

Stochastic Processes

Kevin Chang

March 1, 2022

1 Laplace Transform

- $\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}dt$
- Property
 - $tf(t) \leftrightarrow -F'(s)$
 - $\frac{f(t)}{t} \leftrightarrow \int_s^\infty F(\sigma)d\sigma$
 - $f'(t) \leftrightarrow sF(s) - f(0^-)$
 - $\int_0^t f(\tau)d\tau \leftrightarrow \frac{F(s)}{s}$
 - $e^{at}f(t) \leftrightarrow F(s-a)$
 - $f(t-a)u(t-a) \leftrightarrow e^{-as}F(s)$

2 Moment Generating Function

- Moment Generating Function: $\mathbb{E}[e^{tX}]$
 - Property:
 - * $\mathbb{E}[e^{tX}] = \int_{-\infty}^\infty e^{tx}f_X(x)dx$
 - * $\mathbb{E}[e^{tX}] = \sum_{k=0}^\infty E[X^k] \frac{t^k}{k!}$
 - $e^{tx} = \sum_{k=0}^\infty \frac{(tx)^k}{k!}$
 - $E[e^{tX}] = E[\sum_{k=0}^\infty \frac{(tX)^k}{k!}] = \sum_{k=0}^\infty E[X^k] \frac{t^k}{k!}$
 - * $\frac{d\mathbb{E}[e^{tX}]}{dt} = \mathbb{E}[X]$
 - * $\mathbb{E}[e^{t(aX+b)}] = e^{tb}\mathbb{E}[e^{taX}]$
 - * Not all random variables have Moment generating function
- Characteristic Function: $\mathbb{E}[e^{itX}]$
 - Property:
 - * All random variables have Moment generating function
- Joint Moment Generating Function: $G(x, y) = \mathbb{E}[e^{xX}e^{yY}]$
- Property:
 - (Joint) moment generating function uniquely determines the (joint) CDF
- Example
 - Trapped miner's random walk
 - * Miner has probability of $\frac{1}{3}$ to waste 3 hours in vain, $\frac{1}{3}$ to waste 5 hours in vain, and $\frac{1}{3}$ to spend 2 hours to go out of the mine.
 - * X is the random variables of the hours to go out of the mine
 - * Y_i is the random variables of the hours for the i -th action.
 - * $\mathbb{E}[e^{tX}] = \mathbb{E}[e^{tX}|Y_1=2] + \mathbb{E}[e^{tX}|Y_1=3] + \mathbb{E}[e^{tX}|Y_1=5]$
 - $= \mathbb{E}[e^{2t}] + \mathbb{E}[e^{t(X+3)}] + \mathbb{E}[e^{t(X+5)}]$
 - * Find expectation and variance by joint moment generating function

3 Expectation

- N i.i.d. events, when N is a random variable
 - Suppose N is a integer random variable
 - Suppose $X_1, \dots, X_i, \dots, X_N$ are i.i.d random variables with mean μ and variance σ^2
 - $Y = \sum_{i=1}^N X_i$
 - $\mathbb{E}[Y] = \mathbb{E}[N]\mu$
 - * $\mathbb{E}[Y] = \sum_{n=1}^{\infty} \mathbb{E}[\sum_{i=1}^N X_i | N = n] P[N = n]$
 $= \mu \times \sum_{n=1}^{\infty} n P[N = n] = \mathbb{E}[N]\mu$
 - $\mathbb{E}[Y^2] = \mathbb{E}[N]\mathbb{E}[X^2] + \mathbb{E}[N^2]\mu^2 - \mathbb{E}[N]\mu^2$
 - * $\mathbb{E}[Y^2] = \sum_{n=1}^{\infty} \mathbb{E}[(\sum_{i=1}^N X_i)^2 | N = n] P[N = n] = \sum_{n=1}^{\infty} (n\mathbb{E}[X_i^2] + n(n-1)\mu^2) P[N = n]$
 $= \mathbb{E}[N]\mathbb{E}[X^2] + \mathbb{E}[N^2]\mu^2 - \mathbb{E}[N]\mu^2$
 - $\text{Var}(Y) = \mathbb{E}[N]\sigma^2 + \text{Var}(N)\mu^2$
- Expectation by $P[X > x]$
 - $\mathbb{E}[X] = \sum_x P[X > x]$, when X is a non-negative discrete random variable
 - * $\mathbb{E}[X] = \sum_{x=0}^{\infty} x P[X = x] = \sum_{x=0}^{\infty} \sum_{y=0}^{x-1} P[X = x] = \sum_{y=0}^{\infty} \sum_{x=y+1}^{\infty} P[X = x] = \sum_{y=0}^{\infty} P[X > y]$
 - $\mathbb{E}[X] = \int_0^{\infty} P[X > x] dx$, when X is a non-negative continuous random variable
 - * $\mathbb{E}[X] = \int_0^{\infty} x f_X(x) dx = \int_0^{\infty} \int_0^x f_X(x) dy dx = \int_0^{\infty} \int_y^{\infty} f_X(x) dx dy = \int_0^{\infty} P[X > y] dy$

