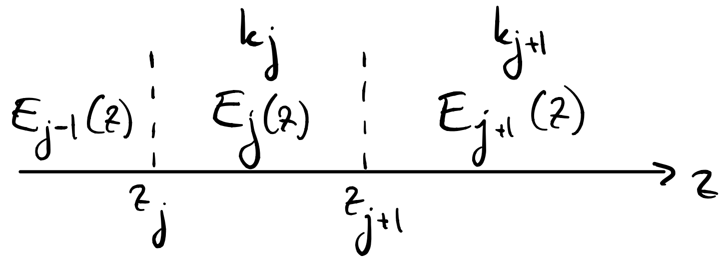

FIGURE 1 – Caption

1 Reflectometry

The propagation of light as a magneto-electric wave of wave vector $\mathbf{k}$ is given by,

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \qquad \mathbf{B}(\mathbf{r},t) = \frac{\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)}{v}$$

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \qquad \omega\mathbf{B}(\mathbf{r},t) = \mathbf{k}\times\mathbf{E}(\mathbf{r},t)$$

In each layer j, there is a wave probagating from the incidence side, and a reflected wave,

$$\begin{aligned}\mathbf{E}_j(\mathbf{r},t) &= \mathbf{E}_{0j}^{(I)} e^{i\left(\mathbf{k}_j^{(I)}\cdot\mathbf{r}-\omega t\right)} + \mathbf{E}_{0j}^{(R)} e^{i\left(\mathbf{k}_j^{(R)}\cdot\mathbf{r}-\omega t\right)} \\ \omega\mathbf{B}_j(\mathbf{r},t) &= \mathbf{k}_j^{(I)}\times\mathbf{E}_{0j}^{(I)} e^{i\left(\mathbf{k}_j^{(I)}\cdot\mathbf{r}-\omega t\right)} + \mathbf{k}_j^{(R)}\times\mathbf{E}_{0j}^{(R)} e^{i\left(\mathbf{k}_j^{(R)}\cdot\mathbf{r}-\omega t\right)}\end{aligned}$$

Wave vector and group speed are related,

$$k_j^{(I,R)} v_j = \omega \qquad k_j^{(I,R)} = |\mathbf{k}_j^{(I,R)}|$$

$$n_j v_j = c \qquad \frac{n_j}{k_j^{(I,R)}} = \frac{c}{\omega} = \frac{\lambda}{2\pi}$$

The interface between layer $j-1$ and j is define as $\mathbf{r}_j$.

$$\mathbf{r}_j = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z_j\hat{\mathbf{z}}$$

The electric field component at the interface need to respect the following equation

$$\begin{aligned}\left(\mathbf{E}_{j-1}(\mathbf{r}_j,t)\right)_{x,y} &= \left(\mathbf{E}_j(\mathbf{r}_j,t)\right)_{x,y} \\ \epsilon_{j-1}\left(\mathbf{E}_{j-1}(\mathbf{r}_j,t)\right)_z &= \epsilon_j\left(\mathbf{E}_j(\mathbf{r}_j,t)\right)_z\end{aligned}$$

while the magnetic field components,

$$\begin{aligned}\frac{1}{\mu_{j-1}}\left(\mathbf{B}_{j-1}(\mathbf{r}_j,t)\right)_{x,y} &= \frac{1}{\mu_j}\left(\mathbf{B}_j(\mathbf{r}_j,t)\right)_{x,y} \\ \left(\mathbf{B}_{j-1}(\mathbf{r}_j,t)\right)_z &= \left(\mathbf{B}_j(\mathbf{r}_j,t)\right)_z\end{aligned}$$

All these equation will have phase factors that must connect at the interface.

$$e^{i\left(\mathbf{k}_j^{(I,R)}\cdot\mathbf{r}_j-\omega t\right)} = e^{-i\omega t}\exp i\left(k_{x,j}^{(I,R)}x + k_{y,j}^{(I,R)}y + k_{z,j}^{(I,R)}z_j\right)$$

The x and y dependance must be the same on each side of each equation, meaning that, in each layer we must have

$$k_{x,j} = k_{x,j}^{(I)} = k_{x,j}^{(R)} \qquad k_{y,j} = k_{y,j}^{(I)} = k_{y,j}^{(R)} \qquad k_{z,j} = k_{z,j}^{(I)} = -k_{z,j}^{(R)}$$

and between planes

$$k_{x,j-1} = k_{x,j} \qquad k_{y,j-1} = k_{y,j}$$

If we define θ_j as the angle between the interface and $\mathbf{k}_j$, we get the following relations (Snell's law).

$$k_{j-1}\cos\theta_{j-1} = k_j\cos\theta_j \qquad \frac{\cos\theta_j}{\cos\theta_{j-1}} = \frac{k_{j-1}}{k_j} = \frac{n_{j-1}}{n_j} = \left(\frac{\epsilon_{j-1}\mu_{j-1}}{\epsilon_j\mu_j}\right)^{1/2}$$

If the first incident beam is k_0 at an angle θ_0, this means,

$$k_0\cos\theta_0 = k_j\cos\theta_j \qquad \frac{k_0}{n_0} = \frac{k_j}{n_j}$$

Polarisation in plane, with $k_y = 0$,

$$\begin{aligned} E_{0x,j}^{(I)} &= E_{0,j}^{(I)}\sin\theta_j & E_{0x,j}^{(R)} &= -E_{0,j}^{(R)}\sin\theta_j \\ E_{0y,j}^{(I)} &= 0 & E_{0y,j}^{(R)} &= 0 \\ E_{0z,j}^{(I)} &= E_{0,j}^{(I)}\cos\theta_j & E_{0z,j}^{(R)} &= E_{0,j}^{(R)}\cos\theta_j \end{aligned}$$

$$\begin{aligned} B_{0x,j}^{(I)} &= 0 & B_{0x,j}^{(R)} &= 0 \\ \omega B_{0y,j}^{(I)} &= k_j^{(I)}E_{0,j}^{(I)} & \omega B_{0y,j}^{(R)} &= k_j^{(R)}E_{0,j}^{(R)} \\ B_{0z,j}^{(I)} &= 0 & B_{0z,j}^{(R)} &= 0 \end{aligned}$$

The relevant equations are

$$\begin{aligned} \sin\theta_{j-1}\left(E_{0,j-1}^{(I)}e^{ik_{z,j-1}z_j} + E_{0,j-1}^{(R)}e^{-ik_{z,j-1}z_j}\right) &= \sin\theta_j\left(E_{0,j}^{(I)}e^{ik_{z,j}z_j} - E_{0,j}^{(R)}e^{-ik_{z,j}z_j}\right) \\ \epsilon_{j-1}\cos\theta_{j-1}\left(E_{0,j-1}^{(I)}e^{ik_{z,j-1}z_j} + E_{0,j-1}^{(R)}e^{-ik_{z,j-1}z_j}\right) &= \epsilon_j\cos\theta_j\left(E_{0,j}^{(I)}e^{ik_{z,j}z_j} + E_{0,j}^{(R)}e^{-ik_{z,j}z_j}\right) \\ \frac{k_{j-1}}{\mu_{j-1}}\left(E_{0,j-1}^{(I)}e^{ik_{z,j-1}z_j} + E_{0,j-1}^{(R)}e^{ik_{z,j-1}z_j}\right) &= \frac{k_j}{\mu_j}\left(E_{0,j}^{(I)}e^{ik_{z,j}z_j} + E_{0,j}^{(R)}e^{ik_{z,j}z_j}\right) \end{aligned}$$

The third one is equivalent to the second one.

$$\mathbf{W}_j \equiv \mathbf{S}_j(z)\cdot\mathbf{E}_j$$

where

$$\mathbf{W}_j(\mathbf{r}) = \begin{pmatrix} E_{0x,j}(\mathbf{r}) \\ \epsilon_j E_{0z,j}(\mathbf{r}) \end{pmatrix} = E_{0,j}(\mathbf{r}) \begin{pmatrix} \sin\theta_j \\ \epsilon_j \cos\theta_j \end{pmatrix} \qquad \mathbf{E}_j = \begin{pmatrix} E_{0,j}^{(I)} \\ E_{0,j}^{(R)} \end{pmatrix} \qquad \mathbf{S}_j(z) = \begin{pmatrix} \alpha_j e^{ik_{z,j}z} & -\alpha_j e^{-ik_{z,j}z} \\ \beta_j e^{ik_{z,j}z} & \beta_j e^{-ik_{z,j}z} \end{pmatrix}$$

where we defined

$$\alpha_j \equiv \sin\theta_j = \left(1 - \frac{n_0^2 \cos^2\theta_0}{n_j^2}\right)^{1/2} \qquad\qquad \beta_j \equiv \epsilon_j \cos\theta_j = \frac{n_j}{\mu_j}\lambda k_0 \cos\theta_0$$

so that,

$$\mathbf{W}_{j-1}(\mathbf{r}_j, t) = \mathbf{W}_j(\mathbf{r}_j, t)$$

$$\det\{\mathbf{S}_j\} = 2\alpha_j\beta_j = 2\epsilon_j \sin\theta_j \cos\theta_j = 2n_j^2 \sin\theta_j \cos\theta_j = 2n_j \sin\theta_j n_0 \cos\theta_0$$

