# Stochastic Processes

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## 1

(a). Suppose that coin 1 has probability 0.7 of coming up heads, and coin 2 has probability 0.6 of coming up heads. If the coin flipped today comes up heads, then we select coin 1 to flip tomorrow, and if it comes up tail, then we select coin 2 to flip tomorrow. If the coin initially flipped is equally likely to be coin 1 or coin 2, then what is the probability that the coin flipped the on the third day after the initial flip coin is 1.
(b). Suppose that the coin flipped on Monday comes up heads. What is the probability that the coin flipped on Friday of the same week also comes up heads?

- $P = \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix}$
- (a) $[0.5\ 0.5]P^3 = [0.6665\ 0.3335]$
- (b) $[1\ 0]P^3 = [0.6667\ 0.3333]$

## 2

A professor continually gives exam to his students. He can give three possible types of exams, and her class is graded as either having done well or badly. Let $p_i$ denote the probability that the class done well on type $i$ exam, and suppose that $p_1 = 0.3, p_2 = 0.6$, and $p_3 = 0.9$. If the class does well on an exam, then the next exam is equally likely to be any of the three types. If the class does badly, then the next exam is always type 1. What proportion of exams are type $i, i = 1, 2, 3$?

- $P = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.6 & 0.2 & 0.2 \\ 0.4 & 0.3 & 0.3 \end{bmatrix}$
- stationary distribution: $p$
  - $p$ is eigenvector corresponding to eigenvalue 1 of $P^T$
  - $p = [\frac{5}{7}\ \frac{1}{7}\ \frac{1}{7}]$

## 3

A DNA nucleotide has any of the 4 values. A standard model for a mutational change of the nucleotide at a spcific location is a Markov chain model that supposes that in going from period to period, the nucleotide does not change with probability $1-3\alpha$ and if it does change then it is equally likely to change to any of the other 3 values, for some $0 < \alpha < \frac{1}{3}$

(a) Show that $P^n_{1,1} = \frac{1}{4} + \frac{3}{4}(1-4\alpha)^n$.

(b) What is the long run proportion of time the chain is in each state?

- $P = \begin{bmatrix} 1-3\alpha & \alpha & \alpha & \alpha \\ \alpha & 1-3\alpha & \alpha & \alpha \\ \alpha & \alpha & 1-3\alpha & \alpha \\ \alpha & \alpha & \alpha & 1-3\alpha \end{bmatrix}$
- (a)
  - $P^n = \frac{1}{4}\begin{bmatrix} 3(1-4\alpha)^n+1 & 1-(1-4\alpha)^n & 1-(1-4\alpha)^n & 1-(1-4\alpha)^n \\ 1-(1-4\alpha)^n & 3(1-4\alpha)^n+1 & 1-(1-4\alpha)^n & 1-(1-4\alpha)^n \\ 1-(1-4\alpha)^n & 1-(1-4\alpha)^n & 3(1-4\alpha)^n+1 & 1-(1-4\alpha)^n \\ 1-(1-4\alpha)^n & 1-(1-4\alpha)^n & 1-(1-4\alpha)^n & 3(1-4\alpha)^n+1 \end{bmatrix}$

    (by Wolfram Alpha)
  - $P^n_{1,1} = \frac{1}{4} + \frac{3}{4}(1-4\alpha)^n$
- (b) it is equally distributed: stationary distribution is $\frac{1}{4}[1,1,1,1]$

## 4

Let $P^n_{ij} = P_i[X_n = j] = P[X_n = j \mid X_0 = i]$.

Let $f^n_{ij} = P[X_n = j, X_{n-1} \neq j, \ldots, X_1 \neq j \mid X_0 = i]$

ie., the probability that the first transition into occurs at time n. $f^0_{ij} = 0$.

**Show that**

$$P^n_{ij} = \sum_{k=0}^{n} f^k_{ij} P^{n-k}_{jj}.$$

- $P^n_{ij} = P[X_n = j|X_0 = i] = P[X_n = j, X_1 = j|X_0 = i] + P[X_n = j, X_1 \neq j|X_0 = i]$

  $= P[X_n = j|X_1 = j, X_0 = i]P[X_1 = j|X_0 = i] + P[X_n = j, X_1 \neq j|X_0 = i]$

  $= P^{n-1}_{jj} f^k_{ij} + P[X_n = j, X_2 = j, X_1 \neq j|X_0 = i] + P[X_n = j, X_2 \neq j, X_1 \neq j|X_0 = i]$

  $= P^{n-1}_{jj} f^2_{ij} + P^{n-2}_{jj} f^2_{ij} + P[X_n = j, X_2 \neq j, X_1 \neq j|X_0 = i]$

  $\ldots$

  $= \sum_{k=0}^{n} P^{n-k}_{jj} f^k_{ij}$

5

Jobs arrive at a processing center in accordance with a Poisson process with rate $\lambda$. However, the center has waiting space for only $N$ jobs and so an arriving job finding $N$ others waiting goes away. At most 1 job per day can be processed, and the processing of this job must start at the beginning of the day. Thus, if there are any jobs waiting for processing at the beginning of a day, then one of them is processed that day, and if no jobs are waiting at the beginning of a day then no jobs are processed that day. Let $X_n$ denote the number of jobs at the center at the beginning of day $n$.

**(a)** Find the transition probabilities of the Markov chain $\{X_n, n \geq 0\}$.

**(b)** Is this chain ergodic? Explain.

**(c)** Write the equations for the stationary probabilities.

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