# Stochastic Processes

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## 1 Laplace Transform

- $\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}dt$
- Property
  - $tf(t) \leftrightarrow -F'(s)$
  - $\frac{f(t)}{t} \leftrightarrow \int_s^\infty F(\sigma)d\sigma$
  - $f'(t) \leftrightarrow sF(s) - f(0^-)$
  - $\int_0^t f(\tau)d\tau \leftrightarrow \frac{F(s)}{s}$
  - $e^{at}f(t) \leftrightarrow F(s-a)$
  - $f(t-a)u(t-a) \leftrightarrow e^{-at}F(s)$

## 2 Moment Generating Function

- Moment Generating Function: $\mathbb{E}[e^{tX}]$
  - Property:
    - $\mathbb{E}[e^{tX}] = \int_{-\infty}^\infty e^{tx}f_X(x)dx$
    - $\mathbb{E}[e^{tX}] = \sum_{k=0}^\infty E[X^k]\frac{t^k}{k!}$
      - $e^{tx} = \sum_{k=0}^\infty \frac{(tx)^k}{k!}$
      - $E[e^{tX}] = E[\sum_{k=0}^\infty \frac{(tX)^k}{k!}] = \sum_{k=0}^\infty E[X^k]\frac{t^k}{k!}$
    - $\frac{d\mathbb{E}[e^{tX}]}{dt} = \mathbb{E}[X]$
    - $\mathbb{E}[e^{t(aX+b)}] = e^tb\mathbb{E}[e^{taX}]$
    - Not all random variables have Moment generating function
- Characteristic Function: $\mathbb{E}[e^{itX}]$
  - Property:
    - All random variables have Moment generating function
- Joint Moment Generating Function: $G(x,y) = \mathbb{E}[e^{xX}e^{yY}]$
- Property:
  - (Joint) moment generating funciton uniquely determines the (joint) CDF
- Example
  - Trapped miner's random walk
    - Miner has probability of $\frac{1}{3}$ to waste 3 hours in vain, $\frac{1}{3}$ to waste 5 hours in vain, and $\frac{1}{3}$ to spend 2 hours to go out of the mine.
    - $X$ is the random variables of the hours to go out of the mine
    - $Y_i$ is the random variables of the hours for the $i$-th action.
    - $\mathbb{E}[e^{tX}] = \mathbb{E}[e^{tX}|Y_1 = 2] + \mathbb{E}[e^{tX}|Y_1 = 3] + \mathbb{E}[e^{tX}|Y_1 = 5]$
      $= \mathbb{E}[e^{2t}] + \mathbb{E}[e^{t(X+3)}] + \mathbb{E}[e^{t(X+5)}]$
    - Find expectation and variance by joint moment generating function

# 3 Expectation

- $N$ i.i.d. events, when $N$ is a random variable
  - Suppose $N$ is a integer random variable
  - Suppose $X_1, \ldots, X_i, \ldots, X_N$ are i.i.d random variables with mean $\mu$ and variance $\sigma^2$
  - $Y = \sum_{i=1}^{N} X_i$
  - $\mathbb{E}[Y] = \mathbb{E}[N]\mu$
    * $\mathbb{E}[Y] = \sum_{n=1}^{\infty} \mathbb{E}[\sum_{i=1}^{N} X_i | N = n] P[N = n]$
      $= \mu \times \sum_{n=1}^{\infty} nP[N = n] = \mathbb{E}[N]\mu$
  - $\mathbb{E}[Y^2] = \mathbb{E}[N]\mathbb{E}[X^2] + \mathbb{E}[N^2]\mu^2 - \mathbb{E}[N]\mu^2$
    * $\mathbb{E}[Y^2] = \sum_{n=1}^{\infty} \mathbb{E}[(\sum_{i=1}^{N} X_i)^2 | N = n] P[N = n] = \sum_{n=1}^{\infty} (n\mathbb{E}[X_i^2] + n(n-1)\mu^2) P[N = n]$
      $= \mathbb{E}[N]\mathbb{E}[X^2] + \mathbb{E}[N^2]\mu^2 - \mathbb{E}[N]\mu^2$
  - $Var(Y) = \mathbb{E}[N]\sigma^2 + Var(N)\mu^2$
- Expectation by $P[X > x]$
  - $\mathbb{E}[X] = \sum_x P[X > x]$, when $X$ is a non-negative discrete random variable
    * $\mathbb{E}[X] = \sum_{x=0}^{\infty} xP[X = x] = \sum_{x=0}^{\infty} \sum_{y=0}^{x-1} P[X = x] = \sum_{y=0}^{\infty} \sum_{x=y+1}^{\infty} P[X = x] = \sum_{y=0}^{\infty} P[X > y]$
  - $\mathbb{E}[X] = \int_0^{\infty} P[X > x]dx$, when $X$ is a non-negative continuous random variable
    * $\mathbb{E}[X] = \int_0^{\infty} x f_X(x)dx = \int_0^{\infty} \int_0^{x} f_X(x)dydx = \int_0^{\infty} \int_y^{\infty} f_X(x)dxdy = \int_0^{\infty} P[X > y]dy$

# 4 Inequality

- Markov Inequality

  Definition:
  - Suppose $X \geq 0$, then $P[X \geq \epsilon] \leq \frac{\mathbb{E}[X]}{\epsilon}$

  Proof:
  1. $\mathbb{E}[X] = \int_0^{\infty} x f_X(x) \geq \int_{\epsilon}^{\infty} x f_X(x) \geq \epsilon \int_{\epsilon}^{\infty} f_X(x) = \epsilon P[X \geq \epsilon]$
  2. $X(\omega) \geq \epsilon \mathbb{1}_{X(\omega) \geq \epsilon}, \forall \omega \in S$
     - Calculate expectation on both side.
     - $\mathbb{E}[X] \geq \epsilon P[X \geq \epsilon]$

  Property:
  - The equality happens when $P[X = k] = 0, \forall k \notin \{0, \epsilon\}$.
- Chebyshev Inequality

  Definition:
  - Suppose $m = \mathbb{E}[X], \sigma^2 = Var(X)$, then $P[|X - m| \geq \epsilon] \leq \frac{\sigma^2}{\epsilon^2}$

  Proof:
  - $P[|X - m| \geq \epsilon] = P[(X - m)^2 \geq \epsilon^2]$
  - $P[(X - m)^2 \geq \epsilon^2] \leq \frac{\mathbb{E}[(X-m)^2]}{\epsilon^2}$ (by Markov Inequality)

  Property:
  - The equality happens when $P[X = k] = 0, \forall k \notin \{m - \epsilon, m, m + \epsilon\}$.
  - Might be tighter than Markov Inequality since it requires $m, \sigma$
- Chernoff Inequality