Polarisation out of plane, with $k_y = 0$,

$$\begin{aligned} E_{0x,j}^{(I)} &= 0 \\ E_{0y,j}^{(I)} &= E_{0,j}^{(I)} \\ E_{0z,j}^{(I)} &= 0 \end{aligned} \qquad\qquad \begin{aligned} E_{0x,j}^{(R)} &= 0 \\ E_{0y,j}^{(R)} &= E_{0,j}^{(R)} \\ E_{0z,j}^{(R)} &= 0 \end{aligned}$$

$$\begin{aligned} \omega B_{0x,j}^{(I)} &= k_j E_{0,j}^{(I)} \sin\theta_j \\ B_{0y,j}^{(I)} &= 0 \\ \omega B_{0z,j}^{(I)} &= k_j E_{0,j}^{(I)} \cos\theta_j \end{aligned} \qquad\qquad \begin{aligned} \omega B_{0x,j}^{(R)} &= -k_j E_{0,j}^{(R)} \sin\theta_j \\ B_{0y,j}^{(R)} &= 0 \\ \omega B_{0z,j}^{(R)} &= k_j E_{0,j}^{(R)} \cos\theta_j \end{aligned}$$

The relevant equations are

$$\begin{aligned} E_{0,j-1}^{(I)} e^{ik_{z,j-1}z_j} + E_{0,j-1}^{(R)} e^{-ik_{z,j-1}z_j} &= E_{0,j}^{(I)} e^{ik_{z,j}z_j} + E_{0,j}^{(R)} e^{-ik_{z,j}z_j} \\ k_{j-1}\sin\theta_{j-1}\left(E_{0,j-1}^{(I)} e^{ik_{z,j-1}z_j} + E_{0,j-1}^{(R)} e^{-ik_{z,j-1}z_j}\right) &= k_j \sin\theta_j \left(E_{0,j}^{(I)} e^{ik_{z,j}z_j} - E_{0,j}^{(R)} e^{-ik_{z,j}z_j}\right) \end{aligned}$$

$$\mathbf{W}_j(\mathbf{r}) = \begin{pmatrix} E_{0y,j}(\mathbf{r}) \\ \omega B_{0x,j}(\mathbf{r}) \end{pmatrix} = E_{0,j}(\mathbf{r}) \begin{pmatrix} 1 \\ k_j \sin\theta_j \end{pmatrix} \qquad \mathbf{E}_j = \begin{pmatrix} E_{0,j}^{(I)} \\ E_{0,j}^{(R)} \end{pmatrix} \qquad \mathbf{S}_j(z) = \begin{pmatrix} e^{ik_{z,j}z} & e^{-ik_{z,j}z} \\ k_{z,j} e^{ik_{z,j}z} & -k_{z,j} e^{-ik_{z,j}z} \end{pmatrix}$$

with

$$k_{z,j} \equiv k_j \sin\theta_j = k_j \alpha_j = \frac{2\pi n_j \alpha_j}{\lambda}$$

For X-Rays both polarisation can pe approximate by the later one.

Lets note that $\det\{\mathbf{S}_j\} = -2k_{z,j}$.
In both polarisation what we have at the interface is,

$$\mathbf{W}_{j-1}(\mathbf{r}_j, t) = \mathbf{W}_j(\mathbf{r}_j, t)$$

In case of $n_j < \cos\theta_j$, $k_{z,j}$ is imaginary. We define $\kappa_z = \mathcal{I}\{k_z\}$ and,

$$\mathbf{S}_j(z) = \begin{pmatrix} e^{ik_{z,j}z} & e^{-ik_{z,j}z} \\ k_{z,j}e^{ik_{z,j}z} & -k_{z,j}e^{-ik_{z,j}z} \end{pmatrix} \qquad \mathbf{S}_j(z) = \begin{pmatrix} e^{-\kappa_{z,j}z} & e^{\kappa_{z,j}z} \\ i\kappa_{z,j}e^{-\kappa_{z,j}z} & -i\kappa_{z,j}e^{\kappa_{z,j}z} \end{pmatrix}$$

$$k_{z,j} \to i\kappa_{z,j}$$

1.1 Interface transfer

We define the transfer matrix as,

$$\mathbf{E}_j = \mathbf{T}_j \cdot \mathbf{E}_{j-1}$$

We can find this matrix with the help of the interface condition,

$$\mathbf{W}_j(\mathbf{r}_j) = \mathbf{W}_{j-1}(\mathbf{r}_j)$$
$$\mathbf{S}_j(\mathbf{r}_j) \cdot \mathbf{E}_j = \mathbf{S}_{j-1}(\mathbf{r}_j) \cdot \mathbf{E}_{j-1}$$

so that,

$$\mathbf{T}_j = \mathbf{S}_j^{-1}(\mathbf{r}_j) \cdot \mathbf{S}_{j-1}(\mathbf{r}_j)$$

Let's note that $\det\{\mathbf{T}_j\} = \frac{k_{j-1}}{k_j}$.

$$\begin{aligned}
\mathbf{T}_j &= \frac{1}{2k_{z,j}} \begin{pmatrix} k_{z,j}e^{-ik_{z,j}z_j} & e^{-ik_{z,j}z_j} \\ k_{z,j}e^{ik_{z,j}z_j} & -e^{ik_{z,j}z_j} \end{pmatrix} \begin{pmatrix} e^{ik_{z,j-1}z_j} & e^{-ik_{z,j-1}z_j} \\ k_{z,j-1}e^{ik_{z,j-1}z_j} & -k_{z,j-1}e^{-ik_{z,j-1}z_j} \end{pmatrix} \\
&= \frac{1}{2k_{z,j}} \begin{pmatrix} (k_{z,j} + k_{z,j-1})e^{-i(k_{z,j}-k_{z,j-1})z_j} & (k_{z,j} - k_{z,j-1})e^{-i(k_{z,j}+k_{z,j-1})z_j} \\ (k_{z,j} - k_{z,j-1})e^{i(k_{z,j}+k_{z,j-1})z_j} & (k_{z,j} + k_{z,j-1})e^{i(k_{z,j}-k_{z,j-1})z_j} \end{pmatrix}
\end{aligned}$$

$$\mathbf{T}_j = \frac{1}{2\kappa_{z,j}} \begin{pmatrix} (\kappa_{z,j} + \kappa_{z,j-1})e^{(\kappa_{z,j}-\kappa_{z,j-1})z_j} & (\kappa_{z,j} - \kappa_{z,j-1})e^{(\kappa_{z,j}+\kappa_{z,j-1})z_j} \\ (\kappa_{z,j} - \kappa_{z,j-1})e^{-(\kappa_{z,j}+\kappa_{z,j-1})z_j} & (\kappa_{z,j} + \kappa_{z,j-1})e^{-(\kappa_{z,j}-\kappa_{z,j-1})z_j} \end{pmatrix}$$

To go from the first interface to the last we just need to apply the matrices in sequence.

$$\mathbf{E}_N = \mathbf{L} \cdot \mathbf{E}_0 \qquad \mathbf{L} = \mathbf{T}_N \cdot \mathbf{T}_{N-1} \cdots \mathbf{T}_2 \cdot \mathbf{T}_1 = \prod_{n=1}^{N} \mathbf{T}_n$$

$$\begin{pmatrix} t \\ 0 \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ r \end{pmatrix}$$

$$r = -\frac{L_{21}}{L_{22}} \qquad t = L_{11} - \frac{L_{12}L_{21}}{L_{22}} = \frac{\det\{\mathbf{L}\}}{L_{22}}$$

Let's note that $\det\{\mathbf{L}\} = \prod_{j=1}^{N} \frac{k_{j-1}}{k_j} = \frac{k_0}{k_N}$ if $k_j > 0$ and 0 otherwise.

1.2 Roughness

$$\begin{aligned}\mathbf{L} &= \int \mathrm{d}z\, p_j(z)\mathbf{T}_N \cdots \mathbf{T}_j(z) \cdot \cdots \cdot \mathbf{T}_1 \\ &= \mathbf{T}_N \cdots \left(\int \mathrm{d}z\, p_j(z)\mathbf{T}_j(z)\right) \cdot \cdots \cdot \mathbf{T}_1\end{aligned}$$

$$p_j(z) = \frac{1}{\sigma_j\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{z-z_j}{\sigma_j}\right)^2\right\}$$

$$\int \mathrm{d}z\, p_j(z) e^{i\left(k_{z,j}\pm k_{z,j-1}\right)z} = e^{i\left(k_{z,j}\pm k_{z,j-1}\right)z_j} e^{-\frac{1}{2}\left(k_{z,j}\pm k_{z,j-1}\right)^2\sigma_j^2}$$

$$\begin{aligned}&\int \mathrm{d}z\, p_j(z)\mathbf{T}_j(z) \\ &= \frac{1}{2k_{z,j}} \begin{pmatrix} \left(k_{z,j}+k_{z,j-1}\right) e^{-i\left(k_{z,j}-k_{z,j-1}\right)z_j} e^{-\frac{1}{2}\left(k_{z,j}-k_{z,j-1}\right)^2\sigma_j^2} & \left(k_{z,j}-k_{z,j-1}\right) e^{-i\left(k_{z,j}+k_{z,j-1}\right)z_j} e^{-\frac{1}{2}\left(k_{z,j}+k_{z,j-1}\right)^2\sigma_j^2} \\ \left(k_{z,j}-k_{z,j-1}\right) e^{i\left(k_{z,j}+k_{z,j-1}\right)z_j} e^{-\frac{1}{2}\left(k_{z,j}+k_{z,j-1}\right)^2\sigma_j^2} & \left(k_{z,j}+k_{z,j-1}\right) e^{i\left(k_{z,j}-k_{z,j-1}\right)z_j} e^{-\frac{1}{2}\left(k_{z,j}-k_{z,j-1}\right)^2\sigma_j^2} \end{pmatrix}\end{aligned}$$

$$\int \mathrm{d}z\, p_j(z)\mathbf{T}_j(z) \approx \frac{1}{2k_{z,j}} \begin{pmatrix} 2k_{z,j} e^{-i\left(k_{z,j}-k_{z,j-1}\right)z_j} & \left(k_{z,j}-k_{z,j-1}\right) e^{-i\left(k_{z,j}+k_{z,j-1}\right)z_j} e^{-\frac{1}{2}\left(k_{z,j}+k_{z,j-1}\right)^2\sigma_j^2} \\ \left(k_{z,j}-k_{z,j-1}\right) e^{i\left(k_{z,j}+k_{z,j-1}\right)z_j} e^{-\frac{1}{2}\left(k_{z,j}+k_{z,j-1}\right)^2\sigma_j^2} & \left(k_{z,j}+k_{z,j-1}\right) e^{i\left(k_{z,j}-k_{z,j-1}\right)z_j} e^{-\frac{1}{2}\left(k_{z,j}-k_{z,j-1}\right)^2\sigma_j^2} \end{pmatrix}$$

