Reaction–Diffusion Exchange Economies with Aggregated Cobb–Douglas Preferences for n $\geq$ 2 Goods

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Abstract

This document extends the two-good zero-sum exchange model originally developed in *Emergent Properties of Distributed Agents with Two-Stage Convex Zero-Sum Optimal Exchange Network* (Tristain 2024) to an economy with $n \geq 2$ goods. The extension becomes feasible by aggregating pairwise Cobb–Douglas preference parameters into a globally consistent Cobb–Douglas exponent vector. This aggregation is valid provided a cycle consistency condition holds. The resulting n-good exchange economy preserves the convexity, zero-sum conservation, and tractability of the original model, while enabling higher-dimensional specialization, pattern formation, and reaction–diffusion dynamics analogous to those described by Turing (Turing 1952).

1 Introduction

The two-stage convex zero-sum exchange model in Tristain 2024 introduces an economy in which agents hold endowments of two goods and subsequently exchange them through a peer to peer engagement within a global market that enforces a conservation law. In the two-good case, the conservation constraint ensures exchange trade from agent i to agent j is equal to exchange trade from agent j to agent i (no further endowment is neither created nor destroyed):

$$e^t_{i,j} + e^t_{j,i} = 0.$$

Each trade is solved via convex optimization using a Cobb–Douglas utility function:

$$u_i(x_1, x_2) = x_1^{\alpha_i} x_2^{1-\alpha_i}, \qquad \alpha_j = 1 - \alpha_i.$$

To reinterpret this system as a *reaction–diffusion economy*, we draw inspiration from Turing's reaction–diffusion dynamics (Turing 1952). In this analogy:

- the **reaction phase** corresponds to each agent applying transformation rules (or production technologies) to its own endowment, and
- the **diffusion phase** corresponds to agents participating in structured exchange with the rest of the system.

This produces endogenous spatial or distributional patterns in the economy, just as chemical species form patterns under reaction–diffusion rules.

To generalize this framework to $n \geq 2$ goods while preserving convexity and analytic tractability, we must generalize the utility functions. The key insight is that pairwise Cobb–Douglas preferences can be aggregated into a globally consistent Cobb–Douglas utility function, enabling the full n-good version of the reaction–diffusion exchange economy.

2 Pairwise Cobb–Douglas Preferences

Let the set of goods be:

$$\mathcal{N} = \{1, 2, \dots, n\}.$$

For each pair $i \neq j$, an agent specifies the pairwise Cobb–Douglas preference:

$$u_{ij}(x_i, x_j) = x_i^{\alpha_{ij}} x_j^{1-\alpha_{ij}}, \qquad \alpha_{ij} \in (0, 1).$$

Symmetry imposes:

$$\alpha_{ji} = 1 - \alpha_{ij}.$$

2.1 Pairwise Ratios

From each pair, define:

$$r_{ij} = \frac{1 - \alpha_{ij}}{\alpha_{ij}} > 0,$$

representing the relative intensity of preference for good j over good i.

3 Aggregation to Global Cobb–Douglas Utility

We seek a global utility representation:

$$u(\mathbf{x}) = \prod_{k=1}^{n} x_k^{\beta_k}, \qquad \beta_k > 0,$$

such that the pairwise marginal rates of substitution match the pairwise preferences.

3.1 Consistency Condition

Pairwise coefficients must satisfy:

$$\frac{\beta_i}{\beta_i + \beta_j} = \alpha_{ij} \qquad \forall i \neq j.$$

Equivalently:

$$\frac{\beta_j}{\beta_i} = r_{ij}.$$

A consistent solution exists *if and only if* every directed cycle $i_1 \to i_2 \to \cdots \to i_k \to i_1$ satisfies:

$$\prod_{s=1}^{k} r_{i_s i_{s+1}} = 1 \qquad (i_{k+1} = i_1).$$

3.2 Constructing $\boldsymbol{\beta}$

Choose a reference good, say $\beta_1 = 1$, then recursively define:

$$\beta_j = r_{1j}\beta_1.$$

Normalize:

$$\tilde{\beta}_k = \frac{\beta_k}{\sum_{j=1}^{n} \beta_j}.$$

This yields a unique (up to scale) global Cobb–Douglas exponent vector.