  Definition:

- Suppose $X_1, \ldots, X_n$ are independent identically distributed Bernoulli random variable with probability $p$ and $X = \sum_{i=1}^{n} X_i$
- $P[X \geq \epsilon] \leq \frac{(pe^t+1-p)^n}{e^{t\epsilon}} \leq \frac{e^{np(e^t-1)}}{e^{t\epsilon}}$
  * $P[X \geq \epsilon] = P[e^{tX} \geq e^{t\epsilon}] \leq \frac{E[e^{tX}]}{e^{t\epsilon}} = \frac{(E[e^{tX_i}])^n}{e^{t\epsilon}} = \frac{(pe^t+1-p)^n}{e^{t\epsilon}} \leq \frac{e^{np(e^t-1)}}{e^{t\epsilon}}$
- $P[X \geq np(1+\epsilon)] \leq (\frac{e^\epsilon}{(1+\epsilon)^{1+\epsilon}})^{np} \leq \begin{cases} e^{\frac{-\epsilon^2 np}{3}} & \text{if } 0 \leq \epsilon \leq 1 \\ e^{\frac{-\epsilon^2 np}{(2+\epsilon)}} & \text{if } \epsilon \geq 1 \end{cases}$
  * Substitude $\epsilon$ with $np(1+\epsilon)$
  * Substitude $t$ with $\log(1+\epsilon)$
  * the last inequality is without proof
- $P[X \leq \epsilon] \leq \frac{(pe^t+1-p)^n}{e^{t\epsilon}} \leq \frac{e^{np(e^t-1)}}{e^{t\epsilon}}$
  * $P[X \leq \epsilon] = P[e^{-tX} \geq e^{-t\epsilon}] \leq \frac{E[e^{-tX}]}{e^{-t\epsilon}} = \frac{(E[e^{-tX_i}])^n}{e^{-t\epsilon}} = \frac{(pe^{-t}+1-p)^n}{e^{-t\epsilon}} \leq \frac{e^{np(e^{-t}-1)}}{e^{-t\epsilon}}$
- $P[X \leq np(1-\epsilon)] \leq (\frac{e^{-\epsilon}}{(1-\epsilon)^{1-\epsilon}})^{np} \leq e^{\frac{-\epsilon^2 np}{2}}$
  * Substitude $\epsilon$ with $np(1-\epsilon)$
  * Substitude $t$ with $-\log(1-\epsilon)$
  * the last inequality is without proof

- Chernoff/ Hoeffding Lemma

  Definition:
  - Suppose $X_1, \ldots, X_n$ are independent distributed random variable and $a_i \leq X_i \leq b_i$
  - Suppose $X = \sum_{i=1}^{n} X_i$ and $\mu = \mathbb{E}[X]$
  - $P[|X-\mu| \geq \epsilon] \leq 2e^{\frac{-2\epsilon^2}{\sum_{i=1}^{n}(b_i-a_i)^2}}$ without proof

- Application:
  - Balls in Bins

    Definition: Throw $n$ balls into $n$ bins, find bounds for the maximum number of balls in all bins
    * $P[$ maximum number of balls in all bins $\geq \epsilon]$
      $= P[\cup_{i=1}^{n}$ number of balls in $i$-th bin $\geq \epsilon]$
      $\leq n \times P[$ number of balls in one bin $\geq \epsilon]$
    * By Markov inequality:
      · $P[$ number of balls in one bin $\geq \epsilon] \leq \frac{1}{\epsilon} \rightarrow$ useless
    * By Chebyshev inequality:
      · $P[$ number of balls in one bin $\geq \epsilon] \leq \frac{(1-\frac{1}{n})}{\epsilon^2}$
      · $P[$ maximum number of balls in all bins $\geq n^{\frac{1}{2}+\epsilon}] \leq \frac{(1-\frac{1}{n})}{n^{2\epsilon}}$
      · when $n \rightarrow \infty$, the maximum number of balls should less than $n^{\frac{1}{2}+\epsilon}$
    * By Chernoff inequality:
      · $P[$ number of balls in one bin $\geq 2\log n] \leq \frac{e^{np(e^t-1)}}{n^{2t}}$
      · $P[$ maximum number of balls in all bins $\geq 2\log n] \leq \frac{e^{np(e^t-1)}}{n^{2t-1}}$
      · when $t$ is a constant $\geq 0.5$ and $n \rightarrow \infty$, the maximum number of balls should less than $2\log n$

# 5 Law of Large Numbers

- $\{X_i\}_{i=1}^{\infty}$ is a sequence of pairwise uncorrelated random variable with
  $\mathbb{E}[X_i] = m$, $Var(X_i) = \sigma_i^2$.
- $M_n = \frac{1}{n}\sum_{i=1}^{n} X_i$
- $M_n \rightarrow m$ almost surely, in mean square and in probability.

# 6 Memoryless

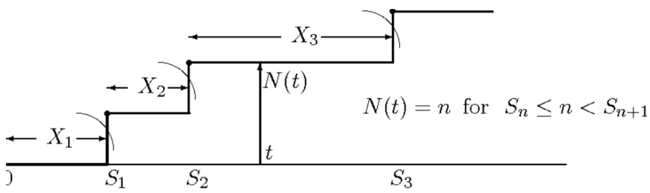
- Definition: $P[X > x_1 + x_2 | X > x_1] = P[X > x_2]$
- Property:
  - Exponential random variable is the only continuous memoryless random variable
  - Bernoulli random variable is the only discrete memoryless random variable

# 7 Famous Random Variable

- Poisson:
  $P[X = k] = \frac{\lambda^k}{k!}\exp(-\lambda)$
  $\mathbb{E}[X] = \sum_{k=0}^{\infty} k\frac{\lambda^k}{k!}\exp(-\lambda) = \sum_{k=0}^{\infty} \lambda\frac{\lambda^{k-1}}{(k-1)!}\exp(-\lambda) = \lambda$
  Interpretation:
  - Cut total time into infinite period in Binomial random variable, $n \to \infty, p \to \frac{\lambda}{n}$
  - $\to P[X = k] = \lim_{n\to\infty} \binom{n}{k}(\frac{\lambda}{n})^k(\frac{n-\lambda}{n})^{n-k} = \frac{\lambda^k}{k!}(1-\frac{\lambda}{n})^n = \frac{\lambda^k}{k!}\exp(-\lambda)$
- Erlang:
  $f_X(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!}, \forall x \in \mathbb{R}$
  $\mathbb{E}[X] = \frac{n}{\lambda}$
  Interpretation:
  - Suppose $X_1, X_2, ..., X_n$ are i.i.d exponential random variable with $\lambda$.
  - $X = \sum_{i=1}^{n} X_i$
  - Proof by induction:
    Suppose $n = 2$, $f_X(x) = \int_0^x \lambda e^{-\lambda t}\lambda e^{-\lambda(x-t)}dt = \lambda^2 x e^{-\lambda x}$

