Stochastic Processes

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Let X be a continuous random variable such that $\mathbb{P}(X < 0) = 0$. Show that $\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > t)dt$

- $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)dx = \int_0^\infty x f_X(x)dx$
  $= \int_0^\infty \int_0^x f_X(x)dtdx = \int_0^\infty \int_t^\infty f_X(x)dxdt = \int_0^\infty P[X > t]dt$

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$p = 1-q$

(i) Let X, Y be independent and identically distributed random variables that take values from the set $\{0, 1, 2, \ldots\}$ such that $\mathbb{P}(X = n) = pq^n$. Find
(a) $\mathbb{P}(X = Y)$ (b) $\mathbb{P}(X \geq 2Y)$
(ii) Show that $\mathbb{P}(X = k|X + Y = n) = \frac{1}{(n+1)}$

- (i)
  - (a)
    - $P[X = Y] = \sum_{n=0}^{\infty} P[X = Y|X = n]P[X = n] = \sum_{n=0}^{\infty} P[Y = n|X = n]P[X = n]$
      $= \sum_{n=0}^{\infty} P[Y = n]P[X = n] = \sum_{n=0}^{\infty} p^2 \times q^{2n} = p^2 \frac{1}{1-q^2}$
  - (b)
    - $P[X \geq 2Y] = \sum_{n=0}^{\infty} P[X \geq 2Y|Y = n]P[Y = n]$
      $= \sum_{n=0}^{\infty} P[X \geq 2n]P[Y = n] = \sum_{n=0}^{\infty} \frac{pq^{2n}}{1-q} pq^n = \frac{p^2q^{3n}}{(1-q)(1-q^3)}$
- (ii)
  - $P[X = k|X + Y = n] = \frac{P[X=k, X+Y=n]}{P[X+Y=n]} = \frac{P[X=k, Y=n-k]}{\sum_{i=0}^{n} P[X=i]P[Y=n-i]}$
    $= \frac{pq^k \times pq^{n-k}}{\sum_{i=0}^{n} pq^i \times pq^{n-i}} = \frac{1}{(n+1)}$

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(i) Let the probability that a family has exactly n children be αp^n when $n \geq 1$, and $p_0 = 1 - \alpha p(1 + p + p^2 + \ldots)$. Suppose that all the sex distributions of n children have the same probability. Show that for $k \geq 1$ the probability that a family has exactly k boys is $2\alpha p^k/(2 - p)^{k+1}$
(ii) Given that a family includes at least one boy, what is the probability that there are two or more?

- (i)

- Suppose the number of the boys is B
- $P[B=k]=\sum_{n=k}^{\infty}\alpha p^n 2^{-n}\binom{n}{k} = \alpha[(\frac{p}{2})^k+\sum_{n=k+1}^{\infty}(\frac{p}{2})^n\binom{n}{k}]$
  $=\alpha[(\frac{p}{2})^k(1-\frac{p}{2})+(\frac{p}{2})^{k+1}+\sum_{n=k+1}^{\infty}(\frac{p}{2})^n\binom{n}{k}]$
  $=\alpha[\binom{k+1}{k+1}(\frac{p}{2})^k(1-\frac{p}{2})+\binom{k+2}{k+1}(\frac{p}{2})^{k+1}(1-\frac{p}{2})+\binom{k+2}{k+1}(\frac{p}{2})^{k+2}+\sum_{n=k+2}^{\infty}(\frac{p}{2})^n\binom{n}{k}]$
  ...
  $=P[B=k+1](1-\frac{p}{2})\frac{2}{p}$
- $P[B=k+1]=P[B=k]\frac{p}{2-p}$
- $P[B=1]=\sum_{n=1}^{\infty}\alpha(\frac{p}{2})^n\binom{n}{1}$
  $=\sum_{n=1}^{\infty}\alpha(\frac{p}{2})^n\binom{n}{1} = \alpha\frac{\frac{p}{2}}{(1-\frac{p}{2})^2}$
- $P[B=k]=P[B=1](\frac{p}{2-p})^{k-1} = \alpha\frac{\frac{p}{2}}{(1-\frac{p}{2})^2}(\frac{p}{2-p})^{k-1} = 2\alpha\frac{p^k}{(2-p)^{k+1}}$

- (ii)
  - $P[B\geq 2|B\geq 1]=\frac{1-P[0\leq B\leq 1]}{1-P[B=0]} = \frac{1-2\alpha(\frac{1}{2-p}+\frac{p}{(2-p)^2})}{1-2\alpha(\frac{1}{2-p})}$
    $=\frac{(2-p)^2-4\alpha}{(2-p)^2-4\alpha+2\alpha p}$

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Let $\{X_n, n=1,2,\ldots\}$ be a sequence of independent Bernoulli random variables such that $\mathbb{P}(X_n=1)=\frac{1}{n}$, $\mathbb{P}(X_n=0)=1-\frac{1}{n}$. Define $A_n=\{X_{n-2}=0, X_{n-1}=1, X_n=1\}$. Show that $\mathbb{P}(A_n \text{ i.o.})=0$.

- $P[A_n]=(1-\frac{1}{n-2})(\frac{1}{n-1})(\frac{1}{n}) = \frac{n-3}{(n-2)(n-1)n}$
- $\sum_{n=1}^{\infty}P[A_n]<\infty \to P[A_n\ f.o.]=1$
- $P[A_n\ i.o.]=1-P[A_n\ f.o.]=0$

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If X and Y are independent exponential random variables with parameters λ_1 and λ_2 respectively, compute the distribution of $Z=\min(X,Y)$. What is the conditional distribution of Z given that $Z=X$?

