E}[N_j(n)]}{n} = \frac{1}{\mu_{jj}}$
- \* if the period of  $X_n = j$  is  $d \rightarrow \lim_{n \rightarrow \infty} p_j^{nd} = \frac{d}{\mu_{jj}}$

Theorem of an irreducible, aperiodic Markov Chain

- \* Either
  - All states have  $\mu_{ii} = \infty$
  - All states have  $\mu_{ii} < \infty$  and  $p_i = \frac{1}{\mu_{ii}}$  is the unique stationary distribution
- \* Proof
  - From if the period of  $X_n = j$  is  $d \rightarrow \lim_{n \rightarrow \infty} p_j^{nd} = \frac{d}{\mu_{jj}}$
  - Proof:  $\lim_{n \rightarrow \infty} p_j^{nd} = \lim_{n \rightarrow \infty} \mathbb{E}[\# \text{ events at } nd]$

Theorem of an irreducible, aperiodic Markov Chain

- \* All states have  $\mu_{ii} < \infty$  and  $p_i = \frac{1}{\mu_{ii}}$  is the unique stationary distribution

$p$  can be calculated as the eigenvector corresponds to eigenvalue 1 of  $P^T$

#### – Detailed Balance

Definition:

- \* Given a distribution  $\pi$
- \*  $\pi_i P_{ij} = \pi_j P_{ji}, \forall i, j$

Property:

- \* distribution  $\pi$  satisfying Detailed Balance is the stationary distribution  $p$
- \* symmetric transition probability matrix  $\rightarrow$  uniform stationary distribution

#### – Reversible

Definition: A Markov Chain with stationary distribution  $p$  is reversible if it satisfies detailed balance

Interpretation

- \* Transitions forward and backward in the stationary distribution have the same probability
- \*  $P[X_{n+1} = j | X_n = i] = P_{ij}$
- \*  $P[X_{n-1} = j | X_n = i] = \frac{P[X_{n-1}=j, X_n=i]}{P[X_n=i]} = \frac{p_j P_{ji}}{p_i} = P_{ij}$

#### – Metropolis Update Rule

Definition

- \* Given a Markov Chain and distribution  $p'$ , find  $P'$  such that  $p'$  is the stationary distribution

Procedure

- \* For each pair  $(i, j)$ ,  $P'_{ij} = P_{ij} \times \min\{1, \frac{p'_j P_{ji}}{p'_i P_{ij}}\}$
- \* construct self loop to satisfy  $\sum_j P'_{ij} = 1$

Proof

- \* To satisfy detailed balance, for each pair  $(i, j)$ , we should set  $p'_i P'_{ij} = \min\{p'_i P_{ij}, p'_j P_{ji}\}$

#### – Distance between Probability Measure

Definition:

- \* Total Variation Distance between  $P_1$  and  $P_2$  is:  $d_{TV}(P_1, P_2) = \frac{1}{2} \sum_{\omega} |P_1[\omega] - P_2[\omega]|$

Interpretation:

- \* consider the distributions as events  $\tau_1, \tau_2$
- \*  $P_i[\omega] = P[\omega | \tau_i]$
- \*  $d_{TV}(P_1, P_2) = \frac{1}{2} \sum_{\omega} |P[\omega | \tau_1] - P[\omega | \tau_2]| = \sum_{\omega} |P[\omega \wedge \tau_1] - P[\omega \wedge \tau_2]|$

#### – Mixing Time

Definition

- \* Mixing time  $\tau$  is the least  $t$  such that for all initial state  $p^0$ ,  $d_{TV}(p, p^0 P^t) \leq \frac{1}{2e}$

Interpretation

- \* the factor  $\frac{1}{2e}$  is set such that  $d_{TV}(p, p^0 P^t) \leq \epsilon$  if  $t \geq \tau \times \log(\frac{1}{\epsilon})$   
Without proof

– Example

Random Walk on Graph

- \* Definition: move from vertex  $i$  to vertex  $j$  with probability  $P_{ij} = \begin{cases} 0 & \text{if } (i, j) \notin E \\ \frac{1}{\text{degree}(i)} & \text{if } (i, j) \in E \end{cases}$
- \* Distribution  $\pi$ ,  $\pi_i = \frac{\text{degree}(i)}{2|E|}$  satisfies detailed balance
- \* If we want stationary distribution to be uniform  $\rightarrow P'_{ij} = \begin{cases} \frac{1}{\text{degree}(i)} & \text{if } \text{degree}(i) \geq \text{degree}(j) \\ \frac{1}{\text{degree}(j)} & \text{if } \text{degree}(i) < \text{degree}(j) \end{cases}$