# 8 Stochastic Processes

- Stochastic Process: a collection of random variable
  Arrival Process: a sequence of arriving event in continuous time




  - $X_i$: the time between the $i$-th event and the $i-1$-th event
  - $S_i$: the time from start to $i$-th event
  - $N(t)$: the number of the arrived event at time $t$
  - $X$ and $S$ Relation:
    - $X_1 = S_1, X_i = S_i - S_{i-1}$
  - $N$ and $S$ Relation:
    - $N(t) < n \leftrightarrow S_{n+1} > t$
    - $N(t) \ge n \leftrightarrow S_n \le t$
    - $N(t) = n \leftrightarrow S_n \le t < S_{n+1}$
    - $N(t) = \max\{n : S_n \le t\}$

- Renewal Process: an arrival process with i.i.d $X_i$
  Delayed Renewal Process: the process becomes a renewal process after several arrivals
  $X_i$ Property
  * if $X_i$ is dependent on the interval states, then $X_i$ might be dependent on $X_{i-1} \to$ not renewal process

  $S_i$ Property
  * $P[\lim_{n\to\infty} S_n = \infty] = 1$
    Proof: $\lim_{n\to\infty} P[S_n = \infty] = \lim_{n\to\infty} P[\sum_{i=1}^{n} X_n = n \times \mathbb{E}[X_i]] = 1$
    Interpretation: infinite events do not take finite time

  $N(t)$ Property
  * for any $t, P[N(t) < \infty] = 1$
    Proof: $P[\lim_{n\to\infty} S_n = \infty] = 1 \to$ for any $t, P[\lim_{n\to\infty} S_{n+1} > t] = 1$
    Interpretation: infinite events do not take finite time
  * $P[\lim_{t\to\infty} N(t) \to \infty] = 1$
    Proof: if $P[\lim_{t\to\infty} N(t) = k] > 0 \to P[X_{k+1} = \infty] > 0$
    Interpretation: finite events do not take infinite time
  * $P[\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mathbb{E}[X_i]}] = 1$
    Proof: $P[\lim_{t\to\infty} \frac{N(t)}{S_{N(t)+1}} \leq \lim_{t\to\infty} \frac{N(t)}{t}] = 1$ and $P[\lim_{t\to\infty} \frac{N(t)}{S_{N(t)+1}} = \frac{1}{\mathbb{E}[X_i]}] = 1$
    $P[\lim_{t\to\infty} \frac{N(t)}{t} \leq \lim_{t\to\infty} \frac{N(t)}{S_{N(t)}}] = 1$ and $P[\lim_{t\to\infty} \frac{N(t)}{S_{N(t)}} = \frac{1}{\mathbb{E}[X_i]}] = 1$

  Inspection Paradox
  * $\mathbb{E}[X_{N(t)+1}] \geq \mathbb{E}[X_i]$: inspection paradox
    Interpretation:
    · $f_{X_{N(t)+1}}(x) = \lambda x f_{X_i}(x)$
    · when selecting $t$ with equal probability, we tend to choose $X_i$ with longer period
  * $P[\lim_{t\to\infty} \frac{1}{t} \int_0^t (S_{N(t)+1} - s) ds = \frac{\mathbb{E}[X_i^2]}{2\mathbb{E}[X_i]}] = 1$
    Proof:
    $P[\lim_{t\to\infty} \frac{1}{t} \sum_{i=i}^{N(t)} \frac{\mathbb{E}[X_i^2]}{2} \leq \lim_{t\to\infty} \frac{1}{t} \int_0^t (S_{N(t)+1} - s) ds] = 1$ and $P[\lim_{t\to\infty} \frac{1}{t} \sum_{i=i}^{N(t)} \frac{\mathbb{E}[X_i^2]}{2} = \frac{\mathbb{E}[X_i^2]}{2\mathbb{E}[X_i]}] = 1$
    $P[\lim_{t\to\infty} \frac{1}{t} \int_0^t (S_{N(t)+1} - s) ds \leq \lim_{t\to\infty} \frac{1}{t} \sum_{i=i}^{N(t)+1} \frac{\mathbb{E}[X_i^2]}{2}] = 1$ and $P[\lim_{t\to\infty} \frac{1}{t} \sum_{i=i}^{N(t)+1} \frac{\mathbb{E}[X_i^2]}{2} = \frac{\mathbb{E}[X_i^2]}{2\mathbb{E}[X_i]}] = 1$
  * $P[\lim_{t\to\infty} \frac{1}{t} \int_0^t (s - S_{N(t)}) ds = \frac{\mathbb{E}[X_i^2]}{2\mathbb{E}[X_i]}] = 1$
    Proof: similar to above
  * $P[\lim_{t\to\infty} \frac{1}{t} \int_0^t X_{N(t)} ds = \frac{\mathbb{E}[X_i^2]}{\mathbb{E}[X_i]}] = 1$
    Proof: $P[\lim_{t\to\infty} \frac{1}{t} \int_0^t X_{N(t)} ds = \lim_{t\to\infty} \frac{1}{t} \int_0^t (S_{N(t)+1} - S_{N(t)}) ds] = 1$
  * $\mathbb{E}[X_{N(t)+1}] = \frac{\mathbb{E}[X_i^2]}{\mathbb{E}[X_i]}$
    Proof: $P[\lim_{t\to\infty} \frac{1}{t} \int_0^t X_{N(t)} ds = \frac{\mathbb{E}[X_i^2]}{\mathbb{E}[X_i]}] = P[\mathbb{E}[X_{N(t)+1}] = \frac{\mathbb{E}[X_i^2]}{\mathbb{E}[X_i]}] = 1$

  Central Limit Theorem
  * $\mu = \mathbb{E}[X_i]$
  * $\sigma = \sqrt{Var(X_i)}$
  * $Z \sim$ Normal(0,1)
  * $\lim_{t\to\infty} P[N(t) \leq \frac{t}{\mu} + k\frac{\sigma\sqrt{t}}{\sqrt{\mu}^3}] = P[Z \leq k]$
    Proof:
    1. Suppose $n(t) = \frac{t}{\mu} + k\frac{\sigma\sqrt{t}}{\sqrt{\mu}^3}$
    2. $P[N(t) \geq n(t)] = P[S_{n(t)} \leq t] = P[\frac{S_{n(t)} - n\mu}{\sigma\sqrt{n}} \leq \frac{t - n\mu}{\sigma\sqrt{n}}]$.
    3. When $t \to \infty$, $\frac{t-n\mu}{\sigma\sqrt{n}} \to k$
    4. By law of large number, $\lim_{t\to\infty} P[\frac{S_{n(t)} - n\mu}{\sigma\sqrt{n}} \leq k] = P[Z \leq k]$

