1 Snoek's limit

$$\left(\mu_0' - 1\right) f_0 = \frac{2}{3}\gamma 4\pi M_s$$

where $4\pi M_s$ is the saturation magnetization and $\gamma \approx 3\,\text{MHz/Oe}$.

2 Reflectometry

2.1 This

$$R(q_z) = R_F \left| \frac{1}{\rho_S} \int \left.\frac{\mathrm{d}\rho}{\mathrm{d}z}\right|_z e^{-iq_z z}\,\mathrm{d}z \right|^2$$

2.2 That

The propagation of light as a magneto-electric wave of wave vector $\mathbf{k}$ is given by,

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \qquad \mathbf{B}(\mathbf{r},t) = \frac{\hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r},t)}{v}$$

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \qquad \omega\mathbf{B}(\mathbf{r},t) = \mathbf{k} \times \mathbf{E}(\mathbf{r},t)$$

In each layer j, there is a wave incident (I) and a reflected (R) wave,

$$\begin{aligned}
\mathbf{E}_j(\mathbf{r},t) &= \left[\mathbf{E}_{0j}^{(\mathrm{I})} \exp\left(i\mathbf{k}_j^{(\mathrm{I})}\cdot\mathbf{r}\right) + \mathbf{E}_{0j}^{(\mathrm{R})} \exp\left(i\mathbf{k}_j^{(\mathrm{R})}\cdot\mathbf{r}\right)\right] e^{-i\omega t} \\
\omega\mathbf{B}_j(\mathbf{r},t) &= \left[\mathbf{k}_j^{(\mathrm{I})} \times \mathbf{E}_{0j}^{(\mathrm{I})} \exp\left(i\mathbf{k}_j^{(\mathrm{I})}\cdot\mathbf{r}\right) + \mathbf{k}_j^{(\mathrm{R})} \times \mathbf{E}_{0j}^{(\mathrm{R})} \exp\left(i\mathbf{k}_j^{(\mathrm{R})}\cdot\mathbf{r}\right)\right] e^{-i\omega t}
\end{aligned}$$

Wave vector and group speed are related,

$$k_j^{(\mathrm{I,R})} v_j = \omega \qquad k_j^{(\mathrm{I,R})} = \left|\mathbf{k}_j^{(\mathrm{I,R})}\right|$$

$$n_j v_j = c \qquad \frac{n_j}{k_j^{(\mathrm{I,R})}} = \frac{c}{\omega} = \frac{\lambda}{2\pi}$$

The interface between layer $j-1$ and j is define as $\mathbf{r}_j$.

$$\mathbf{r}_j = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z_j\hat{\mathbf{z}}$$

The electric field component at the interface need to respect the following equation

$$\begin{aligned}
\left(\mathbf{E}_{j-1}(\mathbf{r}_j,t)\right)_{x,y} &= \left(\mathbf{E}_j(\mathbf{r}_j,t)\right)_{x,y} \\
\epsilon_{j-1}\left(\mathbf{E}_{j-1}(\mathbf{r}_j,t)\right)_z &= \epsilon_j\left(\mathbf{E}_j(\mathbf{r}_j,t)\right)_z
\end{aligned}$$

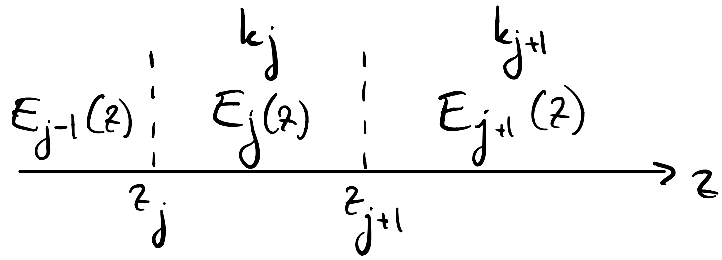

FIGURE 1 – Caption

while the magnetic field components,

$$\frac{1}{\mu_{j-1}}\left(\mathbf{B}_{j-1}(\mathbf{r}_j,t)\right)_{x,y} = \frac{1}{\mu_j}\left(\mathbf{B}_j(\mathbf{r}_j,t)\right)_{x,y}$$
$$\left(\mathbf{B}_{j-1}(\mathbf{r}_j,t)\right)_z = \left(\mathbf{B}_j(\mathbf{r}_j,t)\right)_z$$

Note that, at the frequency of X rays, the relative magnetic permeabilty of matter is unity up to many orders of magnitude, (Snoek's limit)

$$\mu_j \sim \mu_0 \qquad \frac{\Delta\mu}{\mu_0} \ll \frac{\Delta\epsilon}{\epsilon_0}$$

As an example, the continuity of the x component of the electric field gives the following equation,

$$E^{(\mathrm{I})}_{0z,j-1}\exp\left(i\mathbf{k}^{(\mathrm{I})}_{j-1}\cdot\mathbf{r}\right) + E^{(\mathrm{R})}_{0z,j-1}\exp\left(i\mathbf{k}^{(\mathrm{R})}_{j-1}\cdot\mathbf{r}\right) = E^{(\mathrm{I})}_{0z,j}\exp\left(i\mathbf{k}^{(\mathrm{I})}_{j}\cdot\mathbf{r}\right) + E^{(\mathrm{R})}_{0z,j}\exp\left(i\mathbf{k}^{(\mathrm{R})}_{j}\cdot\mathbf{r}\right)$$

