Linear Algebra Review

CS221: Introduction to Artificial Intelligence

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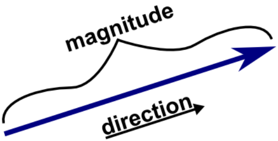
- A quantity that has a magnitude and a direction.



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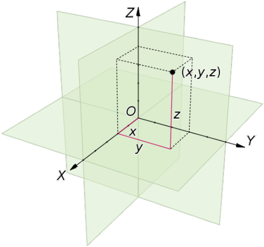
- A **subspace** is a subset of a vector space that is **also** a vector space.

Examples

- Euclidean space $\mathbb{R}^n$.



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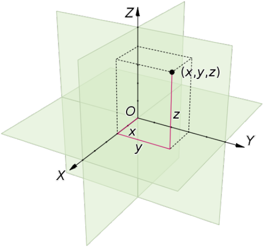
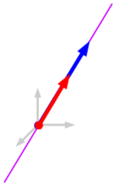


- The **span** of any set of vectors $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$, defined as :

$$\text{span}\left(\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\right) = \{\alpha_1\mathbf{u_1} + \alpha_2\mathbf{u_2} + \cdots + \alpha_n\mathbf{u_n} \mid \alpha_i \in \mathbb{R}\}$$

Subspaces

- A **line** through the origin in $\mathbb{R}^n$(span of a vector).



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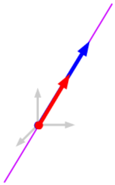
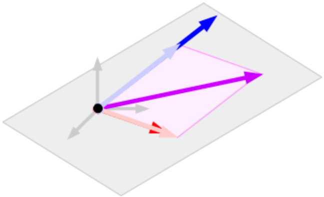


- A **plane** in $\mathbb{R}^n$(span of two vectors).



Linear Independence and Basis of Vector Spaces

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- The dimension of a vector space can be infinite (function spaces).

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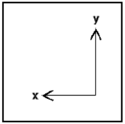
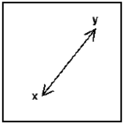
- Examples :
  - Manhattan or ℓ_1 norm : $||\mathbf{u}||_1 = \sum_i |u_i|$.
  - Euclidean or ℓ_2 norm : $||\mathbf{u}||_2 = \sqrt{\sum_i u_i^2}$.



Manhattan



Euclidean

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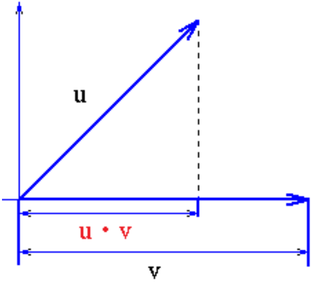

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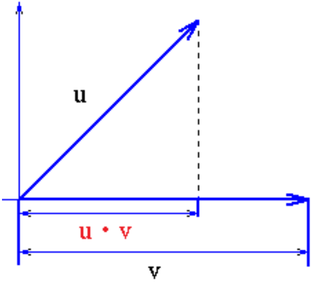
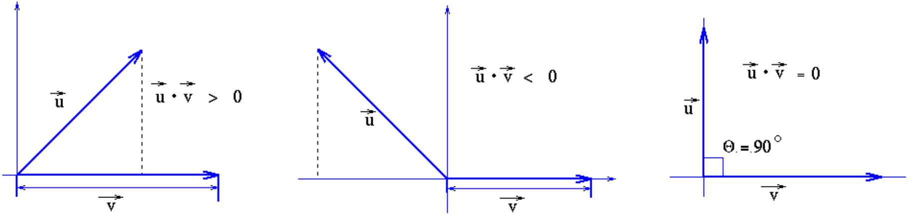


- Related to the Euclidean norm : $\langle \mathbf{u}, \mathbf{u} \rangle = ||\mathbf{u}||_2^2$.

Inner Product

The inner product is a measure of **correlation** between two vectors, scaled by the norms of the vectors :



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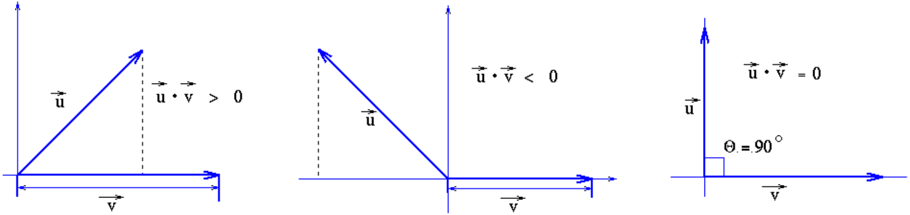