    Interpretation:

· $\frac{t}{\mu}$ is approximately the mean of $N(t)$

· $k\frac{\sigma\sqrt{t}}{\sqrt{\mu^3}}$ is $k\sigma\sqrt{n}$ after dividing by $\mu$, the ratio between $t$ and $N(t)$ and changing $n$ with $\frac{t}{\mu}$

Wald's Identity

* Stopping Times: a random variable $\tau$ s.t. $\{\tau = n\}$ is independent of $\{X_i\}_{i=n+1}^{\infty}$
* Stopping Condition: a condition to stop if we can consider $\tau = \min\{n : \text{ condition}(n) = \top\}$
* Example: $N(t) + 1$ is a stopping times and can be consider $N(t) + 1 = \min\{n : S_n > t\}$
* $\mathbb{E}[\sum_{i=1}^{\tau} X_i] = \mathbb{E}[\tau]\mathbb{E}[X_i]$ if $\mathbb{E}[X_i] < \infty$ and $\mathbb{E}[N] < \infty$
  Proof:
  1. $\mathbb{E}[\sum_{i=1}^{\tau} X_i] = \sum_{i=1}^{\infty} \mathbb{E}[X_i \times \mathbb{1}_{i\le\tau}]$ (by Fubin's Theorem without proof)
     (if $\mathbb{E}[X_i] < \infty$ and $\mathbb{E}[N] < \infty$)
  2. $\sum_{i=1}^{\infty} \mathbb{E}[X_i \times \mathbb{1}_{i\le\tau}] = \mathbb{E}[X_i] \sum_{i=1}^{\infty} \mathbb{E}[\mathbb{1}_{i\le\tau}]$ (by $P[\tau \ge i] = 1 - P[\tau < i]$ is independent of $X_i$)
  3. $\mathbb{E}[X_i] \sum_{i=1}^{\infty} \mathbb{E}[\mathbb{1}_{i\le\tau}] = \mathbb{E}[\tau]\mathbb{E}[X_i]$
* $\lim_{t\to\infty} \frac{\mathbb{E}[N(t)]}{t} = \frac{1}{\mathbb{E}[X_i]}$
  Proof:
  · Suppose $\mu = \mathbb{E}[X_i]$
  · $\frac{\mathbb{E}[N(t)]}{t} = \frac{\mathbb{E}[S_{N(t)+1}]}{t\times\mu} - \frac{1}{t}$ (by considering $N(t) + 1$ as the stopping time)
  · $\lim_{t\to\infty} \frac{\mathbb{E}[N(t)]}{t} \ge \frac{1}{\mu}$ (by $\mathbb{E}[S_{N(t)+1}] > t$)
  · Suppose $\hat{X}_n = \min\{X_n, T\}$, where $T$ is a constant
  · $\frac{\mathbb{E}[N(t)]}{t} \le \frac{\mathbb{E}[\hat{N}(t)]}{t} = \frac{\mathbb{E}[S_{\hat{N}(t)+1}]}{t\times\hat{\mu}} - \frac{1}{t} \le \frac{t+T}{t\times\hat{\mu}} - \frac{1}{t}$
  · $\lim_{n=\sqrt{t},t\to\infty} \frac{\mathbb{E}[N(t)]}{t} \le \frac{1}{\mu}$

Blackwell's Theorem

* $\mathbb{E}[N(t)] = F_{X_i}(t) + \int_0^t \mathbb{E}[N(t-x)] f_{X_i}(t)dt$
  Proof: $\mathbb{E}[N(t)] = \int_0^t \mathbb{E}[N(t)|X_1 = x] f_{X_1}(x)dx$
  $= \int_0^t \mathbb{E}[N(t-x) + 1] f_{X_1}(x)dx = F_{X_i}(t) + \int_0^t \mathbb{E}[N(t-x)] f_{X_i}(t)dt$
* $\mathcal{L}\{\mathbb{E}[N(t)]\}(s) = \frac{\mathcal{L}\{f_{X_i}\}(s)}{s(1-\mathcal{L}\{f_{X_i}\}(s))}$
  Proof: Laplace transform both sides
* Lattice/ Non-Lattice: $N(t)$ is lattice iff $X_i$ only takes on values that are $nd, n \in \mathbb{N}, d \in \mathbb{R}$
* For a non-lattice process: $\lim_{t\to\infty} \mathbb{E}[N(t+\delta) - N(t)] = \frac{\delta}{\mathbb{E}[X_i]}$, for any $\delta$
  Proof: Without Proof
  Interpretation: $\mathbb{E}[N(t)]$ will converge to be linear
* For a lattice process and period $d$: $\lim_{n\to\infty} \mathbb{E}[\#$ events at $t = nd] = \frac{d}{\mathbb{E}[X_i]}$
  Proof: Without Proof
  Interpretation: $\mathbb{E}[N(t)]$ will converge to be stairs with width $d$ and height $\frac{d}{\mathbb{E}[X_i]}$

– Renewal-Reward Process:

Definition

* A renewal process $N(t)$ and $\{R_i\}_{i=1}^{\infty}$ such that $(X_i, R_i)$ are i.i.d.
  ($X_i, R_j, i \ne j$ are independent, but $X_i, R_i$ might be dependent)

Property

* $P[\lim_{t\to\infty} \frac{1}{t} \sum_{i=1}^{N(t)} R_i = \frac{\mathbb{E}[R_i]}{\mathbb{E}[X_i]}] = 1$
  Proof: $P[\lim_{t\to\infty} \frac{1}{t} \sum_{i=1}^{N(t)} R_i = \lim_{t\to\infty} \sum_{i=1}^{N(t)} \frac{R_i}{N(t)} \times \lim_{t\to\infty} \frac{N(t)}{t}] = 1$

– Poisson Process: a renewal process with $X_i \sim$ Exponential($\lambda$)

$S_i$ Property

* $S_i$ is an Erlang random variable
  Erlang is the sum of the Exponential random variables
* Joint Distribution $f_{S_1,\dots,S_n}(s_1, \dots, s_n) = \lambda^n e^{-\lambda s_n}$
  Prove by induction.
  Induce by $f_{S_1,\dots,S_n}(s_1, \dots, s_n) = f_{S_1,\dots,S_{n-1}}(s_1, \dots, s_{n-1}) \times f_{S_n|S_1,\dots,S_{n-1}}(s_n, s_1, \dots, s_{n-1})$

