Emergent Properties of Distributed Agents with Two-Stage Convex Zero-Sum Optimal Exchange Network

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1 Introduction

This paper examines network behaviour of distributed systems under the assumptions of optimal convex programming zero-sum transitions for a two step evaluation network of agents, with stages consisting of:
a) An evaluation of the best exchange outcome for each agent within its neighborhood,
b) The execution of optimal matching pairs driven by the first stage evaluation.
Each agent has access to the evaluation of its immediate neighborhood within a lattice of indices $\{(i, j) \mid 1 \leq i \leq n,\ 1 \leq j \leq m\}$ with canonical base [(1,0), (0,1)] and continuous space of size $n \times m$, where $C_{i,j+1}$ is considered as part of the neighborhood for agent $C_{i,j}$, and $j+1 \equiv (j+1)\ mod\ \max_k$, with $\max_k$ being the maximum number of agents for that row or column in the lattice order.

2 Problem Setting

Let G_t be a two-dimensional lattice G for time t with size $n \times m$. The state of G_t is the collected states of all individual states $s_{i,j}(t) \in \mathbb{R}^2$ of each cell $C_{i,j}$ (where i, j are row and column indices) at time t.

Let $N_{i,j}$ be the neighborhood of a cell $C_{i,j}$, i.e. the set of cells that influence its state. For example, in Von Neumann neighborhood:

$$N_{i,j} = \{C_{i-1,j}, C_{i+1,j}, C_{i,j-1}, C_{i,j+1}\} \tag{2.1}$$

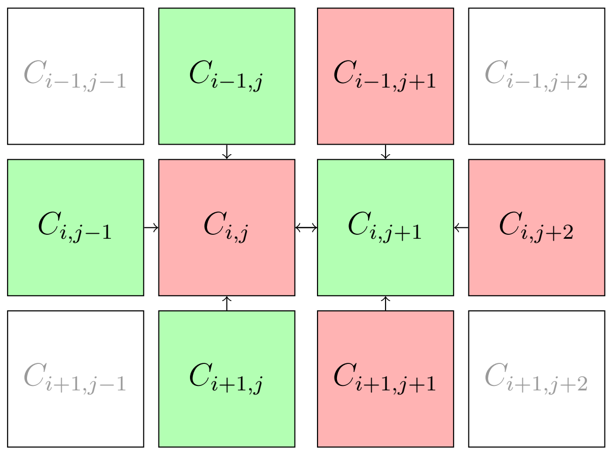

Fig. 1 - Agents $C_{i,j}$, $C_{i,j+1}$ and their neighborhoods (green and red accordingly).

Let $\mathbf{P}_{i,j} \in \mathbb{R}^2$ be the parameters associated with agent $C_{i,j}$.

3 First Stage: Optimal Evaluation Function

3.1 Peer to Peer Convex Programming Evaluation

Let function $o_{\mathbf{P}_{i,j}} : \mathbb{R}^2 \to \mathbb{R}$ be a *strictly convex function* with constant parameters $\mathbf{P}_{i,j}$ such that for all $x_1, x_2 \in \mathbb{R}^2$ and for all $\lambda \in [0,1]$:

$$o_{\mathbf{P}_{i,j}}(\lambda x_1 + (1-\lambda)x_2) < \lambda o_{\mathbf{P}_{i,j}}(x_1) + (1-\lambda) o_{\mathbf{P}_{i,j}}(x_2) \tag{3.1.1}$$

Let $o_{\mathbf{P}_{i,j}}$ be also *strictly monotonically increasing*:

$$\forall x_1, x_2 \in \mathbb{R}^2$$
$$|x_1| < |x_2| \iff o_{\mathbf{P}_{i,j}}(x_1) < o_{\mathbf{P}_{i,j}}(x_2) \tag{3.1.2}$$

3.2 Objective Function

Let's $O_{i,j}$ be the objective function for cell $C_{i,j}$ that depends on parameters $\mathbf{P}_{i,j} \in \mathbb{R}^2$, it's own current state $s_{i,j}(t)$, the parameters and the current states of its neighbors. The result registers the neighbor cell that yields the best outcome for its objective function and its associate state transition, the exchange vector $\vec{e}_{i,j} \in \mathbb{R}^2$:

$$\begin{aligned} O_{i,j}(t+1) &= \mathcal{L}\left(\mathbf{P}_{i,j}, s_{i,j}(t), \{\mathbf{P}_{k,l}, s_{k,l}(t) \mid \forall C_{k,l} \in N_{i,j}\}\right) \\ &= \{\vec{e}_{(i,j)(r,q)} \in \mathbb{R}^2 \mid C_{r,q} \in N_{i,j}\} \end{aligned} \tag{3.2.1}$$