The x and y dependance must be the same for each term of this equation, meaning that, in each layer we must have

$$k_{x,j} = k^{(\mathrm{I})}_{x,j} = k^{(\mathrm{R})}_{x,j} \qquad k_{y,j} = k^{(\mathrm{I})}_{y,j} = k^{(\mathrm{R})}_{y,j}$$

and therefore

$$k_{z,j} = k^{(\mathrm{I})}_{z,j} = -k^{(\mathrm{R})}_{z,j}.$$

Between planes

$$k_{x,j-1} = k_{x,j} \qquad k_{y,j-1} = k_{y,j}$$

If we define θ_j as the angle between the interface and $\mathbf{k}_j$, we get the following relations (Snell's law).

$$k_{j-1}\cos\theta_{j-1} = k_j\cos\theta_j \qquad \frac{\cos\theta_j}{\cos\theta_{j-1}} = \frac{k_{j-1}}{k_j} = \frac{n_{j-1}}{n_j} = \left(\frac{\epsilon_{j-1}\mu_{j-1}}{\epsilon_j\mu_j}\right)^{1/2} \sim \left(\frac{\epsilon_{j-1}}{\epsilon_j}\right)^{1/2}$$

If the first incident beam is k_0 at an angle θ_0, this means,

$$k_0\cos\theta_0 = k_j\cos\theta_j \qquad \frac{k_0}{n_0} = \frac{k_j}{n_j}$$

We will choose the direction of propagation in order to have $k_y = 0$.

$$
\begin{aligned}
k_{z,j} &= k_j \sin\theta_j \\
&= k_0 n_j \left(1 - \cos^2\theta_j\right)^{1/2} \\
&= k_0 \left(n_j^2 - n_j^2 \cos^2\theta_j\right)^{1/2} \\
&= k_0 \left(n_j^2 - n_0^2 \cos^2\theta_0\right)^{1/2}
\end{aligned}
$$

2.3 In plane polarisation

If the polarization of the electric field is in the propagration plane (xz), the components of the electric field for the incident and reflected waves are,

$$
\begin{aligned}
E_{0x,j}^{(\mathrm{I,p})} &= E_{0,j}^{(\mathrm{I,p})} \sin\theta_j & \qquad E_{0x,j}^{(\mathrm{R,p})} &= -E_{0,j}^{(\mathrm{R,p})} \sin\theta_j \\
E_{0y,j}^{(\mathrm{I,p})} &= 0 & E_{0y,j}^{(\mathrm{R,p})} &= 0 \\
E_{0z,j}^{(\mathrm{I,p})} &= E_{0,j}^{(\mathrm{I,p})} \cos\theta_j & E_{0z,j}^{(\mathrm{R,p})} &= E_{0,j}^{(\mathrm{R,p})} \cos\theta_j
\end{aligned}
$$

while the components of the magnetic field are,

$$
\begin{aligned}
B_{0x,j}^{(\mathrm{I,p})} &= 0 & \qquad B_{0x,j}^{(\mathrm{R,p})} &= 0 \\
\omega B_{0y,j}^{(\mathrm{I,p})} &= k_j^{(\mathrm{I,p})} E_{0,j}^{(\mathrm{I,p})} & \omega B_{0y,j}^{(\mathrm{R,p})} &= k_j^{(\mathrm{R,p})} E_{0,j}^{(\mathrm{R,p})} \\
B_{0z,j}^{(\mathrm{I,p})} &= 0 & B_{0z,j}^{(\mathrm{R,p})} &= 0
\end{aligned}
$$

The relevant equations are

$$
\begin{aligned}
\sin\theta_{j-1}\left(E_{0,j-1}^{(\mathrm{I,p})} e^{ik_{z,j-1}z_j} - E_{0,j-1}^{(\mathrm{R,p})} e^{-ik_{z,j-1}z_j}\right) &= \sin\theta_j \left(E_{0,j}^{(\mathrm{I,p})} e^{ik_{z,j}z_j} - E_{0,j}^{(\mathrm{R,p})} e^{-ik_{z,j}z_j}\right) \\
\epsilon_{j-1}\cos\theta_{j-1}\left(E_{0,j-1}^{(\mathrm{I,p})} e^{ik_{z,j-1}z_j} + E_{0,j-1}^{(\mathrm{R,p})} e^{-ik_{z,j-1}z_j}\right) &= \epsilon_j \cos\theta_j \left(E_{0,j}^{(\mathrm{I,p})} e^{ik_{z,j}z_j} + E_{0,j}^{(\mathrm{R,p})} e^{-ik_{z,j}z_j}\right) \\
k_{j-1}\left(E_{0,j-1}^{(\mathrm{I,p})} e^{ik_{z,j-1}z_j} + E_{0,j-1}^{(\mathrm{R,p})} e^{ik_{z,j-1}z_j}\right) &= k_j \left(E_{0,j}^{(\mathrm{I,p})} e^{ik_{z,j}z_j} + E_{0,j}^{(\mathrm{R,p})} e^{ik_{z,j}z_j}\right)
\end{aligned}
$$

Since,

$$
\frac{\epsilon_{j-1}\cos\theta_{j-1}}{\epsilon_j \cos\theta_j} = \frac{k_{j-1}}{k_j} \qquad \rightarrow \qquad \frac{\cos\theta_{j-1} k_j}{\cos\theta_j k_{j-1}} = \frac{\epsilon_j}{\epsilon_{j-1}}
$$

the third equation is equivalent to the second one.