$N(t)$ Property

* $N(t) \sim \text{Poisson}(\lambda t), P[N(t) = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$
  Prove by $P[N(t) = n] = P[S_n \leq t \text{ and } S_{n+1} > t]$
* Conditioned on $N(t) = n$, the set of arrival times $\{s_1, \ldots, s_n\}$ have the same distribution with a set of $n$ sorted i.i.d. Uniform$(0, t)$ random variables
  Prove by $f_{S_1,\ldots,S_n|N(t)}(s_1, \ldots, s_n, n) = \frac{f_{S_1,\ldots,S_n}(s_1,\ldots,s_n)P[X_{n+1}>t-s_n]}{P[N(t)=n]} = \frac{n!}{t^n}$

Property

* $Z$ is the interval from $t$ to the first arrival $\rightarrow Z$ is exponential random variable with same $\lambda$ and independent of $N(t)$ and the arrival time before $t$
  Proof:
  $P[Z > z] = \sum_{n=0}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} P[Z > z|N(t) = n, S_1 = s_1, \ldots, S_n = s_n]ds_1 \ldots ds_n$
  $= \sum_{n=0}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} P[X_{n+1} > z + t - s_n|N(t) = n, S_1 = s_1, \ldots, S_n = s_n]ds_1 \ldots ds_n$
  $= \sum_{n=0}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} P[X_{n+1} > z + t - s_n|X_{n+1} > t - s_n]ds_1 \ldots ds_n = e^{-\lambda z}$
* Stationary Increments: $N(t_1 + t_2) - N(t_1)$ and $N(t_2)$ share the same distribution
  Without Proof
* Independent Increments: $\forall 0 < t_1 < t_2 < \ldots, t_k, N(t_1), N(t_2) - N(t_1), \ldots$ are independent
  Without Proof
* Any arrival process with stationary and independent increments must be a Poisson process
  Without Proof

Exercise

* $\mathbb{E}[S_i|N(t) = n] = \frac{t \times i}{n+1}$
  * $\mathbb{E}[S_i|N(t) = n] = i \times \mathbb{E}[X_1|N(t) = n] = i \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} s_1 \times \frac{n!}{t^n} ds_1 \ldots ds_{n-1}ds_n = \frac{t \times i}{n+1}$
* $\mathbb{E}[\sum_{i=0}^{N(t)} S_i] = \frac{\lambda t^2}{2}$
  * $\mathbb{E}[\sum_{i=0}^{N(t)} S_i] = \sum_{n=0}^{\infty} \mathbb{E}[\sum_{i=0}^{n} S_i|N(t) = n]P[N(t) = n]$
    $= \sum_{n=0}^{\infty} \frac{nt}{2} P[N(t) = n] = \frac{\lambda t^2}{2}$

2D Poisson Process

* Definition:
  * For any region $R$: number of points in $R$ is a Poisson random variable
  * number of points in the non-overlapping region is independent

Combining Poisson Process

* $N^1(t), N^2(t)$ are two independent Poisson process with $\lambda_1, \lambda_2$
* $X_i$ is the first arrival of $X_i^1, X_i^2$
* Property
  * $X_i$ is independent of $\{X_i^1 < X_i^2\}$ and $\{X_i^1 > X_i^2\}$
    Proof: $P[X_1^1 < X_1^2] = \frac{\lambda_1}{\lambda_1+\lambda_2}$
    $P[X_1 > x] = P[X_1^1 > x, X_1^2 > x] = e^{-(\lambda_1+\lambda_2)x}$
    $P[X_1 > x, X_1^1 < X_1^2] = P[X_1 > x]P[X_1^1 < X_1^2]$
  * $X_i$ is a Poisson Process with $\lambda = \lambda_1 + \lambda_2$
  * $\min(X_1, X_2)$ is an exponential random variable with $\lambda = \lambda_1 + \lambda_2$

Splitting Poisson Process

* $N^1(t), N^2(t)$ are two independent Poisson process with $\lambda_1, \lambda_2$
* $N(t)$ is a random process with $\lambda = \lambda_1 + \lambda_2$
  * $N^{1*}(t)$ is the process of the first event
    when $N(t)$ arrives consider it as first event with probability $\frac{\lambda_1}{\lambda_1+\lambda_2}$
  * $N^{2*}(t)$ is the process of the second event
    when $N(t)$ arrives consider it as second event with probability $\frac{\lambda_2}{\lambda_1+\lambda_2}$
* $N^i(t)$ and $N^{i*}(t)$ share the same distribution
* Proof:
  * $B_n(k)$ is a Binomial random variable with $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$
  * $P[N^{1*}(t) = m, N^{2*}(t) = n] = P[N(t) = m + n, B_{m+n}(m)] = P[N^1(t) = m, N^2(t) = n]$

Compound Poisson Process

* $N(t)$ is a Poisson Process
* $A_n$ is a sequence of cost
* $A(t) = \sum_{n=0}^{N(t)} A_n$ is the summation of cost over Poisson Process

Non-Homogeneous Poisson Process

* $N(t) - N(s) \sim$ Poisson($\int_s^t \lambda(x)dx$

Queuing Theory

* Definition: $Arrival\_Process/Service\_Process/number\_of\_services$
  - $M$: memoryless (Poisson) process
  - $D$: deterministic process
  - $G$: general renewal process
* $T$: the random variable of the processing time for each customer
* $Y(t)$: number of cutomers in the service
  - $Y(t) \sim$ Poisson($\lambda \int_0^t P[T > x]dx$)
  - Proof:
    Consider $Y(t)$ is a splitting Poisson Process. Since the distribution for the arrival given $N(t)$ is universal, the probability the arrival is still in service: $\frac{1}{t}\int_0^t P[T > t-x]dx = \frac{1}{t}\int_0^t P[T > x]dx$

# 9 Markov Chain

- Definition
  - Model with states and transition probability matrix
  - States: $\{X_n\}_{n=0}^{\infty}$
  - Transition Probability Matrix: $[P]_{ij} = P[X_{n+1} = j|X_n = i]$
- Terminology
  - $p^n = [P[X_n = 0], P[X_n = 1], \ldots]^T$: distribution at step $n$
  - $T_i = \min\{n \geq 1 : X_n = i\}$: a random variable of the minimum time step to go to state $i$
  - $f_{ij} = P[T_j < \infty|X_0 = i]$: the probability of starting at $i$ and ever reaching $j$
  - $\mu_{ij} = \mathbb{E}[T_j|X_0 = i]$
  - $i \to j$ iff $f_{ij} > 0$: $j$ is reachable from $i$ with probability greater than 0
  - $N_i(n)$: number of visits to $i$ by time $n$
  - Irreducible: $i \leftrightarrow j, \forall$ states $i, j$
  - aperiodic: period of $X_n = i$ is 1, $\forall$ states $i$
- Property
  - Consider a given distribution as an event $\tau : [P[X_n = 0|\tau], P[X_n = 1|\tau], \ldots]^T$
  - Updating distribution
    * $p^n = p^0 P^n$
  - Markovian: transition probability depend only on current state
    * $P[X_{n+1} = j|X_n = i, \ldots, X_0 = x_0] = [P]_{ij}$
  - Stationary Distribution: $p$ s.t. if $p^n = p \to p^{n+1} = p$