4 Inequality

- Markov Inequality

Definition:

- Suppose $X \geq 0$, then $P[X \geq \epsilon] \leq \frac{\mathbb{E}[X]}{\epsilon}$

Proof:

1. $\mathbb{E}[X] = \int_0^{\infty} x f_X(x) dx \geq \int_{\epsilon}^{\infty} x f_X(x) dx \geq \epsilon \int_{\epsilon}^{\infty} f_X(x) dx = \epsilon P[X \geq \epsilon]$
2. $X(\omega) \geq \epsilon \mathbb{1}_{X(\omega) \geq \epsilon}, \forall \omega \in S$
 - Calculate expectation on both side.
 - $\mathbb{E}[X] \geq \epsilon P[X \geq \epsilon]$

Property:

- The equality happens when $P[X = k] = 0, \forall k \notin \{0, \epsilon\}$.

- Chebyshev Inequality

Definition:

- Suppose $m = \mathbb{E}[X], \sigma^2 = \text{Var}(X)$, then $P[|X - m| \geq \epsilon] \leq \frac{\sigma^2}{\epsilon^2}$

Proof:

- $P[|X - m| \geq \epsilon] = P[(X - m)^2 \geq \epsilon^2]$
- $P[(X - m)^2 \geq \epsilon^2] \leq \frac{\mathbb{E}[(X - m)^2]}{\epsilon^2}$ (by Markov Inequality)

Property:

- The equality happens when $P[X = k] = 0, \forall k \notin \{m - \epsilon, m, m + \epsilon\}$.
- Might be tighter than Markov Inequality since it requires m, σ

- Chernoff Inequality

Definition:

- Suppose X_1, \dots, X_n are independent identically distributed Bernoulli random variable with probability p and $X = \sum_{i=1}^n X_i$
- $P[X \geq \epsilon] \leq \frac{(pe^t + 1 - p)^n}{e^{t\epsilon}} \leq \frac{e^{np(e^t - 1)}}{e^{t\epsilon}}$
 - * $P[X \geq \epsilon] = P[e^{tX} \geq e^{t\epsilon}] \leq \frac{E[e^{tX}]}{e^{t\epsilon}} = \frac{(E[e^{tX_i}])^n}{e^{t\epsilon}} = \frac{(pe^t + 1 - p)^n}{e^{t\epsilon}} \leq \frac{e^{np(e^t - 1)}}{e^{t\epsilon}}$
- $P[X \geq np(1 + \epsilon)] \leq (\frac{e^\epsilon}{(1+\epsilon)^{1+\epsilon}})^{np} \leq \begin{cases} e^{-\frac{\epsilon^2 np}{3}} & \text{if } 0 \leq \epsilon \leq 1 \\ e^{-\frac{\epsilon^2 np}{(2+\epsilon)}} & \text{if } \epsilon \geq 1 \end{cases}$
 - * Substitute ϵ with $np(1 + \epsilon)$
 - * Substitute t with $\log(1 + \epsilon)$
 - * the last inequality is without proof
- $P[X \leq \epsilon] \leq \frac{(pe^{-t} + 1 - p)^n}{e^{-t\epsilon}} \leq \frac{e^{np(e^{-t} - 1)}}{e^{-t\epsilon}}$
 - * $P[X \leq \epsilon] = P[e^{-tX} \geq e^{-t\epsilon}] \leq \frac{E[e^{-tX}]}{e^{-t\epsilon}} = \frac{(E[e^{-tX_i}])^n}{e^{-t\epsilon}} = \frac{(pe^{-t} + 1 - p)^n}{e^{-t\epsilon}} \leq \frac{e^{np(e^{-t} - 1)}}{e^{-t\epsilon}}$
- $P[X \leq np(1 - \epsilon)] \leq (\frac{e^{-\epsilon}}{(1-\epsilon)^{1-\epsilon}})^{np} \leq e^{-\frac{\epsilon^2 np}{2}}$
 - * Substitute ϵ with $np(1 - \epsilon)$
 - * Substitute t with $-\log(1 - \epsilon)$
 - * the last inequality is without proof

- Chernoff/ Hoeffding Lemma

Definition:

- Suppose X_1, \dots, X_n are independent distributed random variable and $a_i \leq X_i \leq b_i$
- Suppose $X = \sum_{i=1}^n X_i$ and $\mu = \mathbb{E}[X]$
- $P[|X - \mu| \geq \epsilon] \leq 2e^{\frac{-2\epsilon^2}{\sum_{i=1}^n (b_i - a_i)^2}}$ without proof

- Application:

- Balls in Bins

Definition: Throw n balls into n bins, find bounds for the maximum number of balls in all bins

- * $P[\text{maximum number of balls in all bins} \geq \epsilon]$
 $= P[\cup_{i=1}^n \text{number of balls in } i\text{-th bin} \geq \epsilon]$
 $\leq n \times P[\text{number of balls in one bin} \geq \epsilon]$
- * By Markov inequality:
 - $P[\text{number of balls in one bin} \geq \epsilon] \leq \frac{1}{\epsilon} \rightarrow \text{useless}$
- * By Chebyshev inequality:
 - $P[\text{number of balls in one bin} \geq \epsilon] \leq \frac{(1 - \frac{1}{n})}{\epsilon^2}$
 - $P[\text{maximum number of balls in all bins} \geq n^{\frac{1}{2} + \epsilon}] \leq \frac{(1 - \frac{1}{n})}{n^{2\epsilon}}$
 - when $n \rightarrow \infty$, the maximum number of balls should less than $n^{\frac{1}{2} + \epsilon}$
- * By Chernoff inequality:
 - $P[\text{number of balls in one bin} \geq 2 \log n] \leq \frac{e^{np(e^t - 1)}}{n^{2t}}$
 - $P[\text{maximum number of balls in all bins} \geq 2 \log n] \leq \frac{e^{np(e^t - 1)}}{n^{2t-1}}$
 - when t is a constant ≥ 0.5 and $n \rightarrow \infty$, the maximum number of balls should less than $2 \log n$

5 Law of Large Numbers

- $\{X_i\}_{i=1}^\infty$ is a sequence of pairwise uncorrelated random variable with $\mathbb{E}[X_i] = m$, $\text{Var}(X_i) = \sigma_i^2$.
- $M_n = \frac{1}{n} \sum_{i=1}^n X_i$
- $M_n \rightarrow m$ almost surely, in mean square and in probability.

6 Memoryless

- Definition: $P[X > x_1 + x_2 | X > x_1] = P[X > x_2]$
- Property:
 - Exponential random variable is the only continuous memoryless random variable
 - Bernoulli random variable is the only discrete memoryless random variable

7 Famous Random Variable

- Poisson:

$$P[X = k] = \frac{\lambda^k}{k!} \exp(-\lambda)$$

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} \exp(-\lambda) = \sum_{k=0}^{\infty} \lambda \frac{\lambda^{k-1}}{(k-1)!} \exp(-\lambda) = \lambda$$

Interpretation:

- Cut total time into infinite period in Binomial random variable, $n \rightarrow \infty, p \rightarrow \frac{\lambda}{n}$
- $\rightarrow P[X = k] = \lim_{n \rightarrow \infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} = \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n}\right)^n = \frac{\lambda^k}{k!} \exp(-\lambda)$

- Erlang:

$$f_X(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!}, \forall x \in \mathbb{R}$$

$$\mathbb{E}[X] = \frac{n}{\lambda}$$

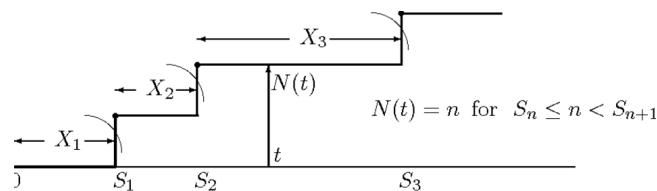
Interpretation:

- Suppose X_1, X_2, \dots, X_n are i.i.d exponential random variable with λ .
- $X = \sum_{i=1}^n X_i$
- Proof by induction:

Suppose $n = 2$, $f_X(x) = \int_0^x \lambda e^{-\lambda t} \lambda e^{-\lambda(x-t)} dt = \lambda^2 x e^{-\lambda x}$

8 Stochastic Processes

- Stochastic Process: a collection of random variable
- Arrival Process: a sequence of arriving event in continuous time



- X_i : the time between the i -th event and the $i - 1$ -th event
- S_i : the time from start to i -th event
- $N(t)$: the number of the arrived event at time t
- X and S Relation:
 - * $X_1 = S_1, X_i = S_i - S_{i-1}$
- N and S Relation:
 - * $N(t) < n \leftrightarrow S_{n+1} > t$
 - * $N(t) \geq n \leftrightarrow S_n \leq t$
 - * $N(t) = n \leftrightarrow S_n \leq t < S_{n+1}$
 - * $N(t) = \max\{n : S_n \leq t\}$