2.4 Out of plane polarisation

If the polarization of the electric field is out of the propagration plane (y), the components of the electric field for the incident and reflected waves are,

$$
\begin{aligned}
E_{0x,j}^{(\mathrm{I,s})} &= 0 & \qquad E_{0x,j}^{(\mathrm{R,s})} &= 0 \\
E_{0y,j}^{(\mathrm{I,s})} &= E_{0,j}^{(\mathrm{I,s})} & E_{0y,j}^{(\mathrm{R,s})} &= E_{0,j}^{(\mathrm{R,s})} \\
E_{0z,j}^{(\mathrm{I,s})} &= 0 & E_{0z,j}^{(\mathrm{R,s})} &= 0
\end{aligned}
$$

while the components of the magnetic field are,

$$
\begin{aligned}
\omega B^{(\mathrm{I,s})}_{0x,j} &= k_j E^{(\mathrm{I,s})}_{0,j} \sin\theta_j & \omega B^{(\mathrm{R,s})}_{0x,j} &= -k_j E^{(\mathrm{R,s})}_{0,j} \sin\theta_j \\
B^{(\mathrm{I,s})}_{0y,j} &= 0 & B^{(\mathrm{R,s})}_{0y,j} &= 0 \\
\omega B^{(\mathrm{I,s})}_{0z,j} &= k_j E^{(\mathrm{I,s})}_{0,j} \cos\theta_j & \omega B^{(\mathrm{R,s})}_{0z,j} &= k_j E^{(\mathrm{R,s})}_{0,j} \cos\theta_j
\end{aligned}
$$

The relevant equations are then,

$$
\begin{aligned}
E^{(\mathrm{I,s})}_{0,j-1} e^{ik_{z,j-1}z_j} + E^{(\mathrm{R,s})}_{0,j-1} e^{-ik_{z,j-1}z_j} &= E^{(\mathrm{I,s})}_{0,j} e^{ik_{z,j}z_j} + E^{(\mathrm{R,s})}_{0,j} e^{-ik_{z,j}z_j} \\
k_{j-1}\sin\theta_{j-1}\left(E^{(\mathrm{I,s})}_{0,j-1} e^{ik_{z,j-1}z_j} - E^{(\mathrm{R,s})}_{0,j-1} e^{-ik_{z,j-1}z_j}\right) &= k_j \sin\theta_j \left(E^{(\mathrm{I,s})}_{0,j} e^{ik_{z,j}z_j} - E^{(\mathrm{R,s})}_{0,j} e^{-ik_{z,j}z_j}\right) \\
k_{j-1}\cos\theta_{j-1}\left(E^{(\mathrm{I,s})}_{0,j-1} e^{ik_{z,j-1}z_j} + E^{(\mathrm{R,s})}_{0,j-1} e^{-ik_{z,j-1}z_j}\right) &= k_j \cos\theta_j \left(E^{(\mathrm{I,s})}_{0,j} e^{ik_{z,j}z_j} + E^{(\mathrm{R,s})}_{0,j} e^{-ik_{z,j}z_j}\right)
\end{aligned}
$$

First and third equations are equivalent because,

$$
\frac{k_{j-1}\cos\theta_{j-1}}{k_j\cos\theta_j} = 1
$$

2.5 Sets of equations

$$
\begin{aligned}
\sin\theta_{j-1}\left(E^{(\mathrm{I,p})}_{0,j-1} e^{ik_{z,j-1}z_j} - E^{(\mathrm{R,p})}_{0,j-1} e^{-ik_{z,j-1}z_j}\right) &= \sin\theta_j \left(E^{(\mathrm{I,p})}_{0,j} e^{ik_{z,j}z_j} - E^{(\mathrm{R,p})}_{0,j} e^{-ik_{z,j}z_j}\right) \\
k_{j-1}\left(E^{(\mathrm{I})}_{0,j-1} e^{ik_{z,j-1}z_j} + E^{(\mathrm{R})}_{0,j-1} e^{ik_{z,j-1}z_j}\right) &= k_j \left(E^{(\mathrm{I})}_{0,j} e^{ik_{z,j}z_j} + E^{(\mathrm{R})}_{0,j} e^{ik_{z,j}z_j}\right)
\end{aligned}
$$

$$
\mathbf{S}^{(\mathrm{p})}_j(z)\cdot\mathbf{E}^{(\mathrm{p})}_j \equiv \begin{pmatrix} \sin\theta_j e^{ik_{z,j}z} & -\sin\theta_j e^{-ik_{z,j}z} \\ k_j e^{ik_{z,j}z} & k_j e^{-ik_{z,j}z} \end{pmatrix} \begin{pmatrix} E^{(\mathrm{I,p})}_{0,j} \\ E^{(\mathrm{R,p})}_{0,j} \end{pmatrix}
$$

$$
\begin{aligned}
E^{(\mathrm{I,s})}_{0,j-1} e^{ik_{z,j-1}z_j} + E^{(\mathrm{R,s})}_{0,j-1} e^{-ik_{z,j-1}z_j} &= E^{(\mathrm{I,s})}_{0,j} e^{ik_{z,j}z_j} + E^{(\mathrm{R,s})}_{0,j} e^{-ik_{z,j}z_j} \\
k_{j-1}\sin\theta_{j-1}\left(E^{(\mathrm{I,s})}_{0,j-1} e^{ik_{z,j-1}z_j} - E^{(\mathrm{R,s})}_{0,j-1} e^{-ik_{z,j-1}z_j}\right) &= k_j \sin\theta_j \left(E^{(\mathrm{I,s})}_{0,j} e^{ik_{z,j}z_j} - E^{(\mathrm{R,s})}_{0,j} e^{-ik_{z,j}z_j}\right)
\end{aligned}
$$

$$
\mathbf{S}^{(\mathrm{s})}_j(z)\cdot\mathbf{E}^{(\mathrm{s})}_j \equiv \begin{pmatrix} e^{ik_{z,j}z} & e^{-ik_{z,j}z} \\ k_j\sin\theta_j e^{ik_{z,j}z} & -k_j\sin\theta_j e^{-ik_{z,j}z} \end{pmatrix} \begin{pmatrix} E^{(\mathrm{I,s})}_{0,j} \\ E^{(\mathrm{R,s})}_{0,j} \end{pmatrix}
$$

