Stochastic Processes

Kevin Chang

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1 Moment Generating Function

- Moment Generating Function: $\mathbb{E}[e^{tX}]$
  - Property:
    * $\mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx$
    * $\mathbb{E}[e^{tX}] = \sum_{k=0}^{\infty} E[X^k]\frac{t^k}{k!}$
      · $e^{tx} = \sum_{k=0}^{\infty} \frac{(tx)^k}{k!}$
      · $E[e^{tX}] = E[\sum_{k=0}^{\infty} \frac{(tX)^k}{k!}] = \sum_{k=0}^{\infty} E[X^k]\frac{t^k}{k!}$
    * $\frac{d\mathbb{E}[e^{tX}]}{dt} = \mathbb{E}[X]$
    * $\mathbb{E}[e^{t(aX+b)}] = e^t b\mathbb{E}[e^{taX}]$
    * Not all random variables have Moment generating function
- Characteristic Function: $\mathbb{E}[e^{itX}]$
  - Property:
    * All random variables have Moment generating function
- Joint Moment Generating Function: $G(x, y) = \mathbb{E}[e^{xX}e^{yY}]$
- Property:
  - (Joint) moment generating funciton uniquely determines the (joint) CDF
- Example
  - Trapped miner's random walk
    * Miner has probability of $\frac{1}{3}$ to waste 3 hours in vain, $\frac{1}{3}$ to waste 5 hours in vain, and $\frac{1}{3}$ to spend 2 hours to go out of the mine.
    * X is the random variables of the hours to go out of the mine
    * Y_i is the random variables of the hours for the i-th action.
    * $\mathbb{E}[e^{tX}] = \mathbb{E}[e^{tX}|Y_1 = 2] + \mathbb{E}[e^{tX}|Y_1 = 3] + \mathbb{E}[e^{tX}|Y_1 = 5]$
      $= \mathbb{E}[e^{2t}] + \mathbb{E}[e^{t(X+3)}] + \mathbb{E}[e^{t(X+5)}]$
    * Find expectation and variance by joint moment generating function

2 Expectation

- Suppose N is a integer random variable
- Suppose $X_1, \ldots, X_i, \ldots, X_N$ are i.i.d random variables with mean μ and variance σ^2
- $Y = \sum_{i=1}^{N} X_i$
- $\mathbb{E}[Y] = \mathbb{E}[N]\mu$
  - $\mathbb{E}[Y] = \sum_{n=1}^{\infty} \mathbb{E}[\sum_{i=1}^{N} X_i|N = n]P[N = n]$
    $= \mu \times \sum_{n=1}^{\infty} nP[N = n] = \mathbb{E}[N]\mu$

- $\mathbb{E}[Y^2] = \mathbb{E}[N]\mathbb{E}[X^2] + \mathbb{E}[N^2]\mu^2 - \mathbb{E}[N]\mu^2$
  - $\mathbb{E}[Y^2] = \sum_{n=1}^{\infty} \mathbb{E}[(\sum_{i=1}^{N} X_i)^2 | N = n] P[N = n] = \sum_{n=1}^{\infty} (n\mathbb{E}[X_i^2] + n(n-1)\mu^2) P[N = n]$
    $= \mathbb{E}[N]\mathbb{E}[X^2] + \mathbb{E}[N^2]\mu^2 - \mathbb{E}[N]\mu^2$
- $Var(Y) = \mathbb{E}[N]\sigma^2 + Var(N)\mu^2$