Stochastic Processes

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Suppose $B_1(t), B_2(t)$ are independent Brownian processes with variance parameters σ_1^2, σ_2^2 respectively. Define: $\forall t,\ X(t) = B_1(t) - B_2(t)$. Derive the mean and autocorrelation functions of $X(t)$.

i.e., $\mathrm{Cov}(X(s), X(t))$

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In Problems 8.1, 8.2, and 8.3, let $\{X(t),\ t \geq 0\}$ denote a Brownian motion process.

8.1. Let $Y(t) = tX(1/t)$.

(a) What is the distribution of $Y(t)$?

(b) Compute $\mathrm{Cov}(Y(s), Y(t))$.

(c) Argue that $\{Y(t),\ t \geq 0\}$ is also Brownian motion.

(d) Let

$$T = \inf\{t > 0: \quad X(t) = 0\}.$$

Using (c) present an argument that

$$P\{T = 0\} = 1.$$

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8.2. Let $W(t) = X(a^2 t)/a$ for $a > 0$. Verify that $W(t)$ is also Brownian motion.

b) Verify that $Y(t) = t^{\beta} X(t^{1-2\beta})$ is not Brownian motion unless $\beta \in \{0, 1\}$.

In particular, plot together $X(t)$ and $\sqrt{t}\, X(1)$.

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(4) For each of the following processes, compute $\mathbb{E} X_t$ and $\mathrm{Cov}(X_s, X_t)$.

($B(t)$ is standard Brownian motion.)

a) $X_t = \int_0^t B(u)\,du$

b) $X_t = \int_0^t u \cdot B(u)\,du$

c) $X_t = \int_0^t u^2 \cdot B(u)\,du$

Note: Since B_u is a Gaussian process, so is each of these X_ts (being sum/integrals of Gaussians). Therefore they are fully determined by the means and covariances.

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