For either polarisation the relation between $\mathbf{E}_j$ and $\mathbf{E}_{j-1}$ is given by the following equation,

$$
\mathbf{S}_{j-1}(z_j)\cdot\mathbf{E}_{j-1} = \mathbf{S}_j(z_j)\cdot\mathbf{E}_j
$$

We then compute $\mathbf{E}_j$ from $\mathbf{E}_{j-1}$ with

$$
\mathbf{E}_j = \mathbf{T}_j\cdot\mathbf{E}_{j-1}
$$

with

$$\mathbf{T}_j \equiv \mathbf{S}_j^{-1}(z_j) \cdot \mathbf{S}_{j-1}(z_j)$$

$$\begin{aligned}
\mathbf{T}_j^{(\mathrm{p})} &= \frac{1}{2k_j \sin\theta_j} \begin{pmatrix} k_j e^{-ik_{z,j}z_j} & \sin\theta_j e^{-ik_{z,j}z_j} \\ -k_j e^{ik_{z,j}z_j} & \sin\theta_j e^{ik_{z,j}z_j} \end{pmatrix} \begin{pmatrix} \sin\theta_{j-1} e^{ik_{z,j-1}z_j} & -\sin\theta_{j-1} e^{-ik_{z,j-1}z_j} \\ k_{j-1} e^{ik_{z,j-1}z_j} & k_{j-1} e^{-ik_{z,j-1}z_j} \end{pmatrix} \\
&= \frac{1}{2k_{z,j}} \begin{pmatrix} \left(\frac{n_j}{n_{j-1}} k_{z,j-1} + \frac{n_{j-1}}{n_j} k_{z,j}\right) e^{-ik_{z,j}^- z_j} & \left(-\frac{n_j}{n_{j-1}} k_{z,j-1} + \frac{n_{j-1}}{n_j} k_{z,j}\right) e^{-ik_{z,j}^+ z_j} \\ \left(-\frac{n_j}{n_{j-1}} k_{z,j-1} + \frac{n_{j-1}}{n_j} k_{z,j}\right) e^{ik_{z,j}^+ z_j} & \left(\frac{n_j}{n_{j-1}} k_{z,j-1} + \frac{n_{j-1}}{n_j} k_{z,j}\right) e^{ik_{z,j}^- z_j} \end{pmatrix} \\
&= \frac{1}{2n_{j-1}n_j k_{z,j}} \begin{pmatrix} \left(n_{j-1}^2 k_{z,j} + n_j^2 k_{z,j-1}\right) e^{-ik_{z,j}^- z_j} & \left(n_{j-1}^2 k_{z,j} - n_j^2 k_{z,j-1}\right) e^{-ik_{z,j}^+ z_j} \\ \left(n_{j-1}^2 k_{z,j} - n_j^2 k_{z,j-1}\right) e^{ik_{z,j}^+ z_j} & \left(n_{j-1}^2 k_{z,j} + n_j^2 k_{z,j-1}\right) e^{ik_{z,j}^- z_j} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\det\left\{\mathbf{T}_j^{(\mathrm{p})}\right\} &= \frac{\left(n_{j-1}^2 k_{z,j} + n_j^2 k_{z,j-1}\right)^2 - \left(n_{j-1}^2 k_{z,j} - n_j^2 k_{z,j-1}\right)^2}{4n_{j-1}^2 n_j^2 k_{z,j}^2} \\
&= \frac{k_{z,j-1}}{k_{z,j}}
\end{aligned}$$

$$\begin{aligned}
\mathbf{T}_j^{(\mathrm{s})} &= \frac{1}{-2k_j \sin\theta_j} \begin{pmatrix} -k_j \sin\theta_j e^{-ik_{z,j}z_j} & -e^{-ik_{z,j}z_j} \\ -k_j \sin\theta_j e^{ik_{z,j}z_j} & e^{ik_{z,j}z_j} \end{pmatrix} \begin{pmatrix} e^{ik_{z,j-1}z_j} & e^{-ik_{z,j-1}z_j} \\ k_{j-1}\sin\theta_{j-1} e^{ik_{z,j-1}z_j} & -k_{j-1}\sin\theta_{j-1} e^{-ik_{z,j-1}z_j} \end{pmatrix} \\
&= \frac{1}{2k_{z,j}} \begin{pmatrix} k_{z,j}^+ e^{-ik_{z,j}^- z_j} & k_{z,j}^- e^{-ik_{z,j}^+ z_j} \\ k_{z,j}^- e^{ik_{z,j}^+ z_j} & k_{z,j}^+ e^{ik_{z,j}^- z_j} \end{pmatrix}
\end{aligned}$$

with

$$k_{z,j}^{\pm} = k_{z,j} \pm k_{z,j-1}$$

Note that,

$$\begin{aligned}
\det\left\{\mathbf{T}_j^{(\mathrm{s})}\right\} &= \frac{\left(k_{z,j} + k_{z,j-1}\right)^2 - \left(k_{z,j} - k_{z,j-1}\right)^2}{4k_{z,j}^2} \\
&= \frac{k_{z,j-1}}{k_{z,j}}
\end{aligned}$$