1.3 Layer Transfer

We define the transfer matrix as,

$$\mathbf{W}_j\left(z_{j+1}\right) = \mathbf{M}_j \cdot \mathbf{W}_j\left(z_j\right)$$

We can find this matrix with the help $\mathbf{E}$

$$\mathbf{S}_j\left(z_{j+1}\right) \cdot \mathbf{E}_j = \mathbf{M}_j \cdot \mathbf{S}_j\left(z_j\right) \cdot \mathbf{E}_j$$

So that,

$$\mathbf{M}_j = \mathbf{S}_j\left(z_{j+1}\right) \cdot \mathbf{S}_j^{-1}\left(z_j\right)$$

$$\begin{aligned}\mathbf{M}_j &= \begin{pmatrix} e^{ik_{z,j}z_{j+1}} & e^{-ik_{z,j}z_{j+1}} \\ k_{z,j}e^{ik_{z,j}z_{j+1}} & -k_{z,j}e^{-ik_{z,j}z_{j+1}} \end{pmatrix} \frac{1}{2k_{z,j}} \begin{pmatrix} k_{z,j}e^{-ik_{z,j}z_j} & e^{-ik_{z,j}z_j} \\ k_{z,j}e^{ik_{z,j}z_j} & -e^{ik_{z,j}z_j} \end{pmatrix} \\ &= \frac{1}{2k_{z,j}} \begin{pmatrix} \cos\left(k_{z,j}d_j\right) & ik_{z,j}^{-1}\sin\left(k_{z,j}d_j\right) \\ ik_{z,j}\sin\left(k_{z,j}d_j\right) & \cos\left(k_{z,j}d_j\right) \end{pmatrix}\end{aligned}$$

To go from the first interface to the last we just need to apply the matrices in sequence.

$$\mathbf{L} = \mathbf{S}_N(z_N)^{-1} \cdot \mathbf{M}_{N-1} \cdot \mathbf{M}_{N-2} \cdots \mathbf{M}_2 \cdot \mathbf{M}_1 \cdot \mathbf{S}_0(z_1)$$

$$r = -e^{i2k_z(a)a}\frac{k_z(b) - k_z(a) + (b-a)k_z(a)k_z(b) - \int_a^b \mathrm{d}z\, k_z^2(z)}{k_z(b) + k_z(a) - (b-a)k_z(a)k_z(b) - \int_a^b \mathrm{d}z\, k_z^2(z)}$$

Explicitly,

$$\begin{aligned}
\mathbf{T}_j &= \frac{1}{2\alpha_j\beta_j}\begin{pmatrix} \beta_j e^{-ik_{z,j}z_j} & \alpha_j e^{-ik_{z,j}z_j} \\ -\beta_j e^{ik_{z,j}z_j} & \alpha_j e^{ik_{z,j}z_j} \end{pmatrix} \cdot \begin{pmatrix} \alpha_{j-1} e^{ik_{z,j-1}z_j} & -\alpha_{j-1} e^{-ik_{z,j-1}z_j} \\ \beta_{j-1} e^{ik_{z,j-1}z_j} & \beta_{j-1} e^{-ik_{z,j-1}z_j} \end{pmatrix} \\
&= \frac{1}{2\alpha_j\beta_j}\begin{pmatrix} \left(\alpha_{j-1}\beta_j + \alpha_j\beta_{j-1}\right) e^{i\left(k_{z,j-1}-k_{z,j}\right)z_j} & \left(-\alpha_{j-1}\beta_j + \alpha_j\beta_{j-1}\right) e^{-i\left(k_{z,j-1}+k_{z,j}\right)z_j} \\ \left(-\alpha_{j-1}\beta_j + \alpha_j\beta_{j-1}\right) e^{i\left(k_{z,j-1}+k_{z,j}\right)z_j} & \left(\alpha_{j-1}\beta_j + \alpha_j\beta_{j-1}\right) e^{-i\left(k_{z,j-1}-k_{z,j}\right)z_j} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{T}(z) &= \frac{1}{2\alpha(z)\beta(z)}\begin{pmatrix} \beta(z) e^{-ik_z(z)z} & \alpha(z) e^{-ik_z(z)z} \\ -\beta(z) e^{ik_z(z)z} & \alpha(z) e^{ik_z(z)z} \end{pmatrix} \cdot \begin{pmatrix} \alpha(z-\delta z) e^{ik_z(z-\delta z)z} & -\alpha(z-\delta z) e^{-ik_z(z-\delta z)z} \\ \beta(z-\delta z) e^{ik_z(z-\delta z)z} & \beta(z-\delta z) e^{-ik_z(z-\delta z)z} \end{pmatrix} \\
&= \frac{1}{2\alpha(z)\beta(z)}\begin{pmatrix} (\alpha(z-\delta z)\beta(z) + \alpha(z)\beta(z-\delta z)) e^{i(k_z(z-\delta z)-k_z(z))z} & (-\alpha(z-\delta z)\beta(z) + \alpha(z)\beta(z-\delta z)) e^{-i(k_z(z-\delta} \\ (-\alpha(z-\delta z)\beta(z) + \alpha(z)\beta(z-\delta z)) e^{i(k_z(z-\delta z)+k_z(z))z} & (\alpha(z-\delta z)\beta(z) + \alpha(z)\beta(z-\delta z)) e^{-i(k_z(z-\delta z)} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\alpha(z-\delta z)\beta(z) + \alpha(z)\beta(z-\delta z) &= 2\alpha(z)\beta(z) - \beta(z)\frac{\mathrm{d}\alpha}{\mathrm{d}z}\delta z - \alpha(z)\frac{\mathrm{d}\beta}{\mathrm{d}z}\delta z \\
&= 2\alpha(z)\beta(z) - \frac{\mathrm{d}(\alpha\beta)}{\mathrm{d}z}\delta z
\end{aligned}$$

$$\begin{aligned}
-\alpha(z-\delta z)\beta(z) + \alpha(z)\beta(z-\delta z) &= \beta(z)\frac{\mathrm{d}\alpha}{\mathrm{d}z}\delta z - \alpha(z)\frac{\mathrm{d}\beta}{\mathrm{d}z}\delta z \\
&= \beta^2(z)\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\alpha}{\beta}\right)\delta z
\end{aligned}$$

$$k_z(z-\delta z) - k_z(z) = -\frac{\mathrm{d}k_z}{\mathrm{d}z}\delta z$$

$$k_z(z-\delta z) + k_z(z) = 2k_z(z) - \frac{\mathrm{d}k_z}{\mathrm{d}z}\delta z$$

$$\begin{aligned}
\mathbf{T}(z) &= \frac{1}{2\alpha(z)\beta(z)}\begin{pmatrix} \left(2\alpha(z)\beta(z) - \frac{\mathrm{d}(\alpha\beta)}{\mathrm{d}z}\delta z\right) e^{-i\left(\frac{\mathrm{d}k_z}{\mathrm{d}z}\delta z\right)z} & \left(\beta^2(z)\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\alpha}{\beta}\right)\delta z\right) e^{-i\left(2k_z(z)-\frac{\mathrm{d}k_z}{\mathrm{d}z}\delta z\right)z} \\ \left(\beta^2(z)\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\alpha}{\beta}\right)\delta z\right) e^{i\left(2k_z(z)-\frac{\mathrm{d}k_z}{\mathrm{d}z}\delta z\right)z} & \left(2\alpha(z)\beta(z) - \frac{\mathrm{d}(\alpha\beta)}{\mathrm{d}z}\delta z\right) e^{i\left(\frac{\mathrm{d}k_z}{\mathrm{d}z}\delta z\right)z} \end{pmatrix} \\
&= \begin{pmatrix} e^{-i\left(\frac{\mathrm{d}k_z}{\mathrm{d}z}\delta z\right)z} & \\ & e^{i\left(\frac{\mathrm{d}k_z}{\mathrm{d}z}\delta z\right)z} \end{pmatrix} + \frac{1}{2\alpha(z)\beta(z)}\begin{pmatrix} \left(-\frac{\mathrm{d}(\alpha\beta)}{\mathrm{d}z}\delta z\right) e^{-i\left(\frac{\mathrm{d}k_z}{\mathrm{d}z}\delta z\right)z} & \left(\beta^2(z)\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\alpha}{\beta}\right)\delta z\right) e^{-i\left(2k_z(z)-\frac{\mathrm{d}k_z}{\mathrm{d}z}\delta z\right)z} \\ \left(\beta^2(z)\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\alpha}{\beta}\right)\delta z\right) e^{i\left(2k_z(z)-\frac{\mathrm{d}k_z}{\mathrm{d}z}\delta z\right)z} & \left(-\frac{\mathrm{d}(\alpha\beta)}{\mathrm{d}z}\delta z\right) e^{i\left(\frac{\mathrm{d}k_z}{\mathrm{d}z}\delta z\right)z} \end{pmatrix} \\
&= \begin{pmatrix} e^{-i\mathrm{d}k_z z} & \\ & e^{i\mathrm{d}k_z z} \end{pmatrix} + \frac{1}{2\alpha(z)\beta(z)}\begin{pmatrix} -\mathrm{d}(\alpha\beta) e^{-i\mathrm{d}k_z z} & \beta^2(z)\mathrm{d}\left(\frac{\alpha}{\beta}\right) e^{-i(2k_z(z)-\mathrm{d}k_z)z} \\ \beta^2(z)\mathrm{d}\left(\frac{\alpha}{\beta}\right) e^{i(2k_z(z)-\mathrm{d}k_z)z} & -\mathrm{d}(\alpha\beta) e^{i\mathrm{d}k_z z} \end{pmatrix}
\end{aligned}$$