4 The n-Good Reaction–Diffusion Exchange Economy

4.1 Endowments and Conservation

Each agent holds an endowment vector:

$$\mathbf{e}_i^t = (e_{i,1}^t, \ldots, e_{i,n}^t) \in \mathbb{R}_+^n,$$

with global conservation:

$$\sum_{i=1}^{m} \mathbf{e}_i^t = \mathbf{E} \quad \forall t.$$

4.2 Reaction Phase

Agents apply linear technologies $\mathcal{R}$, with:

$$\mathbf{e}_i^{t,\text{react}} = \mathbf{e}_i^t + \sum_{r \in \mathcal{R}} x_{i,r}^t (\mathbf{b}_r - \mathbf{a}_r), \qquad \sum_r x_{i,r}^t \mathbf{a}_r \leq \mathbf{e}_i^t.$$

with $\mathbf{a}_r$ being the endowments to be consumed for producing $\mathbf{b}_r$, given the endowment $x_{i,r}$ for agent i.

5 Diffusion (Exchange) Phase

Let $\mathbf{p}^t \in \mathbb{R}_{++}^n$ be the price vector. Agent i chooses $\lambda_i^t \in [0,1]$ and a trade bundle $\mathbf{z}_i^t$:

$$\mathbf{s}_i^t = \lambda_i^t \mathbf{e}_i^{t,\text{react}}, \qquad x_{i,k}^t(\lambda_i^t) = (1 - \lambda_i^t) e_{i,k}^{t,\text{react}} + z_{i,k}^t.$$

Utility:

$$u_i(\mathbf{x}_i^t) = \prod_{k=1}^{n} x_{i,k}^{\tilde{\beta}_k^{(i)}}.$$

Budget:

$$\mathbf{p}^t \cdot \mathbf{z}_i^t \leq \lambda_i^t (\mathbf{p}^t \cdot \mathbf{e}_i^{t,\text{react}}).$$

Demand:

$$z_{i,k}^t(\lambda_i^t) = \frac{\tilde{\beta}_k^{(i)}}{p_k^t} \lambda_i^t (\mathbf{p}^t \cdot \mathbf{e}_i^{t,\text{react}}).$$

Derivative of log-utility in λ_i^t:

$$\frac{d}{d\lambda_i^t}\log u_i(\lambda_i^t) = \sum_{k=1}^{n} \tilde{\beta}_k^{(i)} \frac{-e_{i,k}^{t,\text{react}} + \dfrac{\tilde{\beta}_k^{(i)}}{p_k^t}(\mathbf{p}^t \cdot \mathbf{e}_i^{t,\text{react}})}{(1-\lambda_i^t)e_{i,k}^{t,\text{react}} + \lambda_i^t \dfrac{\tilde{\beta}_k^{(i)}}{p_k^t}(\mathbf{p}^t \cdot \mathbf{e}_i^{t,\text{react}})}.$$

6 Market Clearing

Supply:

$$Q_k^t = \sum_i \lambda_i^t e_{i,k}^{t,\text{react}}.$$

Demand:

$$D_k^t = \sum_i \frac{\tilde{\beta}_k^{(i)}}{p_k^t} \lambda_i^t (\mathbf{p}^t \cdot \mathbf{e}_i^{t,\text{react}}).$$

Clearing requires $D_k^t = Q_k^t$.

7 Conclusion

Aggregating pairwise Cobb–Douglas preferences into global preferences allows the two-good zero-sum model of Tristain 2024 to be extended to an economy with arbitrary dimension n. The resulting reaction–diffusion exchange economy maintains convexity, tractability, and conservation, while opening the door to richer emergent behaviors in higher-dimensional economic systems.

References

Tristain, Diego Correa (2024). *Emergent Properties of Distributed Agents with Two-Stage Convex Zero-Sum Optimal Exchange Network*. GitHub repository. `https://github.com/onedge-network/Emergent_Properties_paper`.

Turing, Alan M. (1952). "The Chemical Basis of Morphogenesis". In: *Philosophical Transactions of the Royal Society of London. Series B* 237.641, pp. 37–72. DOI: `10.1098/rstb.1952.0012`. URL: `https://doi.org/10.1098/rstb.1952.0012`.