$$\mathbf{T}_j^{(\mathrm{p,s})} = \begin{pmatrix} p_j^{(\mathrm{p,s})} e^{-ik_{z,j}^- z_j} & m_j^{(\mathrm{p,s})} e^{-ik_{z,j}^+ z_j} \\ m_j^{(\mathrm{p,s})} e^{ik_{z,j}^+ z_j} & p_j^{(\mathrm{p,s})} e^{ik_{z,j}^- z_j} \end{pmatrix}$$

with,

$$p_j^{(\mathrm{p})} = \frac{k_{z,j} + k_{z,j-1}}{2k_{z,j}} \qquad m_j^{(\mathrm{p})} = \frac{k_{z,j} - k_{z,j-1}}{2k_{z,j}}$$

$$p_j^{(\mathrm{s})} = \frac{n_{j-1}^2 k_{z,j} + n_j^2 k_{z,j-1}}{2n_j n_{j-1} k_{z,j}} \qquad m_j^{(\mathrm{s})} = \frac{n_{j-1}^2 k_{z,j} - n_j^2 k_{z,j-1}}{2n_j n_{j-1} k_{z,j}}$$

We define the following vector and matrix

$$\mathbf{W}_j \equiv \mathbf{S}_j(z) \cdot \mathbf{E}_j$$

where

$$\mathbf{W}_j(\mathbf{r}) = \begin{pmatrix} E_{0x,j}(\mathbf{r}) \\ \epsilon_j E_{0z,j}(\mathbf{r}) \end{pmatrix} = E_{0,j}(\mathbf{r}) \begin{pmatrix} \sin\theta_j \\ \epsilon_j \cos\theta_j \end{pmatrix} \qquad \mathbf{E}_j = \begin{pmatrix} E^{(\mathrm{I})}_{0,j} \\ E^{(\mathrm{R})}_{0,j} \end{pmatrix} \qquad \mathbf{S}_j(z) = \begin{pmatrix} \alpha_j e^{ik_{z,j}z} & -\alpha_j e^{-ik_{z,j}z} \\ \beta_j e^{ik_{z,j}z} & \beta_j e^{-ik_{z,j}z} \end{pmatrix}$$

where we defined

$$\alpha_j \equiv \sin\theta_j = \left(1 - \frac{n_0^2 \cos^2\theta_0}{n_j^2}\right)^{1/2} \qquad\qquad \beta_j \equiv \epsilon_j \cos\theta_j = \frac{n_j}{\mu_j} \lambda k_0 \cos\theta_0$$

so that,

$$\mathbf{W}_{j-1}(\mathbf{r}_j, t) = \mathbf{W}_j(\mathbf{r}_j, t)$$

$$\det\{\mathbf{S}_j\} = 2\alpha_j\beta_j = 2\epsilon_j \sin\theta_j \cos\theta_j = 2n_j^2 \sin\theta_j \cos\theta_j = 2n_j \sin\theta_j n_0 \cos\theta_0$$

Polarisation out of plane, with $k_y = 0$,

$$\begin{aligned} E^{(\mathrm{I})}_{0x,j} &= 0 & \qquad E^{(\mathrm{R})}_{0x,j} &= 0 \\ E^{(\mathrm{I})}_{0y,j} &= E^{(\mathrm{I})}_{0,j} & E^{(\mathrm{R})}_{0y,j} &= E^{(\mathrm{R})}_{0,j} \\ E^{(\mathrm{I})}_{0z,j} &= 0 & E^{(\mathrm{R})}_{0z,j} &= 0 \end{aligned}$$

$$\begin{aligned} \omega B^{(\mathrm{I})}_{0x,j} &= k_j E^{(\mathrm{I})}_{0,j} \sin\theta_j & \qquad \omega B^{(\mathrm{R})}_{0x,j} &= -k_j E^{(\mathrm{R})}_{0,j} \sin\theta_j \\ B^{(\mathrm{I})}_{0y,j} &= 0 & B^{(\mathrm{R})}_{0y,j} &= 0 \\ \omega B^{(\mathrm{I})}_{0z,j} &= k_j E^{(\mathrm{I})}_{0,j} \cos\theta_j & \omega B^{(\mathrm{R})}_{0z,j} &= k_j E^{(\mathrm{R})}_{0,j} \cos\theta_j \end{aligned}$$

The relevant equations are

$$\begin{aligned} E^{(\mathrm{I})}_{0,j-1} e^{ik_{z,j-1}z_j} + E^{(\mathrm{R})}_{0,j-1} e^{-ik_{z,j-1}z_j} &= E^{(\mathrm{I})}_{0,j} e^{ik_{z,j}z_j} + E^{(\mathrm{R})}_{0,j} e^{-ik_{z,j}z_j} \\ k_{j-1}\sin\theta_{j-1}\left(E^{(\mathrm{I})}_{0,j-1} e^{ik_{z,j-1}z_j} + E^{(\mathrm{R})}_{0,j-1} e^{-ik_{z,j-1}z_j}\right) &= k_j \sin\theta_j \left(E^{(\mathrm{I})}_{0,j} e^{ik_{z,j}z_j} - E^{(\mathrm{R})}_{0,j} e^{-ik_{z,j}z_j}\right) \end{aligned}$$

$$\mathbf{W}_j(\mathbf{r}) = \begin{pmatrix} E_{0y,j}(\mathbf{r}) \\ \omega B_{0x,j}(\mathbf{r}) \end{pmatrix} = E_{0,j}(\mathbf{r}) \begin{pmatrix} 1 \\ k_j \sin\theta_j \end{pmatrix} \qquad \mathbf{E}_j = \begin{pmatrix} E^{(\mathrm{I})}_{0,j} \\ E^{(\mathrm{R})}_{0,j} \end{pmatrix} \qquad \mathbf{S}_j(z) = \begin{pmatrix} e^{ik_{z,j}z} & e^{-ik_{z,j}z} \\ k_{z,j} e^{ik_{z,j}z} & -k_{z,j} e^{-ik_{z,j}z} \end{pmatrix}$$

with

$$k_{z,j} \equiv k_j \sin\theta_j = k_j \alpha_j = \frac{2\pi n_j \alpha_j}{\lambda}$$