Random graph coloring

- \* Given a graph with  $V$  vertices, maximum degree  $\Delta$  and  $q$  colors, to color each vertex one color such that adjacent vertex do not share the same color
- \* Assume  $q > 4\Delta$
- \* Markov Chain Transition:
  - Pick random vertex and random color, if the color is changeable then change
- \* Property
  - Aperiodic: there exist self loops
  - Symmetric: symmetric transition
  - Irreducible
- \* Mixing time is  $O(V \log V)$
- Proof:
  - Assume  $X$  is a event s.t. Markov Chain starts with any valid coloring and  $Y$  is a event s.t. Markov Chain starts with uniform distribution
  - Apply same transition on both  $X$  and  $Y$
  - $D_n$  is a random variable for the number of vertices in different colors in  $X$  and  $Y$  at time  $n$
  - Good moves: number of vertices in different colors decrease  $\geq D_n \times (q - 2\Delta) \geq (2\Delta + 1)D_n$   
(vertices with different colors  $\times$  color that is different with any adjacent color in  $X$  and  $Y$ )
  - Bad moves: number of vertices in different colors increase  $\leq (D_n \Delta) \times 2$   
(vertices adjacent to different colors vertices  $\times$  color of the differen colors vertices)
  - $\mathbb{E}[D_{n+1} - D_n] \leq V(1 - \frac{1}{qV})^n$
  - $\mathbb{E}[D_n] \leq V(1 - \frac{1}{qV})^n$
  - $P[D_n \geq 1] \leq V(1 - \frac{1}{qV})^n$

#### • Hidden Markov Chain

- Definition: output is a function of the state
- Interpretation: if the model is not markovian, then reformulate the model as a hidden markov chain by complicating the states and rendering the output as a function of the state

## 10 Continuous Markov Chain

#### • Interpretation

- $v_i$ : coefficient of exponential distribution, where time in state  $i$  before next step is  $\sim \text{Exponential}(v_i)$

#### • Definition

- Model with states and transition rate matrix
- States:  $X(t), \forall 0 \leq t < \infty$
- Transition Probability Matrix  $R$

#### • $P_{ij}(t)$

- Definition:  $P_{ij}(t) = P[X(t) = j | X(0) = i]$

– Chapman-Kolmogorov Equation

\* Definition:  $P(s+t) = P(s) \times P(t)$

\* Proof

$$\begin{aligned} \cdot P_{ij}(s+t) &= P[X(s+t) = j | X(0) = i] \\ &= \sum_k P[X(s+t) = j | X(s) = k, X(0) = i] P[X(s) = k | X(0) = i] \\ &= \sum_k P[X(s+t) = j | X(s) = k] P[X(s) = k | X(0) = i] = \sum_k P_{kj}(t) P_{ik}(s) \end{aligned}$$

– Kolmogorov's Differential Equation

\* Forward:  $\frac{dP(t)}{dt} = P(t)R$

Interpretation:

· Change of distribution at  $t$  equals the distribution at  $t \times R$

Proof:

$$\cdot \frac{dP(t)}{dt} = \lim_{\delta \rightarrow 0} \frac{P(t+\delta) - P(t)}{\delta} = P(t) \lim_{\delta \rightarrow 0} \frac{P(\delta) - P(0)}{\delta} = P(t)R$$

\* Backward:  $\frac{dP(t)}{dt} = RP(t)$

Interpretation:

· Change of distribution at  $t$  equals the distribution at  $t = 0 \times P(t)$

Proof:

$$\cdot \frac{dP(t)}{dt} = \lim_{\delta \rightarrow 0} \frac{P(t+\delta) - P(t)}{\delta} = \lim_{\delta \rightarrow 0} \frac{P(\delta) - P(0)}{\delta} P(t) = RP(t)$$

\* Solution:  $P(t) = e^{Rt}$

•  $R$

– Definition:

\*  $R_{ij} = \frac{dP_{ij}(t)}{dt} |_{t=0}$

\*  $R_{ij} = \begin{cases} -v_i & \text{if } i = j \\ v_i P_{ij} & \text{if } i \neq j \end{cases}$  (if there is no self-transition)

– Interpretation

\*  $\pi R$  is the change of distribution of  $\pi$  (by Kolmogorov's Differential Equation)

\* simulation by transition from state  $i$  to  $j$  when  $e^{-R_{ij}t}$  event arrives

Proof

$$\cdot \frac{dP_{ii}(t)}{dt} = R_{ii}P_{ii}(t) \rightarrow P_{ii}(t) = e^{-R_{ii}t}$$

· simulate the transition out of state  $i$  by  $e^{-R_{ii}t}$  and transition to  $j$  state by probability  $\frac{R_{ij}}{R_{ii}}$  is the same as transition from state  $i$  to  $j$  when  $e^{-R_{ij}t}$  event arrives

Property

· Continuous Markov Chain with same  $R$  are of the same functionality

– Property:

\*  $\sum_j R_{ij} = 0$ : sum of element is a row of  $R$  is 0

• Property

– Self Transition:

\* Since  $R$  defines the Markov Chain, we can modify  $v_i$  to conduct self transition without changing  $R$

– Uniformization:

\* Since  $R$  defines the Markov Chain, we can modify  $v_i$  such that  $v_i$  are the same for all states without changing  $R$

– Stationary Distribution:  $p$  s.t.  $pR = 0, pe^{Rt} = p$

Interpretation:

\*  $\frac{dpP(t)}{dt} = p \frac{dP(t)}{dt} = pRP(t) = 0$

\*  $p$  is the eigenvector of eigenvalue 0 of  $R$ , then  $p$  is the eigenvector of eigenvalue 1 of  $e^{Rt} \rightarrow$  the distribution would not change, if start with  $p$

Trick:

- \* cluster states such that every state in the cluster share the same  $R_{ij}$  to calculate the stationary distribution of the cluster
- \* assume distribution is independent of the cluster and check  $pR = 0$  after the calculation
- Poisson process is a special case of Continuous Markov Chain
  - \*  $v_i = \lambda, \forall i$
  - \*  $i$ -th state transition to  $i + 1$ -th state
- Exploding process: only if  $v_i \rightarrow \infty$ 
  - \* exploding process: traverse infinite states in finite time

## 11 Martingales

### • Definition

- Discrete

General Discrete Martingales

- \*  $\{Z_i\}_{i=0}^{\infty}$  such that
  1.  $\mathbb{E}[|Z_n|] < \infty$
  2.  $\mathbb{E}[Z_{n+1}|Z_0, \dots, Z_n] = Z_n$ 
    - sub-martingales:  $\mathbb{E}[Z_{n+1}|Z_0, \dots, Z_n] \geq Z_n$
    - super-martingales:  $\mathbb{E}[Z_{n+1}|Z_0, \dots, Z_n] \leq Z_n$

Discrete Martingales with respect to  $X_i$

- \*  $\{Z_i\}_{i=0}^{\infty}$  such that
  1.  $\mathbb{E}[|Z_n|] < \infty$
  2.  $\mathbb{E}[Z_{n+1}|X_0, \dots, X_n] = Z_n$ 
    - sub-martingales:  $\mathbb{E}[Z_{n+1}|X_0, \dots, X_n] \geq Z_n$
    - super-martingales:  $\mathbb{E}[Z_{n+1}|X_0, \dots, X_n] \leq Z_n$
- \*  $\mathbb{E}[Z_{n+1}|X_0, \dots, X_n] = Z_n$  implies  $\mathbb{E}[Z_{n+1}|Z_0, \dots, Z_n] = Z_n$ 
  - $Z_n$  is a function of  $X_0, \dots, X_n$
  - $\mathbb{E}[Z_{n+1}|Z_0, \dots, Z_n] = \mathbb{E}[\mathbb{E}[Z_{n+1}|X_0, \dots, X_n, Z_0, \dots, Z_n]|Z_0, \dots, Z_n]$   
 $= \mathbb{E}[\mathbb{E}[Z_{n+1}|X_0, \dots, X_n]|Z_0, \dots, Z_n] = \mathbb{E}[Z_n|Z_0, \dots, Z_n] = Z_n$

- Continuous Martingales with respect to  $N(t)$

- \*  $Y(t)$  such that
  1.  $\mathbb{E}[|Y(t)|] < \infty$
  2.  $\mathbb{E}[Y(t)|\{N(s)|0 \leq s \leq \tau\}] = Y(\tau), \forall \tau \leq t$ 
    - sub-martingales:  $\mathbb{E}[Y(t)|\{N(s)|0 \leq s \leq \tau\}] \geq Y(\tau), \forall \tau \leq t$
    - super-martingales:  $\mathbb{E}[Y(t)|\{N(s)|0 \leq s \leq \tau\}] \leq Y(\tau), \forall \tau \leq t$

### • Property

- $\mathbb{E}[Z_n] = \mathbb{E}[Z_1]$   
 Proof:  $\mathbb{E}[Z_{n+1} - Z_n] = \mathbb{E}[\mathbb{E}[Z_{n+1} - Z_n|Z_0, \dots, Z_n]] = 0$
- $\mathbb{E}[Z_n|\{Z_i|i \in S\}] = Z_{\max_{i \in S} i}$ , where  $\forall i \in S, i < n$   
 Proof:  $\mathbb{E}[Z_n|Z_i] = \mathbb{E}[\mathbb{E}[Z_n|Z_0, \dots, Z_{n-1}]|Z_i] = \mathbb{E}[Z_{n-1}|Z_i]$
- Azuma's Inequality
  - \*  $\mu = \mathbb{E}[Z_0]$
  - \*  $-a_i \leq Z_i - Z_{i-1} \leq b_i$
  - \*  $P[|Z_n - \mu| \geq \delta] \leq 2e^{-\frac{2\delta^2}{\sum_{i=1}^n (b_i + a_i)^2}}$
- Kolmogorov's sub-martingales inequality
  - \*  $P[\sup_{n \geq 1} Z_n \geq a] \leq \frac{\mathbb{E}[Z_1]}{a}$
- Martingales Stopping Theorem