    Property from renewal process
    * consider $X_n = j$ as a event $\to$ Markov Chain becomes a delayed renewal process
    * If $i \leftrightarrow j$ and the model starts from $i$, then following holds
    * $P[\lim_{n\to\infty} \frac{N_j(n)}{n} = \frac{1}{\mu_{jj}}] = 1$
    * $\lim_{n\to\infty} \frac{\mathbb{E}[N_j(n)]}{n} = \frac{1}{\mu_{jj}}$
    * if the period of $X_n = j$ is $d \to \lim_{n\to\infty} p_j^{nd} = \frac{d}{\mu_{jj}}$

Theorem of an irreducible, aperiodic Markov Chain

* Either
  - All states have $\mu_{ii} = \infty$
  - All states have $\mu_{ii} < \infty$ and $p_i = \frac{1}{\mu_{ii}}$ is the unique stationary distribution
* Proof
  - From if the period of $X_n = j$ is $d \to \lim_{n\to\infty} p_j^{nd} = \frac{d}{\mu_{jj}}$
    Proof: $\lim_{n\to\infty} p_j^{nd} = \lim_{n\to\infty} \mathbb{E}[\#\text{ events at } nd]$

Theorem of an irreducible, aperiodic Markov Chain

* All states have $\mu_{ii} < \infty$ and $p_i = \frac{1}{\mu_{ii}}$ is the unique stationary distribution

$p$ can be calculated as the eigenvector corresponds to eigenvalue 1 of $P^T$

- Detailed Balance

  Definition:

  * Given a distribution $\pi$
  * $\pi_i P_{ij} = \pi_j P_{ji}, \forall i, j$

  Property:

  * distribution $\pi$ satisfying Detailed Balance is the stationary distribution $p$
  * symmetric transition probability matrix $\to$ uniform stationary distribution

- Reversible

  Definition: A Markov Chain with stationary distribution $p$ is reversible if it satisfies detailed balance

  Interpretation

  * Transitions forward and backward in the stationary distribution have the same probability
  * $P[X_{n+1} = j | X_n = i] = P_{ij}$
  * $P[X_{n-1} = j | X_n = i] = \frac{P[X_{n-1}=j, X_n=i]}{P[X_n=i]} = \frac{p_j P_{ji}}{p_i} = P_{ij}$

- Metropolis Update Rule

  Definition

  * Given a Markov Chain and distribution $p'$, find $P'$ such that $p'$ is the stationary distribution

  Procedure

  * For each pair $(i, j)$, $P'_{ij} = P_{ij} \times \min\{1, \frac{p'_j P_{ji}}{p'_i P_{ij}}\}$
  * construct self loop to satisfy $\sum_j P'_{ij} = 1$

  Proof

  * To satisfy detailed balance, for each pair $(i, j)$, we should set $p'_i P'_{ij} = \min\{p'_i P_{ij}, p'_j P_{ji}\}$

- Distance between Probability Measure

  Definition:

  * Total Variation Distance between $P_1$ and $P_2$ is: $d_{TV}(P_1, P_2) = \frac{1}{2}\sum_\omega |P_1[\omega] - P_2[\omega]|$

  Interpretation:

  * consider the distributions as events $\tau_1, \tau_2$
  * $P_i[\omega] = P[\omega|\tau_i]$
  * $d_{TV}(P_1, P_2) = \frac{1}{2}\sum_\omega |P[\omega|\tau_1] - P[\omega|\tau_2]| = \sum_\omega |P[\omega \wedge \tau_1] - P[\omega \wedge \tau_2]|$

- Mixing Time

  Definition

  * Mixing time $\tau$ is the least $t$ such that for all initial state $p^0$, $d_{TV}(p, p^0 P^t) \le \frac{1}{2e}$

  Interpretation

  * the factor $\frac{1}{2e}$ is set such that $d_{TV}(p, p^0 P^t) \le \epsilon$ if $t \ge \tau \times \log(\frac{1}{\epsilon})$
    Without proof

- Example

  Random Walk on Graph

* Definition: move from vertex $i$ to vertex $j$ with probability $P_{ij} = \begin{cases} 0 & \text{if } (i,j) \notin E \\ \frac{1}{\text{degree}(i)} & \text{if } (i,j) \in E \end{cases}$
* Distribution $\pi$, $\pi_i = \frac{\text{degree}(x)}{2|E|}$ satisfies detailed balance
* If we want stationary distribution to be uniform $\rightarrow P'_{ij} = \begin{cases} \frac{1}{\text{degree}(i)} & \text{if degree}(i) \geq \text{degree}(j) \\ \frac{1}{\text{degree}(j)} & \text{if degree}(i) < \text{degree}(j) \end{cases}$

Random graph coloring

* Given a graph with $V$ vertices, maximum degree $\Delta$ and $q$ colors, to color each vertex one color such that adjacent vertex do not share the same color
* Assume $q > 4\Delta$
* Markov Chain Transition:
  · Pick random vertex and random color, if the color is changeable then change
* Property
  · Aperiodic: there exist self loops
  · Symmetric: symmetric transition
  · Irreducible
* Mixing time is $O(V \log V)$
  Proof:
  · Assume $X$ is a event s.t. Markov Chain starts with any valid coloring and $Y$ is a event s.t. Markov Chain starts with uniform distribution
  · Apply same transition on both $X$ and $Y$
  · $D_n$ is a random variable for the number of vertices in different colors in $X$ and $Y$ at time $n$
  · Good moves: number of vertices in different colors decrease $\geq D_n \times (q - 2\Delta) \geq (2\Delta + 1)D_n$ (vertices with different colors $\times$ color that is different with any adjacent color in $X$ and $Y$)
  · Bad moves: number of vertices in different colors increase $\leq (D_n\Delta) \times 2$ (vertices adjacent to different colors vertices $\times$ color of the differen colors vertices)
  · $\mathbb{E}[D_{n+1} - D_n] \leq V(1 - \frac{1}{qV})^n$
  · $\mathbb{E}[D_n] \leq V(1 - \frac{1}{qV})^n$
  · $P[D_n \geq 1] \leq V(1 - \frac{1}{qV})^n$