For X-Rays both polarisation can pe approximate by the later one.
Lets note that $\det\{\mathbf{S}_j\} = -2k_{z,j}$.
In both polarisation what we have at the interface is,

$$\mathbf{W}_{j-1}(\mathbf{r}_j, t) = \mathbf{W}_j(\mathbf{r}_j, t)$$

In case of $n_j < \cos\theta_j$, $k_{z,j}$ is imaginary. We define $\kappa_z = \mathcal{I}\{k_z\}$ and,

$$\mathbf{S}_j(z) = \begin{pmatrix} e^{ik_{z,j}z} & e^{-ik_{z,j}z} \\ k_{z,j}e^{ik_{z,j}z} & -k_{z,j}e^{-ik_{z,j}z} \end{pmatrix} \qquad \mathbf{S}_j(z) = \begin{pmatrix} e^{-\kappa_{z,j}z} & e^{\kappa_{z,j}z} \\ i\kappa_{z,j}e^{-\kappa_{z,j}z} & -i\kappa_{z,j}e^{\kappa_{z,j}z} \end{pmatrix}$$

$$k_{z,j} \to i\kappa_{z,j}$$

2.6 Interface transfer

We define the transfer matrix as,

$$\mathbf{E}_j = \mathbf{T}_j \cdot \mathbf{E}_{j-1}$$

We can find this matrix with the help of the interface condition,

$$\mathbf{W}_j(\mathbf{r}_j) = \mathbf{W}_{j-1}(\mathbf{r}_j)$$
$$\mathbf{S}_j(\mathbf{r}_j) \cdot \mathbf{E}_j = \mathbf{S}_{j-1}(\mathbf{r}_j) \cdot \mathbf{E}_{j-1}$$

so that,

$$\mathbf{T}_j = \mathbf{S}_j^{-1}(\mathbf{r}_j) \cdot \mathbf{S}_{j-1}(\mathbf{r}_j)$$

Let's note that $\det\{\mathbf{T}_j\} = \frac{k_{j-1}}{k_j}$.

$$\begin{aligned}\mathbf{T}_j &= \frac{1}{2k_{z,j}} \begin{pmatrix} k_{z,j}e^{-ik_{z,j}z_j} & e^{-ik_{z,j}z_j} \\ k_{z,j}e^{ik_{z,j}z_j} & -e^{ik_{z,j}z_j} \end{pmatrix} \begin{pmatrix} e^{ik_{z,j-1}z_j} & e^{-ik_{z,j-1}z_j} \\ k_{z,j-1}e^{ik_{z,j-1}z_j} & -k_{z,j-1}e^{-ik_{z,j-1}z_j} \end{pmatrix} \\ &= \frac{1}{2k_{z,j}} \begin{pmatrix} (k_{z,j} + k_{z,j-1})e^{-i(k_{z,j}-k_{z,j-1})z_j} & (k_{z,j} - k_{z,j-1})e^{-i(k_{z,j}+k_{z,j-1})z_j} \\ (k_{z,j} - k_{z,j-1})e^{i(k_{z,j}+k_{z,j-1})z_j} & (k_{z,j} + k_{z,j-1})e^{i(k_{z,j}-k_{z,j-1})z_j} \end{pmatrix}\end{aligned}$$

Evanescant solution

$$\mathbf{T}_j = \frac{1}{2\kappa_{z,j}} \begin{pmatrix} (\kappa_{z,j} + \kappa_{z,j-1})e^{(\kappa_{z,j}-\kappa_{z,j-1})z_j} & (\kappa_{z,j} - \kappa_{z,j-1})e^{(\kappa_{z,j}+\kappa_{z,j-1})z_j} \\ (\kappa_{z,j} - \kappa_{z,j-1})e^{-(\kappa_{z,j}+\kappa_{z,j-1})z_j} & (\kappa_{z,j} + \kappa_{z,j-1})e^{-(\kappa_{z,j}-\kappa_{z,j-1})z_j} \end{pmatrix}$$

To go from the first interface to the last we just need to apply the matrices in sequence.

$$\mathbf{E}_N = \mathbf{L} \cdot \mathbf{E}_0 \qquad \mathbf{L} = \mathbf{T}_N \cdot \mathbf{T}_{N-1} \cdots \mathbf{T}_2 \cdot \mathbf{T}_1 = \prod_{n=1}^{N} \mathbf{T}_n$$

$$\begin{pmatrix} t \\ 0 \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ r \end{pmatrix}$$

$$r = -\frac{L_{21}}{L_{22}} \qquad t = L_{11} - \frac{L_{12}L_{21}}{L_{22}} = \frac{\det\{\mathbf{L}\}}{L_{22}}$$

Let's note that $\det\{\mathbf{L}\} = \prod_{j=1}^{N} \frac{k_{j-1}}{k_j} = \frac{k_0}{k_N}$ if $k_j > 0$ and 0 otherwise.