- \* Stopping Times: a random variable  $\tau$  s.t.  $\{\tau = n\}$  is independent of  $\{X_i\}_{i=n+1}^\infty$
- \* Stopping Condition: a condition to stop if we can consider  $\tau = \min\{n : \text{condition}(n) = \top\}$
- \*  $\mathbb{E}[Z_\tau] = \mathbb{E}[Z_0]$  if the either of the following holds
  1.  $P[\tau \leq k] = 1$
  2.  $P[\max_{i \leq \tau} |Z_i| \leq k] = 1$
  3.  $\mathbb{E}[\tau] < k$  and  $\mathbb{E}[|Z_{n+1} - Z_n| | Z_0, \dots, Z_n] < k$

- Application for generating Martingales

- Sum of iid. random variables
  - \*  $\{X_i\}_{i=1}^\infty$  are iid. random variables
  - \*  $Z_n = \sum_{i=1}^n X_i - n\mathbb{E}[X_i]$  is a martingales.
  - \* Proof:  $\mathbb{E}[Z_{n+1} | Z_0, \dots, Z_n] = \mathbb{E}[Z_n + X_{n+1} - \mathbb{E}[X_i] | Z_0, \dots, Z_n] = Z_n$
- Square of sum of iid. random variables
  - \*  $\{X_i\}_{i=1}^\infty$  are iid. random variables and  $\mathbb{E}[X_i] = 0$
  - \*  $Z_n = (\sum_{i=1}^n X_i)^2 - n\mathbb{E}[X_i^2]$  is a martingales.
  - \* Proof:  $\mathbb{E}[Z_{n+1} | Z_0, \dots, Z_n] = \mathbb{E}[Z_n + X_{n+1}^2 + 2X_{n+1}(\sum_{i=1}^n X_i) - \mathbb{E}[X_i^2] | Z_0, \dots, Z_n] = Z_n$
- Product of iid. random variables
  - \*  $\{X_i\}_{i=1}^\infty$  are iid. random variables
  - \*  $Z_n = \frac{\prod_{i=1}^n X_i}{\mathbb{E}[X_i]^n}$  is a martingales.
  - \* Proof:  $\mathbb{E}[Z_{n+1} | Z_0, \dots, Z_n] = \mathbb{E}[Z_n \frac{X_{n+1}}{\mathbb{E}[X_i]} | Z_0, \dots, Z_n] = Z_n$
- Poisson Process
  - \*  $N(t)$  is a poisson process
  - \*  $Y(t) = N(t) - \lambda t$  is a martingales.
  - \* Proof:  $\mathbb{E}[Y(t) | \{N(s) | 0 \leq s \leq \tau\}] = \mathbb{E}[Y(\tau) + Y(t) - Y(\tau) | \{N(s) | 0 \leq s \leq \tau\}]$   
 $= Y(\tau) + \mathbb{E}[N(t) - N(\tau) + \lambda(t - \tau) | \{N(s) | 0 \leq s \leq \tau\}] = Y(\tau)$
- Doob-type Martingales
  - \*  $X, \{Y_i\}_{i=1}^\infty$  are random variables
  - \*  $Z_n = \mathbb{E}[X | Y_1, Y_2, \dots, Y_n]$  is a martingales
  - \* Proof:  $\mathbb{E}[Z_{n+1} | Y_1, \dots, Y_n] = \mathbb{E}[\mathbb{E}[X | Y_1, Y_2, \dots, Y_n, Y_{n+1}] | Y_1, Y_2, \dots, Y_n]$   
 $= \mathbb{E}[X | Y_1, Y_2, \dots, Y_n] = Z_n$

- Example

- Symmetric Random Walk
  - \*  $p = 0.5$
  - \*  $\tau = \min\{i | \sum_{i=0}^n X_i \in \{-a, b\}\}$
  - \*  $Z_n = \sum_{i=0}^n X_i$ , by second rule of Martingales Stopping Theorem:  $\mathbb{E}[Z_\tau] = 0$   
 $\rightarrow P[Z_\tau \text{ at } a] = \frac{b}{a+b}, P[Z_\tau \text{ at } b] = \frac{a}{a+b}$
  - \*  $Z_n = (\sum_{i=0}^n X_i)^2 - n$ , by third rule of Martingales Stopping Theorem:  $\mathbb{E}[Z_\tau] = 0$   
 $\rightarrow \mathbb{E}[\tau] = ab$
- Unbiased Random Walk
  - \*  $\tau = \min\{i | \sum_{i=0}^n X_i \in \{-a, b\}\}$
  - \*  $Z_n = (\frac{1-p}{p})^{\sum_{i=0}^n X_i}$ , by second rule of Martingales Stopping Theorem:  $\mathbb{E}[Z_\tau] = 0$   
 $P[Z_\tau \text{ at } a] = \frac{(\frac{1-p}{p})^b - 1}{(\frac{1-p}{p})^b - (\frac{1-p}{p})^{-a}}, P[Z_\tau \text{ at } b] = \frac{1 - (\frac{1-p}{p})^{-a}}{(\frac{1-p}{p})^b - (\frac{1-p}{p})^{-a}}$
  - \*  $Z_n = \sum_{i=0}^n X_i - n\mathbb{E}[X_0]$ , by third rule of Martingales Stopping Theorem:  $\mathbb{E}[Z_\tau] = 0$   
 $\rightarrow \mathbb{E}[\tau] = \frac{\mathbb{E}[\sum_{i=0}^\tau X_i]}{\mathbb{E}[X_0]}$