- Hidden Markov Chain
  - Definition: output is a function of the state
  - Interpretation: if the model is not markovian, then reformulate the model as a hidden markov chain by complicating the states and rendering the output as a function of the state

# 10 Continuous Markov Chain

- Interpretation
  - $v_i$: coefficient of exponential distribution, where time in state $i$ before next step is $\sim$ Exponential$(v_i)$
- Definition
  - Model with states and transition rate matrix
  - States: $X(t), \forall 0 \leq t < \infty$
  - Transition Probability Matrix $R$
- $P_{ij}(t)$
  - Definition: $P_{ij}(t) = P[X(t) = j | X(0) = i]$
  - Chapman-Kolmogorov Equation
    * Definition: $P(s + t) = P(s) \times P(t)$
    * Proof

- $P_{ij}(s+t) = P[X(s+t) = j|X(0) = i]$
  $= \sum_k P[X(s+t) = j|X(s) = k, X(0) = i]P[X(s) = k|X(0) = i]$
  $= \sum_k P[X(s+t) = j|X(s) = k]P[X(s) = k|X(0) = i] = \sum_k P_{kj}(t)P_{ik}(s)$

- Kolmogorov's Differential Equation
  - Forward: $\frac{dP(t)}{dt} = P(t)R$
    Interpretation:
    - Change of distribution at $t$ equals the distribution at $t \times R$

    Proof:
    - $\frac{dP(t)}{dt} = \lim_{\delta\to 0} \frac{P(t+\delta)-P(t)}{\delta} = P(t)\lim_{\delta\to 0} \frac{P(\delta)-P(0)}{\delta} = P(t)R$
  - Backward: $\frac{dP(t)}{dt} = RP(t)$
    Interpretation:
    - Change of distribution at $t$ equals the distribution at $t = 0 \times P(t)$

    Proof:
    - $\frac{dP(t)}{dt} = \lim_{\delta\to 0} \frac{P(t+\delta)-P(t)}{\delta} = \lim_{\delta\to 0} \frac{P(\delta)-P(0)}{\delta}P(t) = RP(t)$
  - Solution: $P(t) = e^{Rt}$

- $R$
  - Definition:
    - $R_{ij} = \frac{dP_{ij}(t)}{dt}|_{t=0}$
    - $R_{ij} = \begin{cases} -v_i & \text{if } i = j \\ v_iP_{ij} & \text{if } i \neq j \end{cases}$ (if there is no self-transition)
  - Interpretation
    - $\pi R$ is the change of distribution of $\pi$ (by Kolmogorov's Differential Equation)
    - simulation by transition from state $i$ to $j$ when $e^{-R_{ij}t}$ event arrives
      Proof
      - $\frac{dP_{ii}(t)}{dt} = R_{ii}P_{ii}(t) \to P_{ii}(t) = e^{-R_{ii}t}$
      - simulate the transition out of state $i$ by $e^{-R_{ii}t}$ and transition to $j$ state by probability $\frac{R_{ij}}{R_{ii}}$ is the same as transition from state $i$ to $j$ when $e^{-R_{ij}t}$ event arrives

      Property
      - Continuous Markov Chain with same $R$ are of the same functionality
  - Property:
    - $\sum_j R_{ij} = 0$: sum of element is a row of $R$ is 0

- Property
  - Self Transition:
    - Since $R$ defines the Markov Chain, we can modify $v_i$ to conduct self transition without changing $R$
  - Uniformization:
    - Since $R$ defines the Markov Chain, we can modify $v_i$ such that $v_i$ are the same for all states without changing $R$
  - Stationary Distribution: $p$ s.t. $pR = 0, pe^{Rt} = p$
    Interpretation:
    - $\frac{dpP(t)}{dt} = p\frac{dP(t)}{dt} = pRP(t) = 0$
    - $p$ is the eigenvector of eigenvalue 0 of $R$, then $p$ is the eigenvector of eigenvalue 1 of $e^{Rt} \to$ the distribution would not change, if start with $p$

    Trick:
    - cluster states such that every state in the cluster share the same $R_{ij}$ to calculate the stationary distribution of the cluster
    - assume distribution is independent of the cluster and check $pR = 0$ after the calculation
  - Poisson process is a special case of Continuous Markov Chain

    * $v_i = \lambda, \forall i$
    * $i$-th state transition to $i+1$-th state
  - Exploding process: only if $v_i \to \infty$
    * exploding process: traverse infinite states in finite time

# 11 Martingales

- Definition
  - Discrete
    General Discrete Martingales
    * $\{Z_i\}_{i=0}^{\infty}$ such that
      1. $\mathbb{E}[|Z_n|] < \infty$
      2. $\mathbb{E}[Z_{n+1}|Z_0, \ldots, Z_n] = Z_n$
         · sub-martingales: $\mathbb{E}[Z_{n+1}|Z_0, \ldots, Z_n] \geq Z_n$
         · super-martingales: $\mathbb{E}[Z_{n+1}|Z_0, \ldots, Z_n] \leq Z_n$