2.7 Roughness

$$\begin{aligned}\mathbf{L} &= \int \mathrm{d}z\, p_j(z)\mathbf{T}_N \cdots \mathbf{T}_j(z) \cdots \cdot \mathbf{T}_1 \\ &= \mathbf{T}_N \cdots \left(\int \mathrm{d}z\, p_j(z)\mathbf{T}_j(z)\right) \cdots \cdot \mathbf{T}_1\end{aligned}$$

$$p_j(z) = \frac{1}{\sigma_j\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{z-z_j}{\sigma_j}\right)^2\right\}$$

$$\int \mathrm{d}z\, p_j(z) e^{i\left(k_{z,j}\pm k_{z,j-1}\right)z} = e^{i\left(k_{z,j}\pm k_{z,j-1}\right)z_j} e^{-\frac{1}{2}\left(k_{z,j}\pm k_{z,j-1}\right)^2\sigma_j^2}$$

$$\begin{aligned}&\int \mathrm{d}z\, p_j(z)\mathbf{T}_j(z) \\ &= \frac{1}{2k_{z,j}}\begin{pmatrix}\left(k_{z,j}+k_{z,j-1}\right)e^{-i\left(k_{z,j}-k_{z,j-1}\right)z_j}e^{-\frac{1}{2}\left(k_{z,j}-k_{z,j-1}\right)^2\sigma_j^2} & \left(k_{z,j}-k_{z,j-1}\right)e^{-i\left(k_{z,j}+k_{z,j-1}\right)z_j}e^{-\frac{1}{2}\left(k_{z,j}+k_{z,j-1}\right)^2\sigma_j^2} \\ \left(k_{z,j}-k_{z,j-1}\right)e^{i\left(k_{z,j}+k_{z,j-1}\right)z_j}e^{-\frac{1}{2}\left(k_{z,j}+k_{z,j-1}\right)^2\sigma_j^2} & \left(k_{z,j}+k_{z,j-1}\right)e^{i\left(k_{z,j}-k_{z,j-1}\right)z_j}e^{-\frac{1}{2}\left(k_{z,j}-k_{z,j-1}\right)^2\sigma_j^2}\end{pmatrix}\end{aligned}$$

$$\int \mathrm{d}z\, p_j(z)\mathbf{T}_j(z) \approx \frac{1}{2k_{z,j}}\begin{pmatrix} 2k_{z,j}e^{-i\left(k_{z,j}-k_{z,j-1}\right)z_j} & \left(k_{z,j}-k_{z,j-1}\right)e^{-i\left(k_{z,j}+k_{z,j-1}\right)z_j}e^{-\frac{1}{2}\left(k_{z,j}+k_{z,j-1}\right)^2\sigma_j^2} \\ \left(k_{z,j}-k_{z,j-1}\right)e^{i\left(k_{z,j}+k_{z,j-1}\right)z_j}e^{-\frac{1}{2}\left(k_{z,j}+k_{z,j-1}\right)^2\sigma_j^2} & \left(k_{z,j}+k_{z,j-1}\right)e^{i\left(k_{z,j}-k_{z,j-1}\right)z_j}e^{-\frac{1}{2}\left(k_{z,j}-k_{z,j-1}\right)^2\sigma_j^2}\end{pmatrix}$$

2.8 Slope

$$\mathbf{T}_j = \frac{1}{2k_{z,j}}\begin{pmatrix}\left(k_{z,j}+k_{z,j-1}\right)e^{-i\left(k_{z,j}-k_{z,j-1}\right)z_j} & \left(k_{z,j}-k_{z,j-1}\right)e^{-i\left(k_{z,j}+k_{z,j-1}\right)z_j} \\ \left(k_{z,j}-k_{z,j-1}\right)e^{i\left(k_{z,j}+k_{z,j-1}\right)z_j} & \left(k_{z,j}+k_{z,j-1}\right)e^{i\left(k_{z,j}-k_{z,j-1}\right)z_j}\end{pmatrix}$$

$$\begin{aligned}k_{z,j}+k_{z,j-1} &\approx 2k_{z,j} \\ k_{z,j}-k_{z,j-1} &\approx \frac{\mathrm{d}k_z}{\mathrm{d}z}\Delta z_{j,j-1}\end{aligned}$$

$$\mathbf{T}_j = \frac{1}{2k_{z,j}}\begin{pmatrix}2k_{z,j}e^{-i\frac{\mathrm{d}k_z}{\mathrm{d}z}\Delta z_{j,j-1}z_j} & \frac{\mathrm{d}k_z}{\mathrm{d}z}\Delta z_{j,j-1}e^{-i2k_{z,j}z_j} \\ \frac{\mathrm{d}k_z}{\mathrm{d}z}\Delta z_{j,j-1}e^{i2k_{z,j}z_j} & 2k_{z,j}e^{i\frac{\mathrm{d}k_z}{\mathrm{d}z}\Delta z_{j,j-1}z_j}\end{pmatrix}$$

$$\mathbf{T}_j \approx \frac{1}{2k_{z,j}}\begin{pmatrix}2k_{z,j}\left(1-i\frac{\mathrm{d}k_z}{\mathrm{d}z}\Delta z_{j,j-1}z_j+\ldots\right) & \frac{\mathrm{d}k_z}{\mathrm{d}z}\Delta z_{j,j-1}e^{-i2k_{z,j}z_j} \\ \frac{\mathrm{d}k_z}{\mathrm{d}z}\Delta z_{j,j-1}e^{i2k_{z,j}z_j} & 2k_{z,j}\left(1+i\frac{\mathrm{d}k_z}{\mathrm{d}z}\Delta z_{j,j-1}z_j+\ldots\right)\end{pmatrix}$$