## 12 Random Walk

- Definition

$$\begin{aligned}
 - X_i &= \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1-p \end{cases} \\
 - S_n &= \sum_{i=0}^n X_i
 \end{aligned}$$

- The monkey at the cliff

$$P_k = P[\exists n \text{ such that } S_n = k] = \begin{cases} 1 & \text{if } p \geq \frac{1}{2} \\ (\frac{p}{1-p})^k & \text{if } p < \frac{1}{2} \end{cases} \text{ where } k \in \mathbb{N}$$

Proof

$$\begin{aligned}
 * P_k &= P_1^k \text{ by memoryless property} \\
 * P_1 &= p + q \times P_2 \rightarrow P_1 = 1 \text{ or } \frac{p}{1-p} \\
 * \text{if } p &\geq 0.5 \rightarrow P_1 = 1 \text{ } (P_1 \leq 1) \\
 * \text{if } p &< 0.5 \rightarrow P_1 = \frac{p}{1-p} \\
 &\text{Since } P_1 \leq \frac{p}{1-p} \text{ by induction on } n \text{ to } \infty \text{ for } P_1(n) = P[S_n = k]
 \end{aligned}$$

$$\mathbb{E}_k = \mathbb{E}[\min\{n : S_n = k\}] = \begin{cases} \infty & \text{if } p \leq \frac{1}{2} \\ \frac{k}{2p-1} & \text{if } p > \frac{1}{2} \end{cases} \text{ where } k \in \mathbb{N}$$

Proof

$$\begin{aligned}
 * \mathbb{E}_k &= \mathbb{E}_1 \times k \text{ by memoryless property} \\
 * \mathbb{E}_1 &= 1 + 0 \times p + \mathbb{E}_2 \times (1-p) \\
 * \text{if } p &< 0.5 \rightarrow P_1 = \frac{p}{1-p} \rightarrow \mathbb{E}_1 = \infty \\
 * \text{if } p &= 0.5 \rightarrow \mathbb{E}_1 = 1 + \mathbb{E}_1 \text{ (no solution)} \rightarrow \mathbb{E}_1 = \infty \\
 * \text{if } p &> 0.5 \rightarrow \mathbb{E}_1 = \frac{1}{2p-1}
 \end{aligned}$$

$$P_0 = P[\exists n \text{ such that } S_n = 0] = 1 - |2p - 1| \text{ where } k \in \mathbb{N}$$

$$* P_0 = p \times P_{-1} + (1-p) \times P_1$$

$$\mathbb{E}_0 = \mathbb{E}[\min\{n : S_n = 0\}] = \infty$$

$$* \text{if } p \neq \frac{1}{2} \rightarrow P_0 \neq 1 \rightarrow \mathbb{E}_0 = \infty$$

$$* \text{if } p = \frac{1}{2} \rightarrow \mathbb{E}_0 = 1 + \frac{1}{2}\mathbb{E}_{-1} + \frac{1}{2}\mathbb{E}_1 = \infty$$

- The Gambler's Ruin

$$\text{Definition: } \tau = \min\{i | S_n \in \{-a, b\}\}$$

$$A_k = P[S_\tau = b | X_0 = k]$$

$$* A_k = pA_{k+1} + (1-p)A_{k-1}$$

$$A_0 = \begin{cases} \frac{a}{a+b} & \text{if } p = \frac{1}{2} \\ \frac{(\frac{1-p}{p})^a - 1}{(\frac{1-p}{p})^{a+b} - 1} & \text{if } p \neq \frac{1}{2} \end{cases}$$

Solved by previous recursive equation

$$E_k = \mathbb{E}[\tau | X_0 = k]$$

$$* E_k = 1 + pE_{k+1} + (1-p)E_{k-1}$$

$$E_0 = \begin{cases} ab & \text{if } p = \frac{1}{2} \\ \frac{a}{1-2p} - \frac{a+b}{1-2p} \times \frac{(\frac{1-p}{p})^a - 1}{(\frac{1-p}{p})^{a+b} - 1} & \text{if } p \neq \frac{1}{2} \end{cases}$$