A necessary condition of this model is that every optimal evaluation participates of a zero-sum game associated with their optimal exchanges, i.e:

$$\begin{gathered} \forall (i,j),(r,q) \mid i,r \in \{1,\dots,n\},\ j,q \in \{1,\dots,m\} \\ O_{i,j}(t+1) = \{\vec{e}_{(i,j)(r,q)} \mid C_{r,s}\} \\ \wedge\ O_{r,q}(t+1) = \{\vec{e}_{(r,q)(i,j)} \mid C_{i,j}\} \\ \implies \vec{e}_{(i,j)(r,q)} + \vec{e}_{(r,q)(i,j)} = \vec{0} \qquad (3.2.2) \\ \implies \vec{e}_{(i,j)(r,q)} = -\vec{e}_{(r,q)(i,j)} \qquad (3.2.3) \end{gathered}$$

As, by definition, optimal matching pairs are evaluated from their neighborhoods, this pair lies in the intersection of their neighborhoods:

$$\begin{gathered} \forall\ C_{i,j} \in N_{r,q}, \forall C_{r,q} \in N_{i,j} \\ O_{i,j}(t+1) = \{\vec{e}_{(i,j)(r,q)} \mid C_{r,q}\} \\ \wedge\ O_{r,q}(t+1) = \{\vec{e}_{(r,q)(i,j)} \mid C_{i,j}\} \\ \implies \vec{e}_{(i,j)(r,q)} = -\vec{e}_{(r,q)(i,j)} \end{gathered} \tag{3.2.4}$$

In this way, the optimal matches of agents can be calculated locally within their neighborhoods by:

$$\begin{gathered} O_{i,j}(t+1) = \text{maximize} \quad \mathcal{L}\left(\mathbf{P}_{i,j}, s_{i,j}(t), \{\mathbf{P}_{k,l}, s_{k,l}(t) \mid \forall C_{k,l} \in N_{i,j}\}\right) \\ = \max_{\vec{v} \in \mathbb{R}^2}\{\text{maximize} \quad o_{\mathbf{P}_{i,j}}(\vec{v}) \quad \text{subject to} \\ \vec{v}_{(i,j)(k,l)} + \vec{u}_{(k,l)(i,j)} = \vec{0} \quad \text{for} \quad \{o_{\mathbf{P}_{k,l}}(\vec{u}) \mid \forall C_{k,l} \in N_{i,j}\}\} \end{gathered}$$

$$= \{\vec{e}_{(i,j)(r,q)} \in \mathbb{R}^2 \mid C_{r,q} \in N_{i,j}\} \tag{3.2.5}$$

This is, the vector $\vec{e}_{(i,j)(r,q)}$ in all the vectors $\vec{v} = \vec{e}_{(i,j)(r,q)}$ and $\vec{u} = \vec{e}_{(r,q)(i,j)} = -\vec{e}_{(i,j)(r,q)}, \quad \forall C_{r,q} \in N_{i,j}$, that yields the maximum value to the peer to peer convex programming evaluation.

3.3 Uniqueness of Optimal Pair Evaluations

Appendix A provides a demonstration for the uniqueness of the optimal pair:

$$\begin{gathered}\forall\ C_{i,j} \in N_{r,q}, \forall C_{r,q} \in N_{i,j} \\ O_{i,j}(t+1) = \{\vec{e}_{(i,j)(r,q)} \mid C_{r,q}\} \\ \wedge\ O_{r,q}(t+1) = \{\vec{e}_{(r,q)(i,j)} \mid C_{i,j}\} \\ \iff \exists! \vec{e}_{(i,j)(r,q)} \in \mathbb{R}^2, \\ \exists! \vec{e}_{(r,q)(i,j)} \in \mathbb{R}^2 \\ \vec{e}_{(i,j)(r,q)} + \vec{e}_{(r,q)(i,j)} = \vec{0}\end{gathered} \tag{3.3.1}$$

4 Second Stage: Grid State Transition

4.1 Agent State Update

The state of each agent at the next time step is defined by its objective function such that:

$$\begin{gathered}O_{i,j}(t+1) = \{\vec{e}_{(i,j)(r,q)} \mid C_{r,q}\} \\ \wedge\ O_{r,q}(t+1) = \{\vec{e}_{(r,q)(i,j)} \mid C_{i,j}\} \\ \iff s_{i,j}(t+1) = s_{i,j}(t) + \vec{e}_{(i,j)(r,q)} \\ \wedge\ s_{r,q}(t+1) = s_{r,q}(t) + \vec{e}_{(r,q)(i,j)}\end{gathered} \tag{4.1.1}$$

These are the only state transitions that $C_{i,j}$ and $C_{r,q}$ will execute in update G_{t+1}, all others discarded, for all pairs of optimal transactions in their respective neighborhoods.