– Renewal Process: an arrival process with i.i.d X_i

Delayed Renewal Process: the process becomes a renewal process after several arrivals

X_i Property

- * if X_i is dependent on the interval states, then X_i might be dependent on $X_{i-1} \rightarrow$ not renewal process

S_i Property

- * $P[\lim_{n \rightarrow \infty} S_n = \infty] = 1$
Proof: $\lim_{n \rightarrow \infty} P[S_n = \infty] = \lim_{n \rightarrow \infty} P[\sum_{i=1}^n X_i = n \times \mathbb{E}[X_i]] = 1$
Interpretation: infinite events do not take finite time

$N(t)$ Property

- * for any $t, P[N(t) < \infty] = 1$
Proof: $P[\lim_{n \rightarrow \infty} S_n = \infty] = 1 \rightarrow$ for any $t, P[\lim_{n \rightarrow \infty} S_{n+1} > t] = 1$
Interpretation: infinite events do not take finite time
- * $P[\lim_{t \rightarrow \infty} N(t) \rightarrow \infty] = 1$
Proof: if $P[\lim_{t \rightarrow \infty} N(t) = k] > 0 \rightarrow P[X_{k+1} = \infty] > 0$
Interpretation: finite events do not take infinite time
- * $P[\lim_{t \rightarrow \infty} \frac{N(t)}{t} = \frac{1}{\mathbb{E}[X_i]}] = 1$
Proof: $P[\lim_{t \rightarrow \infty} \frac{N(t)}{S_{N(t)+1}} \leq \lim_{t \rightarrow \infty} \frac{N(t)}{t}] = 1$ and $P[\lim_{t \rightarrow \infty} \frac{N(t)}{S_{N(t)+1}} = \frac{1}{\mathbb{E}[X_i]}] = 1$
 $P[\lim_{t \rightarrow \infty} \frac{N(t)}{t} \leq \lim_{t \rightarrow \infty} \frac{N(t)}{S_{N(t)}}] = 1$ and $P[\lim_{t \rightarrow \infty} \frac{N(t)}{S_{N(t)}} = \frac{1}{\mathbb{E}[X_i]}] = 1$

Inspection Paradox

- * $\mathbb{E}[X_{N(t)+1}] \geq \mathbb{E}[X_i]$: inspection paradox
Interpretation:
 - $f_{X_{N(t)+1}}(x) = \lambda x f_{X_i}(x)$
 - when selecting t with equal probability, we tend to choose X_i with longer period
- * $P[\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (S_{N(t)+1} - s) ds = \frac{\mathbb{E}[X_i^2]}{2\mathbb{E}[X_i]}] = 1$
Proof:
 $P[\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=i}^{N(t)} \frac{\mathbb{E}[X_i^2]}{2} \leq \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (S_{N(t)+1} - s) ds] = 1$ and $P[\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=i}^{N(t)} \frac{\mathbb{E}[X_i^2]}{2} = \frac{\mathbb{E}[X_i^2]}{2\mathbb{E}[X_i]}] = 1$
 $P[\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (S_{N(t)+1} - s) ds \leq \lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=i}^{N(t)+1} \frac{\mathbb{E}[X_i^2]}{2}] = 1$ and $P[\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=i}^{N(t)+1} \frac{\mathbb{E}[X_i^2]}{2} = \frac{\mathbb{E}[X_i^2]}{2\mathbb{E}[X_i]}] = 1$
- * $P[\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (s - S_{N(t)}) ds = \frac{\mathbb{E}[X_i^2]}{2\mathbb{E}[X_i]}] = 1$
Proof: similar to above
- * $P[\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t X_{N(t)} ds = \frac{\mathbb{E}[X_i^2]}{\mathbb{E}[X_i]}] = 1$
Proof: $P[\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t X_{N(t)} ds = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (S_{N(t)+1} - S_{N(t)}) ds] = 1$
- * $\mathbb{E}[X_{N(t)+1}] = \frac{\mathbb{E}[X_i^2]}{\mathbb{E}[X_i]}$
Proof: $P[\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t X_{N(t)} ds = \frac{\mathbb{E}[X_i^2]}{\mathbb{E}[X_i]}] = P[\mathbb{E}[X_{N(t)+1}] = \frac{\mathbb{E}[X_i^2]}{\mathbb{E}[X_i]}] = 1$