Solved by previous recursive equation

- Observation

$$S_n = O(n)$$

$$\text{Upperbound: } \lim_{n \rightarrow \infty} P[S_n \leq k\sqrt{n}] = \int_{-\infty}^k \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$\text{Lowerbound: } P[|S_n| \geq k\sqrt{n}] \leq 2e^{-\frac{k^2}{2}}$$

## 13 Brownian Motion

- Standard Brownian Motion

- Interpretation: generalize discrete time and space of random walk to be in continuous time and space

- \*  $S_t = \delta_x (\sum_{i=0}^{\frac{t}{\delta_t}} X_i)$
- \* let  $\delta_x = \sqrt{\delta_t}$  and  $\delta_x \rightarrow 0$
- \*  $\mathbb{E}[S_t] = 0$
- \*  $Var(S_t) = t$ 
  - $Var(S_t) = \delta_x^2 \frac{t}{\delta_t} = t$

- Definition:

- \*  $X(0) = 0$
- \*  $X(t) \sim N(0, \sigma^2 = t)$
- \*  $X(t)$  has independent, stationary increment
  - independent:  $X(t_{i_2}) - X(t_{i_1})$  and  $X(t_{i_1}) - X(t_{i_0})$  are independent
  - stationary:  $X(s+t) - X(t) = X(s)$

- Property

- \* Distribution self-similarity
  - $X(t) \sim N(0, t)$
  - $\sqrt{k}X(\frac{t}{k}) \sim N(0, t)$
- \* Nowhere Differentiable
  - With probability 1,  $X(t)$  is nowhere differentiable
  - $\lim_{\delta_t \rightarrow 0} \frac{X(t+\delta_t) - X(t)}{\delta_t} = \lim_{\delta_t \rightarrow 0} \frac{N(0, \delta_t)}{\delta_t} = \lim_{\delta_t \rightarrow 0} N(0, \frac{1}{\delta_t})$
- \* Unbounded Variation
  - Length of distance  $\rightarrow \infty$  in finite time  $t$
  - $\lim_{n \rightarrow \infty} \sum_{j=1}^n |X(\frac{jt}{n}) - X(\frac{(j-1)t}{n})| = \infty$
  - Proof:  $\lim_{n \rightarrow \infty} \sum_{j=1}^n |X(\frac{jt}{n}) - X(\frac{(j-1)t}{n})| = \lim_{n \rightarrow \infty} \sum_{j=1}^n |X(\frac{t}{n})| = n \times \sqrt{\frac{2}{\pi} \frac{t}{n}} = \infty$

- \* Hitting Time

The Gambler's Ruin

- $\tau = \min\{t \geq 0 : X(t) \in \{-A, B\}\}$
- $P[X(\tau) = A] = \frac{B}{A+B}, P[X(\tau) = B] = \frac{A}{A+B}$   
Prove by Martingales Stopping Theorem on  $X(t)$ :  
 $\rightarrow \mathbb{E}[X(t)] = P[X(\tau) = A]A + P[X(\tau) = B]B = 0$
- $\mathbb{E}[\tau] = AB$   
Prove by Martingales Stopping Theorem on  $X(t)^2 - t$ :  
 $\rightarrow \mathbb{E}[X(t)^2 - t] = P[X(\tau) = A]A^2 + P[X(\tau) = B]B^2 - \mathbb{E}[\tau] = 0$

The monkey at the cliff

- $\tau = \min\{t \geq 0 : X(t) = B\}$
- $P[\tau < \infty] = 1$   
Prove by let  $A = -\infty$  in The Gambler's Ruin
- $P[\tau \leq t] = 2P[X(\tau) \geq B]$   
 $P[\tau \leq t] = P[\tau \leq t \text{ and } X(t) \geq B] + P[\tau \leq t \text{ and } X(t) < B]$   
 $= 2P[\tau \leq t \text{ and } X(t) \geq B] = 2P[X(t) \geq B]$
- $\mathbb{E}[\tau] = \infty$   
Prove by let  $A = -\infty$  in The Gambler's Ruin

- \* Diffusion Equation

- Forward Diffusion Equation:  $\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial^2 f}{\partial x^2}$
- Backward Diffusion Equation:  $\frac{\partial f}{\partial t} = -\frac{1}{2} \frac{\partial^2 f}{\partial x^2}$
- $f(X(t_2) = x | X(t_1) = k)$  satisfies Forward Diffusion equation
- $f(X(t_2) = k | X(t_1) = x)$  satisfies Backward Diffusion equation

- \* Martingales

- $X(t)$  is a martingale
- $X(t)^2 - t$  is a martingale
- $e^{cX(t) - \frac{c^2}{2}t}$  is a martingale