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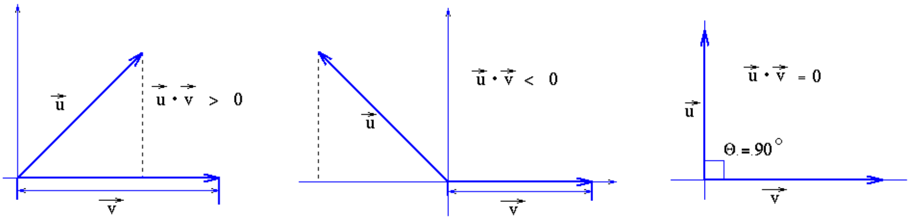

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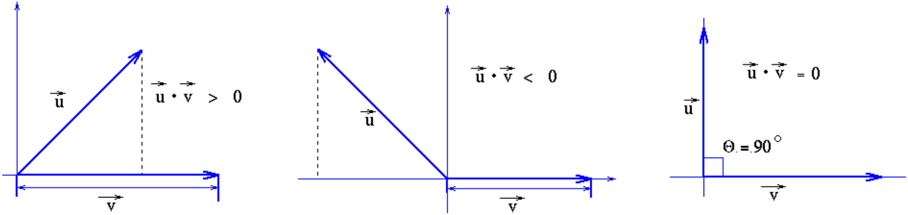

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- If $\langle \mathbf{u}, \mathbf{v} \rangle = 0$, $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal**.

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- We compute $\langle \mathbf{x}, \mathbf{b_1} \rangle$:

$$\begin{aligned}
\langle \mathbf{x}, \mathbf{b_1} \rangle &= \langle \alpha_1 \mathbf{b_1} + \alpha_2 \mathbf{b_2}, \mathbf{b_1} \rangle \\
&= \alpha_1 \langle \mathbf{b_1}, \mathbf{b_1} \rangle + \alpha_2 \langle \mathbf{b_2}, \mathbf{b_1} \rangle \quad \text{By linearity.} \\
&= \alpha_1 + 0 \quad \text{By orthonormality.}
\end{aligned}$$

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- Likewise, $\alpha_2 = \langle \mathbf{x}, \mathbf{b_2} \rangle$.

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  - $\mathcal{L}(\alpha\, \mathbf{u}) = \alpha\, \mathcal{L}(\mathbf{u})$ for any scalar α.
- If the dimension n of $\mathcal{U}$ and m of $\mathcal{V}$ are finite, $\mathcal{L}$ can be represented by an $m \times n$ matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ & & \dots & \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{pmatrix}$$

Matrix Vector Multiplication

$$\mathbf{Au} = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ & & \dots & \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_m \end{pmatrix}$$

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- The inner product for vectors can be represented as $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T\mathbf{v}$.

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  - Every vector in the null space is orthogonal to the rows of $\mathbf{A}$. The **null space** and **row space** of a matrix are **orthogonal**.

Range and Column Space

Another interpretation of the matrix vector product ($\mathbf{A_i}$ is column i of $\mathbf{A}$) :

$$\mathbf{Au} = \begin{pmatrix} \mathbf{A_1} & \mathbf{A_2} & \dots & \mathbf{A_n} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{pmatrix}$$

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- $\mathbf{A}^{-1}$ is the **only** matrix such that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathtt{I}$.

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- Example : Matrices that represent **rotations** are orthogonal.

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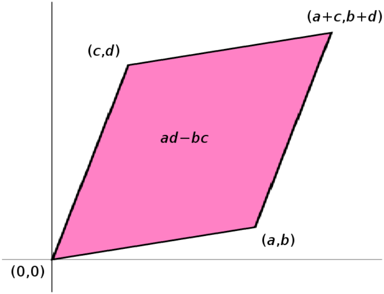

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Eigenvalues and Eigenvectors

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- Eigenvalues and eigenvectors can be **complex valued**, even if all the entries of $\mathbf{A}$ are real.