Central Limit Theorem

- * $\mu = \mathbb{E}[X_i]$
- * $\sigma = \sqrt{\text{Var}(X_i)}$
- * $Z \sim \text{Normal}(0,1)$
- * $\lim_{t \rightarrow \infty} P[N(t) \leq \frac{t}{\mu} + k \frac{\sigma\sqrt{t}}{\sqrt{\mu^3}}] = P[Z \leq k]$
Proof:
 1. Suppose $n(t) = \frac{t}{\mu} + k \frac{\sigma\sqrt{t}}{\sqrt{\mu^3}}$
 2. $P[N(t) \geq n(t)] = P[S_{n(t)} \leq t] = P[\frac{S_{n(t)} - n\mu}{\sigma\sqrt{n}} \leq \frac{t - n\mu}{\sigma\sqrt{n}}]$
 3. When $t \rightarrow \infty, \frac{t - n\mu}{\sigma\sqrt{n}} \rightarrow k$
 4. By law of large number, $\lim_{t \rightarrow \infty} P[\frac{S_{n(t)} - n\mu}{\sigma\sqrt{n}} \leq k] = P[Z \leq k]$

Interpretation:

- $\frac{t}{\mu}$ is approximately the mean of $N(t)$
- $k \frac{\sigma\sqrt{t}}{\sqrt{\mu^3}}$ is $k\sigma\sqrt{n}$ after dividing by μ , the ratio between t and $N(t)$ and changing n with $\frac{t}{\mu}$

Wald's Identity

- * Stopping Times: a random variable τ s.t. $\{\tau = n\}$ is independent of $\{X_i\}_{i=n+1}^\infty$
- * Stopping Condition: a condition to stop if we can consider $\tau = \min\{n : \text{condition}(n) = \top\}$
- * Example: $N(t) + 1$ is a stopping times and can be consider $N(t) + 1 = \min\{n : S_n > t\}$
- * $\mathbb{E}[\sum_{i=1}^\tau X_i] = \mathbb{E}[\tau]\mathbb{E}[X_i]$ if $\mathbb{E}[X_i] < \infty$ and $\mathbb{E}[N] < \infty$
- Proof:
 1. $\mathbb{E}[\sum_{i=1}^\tau X_i] = \sum_{i=1}^\infty \mathbb{E}[X_i \times \mathbb{1}_{i \leq \tau}]$ (by Fubin's Theorem without proof)
(if $\mathbb{E}[X_i] < \infty$ and $\mathbb{E}[N] < \infty$)
 2. $\sum_{i=1}^\infty \mathbb{E}[X_i \times \mathbb{1}_{i \leq \tau}] = \mathbb{E}[X_i] \sum_{i=1}^\infty \mathbb{E}[\mathbb{1}_{i \leq \tau}]$ (by $P[\tau \geq i] = 1 - P[\tau < i]$ is independent of X_i)
 3. $\mathbb{E}[X_i] \sum_{i=1}^\infty \mathbb{E}[\mathbb{1}_{i \leq \tau}] = \mathbb{E}[\tau]\mathbb{E}[X_i]$
- * $\lim_{t \rightarrow \infty} \frac{\mathbb{E}[N(t)]}{t} = \frac{1}{\mathbb{E}[X_i]}$

Proof:

- Suppose $\mu = \mathbb{E}[X_i]$
- $\frac{\mathbb{E}[N(t)]}{t} = \frac{\mathbb{E}[S_{N(t)+1}]}{t \times \mu} - \frac{1}{t}$ (by considering $N(t) + 1$ as the stopping time)
- $\lim_{t \rightarrow \infty} \frac{\mathbb{E}[N(t)]}{t} \geq \frac{1}{\mu}$ (by $\mathbb{E}[S_{N(t)+1}] > t$)
- Suppose $\hat{X}_n = \min\{X_n, T\}$, where T is a constant
- $\frac{\mathbb{E}[N(t)]}{t} \leq \frac{\mathbb{E}[\hat{N}(t)]}{t} = \frac{\mathbb{E}[S_{\hat{N}(t)+1}]}{t \times \hat{\mu}} - \frac{1}{t} \leq \frac{t+T}{t \times \hat{\mu}} - \frac{1}{t}$
- $\lim_{n=\sqrt{t}, t \rightarrow \infty} \frac{\mathbb{E}[N(t)]}{t} \leq \frac{1}{\mu}$