$$\mathbf{T}_j \approx \mathbb{1} + \frac{\mathrm{d}k_z}{\mathrm{d}z}\Delta z_{j,j-1} z_j \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} + \frac{1}{2k_{z,j}}\frac{\mathrm{d}k_z}{\mathrm{d}z}\Delta z_{j,j-1} \begin{pmatrix} 0 & e^{-i2k_{z,j}z_j} \\ e^{i2k_{z,j}z_j} & 0 \end{pmatrix}$$

$$\mathbf{T}_j \approx \mathbb{1} + \Delta k_{z,j,j-1} z_j \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} + \frac{\Delta k_{z,j,j-1}}{2k_{z,j}} \begin{pmatrix} 0 & e^{-i2k_{z,j}z_j} \\ e^{i2k_{z,j}z_j} & 0 \end{pmatrix}$$

$$\mathbf{T}_j \approx 2k_{z,j}\left(\mathbb{1} + \Delta k_{z,j,j-1} z_j \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right) + \Delta k_{z,j,j-1} \begin{pmatrix} 0 & e^{-i2k_{z,j}z_j} \\ e^{i2k_{z,j}z_j} & 0 \end{pmatrix}$$

$$\prod_{j=0}^{N}\mathbf{T}_j = \mathbb{1} + \sum_{j=0}^{N}\frac{1}{2k_{z,j}}\frac{\mathrm{d}k_z}{\mathrm{d}z}\Delta z_{j,j-1} \begin{pmatrix} 0 & e^{-i2k_{z,j}z_j} \\ e^{i2k_{z,j}z_j} & 0 \end{pmatrix} + \mathcal{O}(\Delta z^2)$$

$$\prod_{j=0}^{N}\mathbf{T}_j = \mathbb{1} + \int_{\text{top}}^{\text{bottom}} \mathrm{d}z\, \frac{1}{2k_z(z)}\frac{\mathrm{d}k_z}{\mathrm{d}z} \begin{pmatrix} 0 & e^{-i2k_z(z)z} \\ e^{i2k_z(z)z} & 0 \end{pmatrix}$$

$$\prod_{j=0}^{N}\mathbf{T}_j = \mathbb{1} + \int \mathrm{d}z\, \frac{1}{2k_z(z)}\frac{\mathrm{d}k_z}{\mathrm{d}z} \begin{pmatrix} 0 & e^{-i2k_z(z)z} \\ e^{i2k_z(z)z} & 0 \end{pmatrix}$$

$$\prod_{j=0}^{N}\mathbf{T}_j = \mathbb{1} + \int \mathrm{d}z\, \frac{1}{2k_z(z)}\frac{\mathrm{d}k_z}{\mathrm{d}z} \begin{pmatrix} 0 & e^{-i2k_z(z)z} \\ e^{i2k_z(z)z} & 0 \end{pmatrix}$$

$$\prod_{j=0}^{N}\mathbf{T}_j = \mathbb{1} + \int_{k_{z,\text{top}}}^{k_{z,\text{bottom}}} \frac{\mathrm{d}k_z}{2k_z} \begin{pmatrix} 0 & e^{-i2k_z z(k_z)} \\ e^{i2k_z z(k_z)} & 0 \end{pmatrix}$$

2.9 Layer Transfer

We define the transfer matrix as,

$$\mathbf{W}_j(z_{j+1}) = \mathbf{M}_j \cdot \mathbf{W}_j(z_j)$$

We can find this matrix with the help $\mathbf{E}$

$$\mathbf{S}_j(z_{j+1}) \cdot \mathbf{E}_j = \mathbf{M}_j \cdot \mathbf{S}_j(z_j) \cdot \mathbf{E}_j$$

So that,

$$\mathbf{M}_j = \mathbf{S}_j(z_{j+1}) \cdot \mathbf{S}_j^{-1}(z_j)$$

$$
\begin{aligned}
\mathbf{M}_j &= \begin{pmatrix} e^{ik_{z,j}z_{j+1}} & e^{-ik_{z,j}z_{j+1}} \\ k_{z,j}e^{ik_{z,j}z_{j+1}} & -k_{z,j}e^{-ik_{z,j}z_{j+1}} \end{pmatrix} \frac{1}{2k_{z,j}} \begin{pmatrix} k_{z,j}e^{-ik_{z,j}z_j} & e^{-ik_{z,j}z_j} \\ k_{z,j}e^{ik_{z,j}z_j} & -e^{ik_{z,j}z_j} \end{pmatrix} \\
&= \frac{1}{2k_{z,j}} \begin{pmatrix} \cos\left(k_{z,j}d_j\right) & ik_{z,j}^{-1}\sin\left(k_{z,j}d_j\right) \\ ik_{z,j}\sin\left(k_{z,j}d_j\right) & \cos\left(k_{z,j}d_j\right) \end{pmatrix}
\end{aligned}
$$

To go from the first interface to the last we just need to apply the matrices in sequence.

$$
\mathbf{L} = \mathbf{S}_N(z_N)^{-1} \cdot \mathbf{M}_{N-1} \cdot \mathbf{M}_{N-2} \cdots \mathbf{M}_2 \cdot \mathbf{M}_1 \cdot \mathbf{S}_0(z_1)
$$

$$
r = -e^{i2k_z(a)a} \frac{k_z(b) - k_z(a) + (b-a)k_z(a)k_z(b) - \int_a^b \mathrm{d}z\, k_z^2(z)}{k_z(b) + k_z(a) - (b-a)k_z(a)k_z(b) - \int_a^b \mathrm{d}z\, k_z^2(z)}
$$