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- $\mathbf{P}$ is full rank. Inverting it yields the **eigendecomposition** of $\mathbf{A}$:

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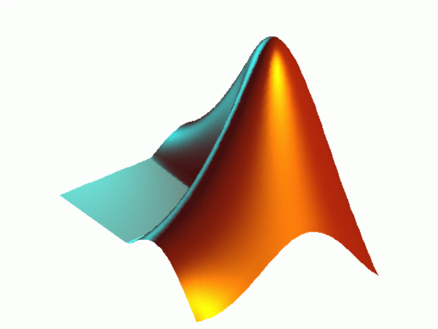
- $|\mathbf{A}| = |\Lambda| = \prod_{i=1}^{n} \lambda_i$.
- The **rank** of **A** is equal to the **number of nonzero eigenvalues**.
- If λ is a nonzero eigenvalue of $\mathbf{A}$, $\frac{1}{\lambda}$ is an eigenvalue of $\mathbf{A}^{-1}$ with the **same eigenvector**.
- The eigendecomposition makes it possible to compute **matrix powers** efficiently :
  $\mathbf{A}^m = \left(\mathbf{P}\Lambda\mathbf{P}^{-1}\right)^m = \mathbf{P}\Lambda\mathbf{P}^{-1}\ \mathbf{P}\Lambda\mathbf{P}^{-1}\ \ldots\mathbf{P}\Lambda\mathbf{P}^{-1} = \mathbf{P}\Lambda^m\mathbf{P}^{-1}$.

Matlab Example



Eigendecomposition of a Symmetric Matrix

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- Consequently, the eigendecomposition becomes : $\mathbf{A} = \mathbf{U}\Lambda\mathbf{U}^T$ for Λ real and $\mathbf{U}$ orthogonal.
- The eigenvectors of $\mathbf{A}$ are an orthonormal **basis** for the **column space** (and the row space, since they are equal).

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- It would be great to generalize this to all matrices...

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- In statistics analyzing data with the singular value decomposition is called **Principal Component Analysis**.

Compression using the Singular Value Decomposition

- The singular value decomposition can be viewed as a sum of rank 1 matrices :

$$\mathbf{A} = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T \cdots + \sigma_R \mathbf{u}_R \mathbf{v}_R^T$$

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- This is a form of **lossy compression**.

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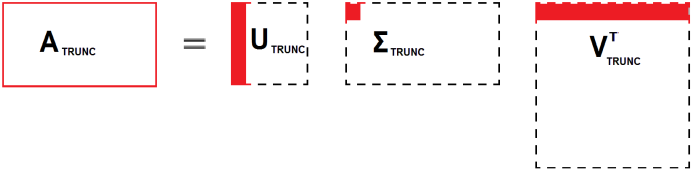
- Truncating the singular value decomposition allows us to represent the matrix with **less parameters**.



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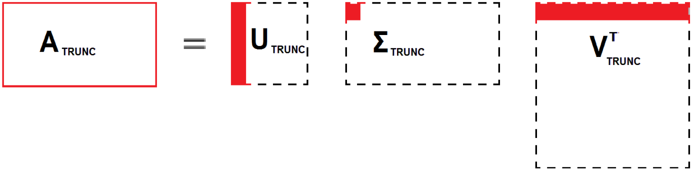

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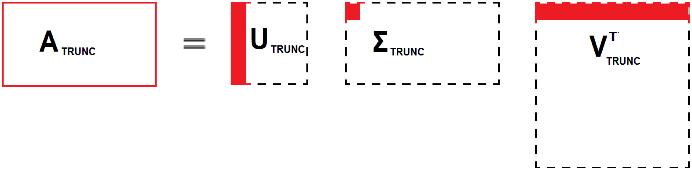

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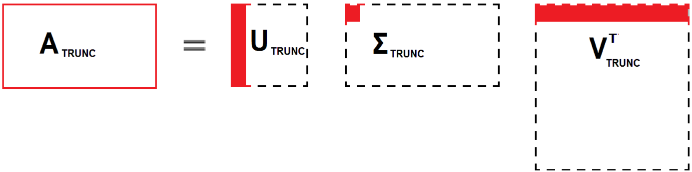

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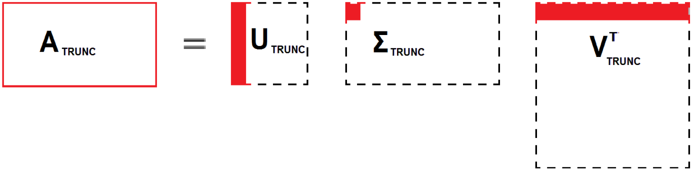

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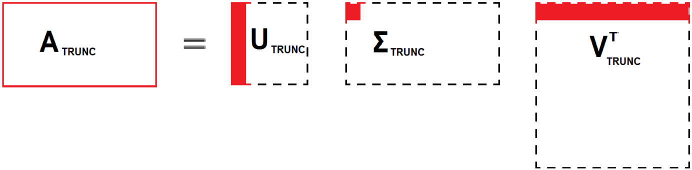

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  - Rank 80 approximation : $512 \times 80 + 80 + 512 \times 80 =$**82 000**