$$\mathbf{T}(z) = \mathbb{I} + \frac{1}{2\alpha(z)\beta(z)} \begin{pmatrix} -\mathrm{d}(\alpha\beta) & \beta^2(z)\mathrm{d}\left(\frac{\alpha}{\beta}\right)e^{-i2k_z(z)z} \\ \beta^2(z)\mathrm{d}\left(\frac{\alpha}{\beta}\right)e^{i2k_z(z)z} & -\mathrm{d}(\alpha\beta) \end{pmatrix}$$

$$\delta\mathbf{T} = \frac{1}{2\alpha(z)\beta(z)} \begin{pmatrix} -\mathrm{d}(\alpha\beta) & \beta^2(z)\mathrm{d}\left(\frac{\alpha}{\beta}\right)e^{-i2k_z(z)z} \\ \beta^2(z)\mathrm{d}\left(\frac{\alpha}{\beta}\right)e^{i2k_z(z)z} & -\mathrm{d}(\alpha\beta) \end{pmatrix}$$

$$\prod_{z_j} \mathbf{T}(z_j) = \prod_{z_j} \left(\mathbb{I} + \delta\mathbf{T}(z_j)\right) \approx \mathbb{I} + \sum_{z_j} \delta\mathbf{T}(z_j) + \mathcal{O}(\delta z^2)$$

For a sample with multiple interfaces

$$\mathbf{E}_N = \left(\prod_{n=1}^{N} \cdot \mathbf{T}_n\right) \cdot \mathbf{E}_0$$

Explicitly,

$$E_{N,z} = T_{N,zy}T_{N-1,yx}T_{N-2,xw}\dots T_{3,dc}T_{2,cb}T_{1,ba}E_{0,a}$$

This is a matrix screw product
The second polarisation

$$\mathbf{M}_{j,j+1} = \begin{pmatrix} \cos\left(k_{z,j}d_j\right) & k_{z,j}^{-1}\sin\left(k_{z,j}d_j\right) \\ k_{z,j}\sin\left(k_{z,j}d_j\right) & \cos\left(k_{z,j}d_j\right) \end{pmatrix}$$

$$\mathbf{M}(z) = \begin{pmatrix} \cos\left(k_z(z)d\right) & k_z(z)^{-1}\sin\left(k_z(z)d\right) \\ k_z(z)\sin\left(k_z(z)d\right) & \cos\left(k_z(z)d\right) \end{pmatrix}$$

$$\begin{aligned}\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}z} &= \begin{pmatrix} -\sin\left(k_z(z)d\right)\frac{\mathrm{d}k_z}{\mathrm{d}z}d & -k_z(z)^{-2}\sin\left(k_z(z)d\right)\frac{\mathrm{d}k_z}{\mathrm{d}z} + k_z(z)^{-1}\cos\left(k_z(z)d\right)\frac{\mathrm{d}k_z}{\mathrm{d}z}d \\ \sin\left(k_z(z)d\right)\frac{\mathrm{d}k_z}{\mathrm{d}z} + k_z(z)\cos\left(k_z(z)d\right)\frac{\mathrm{d}k_z}{\mathrm{d}z}d & -\sin\left(k_z(z)d\right)\frac{\mathrm{d}k_z}{\mathrm{d}z}d \end{pmatrix} \\ &= \begin{pmatrix} -\sin\left(k_z(z)d\right) & k_z(z)^{-1}\cos\left(k_z(z)d\right) \\ k_z(z)\cos\left(k_z(z)d\right) & -\sin\left(k_z(z)d\right) \end{pmatrix}\frac{\mathrm{d}k_z}{\mathrm{d}z}d + \begin{pmatrix} 0 & -k_z(z)^{-2}\sin\left(k_z(z)d\right) \\ \sin\left(k_z(z)d\right) & 0 \end{pmatrix}\frac{\mathrm{d}k_z}{\mathrm{d}z}\end{aligned}$$

$$\lim_{\delta z\to 0} \mathbf{M}(z) = \begin{pmatrix} 1 & \delta z \\ k_z^2(z)\delta z & 1 \end{pmatrix} = \mathbb{I} + \delta\mathbf{M}(z)$$

$$\delta\mathbf{M}(z) = \begin{pmatrix} 0 & 1 \\ k_z^2(z) & 0 \end{pmatrix}\delta z$$

$$\prod_{z_j} \mathbf{M}(z_j) = \prod_{z_j} \left(\mathbb{I} + \delta\mathbf{M}(z_j)\right) \approx \mathbb{I} + \sum_{z_j} \delta\mathbf{M}(z_j) + \mathcal{O}(\delta z^2)$$

$$\sum_{z_j} \delta \mathbf{M}(z_j) = \int_a^b \mathrm{d}z \begin{pmatrix} 0 & 1 \\ k_z^2(z) & 0 \end{pmatrix}$$

$$\begin{aligned} k_z^2(z) &= \frac{(2\pi)^2 n^2(z) \alpha^2(z)}{\lambda^2} = \frac{(2\pi)^2 n^2(z)}{\lambda^2} \left(1 - \frac{\cos^2 \theta_0}{n^2(z)} \right) \\ &= \frac{(2\pi)^2}{\lambda^2} \left[n^2(z) - \cos^2 \theta_0 \right] \end{aligned}$$

$$\begin{aligned} \mathbf{T} &= \mathbf{S}^{-1}(b) \cdot \begin{pmatrix} 1 & b-a \\ \int_a^b \mathrm{d}z\, k_z^2(z) & 1 \end{pmatrix} \mathbf{S}(a) \\ &= \frac{1}{2k_z(b)} \begin{pmatrix} k_z(b) e^{-ik_z(b)b} & e^{-ik_z(b)b} \\ k_z(b) e^{ik_z(b)b} & -e^{ik_z(b)b} \end{pmatrix} \begin{pmatrix} 1 & b-a \\ \int_a^b \mathrm{d}z\, k_z^2(z) & 1 \end{pmatrix} \begin{pmatrix} e^{ik_z(a)a} & e^{-ik_z(a)a} \\ k_z(a) e^{ik_z(a)a} & -k_z(a) e^{-ik_z(a)a} \end{pmatrix} \\ &= \frac{1}{2k_z(b)} \begin{pmatrix} k_z(b) e^{-ik_z(b)b} & e^{-ik_z(b)b} \\ k_z(b) e^{ik_z(b)b} & -e^{ik_z(b)b} \end{pmatrix} \begin{pmatrix} e^{ik_z(a)a} + (b-a) k_z(a) e^{ik_z(a)a} & e^{-ik_z(a)a} - (b-a) k_z(a) e^{-ik_z(a)a} \\ e^{ik_z(a)a} \int_a^b \mathrm{d}z\, k_z^2(z) + k_z(a) e^{ik_z(a)a} & e^{-ik_z(a)a} \int_a^b \mathrm{d}z\, k_z^2(z) - k_z(a) e^{-ik_z(a)a} \end{pmatrix} \\ &= \frac{1}{2k_z(b)} \begin{pmatrix} k_z(b) e^{-ik_z(b)b} & e^{-ik_z(b)b} \\ k_z(b) e^{ik_z(b)b} & -e^{ik_z(b)b} \end{pmatrix} \begin{pmatrix} e^{ik_z(a)a} (1 + (b-a) k_z(a)) & e^{-ik_z(a)a} (1 - (b-a) k_z(a)) \\ e^{ik_z(a)a} \left(\int_a^b \mathrm{d}z\, k_z^2(z) + k_z(a) \right) & e^{-ik_z(a)a} \left(\int_a^b \mathrm{d}z\, k_z^2(z) - k_z(a) \right) \end{pmatrix} \\ &= \frac{1}{2k_z(b)} \left(\begin{matrix} k_z(b) e^{-ik_z(b)b} e^{ik_z(a)a} (1 + (b-a) k_z(a)) + e^{-ik_z(b)b} e^{ik_z(a)a} \left(\int_a^b \mathrm{d}z\, k_z^2(z) + k_z(a) \right) & k_z(b) e^{-ik_z(b)b} e^{-ik_z(a)a} (1 - \\ k_z(b) e^{ik_z(b)b} e^{ik_z(a)a} (1 + (b-a) k_z(a)) - e^{ik_z(b)b} e^{ik_z(a)a} \left(\int_a^b \mathrm{d}z\, k_z^2(z) + k_z(a) \right) & k_z(b) e^{ik_z(b)b} e^{-ik_z(a)a} (1 - \end{matrix} \right. \\ &= \frac{1}{2k_z(b)} \left(\begin{matrix} e^{-ik_z(b)b} e^{ik_z(a)a} \left(k_z(b) (1 + (b-a) k_z(a)) + k_z(a) + \int_a^b \mathrm{d}z\, k_z^2(z) \right) & e^{-ik_z(b)b} e^{-ik_z(a)a} \left(k_z(b) (1 - (b-a) k_z(a \right. \\ e^{ik_z(b)b} e^{ik_z(a)a} \left(k_z(b) (1 + (b-a) k_z(a)) - k_z(a) - \int_a^b \mathrm{d}z\, k_z^2(z) \right) & e^{ik_z(b)b} e^{-ik_z(a)a} \left(k_z(b) (1 - (b-a) k_z(a) \right. \end{matrix} \right. \end{aligned}$$

$$r = -\frac{L_{21}}{L_{22}} \qquad\qquad t = L_{11} - \frac{L_{12} L_{21}}{L_{22}}$$

$$r = -e^{i2k_z(a)a} \frac{k_z(b) - k_z(a) + (b-a) k_z(a) k_z(b) - \int_a^b \mathrm{d}z\, k_z^2(z)}{k_z(b) + k_z(a) - (b-a) k_z(a) k_z(b) - \int_a^b \mathrm{d}z\, k_z^2(z)}$$