Blackwell's Theorem

- * $\mathbb{E}[N(t)] = F_{X_i}(t) + \int_0^t \mathbb{E}[N(t-x)]f_{X_i}(t)dt$
- Proof: $\mathbb{E}[N(t)] = \int_0^t \mathbb{E}[N(t)|X_1 = x]f_{X_1}(x)dx$
 $= \int_0^t \mathbb{E}[N(t-x) + 1]f_{X_1}(x)dx = F_{X_i}(t) + \int_0^t \mathbb{E}[N(t-x)]f_{X_i}(t)dt$
- * $\mathcal{L}\{\mathbb{E}[N(t)]\}(s) = \frac{\mathcal{L}\{f_{X_i}\}(s)}{s(1-\mathcal{L}\{f_{X_i}\}(s))}$
- Proof: Laplace transform both sides
- * Lattice/ Non-Lattice: $N(t)$ is lattice iff X_i only takes on values that are $nd, n \in \mathbb{N}, d \in \mathbb{R}$
- * For a non-lattice process: $\lim_{t \rightarrow \infty} \mathbb{E}[N(t+\delta) - N(t)] = \frac{\delta}{\mathbb{E}[X_i]}$, for any δ
- Proof: Without Proof
- Interpretation: $\mathbb{E}[N(t)]$ will converge to be linear
- * For a lattice process and period d : $\lim_{n \rightarrow \infty} \mathbb{E}[\# \text{ events at } t = nd] = \frac{d}{\mathbb{E}[X_i]}$
- Proof: Without Proof
- Interpretation: $\mathbb{E}[N(t)]$ will converge to be stairs with width d and height $\frac{d}{\mathbb{E}[X_i]}$

– Renewal-Reward Process:

Definition

- * A renewal process $N(t)$ and $\{R_i\}_{i=1}^\infty$ such that (X_i, R_i) are i.i.d.
 $(X_i, R_j, i \neq j)$ are independent, but X_i, R_i might be dependent

Property

- * $P[\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=1}^{N(t)} R_i = \frac{\mathbb{E}[R_i]}{\mathbb{E}[X_i]}] = 1$
- Proof: $P[\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i=1}^{N(t)} R_i = \lim_{t \rightarrow \infty} \sum_{i=1}^{N(t)} \frac{R_i}{N(t)} \times \lim_{t \rightarrow \infty} \frac{N(t)}{t}] = 1$

– Poisson Process: a renewal process with $X_i \sim \text{Exponential}(\lambda)$

S_i Property

- * S_i is an Erlang random variable
 Erlang is the sum of the Exponential random variables
- * Joint Distribution $f_{S_1, \dots, S_n}(s_1, \dots, s_n) = \lambda^n e^{-\lambda s_n}$
 Prove by induction.
 Induce by $f_{S_1, \dots, S_n}(s_1, \dots, s_n) = f_{S_1, \dots, S_{n-1}}(s_1, \dots, s_{n-1}) \times f_{S_n|S_1, \dots, S_{n-1}}(s_n, s_1, \dots, s_{n-1})$

$N(t)$ Property

- * $N(t) \sim \text{Poisson}(\lambda t)$, $P[N(t) = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$
Prove by $P[N(t) = n] = P[S_n \leq t \text{ and } S_{n+1} > t]$
- * Conditioned on $N(t) = n$, the set of arrival times $\{s_1, \dots, s_n\}$ have the same distribution with a set of n sorted i.i.d. $\text{Uniform}(0, t)$ random variables
Prove by $f_{S_1, \dots, S_n | N(t)}(s_1, \dots, s_n, n) = \frac{f_{S_1, \dots, S_n}(s_1, \dots, s_n) P[X_{n+1} > t - s_n]}{P[N(t) = n]} = \frac{n!}{t^n}$

Property

- * Z is the interval from t to the first arrival $\rightarrow Z$ is exponential random variable with same λ and independent of $N(t)$ and the arrival time before t
Proof:
$$\begin{aligned} P[Z > z] &= \sum_{n=0}^{\infty} \int_0^{\infty} \dots \int_0^{\infty} P[Z > z | N(t) = n, S_1 = s_1, \dots, S_n = s_n] ds_1 \dots ds_n \\ &= \sum_{n=0}^{\infty} \int_0^{\infty} \dots \int_0^{\infty} P[X_{n+1} > z + t - s_n | N(t) = n, S_1 = s_1, \dots, S_n = s_n] ds_1 \dots ds_n \\ &= \sum_{n=0}^{\infty} \int_0^{\infty} \dots \int_0^{\infty} P[X_{n+1} > z + t - s_n | X_{n+1} > t - s_n] ds_1 \dots ds_n = e^{-\lambda z} \end{aligned}$$
- * Stationary Increments: $N(t_1 + t_2) - N(t_1)$ and $N(t_2)$ share the same distribution
Without Proof
- * Independent Increments: $\forall 0 < t_1 < t_2 < \dots, t_k, N(t_1), N(t_2) - N(t_1), \dots$ are independent
Without Proof
- * Any arrival process with stationary and independent increments must be a Poisson process
Without Proof