- $P[Z\leq z]$
  - $P[Z>z]=P[X>z,Y>z]=e^{-(\lambda_1+\lambda_2)z}$
  - $P[Z\leq z]=1-P[Z>z]=1-e^{-(\lambda_1+\lambda_2)z}$
- $P[Z\leq z|Z=X]$
  - $P[Z>z|Z=X]=\frac{P[Y>X>z]}{P[Z=X]} = \frac{\int_z^{\infty}e^{-\lambda_2 x}\lambda_1 e^{-\lambda_1 x}dx}{\int_0^{\infty}e^{-\lambda_2 x}\lambda_1 e^{-\lambda_1 x}dx} = e^{-(\lambda_1+\lambda_2)z}$
  - $P[Z\leq z|Z=X]=1-P[Z>z|Z=X]=1-e^{-(\lambda_1+\lambda_2)z}$

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The conditional variance of X, given Y, is defined by

$$\mathrm{Var}(X|Y) = \mathbb{E}\left[(X - \mathbb{E}[X|Y])^2|Y\right].$$

Show that

$$\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X|Y)] + \mathrm{Var}(\mathbb{E}[X|Y]).$$

- $Var(X|Y) = \mathbb{E}[(X - \mathbb{E}[X|Y])^2|Y] = \mathbb{E}[X^2 + \mathbb{E}[X|Y]^2 - 2X\mathbb{E}[X|Y]|Y]$
  $= \mathbb{E}[X^2|Y] + \mathbb{E}[\mathbb{E}[X|Y]^2|Y] - 2\mathbb{E}[X|Y] \times \mathbb{E}[X|Y] = \mathbb{E}[X^2|Y] - \mathbb{E}[X|Y]^2$
- $\mathbb{E}[Var(X|Y)] = \mathbb{E}[\mathbb{E}[X^2|Y] - \mathbb{E}[X|Y]^2] = \mathbb{E}[X^2] - \mathbb{E}[\mathbb{E}[X|Y]^2]$
- $Var(\mathbb{E}[X|Y]) = \mathbb{E}[\mathbb{E}[X|Y]^2] - \mathbb{E}[\mathbb{E}[X|Y]]^2 = \mathbb{E}[\mathbb{E}[X|Y]^2] - \mathbb{E}[X]^2$
- $\mathbb{E}[Var(X|Y)] + Var(\mathbb{E}[X|Y]) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = Var(X)$

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An urn contains a white balls and b black balls. After a ball is drawn, it is returned to the urn if it is white; but if it is black, it is replaced by a white ball from another urn. Let M_n denote the expected number of white balls in the urn after the foregoing operation has been repeated n times.

(i) Derive the recursive equation

$$M_{n+1} = \left(1 - \frac{1}{a+b}\right) M_n + 1.$$

(ii) Use part (i) to prove that

$$M_n = a + b - b\left(1 - \frac{1}{a+b}\right)^n.$$

- (i)
  - Suppose the number of white balls in the urn after the n times foregoing operation is X_n
  - $P[X_{n+1} = k] = P[X_n = k]\frac{k}{a+b} + P[X_n = k-1]\frac{a+b-k+1}{a+b}$
  - $\mathbb{E}[X_{n+1}] = \sum_{k=a}^{a+b} kP[X_{n+1}] = \sum_{k=a}^{a+b} k(P[X_n = k]\frac{k}{a+b} + P[X_n = k-1]\frac{a+b-k+1}{a+b})$
    $= \sum_{k=a}^{a+b} P[X_n = k]\frac{k^2}{a+b} + \sum_{k=a}^{a+b} P[X_n = k](k+1)\frac{a+b-k}{a+b}) = \sum_{k=a}^{a+b} P[X_n = k](k - \frac{k}{a+b} + 1)$
    $= \mathbb{E}[X_n](1 - \frac{1}{a+b}) + 1$
  - $M_{n+1} = (1 - \frac{1}{a+b})M_n + 1$
- (ii)
  - $M_0 = a$
  - $M_{n+1} = (1 - \frac{1}{a+b})M_n + 1$
  - $M_n = 1 + (1 - \frac{1}{a+b}) + \cdots + (1 - \frac{1}{a+b})^{n-1} + a(1 - \frac{1}{a+b})^n = \frac{(1-\frac{1}{a+b})^n - 1}{(1-\frac{1}{a+b}) - 1} + a(1 - \frac{1}{a+b})^n$
    $= a + b - b(1 - \frac{1}{a+b})^n$

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2.3. For a Poisson process show, for $s < t$, that

$$P\{N(s) = k \mid N(t) = n\} = \binom{n}{k}\left(\frac{s}{t}\right)^k\left(1 - \frac{s}{t}\right)^{n-k}, \qquad k = 0, 1, \ldots, n.$$

- $P[N(s) = k|N(t) = n] = \frac{P[N(s)=k, N(t-s)=n-k]}{P[N(t)=n]} = \frac{\frac{(\lambda s)^k}{k!}e^{-\lambda s} \times \frac{(\lambda(t-s))^{n-k}}{(n-k)!}e^{-\lambda(t-s)}}{\frac{(\lambda t)^n}{n!}e^{-\lambda t}}$
  $= \binom{n}{k}(\frac{s}{t})^k(1 - \frac{s}{t})^{n-k}$