We new define the propagation matrix as,

$$\mathbf{W}_j(\mathbf{r}_{j+1},t) = \mathbf{M}_{j,j+1}\cdot\mathbf{W}_j(\mathbf{r}_j,t)$$

It can be easily shown that,

$$\mathbf{M}_{j,j+1} = \mathbf{S}_j(\mathbf{r}_{j+1},t)\cdot\mathbf{S}_j^{-1}(\mathbf{r}_j,t)$$

The the parallele polarisation this yield,

$$\begin{aligned}
\mathbf{M}^{(1)}_{j,j+1} &= \frac{1}{2\alpha_j\beta_j}\begin{pmatrix}\alpha_j e^{ik_{z,j}z_{j+1}} & -\alpha_j e^{-ik_{z,j}z_{j+1}}\\ \beta_j e^{ik_{z,j}z_{j+1}} & \beta_j e^{-ik_{z,j}z_{j+1}}\end{pmatrix}\cdot\begin{pmatrix}\beta_j e^{-ik_{z,j}z_j} & \alpha_j e^{-ik_{z,j}z_j}\\ -\beta_j e^{ik_{z,j}z_j} & \alpha_j e^{ik_{z,j}z_j}\end{pmatrix}\\
&= \frac{1}{2\alpha_j\beta_j}\begin{pmatrix}\alpha_j\beta_j\left(e^{ik_{z,j}(z_{j+1}-z_j)}+e^{-ik_{z,j}(z_{j+1}-z_j)}\right) & \alpha_j^2\left(e^{ik_{z,j}(z_{j+1}-z_j)}-e^{-ik_{z,j}(z_{j+1}-z_j)}\right)\\ \beta_j^2\left(e^{ik_{z,j}(z_{j+1}-z_j)}-e^{-ik_{z,j}(z_{j+1}-z_j)}\right) & \alpha_j\beta_j\left(e^{ik_{z,j}(z_{j+1}-z_j)}+e^{-ik_{z,j}(z_{j+1}-z_j)}\right)\end{pmatrix}\\
&= \begin{pmatrix}\cos(k_{z,j}d_j) & \frac{\alpha_j}{\beta_j}\sin(k_{z,j}d_j)\\ \frac{\beta_j}{\alpha_j}\sin(k_{z,j}d_j) & \cos(k_{z,j}d_j)\end{pmatrix}
\end{aligned}$$

The second polarisation

$$\begin{aligned}
\mathbf{M}^{(2)}_{j,j+1} &= \frac{1}{-2k_{z,j}}\begin{pmatrix}e^{ik_{z,j}z_{j+1}} & e^{-ik_{z,j}z_{j+1}}\\ k_{z,j}e^{ik_{z,j}z_{j+1}} & -k_{z,j}e^{-ik_{z,j}z_{j+1}}\end{pmatrix}\cdot\begin{pmatrix}-k_{z,j}e^{-ik_{z,j}z_j} & -e^{-ik_{z,j}z_j}\\ -k_{z,j}e^{ik_{z,j}z_j} & e^{ik_{z,j}z_j}\end{pmatrix}\\
&= \begin{pmatrix}\cos(k_{z,j}d_j) & k_{z,j}^{-1}\sin(k_{z,j}d_j)\\ k_{z,j}\sin(k_{z,j}d_j) & \cos(k_{z,j}d_j)\end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\gamma^{(1)} &= \frac{\beta}{\alpha} = \frac{\epsilon\cos\theta}{\sin\theta} = \frac{n^2\cos\theta}{\mu\sin\theta} = \frac{\lambda^2k^2\cos\theta}{\mu\sin\theta}\\
\gamma^{(2)} &= k\sin\theta
\end{aligned}$$

If it goes through,

$$\mathbf{E}_0 = A\begin{pmatrix}1\\ r\end{pmatrix} \qquad\qquad \mathbf{E}_2 = A\begin{pmatrix}t\\ 0\end{pmatrix}$$

$$\mathbf{W}_{N+1}(z_N) = \mathbf{W}_N(z_N) = \mathbf{M}_{0,N}\cdot\mathbf{W}_0(z_0)$$

$$\mathbf{E}_N = \mathbf{S}_{N+1}^{-1}(z_N)\cdot\mathbf{M}_{0,N}\cdot\mathbf{S}_0(z_0)\cdot\mathbf{E}_0$$

$$\mathbf{L} = \mathbf{S}_{N+1}^{-1}(z_N)\cdot\mathbf{M}_{0,N}\cdot\mathbf{S}_0(z_0)$$

$$\begin{pmatrix}t\\ 0\end{pmatrix} = \begin{pmatrix}L_{11} & L_{12}\\ L_{21} & L_{22}\end{pmatrix}\cdot\begin{pmatrix}1\\ r\end{pmatrix}$$

$$r = -\frac{L_{21}}{L_{22}} \qquad\qquad t = L_{11} - \frac{L_{12}L_{21}}{L_{22}}$$

If a plane reflects,

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ r \end{pmatrix}$$

$$r = -\frac{L_{11}}{L_{12}} \qquad\qquad r = -\frac{L_{21}}{L_{22}}$$

$$\mathbf{M}_{0,1} = \mathbf{S}_1(z_1) \cdot \mathbf{S}_1^{-1}(z_0)$$

Test 1 layer, case 1.

$$\begin{aligned}
\mathbf{SMS} &= \mathbf{S}_2^{-1}(z_1) \cdot \mathbf{M}_{0,1} \cdot \mathbf{S}_0(z_0) \\
&= \frac{1}{\alpha_0\beta_0} \begin{pmatrix} \beta_0 e^{-ik_{z,0}z_1} & \alpha_0 e^{-ik_{z,0}z_1} \\ -\beta_0 e^{ik_{z,0}z_1} & \alpha_0 e^{ik_{z,0}z_1} \end{pmatrix} \cdot \begin{pmatrix} \cos\left(k_{z,1}d_1\right) & \frac{\alpha_1}{\beta_1}\sin\left(k_{z,1}d_1\right) \\ \frac{\beta_1}{\alpha_1}\sin\left(k_{z,1}d_1\right) & \cos\left(k_{z,1}d_1\right) \end{pmatrix} \cdot \begin{pmatrix} \alpha_0 e^{ik_{z,0}z_0} & -\alpha_0 e^{-ik_{z,0}z_0} \\ \beta_0 e^{ik_{z,0}z_0} & \beta_0 e^{-ik_{z,0}z_0} \end{pmatrix}
\end{aligned}$$