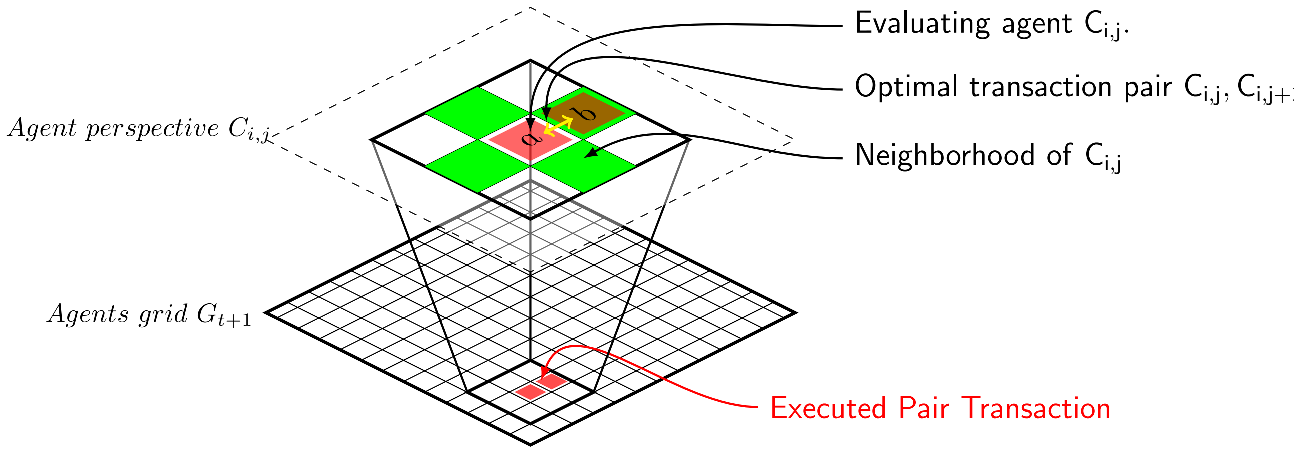

Fig. 2 - Agent $C_{i,j}$ evaluates its Von Neumann neighborhood and registers the exchange that has the best outcome for its objective function. For a transaction to be executed, it needs to evaluate as the optimal for both agents $C_{i,j}, C_{i,j+1}$.

5 Algorithm and Simulation Procedure

Algorithm 1 Distributed Convex Optimization for Multiple Agents

Initialize agent parameters $\mathbf{P}^0_{i,j}$ and state $s^0_{i,j}$for all agents $\{(i,j) \mid 1 \leq i \leq n,\ 1 \leq j \leq m\}$ in grid $G_0^{n\times m}$.
Initialize registers $\mathbf{S1}^{n\times m}$ and $\mathbf{S2}$
for each iteration $t = 0, 1, \ldots, T$ **do**
  Initialize a new grid G_{t+1} of the same size as G_t
  for each agent $C_{i,j}$ **do**
    Get neighborhood $N_{i,j}$ of agent $C_{i,j}$
    for each agent $C_{k,l}$ in $N_{i,j}$ **do**
      Get parameters $\mathbf{P}_{k,l}$
      solve the local optimization problem:

$$\tilde{\mathbf{e}}^{t+1}_{(i,j)(k,l)} = \arg\max_{\tilde{\mathbf{e}}_{(i,j)(k,l)}} \left(o_{\mathbf{P}_{i,j}}(\vec{e}_{(i,j)(k,l)}) + \lambda_{(i,j)(k,l)}\left(\vec{e}_{(i,j)(k,l)} + \vec{e}_{(k,l)(i,j)}\right)\right)$$

where $\lambda_{(i,j)(k,l)}$ is the dual variable associated with the coupling constraint and $\vec{e}_{(k,l)(i,j)}$ is the optimal dual solution from the perspective of agent $C_{k,l}$, this is:

$$\tilde{\mathbf{e}}^{t+1}_{(k,l)(i,j)} = \arg\max_{\tilde{\mathbf{e}}_{ki,l)(i,j)}} \left(o_{\mathbf{P}_{k,l}}(\vec{e}_{(k,l)(i,j)}) + \lambda_{(k,l)(i,j)}\left(\vec{e}_{(k,l)(i,j)} + \vec{e}_{(i,j)(k,l)}\right)\right)$$

where, by the constraint given:

$$\tilde{\mathbf{e}}^{t+1}_{(i,j)(k,l)} + \tilde{\mathbf{e}}^{t+1}_{(k,l)(i,j)} = \vec{0}$$