\* Zeros

Definition:  $P[X(t) = 0, t_0 < t < t_1] = \frac{2}{\pi} \cos^{-1}(\sqrt{\frac{t_0}{t_1}})$

- Prove by  $P[X(t) = 0, t_0 < t < t_1] = \int_{-\infty}^{\infty} f_{X(t_0)}(x_1) P[T_{-x} \leq t_1 - t_0] dx_1$

Property

- $P[X(t) = 0, 0 < t < t_1] = 1, \forall t_1 > 0$
- $P[\inf\{t > 0 : X(t) = 0\} = 0] = 1$
- $P[\text{there are infinitely many zeros in } [0, t]] = 1$

– Brownian Bridge

\* Definition: the distribution of  $t_1$  given the result of the future  $X(t_2)$

\* Property

$$f_{X(t_1)|X(t_2)}(x_1, x_2) = \frac{f_{X(t_1), X(t_2)}(x_1, x_2)}{f_{X(t_2)}(x_2)} \sim N(\frac{t_1}{t_2}x_2, \frac{t_1(t_2-t_1)}{t_2})$$

- let  $s = t_2 - t_1$

$$\begin{aligned} f_{X(t_1)|X(t_2)}(x_1, x_2) &= \frac{f_{X(t_1), X(t_2)}(x_1, x_2)}{f_{X(t_2)}(x_2)} \\ &= \frac{f_{X(t_1), X(s)}(x_1, x_2 - x_1)}{f_{X(t_2)}(x_2)} \quad (\text{By transformation of 2-D random variables}) \end{aligned}$$

$$= \frac{f_{X(t_1)}(x_1) f_{X(s)}(x_2 - x_1)}{f_{X(t_2)}(x_2)} = \frac{\frac{1}{\sqrt{2\pi t_1}} e^{-\frac{x_1^2}{2t_1}} \frac{1}{\sqrt{2\pi(t_2-t_1)}} e^{-\frac{(x_2-x_1)^2}{2(t_2-t_1)}}}{\frac{1}{\sqrt{2\pi t_2}} e^{-\frac{x_2^2}{2t_2}}}$$

$$= \frac{1}{\sqrt{2\pi \frac{t_1(t_2-t_1)}{t_2}}} e^{-\frac{(x_1 - \frac{t_1}{t_2}x_2)^2}{2 \frac{t_1(t_2-t_1)}{t_2}}} \rightarrow X(t_1) - \frac{t_1}{t_2}X(t_2) \sim N(0, \frac{t_1(t_2-t_1)}{t_2})$$

- $\mathbb{E}[X(t_1)|X(t_2)] = \frac{t_1}{t_2}X(t_2)$
- $\text{Var}(X(t_1)|X(t_2)) = \frac{t_1(t_2-t_1)}{t_2}$
- $Y(t_1) = X(t_1) - \frac{t_1}{t_2}X(t_2)$  share the same distribution as  $X(t_1)|X(t_2) = 0$

$$\text{Cov}(X(t_1), X(t_2)|X(t_3)) = \frac{t_1(t_3-t_2)}{t_3}$$

$$\begin{aligned} &\cdot \text{Cov}(X(t_1), X(t_2)|X(t_3)) \\ &= \mathbb{E}[X(t_1)X(t_2)|X(t_3)] - \mathbb{E}[X(t_1)|X(t_3)] \times \mathbb{E}[X(t_2)|X(t_3)] \\ &= \mathbb{E}[X(t_1)^2 + X(t_1)(X(t_2) - X(t_1))|X(t_3)] - \frac{t_1 t_2}{t_3^2} X(t_3)^2 \\ &= \mathbb{E}[X(t_1)(X(t_2) - X(t_1))|X(t_3)] + \frac{t_1(t_1-t_2)}{t_3^2} X(t_3)^2 + \frac{t_1(t_3-t_1)}{t_3} \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \frac{t_1(t_3-t_1)}{t_3}}} e^{-\frac{(x_1 - \frac{t_1}{t_3}X(t_3))^2}{2 \frac{t_1(t_3-t_1)}{t_3}}} \mathbb{E}[x_1(X(t_2) - x_1)|X(t_3), X(t_1) = x_1] dx_1 + \frac{t_1(t_1-t_2)}{t_3^2} X(t_3)^2 + \frac{t_1(t_3-t_1)}{t_3} \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \frac{t_1(t_3-t_1)}{t_3}}} e^{-\frac{(x_1 - \frac{t_1}{t_3}X(t_3))^2}{2 \frac{t_1(t_3-t_1)}{t_3}}} x_1(-x_1 + X(t_3)) \frac{t_2-t_1}{t_3-t_1} dx_1 + \frac{t_1(t_1-t_2)}{t_3^2} X(t_3)^2 + \frac{t_1(t_3-t_1)}{t_3} \\ &= \frac{t_1(t_2-t_1)}{t_3^2} X(t_3)^2 - \frac{t_1(t_2-t_1)}{t_3} + \frac{t_1(t_1-t_2)}{t_3^2} X(t_3)^2 + \frac{t_1(t_3-t_1)}{t_3} \\ &= \frac{t_1(t_3-t_2)}{t_3} \end{aligned}$$