Iterative

$$\begin{aligned}
\mathbf{M}^{(2)}_{j-1,j+1} &= \begin{pmatrix} \cos\left(k_{z,j}d_j\right) & k_{z,j}^{-1}\sin\left(k_{z,j}d_j\right) \\ k_{z,j}\sin\left(k_{z,j}d_j\right) & \cos\left(k_{z,j}d_j\right) \end{pmatrix} \cdot \begin{pmatrix} \cos\left(k_{z,j-1}d_{j-1}\right) & k_{z,j-1}^{-1}\sin\left(k_{z,j-1}d_{j-1}\right) \\ k_{z,j-1}\sin\left(k_{z,j-1}d_{j-1}\right) & \cos\left(k_{z,j-1}d_{j-1}\right) \end{pmatrix} \\
&= \begin{pmatrix} \cos\left(k_{z,j}d_j\right)\cos\left(k_{z,j-1}d_{j-1}\right) + \frac{k_{z,j-1}}{k_{z,j}}\sin\left(k_{z,j}d_j\right)\sin\left(k_{z,j-1}d_{j-1}\right) & k_{z,j-1}^{-1}\cos\left(k_{z,j}d_j\right)\sin\left(k_{z,j-1}d_{j-1}\right) + k \\ k_{z,j-1}\cos\left(k_{z,j}d_j\right)\sin\left(k_{z,j-1}d_{j-1}\right) + k_{z,j}\sin\left(k_{z,j}d_j\right)\cos\left(k_{z,j-1}d_{j-1}\right) & \cos\left(k_{z,j}d_j\right)\cos\left(k_{z,j-1}d_{j-1}\right) + \frac{k_{z,j}}{k_{z,j-1}} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{M}_{0,N} &= \mathbf{M}_{N-1,N} \cdot \mathbf{M}_{0,N-1} \\
&= \begin{pmatrix} \cos\left(k_{z,N}d_N\right) & k_{z,N}^{-1}\sin\left(k_{z,N}d_N\right) \\ k_{z,N}\sin\left(k_{z,N}d_N\right) & \cos\left(k_{z,N}d_N\right) \end{pmatrix} \cdot \begin{pmatrix} a_{N-1} & b_{n-1} \\ c_{n-1} & a_{N-1} \end{pmatrix} \\
&= \begin{pmatrix} a_{N-1}\cos\left(k_{z,N}d_N\right) + c_{n-1}k_{z,N}^{-1}\sin\left(k_{z,N}d_N\right) & b_{N-1}\cos\left(k_{z,N}d_N\right) + a_{n-1}k_{z,N}^{-1}\sin\left(k_{z,N}d_N\right) \\ a_{N-1}k_{z,N}\sin\left(k_{z,N}d_N\right) + c_{n-1}\cos\left(k_{z,N}d_N\right) & b_{N-1}k_{z,N}\sin\left(k_{z,N}d_N\right) + a_{n-1}\cos\left(k_{z,N}d_N\right) \end{pmatrix} \\
&= \begin{pmatrix} a_{N-1}\left(1 - \frac{k_{z,N}^2 d_N^2}{2}\right) + c_{n-1}d_N & b_{N-1}\left(1 - \frac{k_{z,N}^2 d_N^2}{2}\right) + a_{n-1}d_N \\ a_{N-1}k_{z,N}^2 d_N + c_{n-1}\left(1 - \frac{k_{z,N}^2 d_N^2}{2}\right) & b_{N-1}k_{z,N}^2 d_N + a_{n-1}\left(1 - \frac{k_{z,N}^2 d_N^2}{2}\right) \end{pmatrix}
\end{aligned}$$

$$a_n = a_{n-1}\left(1 - \frac{k_{z,N}^2 d_N^2}{2}\right) + c_{n-1}d_n \qquad\qquad b_n = b_{n-1}\left(1 - \frac{k_{z,N}^2 d_N^2}{2}\right) + a_{n-1}d_n$$

$$c_{n-1} = \frac{a_n - a_{n-1}}{d_n} + a_{n-1}\frac{k_{z,N}^2 d_N}{2} \qquad\qquad a_{n-1} = \frac{b_n - b_{n-1}}{d_n}$$

$$c(x) = \frac{\mathrm{d}a}{\mathrm{d}x}$$

$$
\begin{aligned}
\mathbf{W}_0(z) &= \mathbf{S}_j(z) \cdot \mathbf{E}_0 \\
&= \begin{pmatrix} A\alpha_0 e^{ik_{z,0}z} - rA\alpha_0 e^{-ik_{z,0}z} \\ A\beta_0 e^{ik_{z,0}z} + rA\beta_0 e^{-ik_{z,0}z} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{W}_2(z) &= \mathbf{S}_j(z) \cdot \mathbf{E}_2 \\
&= \begin{pmatrix} tA\alpha_0 e^{ik_{z,0}z} \\ tA\beta_0 e^{ik_{z,0}z} \end{pmatrix}
\end{aligned}
$$

$$
\mathbf{M}^{(1)} = \begin{pmatrix} \cos\left(k_{z,1}d_1\right) & \frac{\alpha_1}{\beta_1}\sin\left(k_{z,1}d_1\right) \\ \frac{\beta_1}{\alpha_1}\sin\left(k_{z,1}d_1\right) & \cos\left(k_{z,1}d_1\right) \end{pmatrix}
$$

$$
\mathbf{W_2}(z_1) = \mathbf{M} \cdot \mathbf{W_0}(z_0)
$$

$$
\begin{pmatrix} tA\alpha_0 e^{ik_{z,0}z} \\ tA\beta_0 e^{ik_{z,0}z} \end{pmatrix} = \begin{pmatrix} \cos\left(k_{z,1}d_1\right) & \frac{\alpha_1}{\beta_1}\sin\left(k_{z,1}d_1\right) \\ \frac{\beta_1}{\alpha_1}\sin\left(k_{z,1}d_1\right) & \cos\left(k_{z,1}d_1\right) \end{pmatrix} \cdot \begin{pmatrix} A\alpha_0 e^{ik_{z,0}z} - rA\alpha_0 e^{-ik_{z,0}z} \\ A\beta_0 e^{ik_{z,0}z} + rA\beta_0 e^{-ik_{z,0}z} \end{pmatrix}
$$

$$
\begin{pmatrix} t\alpha_0 e^{ik_{z,0}z} \\ t\beta_0 e^{ik_{z,0}z} \end{pmatrix} = \begin{pmatrix} \cos\left(k_{z,1}d_1\right) & \frac{\alpha_1}{\beta_1}\sin\left(k_{z,1}d_1\right) \\ \frac{\beta_1}{\alpha_1}\sin\left(k_{z,1}d_1\right) & \cos\left(k_{z,1}d_1\right) \end{pmatrix} \cdot \begin{pmatrix} \alpha_0 e^{ik_{z,0}z} - rA\alpha_0 e^{-ik_{z,0}z} \\ \beta_0 e^{ik_{z,0}z} + rA\beta_0 e^{-ik_{z,0}z} \end{pmatrix}
$$

Index of refraction for X-Rays is smaller then 1,

$$n = 1 - \delta = 1 - \frac{1}{2\pi} r_e N \lambda^2$$

where $r_e = 2.818 \times 10^{-15}$ m is the electron radius, N is the electronic density, and λ is the wavelength.

$$v = \frac{c}{n} = \frac{\lambda}{n\nu} = \frac{2\pi}{n\nu k}$$

$$Z_j = -\frac{i}{k_j} \qquad\qquad Z_j^{-1} = ik_j$$

$$E_j(z) = E_j^{(r)} e^{ik_j z} + E_j^{(l)} e^{-ik_j z}$$

$$H_j(z) = \frac{1}{Z_j}\left(E_j^{(r)} e^{ik_j z} - E_j^{(l)} e^{-ik_j z}\right)$$

$$\mathbf{W}_j(z) \equiv \begin{pmatrix} E_j(z) \\ H_j(z) \end{pmatrix}$$

$$E_{j-1}(z_j) = E_j(z_j) \qquad\qquad H_{j-1}(z_j) = H_j(z_j)$$

This can also be writen,

$$\mathbf{W}_{j-1}(z_j) = \mathbf{W}_j(z_j)$$

where,

$$\begin{aligned} E_{j-1}^{(r)} e^{ik_{j-1}z_j} + E_{j-1}^{(l)} e^{-ik_{j-1}z_j} &= E_j^{(r)} e^{ik_j z_j} + E_j^{(l)} e^{-ik_j z_j} \\ \frac{1}{Z_{j-1}}\left(E_{j-1}^{(r)} e^{ik_{j-1}z_j} - E_j^{(l)} e^{-ik_{j-1}z_j}\right) &= \frac{1}{Z_j}\left(E_j^{(r)} e^{ik_j z_j} - E_j^{(l)} e^{-ik_j z_j}\right) \end{aligned}$$

$$\mathbf{S}_{j-1}(z_j) \cdot \mathbf{E}_{j-1} = \mathbf{S}_j(z_j) \cdot \mathbf{E}_j$$

with,

$$\mathbf{S}_j(z) = \begin{pmatrix} e^{ik_j z} & e^{-ik_j z} \\ Z_j^{-1} e^{ik_j z} & -Z_j^{-1} e^{-ik_j z} \end{pmatrix} \qquad\qquad \mathbf{E}_j = \begin{pmatrix} E_j^{(r)} \\ E_j^{(l)} \end{pmatrix}$$

$$\mathbf{W}_j(z) = \mathbf{S}_j(z) \cdot \mathbf{E}_j$$

We define the propagation matrix as,

$$\mathbf{W}_j(z_{j+1}) = \mathbf{M}_{j,j+1} \cdot \mathbf{W}_j(z_j)$$

$$\mathbf{S}_j(z_{j+1}) \cdot \mathbf{E}_j = \mathbf{M}_{j,j+1} \cdot \mathbf{S}_j(z_j) \cdot \mathbf{E}_j$$

It can be easily shown that,

$$\mathbf{M}_{j,j+1} = \mathbf{S}_j(z_{j+1}) \cdot \mathbf{S}_j^{-1}(z_j)$$

$$\mathbf{S}_j^{-1}(z) = \frac{Z_j}{2} \begin{pmatrix} Z_j^{-1} e^{-ik_j z} & e^{-ik_j z} \\ Z_j^{-1} e^{ik_j z} & -e^{ik_j z} \end{pmatrix}$$

$$\mathbf{M}_{j,j+1} = \frac{Z_j}{2} \begin{pmatrix} e^{ik_j z_{j+1}} & e^{-ik_j z_{j+1}} \\ Z_j^{-1} e^{ik_j z_{j+1}} & -Z_j^{-1} e^{-ik_j z_{j+1}} \end{pmatrix} \cdot \begin{pmatrix} Z_j^{-1} e^{-ik_j z_j} & e^{-ik_j z_j} \\ Z_j^{-1} e^{ik_j z_j} & -e^{ik_j z_j} \end{pmatrix}$$

$$d_{j,j+1} = z_{j+1} - z_j$$

$$\begin{aligned} \mathbf{M}_{j,j+1} &= \frac{Z_j}{2} \begin{pmatrix} Z_j^{-1}\left(e^{ik_j d_{j,j+1}} + e^{-ik_j d_{j,j+1}}\right) & e^{ik_j d_{j,j+1}} - e^{-ik_j d_{j,j+1}} \\ Z_j^{-2}\left(e^{ik_j d_{j,j+1}} - e^{-ik_j d_{j,j+1}}\right) & Z_j^{-1}\left(e^{ik_j d_{j,j+1}} + e^{-ik_j d_{j,j+1}}\right) \end{pmatrix} \\ &= \begin{pmatrix} \cos\left(k_j d_{j,j+1}\right) & iZ_j \sin\left(k_j d_{j,j+1}\right) \\ iZ_j^{-1} \sin\left(k_j d_{j,j+1}\right) & \cos\left(k_j d_{j,j+1}\right) \end{pmatrix} \end{aligned}$$