      register solution $\{\tilde{\mathbf{e}}^{t+1}_{(i,j)(k,l)} \mid C_{k,l} \in N_{i,j}\}$ in $\mathbf{S1}^{n\times m}$
    end for
    Obtain $O_{i,j}(t+1) = \max_{\tilde{\mathbf{e}}^{t+1}_{(i,j)(k,l)}} \{\tilde{\mathbf{e}}^{t+1}_{(i,j)(k,l)} \mid C_{k,l} \in N_{i,j}\} \in \mathbf{S1}^{n\times m}$
    if $\exists O_{k,l}(t+1) \in \mathbf{S1}^{n\times m} \wedge O_{k,l} = \{\tilde{\mathbf{e}}^{t+1}_{(k,l)(i,j)} \mid C_{i,j} \in N_{k,l}\}$ **then**
      Register $(O_{i,j}(t+1), O_{k,l}(t+1))$ in $\mathbf{S2}$
    end if
  end for
  for $(O_{i,j}(t+1), O_{k,l}(t+1)) \in \mathbf{S2}$ **do**
    Update agent states $s^{t+1}_{i,j}, s^{t+1}_{k,l}$ adding the exchange vector to the previous state:

$$s^{t+1}_{i,j} = s^t_{i,j} + \tilde{\mathbf{e}}^{t+1}_{(i,j)(k,l)}$$
$$s^{t+1}_{k,l} = s^t_{i,j} + \tilde{\mathbf{e}}^{t+1}_{(k,l)(i,j)}$$

  end for
end for

A Appendix: Uniqueness of Optimal Pair Evaluations

This appendix provides detailed mathematical demonstration of the theorem stated in section 3.3 "Uniqueness of Optimal Pair Evaluations".

A.1 Uniqueness of the Peer to Peer Convex Programming Evaluation

Let $o_{\mathbf{P}_{i,j}}(\vec{v})$ be the objective function for the convex programming evaluation:

$$\text{maximize} \quad o_{\mathbf{P}_{i,j}}(\vec{v}) \quad \text{subject to} \quad \vec{v}_{(i,j)(k,l)} + \vec{u}_{(k,l)(i,j)} = 0 \tag{A.1.1}$$
$$\text{for} \quad \{o_{\mathbf{P}_{k,l}}(\vec{u}) \mid \forall C_{k,l} \in N_{i,j}\}$$

Let $\vec{e}_{(i,j)(r,q)}$ and $\vec{e}_{(i,j)(r',q')}$ be solutions for the *strictly convex programming evaluation* for the $(C_{i,j}, C_{r,q})$ pair of agents. This is:

$$o_{\mathbf{P}_{k,l}}(\vec{e}_{(i,j)(r,q)}) \geq o_{\mathbf{P}_{k,l}}(\vec{u}) \quad \text{for} \quad \{o_{\mathbf{P}_{k,l}}(\vec{u}) \mid \forall C_{k,l} \in N_{i,j}\}$$
$$\wedge \quad o_{\mathbf{P}_{k,l}}(\vec{e}_{(i,j)(r',q')}) \geq o_{\mathbf{P}_{k,l}}(\vec{u}) \quad \text{for} \quad \{o_{\mathbf{P}_{k,l}}(\vec{u}) \mid \forall C_{k,l} \in N_{i,j}\} \tag{A.1.2}$$
$$\implies o_{\mathbf{P}_{k,l}}(\vec{e}_{(i,j)(r,q)}) \geq o_{\mathbf{P}_{k,l}}(\vec{e}_{(i,j)(r',q')})$$
$$\wedge \quad o_{\mathbf{P}_{k,l}}(\vec{e}_{(i,j)(r,q)}) \leq o_{\mathbf{P}_{k,l}}(\vec{e}_{(i,j)(r',q')}) \tag{A.1.3}$$
$$\implies o_{\mathbf{P}_{k,l}}(\vec{e}_{(i,j)(r,q)}) = o_{\mathbf{P}_{k,l}}(\vec{e}_{(i,j)(r',q')}) \tag{A.1.4}$$

Let :

$$z := \alpha o_{\mathbf{P}_{i,j}}(\vec{e}_{(i,j)(r,q)}) + (1-\alpha) o_{\mathbf{P}_{i,j}}(\vec{e}_{(i,j)(r',q')}) \tag{A.1.5}$$

Then by equation (A.1.4):

$$z := o_{\mathbf{P}_{i,j}}(\vec{e}_{(i,j)(r,q)}) = o_{\mathbf{P}_{i,j}}(\vec{e}_{(i,j)(r',q')}) \tag{A.1.6}$$

z is maximum to the convex program (equation A.1.1) and equal to $o_{\mathbf{P}_{i,j}}(\vec{e}_{(i,j)(r,q)}) = o_{\mathbf{P}_{i,j}}(\vec{e}_{(i,j)(r',q')})$.
The value for the optimization program is unique q.e.d.