Exercise

- * $\mathbb{E}[S_i | N(t) = n] = \frac{t \times i}{n+1}$
 $\cdot \mathbb{E}[S_i | N(t) = n] = i \times \mathbb{E}[X_1 | N(t) = n] = i \int_0^t \int_0^{s_n} \dots \int_0^{s_2} s_1 \times \frac{n!}{t^n} ds_1 \dots ds_{n-1} ds_n = \frac{t \times i}{n+1}$
- * $\mathbb{E}[\sum_{i=0}^{N(t)} S_i] = \frac{\lambda t^2}{2}$
 $\cdot \mathbb{E}[\sum_{i=0}^{N(t)} S_i] = \sum_{n=0}^{\infty} \mathbb{E}[\sum_{i=0}^n S_i | N(t) = n] P[N(t) = n]$
 $= \sum_{n=0}^{\infty} \frac{n t}{2} P[N(t) = n] = \frac{\lambda t^2}{2}$

2D Poisson Process

- * Definition:
 - For any region R : number of points in R is a Poisson random variable
 - number of points in the non-overlapping region is independent

Combining Poisson Process

- * $N^1(t), N^2(t)$ are two independent Poisson process with λ_1, λ_2
- * X_i is the first arrival of X_i^1, X_i^2
- * Property
 - X_i is independent of $\{X_i^1 < X_i^2\}$ and $\{X_i^1 > X_i^2\}$
Proof: $P[X_1^1 < X_1^2] = \frac{\lambda_1}{\lambda_1 + \lambda_2}$
 $P[X_1 > x] = P[X_1^1 > x, X_1^2 > x] = e^{-(\lambda_1 + \lambda_2)x}$
 $P[X_1 > x, X_1^1 < X_1^2] = P[X_1 > x] P[X_1^1 < X_1^2]$
 - X_i is a Poisson Process with $\lambda = \lambda_1 + \lambda_2$

Splitting Poisson Process

- * $N^1(t), N^2(t)$ are two independent Poisson process with λ_1, λ_2
- * $N(t)$ is a random process with $\lambda = \lambda_1 + \lambda_2$
 - $N^{1*}(t)$ is the process of the first event
when $N(t)$ arrives consider it as first event with probability $\frac{\lambda_1}{\lambda_1 + \lambda_2}$
 - $N^{2*}(t)$ is the process of the second event
when $N(t)$ arrives consider it as second event with probability $\frac{\lambda_2}{\lambda_1 + \lambda_2}$
- * $N^i(t)$ and $N^{i*}(t)$ share the same distribution
- * Proof:
 - $B_n(k)$ is a Binomial random variable with $p = \frac{\lambda_1}{\lambda_1 + \lambda_2}$
 - $P[N^{1*}(t) = m, N^{2*}(t) = n] = P[N(t) = m + n, B_{m+n}(m)] = P[N^1(t) = m, N^2(t) = n]$

Compound Poisson Process

- * $N(t)$ is a Poisson Process

- * A_n is a sequence of cost
- * $A(t) = \sum_{n=0}^{N(t)} A_n$ is the summation of cost over Poisson Process

Non-Homogeneous Poisson Process

- * $N(t) - N(s) \sim \text{Poisson}(\int_s^t \lambda(x) dx)$

Queuing Theory

- * Definition: *Arrival_Process/Service_Process/number_of_services*
 - M : memoryless (Poisson) process
 - D : deterministic process
 - G : general renewal process
- * T : the random variable of the processing time for each customer
- * $Y(t)$: number of cutomers in the service
 - $Y(t) \sim \text{Poisson}(\lambda \int_0^t P[T > x] dx)$
 - Proof:

Consider $Y(t)$ is a splitting Poisson Process. Since the distribution for the arrival given $N(t)$ is universal, the probability the arrival is still in service: $\frac{1}{t} \int_0^t P[T > t-x] dx = \frac{1}{t} \int_0^t P[T > x] dx$

9 Markov Chain

- Definition

- Model with states and transition probability matrix
- States: $\{X_i\}_{i=0}^{\infty}$
- Transition Probability Matrix: $[P]_{ij} = P[X_{n+1} = j | X_n = i]$

- Terminology

- p^n : distribution at step n
- $T_i = \min\{n \geq 1 : X_n = i\}$
- $f_{ij} = P[T_j < \infty | X_0 = i]$: the probability of starting at i and ever reaching j
- $\mu_{ij} = \mathbb{E}[T_j | X_0 = i]$
- $i \rightarrow j$ iff $f_{ij} > 0$: j is reachable from i with probability greater than 0
- $N_i(n)$: number of visits to i by time n
- Irreducible: $i \leftrightarrow j, \forall$ states i, j
- aperiodic: period of $X_n = i$ is 1, \forall states i