    Discrete Martingales with respect to $X_i$
    * $\{Z_i\}_{i=0}^{\infty}$ such that
      1. $\mathbb{E}[|Z_n|] < \infty$
      2. $\mathbb{E}[Z_{n+1}|X_0, \ldots, X_n] = Z_n$
         · sub-martingales: $\mathbb{E}[Z_{n+1}|X_0, \ldots, X_n] \geq Z_n$
         · super-martingales: $\mathbb{E}[Z_{n+1}|X_0, \ldots, X_n] \leq Z_n$
    * $\mathbb{E}[Z_{n+1}|X_0, \ldots, X_n] = Z_n$ implies $\mathbb{E}[Z_{n+1}|Z_0, \ldots, Z_n] = Z_n$
      · $Z_n$ is a function of $X_0, \ldots, X_n$
      · $\mathbb{E}[Z_{n+1}|Z_0, \ldots, Z_n] = \mathbb{E}[\mathbb{E}[Z_{n+1}|X_0, \ldots, X_n, Z_0, \ldots, Z_n]|Z_0, \ldots, Z_n]$
        $= \mathbb{E}[\mathbb{E}[Z_{n+1}|X_0, \ldots, X_n]|Z_0, \ldots, Z_n] = \mathbb{E}[Z_n|Z_0, \ldots, Z_n] = Z_n$
  - Continuous Martingales with respect to $N(t)$
    * $Y(t)$ such that
      1. $\mathbb{E}[|Y(t)|] < \infty$
      2. $\mathbb{E}[Y(t)|\{N(s)|0 \leq s \leq \tau\}] = Y(\tau), \forall \tau \leq t$
         · sub-martingales: $\mathbb{E}[Y(t)|\{N(s)|0 \leq s \leq \tau\}] \geq Y(\tau), \forall \tau \leq t$
         · super-martingales: $\mathbb{E}[Y(t)|\{N(s)|0 \leq s \leq \tau\}] \leq Y(\tau), \forall \tau \leq t$
- Property
  - $\mathbb{E}[Z_n] = \mathbb{E}[Z_1]$
    Proof: $\mathbb{E}[Z_{n+1} - Z_n] = \mathbb{E}[\mathbb{E}[Z_{n+1} - Z_n|Z_0, \ldots, Z_n]] = 0$
  - $\mathbb{E}[Z_n|\{Z_i|i \in S\}] = Z_{\max_{i \in S} i}$, where $\forall i \in S, i < n$
    Proof: $\mathbb{E}[Z_n|Z_i] = \mathbb{E}[\mathbb{E}[Z_n|Z_0, \ldots, Z_{n-1}]|Z_i] = \mathbb{E}[Z_{n-1}|Z_i]$
  - Azuma's Inequality
    * $\mu = \mathbb{E}[Z_0]$
    * $-a_i \leq Z_i - Z_{i-1} \leq b_i$
    * $P[|Z_n - \mu| \geq \delta] \leq 2e^{-\frac{2\delta^2}{\sum_{i=1}^{n}(b_i + a_i)^2}}$
  - Kolmogorov's sub-martingales inequality
    * $P[\sup_{n \geq 1} Z_n \geq a] \leq \frac{\mathbb{E}[Z_1]}{a}$
  - Martingales Stopping Theorem
    * Stopping Times: a random variable $\tau$ s.t. $\{\tau = n\}$ is independent of $\{X_i\}_{i=n+1}^{\infty}$
    * Stopping Condition: a condition to stop if we can consider $\tau = \min\{n : \text{ condition}(n) = \top\}$
    * $\mathbb{E}[Z_\tau] = \mathbb{E}[Z_0]$ if the either of the following holds
      1. $P[\tau \leq k] = 1$

2. $P[\max_{i \le \tau} |Z_\tau| \le k] = 1$
3. $\mathbb{E}[\tau] < k$ and $\mathbb{E}[|Z_{n+1} - Z_n||Z_0, \ldots, Z_n] < k$

- Application for generating Martingales
  - Sum of iid. random variables
    * $\{X_i\}_{i=1}^{\infty}$ are iid. random variables
    * $Z_n = \sum_{i=1}^{n} X_i - n\mathbb{E}[X_i]$ is a martingales.
    * Proof: $\mathbb{E}[Z_{n+1}|Z_0, \ldots, Z_n] = \mathbb{E}[Z_n + X_{n+1} - \mathbb{E}[X_i]|Z_0, \ldots, Z_n] = Z_n$
  - Squre of sum of iid. random variables
    * $\{X_i\}_{i=1}^{\infty}$ are iid. random variables and $\mathbb{E}[X_i] = 0$
    * $Z_n = (\sum_{i=1}^{n} X_i)^2 - n\mathbb{E}[X_i^2]$ is a martingales.
    * Proof: $\mathbb{E}[Z_{n+1}|Z_0, \ldots, Z_n] = \mathbb{E}[Z_n + X_{n+1}^2 + 2X_{n+1}(\sum_{i=1}^{n} X_i) - \mathbb{E}[X_i^2]|Z_0, \ldots, Z_n] = Z_n$
  - Product of iid. random variables
    * $\{X_i\}_{i=1}^{\infty}$ are iid. random variables
    * $Z_n = \frac{\prod_{i=1}^{n} X_i}{\mathbb{E}[X_i]^n}$ is a martingales.
    * Proof: $\mathbb{E}[Z_{n+1}|Z_0, \ldots, Z_n] = \mathbb{E}[Z_n(\frac{X_{n+1}}{\mathbb{E}[X_i]})|Z_0, \ldots, Z_n] = Z_n$
  - Poisson Process
    * $N(t)$ is a poisson process
    * $Y(t) = N(t) - \lambda t$ is a martingales.
    * Proof: $\mathbb{E}[Y(t)|\{N(s)|0 \le s \le \tau\}] = \mathbb{E}[Y(\tau) + Y(t) - Y(\tau)|\{N(s)|0 \le s \le \tau\}]$
      $= Y(\tau) + \mathbb{E}[N(t) - N(\tau) + \lambda(t - \tau)|\{N(s)|0 \le s \le \tau\}] = Y(\tau)$
  - Doob-type Martingales
    * $X, \{Y_i\}_{i=1}^{\infty}$ are random variables
    * $Z_n = \mathbb{E}[X|Y_1, Y_2, \ldots, Y_n]$ is a martingales
    * Proof: $\mathbb{E}[Z_{n+1}|Y_1, \ldots, Y_n] = \mathbb{E}[\mathbb{E}[X|Y_1, Y_2, \ldots, Y_n, Y_{n+1}]|Y_1, Y_2, \ldots, Y_n]$
      $= \mathbb{E}[X|Y_1, Y_2, \ldots, Y_n] = Z_n$

- Example

  Random Walk
  - $X_i = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1-p \end{cases}$
  - Symmetric Random Walk
    * $p = 0.5$
    * $\tau = \min\{i| \sum_{i=0}^{n} X_i \in \{-a, b\}\}$
    * $Z_n = \sum_{i=0}^{n} X_i$, by second rule of Martingales Stopping Theorem: $\mathbb{E}[Z_\tau] = 0$
      $\rightarrow P[Z_\tau \text{ at } a] = \frac{b}{a+b}, P[Z_\tau \text{ at } b] = \frac{a}{a+b}$
  - Unbiased Random Walk
    * $\tau = \min\{i| \sum_{i=0}^{n} X_i \in \{-a, b\}\}$
    * $Z_n = (\frac{1-p}{p})^{\sum_{i=0}^{n} X_i}$, by second rule of Martingales Stopping Theorem: $\mathbb{E}[Z_\tau] = 0$
      $P[Z_\tau \text{ at } a] = \frac{(\frac{1-p}{p})^b - 1}{(\frac{1-p}{p})^b - (\frac{1-p}{p})^{-a}}, P[Z_\tau \text{ at } b] = \frac{1 - (\frac{1-p}{p})^{-a}}{(\frac{1-p}{p})^b - (\frac{1-p}{p})^{-a}}$
    * $Z_n = \sum_{i=0}^{n} X_i - n\mathbb{E}[X_0]$, by third rule of Martingales Stopping Theorem: $\mathbb{E}[Z_\tau] = 0$
      $\rightarrow \mathbb{E}[\tau] = \frac{\mathbb{E}[\sum_{i=0}^{\tau} X_i]}{\mathbb{E}[X_0]}$