• Brownian Motion with drift

– Interpretation: generalize discrete time and space of biased random walk to be in continuous time and space

$$* X_i = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1-p \end{cases}$$

$$* S_t = \delta_x \left( \sum_{i=0}^{\frac{t}{\delta_t}} X_i \right)$$

$$* \text{let } \delta_x = \sqrt{\delta_t}, p = \frac{1+\mu\sqrt{\delta_t}}{2}, \text{ and } \delta_x \rightarrow 0$$

$$* \mathbb{E}[S_t] = \mu t$$

$$\cdot \mathbb{E}[S_t] = \delta_x \frac{t}{\delta_t} (2p - 1) = \mu t$$



- \*  $Var(S_t) = t$ 
  - $Var(S_t) = \delta_x^2 \frac{t}{\delta_t} (1 - (2p - 1)^2) = t$
- Definition:
  - \*  $X(t)$  is Standard Brownian Motion
  - \*  $Y(t) = X(t) + \mu t$
- Property
  - \* Hitting Time
    - The Gambler's Ruin
      - $\tau = \min\{t \geq 0 : Y(t) \in \{-A, B\}\}$
      - $P[Y(t) = A] = \frac{e^{-2\mu B} - 1}{e^{-2\mu B} - e^{2\mu A}}, P[Y(t) = B] = \frac{1 - e^{2\mu A}}{e^{-2\mu B} - e^{2\mu A}}$
      - Prove by Martingales Stopping Theorem on  $e^{cX(t) - \frac{c^2}{2}t}$  and  $c = -2\mu$ :
        - $\rightarrow \mathbb{E}[e^{cX(t) - \frac{c^2}{2}t}] = \mathbb{E}[e^{-2\mu Y(t)}] = 1$
      - $\mathbb{E}[\tau] = \frac{1}{\mu}(P[Y(t) = B] \times (A + B) - A)$
      - Prove by Martingales Stopping Theorem on  $X(t)$ :
        - $\rightarrow \mathbb{E}[X(t)] = P[Y(t) = B]\mathbb{E}[B - \mu t | Y(t) = B] + P[Y(t) = A]\mathbb{E}[-A - \mu t | Y(t) = A] = 0$
    - The monkey at the cliff
      - $\tau = \min\{t \geq 0 : X(t) = B\}$
      - $P[\tau < \infty] = \begin{cases} e^{2\mu B} & \text{if } \mu < 0 \\ 1 & \text{if } \mu \geq 0 \end{cases}$
      - Prove by let  $A = -\infty$  in The Gambler's Ruin
- Gaussian Process
  - Definition: A stochastic process  $\{X(t) : t \geq 0\}$  such that for every  $\{t_i\}_{i=1}^n, [X(t_1), X(t_2), \dots, X(t_n)]$  is a joint Gaussian distribution
    - \* Defined by
      - $\mathbb{E}[X(t)], \forall t$
      - $Cov(X(s), X(t)), \forall s, t$
  - Property
    - \* Standard Brownian Motion is a Gaussian Process with  $\mathbb{E}[X(t)] = 0, Cov(X(s), X(t)) = \min\{s, t\}$ 
      - $Cov(X(s), X(t)) = \min(s, t)$  (by  $X(t) = X(s) + X(t - s)$  if  $t > s$ )
- Geometric Brownian Motion
  - Definition:
    - \*  $Y(t) = e^{\sigma X(t)}$
  - Property:
    - \*  $\mathbb{E}[Y(t)] = e^{\frac{\sigma^2 t}{2}}$
    - \*  $Var[Y(t)] = e^{\sigma^2 t}$
- Brownian Motion reflected at the origin
  - Definition:
    - \*  $Z(t) = |X(t)|$
  - Property
    - \*  $P[Z(t) \geq x] = \frac{2}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$ 
      - same distribution as Maximum Brownian Motion
- Maximum Brownian Motion
  - Definition:
    - \*  $Z(t) = \max_{0 \leq s \leq t} X(s)$
  - Property

- \*  $P[Z(t) \geq x] = \frac{2}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$
- $P[Z(t) \geq x] = P[T_x \leq t] = \frac{2}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$
- same distribution as Brownian Motion reflected at the origin

- Tricks

- Create  $Y_1, Y_2 \sim N(0, 1)$  and  $Cov(Y_1, Y_2) = \cos \theta$ 
  - \*  $X_1, X_2 \sim N(0, 1)$  and independent
  - \*  $Y_1 = X_1$
  - \*  $Y_2 = \cos \theta \times X_1 + \sin \theta \times X_2$