- Property

- Updating distribution
 - * $p^n = p^0 P^n$
- Markovian: transition probability depend only on current state
 - * $P[X_{n+1} = j | X_n = i, \dots, X_0 = x_0] = [P]_{ij}$
- Stationary Distribution: p

Property from renewal process

- * consider $X_n = j$ as a event \rightarrow Markov Chain becomes a delayed renewal process
- * If $i \leftrightarrow j$ and the model starts from i , then following holds
- * $P[\lim_{n \rightarrow \infty} \frac{N_j(n)}{n} = \frac{1}{\mu_{jj}}] = 1$
- * $\lim_{n \rightarrow \infty} \frac{\mathbb{E}[N_j(n)]}{n} = \frac{1}{\mu_{jj}}$
- * if the period of $X_n = j$ is $d \rightarrow \lim_{n \rightarrow \infty} p_j^{nd} = \frac{d}{\mu_{jj}}$

Theorem of an irreducible, aperiodic Markov Chain

- * Either

- All states have $\mu_{ii} = \infty$
- All states have $\mu_{ii} < \infty$ and $p_i = \frac{1}{\mu_{ii}}$ is the unique stationary distribution
- * Proof
 - From if the period of $X_n = j$ is $d \rightarrow \lim_{n \rightarrow \infty} p_j^{nd} = \frac{d}{\mu_{jj}}$

Theorem of an irreducible, aperiodic Markov Chain

- * All states have $\mu_{ii} < \infty$ and $p_i = \frac{1}{\mu_{ii}}$ is the unique stationary distribution

p can be calculated as the eigenvector corresponds to eigenvalue 1 of P^T

– Detailed Balance

Definition:

- * Given a distribution π
- * $\pi_i P_{ij} = \pi_j P_{ji}, \forall i, j$

Property:

- * distribution π satisfying Detailed Balance is the stationary distribution p

– Reversible

Definition: A Markov Chain with stationary distribution p is reversible if it satisfies detailed balance

Interpretation

- * Transitions forward and backward in the stationary distribution have the same probability
- * $P[X_{n+1} = j | X_n = i] = P_{ij}$
- * $P[X_{n-1} = j | X_n = i] = \frac{P[X_{n-1}=j, X_n=i]}{P[X_n=i]} = \frac{p_j P_{ji}}{p_i} = P_{ij}$

– Metropolis Update Rule

Definition

- * Given a Markov Chain and distribution p' , find P' such that p' is the stationary distribution

Procedure

- * For each pair (i, j) , $P'_{ij} = P_{ij} \times \min\{1, \frac{p'_j P_{ji}}{p'_i P_{ij}}\}$
- * construct self loop to satisfy $\sum_j P'_{ij} = 1$

Proof

- * To satisfy detailed balance, for each pair (i, j) , we should set $p'_i P'_{ij} = \min\{p'_i P_{ij}, p'_j P_{ji}\}$

– Distance between Probability Measure

Definition:

- * Total Variation Distance between P_1 and P_2 is: $d_{TV}(P_1, P_2) = \frac{1}{2} \sum_{\omega} |P_1[\omega] - P_2[\omega]|$

– Mixing Time

Definition

- * Mixing time τ is the least t such that for all initial state p^0 , $d_{TV}(p, p^0 P^t) \leq \frac{1}{2\epsilon}$

Interpretation

- * the factor $\frac{1}{2\epsilon}$ is set such that $d_{TV}(p, p^0 P^t) \leq \epsilon$ if $t \geq \tau \times \log(\frac{1}{\epsilon})$

– Example

Random Walk on Graph

- * Definition: move from vertex i to vertex j with probability $P_{ij} = \begin{cases} 0 & \text{if } (i, j) \notin E \\ \frac{1}{\text{degree}(i)} & \text{if } (i, j) \in E \end{cases}$
- * Distribution π , $\pi_i = \frac{\text{degree}(i)}{2|E|}$ satisfies detailed balance
- * If we want stationary distribution to be uniform $\rightarrow P'_{ij} = \begin{cases} \frac{1}{\text{degree}(i)} & \text{if } \text{degree}(i) \geq \text{degree}(j) \\ \frac{1}{\text{degree}(j)} & \text{if } \text{degree}(i) < \text{degree}(j) \end{cases}$

• Hidden Markov Chain

- Definition: output is a function of the state
- Interpretation: if the model is not markovian, then reformulate the model as a hidden markov chain by complicating the states and rendering the output as a function of the state