If there is N layer

$$\mathbf{W}_N(z_N) = \mathbf{M}_{0,N} \cdot \mathbf{W}_0(z_0)$$

$$\boldsymbol{\mathcal{M}} = \mathbf{M}_{0,N} = \mathbf{M}_{N-1,N} \cdot \ldots \cdot \mathbf{M}_{1,2} \cdot \mathbf{M}_{0,1}$$

The wave above the layer is,

$$E_0(z) = Ae^{ik_0 z} + rAe^{-ik_0 z}$$

as the wave below,

$$E_N(z) = tAe^{ik_N z}$$

$$\mathbf{W}_0(0) = A \begin{pmatrix} 1 & 1 \\ Z_0^{-1} & -Z_0^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ r \end{pmatrix} = \frac{A}{Z_0} \begin{pmatrix} Z_0(1+r) \\ 1-r \end{pmatrix}$$

$$\mathbf{W}_N(L) = A \begin{pmatrix} e^{ik_N L} & e^{-ik_N L} \\ Z_N^{-1} e^{ik_N L} & -Z_N^{-1} e^{-ik_N L} \end{pmatrix} \begin{pmatrix} t \\ 0 \end{pmatrix} = \frac{Ate^{ik_N L}}{Z_N} \begin{pmatrix} Z_N \\ 1 \end{pmatrix}$$

$$\frac{Z_0 te^{ik_N L}}{Z_N} \begin{pmatrix} Z_N \\ 1 \end{pmatrix} = \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{pmatrix} \cdot \begin{pmatrix} Z_0(1+r) \\ 1-r \end{pmatrix}$$

$$\begin{aligned} Z_0 te^{ik_N L} &= \mathcal{M}_{11} Z_0 (1+r) + \mathcal{M}_{12}(1-r) \\ Z_0 te^{ik_N L} &= \mathcal{M}_{21} Z_0 Z_N (1+r) + \mathcal{M}_{22} Z_N (1-r) \end{aligned}$$

$$\begin{aligned} Z_0 te^{ik_N L} &= \mathcal{M}_{11} Z_0 + \mathcal{M}_{12} + r(\mathcal{M}_{11} Z_0 - \mathcal{M}_{12}) \\ Z_0 te^{ik_N L} &= \mathcal{M}_{21} Z_0 Z_N + \mathcal{M}_{22} Z_N + r(\mathcal{M}_{21} Z_0 Z_N - \mathcal{M}_{22} Z_N) \end{aligned}$$

$$\begin{aligned} Z_0 te^{ik_N L} - r(\mathcal{M}_{11} Z_0 - \mathcal{M}_{12}) &= \mathcal{M}_{11} Z_0 + \mathcal{M}_{12} \\ Z_0 te^{ik_N L} - rZ_N(\mathcal{M}_{21} Z_0 - \mathcal{M}_{22}) &= Z_N(\mathcal{M}_{21} Z_0 + \mathcal{M}_{22}) \end{aligned}$$

$$\begin{pmatrix} Z_0 e^{ik_N L} & \mathcal{M}_{12} - \mathcal{M}_{11} Z_0 \\ Z_0 e^{ik_N L} & Z_N(\mathcal{M}_{22} - \mathcal{M}_{21} Z_0) \end{pmatrix} \begin{pmatrix} t \\ r \end{pmatrix} = \begin{pmatrix} \mathcal{M}_{11} Z_0 + \mathcal{M}_{12} \\ Z_N(\mathcal{M}_{21} Z_0 + \mathcal{M}_{22}) \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} t \\ r \end{pmatrix} &= \frac{1}{Z_0 e^{ik_N L}(\mathcal{M}_{22} Z_N - \mathcal{M}_{21} Z_0 Z_N) - Z_0 e^{ik_N L}(\mathcal{M}_{12} - \mathcal{M}_{11} Z_0)} \\ &\quad \times \begin{pmatrix} Z_N(\mathcal{M}_{22} - \mathcal{M}_{21} Z_0) & \mathcal{M}_{11} Z_0 - \mathcal{M}_{12} \\ -Z_0 e^{ik_N L} & Z_0 e^{ik_N L} \end{pmatrix} \begin{pmatrix} \mathcal{M}_{11} Z_0 + \mathcal{M}_{12} \\ Z_N(\mathcal{M}_{21} Z_0 + \mathcal{M}_{22}) \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} t \\ r \end{pmatrix} &= \frac{1}{Z_0 e^{ik_N L}} \frac{1}{Z_N(\mathcal{M}_{22} - \mathcal{M}_{21} Z_0) - (\mathcal{M}_{12} - \mathcal{M}_{11} Z_0)} \\ &\quad \times \begin{pmatrix} Z_N(\mathcal{M}_{22} - \mathcal{M}_{21} Z_0)(\mathcal{M}_{11} Z_0 + \mathcal{M}_{12}) + Z_N(\mathcal{M}_{11} Z_0 - \mathcal{M}_{12})(\mathcal{M}_{21} Z_0 + \mathcal{M}_{22}) \\ -Z_0 e^{ik_N L}(\mathcal{M}_{11} Z_0 + \mathcal{M}_{12}) + Z_0 e^{ik_N L}(\mathcal{M}_{21} Z_0 Z_N + \mathcal{M}_{22} Z_N) \end{pmatrix} \end{aligned}$$

$$\begin{aligned} & Z_N(\mathcal{M}_{22} - \mathcal{M}_{21} Z_0)(\mathcal{M}_{11} Z_0 + \mathcal{M}_{12}) + Z_N(\mathcal{M}_{22} + \mathcal{M}_{21} Z_0)(\mathcal{M}_{11} Z_0 - \mathcal{M}_{12}) \\ &= 2 Z_N Z_0 (\mathcal{M}_{22} \mathcal{M}_{11} - \mathcal{M}_{21} \mathcal{M}_{12}) \\ &= 2 Z_N Z_0 \det\{\boldsymbol{\mathcal{M}}\} \end{aligned}$$

$$t = \frac{2ie^{-ik_N L} k_0 \det\{\boldsymbol{\mathcal{M}}\}}{-\mathcal{M}_{21} + k_0 k_N \mathcal{M}_{12} + i(k_0 \mathcal{M}_{22} + k_N \mathcal{M}_{11})}$$

$$r = \frac{\mathcal{M}_{21} + k_0 k_N \mathcal{M}_{12} + i(k_0 \mathcal{M}_{22} - k_N \mathcal{M}_{11})}{-\mathcal{M}_{21} + k_0 k_N \mathcal{M}_{12} + i(k_0 \mathcal{M}_{22} + k_N \mathcal{M}_{11})}$$

Let's write it down in fonction of z for a slice d thick.

$$\begin{aligned}
\mathbf{T}(z) &= \frac{1}{2k_z(z_+)} \begin{pmatrix} k_z(z_+)e^{-ik_z(z_+)z} & e^{-ik_z(z_+)z} \\ k_z(z_+)e^{ik_z(z_+)z} & -e^{ik_z(z_+)z} \end{pmatrix} \begin{pmatrix} e^{ik_z(z_-)z} & e^{-ik_z(z_-)z} \\ k_z(z_-)e^{ik_z(z_-)z} & -k_z(z_-)e^{-ik_z(z_-)z} \end{pmatrix} \\
&= \frac{1}{2k_z(z_+)} \begin{pmatrix} 2k_z(z)e^{-ik'_z(z)\Delta z z} & k'_z(z)\Delta z e^{-i2k_z(z)z} \\ k'_z(z)\Delta z e^{i2k_z(z)z} & 2k_z(z)e^{ik'_z(z)\Delta z z} \end{pmatrix}
\end{aligned}$$

$$k_z(z_+) + k_z(z_-) = 2k_z(z) \qquad\qquad k_z(z_+) - k_z(z_-) = \frac{\mathrm{d}k_z}{\mathrm{d}z}\Delta z = k'_z(z)\Delta z$$

$$\begin{aligned}
\lim_{\Delta z \to 0} \mathbf{T}(z) &= \frac{1}{2k_z(z)} \begin{pmatrix} 2k_z(z)(1 - ik'_z(z)\Delta z z) & k'_z(z)\Delta z e^{-i2k_z(z)z} \\ k'_z(z)\Delta z e^{i2k_z(z)z} & 2k_z(z)(1 + ik'_z(z)\Delta z z) \end{pmatrix} \\
&= \mathbb{I} + \frac{1}{2k_z(z)} \begin{pmatrix} -i2k_z(z)k'_z(z)z & k'_z(z)e^{-i2k_z(z)z} \\ k'_z(z)e^{i2k_z(z)z} & i2k_z(z)k'_z(z)z \end{pmatrix} \Delta z \\
&= \mathbb{I} + \frac{1}{2k_z(z)} \begin{pmatrix} -i2k_z(z)z & e^{-i2k_z(z)z} \\ e^{i2k_z(z)z} & i2k_z(z)z \end{pmatrix} k'_z(z)\Delta z
\end{aligned}$$

$$\delta\mathbf{T} = \frac{1}{2k_z(z)} \begin{pmatrix} -i2k_z(z)z & e^{-i2k_z(z)z} \\ e^{i2k_z(z)z} & i2k_z(z)z \end{pmatrix} k'_z(z)\delta z$$

$$\prod_{z_j} \mathbf{T}(z_j) = \prod_{z_j} \left(\mathbb{I} + \delta\mathbf{T}(z_j)\right) \approx \mathbb{I} + \sum_{z_j} \delta\mathbf{T}(z_j) + \mathcal{O}(\delta z^2)$$

$$k_z(z) = \frac{2\pi}{\lambda}\left(n^2(z) - n_0^2 \cos^2\theta_0\right)^{1/2}$$

$$\frac{\mathrm{d}k_z}{\mathrm{d}z} = \left(\frac{2\pi}{\lambda}\right)^2 \frac{n(z)}{k_z(z)} \frac{\mathrm{d}n}{\mathrm{d}z}$$

$$\begin{aligned}
\mathbf{T}_j &= \frac{1}{2k_{z,j}} \begin{pmatrix} k_{z,j}e^{-ik_{z,j}z_j} & e^{-ik_{z,j}z_j} \\ k_{z,j}e^{ik_{z,j}z_j} & -e^{ik_{z,j}z_j} \end{pmatrix} \begin{pmatrix} e^{ik_{z,j-1}z_j} & e^{-ik_{z,j-1}z_j} \\ k_{z,j-1}e^{ik_{z,j-1}z_j} & -k_{z,j-1}e^{-ik_{z,j-1}z_j} \end{pmatrix} \\
&= \frac{1}{2k_{z,j}} \begin{pmatrix} (k_{z,j} + k_{z,j-1})e^{-i(k_{z,j}-k_{z,j-1})z_j} & (k_{z,j} - k_{z,j-1})e^{-i(k_{z,j}+k_{z,j-1})z_j} \\ (k_{z,j} - k_{z,j-1})e^{i(k_{z,j}+k_{z,j-1})z_j} & (k_{z,j} + k_{z,j-1})e^{i(k_{z,j}-k_{z,j-1})z_j} \end{pmatrix}
\end{aligned}$$

$$\mathbf{T}_j \approx \begin{pmatrix} e^{-i(k_{z,j}-k_{z,j-1})z_j} & \frac{k_{z,j}-k_{z,j-1}}{k_{z,j}+k_{z,j-1}} e^{-i(k_{z,j}+k_{z,j-1})z_j} \\ \frac{k_{z,j}-k_{z,j-1}}{k_{z,j}+k_{z,j-1}} e^{i(k_{z,j}+k_{z,j-1})z_j} & e^{i(k_{z,j}-k_{z,j-1})z_j} \end{pmatrix}$$

$$\mathbf{T}_{j+1} = \frac{1}{2k_{z,j+1}} \begin{pmatrix} \left(k_{z,j+1} + k_{z,j}\right) e^{-i\left(k_{z,j+1} - k_{z,j}\right) z_{j+1}} & \left(k_{z,j+1} - k_{z,j}\right) e^{-i\left(k_{z,j+1} + k_{z,j}\right) z_{j+1}} \\ \left(k_{z,j+1} - k_{z,j}\right) e^{i\left(k_{z,j+1} + k_{z,j}\right) z_{j+1}} & \left(k_{z,j+1} + k_{z,j}\right) e^{i\left(k_{z,j+1} - k_{z,j}\right) z_{j+1}} \end{pmatrix}$$

$$\mathbf{T}_{j+1} \cdot \mathbf{T}_j = \frac{1}{4k_{z,j}k_{z,j+1}} \Bigg(\begin{matrix} \left(k_{z,j+1} + k_{z,j}\right)\left(k_{z,j} + k_{z,j-1}\right) e^{-i\left(k_{z,j+1} - k_{z,j}\right) z_{j+1}} e^{-i\left(k_{z,j} - k_{z,j-1}\right) z_j} + \left(k_{z,j+1} - k_{z,j}\right)\left(k_{z,j} - k_{z,j-1}\right) e^{-i\big(k_{z,j} \\ \left(k_{z,j+1} - k_{z,j}\right)\left(k_{z,j} + k_{z,j-1}\right) e^{i\left(k_{z,j+1} + k_{z,j}\right) z_{j+1}} e^{-i\left(k_{z,j} - k_{z,j-1}\right) z_j} + \left(k_{z,j+1} + k_{z,j}\right)\left(k_{z,j} - k_{z,j-1}\right) e^{i\big(k_{z,j+1}} \end{matrix}$$

$$\begin{aligned}
11 &= \left(k_{z,j+1} + k_{z,j}\right)\left(k_{z,j} + k_{z,j-1}\right) e^{-i\left(k_{z,j+1} - k_{z,j}\right) z_{j+1}} e^{-i\left(k_{z,j} - k_{z,j-1}\right) z_j} \\
&\quad + \left(k_{z,j+1} - k_{z,j}\right)\left(k_{z,j} - k_{z,j-1}\right) e^{-i\left(k_{z,j+1} + k_{z,j}\right) z_{j+1}} e^{i\left(k_{z,j} + k_{z,j-1}\right) z_j} \\
12 &= \left(k_{z,j+1} + k_{z,j}\right)\left(k_{z,j} - k_{z,j-1}\right) e^{-i\left(k_{z,j+1} - k_{z,j}\right) z_{j+1}} e^{-i\left(k_{z,j} + k_{z,j-1}\right) z_j} \\
&\quad + \left(k_{z,j+1} - k_{z,j}\right)\left(k_{z,j} + k_{z,j-1}\right) e^{-i\left(k_{z,j+1} + k_{z,j}\right) z_{j+1}} e^{i\left(k_{z,j} - k_{z,j-1}\right) z_j} \\
21 &= \left(k_{z,j+1} - k_{z,j}\right)\left(k_{z,j} + k_{z,j-1}\right) e^{i\left(k_{z,j+1} + k_{z,j}\right) z_{j+1}} e^{-i\left(k_{z,j} - k_{z,j-1}\right) z_j} \\
&\quad + \left(k_{z,j+1} + k_{z,j}\right)\left(k_{z,j} - k_{z,j-1}\right) e^{i\left(k_{z,j+1} - k_{z,j}\right) z_{j+1}} e^{i\left(k_{z,j} + k_{z,j-1}\right) z_j} \\
22 &= \left(k_{z,j+1} - k_{z,j}\right)\left(k_{z,j} - k_{z,j-1}\right) e^{i\left(k_{z,j+1} + k_{z,j}\right) z_{j+1}} e^{-i\left(k_{z,j} + k_{z,j-1}\right) z_j} \\
&\quad + \left(k_{z,j+1} + k_{z,j}\right)\left(k_{z,j} + k_{z,j-1}\right) e^{i\left(k_{z,j+1} - k_{z,j}\right) z_{j+1}} e^{i\left(k_{z,j} - k_{z,j-1}\right) z_j}
\end{aligned}$$

$$\begin{aligned}
11 &= \left(k_{z,j+1} + k_{z,j}\right)\left(k_{z,j} + k_{z,j-1}\right) e^{-i\left(k_{z,j+1} z_{j+1} - k_{z,j}\left(z_{j+1} - z_j\right) - k_{z,j-1} z_j\right)} \\
&\quad + \left(k_{z,j+1} - k_{z,j}\right)\left(k_{z,j} - k_{z,j-1}\right) e^{-i\left(k_{z,j+1} z_{j+1} + k_{z,j}\left(z_{j+1} - z_j\right) - k_{z,j-1} z_j\right)} \\
12 &= \left(k_{z,j+1} + k_{z,j}\right)\left(k_{z,j} - k_{z,j-1}\right) e^{-i\left(k_{z,j+1} z_{j+1} - k_{z,j}\left(z_{j+1} - z_j\right) + k_{z,j-1} z_j\right)} \\
&\quad + \left(k_{z,j+1} - k_{z,j}\right)\left(k_{z,j} + k_{z,j-1}\right) e^{-i\left(k_{z,j+1} z_{j+1} + k_{z,j}\left(z_{j+1} - z_j\right) + k_{z,j-1} z_j\right)} \\
21 &= \left(k_{z,j+1} - k_{z,j}\right)\left(k_{z,j} + k_{z,j-1}\right) e^{i\left(k_{z,j+1} + k_{z,j}\right) z_{j+1}} e^{-i\left(k_{z,j} - k_{z,j-1}\right) z_j} \\
&\quad + \left(k_{z,j+1} + k_{z,j}\right)\left(k_{z,j} - k_{z,j-1}\right) e^{i\left(k_{z,j+1} - k_{z,j}\right) z_{j+1}} e^{i\left(k_{z,j} + k_{z,j-1}\right) z_j} \\
22 &= \left(k_{z,j+1} - k_{z,j}\right)\left(k_{z,j} - k_{z,j-1}\right) e^{i\left(k_{z,j+1} + k_{z,j}\right) z_{j+1}} e^{-i\left(k_{z,j} + k_{z,j-1}\right) z_j} \\
&\quad + \left(k_{z,j+1} + k_{z,j}\right)\left(k_{z,j} + k_{z,j-1}\right) e^{i\left(k_{z,j+1} - k_{z,j}\right) z_{j+1}} e^{i\left(k_{z,j} - k_{z,j-1}\right) z_j}
\end{aligned}$$